Likelihood

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Overview

Generating Functions

Analytic view of probability

- Recall the Taylor's series expansion of a function of one variable around h.
$T_a(x) \equiv f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + ... \equiv \sum_{n=0}^{\infty} f^n(a)\frac{(x-a)^n}{n!}$
$T_h(x) \equiv f(x+h) = f(h) + f'(h)\frac{(x)}{1!} + f''(h)\frac{(x)^2}{2!} + f'''(h)\frac{(x)^3}{3!} + ... \equiv \sum_{n=0}^{\infty} f^n(h)\frac{(x)^n}{n!}$
- Taylor's series describes an approximation technique for a function which is continuously differentiable to nth degree.

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- Taylor's series describes an approximation technique for a function which is continuously differentiable to nth degree.
- Now consider $M_X(t) = E[e^{tX}]$. Recall $E(f(x)) = \sum f(x)p(x)$.

MGF and CGF

Analytic view of probability contd.

- The function $M_X(t)$ computes *all* the moments of X and thus completely specifies a distribution, just as the Taylor's series completely specifies a function to an arbitrary extent.
- To obtain any given moment from the function, we only need to differentiate it w.r.t. t.
- It can be helpful in getting a particular moment and for proofs of certain theorems.

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- To obtain any given moment from the function, we only need to differentiate it w.r.t. t.
- It can be helpful in getting a particular moment and for proofs of certain theorems.
- Characteristic Function of X is $E[e^{itX}]$.
- It always exists for a continuous probability density while an MGF may or may not exist.

MGF of the Gaussian

MGF for the univariate Gaussian

As an example we'll find the MGF of the univariate Gaussian distribution.

- Recall, $M_X(t) = E[e^{tX}]$. For a gaussian $p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$
- So $M_{X=0}(t) = \int\limits_{-\infty}^{\infty} e^{tx} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx$

$$= \int\limits_{-\infty}^{\infty} e^{tx} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}z^2 + t\sigma z} dz$$

MGF of the Gaussian

MGF for the Gaussian contd.

- $$M_{X=0}(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - t\sigma)^2} dz$$
  $$= e^{\mu t + \frac{1}{2}t^2\sigma^2}$$
- $M_{X=\mu}(t) = E(e^{t(x-\mu)}) = e^{\frac{1}{2}t^2\sigma^2}$
- Now that we have the MGF, any moment of the distribution is: For $X \equiv \mathcal{N}(\mu, \sigma^2) E(X^n) = M^{\{n\}}(t)|_{t=0} = \frac{d^n}{dx^n} e^{\mu t + \frac{1}{2}t^2\sigma^2}|_{t=0}$
- E.g. $E(X) = M'(t)|_{t=0} = [(\mu + t\sigma^2)e^{\mu t + \frac{1}{2}t^2\sigma^2}]|_{t=0} = \mu$
  $E(X^2) = M''(t)|_{t=0} =$
  $[\sigma^2 e^{\mu t + \frac{1}{2}t^2\sigma^2} + (\mu + t\sigma^2)^2 e^{\mu t + \frac{1}{2}t^2\sigma^2}]|_{t=0} = \mu^2 + \sigma^2$
- From which we can get
  $Var(X) = E(X^2) - E^2(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

Some useful identities

Markov's and Chebyshev's inequalities

- We start with Markov's inequality
  $P(X \geq a) \leq E(X)/a$
  The proof is fairly simple.

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- Substituting $g(X)$ for X, we get $P(X \geq a) \leq E(g(X))/a$
- If we put $g(X) = (X - \bar{x})^2$, then
  $P((X - \bar{x})^2 \geq \epsilon^2) \leq \frac{E[(X-\bar{x})^2]}{\epsilon^2}$.
  Or, $P(|X - \bar{x}| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}$.

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  $P((X - \bar{x})^2 \geq \epsilon^2) \leq \frac{E[(X-\bar{x})^2]}{\epsilon^2}$.
  Or, $P(|X - \bar{x}| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}$.
  Or, $1 - P(|X - \bar{x}| \leq \epsilon) \leq \frac{Var(X)}{\epsilon^2}$.
  Or, $P(|X - \bar{x}| \leq \epsilon) \geq 1 - \frac{Var(X)}{\epsilon^2}$
- This is also known as weak law of large numbers. (How?)

