1. We draw the top 7 cards from a well-shuffled standard 52-card deck. Find the probability that:

(a) The 7 cards include exactly 3 aces.
(b) The 7 cards include exactly 2 kings.
(c) The probability that the 7 cards include exactly 3 aces. or exactly 2 kings, or both.

Solution:

(a) There are four aces in a pack of 52 cards. We want exactly 3 aces. We can select 3 aces in 4C3 ways. The remaining four cards can be selected in 48C4 ways.

Lastly, the sample space will be 52C7.

Therefore, the Probability that the 7 cards include exactly 3 aces is:

4C3 * 48C4 / 52C7

= 4 * 194,580 / 133,784,560
= 0.0058177

(b) There are four kings in a pack of 52 cards. We want 2 kings. We can select them in 4C2 ways.
The remaining five cards can be selected in 48C5 ways.

Lastly, the sample space will be 52C7.

Therefore, the probability that the 7 cards include 2 kings is:

4C2 * 48C5 / 52C7

= 6 * 1,712,304 / 133,784,560

= 10,273,824 / 133,784,560

= 0.076

(c) We will make use of subproblems (a) & (b)

The probability that the 7 cards include exactly 3 aces. or exactly 2 kings, or both can be shown as:

P(Exactly 3 aces) + P(exactly 2 kings) - P(Exactly 3 aces and exactly 2 kings)

We know the probabilities for exactly 3 aces and 2 kings. Let us calculate a probability for exactly 3 aces 2 kings.

We can select 3 aces and 2 kings in 4C3 * 4C2 ways and we know the sample space will be 52C7.

Therefore, the probability that the 7 cards include exactly 3 aces and 2 kings will be

4C3 * 4C2 * 44C2 / 52C7

= 4 * 6 * 946 / 133,784,560
= 0.0001697

Thus, the probability that the 7 cards include exactly 3 aces. or exactly 2 kings, or both will be

0.0058177 + 0.076 - 0.0001697

= 0.081648

2. Alice and Bob have 2n+1 coins, each coin with a probability of heads equal to 1/2. Bob tosses n+1 coins, while Alice tosses the remaining n coins. Assuming independent coin tosses, show that the probability that after all coins have been tossed, Bob will have gotten more heads than Alice is 1/2.

Solution:

Alice has n coins and bob has n + 1 coins.

Let us consider bob and Alice has tossed n coins.

In this case, there are 3 possibilities:

i. Bob has more heads than Alice.
ii. Alice has more heads than Bob
Iii. Bob and Alice have the same number of heads.

We will not consider possibility number 2 because in the question they have asked the probability of Bob getting more heads than Alice.

Let us consider the probability of Bob has more heads than Alice is p. Also, from the observation, we can See that the probability of Alice getting more heads than bob will also be p because they are tossing the same number of coins and the probability of getting heads in each toss is also the same i.e. 1 / 2.

According to the total probability law, the probability of getting a tie will be (1 - 2p).

Let W be an event where Bob has more heads than Alice.

Now in possibility no 1 since bob has more heads than Alice it doesn’t matter whether Bob gets more heads or not.

But for the third possibility, it is necessary for the bob to get a head in the last toss.

Therefore

P(W) = P(Head or tail in last toss) * (Probability of bob getting more heads than Alice in n toss) +
P(Getting head in last toss for bob) * (Probability of bob getting heads equal to Alice)

$= 1 * p + 1 / 2 (1 - 2p)$

$= p + 1 / 2 - p$
$= 1 / 2$

Hence proved

3. We are given three coins: one has heads in both faces, the second has tails in both faces, and the third has a head in one face and a tail in the other. We choose a coin at random, toss it, and the result is heads. What is the probability that the opposite face is tails?

