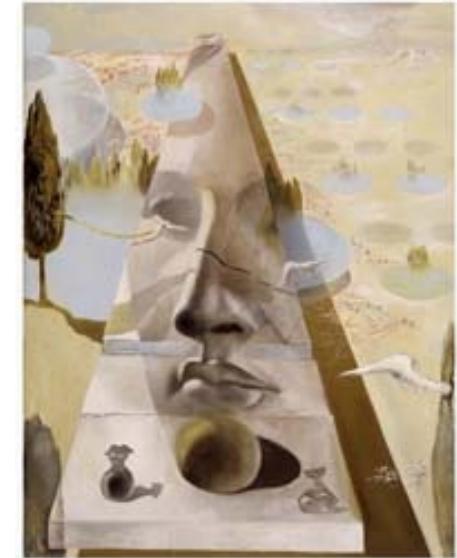


# Lecture 8

# Fitting and Matching

- Problem formulation
- Least square methods
- RANSAC
- Hough transforms
- Multi-model fitting
- Fitting helps matching!



#### Reading:

- [HZ] Chapter: 4 "Estimation – 2D projective transformation"  
Chapter: 11 "Computation of the fundamental matrix F"  
[FP] Chapter:10 "Grouping and model fitting"

Some slides of this lectures are courtesy of profs. S. Lazebnik & K. Grauman

# Fitting

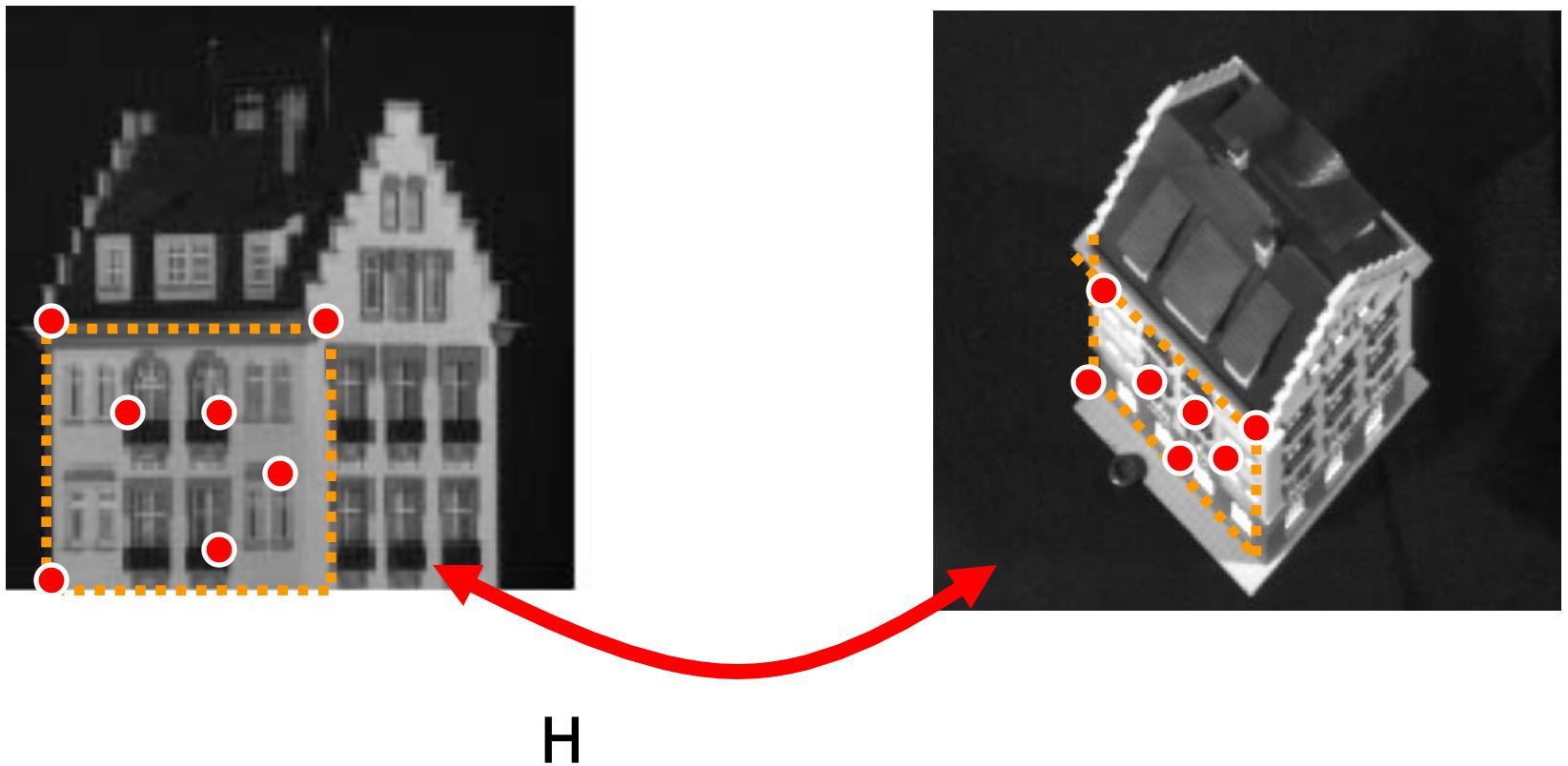
## Goals:

- Choose a parametric model to fit a certain quantity from data
  - Estimate model parameters
- 
- Lines
  - Curves
  - Homographic transformations
  - Fundamental matrices
  - Shape models

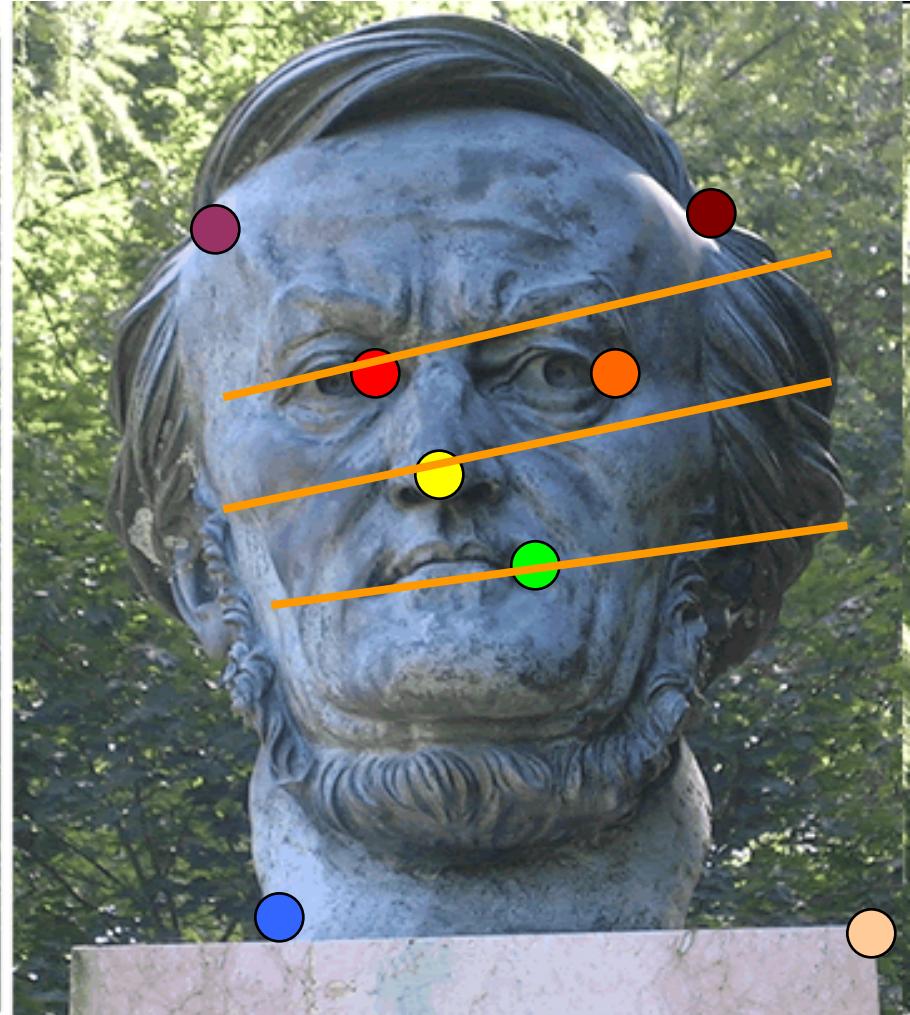
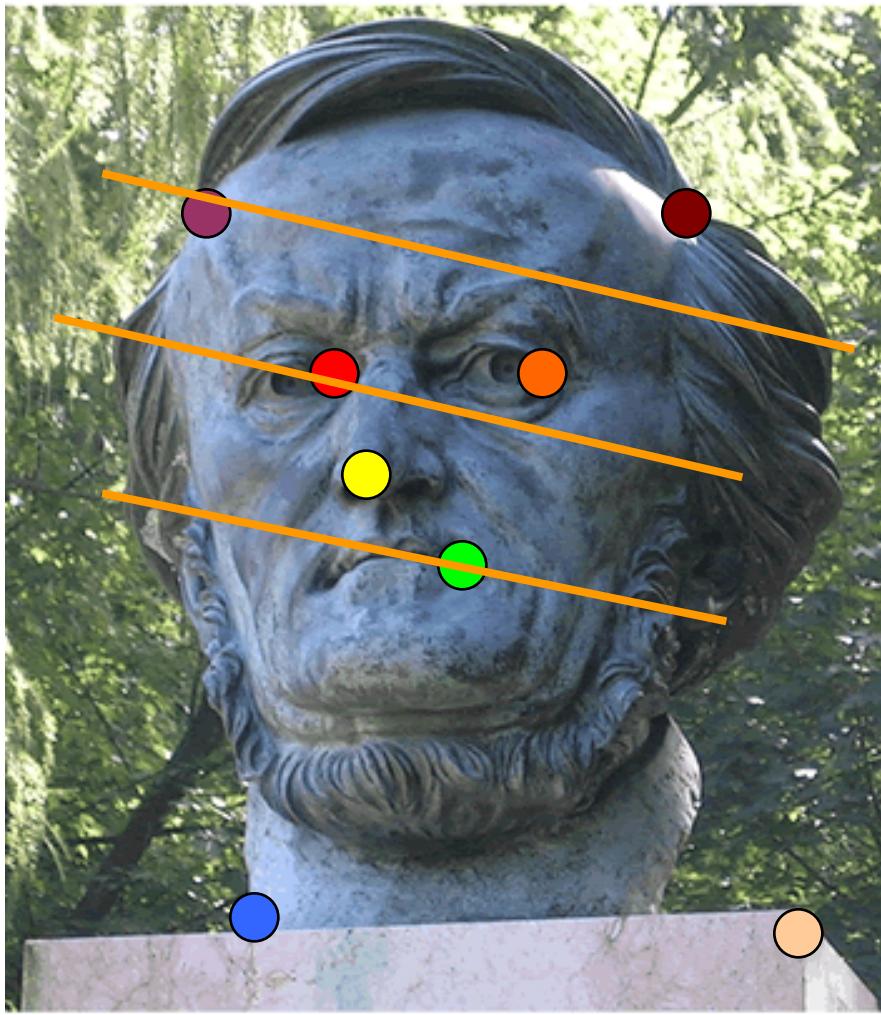
# Example: fitting lines (for computing vanishing points)



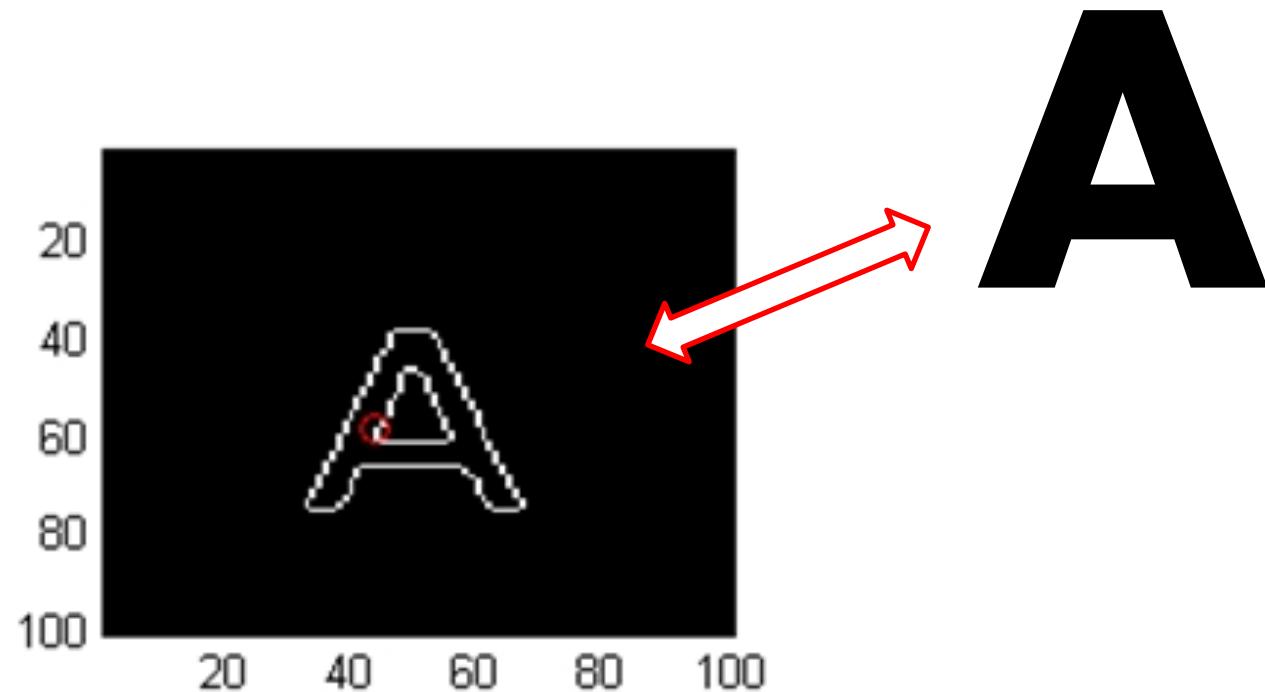
# Example: Estimating an homographic transformation



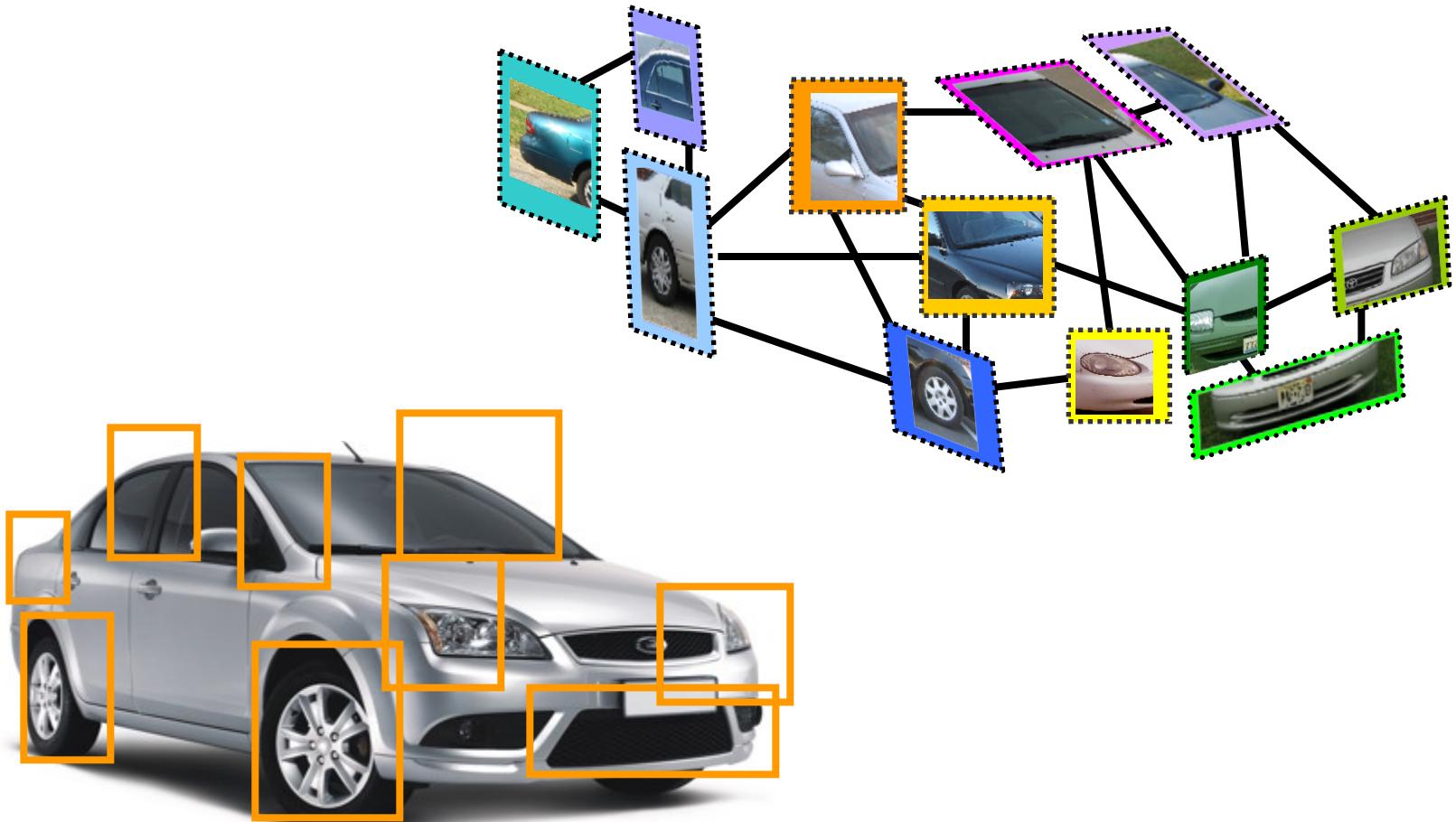
# Example: Estimating F



# Example: fitting a 2D shape template



# Example: fitting a 3D object model



Fitting, matching and recognition  
are interconnected problems

# Fitting

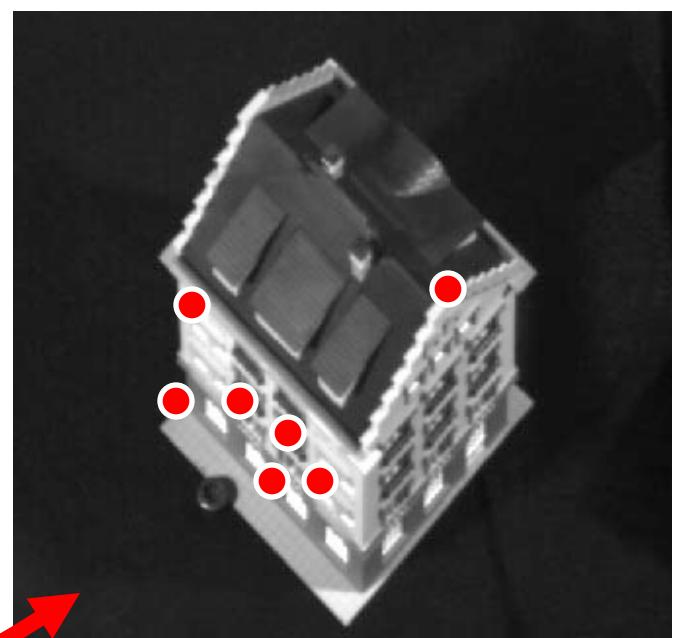
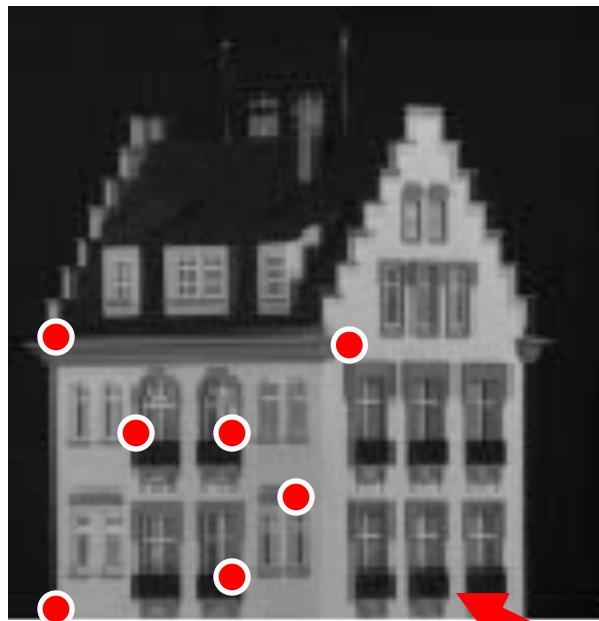
## Critical issues:

