OM HW 4

April 18, 2020

1 (1) Solve Ax = B

1.1

Solve $Ax = b$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$

(i) Is this system consistent?

(ii) Is the solution you have found, exact, least norm, or least square?

(iii) If the solution is not exact, what is the quantity being minimized?

1.2

In this problem, you will use the least square solution for $Ax = B$ to solve linear regression, by formulating linear regression as a matrix equation.

The data points are $\{(0, 6), (1, 0), (2, 0)\}$.

(i) What is the equation of a general line in slope intercept form?

(ii) If your data points lied on a line, what equations would hold?

(iii) Rewrite the system of equations from the previous part in matrix form ($Ax = B$). What are the elements of the matrices A, B, and x? What is the unknown matrix?

(iv) Solve your matrix equation to get the line of best fit. What is the equation of this line?

(v) Can you give a case where this slope intercept formulation would fail to yield a correct solution?

2 (2) Solve Ax=b

2.1

Solve the following system of linear equations
$$\begin{bmatrix} 2 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(i) Is this system overconstrained or underconstrained?

(ii) What quantity is being minimized to find a solution for this system?

(iii) Is this a least norm or least squares problem?

3 Simplex: Pivot Theory

3.1

Read each of the following statements carefully and decide Whether it is true or false. Briefly justify your answer.

(a) If a tie occurs in the pivot row choice during a pivot step while solving an LP by the Simplex Method, the basic feasible solution obtained after this pivot step is degenerate.

(b) In solving an LP by the Simplex Method, a different basic feasible solution is generated after every pivot step.

(c) The total number of optimal solutions of an LP is always finite since the total number of different bases is finite.

3.2

The standard Simplex Method determines the incoming column by selecting the column that maximizes the decrease in the objective function per unit increase of the incoming variable. In terms of the given canonical form, indicate how the standard Simplex Method determines the pivot row and pivot column.

3.3

A pivoting method is scale-invariant if the selection of pivot row and column remains unchanged under any positive scaling of some (or all) of the variables. A positive scaling of a variable x_j is of the form $x_j^{'} = d_j x_j$ where $d_j > 0$. Is the standard Simplex Method scale-invariant? Justify your answer.

3.4

An alternative pivoting method is to choose the pivot row and column so as to maximize the decrease in the objective function at each pivot step while maintaining feasibility. In terms of the canonical form, describe how this method determines the pivot row and pivot column.

3.5

Suppose that for an LP in standard form, the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has rank $r < m$, i.e., it has $m - r$ redundant equations.

1. Show that there will be at least $k = m - r$ artificial variables left in the tableau at the end of Phase I.

4 Convexity

4.1

Prove (or give a counter example) that a polygon with all interior angles less than 180 degrees is always convex.

4.2

Suggest (and formally verify)

(i) a function that is both convex and concave

(ii) A function that is neither convex nor concave.

4.3

The basic definition of a convex set C is: $\forall x, y \in C, \forall t \in [0,1] : tx + (1-t)y \in C$.

Starting from this basic definition show that the following holds: $\forall\{x_1, \ldots, x_k\} \in C, \forall\{\theta_1, \ldots, \theta_k\} \in [0,1] s.t. \sum_{i=1}^{k} \theta_i = 1 : \sum_{i=1}^{k} \theta_i x_i \in C$. (Hint: Use induction on k)

4.4

Verify whether the following set is convex or not:

$$C = \{(x,t) \in R^{n+1}, \|x\| \leq t\}$$

5 SVM

5.1

What is the advantge of finding the dual formulation of the **SVM**. Also, write the dual for the standard max-margin the formulation which allows for some points to be misclassified as long as the margin is good enough.

5.2

What is the primal objective of an **SVM** to separate two classes of points which are linearly separable by a hyper plane? Now, derive the corresponding Lagrangian dual.

5.3

Show that if there exists one hyper plane which separates two classes of points, we can find infinite more Which do the same. Give some ideas on how to judge how good a hyper plane is. Write the primal formulation for one of them. How do you change the formulation if you want to have the margin greater than some given number, While it is undesirable but not forbidden to have points of one class on the opposite side of the hyper plane?

6 KKT

6.1

How do you handle linearly dependent inequality conditions when solving a problem using the method of Lagrange multipliers? How do you handle nonnegativity constraints when using the **KKT** conditions? When are the **KKT** conditions necessary for optimality?