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Weak law of large numbers

- Well that exists not for $P(X)$ but $P(E(X))$.
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  $Var(X) = E[(X - \bar{X})^2]$

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  $$\begin{aligned} Var(X) &= E[(X - \bar{X})^2] \\ &= E(X^2) - \mu^2 \\ &= \frac{1}{n^2}E(\sum x_i x_j) - \mu^2 \end{aligned}$$

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$$
\begin{aligned}
Var(X) &= E[(X-\bar{X})^2] \\
&= E(X^2) - \mu^2 \\
&= \frac{1}{n^2}E(\sum x_i x_j) - \mu^2 \\
&= \frac{1}{n^2}[\sum E(x_i x_j) - n^2\mu^2] \\
&= \frac{1}{n^2}\sum[E(x_i x_j) - \mu^2] \\
&= \frac{1}{n^2}[\sum(E(x_i^2) - \mu^2) + \sum_{x_i \neq x_j}(E(x_i x_j) - \mu^2)] \\
&= \frac{1}{n^2}[n\sigma^2 + 0] \\
&= \frac{1}{n}\sigma^2
\end{aligned}
$$

Some useful identities

Weak law of large numbers contd.

- From the first variant of Chebyshev's inequality
  $P(|X - \bar{X}| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$
  $\lim\limits_{n \to \infty} P(|X - \bar{X}| \geq \epsilon) = 0$
- Also see this link[1]
- Which method looks more correct?
- Are both the same?

[1] *Weak Law of Large Numbers at Woflram Mathworld.* http://mathworld.wolfram.com/WeakLawofLargeNumbers.html. Accessed: 2018-05-30.

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  $P(|X - \bar{X}| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$
  $\lim_{n \to \infty} P(|X - \bar{X}| \geq \epsilon) = 0$
- Also see this link[1]
- Which method looks more correct?
- Are both the same?
- Prove Bernoulli's Inequality:
  If X is distributed according the a binomial (Bernoulli) distribution, with the probability of succes p, then for some $\epsilon, \delta, \exists N$ depending on ϵ, δ such that
  $P[|\frac{X}{n} - p| < \epsilon] > 1 - \delta,$ for $n > N$

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Some Limit Theorems

There's a family of Central Limit Theorems

- You can see the wikipedia page[2] for small list of them.
- We'll mention two and prove one of them.
- Lindberg Levy CLT:
  For $\{X_1, X_2, ..., X_n\}$ sequence of (non-Normal)iid random variables with $E(X) = \mu$ and $Var[X_i] = \sigma^2 < \infty$
  $$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu\right) \xrightarrow{d} N\left(0, \sigma^2\right).$$

[2] *Central Limit Theorems.* https://en.wikipedia.org/wiki/Central_limit_theorem. Accessed: 2018-06-01.

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  $$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu\right) \xrightarrow{d} N\left(0, \sigma^2\right).$$
- We'll prove and alternative statement, that for $Z = \frac{(\bar{X}-\mu)}{\sigma/\sqrt{n}}$, $Z \xrightarrow{d} N(0, \sigma^2)$

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Some Limit Theorems

Lindberg Levy CLT

- Proof:
  $M_z(t) = M_{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}(t) = E(\frac{(\bar{X}-\mu)t}{\sigma/\sqrt{n}})$

Some Limit Theorems

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- Proof:
$$M_z(t) = M_{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}(t) = E(\frac{(\bar{X}-\mu)t}{\sigma/\sqrt{n}})$$
$$= \int_{-\infty}^{\infty} e^{\frac{(\bar{X}-\mu)t}{\sigma/\sqrt{n}}} P(\bar{X})d\bar{X}$$

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- Proof:

$$M_z(t) = M_{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}(t) = E(\frac{(\bar{X}-\mu)t}{\sigma/\sqrt{n}})$$

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$$= \int_{-\infty}^{\infty} e^{\frac{\bar{X}t-\mu t}{\sigma/\sqrt{n}}} P(\bar{X})d\bar{X}$$

Some Limit Theorems

Lindberg Levy CLT

- Proof:

$$
\begin{aligned}
M_z(t) = M_{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}(t) = E\left(\frac{(\bar{X}-\mu)t}{\sigma/\sqrt{n}}\right) \\
&= \int_{-\infty}^{\infty} e^{\frac{(\bar{X}-\mu)t}{\sigma/\sqrt{n}}} P(\bar{X}) d\bar{X} \\
&= \int_{-\infty}^{\infty} e^{\frac{\bar{X}t-\mu t}{\sigma/\sqrt{n}}} P(\bar{X}) d\bar{X} \\
&= \int_{-\infty}^{\infty} e^{\frac{\bar{X}t\sqrt{n}}{\sigma}} e^{\frac{-\mu t\sqrt{n}}{\sigma}} P(\bar{X}) d\bar{X}
\end{aligned}
$$