- noisy data
- outliers
- missing data
- Intra-class variation

# Critical issues: noisy data

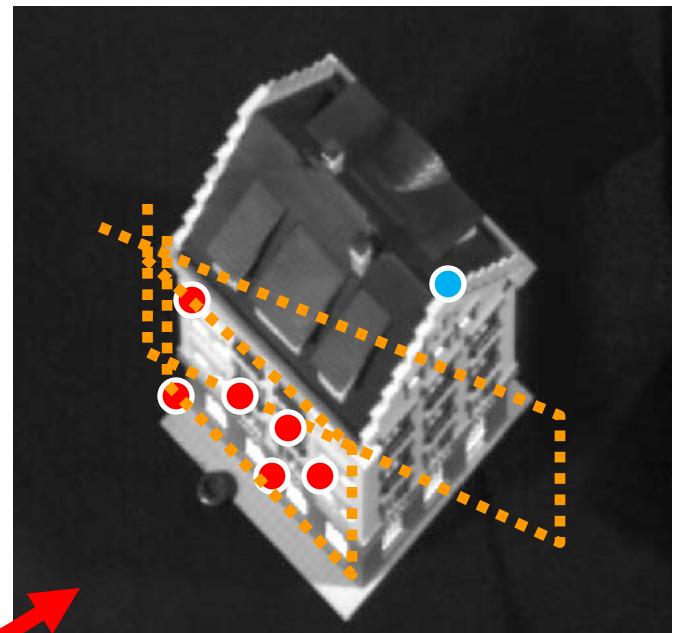
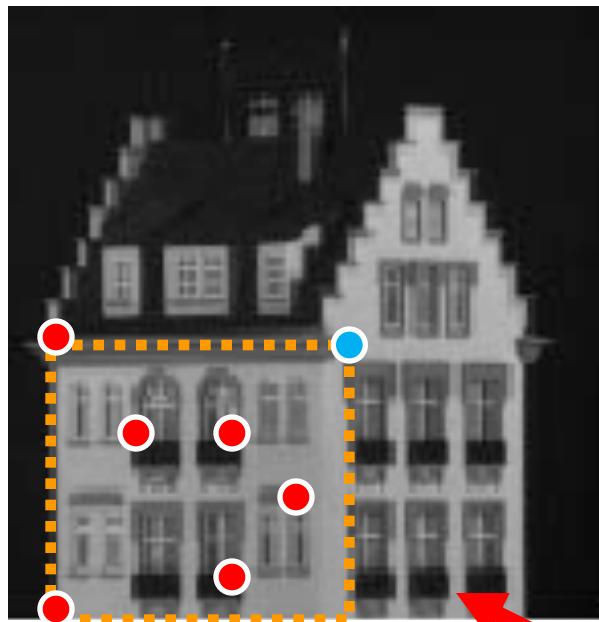


# Critical issues: outliers



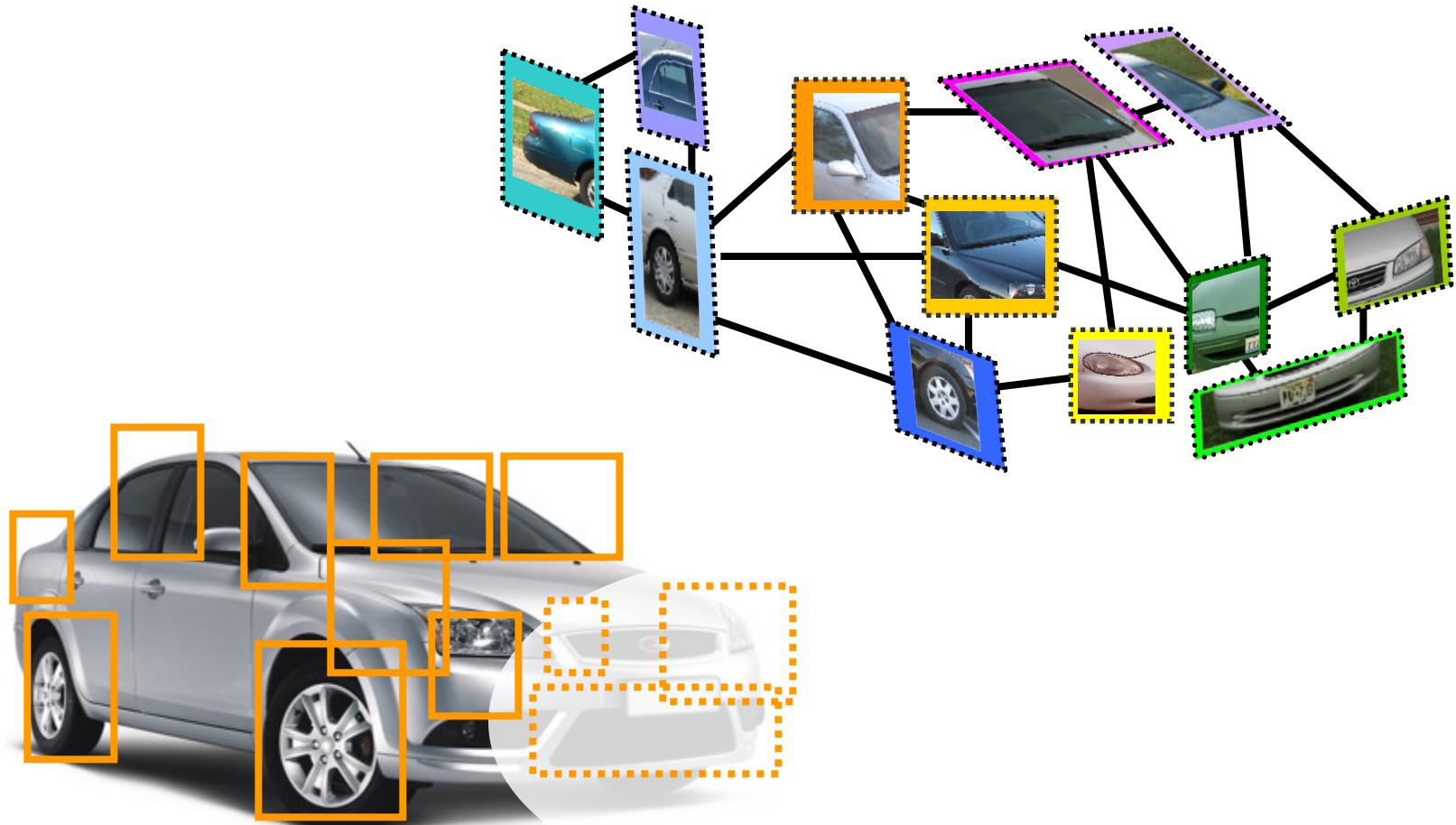
What's H?

# Critical issues: outliers

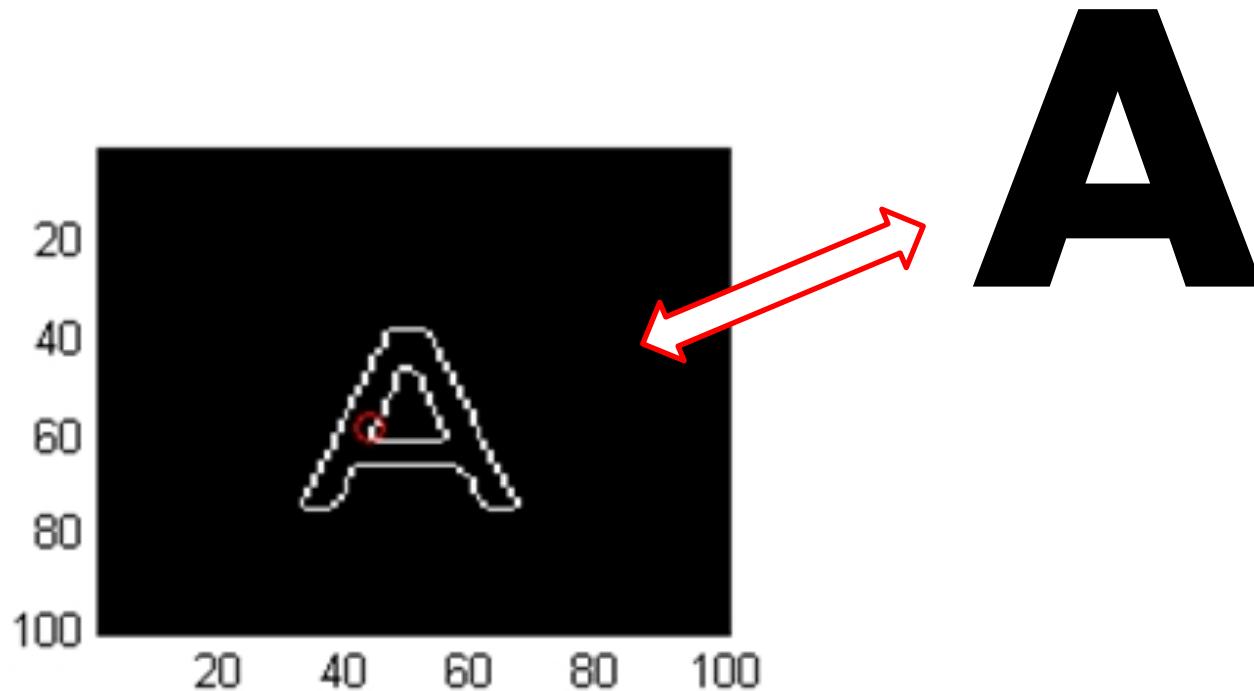


What's H? ☹

# Critical issues: missing data (occlusions)



# Critical issues: noisy data (intra-class variability)



# Fitting

**Goal:** Choose a parametric model to fit a certain quantity from data

## Techniques:

- Least square methods
- RANSAC
- Hough transform
- EM (Expectation Maximization) [not covered]

# Least squares methods

## - fitting a line -

- Data:  $(x_1, y_1), \dots, (x_n, y_n)$

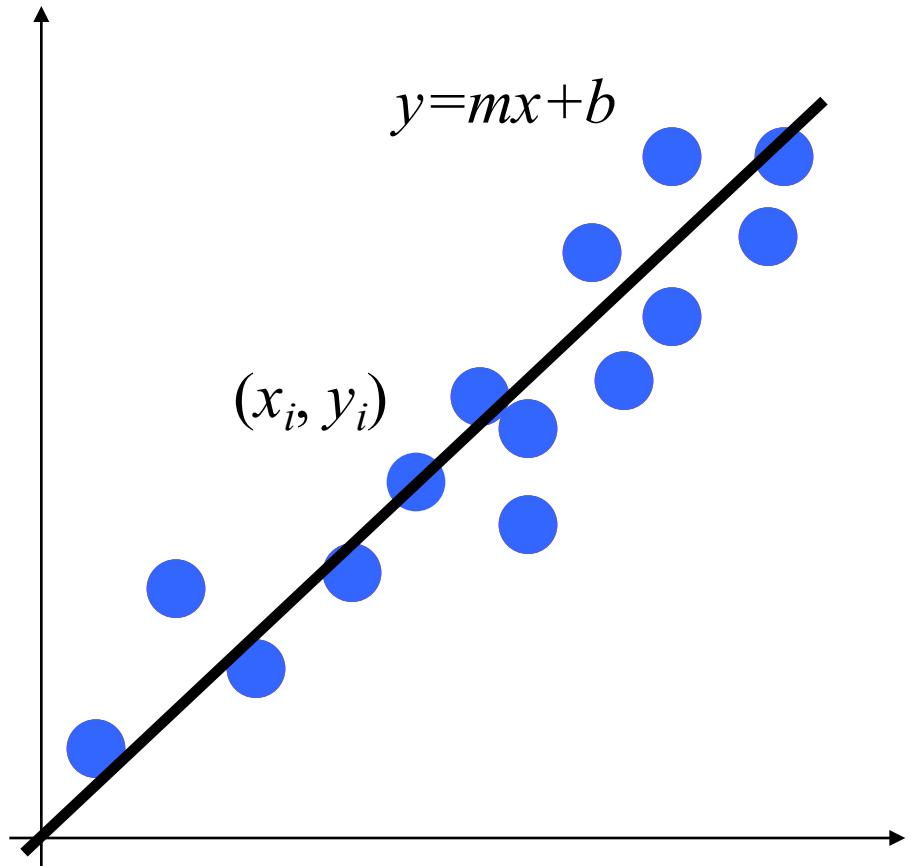
- Model of the line:

$$y_i - mx_i - b = 0 \quad [\text{Eq. 1}]$$

- Parameters:  $m, b$

- Find  $(m, b)$  to minimize fitting error (**residual**):

$$E = \sum_{i=1}^n (y_i - mx_i - b)^2 \quad [\text{Eq. 2}]$$



# Least squares methods

- fitting a line -

$$E = \sum_{i=1}^n (y_i - mx_i - b)^2 \quad [\text{Eq. 2}]$$

$$E = \sum_{i=1}^n \left( y_i - \begin{bmatrix} x_i & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} \right)^2 = \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} \right\|^2 = \|Y - Xh\|^2 \quad [\text{Eq. 3}]$$

$$= (Y - Xh)^T (Y - Xh) = Y^T Y - 2(Xh)^T Y + (Xh)^T (Xh) \quad [\text{Eq. 4}]$$

Find  $h = [m, b]^T$  that minimizes E

$$\frac{dE}{dh} = -2X^T Y + 2X^T Xh = 0 \quad [\text{Eq. 5}]$$

$$X^T Xh = X^T Y \quad [\text{Eq. 7}]$$

Normal equation

$$h = (X^T X)^{-1} X^T Y \quad [\text{Eq. 6}]$$

# Least squares methods

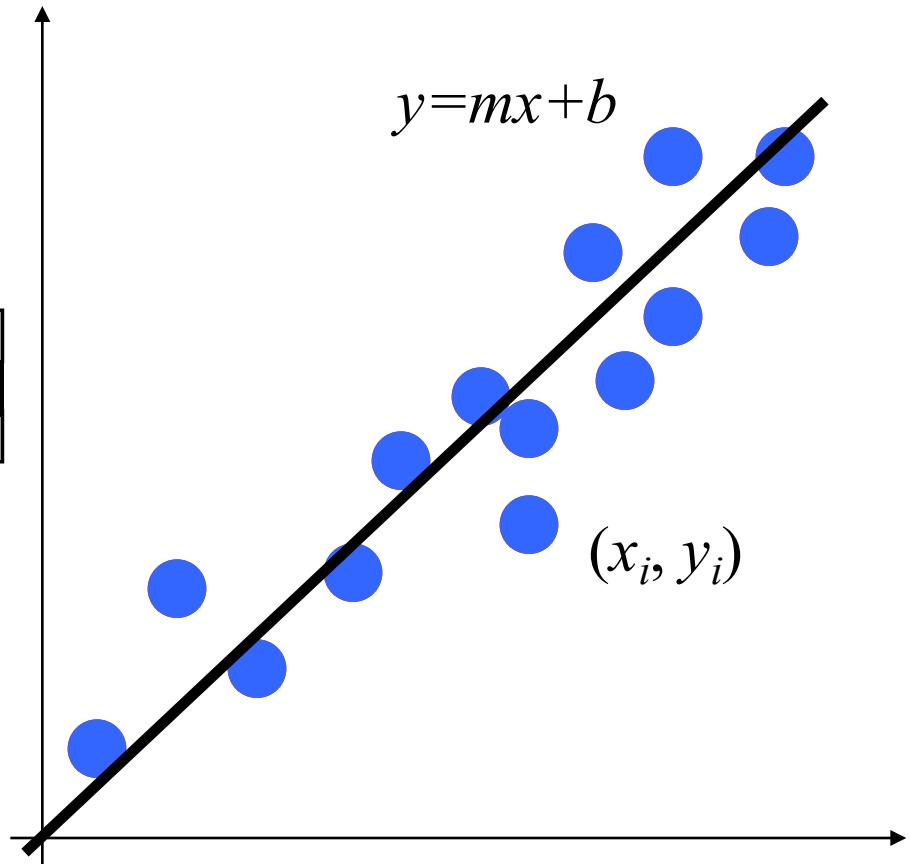
- fitting a line -

$$E = \sum_{i=1}^n (y_i - mx_i - b)^2$$

$$h = (X^T X)^{-1} X^T Y$$

[Eq. 6]

$$h = \begin{bmatrix} m \\ b \end{bmatrix}$$



Issues?

- Fails completely for vertical lines

# Least squares methods

## - fitting a line -

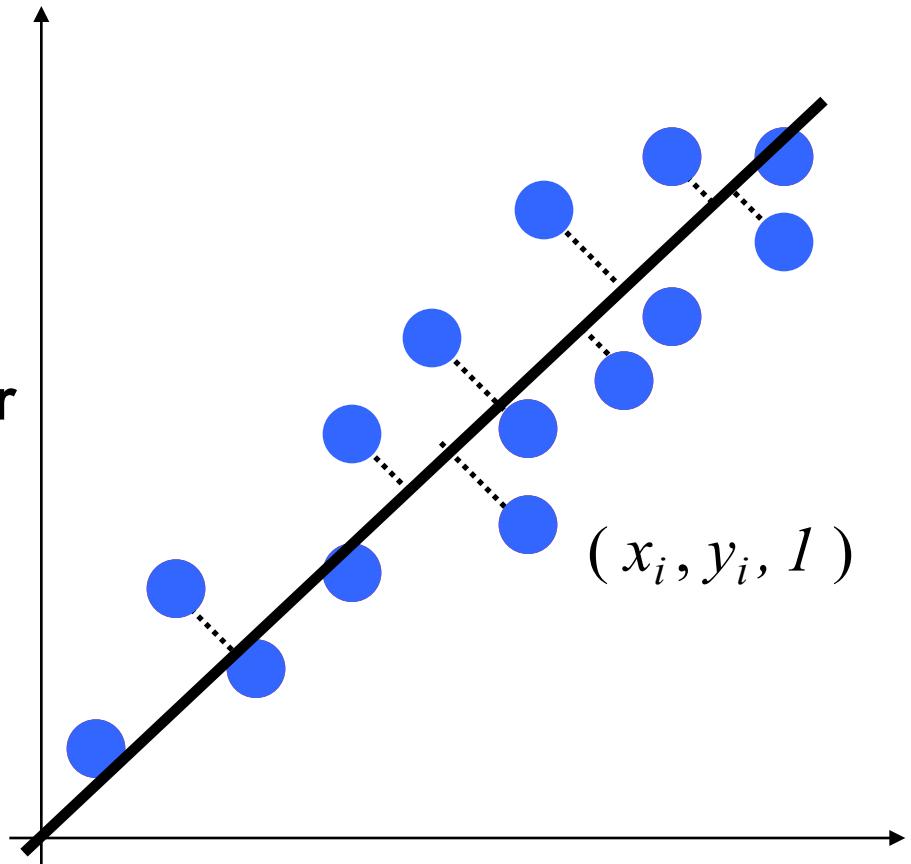
- Distance between point  $(x_i, y_i, 1)$  and line  $(a, b, d)$
- Find  $(a, b, d)$  to minimize the sum of squared perpendicular distances

