1. Linear Algebra

1.1 (i) To verify: $\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2)$

Sol^n: Let a and b be n-dimensional vector.

$a = [a_1\ a_2 \ldots a_n]^T$, $b = [b_1\ b_2 \ldots b_n]^T$

$$a+b = [(a_1+b_1)\ (a_2+b_2) + \ldots (a_n+b_n)]^T$$

and $\|a+b\|^2 = \left(\sqrt[2]{(a_1+b_1)^2 + (a_2+b_2)^2 + \ldots (a_n+b_n)^2}\right)^2$

or $= (a_1+b_1)^2 + (a_2+b_2)^2 + \ldots (a_n+b_n)^2$ — ①

$$a-b = [(a_1-b_1)\ (a_2-b_2) \ldots (a_n-b_n)]^T$$

and $\|a-b\|^2 = \left(\sqrt[2]{(a_1-b_1)^2 + (a_2-b_2)^2 + \ldots (a_n-b_n)^2}\right)^2$

or $= (a_1-b_1)^2 + (a_2-b_2)^2 + \ldots (a_n-b_n)^2$ — ②

We get LHS by adding ① and ②

$\therefore$ LHS $= (a_1+b_1)^2 + (a_1-b_1)^2 + (a_2+b_2)^2 + (a_2-b_2)^2 + \ldots (a_n+b_n)^2 + (a_n-b_n)^2$

$= 2(a_1^2 + a_2^2 + \ldots a_n^2) + 2(b_1^2 + b_2^2 + \ldots b_n^2)$ —③

$= \cancel{2(\|a\|^2}$

~~RHS~~ $\|a\|^2 = a_1^2 + a_2^2 + \ldots a_n^2$

$\|b\|^2 = b_1^2 + b_2^2 + \ldots b_n^2$

$\therefore$ ③ ~~As~~ can be written as,

$= 2\|a\|^2 + 2\|b\|^2$

$= 2(\|a\|^2 + \|b\|^2)$

$=$ RHS.

Hence verified.

1.1 (ii) To verify: $(a+b)^T(a-b) = ||a||^2 - ||b||^2$

Let a and b be n-dimensional vectors defined as,

$a = [a_1 \; a_2 \; \cdots \; a_n]^T$, $b = [b_1 \; b_2 \; \cdots \; b_n]^T$

$(a+b)^T = [(a_1+b_1) \; (a_2+b_2) \; \cdots \; (a_n+b_n)]$ — ①

and $(a-b) = [(a_1-b_1) \; (a_2-b_2) \; \cdots \; (a_n-b_n)]^T$ — ②

$LHS = (a+b)^T(a-b)$

$= [(a_1+b_1) \; (a_2+b_2) \cdots (a_n+b_n)] \; [(a_1-b_1) \; (a_2-b_2) \cdots (a_n-b_n)]^T$

[from ① and ②]

$= (a_1+b_1)(a_1-b_1) + (a_2+b_2)(a_2-b_2) + \cdots (a_n+b_n)(a_n-b_n)$

$= a_1^2 - b_1^2 + a_2^2 - b_2^2 + \cdots a_n^2 - b_n^2$

$= (a_1^2 + a_2^2 + \cdots a_n^2) - (b_1^2 + b_2^2 + \cdots b_n^2)$ — ③

Now, $||a||^2 = (a_1^2 + a_2^2 + \cdots a_n^2)$

$||b||^2 = (b_1^2 + b_2^2 + \cdots b_n^2)$

$\therefore$ ③ becomes,

$= ||a||^2 - ||b||^2$

$= RHS$

Hence verified.

1.2 Given: A matrix B is symmetric if $B = B^T$.

To prove: (i) For any square matrix B, $B + B^T$ is symmetric.

(ii) If A is invertible, then $(A^{-1})^T = (A^T)^{-1}$.

(i) Let $C = B + B^T$.

We need to prove that $C = C^T$, i.e C is symmetric.

Consider RHS, $C^T = (B + B^T)^T$

$= B^T + (B^T)^T$ [~~$(A+B)^T$~~ Property: $(A+B)^T = A^T + B^T$]

$= B^T + B$ [Property: $(A^T)^T = A$]

$= B + B^T$ [Commutative under ~~addi~~ matrix addition].

$C^T = C$

Hence proved.

