1. e^{x-y}. Maths

Let ~~function~~ function f be $f = e^{x-y}$

~~There~~ The first order partial derivates of f is $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$.
We Find -

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{x-y})$$

$$e^{x-y} \frac{\partial}{\partial x}(x-y)$$

$$e^{x-y}(1-0)$$

$$= e^{x-y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(e^{x-y})$$

$$= e^{x-y}\frac{\partial}{\partial y}(x-y)$$

$$= e^{x-y}(0-1)$$

$$= -e^{x-y}.$$

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$\therefore$ The first order derivates of e^{x-y} are $\frac{\partial f}{\partial x} = e^{x-y}$, $\frac{\partial f}{\partial y} = -e^{x-y}$

2. Euler's theorem on Homogenous function -
If $z = f(x,y)$ is a homogenous function of (x,y) of Degree 'n' then
$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ for all $x, y \in$ domain of the function.

3. Neighbourhood of a point (a,b)
Let δ be any positive integer the points (x,y) such that $a-\delta \le x \le a+\delta$
$b-\delta \le y \le b+\delta$.
Determined in a square bounded by the lines for $x = a-\delta, a+\delta$;
$y = b-\delta, b+\delta$. this square is called a neighbourhood of the point (a,b)
$\therefore$ The set $\{(x,y) : a-\delta \le x \le a+\delta,\ b-\delta \le y \le b+\delta\}$ is the neighbourhood of the point (a,b).

Section B.

Question 1.

A. $U = \log(x^2+y^2+z^2)$ prove that $x\frac{\partial^2 U}{\partial y \partial x} = y\frac{\partial^2 U}{\partial x \partial y}$

Sol Given $U = \log(x^2+y^2+z^2)$

Then

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)\right)$$

$$= \frac{1}{x^2+y^2+z^2}\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= \frac{1}{x^2+y^2+z^2}(2x+0+0)$$

$$= \frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(\log(x^2+y^2+z^2)\right)$$

$$= \frac{1}{x^2+y^2+z^2}\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= \frac{2y}{x^2+y^2+z^2}$$

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2+z^2}\right)$$

$$= 2x\left(\frac{-1}{(x^2+y^2+z^2)^2}\right)\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= \frac{-2x}{(x^2+y^2+z^2)^2}(0+2y+0)$$

$$= \frac{-4xy}{(x^2+y^2+z^2)^2}$$

~~Take~~ Mult x on both side

$$x\frac{\partial^2 U}{\partial y \partial x} = x\left(\frac{-4xy}{(x^2+y^2+z^2)^2}\right)$$

$$= \frac{-4x^2y}{\cancel{x}(x^2+y^2+z^2)^2} = \frac{-4xy}{(x^2+y^2+z^2)^2}$$

$$\frac{1}{x} = \frac{-1}{x^2}$$

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2+z^2}\right)$$

$$2y\left(\frac{-1}{(x^2+y^2+z^2)^2}\right)\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$\frac{-2y}{(x^2+y^2+z^2)^2}(2x+0+0)$$

$$\frac{-4xy}{(x^2+y^2+z^2)^2}$$

. Mult y on both side

$$y\frac{\partial^2 U}{\partial x \partial y} = y\left(\frac{-4xy}{(x^2+y^2+z^2)^2}\right)$$

$$y\frac{\partial^2 U}{\partial x \partial y} = \frac{-4xy^2}{y(x^2+y^2+z^2)^2}$$

$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ Hence proved.

B. Discuss Maxima & Minima of x^2+xy.

Sol Let the given function be $f(x,y)$ then $f = x^2+xy$.

Now will get values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+xy)$$
$$= 2x+(1)y$$
$$= 2x+y.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+xy)$$
$$= 0+x(1)$$
$$= x.$$

Now take $\frac{\partial f}{\partial x} = 0$,

$\cancel{\partial f}$ $2x+y=0 \rightarrow (1)$.

$\frac{\partial f}{\partial y} = 0$, $x=0 \rightarrow (2)$

substitute 2 in (1)

$$2(0)+y=0$$
$$0+y=0 \quad \therefore y=0.$$

$\therefore (0,0)$ is the stationary point.