[Eq. 8]

$$E = \sum_{i=1}^n (ax_i + by_i + d)^2$$

$$\boxed{A} \boxed{h} = 0 \quad [\text{Eq. 9}]$$

data    model parameters



# Least squares methods

- fitting a line -

$A h = 0$      $A$  is rank deficient

Minimize  $\| A h \|$     subject to     $\| h \| = 1$

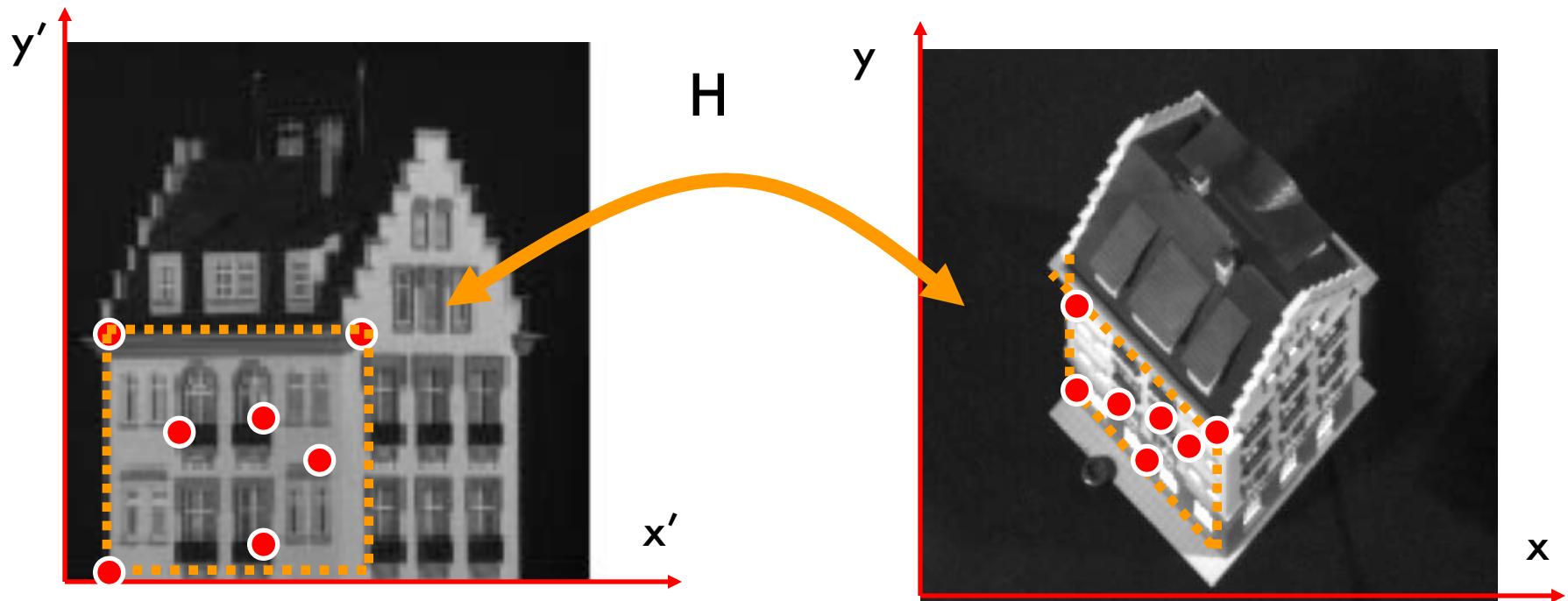
$$A = UDV^T$$

$h$  = last column of  $V$

See [HZ], sec. A5.3 - page 593

# Least squares methods

## - fitting an homography -



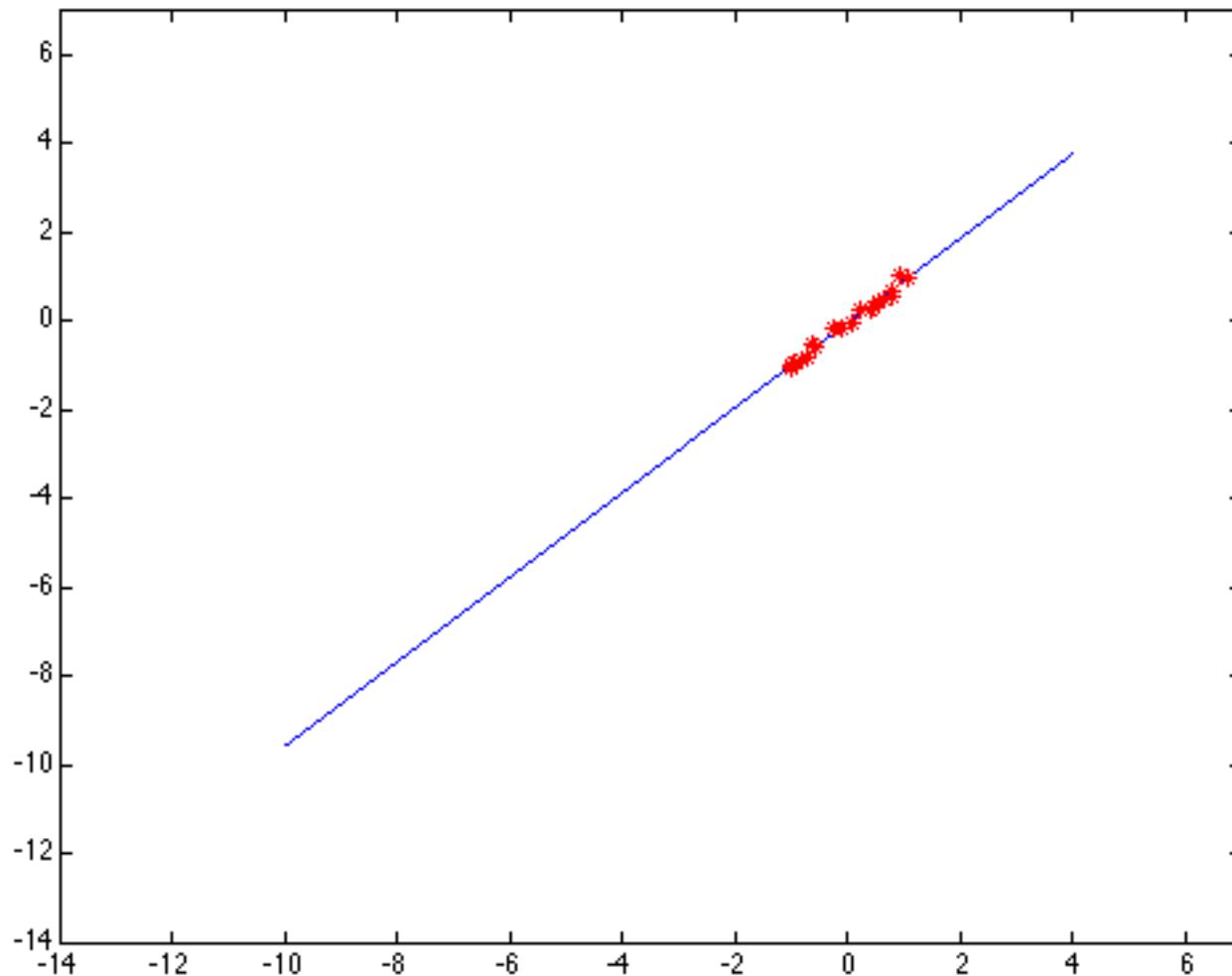
$$\boxed{A} \boxed{h} = 0 \quad [\text{Eq. 10}]$$

data    model parameters

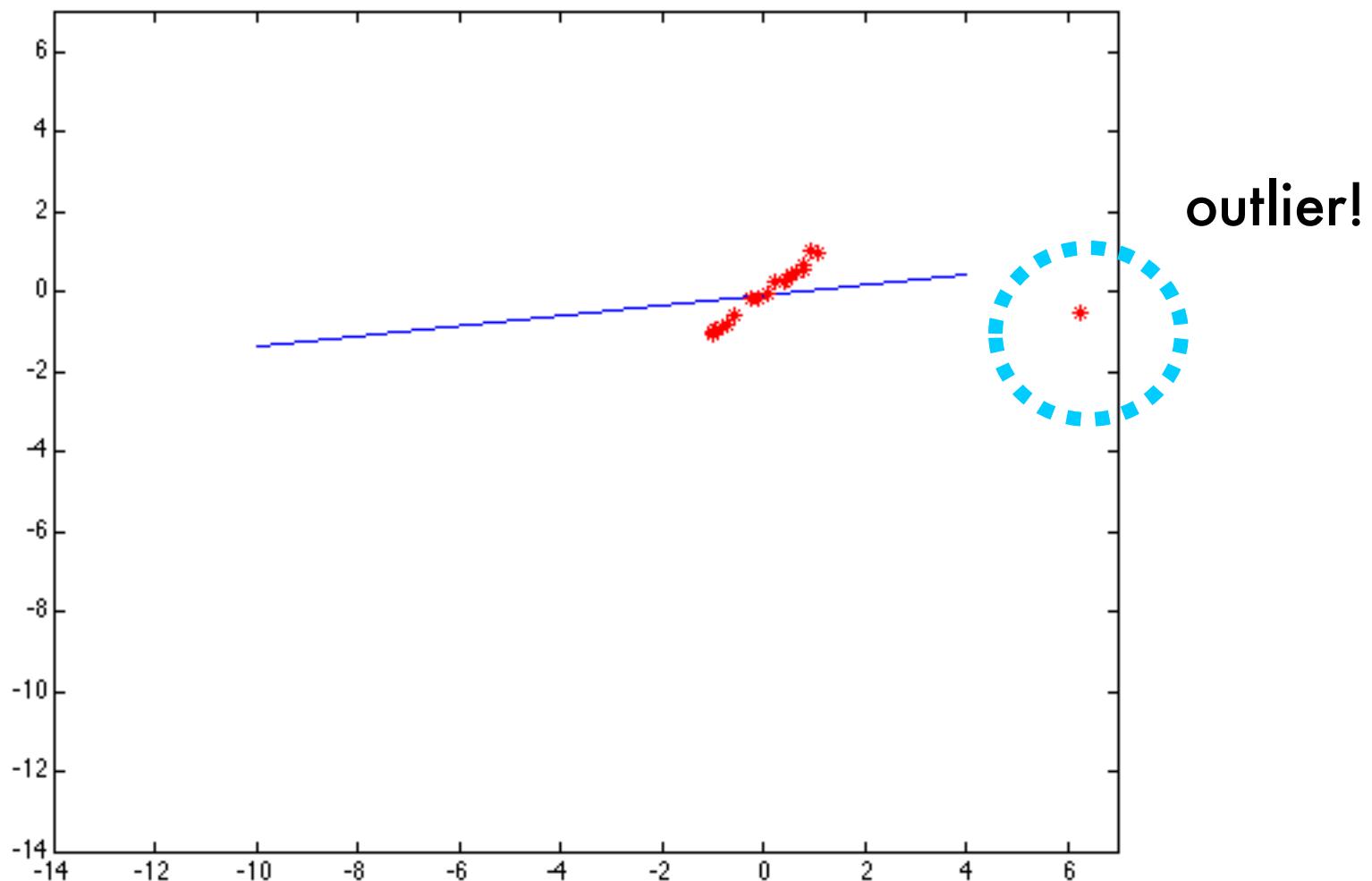
See HZ

- Sec 4.1 for details (DLT algorithm)
- Sec 4.1.2 (or APPENDIX)

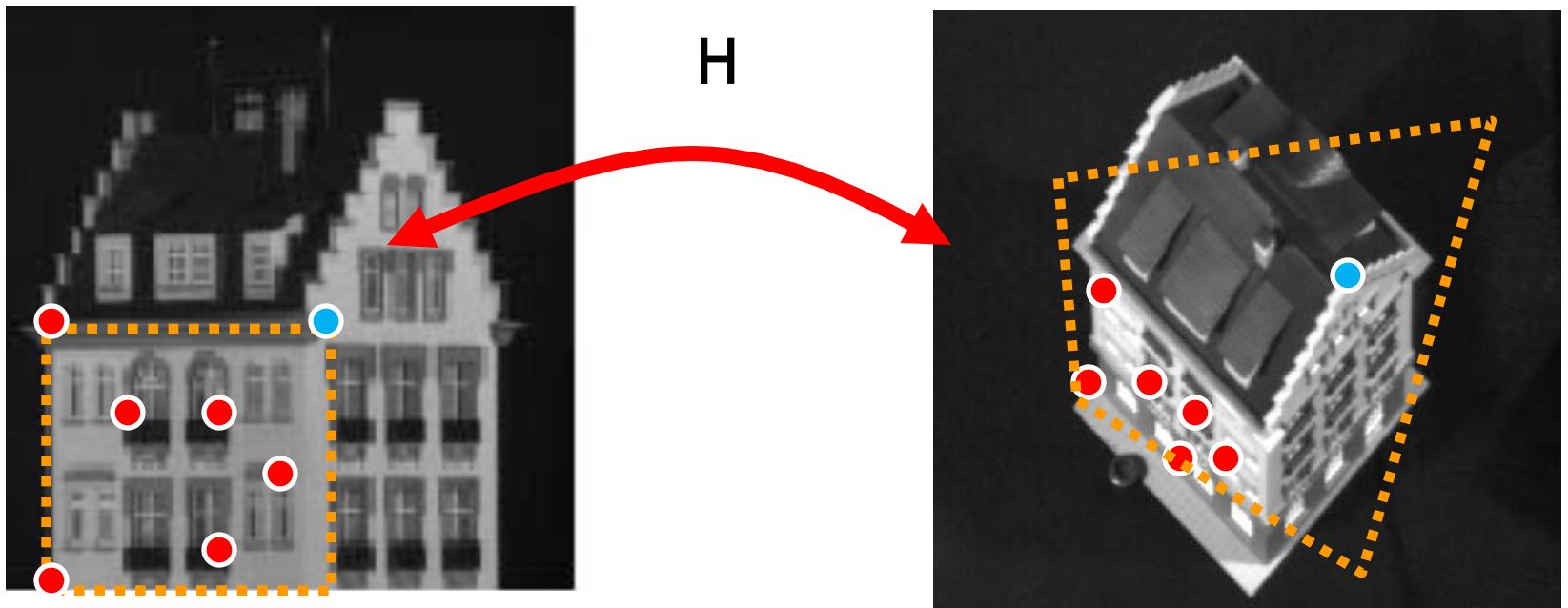
# Least squares: Robustness to noise



# Least squares: Robustness to noise



# Critical issues: outliers



**CONCLUSION:** Least square is not robust w.r.t. outliers

# Least squares: Robust estimators

Instead of minimizing  $E = \sum_{i=1}^n (ax_i + by_i - d)^2$  [Eq. 8]

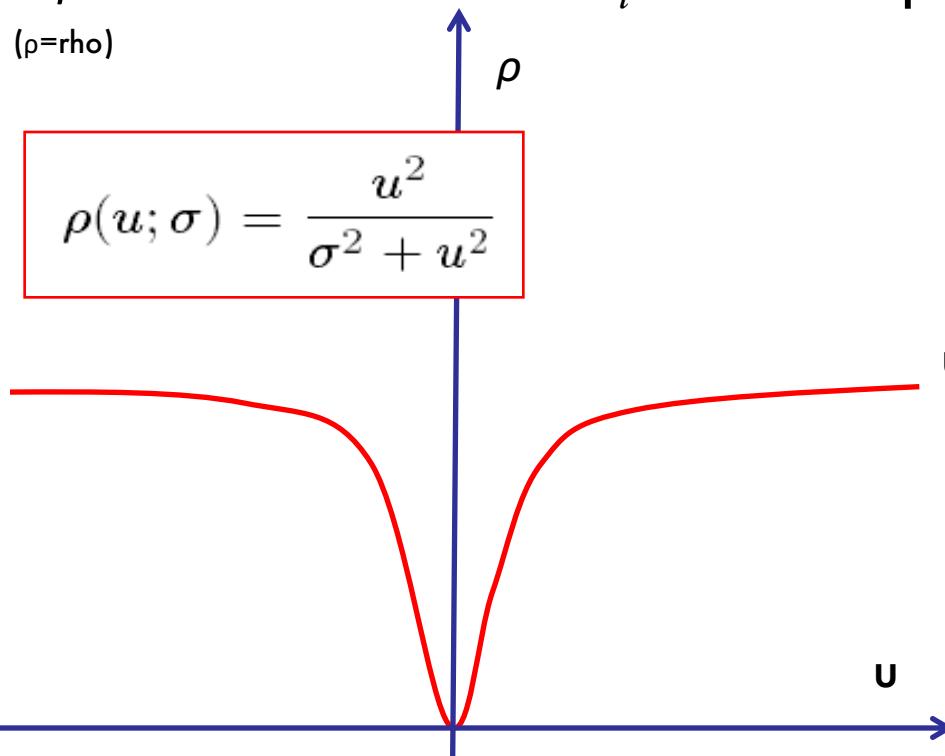
We minimize

$$E = \sum_i \rho(u_i; \sigma) \quad [\text{Eq. 11}]$$

$$u_i = ax_i + by_i - d$$

- $u_i$  = error (residual) of  $i^{\text{th}}$  point w.r.t. model parameters  $h = (a, b, d)$
- $\rho$  = robust function of  $u_i$  with scale parameter  $\sigma$

[Eq.]



**Robust function  $\rho$ :**

- When  $u$  is large,  $\rho$  saturates to 1
- When  $u$  is small,  $\rho$  is a function of  $u^2$

In conclusion:

- Favors a configuration with small residuals
- Penalizes large residuals

# Least squares: Robust estimators

Instead of minimizing  $E = \sum_{i=1}^n (ax_i + by_i - d)^2$  [Eq. 8]