(ii) Let $B = A^{-1}$ [given A is invertible]

Then $B^T = (A^{-1})^T$ - (1)

~~Consid~~ Also, $AB = A(A^{-1}) = I$

Take transpose, $(AB)^T = I^T$

or $B^T A^T = I$

or $B^T = I(A^T)^{-1}$ [Post-multiply by $(A^T)^{-1}$]

or $B^T = (A^T)^{-1}$

or $(A^{-1})^T = (A^T)^{-1}$ [Since $B = A^{-1}$]

Hence proved.

1.3 To prove that L1 and L2 norms are equivalent i.e. for constants $c_1, c_2 \in \mathbb{R}$ such that $0 \le c_1 \le c_2$,

$$c_1 \|x\|_2 \le \|x\|_1 \le c_2 \|x\|_2 \quad \forall x.$$

Sol^n: Consider L1 norm squared,

$$\|x\|_1^2 = \left(\sum_{i=1}^{n} |x_i|\right)^2$$

$$= \sum_{i=1}^{n} |x_i|^2 + \sum_{i=1}^{n} \sum_{j!=i} |x_i||x_j| \quad - (1)$$

We know that arithmetic mean of 2 nos $\ge$ Geometric mean.

i.e for any x_i^2, x_j^2,

$$\frac{x_i^2 + x_j^2}{2} \ge \sqrt{x_i^2 x_j^2}$$

or

$$\frac{x_i^2 + x_j^2}{2} \ge |x_i||x_j|$$

Substitute this in (1) we get,

$$\|x\|_1^2 \le \sum_{i=1}^{n} |x_i|^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j!=i} \left(|x_i|^2 + |x_j|^2\right)$$

$$\le \sum_{i=1}^{n} |x_i|^2 + (n-1) \sum_{i=1}^{n} |x_i|^2$$

$$\le n \|x\|_2^2$$

or $\|x\|_1 \le \sqrt{n} \|x\|_2$

$\therefore \boxed{c_2 = \sqrt{n}}$

Also, $\|x\|_1^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i||x_j|$

$$\ge \sum_{i=1}^{n} |x_i|^2$$

$$\ge (\|x\|_2)^2$$

$\Rightarrow \boxed{c_1 = 1}$

Hence $c_1\|x\|_2 \le \|x\|_1 \le c_2 \|x\|_2$ where $c_1 = 1$ and $c_2 = \sqrt{n}$ & $0 \le c_1 \le c_2$

2. Linear programing

2.1 To minimize $z = 5x_1 + 2x_2$

- The points marked on the graph are,

$P_1(4,0)$, $P_2(2.4, 0.8)$, $P_3(0.429, 3.429)$, $P_4(0, 6)$.

- z is minimum at one of these points.