~~To find At (0~~

$$\Rightarrow r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x+y)$$
$$= 2+0 = 2.$$

$$s = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x+y) = 0+1$$
$$= 1.$$

$$t = \frac{\partial^2 f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x) = 0.$$

then $rt - s^2 = 2(0) - (1)^2$
$$= 0-1$$
$$= -1$$

$\therefore rt - s^2 < 0, r > 0$. $\therefore$ F has no extreme values. At (0,0)

Question 2

Verify euler's theorem for $z = x^n \log\left(\frac{y}{x}\right)$

Sol. Given $z = x^n \log\left(\frac{y}{x}\right)$

Clearly z is a homogeneous function of degree n to verify euler theorem we have to prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$.

$\because = x^n[\log y - \log x]$ $\left[\because \log\frac{m}{n} = \log m - \log n\right]$.

Now.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^n(\log y - \log x)\right)$$ [$\because$ we apply uv here as there are x in term).

$$= x^n\frac{\partial}{\partial x}(\log y - \log x) + (\log y - \log x)\frac{\partial}{\partial x}(x^n)$$

$$= x^n\left(0 - \frac{1}{x}\right) + (\log y - \log x)\, nx^{n-1}$$

$$\left[nx^{n-1} = n\left(\frac{x^n}{x}\right)\right]$$

$$\frac{x\partial z}{\partial x} = x\left[\left(\frac{-x^n}{x}\right) + (\log y - \log x)\,n\left(\frac{x^n}{x}\right)\right]$$

$$= -x^n + (\log y - \log x)(nx^n).$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^n(\log y - \log x)\right]$$

$$= x^n\frac{\partial}{\partial y}(\log y - \log x)$$

$$= x^n\left(\frac{1}{y} - 0\right)$$

$$= \frac{x^n}{y}.$$

$$y\frac{\partial z}{\partial y} = y\frac{x^n}{y} \qquad y\frac{\partial z}{\partial y} = x^n.$$

$$\therefore x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y} = \cancel{x^n} + \cancel{(-x^n)} + (\log y - \log x)(nx^n)$$

$$= (\log y - \log x)(nx^n)$$

$$= nx^n(\log y - \log x) \qquad [\because x^n(\log y - \log x) = z].$$

$$= nz$$

Hence euler theorem is verified

i.e $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$

6 find $\frac{dy}{dx}$ for f $x^3+y^3=3axy$.

Given $x^3+y^3=3axy$, $x^3+y^3-3axy=0$.

We know

let the given fn be $f(x,y)=x^3+y^3-3axy=0$.

~~We know that~~ $\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^3+y^3-3axy)$

$=3x^2+0-3ay(1)$.

$=3x^2-3ay$

$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^3+y^3-3axy)$

$=0+3y^2-3ax(1)$

$=3y^2-3ax$.

$\therefore$ we know that for fn $f(x,y)=0$, then $\frac{dy}{dx}=\frac{-f_x}{f_y}$.

$$\frac{dy}{dx}=\frac{-(3x^2-3ay)}{3y^2-3ax}=\frac{-\cancel{3}(x^2-ay)}{\cancel{3}(y^2-ax)}$$

$$=\frac{-x^2-ay}{y^2-ax}=\frac{ay-x^2}{y^2-ax}.$$

Question 3.

b $u=x^2-y^2$, $x=2r-3s+4$, $y=-r+8s-5$. find $\frac{\partial u}{\partial r}$.

Sol. We know that $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$.

Given $u=x^2-y^2$.

$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(x^2-y^2)=2x-0=2x$

$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}(x^2-y^2)=0-2y=-2y$.

also $x=2r-3s+4$.

$\frac{\partial x}{\partial r}=\frac{\partial}{\partial r}(2r-3s+4)=2(1)-0+0=2$.

$y=-r+8s-5$

$\frac{\partial y}{\partial r}=\frac{\partial}{\partial r}(-r+8s-5)=-1+0-0=-1$

$$-\frac{\partial u}{\partial u} = (2x)(2) + (-2)(-1) = 4x+2y.$$

Q. If $u = \log\left[\frac{x^4+y^4}{x+y}\right]$ show the $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

Sol Given $u = \log\left[\frac{x^4+y^4}{x+y}\right]$

$$e^u = \frac{x^4+y^4}{x+y} = z$$

$$= \frac{x^{\cancel{4}\,3}\left[1+\frac{y^4}{x^4}\right]}{x\left[1+\frac{y}{x}\right]} = x^3\phi\left[\frac{y}{x}\right].$$

By eulers theorem we get.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3z.$$

$$\Rightarrow x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 3e^u$$

$$\Rightarrow xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u$$

$$\cancel{e^u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 3\cancel{e^u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$

Hence proved.