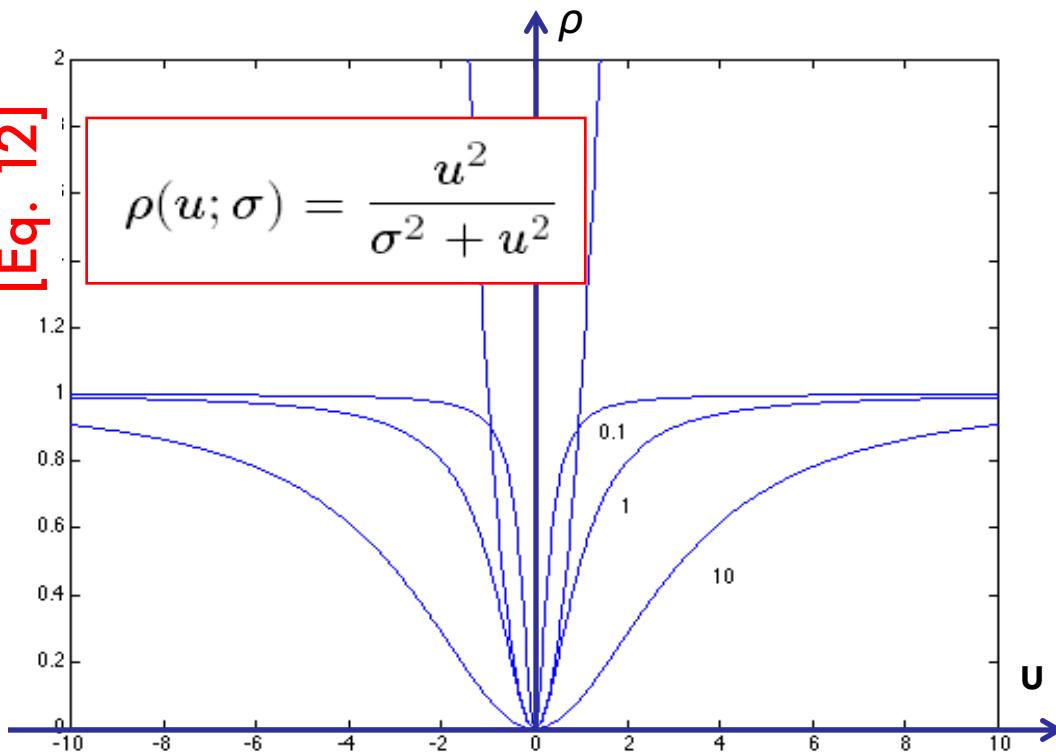
We minimize

$$E = \sum_i \rho(u_i; \sigma) \quad [\text{Eq. 11}]$$

$$u_i = ax_i + by_i - d$$

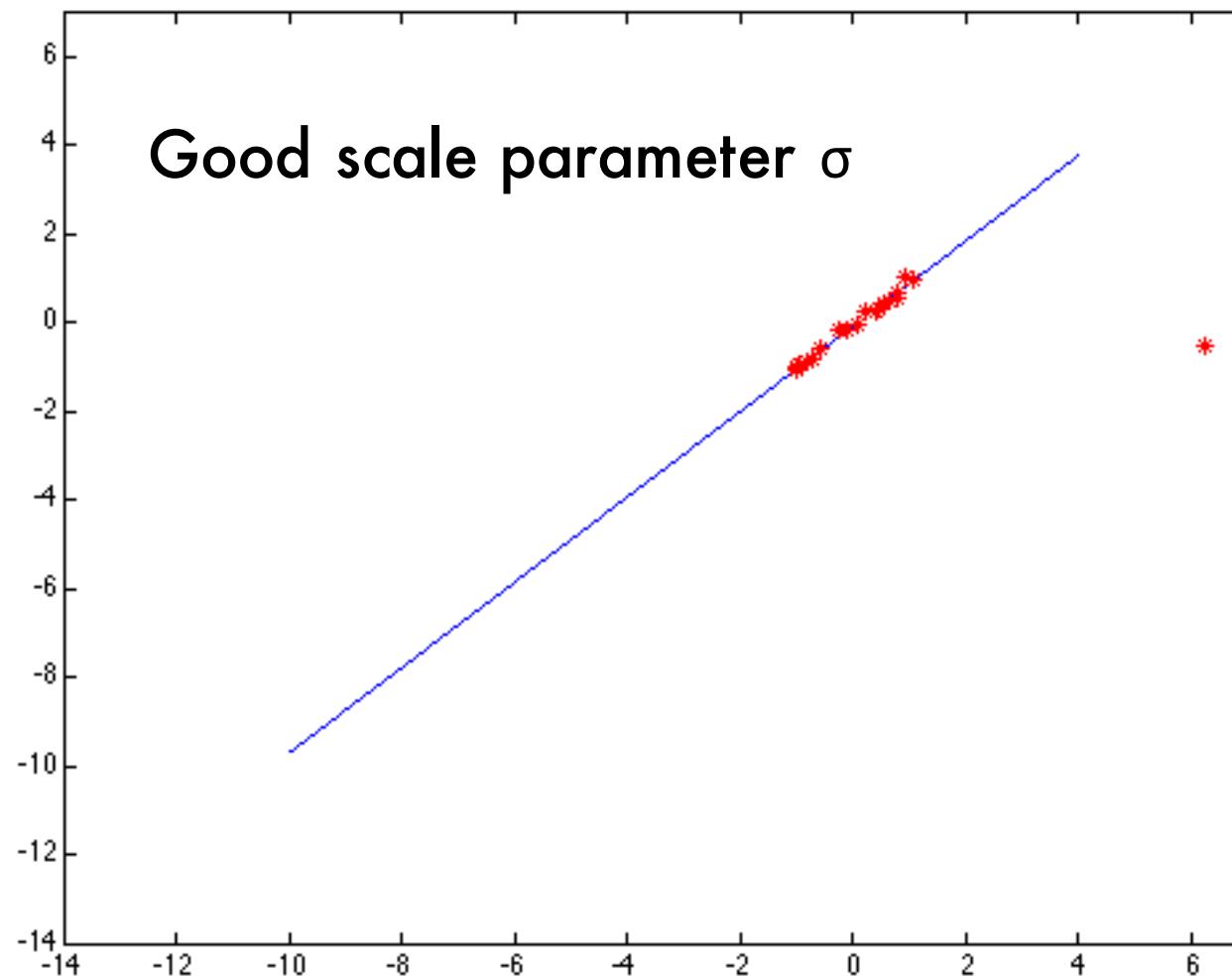
- $u_i$  = error (residual) of  $i^{\text{th}}$  point w.r.t. model parameters  $h = (a, b, d)$
- $\rho$  = robust function of  $u_i$  with scale parameter  $\sigma$

[Eq. 12]



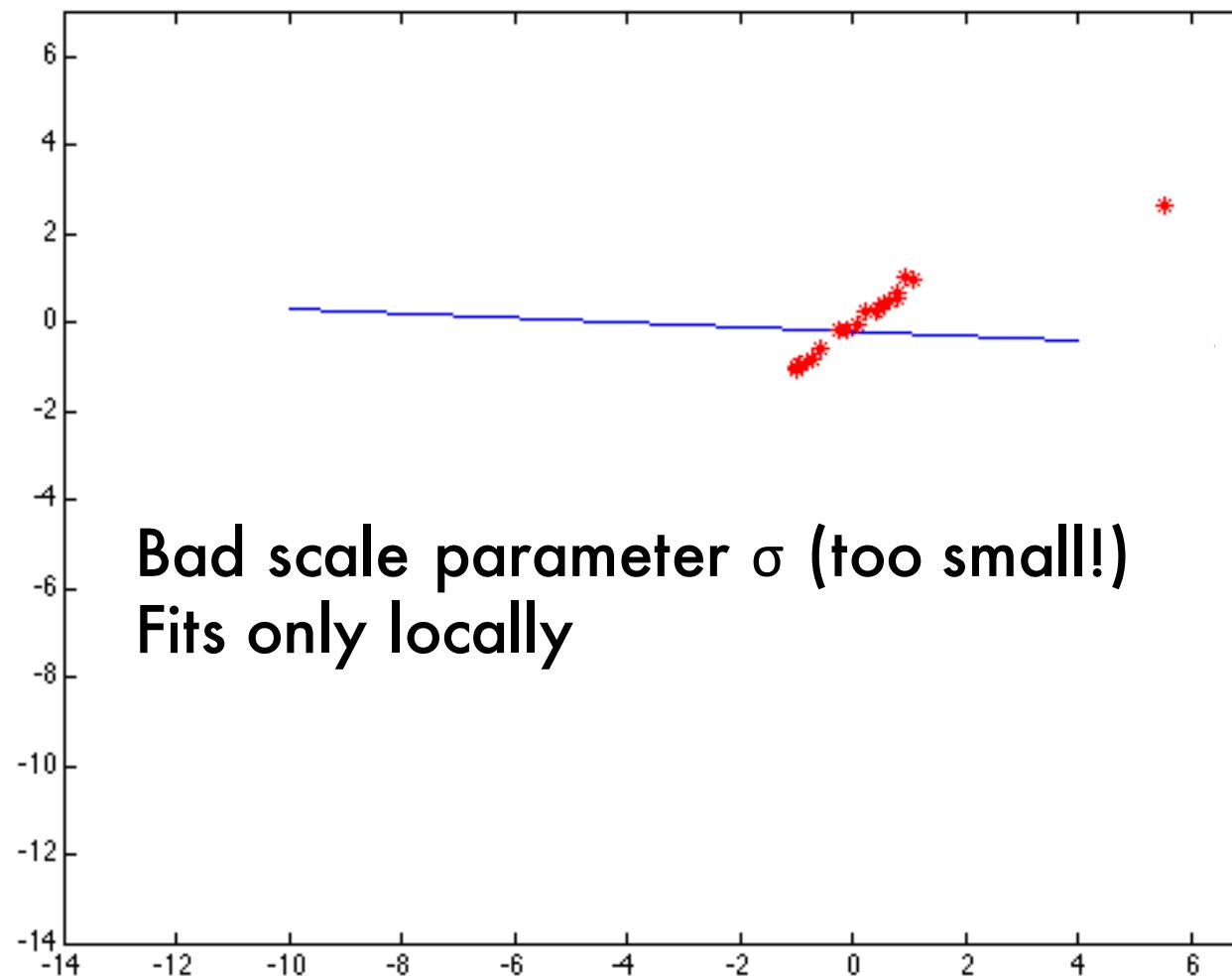
- Small sigma  $\rightarrow$  highly penalize large residuals
- Large sigma  $\rightarrow$  mildly penalize large residual (like LSQR)

# Least squares: Robust estimators

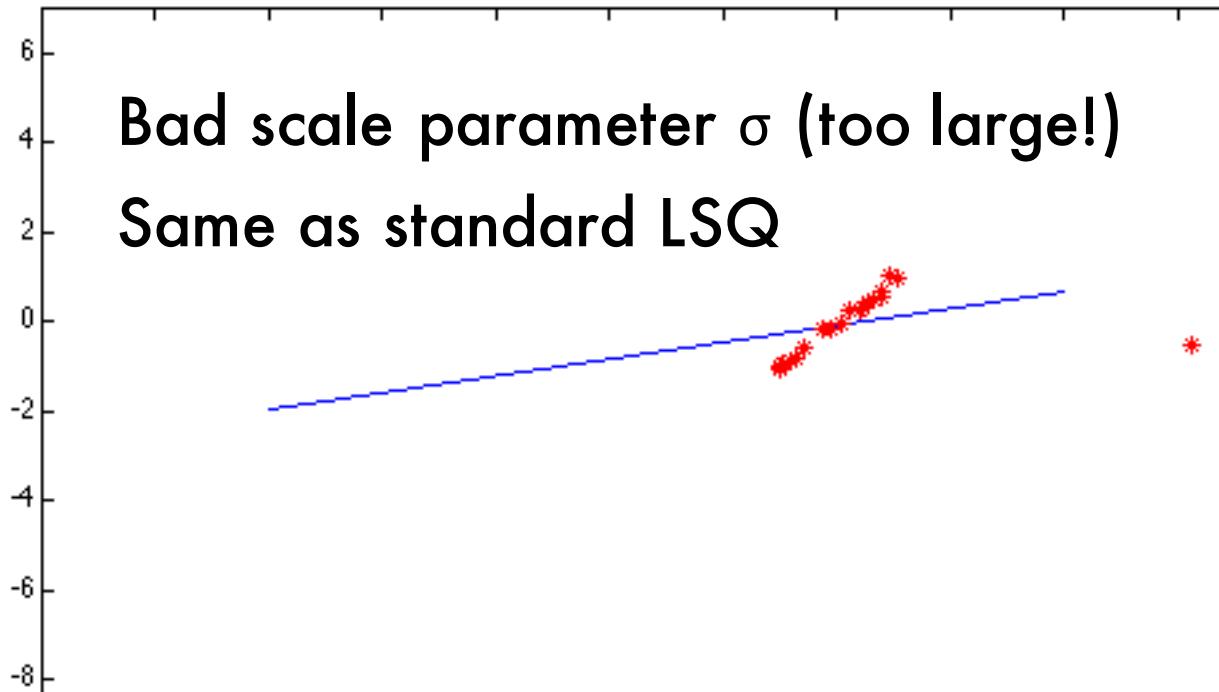


The effect of the outlier is eliminated

# Least squares: Robust estimators



# Least squares: Robust estimators



- **CONCLUSION:** Robust estimator useful if prior info about the distribution of points is known
- Robust fitting is a nonlinear optimization problem (iterative solution)
- Least squares solution provides good initial condition

# Fitting

**Goal:** Choose a parametric model to fit a certain quantity from data

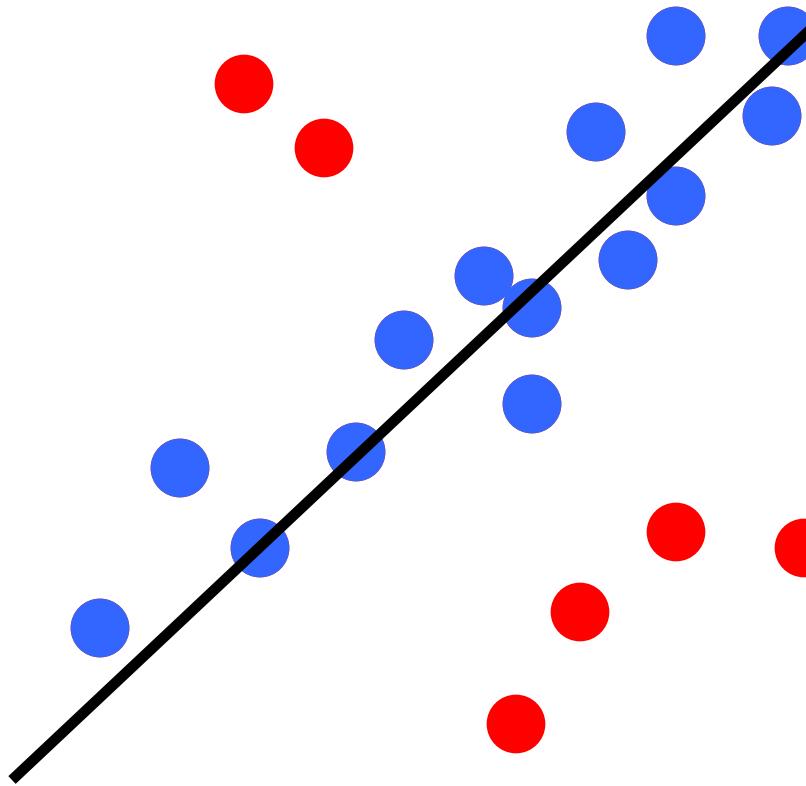
**Techniques:**

- Least square methods
- RANSAC
- Hough transform

# Basic philosophy (voting scheme)

- Data elements are used to vote for one (or multiple) models
- Robust to outliers and missing data
- Assumption 1: Noisy data points will not vote consistently for any single model (“few” outliers)
- Assumption 2: There are enough data points to agree on a good model (“few” missing data)

## Example: Line fitting

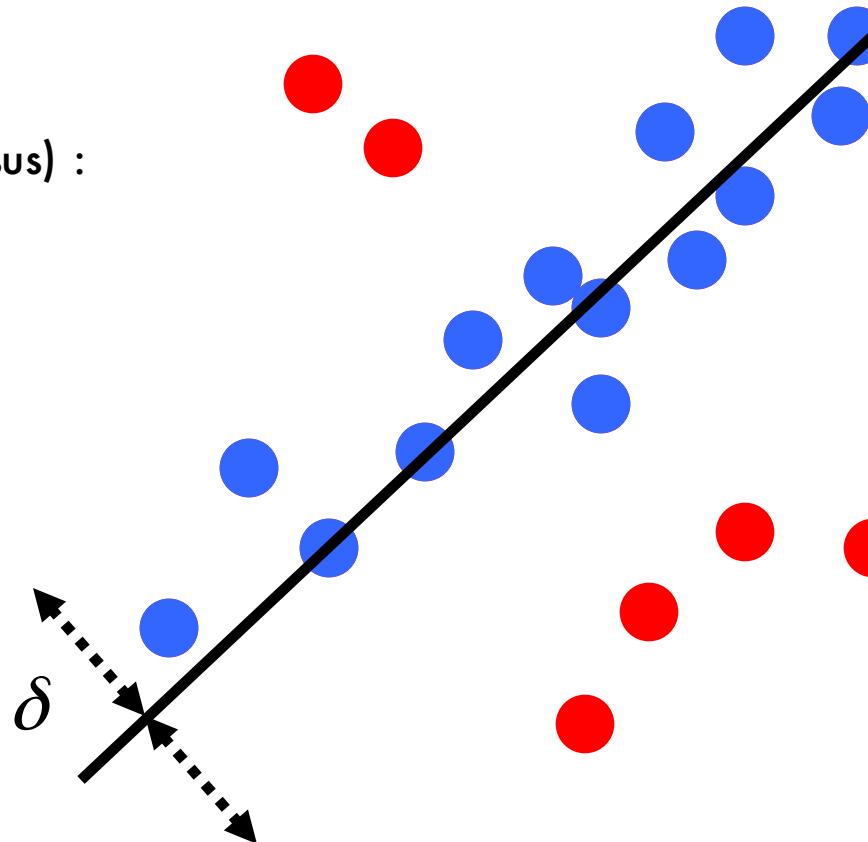


- Enough “good” data points supporting the line model in presence of noise
- “Few” outliers compared to the “good” data points – these few outliers won’t “consistently” vote for a line model

# RANSAC

(RANdom SAmple Consensus) :

Fischler & Bolles in '81.



$$\pi : P \rightarrow \{I, O\}$$

such that:

$$r(I, h) < \delta, \quad \forall I \in I$$

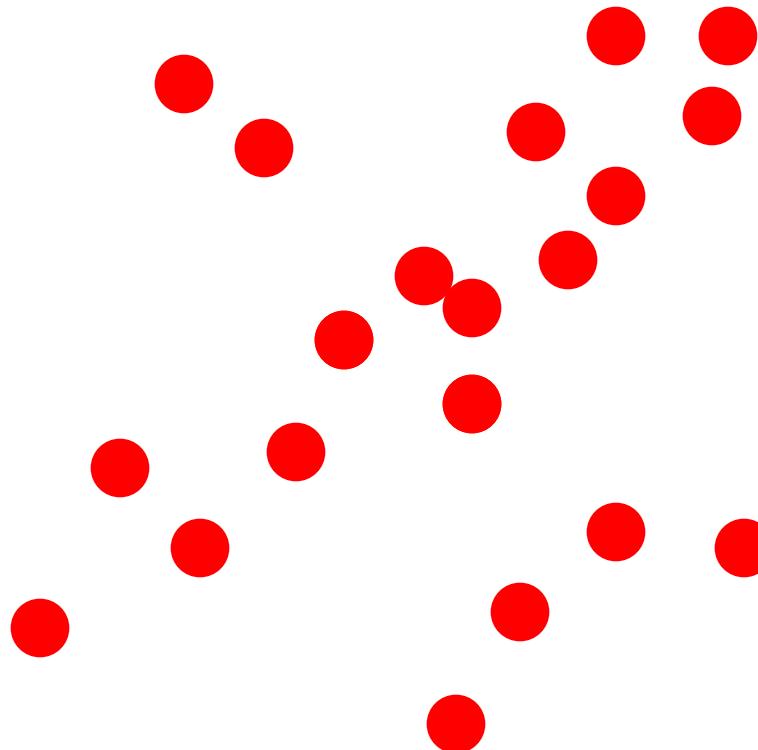
[Eq. 12]

$$\min_{\pi} |O|$$

Model parameters  $a, b, d$

$$r(I, h) = \text{residual} = \sum_{i=1}^n (ax_i + by_i + d)^2$$

# RANSAC

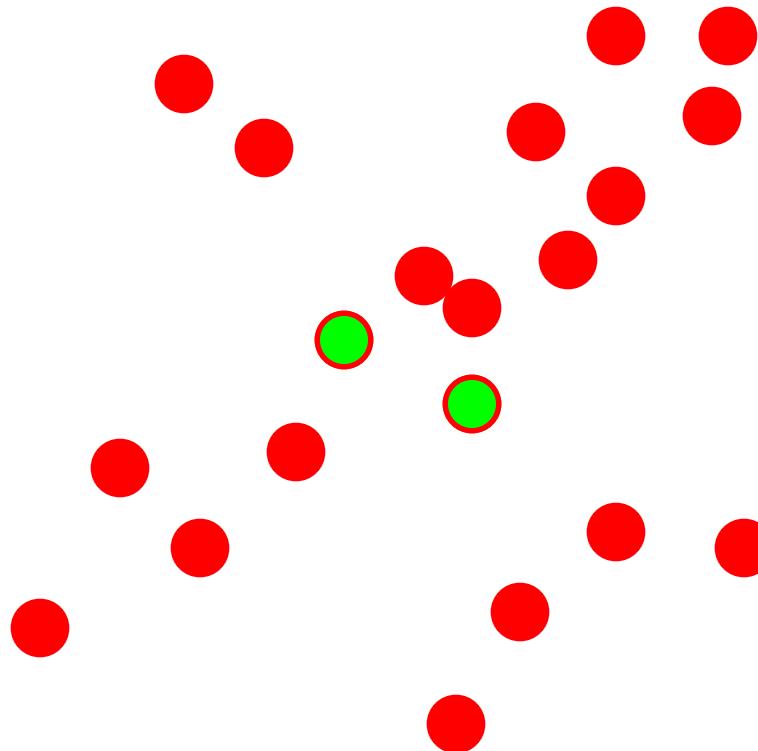


$P$  = Sample set = set of points in 2D

## Algorithm:

1. Select random sample of minimum required size to fit model
  2. Compute a putative model from sample set
  3. Compute the set of inliers to this model from whole data set
- Repeat 1-3 until model with the most inliers over all samples is found

