Design
&
Analysis of Algorithms

Function Comparison

Exercises

Given two functions f(n) and g(n). Find out the asymptotic relationship between them in each of the following problems.

1. If $\mathbf{f(n) = 2^n}$ & $\mathbf{g(n) = n^2}$, check their asymptotic relation

- Before applying substitution of values, take log on both sides and reduce the function to the maximum.
- Applying log on both sides, we get

$f(n) = \mathbf{\log 2^n}$ $\qquad\qquad\qquad$ $g(n) = \mathbf{\log n^2}$

Now, By using the formula,

$\log a^b = b \log a$, we get

$n \log 2$ **$2 \log n$**

We know, $\log 2 = 1$ (ie. $\log_2 2 = 1$)

n **$2 \log n$**

Next, Substitute the value of $n = 2^{100}$, we get

2^{100} **$2 \log 2^{100}$**

$= 2 * 100 \log 2$ (using log formula)

2^{100} **$= 200$**

- Thus, we can say that f(n) is having a higher value than g(n)

- Therefore, **$f(n) > g(n)$**

  ➔ **$f(n) = \Omega\ g(n)$**

2. $f(n) = 3^n$ & $g(n) = 2^n$

- Taking log on both sides,

$$\log 3^n \qquad\qquad \log 2^n$$

Applying log formula **($\log a^b = b \log a$)**

$$n \log 3 \qquad\qquad n \log 2$$

Cancelling **n** on both sides since they are common, we get

$$\log 3 \qquad\qquad \log 2$$

Here, we can conclude asymptotically both are equal , but value wise $f(n) = \log 3$ is greater.

Thus we can write, $f(n)>g(n)$ ➔ $f(n) = \Omega\, g(n)$

3. $f(n) = n^2$ & $g(n) = n\log n$

We can write this as

$n * n$ $\quad\quad$ $n \log n$

Cancelling out the common term, we get

n $\quad\quad$ $\log n$

From this itself, we can conclude **$f(n) > g(n)$**

But if needed, we can substitute **n = any value**

Here, let's put **$n = 2^{100}$** , we get

2^{100} $\quad\quad$ $\log 2^{100} \rightarrow 100 * \log 2$

$\mathbf{2^{100}}$ $\quad\quad$ $\rightarrow 100$

Therefore , we can say, **$f(n) = \Omega\, g(n)$**

4. $f(n) = (n + k)^m \quad g(n) = n^m$

- By using the formula, $\mathbf{(a+b)^n} = \mathbf{a^n + \ldots}$, we take only the highest power = $\mathbf{a^n}$

- So, $f(n) = \mathbf{n^m}$ & $g(n) = \mathbf{n^m}$
- Thus, we see the relation as $\mathbf{f(n) = g(n)}$

$$\Rightarrow \mathbf{f(n) = \theta\, g(n)}$$

Homework

1. $f(n) = 2^{2n}$ & $g(n) = 2^n$
2. $f(n) = n^{\log n}$ & $g(n) = 2^n$
3. $f(n) = 2^{n+1}$ & $g(n) = 2^n$

Exercises

5. $f(n) = n$ & $g(n) = (\log n)^{100}$

Applying log, we get

$\log n$	$100 \log \log n$

Substitute $n = 2^{128}$

We get, $\log 2^{128}$	$100 \log \log 2^{128}$

Applying log formula $(\log a^b = b \log a)$

128	$100 \log 128$

Some values to remember

log128 = (log 128 / log 2)= 7

Similarly, log1024 = (log1024/log2) = 10

log 2 = 1

log 4 = 2

log 128 =7

log 256 = 8

log 1024 = 10

Exercise 5 – continues…

128 **100 x 7 = 700**

Now, we can say **$f(n) < g(n)$**

Again to recheck, let us substitute **$n = 2^{1024}$**

We get, **$\log 2^{1024}$** **$100 \times \log \log 2^{1024}$**

= 1024 **$100 \times \log 1024$**

= 1024 **$100 \times 10 = 1000$**

Now, f(n) is found to be greater. i.e. after some huge input value, we can see n is growing max. Thus, we can conclude that $f(n) > g(n)$

$$\Rightarrow f(n) = \Omega\, g(n)$$

6. $f(n) = n^{\log n}$ & $g(n) = n\log n$

- Since there is no common term to cancel, we can apply log to reduce it

log n log n **log (n logn)**

We can use the formula, **log(ab) = log a + log b**

log n log n **log n + log log n**

Now, Substitute $n = 2^{1024}$

$\log 2^{1024} \times \log 2^{1024}$ **$\log 2^{1024} + \log\log 2^{1024}$**

1024 x1024 **1024 + log 1024**

1024 x 1024 **1024 + 10 = 1034**

Thus, f(n) > g(n)

