Week 2: Risk and Reward: Modeling High Uncertainty Settings

- High-Uncertainty Settings: Stock Price Example
- Probability Distributions: Scenario Approach
- Parameters of the Probability Distributions: Expected Value, Variance, Standard Deviation
- Uncertainty and Risk

Session 1

Week 2: Risk and Reward: Modeling High Uncertainty Settings

- Common Scenarios for Multiple Random Variables
- Risk Reduction Example: Investing in a Pair of Stocks
- Calculating and Interpreting Correlation Values

Session 2

- Using Scenarios for Optimizing Under High Uncertainty: Portfolio Selection Problem
- Sensitivity Analysis and Efficient Frontier

Investing in a Single Stock: Reward and Risk

- Model of a Future Return on Stock A

Scenario	Probability
-0.00024	0.05
0.01760	0.05
-0.02114	0.05
-0.01178	0.05
-0.01515	0.05
-0.00353	0.05
-0.01772	0.05
-0.02345	0.05
0.03562	0.05
0.03108	0.05
0.01557	0.05
0.00073	0.05
-0.02188	0.05
0.02063	0.05
0.03044	0.05
0.01276	0.05
0.01214	0.05
0.00138	0.05
-0.00507	0.05
0.01134	0.05

Scenario 1: Return $R_1 = -0.00024$, occurring with probability $p_1 = 0.05$

- 40 parameters provide a ***complete description*** of this distribution

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0.00138	0.05
-0.00507	0.05
0.01134	0.05

Expected Value of R:

$E(R) = p_1*R_1 + p_2*R_2 + \ldots + p_{19}*R_{19} + p_{20}*R_{20} \approx 0.003467$

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Reward

Expected Value of R:

$$E(R) = p_1{*}R_1 + p_2{*}R_2 + \ldots + p_{19}{*}R_{19} + p_{20}{*}R_{20} \approx 0.003467$$

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Expected Value of R:

$$E(R) = p_1{*}R_1 + p_2{*}R_2 + \ldots + p_{19}{*}R_{19} + p_{20}{*}R_{20} \approx 0.003467$$

Standard Deviation of R:

$$SD(R) = \sqrt{Var(R)}$$

$$= \sqrt{p_1{*}(R_1 - E(R))^2 + \ldots + p_{20}{*}(R_{20} - E(R))^2}$$

Investing in a Single Stock: Reward and Risk

- Model of a Future Return on Stock A

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$$= \sqrt{p_1*(R_1 - E(R))^2 + \ldots + p_{20}*(R_{20} - E(R))^2}$$

Risk

Investing in a Single Stock: Reward and Risk

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Reward

Expected Value of R:

$$E(R) = p_1 * R_1 + p_2 * R_2 + \ldots + p_{19} * R_{19} + p_{20} * R_{20} \approx 0.003467$$

Standard Deviation of R:

$$SD(R) = \sqrt{Var(R)}$$

$$= \sqrt{p_1 * (R_1 - E(R))^2 + \ldots + p_{20} * (R_{20} - E(R))^2}$$

Risk

Common Scenarios for Multiple Random Variables

- Consider stocks X and Y with returns “tomorrow” described by two equally probable scenarios

Scenario	Return on Stock X	Return on Stock Y	Probability
1	0.004	0.003	0.5
2	-0.002	-0.001	0.5

- Each scenarios can represent actual returns of Stock X and Stock Y observed on the same trading day in the past

- For example, under Scenario 1, the return on Stock X, R_X , is 0.004 and, simultaneously, the return on Stock Y, R_Y, is 0.003

Common Scenarios for Multiple Random Variables

- Consider stocks X and Y with returns “tomorrow” described by two equally probable scenarios

Scenario	Return on Stock X	Return on Stock Y	Probability
1	0.004	0.003	0.5
2	-0.002	-0.001	0.5

- Expected value of the R_X is $E(R_X)$ = 0.5*0.004+0.5*(-0.002) = 0.001, standard deviation of R_X is $SD(R_X)$ =
$$\sqrt{0.5*(0.004-0.001)^2+0.5*(-0.002-0.001)^2}=0.003$$

- Expected value of the R_Y is $E(R_Y)$ = 0.5*0.003+0.5*(-0.001) = 0.001, standard deviation of R_Y is $SD(R_Y)$ =
$$\sqrt{0.5*(0.003-0.001)^2+0.5*(-0.001-0.001)^2}=0.002$$

Risk Reduction Example: Investing in a Pair of Stocks

- A company invests $50,000 into each of the stocks **X** and **Y** “today”

Scenario	**Return on Stock X**	**Return on Stock Y**	**Probability**
1	0.004	0.003	0.5
2	-0.002	-0.001	0.5

- How much profit will this investment bring “tomorrow”?