# RANSAC

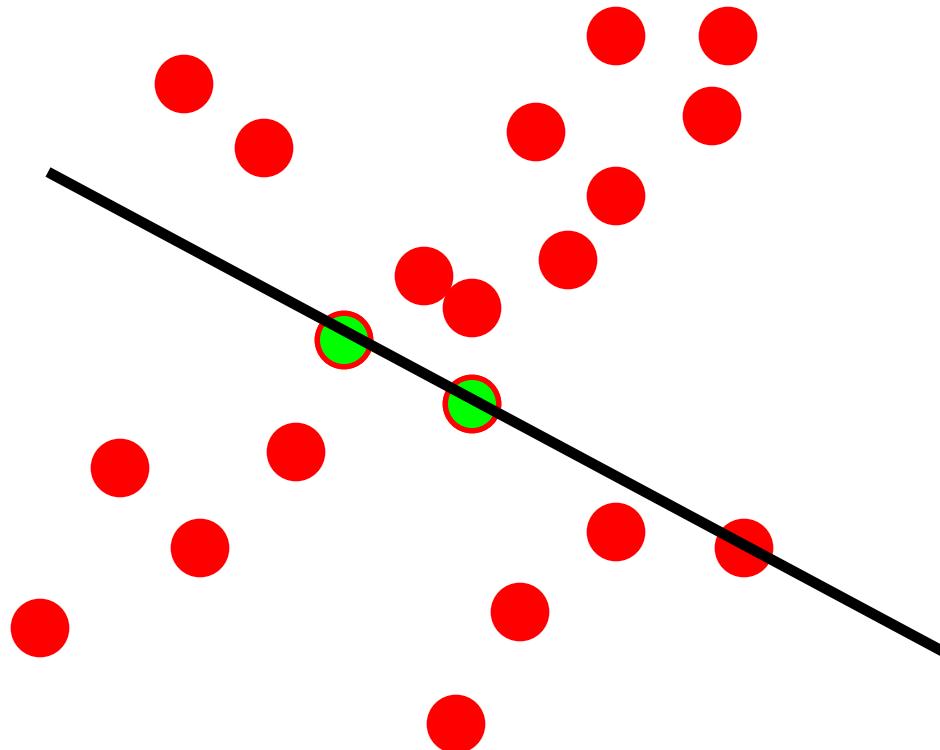


**P** = Sample set = set of points in 2D

## Algorithm:

1. Select random sample of minimum required size to fit model [?]
  2. Compute a putative model from sample set
  3. Compute the set of inliers to this model from whole data set
- Repeat 1-3 until model with the most inliers over all samples is found

# RANSAC

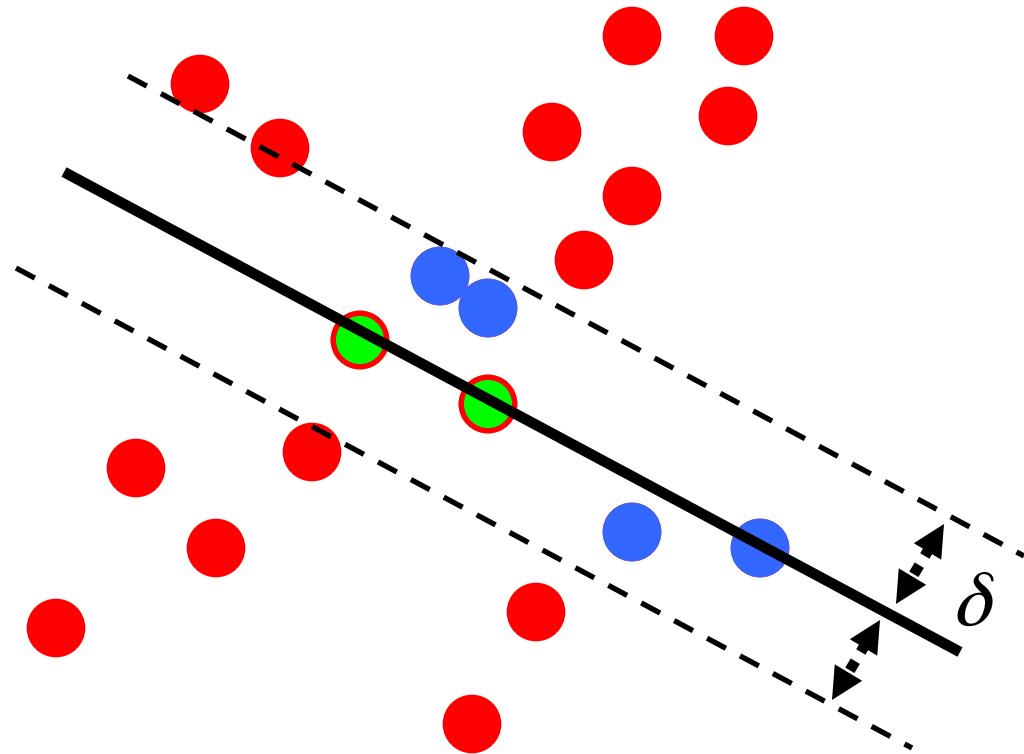


**P** = Sample set = set of points in 2D

## Algorithm:

1. Select random sample of minimum required size to fit model [?]
  2. Compute a putative model from sample set
  3. Compute the set of inliers to this model from whole data set
- Repeat 1-3 until model with the most inliers over all samples is found

# RANSAC



$P$  = Sample set = set of points in 2D

$$|O| = ? = 14$$

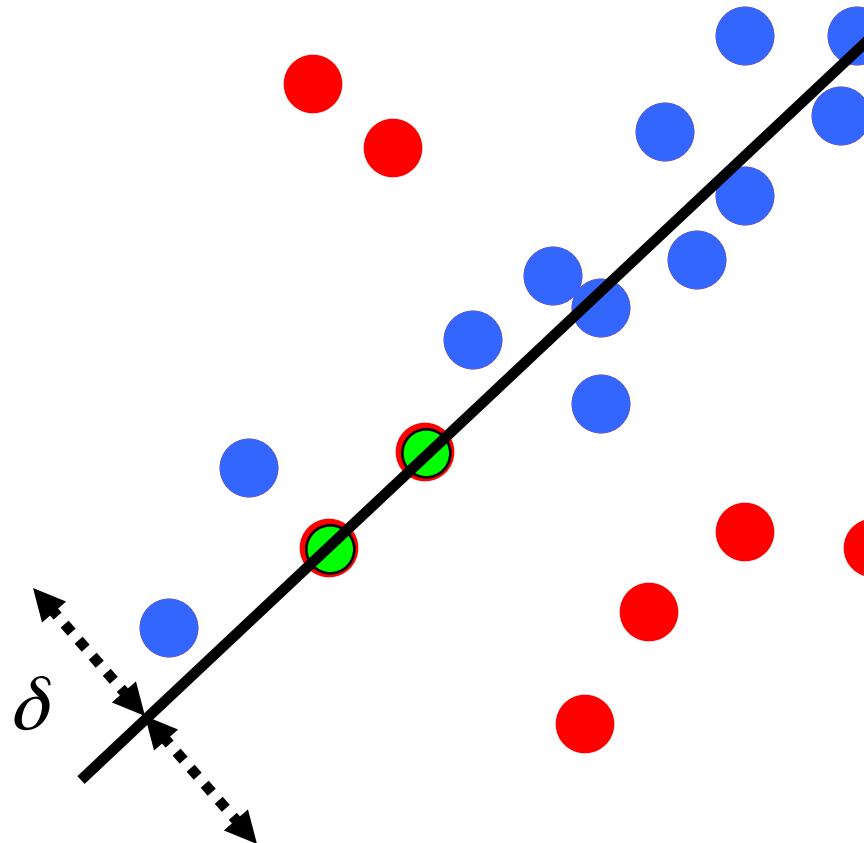
$$|I| = ? = 6$$

Algorithm:

1. Select random sample of minimum required size to fit model [?]
2. Compute a putative model from sample set
3. Compute the set of inliers to this model from whole data set

Repeat 1-3 until model with the most inliers over all samples is found

# RANSAC



$$\begin{aligned} |O| &= 6 \\ |I| &= 14 \end{aligned}$$

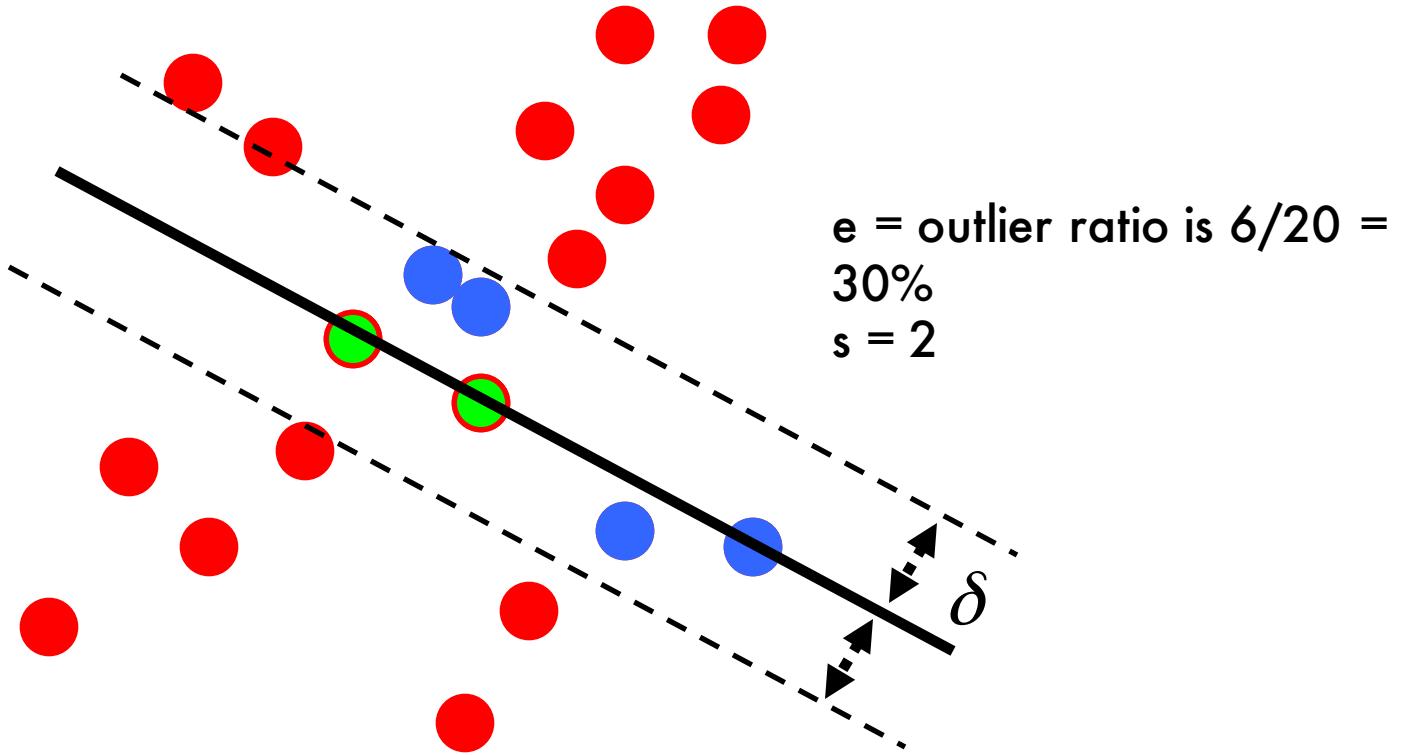
Algorithm:

1. Select random sample of minimum required size to fit model [?]
  2. Compute a putative model from sample set
  3. Compute the set of inliers to this model from whole data set
- Repeat 1-3 until model with the most inliers over all samples is found

# How many samples?

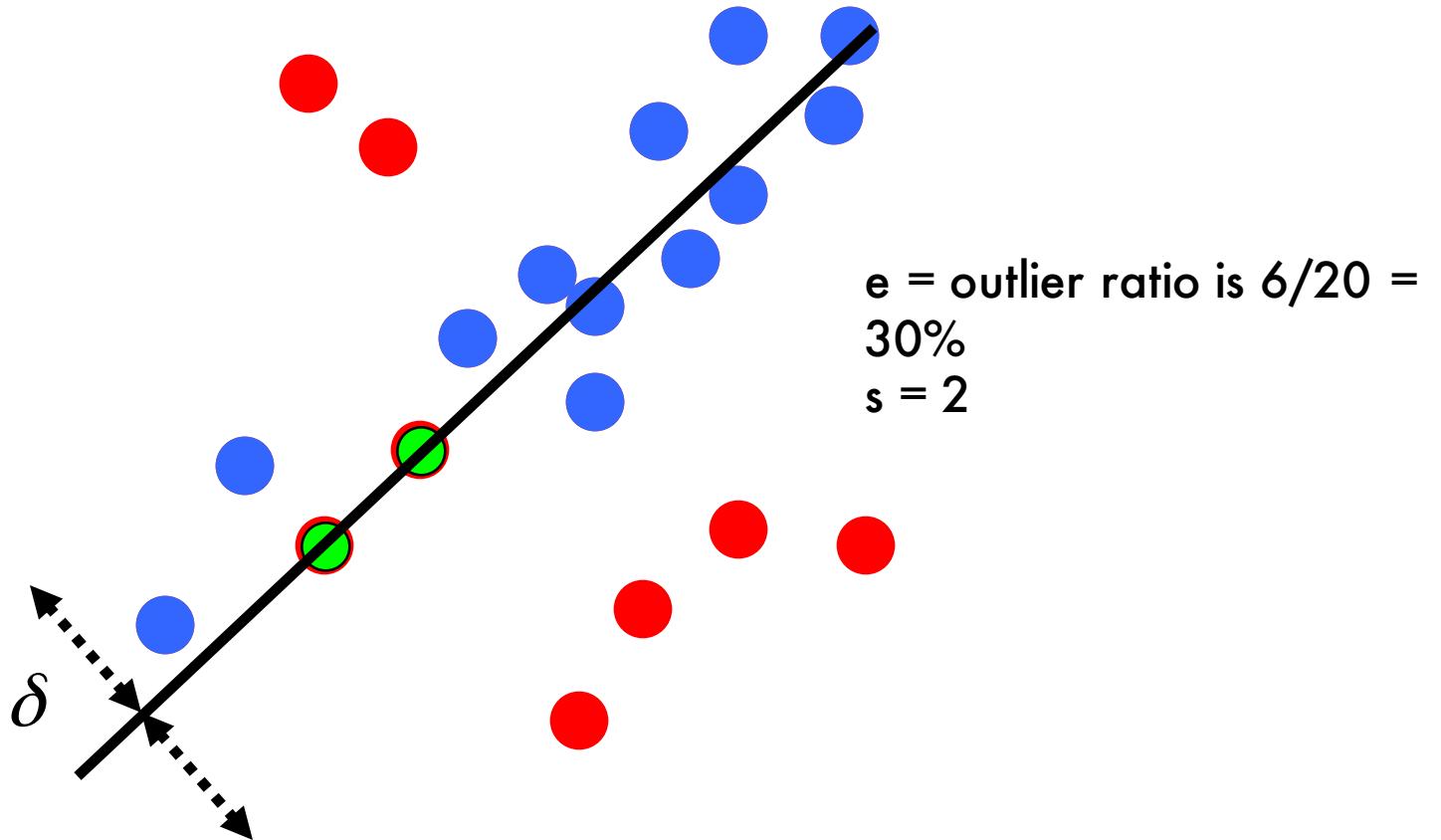
- Computationally unnecessary (and infeasible) to explore the entire sample space
- **N samples are sufficient**
- N = number of samples required to ensure, with a probability p, that at least one random sample produces an inlier set that is free from “real” outliers
- Function of s and e:
  - e = outlier ratio
  - s = minimum number of data points needed to fit the model
- Usually, p=0.99

# Example



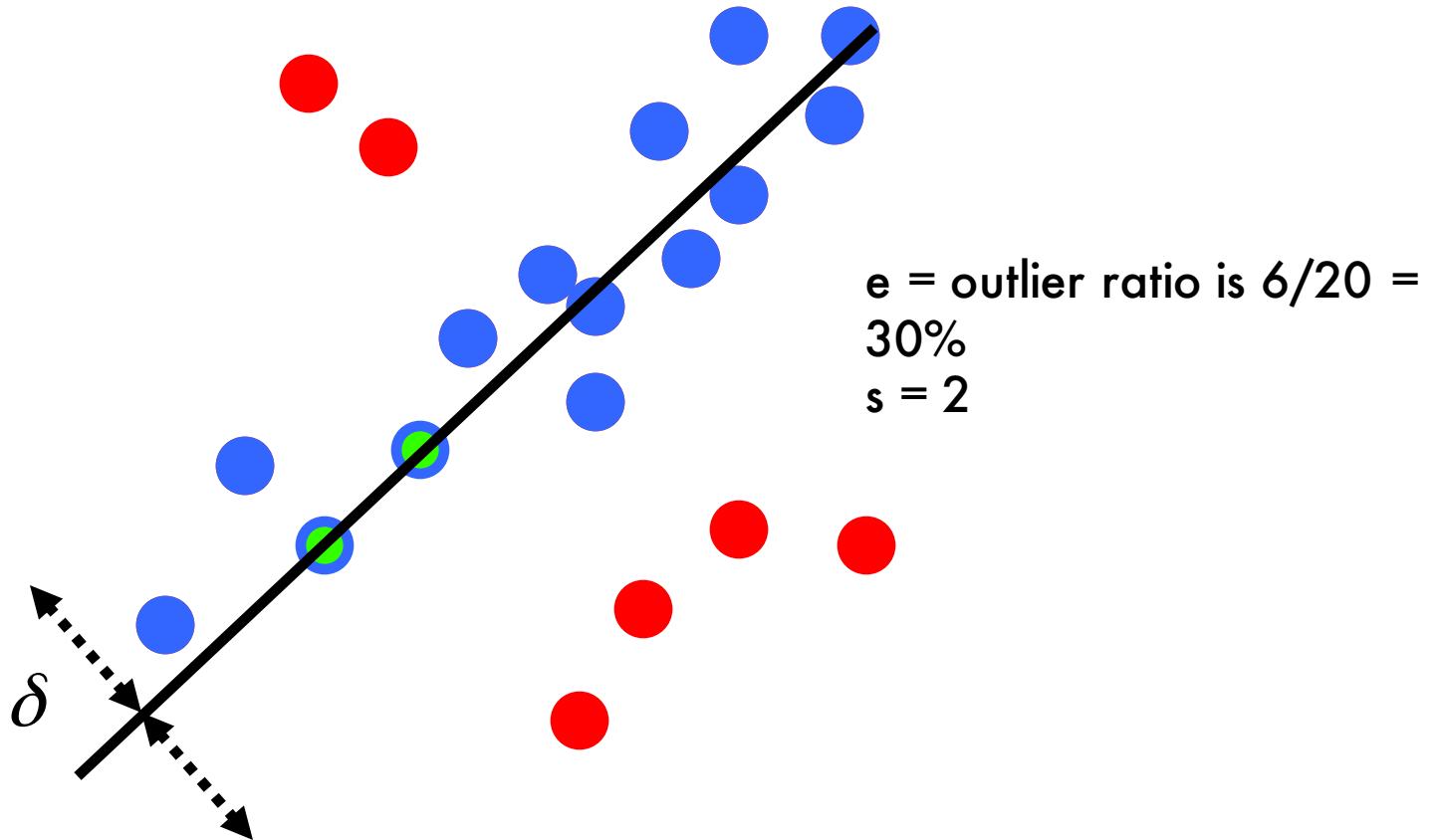
- Here a random sample is given by two green points
- The estimated inlier set is given by the green+blue points
- How many “real” outliers we have here?

# Example



- Random sample is given by two green points
- The estimated inlier set is given by the green+blue points
- How many “real” outliers we have here?