Now, just to recheck, let us substitute

$$n = 2^{2^{20}}$$

We get,

$$\log 2^{2^{20}} \text{ x } \log 2^{2^{20}} \qquad\qquad \log 2^{2^{20}} + \log\log 2^{2^{20}}$$

$$2^{20} \text{ x } 2^{20} \qquad\qquad 2^{20} + \log 2^{20}$$

$$2^{20} \text{ x } 2^{20} \qquad\qquad 2^{20} + 20$$

Thus, we can say that , $f(n) > g(n)$

$$\Rightarrow f(n) = \Omega\, g(n)$$

7. $f(n) = \sqrt{\log n}$ & $g(n) = \log\log n$

- We can write $\sqrt{\log n} = (\log n)^{1/2}$
- Now, Applying **log** on both sides , we get

$$\log (\log n)^{1/2} \qquad \log\log\log n$$

$$\tfrac{1}{2} \log\log n \qquad \log\log\log n$$

Now, by substituting $n = 2^{2^{20}}$

$$\tfrac{1}{2} \log\log 2^{2^{20}} \qquad \log\log\log 2^{2^{20}}$$

$$\tfrac{1}{2} \log 2^{20} \qquad \log\log 2^{20}$$

$$\tfrac{1}{2} \times 20 \qquad \log 20$$

$$= 10 \qquad = 4.32 \approx 4$$

Thus, $f(n) > g(n) \Rightarrow f(n) = \Omega\, g(n)$

8. $f(n) = n^{\sqrt{n}}$ & $g(n) = n^{\log n}$

Applying **log**, we get

$$\sqrt{n} \log n \qquad \log n \log n$$

Cancelling the common term log n, we get

$$\sqrt{n} \qquad \log n$$

This can be written as,

$$(n)^{1/2} \qquad \log n$$

Applying **log** again, we get

$$\tfrac{1}{2} \log n \qquad \log \log n$$

Now, substituting **$n = 2^{128}$**

We get, **$\frac{1}{2} \log 2^{128}$** **$\log \log 2^{128}$**

$\frac{1}{2} \times 128$ **$\log 128$**

64 **7**

Thus, we can conclude that,

$f(n) > g(n) \Rightarrow f(n) = \Omega\, g(n)$

Also, try with 2^{256} and see the result

9. $f(n) = n^2 \log n$ & $g(n) = n(\log n)^{10}$

Applying log on both sides,

$\log(n^2 \log n)$ **$\log[n(\log n)^{10}]$**

By applying the formula ,

$\log(ab) = \log a + \log b$ **,** we get

$\log n^2 + \log \log n$ **$\log n + \log (\log n)^{10}$**

This can be written again as:-

$2\log n + \log \log n$ **$\log n + 10 \log \log n$**

From here, we can conclude **2logn** is greater than **logn** but again a confusion arises since **10loglog** is greater than **loglogn.**

But to give more proof, let us substitute **$n = 2^{128}$**

Now we get,

$2 \log 2^{128} + \log \log 2^{128}$	$\log 2^{128} + 10 \log \log 2^{128}$
$2 \times 128 + \log 128$	$128 + 10 \log 128$
$256 + 7$	$128 + 10 \times 7$
263	198

Now it can be clearly said **$f(n) > g(n)$**

$$\Rightarrow f(n) = \Omega\, g(n)$$

10. $f(n) = 2^{\log n}$ & $g(n) = n^{\sqrt{n}}$

Applying log on both sides,

$$\log 2^{\log n} \qquad \log n^{\sqrt{n}}$$

$$\log n \log 2 \qquad \sqrt{n} \log n$$

Cancelling the common term **logn** from both side we get

$$\log 2 \qquad \sqrt{n}$$

$$1 \qquad \sqrt{n}$$

Now, it is very clear that $\sqrt{n}$ is greater than **1**

Thus, $f(n) < g(n) \Rightarrow f(n) = O\, g(n)$

11. $f(n) = n^{\log n}$ & $g(n) = 2^{\sqrt{n}}$

Applying log on both sides, we get

$\log n^{\log n}$	$\log 2^{\sqrt{n}}$
$\log n \log n$	$\sqrt{n} \log 2$
$(\log n)^2$	$\sqrt{n} \Rightarrow n^{1/2}$