Risk Reduction Example: Investing in a Pair of Stocks

- A company invests \$50,000 into each of the stocks **X** and **Y** "today"

Scenario	**Return on Stock X**	**Return on Stock Y**	**Probability**
1	0.004	0.003	0.5
2	-0.002	-0.001	0.5

- If Scenario 1 is realized tomorrow, the company's profit will be \$50,000*(0.004)+\$50,000*(0.003) = \$200+\$150 = \$350

- If Scenario 2 is realized tomorrow, the company's profit will be \$50,000*(-0.002)+\$50,000*(-0.001) = -\$100+(-\$50) = -\$150

- Expected profit is 0.5*\$350+0.5*(-\$150) = **\$100**, and the standard deviation of profit is $\sqrt{0.5*(\$350-\$100)^2+0.5*(-\$150-\$100)^2}=\mathbf{\$250}$

Risk Reduction Example: Investing in a Pair of Stocks

- Now, consider stock Z with returns “tomorrow” described by two equally probable scenarios

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

Risk Reduction Example: Investing in a Pair of Stocks

- Now, consider stock Z with returns "tomorrow" described by two equally probable scenarios

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
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- Stock Z has returns that are identical to those for Stock Y, but in different scenarios

Risk Reduction Example: Investing in a Pair of Stocks

- Now, consider stock Z with returns “tomorrow” described by two equally probable scenarios

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

- Stock Z has returns that are identical to those for Stock Y, but in different scenarios

- Expected value of the R_Z is $E(R_Z) = 0.5*(-0.001)+0.5*(0.003) = 0.001$, standard deviation of R_Z is $SD(R_Z) =$

$$\sqrt{0.5*(-0.001-0.001)^2+0.5*(0.003-0.001)^2}=0.002$$

Risk Reduction Example: Investing in a Pair of Stocks

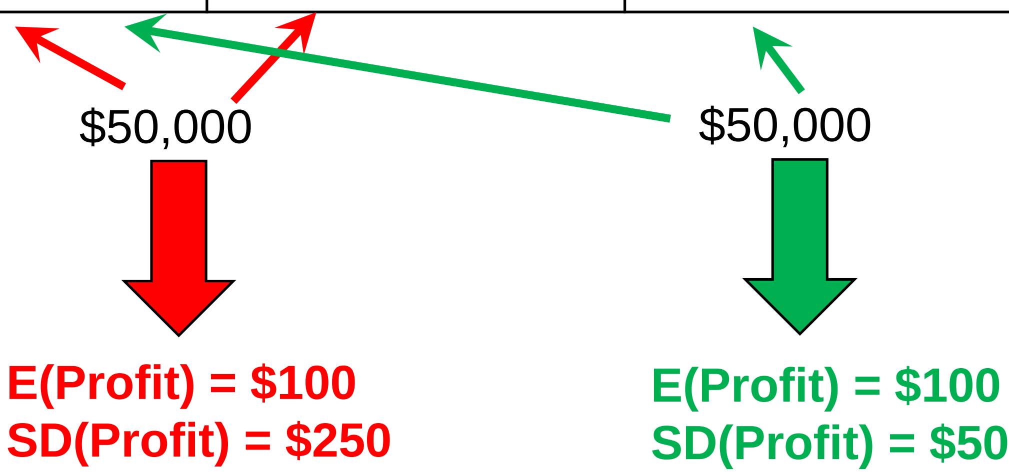
- A company invests $50,000 into each of the stocks X and Z "today"

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

- If Scenario 1 is realized tomorrow, the company's profit will be $50,000*(0.004)+$50,000*(-0.001) = $200+(-$50) = $150
- If Scenario 2 is realized tomorrow, the company's profit will be $50,000*(-0.002)+$50,000*(0.003) = -$100+$150 = $50
- **Expected profit** is 0.5*$150+0.5*$50 = $75 + $25 = **$100**, the **standard deviation of profit** is $\sqrt{0.5*(\$150-\$100)^2+0.5*(\$50-\$100)^2}=\mathbf{\$50}$

Side-by-Side Comparison: X and Y vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5
E(R)	0.001	0.001	0.001	
SD(R)	0.003	0.002	0.002	



- If company replaces Stock Y by Stock Z (stock with the same risk-reward "profile") in the portfolio, the portfolio risk will be drastically reduced

Side-by-Side Comparison: X only vs. X and Y

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

$100,000

E(Profit) = $100
SD(Profit) = $300

E(Profit) = 0.5*$100,000*0.004
+0.5*$100,000*(-0.002) = $100

SD(Profit) =

$$\sqrt{0.5*(\$400-\$100)^2+0.5*(-\$200-\$100)^2}=\$300$$

Side-by-Side Comparison: Y only vs. X and Y

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

$100,000

E(Profit) = $100
SD(Profit) = $200

E(Profit) = 0.5*$100,000*0.003
+0.5*$100,000*(-0.001) = $100

SD(Profit) =

$$\sqrt{0.5*(\$300-\$100)^2+0.5*(-\$100-\$100)^2}=\$200$$

Side-by-Side Comparison: X or Y only vs. X and Y

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

$50,000

E(Profit) = $100
SD(Profit) = $250

If company splits $100,000 equally among Stocks X and Y, it will get the same expected return as Stocks X or Y, and the standard deviation of returns of $250, **between** the standard deviation values for Stock X and Stock Y