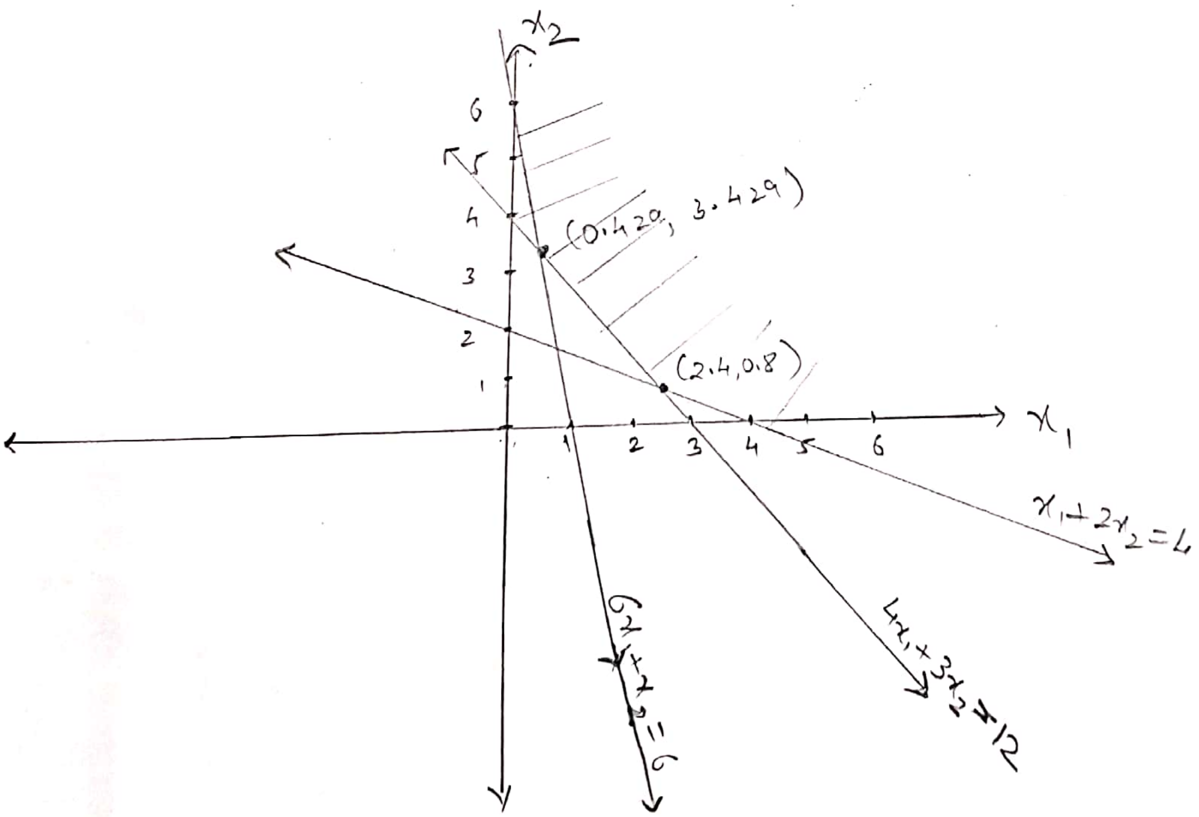
At P_1, $z = 5(4) + 2(0) = 20$

At P_2, $z = 5\cdot(2.4) + 2(0.8) = 13.6$

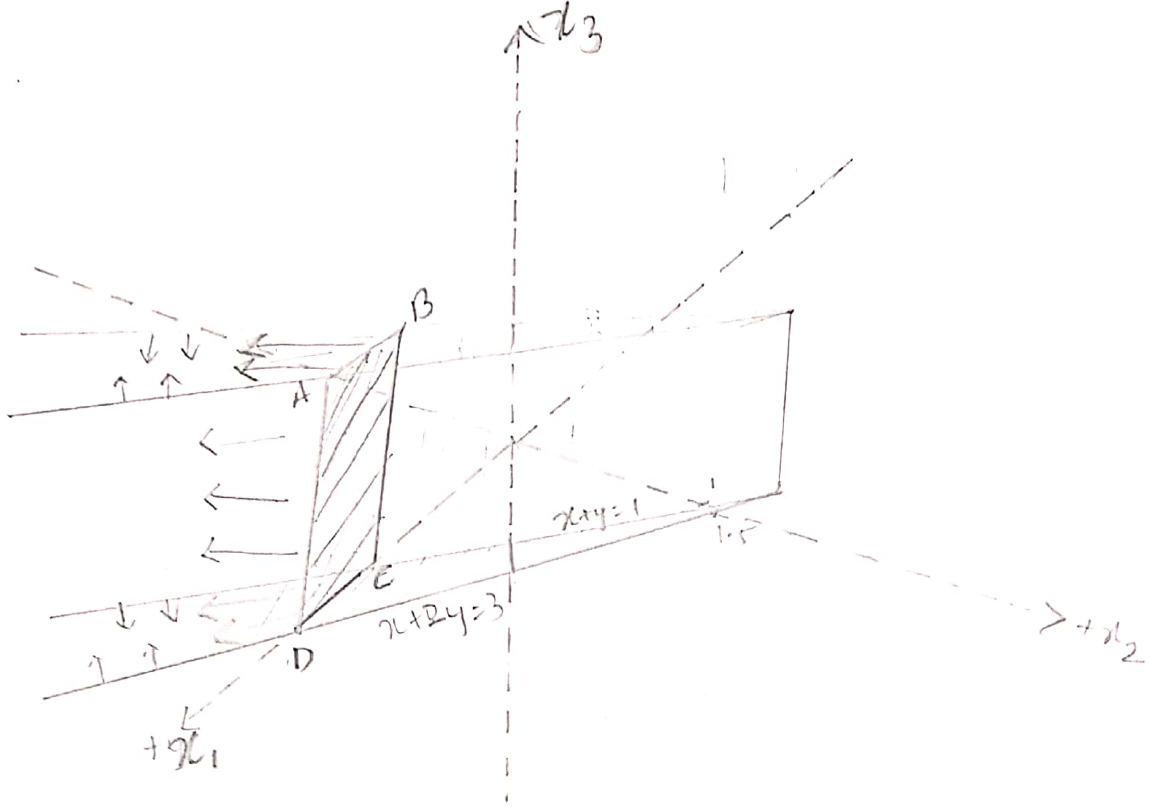
At P_3, $z = 5(0.429) + 2(3.429) = 9.003$

At P_4, $z = 5(0) + 2(6) = 12$

Therefore, $z = 5x_1 + 2x_2$ is minimum at $P_2(0.429, 3.429)$.



