Akshay Tumkur Renukaprasad
MTK number : 03694060

Problem 1

Let the function be denoted by g(θ)

$g(\theta) = \theta^t(1-\theta)^h$

Differentitating w.r.t to θ we get the first derivative

$g'(\theta) = t\theta^{t-1}(1-\theta)^h - h\theta^t(1-\theta)^{h-1}$

Differentitating w.r.t to θ again, we get the second derivate:

$g''(\theta) = t(t-1)\theta^{t-2}(1-\theta)^h - 2th\theta^{t-1}(1-\theta)^{h-1} + h(h-1)\theta^t(1-\theta)^{h-2}$

Now we let f(θ)=log g(θ) and find the 1st and 2nd derivatives

$f(\theta) = log(g(\theta)) = \text{t } \log(\theta) + \text{h } \log(1\text{-}\theta)$

$f'(\theta) = \frac{t}{\theta} - \frac{h}{1-\theta}$

and $f''(\theta) = \frac{h}{(1-\theta)^2} - \frac{t}{\theta^2}$

Problem 2

We find points on f(θ) = log g(θ) by equating f'(θ) to 0

$$\Rightarrow \frac{t}{\theta} - \frac{h}{1-\theta} = 0$$

$$\Rightarrow \frac{t-(t+h)\theta}{\theta(1-\theta} = 0$$

$$\Rightarrow t - (t+h)\theta = 0$$

$$\Rightarrow \theta = \frac{t}{t+h} \qquad (1)$$

We substitute the value of θ in f''(θ) with that in equation (1) and get

$f''(\theta) = \frac{(t+h)^2}{h} - \frac{(t+h)^2}{t} = \frac{t-h}{th}(t+h)^2 <$ 0 if h > t

Again replacing the value of θ in g'(θ) with that in equation (1) we get

$$g'(\theta) = t\left[\frac{t}{t+h}\right]^{t-1}\left[1-\frac{t}{t+h}\right]^{h} - h\left[\frac{t}{t+h}\right]^{t}\left[1-\frac{t}{t+h}\right]^{h-1}$$

$$g'(\theta) = \frac{t^t h^h}{(t+h)^{t+h-1}} - \frac{t^t h^h}{(t+h)^{t+h-1}} = 0 \qquad (2)$$

Hence, the critical points are same.
Now we find if it is a maximum or not.
Now g"((θ) is given by:

$$g''(\theta) = t(t-1)\theta^{t-2}(1-\theta)^{h} - 2th\theta^{t-1}(1-\theta)^{h-1} + h(h-1)\theta^{t}(1-\theta)^{h-2}$$

$$g''\left(\frac{t}{t+h}\right) = t(t-1)\left(\frac{t}{t+h}\right)^{t-2}\left(1-\frac{t}{t+h}\right)^{h} - 2th\left(\frac{t}{t+h}\right)^{t-1}\left(1-\frac{t}{t+h}\right)^{h-1} + h(h-1)\left(\frac{t}{t+h}\right)^{t}\left(1-\frac{t}{t+h}\right)_{h-2}$$

On solving further we get,

$$g''\left(\frac{t}{t+h}\right) = -\frac{t^{t-1}h^{h-1}}{(t+h)^{t+h-1}} < 0 \text{ as t,h} \in \mathbb{N}$$

Therefore, log function retains the critical points of the main function

Problem 3

We know that θ_{MAP} is the maximum of the p(θ = x | D) and is given by':

$$\theta_{MAP} = \frac{N_T+a-1}{N+a+b-1} \text{ and } \theta_{MLE} = \frac{N_T}{N}$$

If $\theta_{MAP} = \theta_{MLE}$ then a=b=1
This means that the prior distribution is uniform i.e there exists a prior p(θ) such that the result holds. Also, we know that such a prior will always exist. Hence, it is true that θ_{MLE} is a special case of θ_{MAP}

Problem 4

Now $\theta_{MLE} = \frac{m}{m+l}$ and
$E_{\mathrm{PR}}[\theta|a,b] = \frac{a}{a+b}$

The consequent posterior distribution is given by the Beta distribution Beta(x | a+m, b+l) and the mean of this is:

$$E_{\mathrm{PS}}[X] = \frac{a+m}{a+b+m+l} = \frac{a}{a+b+m+l} + \frac{m}{a+b+m+l}$$

Let $0 \leq \lambda \leq 1$ be $\frac{a+b}{a+b+m+l}$

Then we get,

$$\lambda E_{\mathrm{PR}} = \left[\frac{a+b}{a+b+m+l}\right]\frac{a}{a+b} = \frac{a}{a+b+m+l} \qquad (4)$$

and $(1-\lambda)\theta_{\text{MLE}} = \left[\frac{m+l}{a+b+m+l}\right]\frac{m}{m+l} = \frac{m}{a+b+m+l}$ (5)

On adding equation (4) and (5) we have, λE_{PR} +

$(1-\lambda)\theta_{\text{MLE}} = E_{\text{PS}}$

Problem 5

The Poisson Distribution of X is given by

$$P(X = k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Now, for n i.i.d samples for X the probability is given by:

$$P(D|\lambda) = \prod_{i=1}^{n} \frac{\lambda^{k_i} e^{-\lambda}}{k_i!}$$

$$\Rightarrow P(D|\lambda) = e^{-n\lambda} \prod_{i=1}^{n} \frac{\lambda^{k_i}}{k_i!} \qquad (6)$$

Taking log of equation (6) and denoting it by f(λ) we have,

$$f(\lambda) = -n\lambda + \Big(\sum_{i=1}^{n} k_i\Big) \log\lambda + \log\Big(\sum_{i=1}^{n} k_i\Big)$$

Differentiating w.r.t to λ and equating to 0 we get

$$f'(\lambda) = -n + \frac{\sum_{i=1}^{n} k_i}{\lambda} = 0$$

$$\Rightarrow \lambda = \frac{\sum_{i=1}^{n} k_i}{n}$$

Therefore we have $\theta_{\text{MLE}} = \frac{\sum_{i=1}^{n} k_i}{n}$

Now we let the prior have a Gamma distribution with constants α and β

$$P(\lambda|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma\alpha}\lambda^{\alpha\text{ -}1}e^{-\lambda\beta}$$

Now the posterior distribution is given by:

$$P(\lambda|D) = \frac{P(D|\lambda).P(\lambda)}{P(D)}$$

On replacing values we get,

$$P(\lambda|D) = \frac{1}{P(D)}.\frac{e^{-n\lambda}\lambda^{\sum k_i}}{\prod k_i}.\frac{\beta^\alpha}{\Gamma\alpha}\lambda^{\alpha\text{ -}1}e^{-\lambda\beta}$$

$\Rightarrow P(\lambda|D) = che^{(-n+\beta)\lambda}.\lambda^{(\alpha -1 + Pk_i)}$i where c is a constant

Again taking log of the above function we get,

$(-n+\beta)\lambda + (\alpha - 1 + {}^{X}k_i)\log\lambda$

On finding the derivative and equating it to 0 we get,

$$\lambda = \frac{\alpha-1+\sum k_i}{(-n+\beta)} = \theta_{\text{MAP}}$$