# Example



$N$  is the number of times we need to sample my data (and thus repeat the steps 1-3 in the previous slides) before I find the configuration above with probability  $p$ . Again this is function of  $e$  and  $s$  as well.

# How many samples?

- Number  $N$  of samples required to ensure, with a probability  $p$ , that at least one random sample produces an inlier set that is free from “real” outliers for a given  $s$  and  $e$ .
- E.g.,  $p=0.99$

$$N = \log(1 - p) / \log\left(1 - (1 - e)^s\right)$$
 [Eq. 13]

s	proportion of outliers $e$							
	5%	10%	20%	25%	30%	40%	50%	
2	2	3	5	6	7	11	17	
3	3	4	7	9	11	19	35	
4	3	5	9	13	17	34	72	
5	4	6	12	17	26	57	146	
6	4	7	16	24	37	97	293	
7	4	8	20	33	54	163	588	
8	5	9	26	44	78	272	1177	

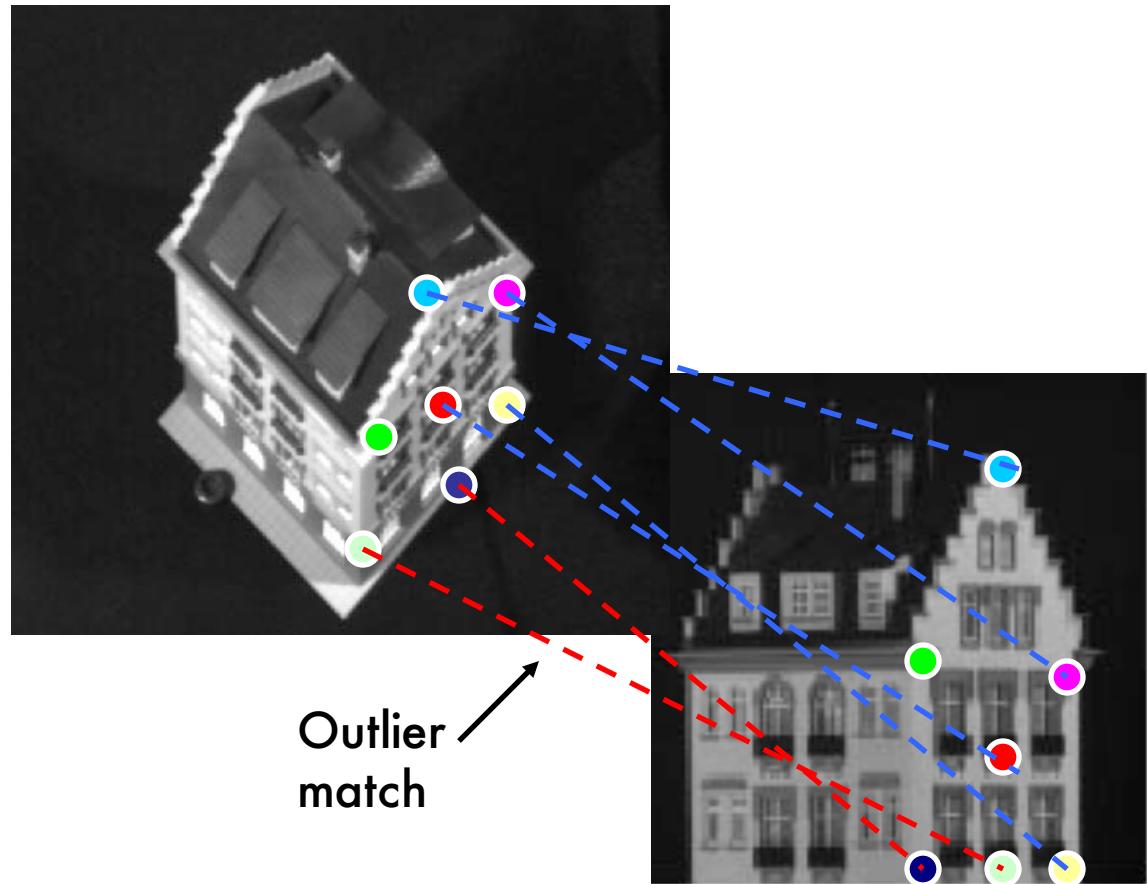
$e$  = outlier ratio

$s$  = minimum number needed to fit the model

Note: this table assumes “negligible” measurement noise

# Estimating $H$ by RANSAC

- $H \rightarrow 8$  DOF
- Need 4 correspondences



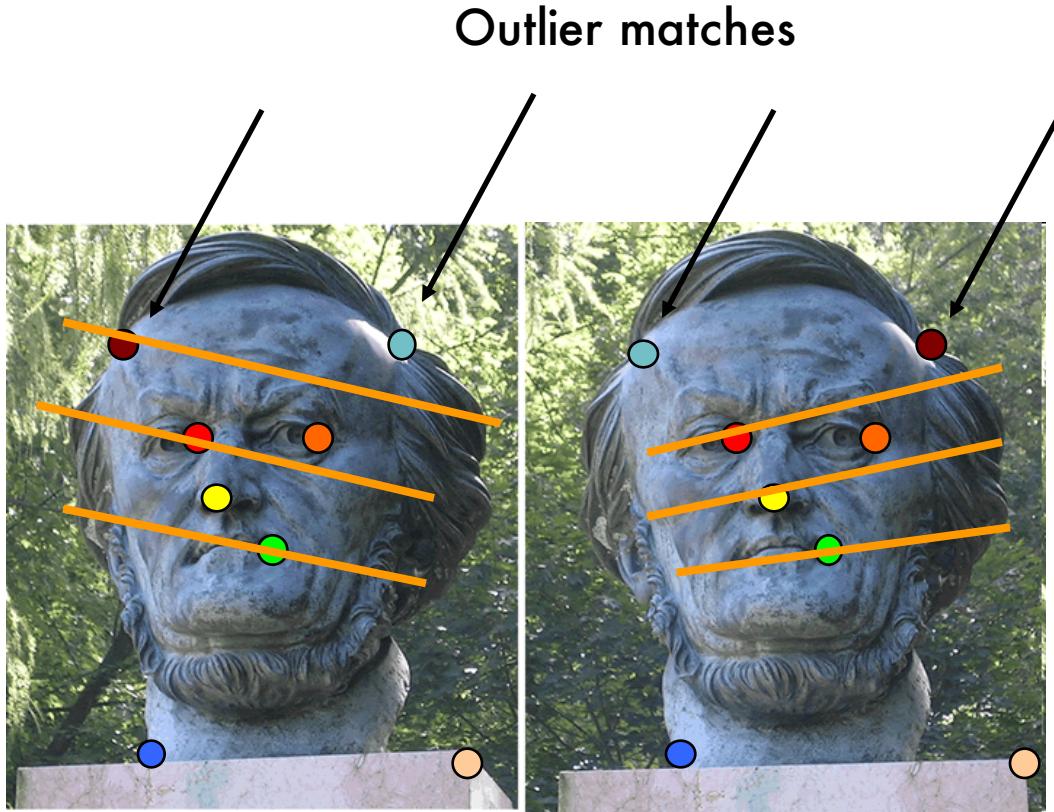
$P$  = Sample set = set of matches between 2 images

Algorithm:

1. Select a random sample of minimum required size [?]
  2. Compute a putative model from these
  3. Compute the set of inliers to this model from whole sample space
- Repeat 1-3 until model with the most inliers over all samples is found

# Estimating F by RANSAC

- $F \rightarrow 7$  DOF
- Need 7 (8) correspondences



$P$  = Sample set = set of matches between 2 images

Algorithm:

1. Select a random sample of minimum required size [?]
  2. Compute a putative model from these
  3. Compute the set of inliers to this model from whole sample space
- Repeat 1-3 until model with the most inliers over all samples is found

# RANSAC - conclusions

**Good:**

- Simple and easily implementable
- Successful in different contexts

**Bad:**

- Many parameters to tune
- Trade-off accuracy-vs-time
- Cannot be used if ratio inliers/outliers is too small

# Fitting

**Goal:** Choose a parametric model to fit a certain quantity from data

**Techniques:**

- Least square methods
- RANSAC
- Hough transform

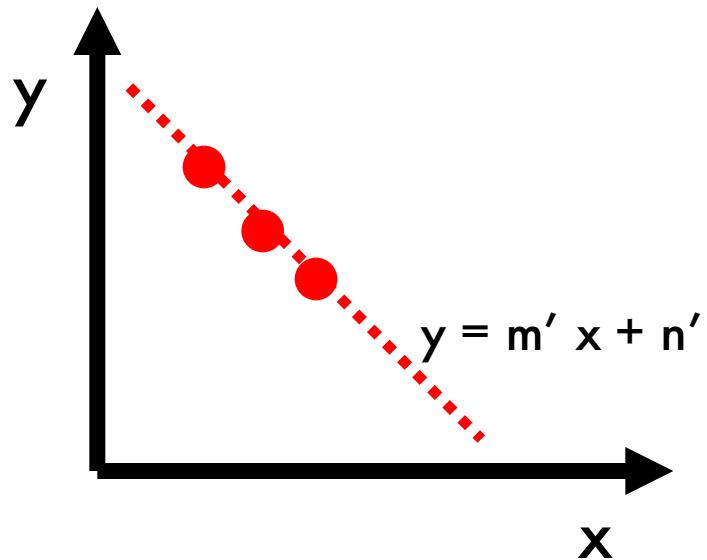
# Hough transform

P.V.C. Hough, *Machine Analysis of Bubble Chamber Pictures*, Proc. Int. Conf. High Energy Accelerators and Instrumentation, 1959

# Hough transform

P.V.C. Hough, *Machine Analysis of Bubble Chamber Pictures*, Proc. Int. Conf. High Energy Accelerators and Instrumentation, 1959

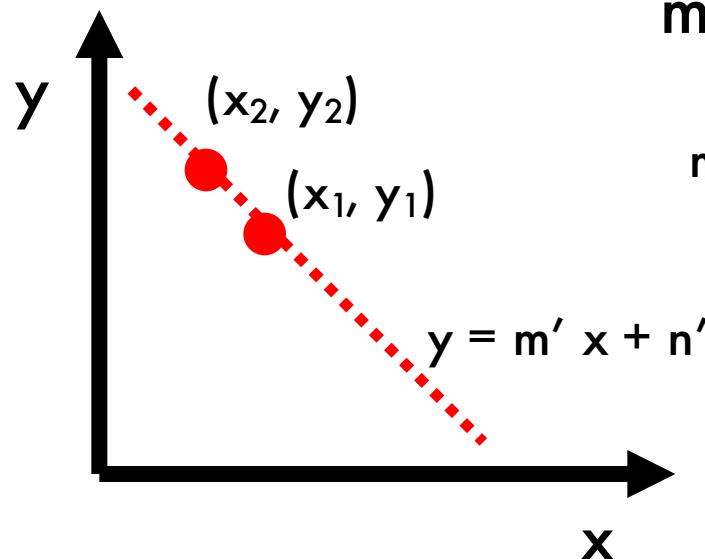
Given a set of points, find the line parameterized by  $m, n$  that explains the data points best: that is,  $m = m'$  and  $n = n'$



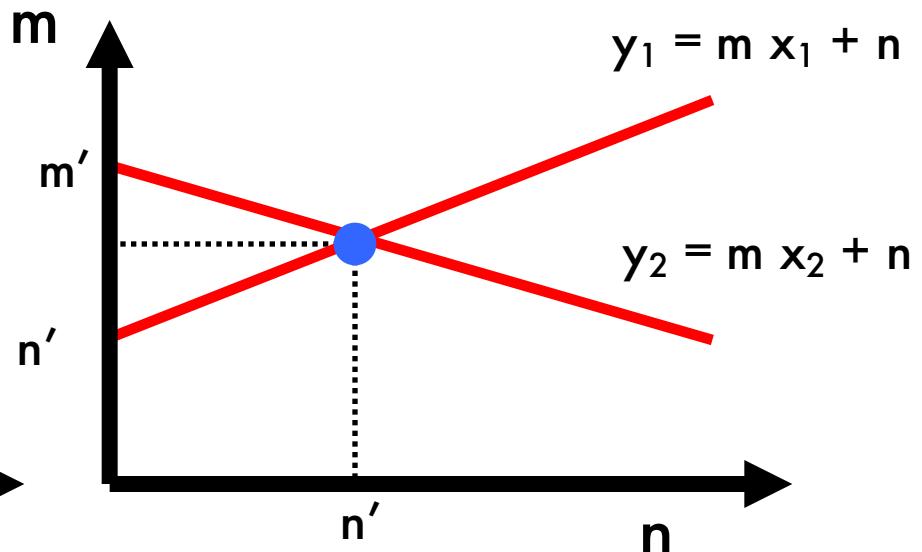
# Hough transform

P.V.C. Hough, *Machine Analysis of Bubble Chamber Pictures*, Proc. Int. Conf. High Energy Accelerators and Instrumentation, 1959

Given a set of points, find the line parameterized by  $m, n$  that explains the data points best: that is,  $m = m'$  and  $n = n'$



Original space where the data points are



Hough space defined by the parameters of the model we want to fit (i.e.,  $m, n$ )

# Hough transform

P.V.C. Hough, *Machine Analysis of Bubble Chamber Pictures*, Proc. Int. Conf. High Energy Accelerators and Instrumentation, 1959

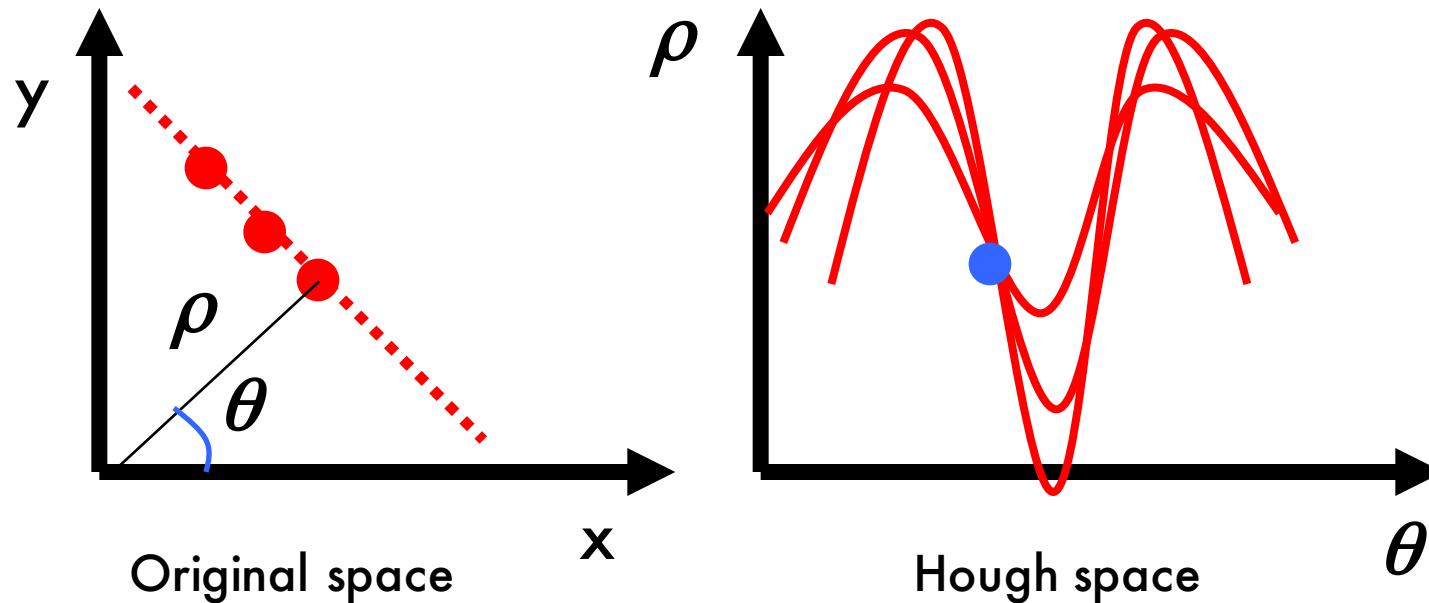
**Any Issue?** The parameter space  $[m,n]$  is unbounded...

# Hough transform

P.V.C. Hough, *Machine Analysis of Bubble Chamber Pictures*, Proc. Int. Conf. High Energy Accelerators and Instrumentation, 1959

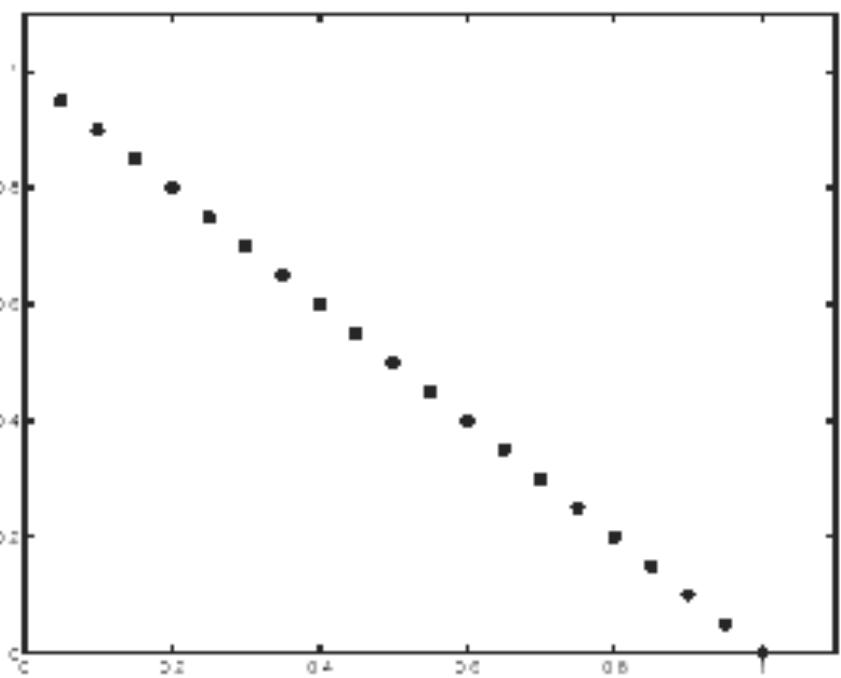
**Any Issue?** The parameter space  $[m,n]$  is unbounded...

