Infinite Series

Series: If $u_1, u_2, u_3, \ldots u_n, \ldots$ is a sequence

Then $u_1 + u_2 + u_3 + \ldots = \sum_{n=1}^{\infty} u_n$ or $\sum u_n$ is called a series.

i.e if $\{u_n\}$ is a sequence, Then $\sum u_n$ is called a series.

(i) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$

(ii) $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} + \ldots =$

(iii) $1 + 2 + 3 + 4 + \ldots + n + \ldots$

(iv) $2 - 2 + 2 - 2 + 2 - 2 + \ldots$

Types of Series:-

1. Convergent series
2. Divergent series
3. Oscillatory series.

Partial Sum: The sum of first n terms of a series is called nth partial sum

i.e If $\sum u_n = u_1 + u_2 + u_3 + \ldots$ is a series

Then $S_n = u_1 + u_2 + u_3 + \ldots + u_n$ is called nth partial sum of the series $\sum u_n$.

If S be the sum of all terms of the series $\sum u_n$, Then

$$\lim_{n\to\infty} S_n = S$$

Series of positive terms:

If u_n is positive for all $n \in N$,
i.e. every term of a series is a +ve real number, then the series
$\sum u_n = u_1 + u_2 + u_3 + \cdots$ is called a series of +ve terms.

(i) $\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

(ii) $\sum \frac{1}{2n+1} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$

but (iii) $\sum (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$
is not a +ve terms series.

Convergent series:

A series $\sum u_n$ is said to be convergent if the sequence $\{u_n\}$ is convergent.

i.e. A series $\sum u_n$ is said to be convergent if $\lim_{n\to\infty} S_n$ = a finite & unique

Where $S_n = u_1 + u_2 + u_3 + \cdots + u_n$.

i.e. The sequence $\{S_n\} = S_1, S_2, S_3, \cdots S_n, \cdots$ is cgt

Where $S_1 = u_1$

$S_2 = u_1 + u_2$

$S_3 = u_1 + u_2 + u_3$

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$

Series

(i) Test the convergence of the sequence

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots+\frac{1}{2^{n-1}}+\cdots$$

Solution: Here $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}$

$$= \frac{1\cdot\{1-(\frac{1}{2})^n\}}{1-\frac{1}{2}}$$

{ Given series are in G.P. Where first term $= a = 1$, common ratio $= r = \frac{1}{2} < 1$, $S_n = \frac{a(1-r^n)}{1-r}$

$$= \frac{1-\frac{1}{2^n}}{\frac{1}{2}}$$

$$S_n = 2\left(1-\frac{1}{2^n}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right) = 2\left(1-\frac{1}{\infty}\right) = 2(1-0) = 2$$

$\lim_{n\to\infty} S_n = 2$ (finite & unique)

Hence $\Sigma \frac{1}{2^{n-1}}$ is convergent & converges to 2

i.e its limit $l = 2$

(ii) Test the convergency of the series

$$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots$$

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(2) Divergent Series:

A series Σu_n is said to be divergent if the sequence $\{u_n\}$ is divergent

i.e $\lim_{n\to\infty} S_n = \pm\infty$

① Test the convergency of the series
$1+2+3+4+\cdots$

Solution: $S_n = 1+2+3+\cdots+n$

$= \frac{n(n+1)}{2}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$.

Hence the given series Σn is divergent.

② $1+2+2^2+2^3+\cdots+2^{n-1}+\cdots$ is dgt.

Oscillatory Series:

A series Σu_n is said to be oscillatory if $\lim_{n\to\infty} S_n$ = not unique

i.e if Σu_n is neither cgt or dgt.

① $\Sigma(-1)^n = -1+1-1+1-1+1\cdots$

$S_1 = -1$, $S_2 = -1+1 = 0$, $S_3 = -1+1-1 = -1$

$\{S_n\} = \{-1, 0, -1, 0, -1, \cdots\}$

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Theorem (1) Necessary condition for convergent Series: of positive terms:

A series Σu_n of +ve terms is convergent Then $u_n \to 0$ as $n \to \infty$

i.e $\lim_{n\to\infty} u_n = 0$. but the converse need not be true

& if $\lim_{n\to\infty} u_n \neq 0$, then Σu_n must be divergent.

(1) $\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Here $u_n = \frac{1}{n}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

but $\Sigma \frac{1}{n}$ is not convergent (By p-series test)

(2) Test the convergency of the series

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$$

Here $u_n = \left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$$

Hence $\Sigma \left(1+\frac{1}{n}\right)^n$ is not cgt.

(3) $\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}} + \frac{4}{\sqrt{17}} + \cdots + \frac{2^n}{\sqrt{4^n+1}} + \cdots$

is not cgt.

Σu_n

Note: (1) A series of finite +ve terms is convergent if $\exists$ a number K such that

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n < K, \ \forall n \in N$$

if there exists no such no K

Then Σu_n is divergent.