Some Limit Theorems

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\begin{aligned}
M_z(t) = M_{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}(t) &= E\left(\frac{(\bar{X}-\mu)t}{\sigma/\sqrt{n}}\right) \\
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&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} \int_{-\infty}^{\infty} e^{\frac{\bar{X}t\sqrt{n}}{\sigma}} P(\bar{X}) d\bar{X}
\end{aligned}
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&= \int_{-\infty}^{\infty} e^{\frac{\bar{X}t\sqrt{n}}{\sigma}} e^{\frac{-\mu t\sqrt{n}}{\sigma}} P(\bar{X})d\bar{X} \\
&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} \int_{-\infty}^{\infty} e^{\frac{\bar{X}t\sqrt{n}}{\sigma}} P(\bar{X})d\bar{X} \\
&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{\frac{\bar{X}t\sqrt{n}}{\sigma}}]
\end{aligned}
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&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} \int_{-\infty}^{\infty} e^{\frac{\bar{X}t\sqrt{n}}{\sigma}} P(\bar{X}) d\bar{X} \\
&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{\frac{\bar{X}t\sqrt{n}}{\sigma}}] \\
&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{\frac{(\frac{\sum X_i}{n})t\sqrt{n}}{\sigma}}]
\end{aligned}
$$

Some Limit Theorems

Lindberg Levy CLT

- Proof contd:

$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{\frac{(\frac{\sum X_i}{n})t\sqrt{n}}{\sigma}}]$$

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$$= e^{\frac{-\mu t \sqrt{n}}{\sigma}} E[e^{\frac{(\frac{\sum X_i}{n}) t \sqrt{n}}{\sigma}}]$$

$$= e^{\frac{-\mu t \sqrt{n}}{\sigma}} E[e^{X_1 \frac{t \sqrt{n}}{n\sigma}} e^{X_2 \frac{t \sqrt{n}}{n\sigma}} ... e^{X_n \frac{t \sqrt{n}}{n\sigma}}]$$

Some Limit Theorems

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$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{X_1 \frac{t\sqrt{n}}{n\sigma}}] E[e^{X_2 \frac{t\sqrt{n}}{n\sigma}}] ... E[e^{X_n \frac{t\sqrt{n}}{n\sigma}}]$$

Why?

Some Limit Theorems

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$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{X_1 \frac{t\sqrt{n}}{n\sigma}}] E[e^{X_2 \frac{t\sqrt{n}}{n\sigma}}] ... E[e^{X_n \frac{t\sqrt{n}}{n\sigma}}]$$

X_i are i.i.d.! $E(X,Y) = E(X)E(Y)$ if X, Y are independent

Some Limit Theorems

Lindberg Levy CLT

- Proof contd:

$$
\begin{aligned}
&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{\frac{(\frac{\sum X_i}{n})t\sqrt{n}}{\sigma}}] \\
&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{X_1 \frac{t\sqrt{n}}{n\sigma}} e^{X_2 \frac{t\sqrt{n}}{n\sigma}} ... e^{X_n \frac{t\sqrt{n}}{n\sigma}}] \\
&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{X_1 \frac{t\sqrt{n}}{n\sigma}}] E[e^{X_2 \frac{t\sqrt{n}}{n\sigma}}] ... E[e^{X_n \frac{t\sqrt{n}}{n\sigma}}] \\
&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E^n[e^{X \frac{t}{\sqrt{n}\sigma}}]
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&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E^n[e^{X\frac{t}{\sqrt{n}\sigma}}] \\
&= e^{\frac{-\mu t\sqrt{n}}{\sigma}} [M_X(\frac{t}{\sqrt{n}\sigma})]^n
\end{aligned}
$$

Some Limit Theorems

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$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} [M_X(\frac{t}{\sqrt{n}\sigma})]^n$$

Or, $\ln M_z(t) = \frac{-\mu t\sqrt{n}}{\sigma} + n\ln[M_X(\frac{t}{\sqrt{n}\sigma})]$

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Or, $\ln M_z(t) = \frac{-\mu t\sqrt{n}}{\sigma} + n \ln[M_X(\frac{t}{\sqrt{n}\sigma})]$

$$= \frac{-\mu t\sqrt{n}}{\sigma} + n \ln E[e^{\frac{tX}{\sqrt{n}\sigma}}]$$

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$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{X_1\frac{t\sqrt{n}}{n\sigma}} e^{X_2\frac{t\sqrt{n}}{n\sigma}} ... e^{X_n\frac{t\sqrt{n}}{n\sigma}}]$$

