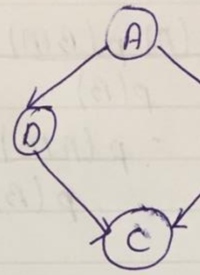
Q.3) Bayesian Network

a) Network :-



Equation I, $A \perp C \mid B, D$.

We have two paths to reach C from A.

1) $A \longrightarrow D \longrightarrow C$ Head-tail path.

$$p(A, D, C) = p(A)\, p(D \mid A)\, p(C \mid D)$$

When D is observed,

$$p(A, C \mid D) = \frac{p(A, D, C)}{p(D)} = \frac{p(A)\, p(D \mid A)\, p(C \mid D)}{p(D)}$$

$$= \frac{p(A, D)\, p(C \mid D)}{p(D)} = \frac{p(A \mid D)\, p(D)\, p(C \mid D)}{p(D)}$$

$$= p(A \mid D)\, p(C \mid D).$$

This indicates that,

$A \perp C \mid D$ is true.

2) $A \longrightarrow B \longrightarrow C$ Head-tail path.

$$p(A, B, C) = p(A)\, p(B \mid A)\, p(C \mid B)$$

When B is observed.

$$p(A, C \mid B) = \frac{p(A, B, C)}{p(B)} = \frac{p(A)\, p(B \mid A)\, p(C \mid B)}{p(B)}$$

$$= \frac{p(A, B)\, p(C \mid B)}{p(B)} = \frac{p(A \mid B)\, p(B)\, p(C \mid B)}{p(B)}$$

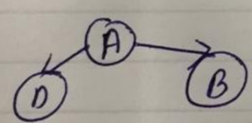
$$= p(A \mid B)\, p(C \mid B).$$

This indicates that $A \perp C \mid B$ is true.

Therefore Equation I :- $A \perp C \mid B, D$ is true.

Equation II, $B \perp D \mid A, C$

We have two path to ~~stan~~ reach D from B

1) 

Tail - Tail path

$$p(A, B, D) = p(A)\, p(D \mid A)\, p(B \mid D)$$

When A is observed,

$$p(B, D \mid A) = \frac{p(A, B, D)}{p(A)} = \frac{p(A)\, p(D \mid A)\, p(B \mid A)}{p(A)}$$

$$= p(D \mid A)\, p(B \mid A)$$

This indicates that $B \perp D \mid C$ is true.

2) $B \rightarrow C \leftarrow D$ Head - Head path.

$p(B,C,D) = p(B)\,p(D)\,p(C \mid B,D)$.

When C is observed,

$$p(B,D \mid C) = \frac{p(B,C,D)}{p(C)} = \frac{p(B)\,p(D)\,p(C \mid B,D)}{p(C)}$$

We have,

$$p(B,D) = \sum_{C} p(B,D,C) = p(B)\,p(D) \sum_{C} \underbrace{p(C \mid B,D)}_{1}$$

$$= p(B)\,p(D).$$

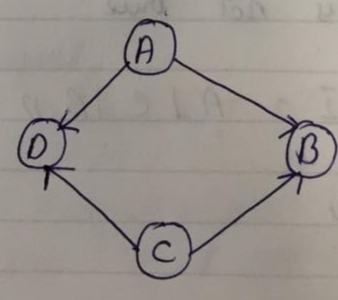
$\because B \perp D \mid \phi$, they are marginally independent

$\therefore p(B,D \mid C) \neq p(B \mid C)\,p(D \mid C)$

$\therefore B \perp D \mid C$ is not true

Therefore Equation II :- $B \perp D \mid A, C$ is false.

b) Network :-



Equation I :-

$A \perp C \mid B, D$.

We have two paths to reach to C from A.

1) $A \rightarrow B \leftarrow C$ Head-Head path.

$$p(A, B, C) = p(A)\, p(C)\, p(B \mid A, C)$$

When B is observed,

$$p(A, C \mid B) = \frac{p(A, B, C)}{p(B)} = \frac{p(A)\, p(C)\, p(B \mid A, C)}{p(B)}$$

We have,

$$p(A, C) = \sum_B p(A, B, C) = p(A)\, p(C) \underbrace{\sum_B p(B \mid A, C)}_{1}$$

$$= p(A)\, p(C)$$

$\therefore A \perp C \mid \phi$, they are marginally independent.

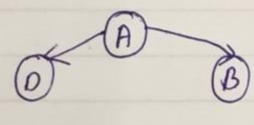
$$p(A, C \mid B) \neq p(A \mid B)\, p(C \mid B)$$

$\therefore$ $A \perp C \mid B$ is not true.

Therefore Equation I :- $A \perp C \mid B, D$ is false.

Equation II, $B \perp D \mid A, C$.

We have two paths to reach D from B.

1) 

Tail- Tail path

$$p(A,B,D) = p(A)\, p(D|A)\, p(B|A).$$

When A is observed,

$$p(B,D \perp A) = \frac{p(A,B,D)}{p(A)} = \frac{p(A)\, p(D|A)\, p(B|A)}{p(A)}$$

$$= p(D|A)\, p(B|A)$$

This indicate that $B \perp D \mid A$ is true.

2) 

Tail - Tail path.

$$p(B,C,D) = p(C)\, p(B|C)\, p(D|C)$$

When C is observed,

$$p(B,D|C) = \frac{p(B,C,D)}{p(C)} = \frac{p(C)\, p(B|C)\, p(D|C)}{p(C)}$$

$$= p(B|C)\, p(D|C)$$

This indicates that $B \perp D \mid C$ is true.

Therefore Equation II :- $B \perp D \mid A, C$ is true.