(i) $\displaystyle \Sigma \frac{1}{\lfloor n} = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \cdots$

$$S_n = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \cdots + \frac{1}{\lfloor n}$$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}.$$

$$= \frac{1 \cdot \left\{1 - \left(\frac{1}{2}\right)^n\right\}}{1 - \frac{1}{2}} . \quad \left(\text{Given series are in G.P}\right)$$

$$= 2\left(1 - \frac{1}{2^n}\right)$$

$$\lim_{n\to\infty} S_n < \lim_{n\to\infty} 2\left(1 - \frac{1}{2^n}\right) = 2(1-0) = 2$$

$$\lim_{n\to\infty} S_n < 2, \ \forall n \in N.$$

$\displaystyle \Sigma \frac{1}{\lfloor n}$ is convergent.

(ii) The series $1 + \frac{1}{3} + \frac{1}{3} + \cdots$ is dgt

as $S_n > \frac{n}{3} \to \infty$.

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Geometric Series

A series of the form $1+r+r^2+r^3+\cdots$ is called a geometric series.

Convergency of Geometric Series

$$S_n = 1+r+r^2+r^3+\cdots+r^{n-1}$$

$$= \frac{1-r^n}{1-r} \text{ or } \frac{r^n-1}{r-1} \quad (r<1 \text{ or } r>1)$$

Case I When $|r|<1$,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^n-1}{r-1}$$

$$= \frac{1}{1-r} \quad \text{as } \lim_{n\to\infty} r^n = 0 \text{ if } |r|<1$$

$\therefore$ The series $\sum r^{n-1}$ is cgt

Case II $\sum r^{n-1}$ is divergent if $r \geq 1$

Case III $\sum r^{n-1}$ is oscillatory if $r \leq -1$

i.e A geometric series $\sum r^{n-1} = 1+r+r^2+r^3+\cdots$ is

(I) convergent if $|r|<1$
(II) divergent if $r \geq 1$
(III) oscillatory if $r \leq -1$

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① Test the convergency of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

i.e $\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\cdots$

Solution: Given series is a geometric series & $r=\frac{1}{2}<1$

$\therefore$ The given series is cgt.

② Test the convergence of the series

$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots$ $\rightarrow$ cgt as $r=\frac{1}{3}<1$

③ Test the convergence of the series

$1+2+2^2+2^3+\cdots$ $\rightarrow$ dgt as $r=2>1$

Auxilliary series test (p-series test).

A series of the form (type)

$\sum\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$ is

(i) convergent if $p>1$

& (ii) divergent if $p\leq 1$

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① $\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is dgt ~~convergent~~ series as $p = 1$

② $\Sigma \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is a convergent series as $p = 2 > 1$

③ $\Sigma \frac{1}{n^{3/2}} = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \frac{1}{4^{3/2}} + \cdots$ is a convergent series as $p = \frac{3}{2} > 1$

④ $\Sigma \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ is a divergent series as $p = \frac{1}{2} < 1$

Method for finding the auxilliary series:

① If $u_n = \frac{n^2+1}{n^3+2}$

take auxilliary series, $v_n = \frac{n^2}{n^3} = \frac{1}{n}$

Here $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent

Then Σu_n is also divergent

Auxilliary series, ~~v_n The highest degree~~

$$v_n = \frac{\text{The highest degree term in Nr of } u_n}{\text{The highest degree term in Dr of } u_n}$$

(2) If $u_n = \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$, Then

$$v_n = \frac{\sqrt{n^2}}{\sqrt{n^3}} = \frac{n}{n^{3/2}} = \frac{1}{\sqrt{n}}$$

Here $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ is dgt as $p = \frac{1}{2} < 1$

Then Σu_n is also dgt.

(3) If $u_n = \frac{\sqrt{n^2+1}}{n^3}$, Then

$$v_n = \frac{\sqrt{n^2}}{n^3} = \frac{n}{n^3} = \frac{1}{n^2}$$

Here $\Sigma v_n = \Sigma \frac{1}{n^2}$ is cgt

Then Σu_n is also cgt.

First Comparision test:

If Σu_n & Σv_n be two ~~terms~~ series of positive terms such that ~~such that~~ $|u_n| \le |v_n|, \ \forall n$

(i) If Σv_n is cgt, Then Σu_n is also cgt.

(ii) If Σu_n is dgt, Then Σv_n is dgt.

Example (1) check the convergence of $\Sigma \frac{1}{n^2} \sin \frac{1}{n}$

Solution: We have

$$\left|\frac{1}{n^2} \sin \frac{1}{n}\right| \le \frac{1}{n^2}$$

Here $\Sigma \frac{1}{n^2}$ is

So $\Sigma \frac{1}{n^2}$ si

(2) Test the co

Solution: We h

$n^2 - \cos^2$

$\Rightarrow \frac{1}{n^2 - \cos}$

$\Rightarrow \frac{n}{n^2 - c}$

$\Rightarrow \frac{n}{n^2 - c}$

Here $\Sigma \frac{1}{n}$

So $\Sigma \frac{n}{n^2 -}$

(3) Test the c

Solution: $n^4 < n$

$\Rightarrow \frac{1}{n^4+5} <$

$\Rightarrow \frac{n^2+2}{n^4+5} <$

$\Rightarrow \frac{n^2+2}{n^4+5}$

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Here $\sum \frac{1}{n^2}$ is cgt

So $\sum \frac{1}{n^2} \sin \frac{1}{n}$ is also cgt.