$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E[e^{X_1\frac{t\sqrt{n}}{n\sigma}}]E[e^{X_2\frac{t\sqrt{n}}{n\sigma}}]...E[e^{X_n\frac{t\sqrt{n}}{n\sigma}}]$$

$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} E^n[e^{X\frac{t}{\sqrt{n}\sigma}}]$$

$$= e^{\frac{-\mu t\sqrt{n}}{\sigma}} [M_X(\frac{t}{\sqrt{n}\sigma})]^n$$

Or, $\ln M_z(t) = \frac{-\mu t\sqrt{n}}{\sigma} + n\ln[M_X(\frac{t}{\sqrt{n}\sigma})]$

$$= \frac{-\mu t\sqrt{n}}{\sigma} + n\ln E[e^{\frac{tX}{\sqrt{n}\sigma}}]$$

$$= \frac{-\mu t\sqrt{n}}{\sigma} + n\ln\Big[\int \Big(1 + \frac{t}{\sqrt{n}\sigma}\frac{X}{1!} + (\frac{t}{\sqrt{n}\sigma})^2\frac{X^2}{2!} + ...\Big)P(X)dX\Big]$$

Some Limit Theorems

Lindberg Levy CLT

- Proof contd:
  So, $\ln M_z(t) =$
  $$\frac{-\mu t\sqrt{n}}{\sigma} + n\ln\Big[\int\Big(1 + \frac{t}{\sqrt{n}\sigma}\frac{X}{1!} + \Big(\frac{t}{\sqrt{n}\sigma}\Big)^2\frac{X^2}{2!} + ...\Big)P(X)dX\Big]$$

Some Limit Theorems

Lindberg Levy CLT

- Proof contd:
  So, $\ln M_z(t) =$
  $$\frac{-\mu t\sqrt{n}}{\sigma} + n\ln\left[\int\left(1 + \frac{t}{\sqrt{n}\sigma}\frac{X}{1!} + \left(\frac{t}{\sqrt{n}\sigma}\right)^2\frac{X^2}{2!} + \ldots\right)P(X)dX\right]$$
  $$= \frac{-\mu t\sqrt{n}}{\sigma} + n\ln\left[1 + \frac{t}{\sqrt{n}\sigma}\mu_1 + \left(\frac{t}{\sqrt{n}\sigma}\right)^2\mu_2/2! + \ldots\right]$$

Some Limit Theorems

Lindberg Levy CLT

- Proof contd:

  So, $\ln M_z(t) =$

  $$\frac{-\mu t\sqrt{n}}{\sigma} + n\ln\left[\int\left(1 + \frac{t}{\sqrt{n}\sigma}\frac{X}{1!} + \left(\frac{t}{\sqrt{n}\sigma}\right)^2\frac{X^2}{2!} + ...\right)P(X)dX\right]$$

  $$= \frac{-\mu t\sqrt{n}}{\sigma} + n\ln\left[1 + \tfrac{t}{\sqrt{n}\sigma}\mu_1 + \left(\tfrac{t}{\sqrt{n}\sigma}\right)^2\mu_2/2! + ...\right]$$

  $$= \frac{-\mu t\sqrt{n}}{\sigma} + n\left[\left(\tfrac{t}{\sqrt{n}\sigma}\mu_1 + \left(\tfrac{t}{\sqrt{n}\sigma}\right)^2\mu_2/2 + ...\right) - \tfrac{1}{2}\left(\tfrac{t}{\sqrt{n}\sigma}\mu_1 + ...\right)^2 + ...\right]$$

Some Limit Theorems

Lindberg Levy CLT

- Proof contd:

  So, $\ln M_z(t) =$

$$\frac{-\mu t\sqrt{n}}{\sigma} + n\ln\Big[\int\Big(1 + \frac{t}{\sqrt{n}\sigma}\frac{X}{1!} + \Big(\frac{t}{\sqrt{n}\sigma}\Big)^2\frac{X^2}{2!} + ...\Big)P(X)dX\Big]$$
$$= \frac{-\mu t\sqrt{n}}{\sigma} + n\ln\big[1 + \tfrac{t}{\sqrt{n}\sigma}\mu_1 + \big(\tfrac{t}{\sqrt{n}\sigma}\big)^2\mu_2/2! + ...\big]$$
$$= \frac{-\mu t\sqrt{n}}{\sigma} + n\Big[\big(\tfrac{t}{\sqrt{n}\sigma}\mu_1 + \big(\tfrac{t}{\sqrt{n}\sigma}\big)^2\mu_2/2 + ...\big) - \tfrac{1}{2}\big(\tfrac{t}{\sqrt{n}\sigma}\mu_1 + ...\big)^2 + ...\Big]$$
$$= \frac{-\mu t\sqrt{n}}{\sigma} + \frac{\sqrt{n}t\mu}{\sigma} + \frac{1}{2}\frac{t^2}{\sigma^2}(\mu_2 - \mu^2)$$