Solution:

There are 3 coins

Coin 1 => HH (heads on both the sides)

Coin 2 => TT (tails on both the sides)

Coin 3 => HT (Head on one side and tail on the other)

We will first list down our events:

P(H) => Probability of getting heads

P(T) => Probability of getting tails

P(HH) => Probability of selecting HH coin

P(TT) => Probability of selecting TT coin

P(HT) => Probability of selecting HT coin

We know that $P(HH) = P(TT) = P(HT) = 1 / 3$

We have to find $P(HT / H)$ which can be given as

$= P(H / HT) * P(HT) / P(H)$

We know that $P(HT) = 1 /3$ and $P(H / HT) = 1 / 2$
We have to find $P(H)$

By using the Total Probability theorem, we can write down P(H) as

$P(H / HH) * P(HH) + P(H / TT) * P(TT) + P(H / HT) * P(HT)$

$= 1 * 1 /3 + 0 * 1 / 3 + 1 / 2 * 1 /3$

= 1 / 3 + 1 / 6

= 1 / 2

Therefore,P(HT / H)

= (1 / 2 * 1 / 3) / (1 / 2)

= 1 / 3

4. Each of the k jars contains m white and n black balls. A ball is randomly chosen from jar 1 and transferred jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e. it is m/(m+n).

Solution:

There are total k jars.

Let us define our events.

P(Wk) => Probability of getting a white ball from jar 1….k
P(Bk) => Probability of getting a black ball from jar 1….k

Probability of selecting a white ball from any of the jar provided you are not transferring the ball from the previous jar is = m / m + n
Probability of selecting a white ball from any of the jar provided you are not transferring the ball from the previous jar is = n / m + n

Now, the probability of selecting a white ball from 1 jar will be
P(W1) = m / m + n

We have to prove P(W1) = P(Wk) meaning P(Wk) = m / m + n

Now, when we transfer a ball from (k - 1)th jar to kth jar there will be 2 possibilities:

(a) You have transferred a white ball in kth jar
(b) You have transferred a black ball in kth jar

Therefore:
P(WK) = P(W(K-1)) * P(WK/WK-1) + P(B(K-1)) * P(WK/B(K-1))

where

P(W(k-1)) => Probability of selecting a white ball from k-1 th jar

P(WK/WK-1) => Probability of selecting white ball given that you have selected white ball from k-1 th jar.

$P(B(K-1))$ => Probability of selecting black ball from k-1 th jar

$P(Wk/(Bk-1))$ => Probability of selecting white ball given that you have selected black ball from k-1 th jar.

$P(W(k-1)) = m / (m + n)$

$P(WK/WK-1) = m + 1 / (m + n + 1)$

$P(B(K-1)) = n / (m + n)$

$P(Wk/(Bk-1)) = m / (m + n + 1)$

Therefore $P(Wk)$

$= (m / (m + n)) * (m + 1) / (m + n + 1) + n / (m + n) * m / (m + n + 1)$

$= (m / (m + n)) * [m+1+n / m + n + 1]$

$= m / (m + n)$

$= P(W1)$

Hence proved.

5. A power utility can supply electricity to a city from n different power plants.Power plant i fails with probability pi, independent of the others.
(a) Suppose that any one plant can produce enough electricity to supply the entire city.
What is the probability that the city will experience a black-out?
b) Suppose that two power plants are necessary to keep the city from a black-out.
Find the probability that the city will experience a black-out.

Solution:

(a)
Let X be an event where there is a complete blackout.

It is given that any one of the power plant is enough to provide entire electricity.

So, there would be complete black out if all the power plants stop working

It is given that power plant i fails with probability pi i.e 1st power plant fails with probability p1,2nd with p2,3 with p3 so on and so forth until nth power plant with pn.

Therefore the probability that the city will experience a black-out will be

$P(X = n) = p1 * p2 * p3 \ldots. Pn.$

(b)Let X be an event where there is a complete blackout.

It is given that two power plant are required to provide entire electricity.

For a complete blackout there will be 2 possibilities:

i. Entire power plants stops working
ii. Only one of the power plant is working and rest of the power plants are not working.

In probability terms it can be written as.

$P(X=n) + P(X = n - 1)$

Where $P(X = n)$ => N power plants failed

$P(X = n - 1)$ => N - 1 power plants failed.

We have calculated $P(X = n)$ from the above subproblem

For $P(X = n - 1)$ it will be

$(1 - p1) * p2 * p3 \ldots pn + (1 - p2) * p1 * p3 \ldots pn + \ldots\ldots + (1 - pn) * p1 * p2 * p3 .. p(n-1)$

Thus the probability that city will experience a complete blackout will be

$P(X=n) + P(X = n - 1)$

$= p1 * p2 * p3 \ldots pn + (1 - p1) * p2 * p3 \ldots pn + (1 - p2) * p1 * p3 \ldots pn + \ldots\ldots +$
$(1 - pn) * p1 * p2 * p3 .. p(n-1)$