Applying log again, we get

$\log (\log n)^2$	$\log n^{1/2}$
$2 \log \log n$	$\frac{1}{2} \log n$

Now, substitute $n = 2^{128}$

$2 \log \log 2^{128}$	**$\frac{1}{2} \log 2^{128}$**
$2 \log 128$	**$\frac{1}{2} \times 128$**
$2 \times 7 = 14$	**64**

Thus, we can conclude that

$$f(n) < g(n) \Rightarrow f(n) = O\,g(n)$$

12. $f(n) = 3n^{\sqrt{n}}$ & $g(n) = 2^{\sqrt{n}\log n}$

Here, before applying log, let us try to use another formula on **g(n)**

we know, $\log a^b = b \log a$, then we can write **g(n)** as

$$2^{\sqrt{n}\log n} \Rightarrow 2^{\log n^{\sqrt{n}}}$$

Now, again by using the formula**,**

$$a^{\log_c b} = b^{\log_c a}$$

g(n) will be now in the form,

$$(n^{\sqrt{n}})^{\log 2}$$

$$\Rightarrow n^{\sqrt{n}}$$

Now, we have

$$f(n) = 3n^{\sqrt{n}} \quad \text{and} \quad g(n) = n^{\sqrt{n}}$$

From here we can conclude **$f(n) > g(n)$** either by cancelling the common term or by just looking into its value.

Thus, **$f(n) = \Omega\, g(n)$**

Homework

1. Given four functions

$$f1 = 2^n$$

$$f2 = n^{3/2}$$

$$f3 = n\log n$$

$$f4 = n^{\log n}$$

Compare each function with the other and find their sequence in asymptotic order of worst case to best case.

Conversion of Base in Log

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Example -1

$$\log_6 (1/3) = \frac{\log_{10} (1/3)}{\log_{10} 6}$$

$$= -\,0.6131$$

Example -2

$$\log_{3/5}(6/7) = \frac{\log_{10}(6/7)}{\log_{10}(3/5)}$$

$$= 0.3017$$

Example -3

$$\log_{\sqrt{5}} (10) = \frac{\log_{10} (10)}{\log_{10} (\sqrt{5})}$$

$$= 2.86135$$

Example - 4

$$\log_9 27 = \frac{\log_3 27}{\log_3 9}$$

$$= 1.5$$

Problem -1

- **What is the asymptotic relationship between the functions**

$$n^3\log_2 n \quad \text{and} \quad 3n\log_8 n$$

Solution

We have

$$f(n) = n^3\log_2 n \quad \text{and} \quad g(n) = 3n\log_8 n$$

Now, let us simplify **g(n)** as

$$3n \times \log_8 n$$

$$3n \times \frac{\log_2 n}{\log_2 8}$$

$$= 3n \times \frac{\log n}{3}$$

Now, we have

$$\mathbf{f(n) = n^3\log_2 n \qquad \& \qquad g(n) = 3n \times \frac{\log n}{3}}$$

$$\mathbf{n^3\log n \qquad\qquad\qquad n \log n}$$

Cancelling out the common terms, we get

$$\mathbf{n^2 \qquad\qquad\qquad 1}$$

Now it can be clearly said **f(n) > g(n)**

$$\Rightarrow \mathbf{f(n) = \Omega\, g(n)}$$

Problem -2

$f(n) = \log_2 n$ and $g(n) = \log_8 n$

This can be written as,

$$\log_2 n \qquad\qquad \frac{\log_2 n}{\log_2 8}$$

$$\log_2 n \qquad\qquad 1/3 \times \log_2 n$$

Cancelling out the common term, we get

$$1 \qquad\qquad 1/3$$

Thus, we can say $f(n) > g(n)$

$\Rightarrow f(n) = \Omega\, g(n)$

Problem - 3

$$f(n) = \log_2 n^{\log_2 17} \quad \& \quad g(n) = \log_2 17^{\log_2 n}$$

Solution

Here, let us write **f(n)** as,

$$\log_2 17 \log_2 n$$ **[by using $\log a^b = b \log a$]**

Similarily, **g(n)** as

$$\log_2 n \log_2 17$$ **[by using $\log a^b = b \log a$]**

Now we have,

$$\log_2 17 \log_2 n \qquad\qquad \log_2 n \log_2 17$$

Cancelling out the common term, we get

$$1 \qquad\qquad 1$$

Thus, we can conclude that $f(n) = g(n)$

$$\Rightarrow f(n) = \theta\, g(n)$$