6.2

What are the **KKT** conditions? For what cases are these constraints sufficient for optimality?

7 Knapsack

7.1

For the following LP, express the optimal value and the optimal solution in term of the problem parameters. If the optimal solution is not unique, it is sufficient to give one optimal solution.

The variable is $\mathbf{x} \in \mathbb{R}^n$.

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & 1^T\mathbf{x} = k, \\ & 0 \leq x_i \leq 1 \quad i = 1 \ldots n \end{aligned}$$

k is an integer with $1 \leq k \leq n$.

7.2

For the following LP, express the optimal value and the optimal solution in term of the problem parameters. If the optimal solution is not unique, it is sufficient to give one optimal solution.

The variable is $\mathbf{x} \in \mathbb{R}^n$.

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & 1^T\mathbf{x} \leq k, \\ & 0 \leq x_i \leq 1 \quad i = 1 \ldots n \end{aligned}$$

k is an integer with $1 \leq k \leq n$.

7.3

For the following LP, express the optimal value and the optimal solution in term of the problem parameters. If the optimal solution is not unique, it is sufficient to give one optimal solution.

The variable is $\mathbf{x} \in \mathbb{R}^n$.

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{d}^T\mathbf{x} \geq \alpha, \\ & 0 \leq x_i \leq 1 \quad i = 1 \ldots n \end{aligned}$$

α and the components of $\mathbf{d}$ are positive.

7.4

For the following LP, express the optimal value and the optimal solution in term of the problem parameters. If the optimal solution is not unique, it is sufficient to give one optimal solution.

The variable is $\mathbf{x} \in \mathbb{R}^n$.

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{d}^T\mathbf{x} \leq k, \\ & 0 \leq x_i \leq 1 \quad i = 1 \ldots n \end{aligned}$$

k is a constant. Components of $\mathbf{d}$ are positive.

8 Sequence of Basic Feasible Solutions

8.1

Find the sequence of Basic Feasible Solutions to arrive at the optimal point for the following Linear program:

$$\begin{aligned} \text{maximize} \quad & z = 2x_1 + 3x_2 \\ \text{subject to} \quad & 5x_1 + 25x_2 \leq 40, \\ & x_1 + 3x_2 \geq 20, \\ & x_1 + x_2 = 20, \\ & x_1, x_2 \geq 0 \end{aligned}$$

8.2

Find the sequence of Basic Feasible Solutions and arrive at the optimal point.

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + 4x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 6, \\ & 2x_1 + 3x_2 \leq 9, \\ & x_1, x_2 \geq 0 \end{aligned}$$

8.3

Find the sequence of Basic Feasible Solutions and arrive at the optimal point.

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & -2x_1 + 2x_2 \geq 1, \\ & -8x_1 + 10x_2 \leq 13, \\ & x_1, x_2 \geq 0 \end{aligned}$$

8.4

Find the sequence of Basic Feasible Solutions and arrive at the optimal point.

$$\begin{aligned} \text{maximize} \quad & z = 2x_1 + 10x_2 \\ \text{subject to} \quad & 5x_1 + 2x_2 \leq 15, \\ & 2x_1 + x_2 \leq 20, \\ & x_1, x_2 \geq 0 \end{aligned}$$

9 (1) Linear Algebra

9.1

Show that the eigenvalues of a Hermitian matrix are equal.

9.2

If all the row-sums of a square matrix $\mathbf{A}$ have the same value (say $\mathbf{K}$), Show that $\mathbf{K}$ is an eigenvalue of $\mathbf{A}$.

9.3

Show that a square matrix $\mathbf{A}$ is singular if and only if 0 is an eigenvalue of $\mathbf{A}$.

10 (2) Linear Algebra

10.1

What can we say about matrix $\mathbf{B}$ if $\mathbf{AB} = \mathbf{BC}$ and matrices $\mathbf{A}$ and $\mathbf{C}$ have no common eigenvalues?

10.2

If $\mathbf{A}$ is a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ and x is a vector such that $\|\mathbf{x}\|_2 = 1$, prove that:

$$\min \mathbf{x}^T \mathbf{A} \mathbf{x} = \lambda_n$$

10.3

If A is a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ and x is a vector such that $\|\mathbf{x}\|_2 = 1$, show that

$$\mathbf{x}^T \mathbf{A} \mathbf{x} \leq \lambda_1$$

For what values of $\mathbf{x}$ does equality occur?