Use a polar representation for the parameter space

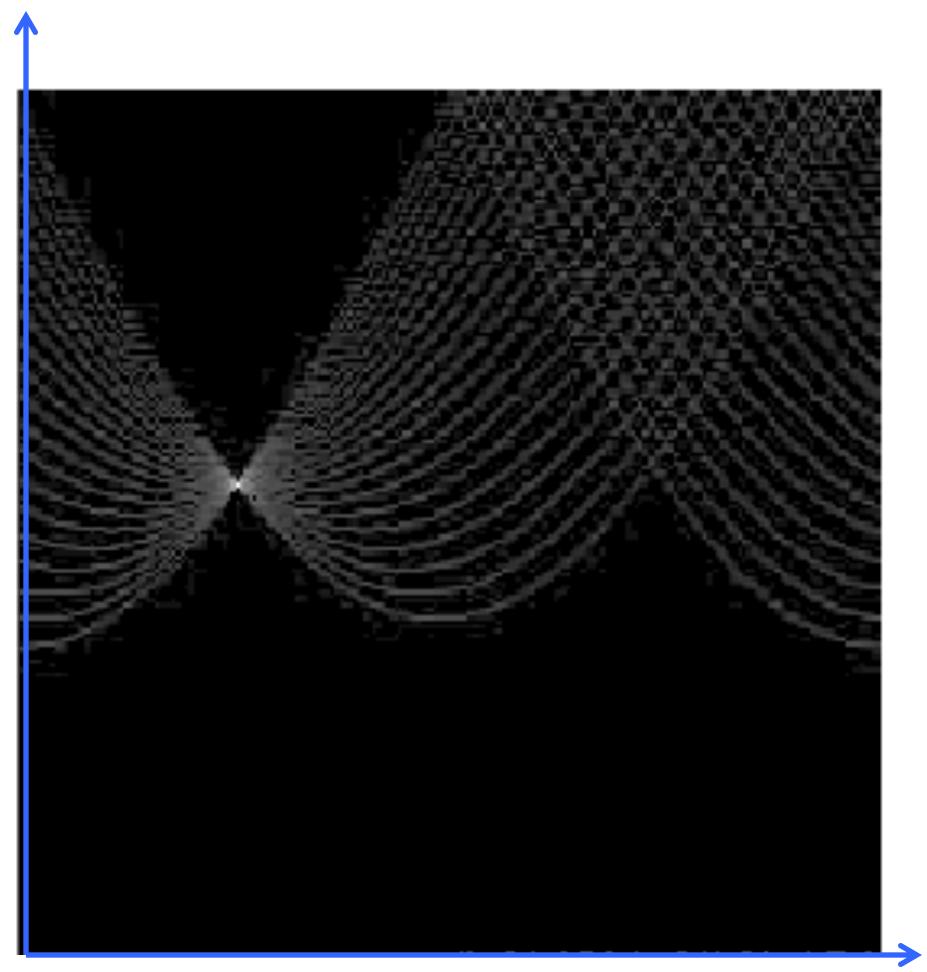


$$x \cos \theta + y \sin \theta = \rho \quad [\text{Eq. 13}]$$

# Hough transform - experiments



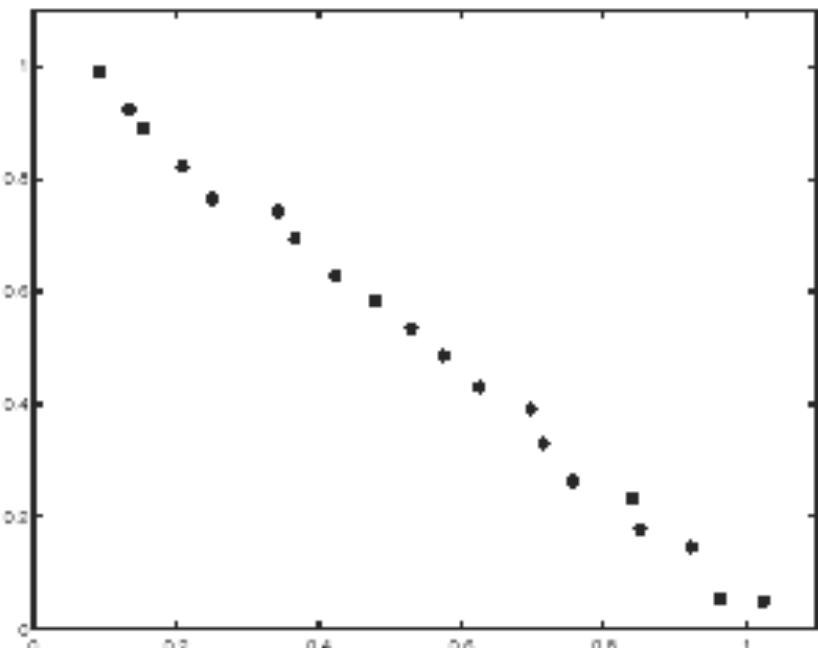
Original space



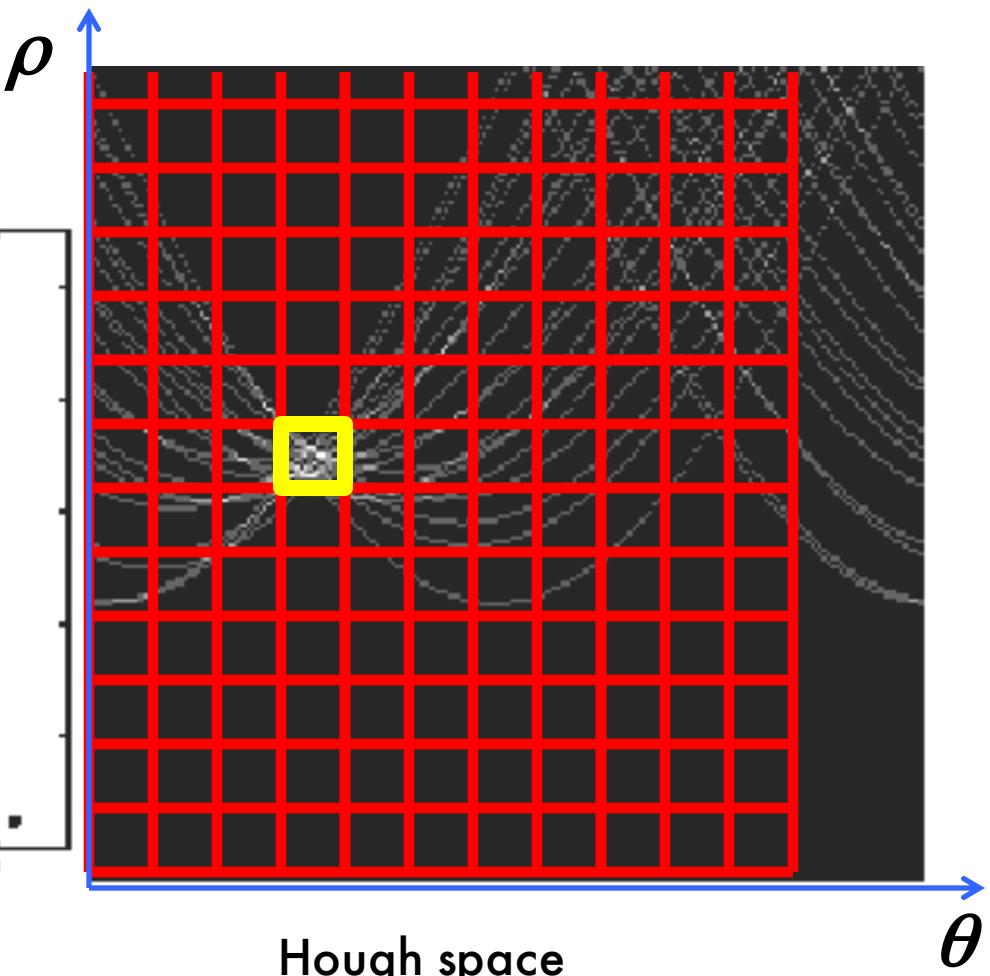
Hough space

# Hough transform - experiments

Noisy data



Original space

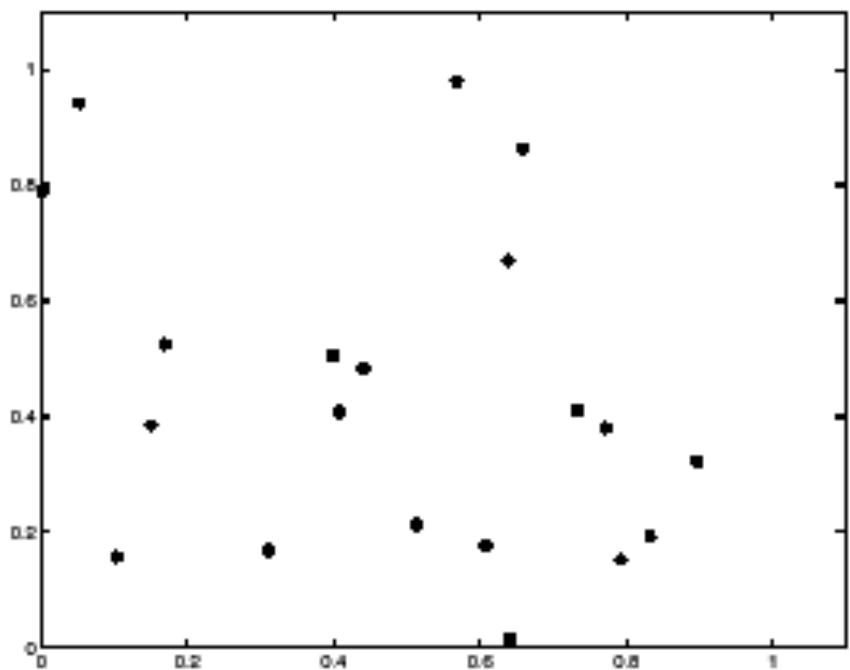


Hough space

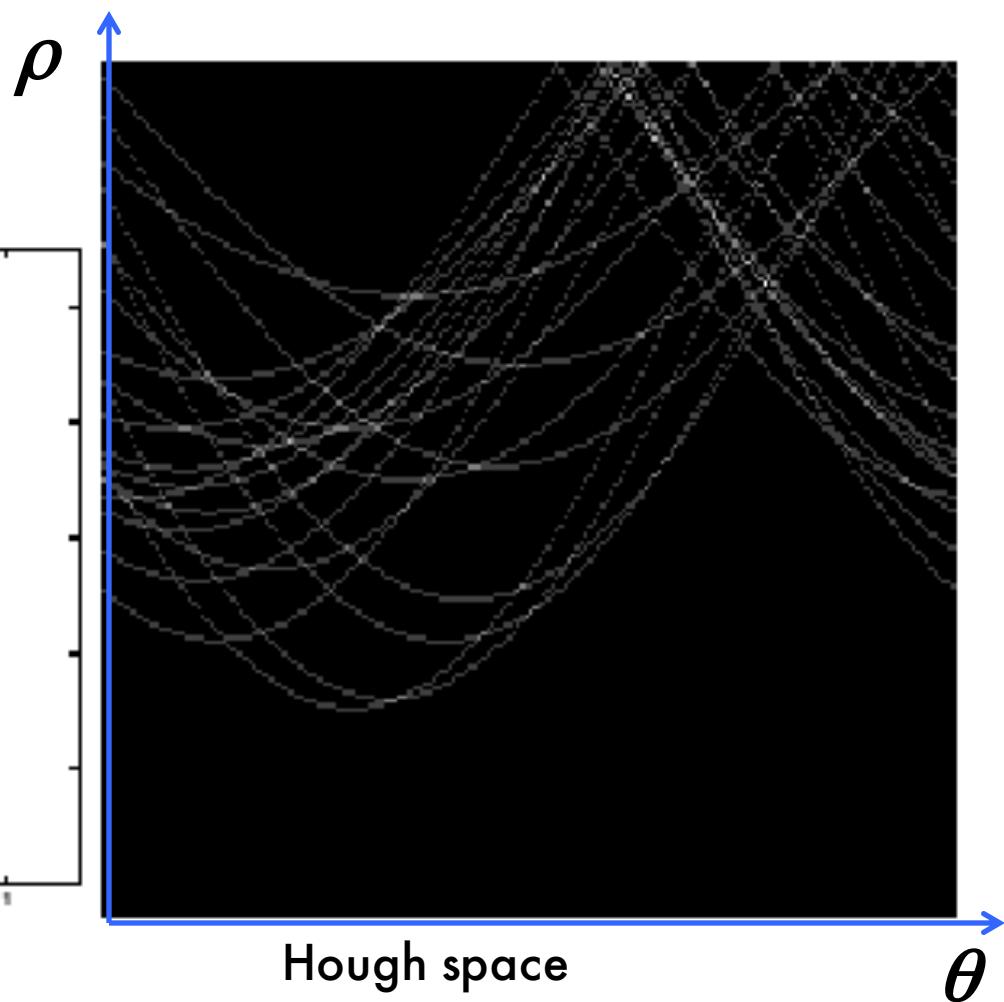
How to compute the intersection point? In presence of noise!

IDEA: introduce a grid a count intersection points in each cell

# Hough transform - experiments



Original space



Hough space

Issue: spurious peaks due to uniform noise

# Hough transform - conclusions

- All points are processed independently, so can cope with occlusion/outliers
- Some robustness to noise: noise points unlikely to contribute consistently to any single cell

Bad:

- Spurious peaks due to uniform noise
- Trade-off noise-grid size (hard to find sweet point)
- Doesn't handle well high dimensional models

# Applications – lane detection



Courtesy of Minchae Lee

# Applications – computing vanishing points



# Generalized Hough transform

[more on forthcoming lectures]

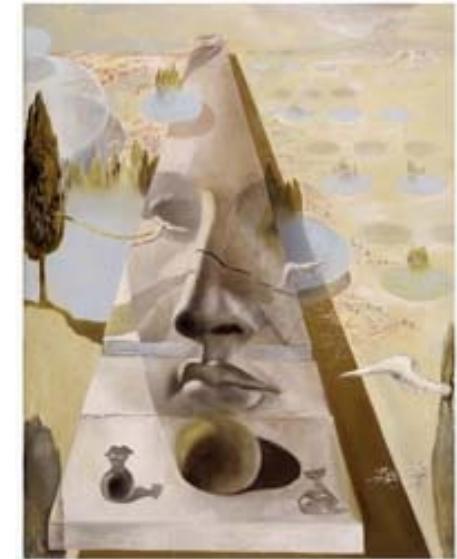
D. Ballard, Generalizing the Hough Transform to Detect Arbitrary Shapes, Pattern Recognition 13(2), 1981

- Parameterize a shape by measuring the location of its parts and shape centroid
  - Given a set of measurements, cast a vote in the Hough (parameter) space
  - Used in object recognition! (the implicit shape model)

B. Leibe, A. Leonardis, and B. Schiele, [Combined Object Categorization and Segmentation with an Implicit Shape Model](#), ECCV Workshop on Statistical Learning in Computer Vision 2004

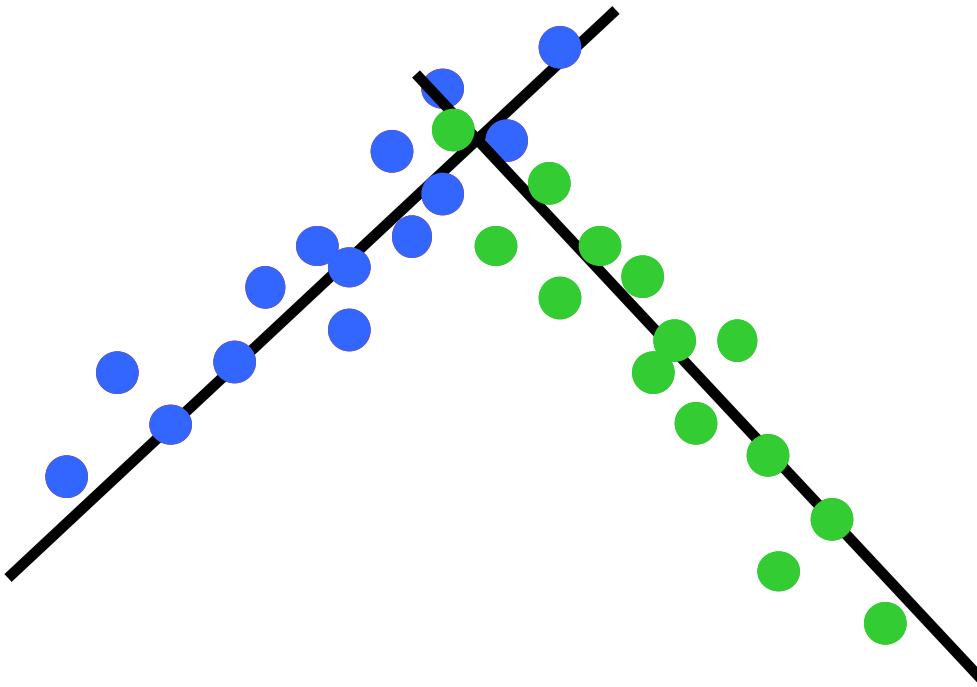
# Lecture 9

## Fitting and Matching



- Problem formulation
- Least square methods
- RANSAC
- Hough transforms
- Multi-model fitting
- Fitting helps matching!

# Fitting multiple models



- Incremental fitting
- E.M. (probabilistic fitting)
- Hough transform

# Incremental line fitting

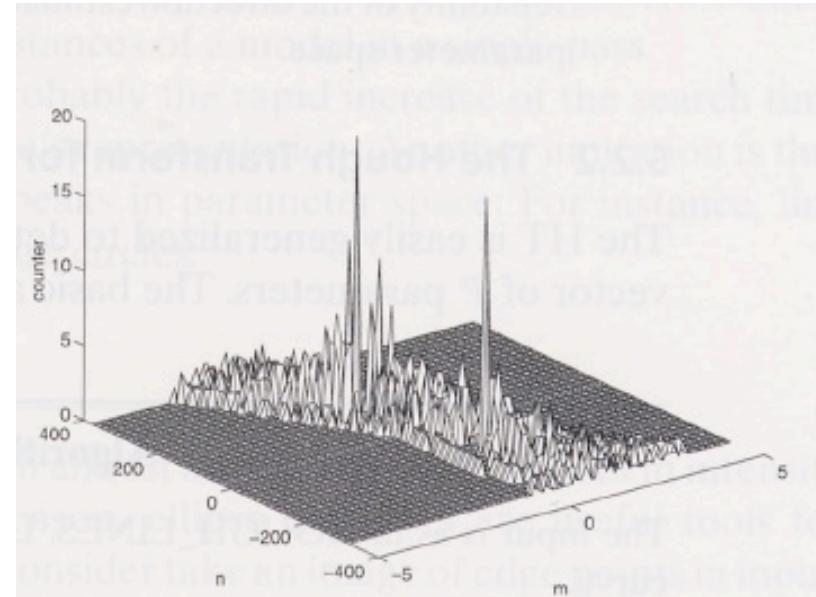
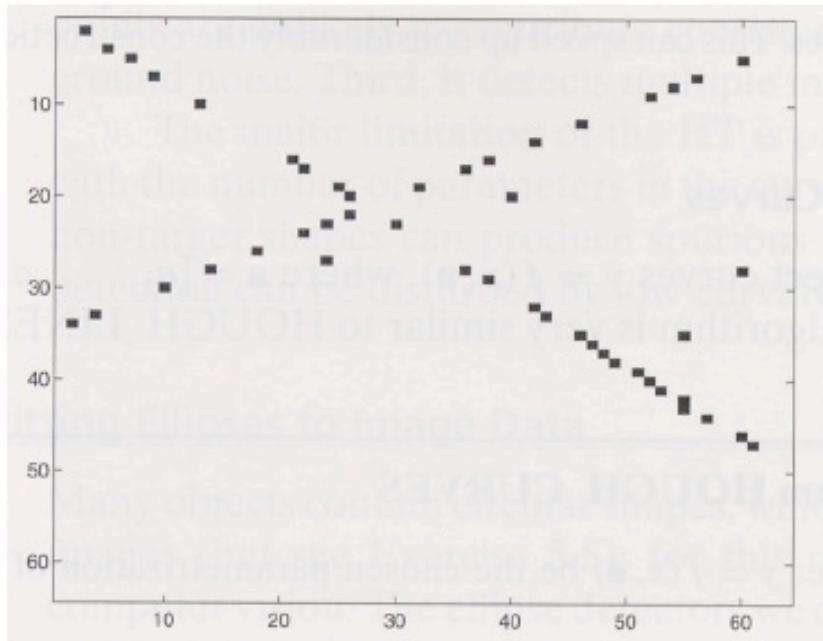
Scan data point sequentially (using locality constraints)

Perform following loop:

1. Select N point and fit line to N points
  2. Compute residual  $R_N$
  3. Add a new point, re-fit line and re-compute  $R_{N+1}$
  4. Continue while line fitting residual is small enough,
- When residual exceeds a threshold, start fitting new model (line)

# Hough transform

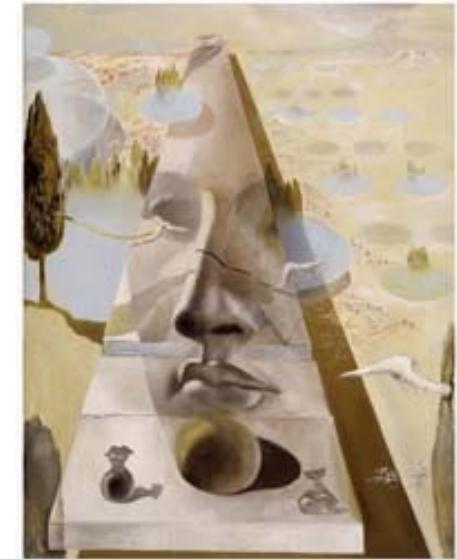
Courtesy of unknown



Same cons and pros as before...