(2) Test the convergency of $\sum \frac{n}{n^2 - \cos^2 n}$

Solution: We have

$$n^2 - \cos^2 n \le n^2$$

$$\Rightarrow \frac{1}{n^2 - \cos^2 n} \ge \frac{1}{n^2}$$

$$\Rightarrow \frac{n}{n^2 - \cos^2 n} \ge \frac{n}{n^2}$$

$$\Rightarrow \frac{n}{n^2 - \cos^2 n} \ge \frac{1}{n}$$

Here $\sum \frac{1}{n}$ is dgt

So $\sum \frac{n}{n^2 - \cos^2 n}$ is also dgt.

(3) Test the convergence of $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$

Solution: $\because n^4 < n^4 + 5$

$$\Rightarrow \frac{1}{n^4+5} < \frac{1}{n^4}$$

$$\Rightarrow \frac{n^2+2}{n^4+5} < \frac{n^2+2}{n^4}$$

$$\Rightarrow \frac{n^2+2}{n^4+5} < \frac{1}{n^2} + 2\frac{1}{n^4}$$

Here $\sum \frac{1}{n^2}$ is cgt as $p = 2 > 1$

& $\sum \frac{1}{n^4}$ is cgt as $p = 4 > 1$

$\Rightarrow \sum \frac{1}{n^2} + 2 \sum \frac{1}{n^4}$ is also cgt.

$\Rightarrow \sum \frac{n^2+2}{n^4+5}$ is also cgt.

Second comparision test (limit form)

If $\sum u_n$ & $\sum v_n$ be two series of positive terms such that

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l \text{ (finite \& non-zero)}$$

Then $\sum u_n$ & $\sum v_n$ ~~both~~ are either both convergent or both divergent.

Test the convergency of the following series

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+n}}$ (ii) $\sum_{n=1}^{\infty} \frac{n}{(n+1)^3}$ (iii)

(iii) $\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots \infty$

(iv) $\frac{1}{4\cdot 7\cdot 10} + \frac{4}{7\cdot 10\cdot 13} + \frac{9}{10\cdot 13\cdot 16} + \cdots \infty$

(v) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots + \cancel{\infty}$

(XIV) $\sum_{\infty} \sin \frac{1}{n}$ –

(VI) $\sum_{n=1} \frac{1}{\sqrt{n}+\sqrt{\ldots}}$

(VIII) $\sum_{n=1}^{\infty} \frac{2n^3+\ldots}{4n^5+\ldots}$

(X) $\sum_{n=1}^{\infty} (\sqrt{n^2+1}\ldots$

Solution: (i) $\sum_{n=1}^{\infty}$

Here $u_n =$

~~Now~~ ~~But $\sum v_n$~~

Now $\lim_{n\to\infty} \frac{u_n}{v_n} =$

$= \text{Lt}$

$\lim_{n\to\infty} \frac{u_n}{v_n} =$

But $\sum v_n = \sum$

as $p =$

so $\sum u_n$ is

(XIV) $\sum\limits_{\infty} \sin\frac{1}{n}$ — dgt

(vi) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ → dgt (vii) $\sum\limits_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n-1}}$ → dgt

(viii) $\sum\limits_{n=1}^{\infty} \frac{2n^3+5}{4n^5+1}$ → cgt (IX) $\sum \frac{\sqrt{n}}{\sqrt{n^3+1}}$ → dgt

(X) $\sum\limits_{n=1}^{\infty} (\sqrt{n^2+1}-n)$ → dgt (XI) $\sum(\sqrt{n^3+1}-\sqrt{n^3-1})$ → cgt

(XII) $\sum(\sqrt{n^3+1}-\sqrt{n^3})$ → cgt

Solution: ① $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n^5+n}}$ (XIII) $\sum(\sqrt{n^4+1}-\sqrt{n^4-1})$

Here $u_n = \frac{1}{\sqrt{n^5+n}}$ & $v_n = \frac{1}{n^{5/2}}$

~~Now But $\sum u_n$~~ $\cancel{\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n^5+n}} \times \frac{n^{5/2}}{1}}$

Now $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n^5+n}} \times \frac{n^{5/2}}{1}$

$= \lim\limits_{n\to\infty} \frac{\cancel{n^{5/2}}}{\cancel{n^{5/2}}\left(\sqrt{1+\frac{1}{n^4}}\right)} = \frac{1}{\sqrt{1+0}} = 1$

$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0$

But $\sum v_n = \sum \frac{1}{n^{5/2}}$ is a cgt series.

as $p = \frac{5}{2} > 1$

So $\sum u_n$ is also cgt.