2.2



minimize $z = c_1x_1 + c_2x_2 + c_3x_3$

subject to
$$x_1 + x_2 \geq 1$$
$$x_1 + 2x_2 \leq 3$$
$$x_1 \geq 0 ;\ x_2 \leq 0$$
$$-1 \leq x_3 \leq 1$$

Optimal value when :-

(i) $c = (-1, 0, 1)$

The objective f^n becomes, $z = -x_1 + x_3$

∴ The corner points shown in graph & the value of z at

A, $z = -3+1 = -2$

at B, $z = -1+1 = 0$

at C, $z = -1+(-1) = -2$

at D, $z = -3-1 = -4$

∴ Minimum value is $z = -4$ at $D(3, 0, -1)$.

(ii) $c = (0, 1, 0)$

∴ The objective f^n becomes, min $z = x_2$

And min. value of $z = -\infty$.

(iii) $c = (0, 0, -1)$, The objective f^n becomes, min. $z = -x_3$

at A, $z = -1$, at C, $z = 1$ ∴ min value of $z = -1$ at

at B, $z = -1$, at D, $z = 1$ $A(3, 0, 1)$ and $B(1, 0, 1)$

Scanned by CamScanner

2.3 Transportation problem : LP formulation

Let x_{ij} be the number of cases to ship from cannery $i = 1, 2$ to warehouse $j = a, b, c$.

Therefore, the decision variable is,

$$x_{ij} \geq 0 \quad , \quad i = 1, 2 \quad , \quad j = a, b, c$$

(i) Considering the constraints on availability :-

The number of cases shipped out of each cannery i cannot be greater than the number of cases available.

Therefore, $\sum_{j=a,b,c} x_{1j} \leq 250$ or $x_{1a} + x_{1b} + x_{1c} \leq 250$ — ①

and $\sum_{j=a,b,c} x_{2j} \leq 450$ or $x_{2a} + x_{2b} + x_{2c} \leq 450$

(ii) Considering constraints on demand :

The amount demanded at each warehouse must be equal to the amount shipped from each cannery to the warehouse, This demand be met exactly.

Therefore,

$$x_{1a} + x_{2a} = 200$$

$$x_{1b} + x_{2b} = 200 \quad — ②$$

$$x_{1c} + x_{2c} = 200$$

The target is to minimize the transportation of two types of canneries from three warehouses.

Let z be this cost of transportation.

$$\therefore \quad z = 3.4x_{1a} + 2.2x_{1b} + 2.9x_{1c} + 3.4x_{2a} + 2.4x_{2b} + 2.5x_{2c}$$

Hence the LP formulation of above problem is,

$$\min \; Z = 3.4x_{1a} + 2.2x_{1b} + 2.9x_{1c} + 3.4x_{2a} + 2.4x_{2b} + 2.5x_{2c}$$

subject to

$$x_{1a} + x_{1b} + x_{1c} \leq 250$$

$$x_{2a} + x_{2b} + x_{2c} \leq 450$$

$$x_{1a} + x_{2a} = 200$$

$$x_{1b} + x_{2b} = 200$$

$$x_{1c} + x_{2c} = 200$$

and $x_{1a} \geq 0$, $x_{2a} \geq 0$, $x_{3a} \geq 0$, $x_{1b} \geq 0$, $x_{2b} \geq 0$, $x_{3b} \geq 0$, $x_{1c} \geq 0$,
$x_{2c} \geq 0$, $x_{3c} \geq 0$.

3. Graph Theory, Computational Complexity

3.1 Given: complete graph with $n=7$ vertices.

To find: Number of 5-length paths from vertex 4 to vertex 7.

Sol^n: Let A be adjecency matrix of complete graph K_7. Therefore A is 7×7 matrix with entries

$$A_{ij} = \begin{cases} 1 & i \neq j \\ 0 & i = j \end{cases}$$

The m^{th} power of adjacency matrix, A^m, gives the number of paths of length exactly m between any i and j node.

Hence we need to find the entry A^5_{47} i.e $i=4$, $j=7$ entry of matrix A^5.

Let J be 7×7 ~~ent~~ matrix with all entries as 1.