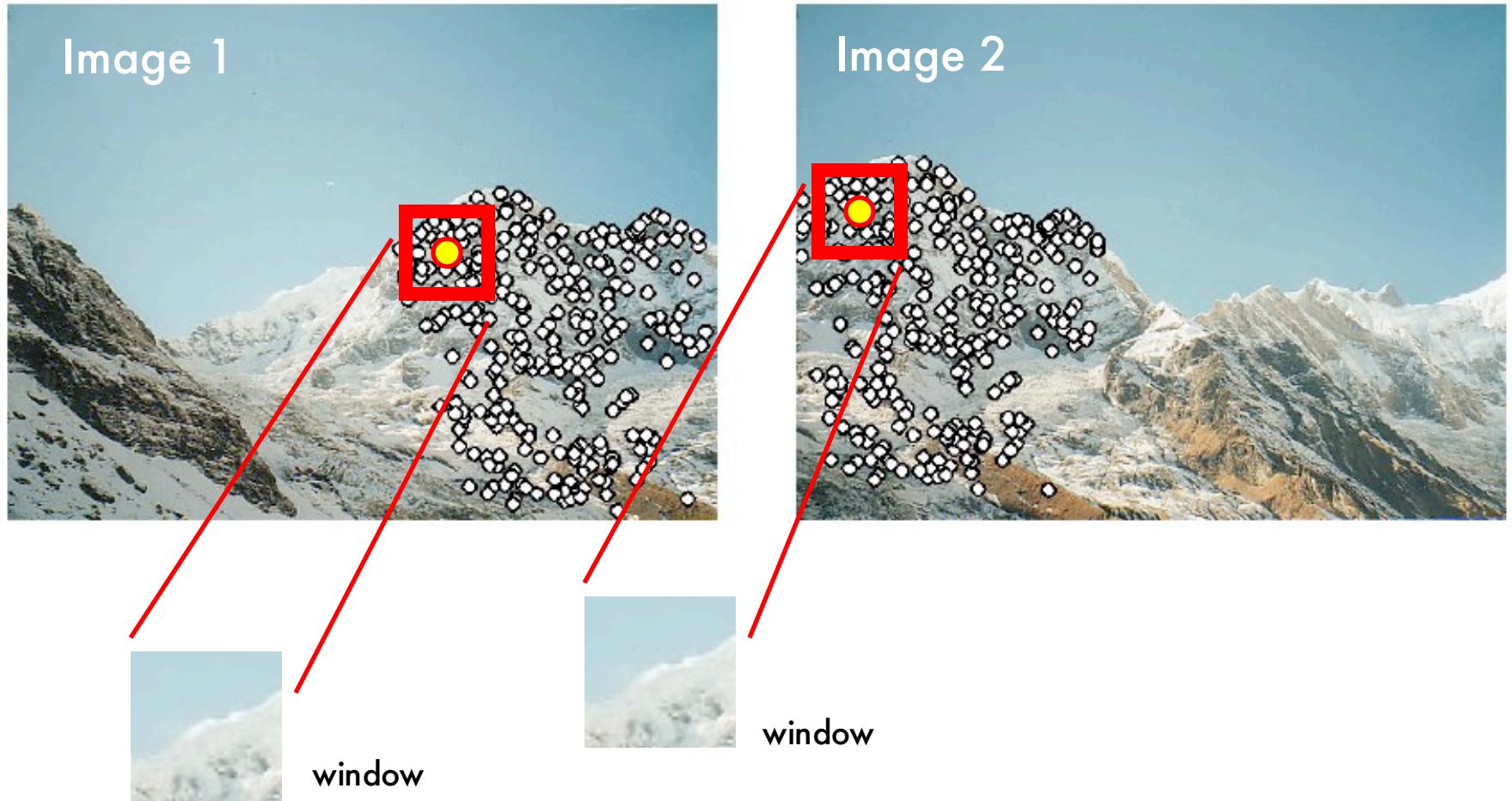
# Lecture 9

## Fitting and Matching



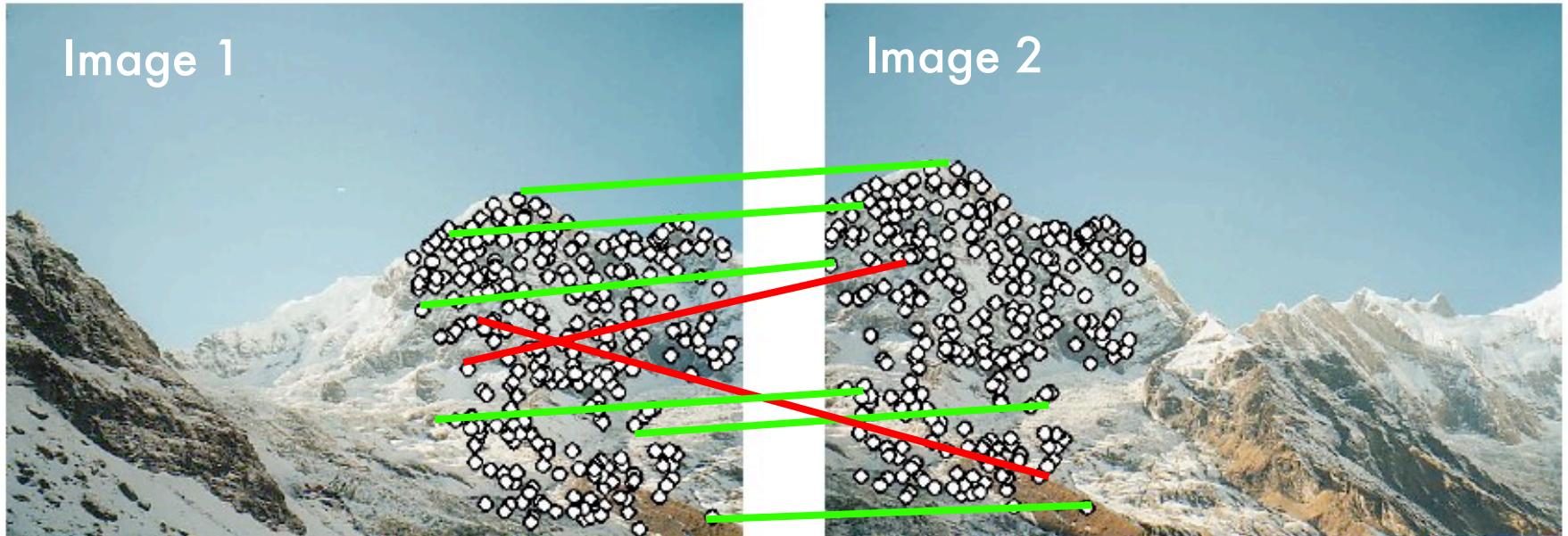
- Problem formulation
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# Fitting helps matching!



Features are matched (for instance, based on correlation)

# Fitting helps matching!



**Matches based on appearance only**  
Green: good matches  
Red: bad matches

## Idea:

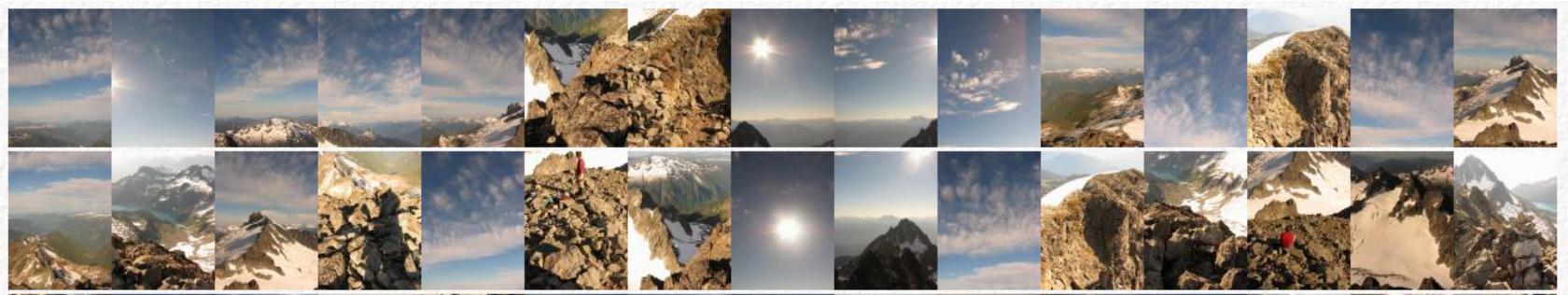
- Fitting an homography  $H$  (by RANSAC) mapping features from images 1 to 2
- Bad matches will be labeled as outliers (hence rejected)!

# Fitting helps matching!



# Recognising Panoramas

M. Brown and D. G. Lowe. Recognising Panoramas. In Proceedings of the 9th International Conference on Computer Vision - ICCV2003



**Next lecture:  
Feature detectors and descriptors**



# Least squares methods

- fitting a line -

$$Ax = b$$

- More equations than unknowns
- Look for solution which minimizes  $\|Ax-b\| = (Ax-b)^T(Ax-b)$
- Solve  $\frac{\partial(Ax-b)^T(Ax-b)}{\partial x_i} = 0$
- LS solution

$$x = (A^T A)^{-1} A^T b$$

# Least squares methods

- fitting a line -

**Solving**  $x = (A^t A)^{-1} A^t b$

$$A^+ = (A^t A)^{-1} A^t \quad = \text{pseudo-inverse of } A$$

$$A = U \sum V^t \quad = \text{SVD decomposition of } A$$

$$A^{-1} = V \sum^{-1} U^T$$

$$A^+ = V \sum^+ U^T$$

with  $\sum^+$  equal to  $\sum^{-1}$  for all nonzero singular values and zero otherwise

# Least squares methods

## - fitting an homography -

$$h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' - x' = 0$$

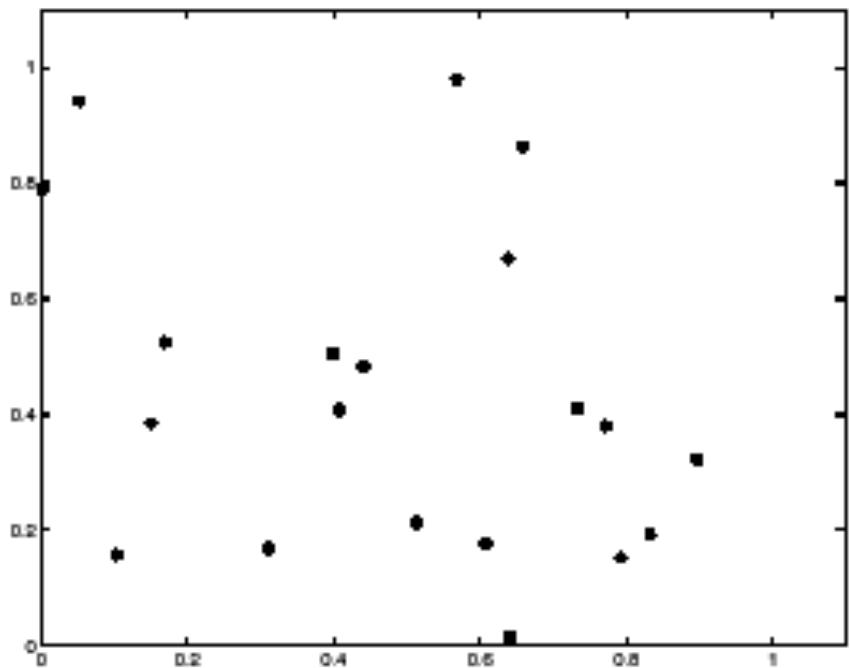
$$h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' - y' = 0$$

From  $n \geq 4$  corresponding points:

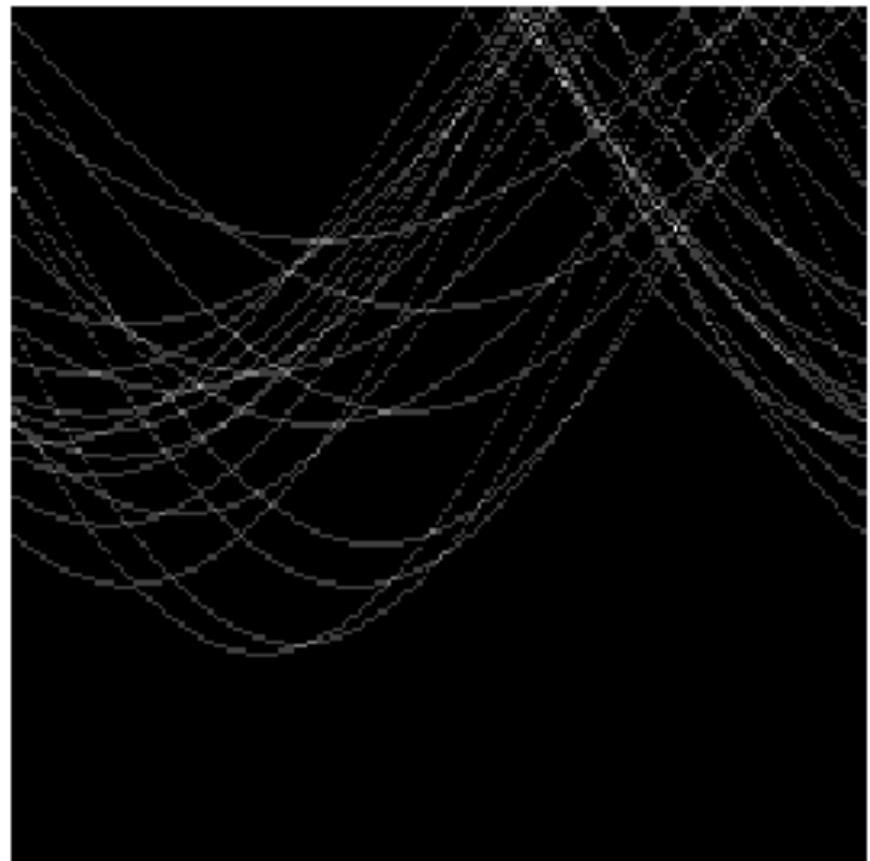
$$A h = 0$$

$$\left( \begin{array}{ccccccc} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 & -y'_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & -y_2x'_2 & -x'_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & -y_2y'_2 & -y'_2 \\ \vdots & \vdots \\ x_n & y_n & 1 & 0 & 0 & 0 & -x_nx'_n & -y_nx'_n & -x'_n \\ 0 & 0 & 0 & x_n & y_n & 1 & -x_ny'_n & -y_ny'_n & -y'_n \end{array} \right) \begin{bmatrix} h_{1,1} \\ h_{1,2} \\ \vdots \\ h_{3,3} \end{bmatrix} = 0$$

# Hough transform - experiments



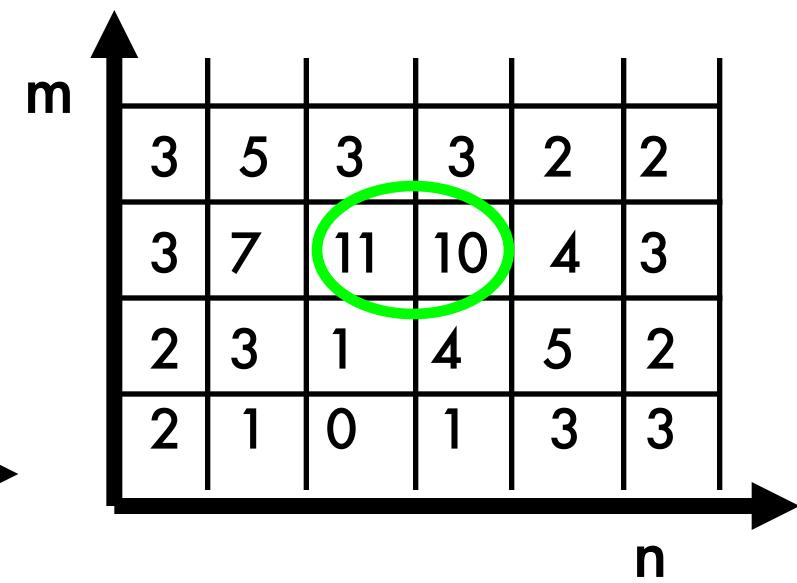
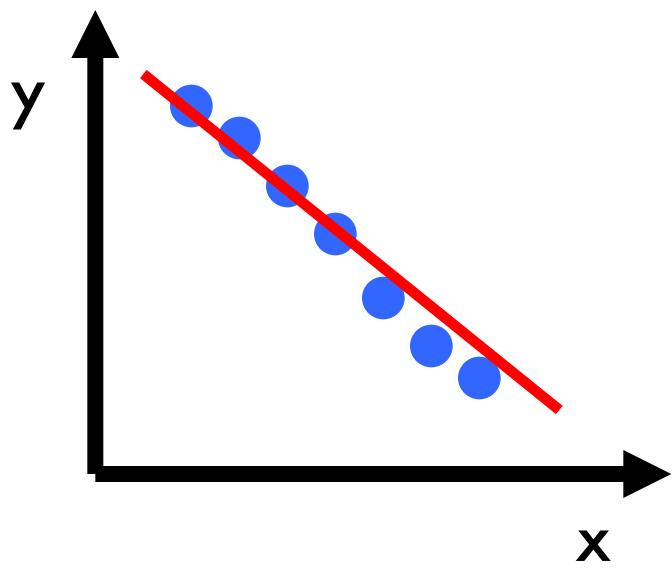
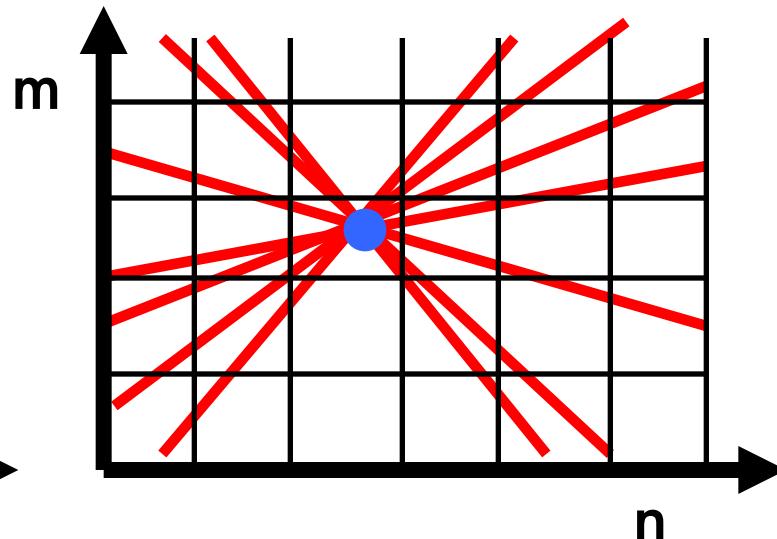
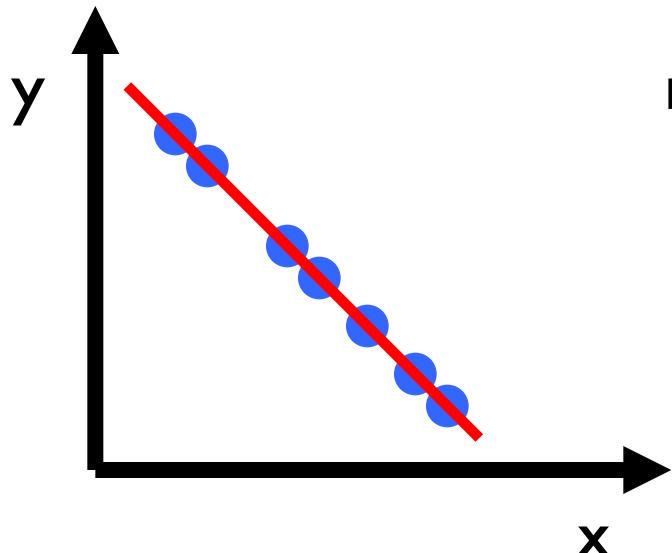
features



votes

Issue: spurious peaks due to uniform noise

# Hough transform



# Fitting helps matching!



Images courtesy of Brandon Lloyd