Some Limit Theorems

Lindberg Levy CLT

- Proof contd:

  So, $\ln M_z(t) =$

  $$\frac{-\mu t\sqrt{n}}{\sigma} + n\ln\Big[\int\Big(1+\frac{t}{\sqrt{n}\sigma}\frac{X}{1!}+\Big(\frac{t}{\sqrt{n}\sigma}\Big)^2\frac{X^2}{2!}+\ldots\Big)P(X)dX\Big]$$

  $$= \frac{-\mu t\sqrt{n}}{\sigma} + n\ln\big[1+\tfrac{t}{\sqrt{n}\sigma}\mu_1+\big(\tfrac{t}{\sqrt{n}\sigma}\big)^2\mu_2/2!+\ldots\big]$$

  $$= \frac{-\mu t\sqrt{n}}{\sigma} + n\big[\big(\tfrac{t}{\sqrt{n}\sigma}\mu_1+\big(\tfrac{t}{\sqrt{n}\sigma}\big)^2\mu_2/2+\ldots\big)-\tfrac{1}{2}\big(\tfrac{t}{\sqrt{n}\sigma}\mu_1+\ldots\big)^2+\ldots\big]$$

  $$= \frac{-\mu t\sqrt{n}}{\sigma} + \frac{\sqrt{n}t\mu}{\sigma} + \frac{1}{2}\frac{t^2}{\sigma^2}(\mu_2-\mu^2)$$

  $$= \frac{t^2}{2}$$

Some Limit Theorems

Lindberg Levy CLT

- Proof contd:

  So, $\ln M_z(t) =$

  $$\frac{-\mu t\sqrt{n}}{\sigma} + n\ln\Big[\int\Big(1 + \frac{t}{\sqrt{n}\sigma}\frac{X}{1!} + \Big(\frac{t}{\sqrt{n}\sigma}\Big)^2\frac{X^2}{2!} + ...\Big)P(X)dX\Big]$$
  $$= \frac{-\mu t\sqrt{n}}{\sigma} + n\ln\big[1 + \tfrac{t}{\sqrt{n}\sigma}\mu_1 + \big(\tfrac{t}{\sqrt{n}\sigma}\big)^2\mu_2/2! + ...\big]$$
  $$= \frac{-\mu t\sqrt{n}}{\sigma} + n\big[\big(\tfrac{t}{\sqrt{n}\sigma}\mu_1 + \big(\tfrac{t}{\sqrt{n}\sigma}\big)^2\mu_2/2 + ...\big) - \tfrac{1}{2}\big(\tfrac{t}{\sqrt{n}\sigma}\mu_1 + ...\big)^2 + ...\big]$$
  $$= \frac{-\mu t\sqrt{n}}{\sigma} + \frac{\sqrt{n}t\mu}{\sigma} + \frac{1}{2}\frac{t^2}{\sigma^2}(\mu_2 - \mu^2)$$
  $$= \frac{t^2}{2}$$

  Or, $M_z(t) = e^{\frac{t^2}{2}}$, which is of the form for MGF of a standard normal.

Lyapunov's CLT

A more general CLT

- Lyapunov's CLT generalizes the notion of the CLT to the case where X_i are allowed to have different distributions.
  For $\{X_1, X_2, ..., X_n\}$ a sequence of independent random variables, each with finite mean μ_i and variance σ_i^2, let,
  $s_n^2 = \sum_{i=1}^{n} \sigma_i^2$,
  if for some $\delta > 0$, *Lyapunov's Condition* holds, i.e.,
  $$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} \mathrm{E}\left[|X_i - \mu_i|^{2+\delta}\right] = 0$$
  then, the sum $\frac{X_i - \mu_i}{s_n}$ converges in distribution. Or,
  $$\frac{1}{s_n} \sum_{i=1}^{n} (X_i - \mu_i) \xrightarrow{d} N(0,1), \text{ as } n \to \infty.$$

References I

[1] *Central Limit Theorems*. https://en.wikipedia.org/wiki/Central_limit_theorem. Accessed: 2018-06-01 (22, 23).

[2] *Weak Law of Large Numbers at Woflram Mathworld*. http://mathworld.wolfram.com/WeakLawofLargeNumbers.html. Accessed: 2018-05-30 (20, 21).