$\therefore$ 'A' can be written as, $A = J - I$ where $I \rightarrow$ identy matrix of 7×7

Also, ~~the~~ ~~J^m~~ each entry in J^m is 7^{m-1} and ~~each~~ $I^m = I \ \forall m$.

$\therefore \quad A^5 = (J-I)^5$

Using the binomial expansion, $(x-y)^n$ is given by

$${}^nC_0 x^{n-1} - {}^nC_1 x^{n-2} y + {}^nC_2 x^{n-3} y^2 - \dots \dots {}^nC_{n-1} x y^{n-1} + {}^nC_n y^{n-1}$$

$\therefore$ Any element in $(J-I)^5$ is

$${}^5C_0 (7)^{5-1} - {}^5C_1 (7)^{5-2} (1) + {}^5C_2 (7)^{5-3} (1)^2 - {}^5C_3 (7)^{5-4} (1)^3 + {}^5C_4 (7)^{5-5} - {}^5C_5 (1)^4$$

$$= 2401 - 1715 + 490 - 70 + 5$$

$$= 1111$$

$\therefore$ Number of 5-length paths between nodes 4 and 7 is 1111.

3.2 <u>Given:</u> f and g are unbounded, monotonically increasing function on $\mathbb{R}$.

<u>To verify:</u> If $f \in O(g) \Rightarrow \log(f) \in O(\log(g))$

Let $f(n) = O(g(n))$ for $n > n_0$

By definition of Big-O, there exist c and n_0 such that

$$f(n) \leq c\, g(n) \quad \forall n > n_0 \quad - \text{①}$$

Since f and g are monotonically increasing function, taking log on both sides of ① still holds the inequality.

$$\therefore \quad \log f(n) \leq \log(c\, g(n))$$

or $$\log f(n) \leq \log c + \log(g(n)), \quad \forall n > n_0 \quad - \text{②}$$

Since c and ~~n_0~~ n_0 are constants, there must be a const. c' such that

$$c' \geq \frac{\log c}{\log(g(n_0))} + 1$$

or $$(c'-1)\log(g(n_0)) \geq \cancel{c}\, \log c \quad - \text{③}$$

$$\therefore \quad \log f(n) \leq \log + \log(g(n))$$

$$\leq (c'-1)\log(g(n_0)) + \log(g(n)) \quad [\text{From ③}]$$

$$\leq (c'-1)\log(g(n)) + \log(g(n)) \quad [\text{Since } g(n) > g(n_0)\ \forall n > n_0 \text{ as } g \text{ is monotonically increasing}]$$

$$\Rightarrow \quad \log f(n) \leq c' \log(g(n)), \quad \forall n > n_0$$

or $$\log f(n) = O(\log(g(n))), \quad \forall n > n_0 \quad [\text{By definition of Big-O}]$$

Hence, the implication holds.

3.3 To verify: $\log(n!) \in \Theta(n\log n)$

or $c_2 n\log n \le \log(n!) \le c_1 n\log n$ for some const. c_1, c_2

(i) Proving RHS inequality:

$\log(n!) = \log(n \times (n-1) \times \ldots \times 1)$

$= \log n + \log(n-1) + \log(n-2) + \ldots \log(1)$

$\le \log n + \log n + \ldots \log(n)$ [Replacing n elements by $\log n$]

$\le n\log(n)$

or $\log(n!) = c_1 n\log(n)$ for some constant c_1. - 1

Hence RHS inequality proved.

(ii) Proving LHS inequality:-

$\log(n!) = \log(n \times (n-1) \times \ldots \times 1)$

$= \log(n) + \log(n-1) + \ldots + ~~\log~~ \log(\frac{n}{2}) + \log(\frac{n}{2}-1) + \ldots \log 1$

Replacing first $\frac{n}{2}$ elements of the sum by $\log(n/2)$

$\therefore \log(n!) \ge \log(\frac{n}{2}) + \log(n/2) + \ldots \log(n/2) + \log(\frac{n}{2}-1) + \ldots \log 1$

[Since $\log(n-i) \ge \log(\frac{n}{2}) \; \forall i \in [0, \frac{n}{2}]$]

$\Rightarrow \log(n!) \ge \frac{n}{2}\log(\frac{n}{2})$ [Removing last $\frac{n}{2}$ elements, & the inequality still holds]

$\therefore \log(n!) = c_2 n\log n$, for some const. c_2. - 2

Hence LHS inequality proved.

~~Hen~~ Therefore from 1 & 2, and ~~log(n!)~~ by definition of big-Θ,

$\log(n!) \in \Theta(n\log n)$.