Side-by-Side Comparison: X or Z only vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

$100,000

E(Profit) = $100
SD(Profit) = $300

E(Profit) = 0.5*$100,000*0.004 +0.5*$100,000*(-0.002) = $100

SD(Profit) =

$$\sqrt{0.5*(\$400-\$100)^2+0.5*(-\$200-\$100)^2}=\$300$$

Side-by-Side Comparison: X or Z only vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

$100,000

E(Profit) = $100
SD(Profit) = $300

E(Profit) = 0.5*$100,000*(-0.001) +0.5*$100,000*(0.003) = $100

SD(Profit) =

$$\sqrt{0.5*(-\$100-\$100)^2+0.5*(\$300-\$100)^2}=\$200$$

Side-by-Side Comparison: X and Y vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

$50,000

E(Profit) = $100
SD(Profit) = $50

If company splits $100,000 equally among Stocks X and Z, it will get the same expected return as Stock X or Stock Z, and the standard deviation of returns of $50, **lower** than those of either Stock X or Stock Z

Side-by-Side Comparison: X and Y vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5
E(R)	0.001	0.001	0.001	
SD(R)	0.003	0.002	0.002	

$50,000 $50,000

E(Profit) = $100
SD(Profit) = $250

E(Profit) = $100
SD(Profit) = $50

- Risk reduction can be achieved when combining random variables that "interact" in a particular way

Elimination of Risk: Y and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

$50,000

E(Profit) = $100
SD(Profit) = $0

Profit Under Scenario 1
= $50,000*(0.003)
+$50,000*(-0.001) = $100

Profit Under Scenario 2
= $50,000*(-0.001)
+$50,000*(0.003) = $100

Side-by-Side Comparison: X and Y vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

- Why combining X and Y is not as beneficial for risk reduction as combining X and Z?

Side-by-Side Comparison: X and Y vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

Side-by-Side Comparison: X and Y vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5
E(R)	**0.001**	**0.001**	0.001	
SD(R)	0.003	0.002	0.002	

- In Scenario 1, returns for X and Y simultaneously ***"rise" above*** their respective expected values
- In Scenario 2, returns for X and Y simultaneously ***"drop" below*** their respective expected values
- Random variables that, on average, "move in unison" are said to be "positively correlated"

Side-by-Side Comparison: X and Y vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5
E(R)	**0.001**	0.001	**0.001**	
SD(R)	0.003	0.002	0.002	

- In Scenario 1, return for X **"*rises" above*** its expected value, while the return for Z ***"drops" below*** its expected value

Side-by-Side Comparison: X and Y vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5
E(R)	**0.001**	0.001	**0.001**	
SD(R)	0.003	0.002	0.002	

- In Scenario 1, return for X "***rises" above*** its expected value, while the return for Z ***"drops" below*** its expected value
- In Scenario 2, return for X ***"drops" below*** its expected value, while the return for Z "***rises" above*** its expected value
- Random variables that, on average, "move in opposite directions" are said to be "negatively correlated"

Correlation: Definition

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

- Correlation between random variables A and B is defined as

$$\text{Corr}(A,B) = \frac{E(A*B) - E(A)*E(B)}{SD(A)*SD(B)}$$

Corr(A,B) is the same as Corr(B,A)

- In order to calculate the correlation between random variables A and B, we need to calculate their individual expected values and standard deviations, and, in addition, the expected value of the product of A and B

Calculating Correlation Between R_X and R_Y

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

- Expected value of the product of R_X and R_Y

$$E(R_X*R_Y) = 0.5*(0.004*0.003)+0.5*(-0.002)*(-0.001)$$
$$= 5*12*10^{-7}+5*2*10^{-7} = 7*10^{-6} = 0.000007$$

R_X and R_Y are "perfectly correlated"

- Correlation between R_X and R_Y

$$Corr(R_X*R_Y) = \frac{E(R_X*R_Y)-E(Rx)*E(R_Y)}{SD(Rx)*SD(R_Y)} = \frac{7*10^{-6}-0.001*0.001}{0.003*0.002} = 1$$

Calculating Correlation Between R_X and R_Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

R_X and R_Z are "perfectly anti-correlated"

- Expected value of the product of R_X and R_Z

$$E(R_X{*}R_Z) = 0.5{*}(0.004){*}(-0.001)+0.5{*}(-0.002){*}(0.003)$$
$$= -5{*}4{*}10^{-7}-5{*}6{*}10^{-7} = -5{*}10^{-6} = -0.000005$$

- Correlation between R_X and R_Z

$$\mathrm{Corr}(R_X{*}R_Z) = \frac{E(R_X * R_Z) - E(R_X) * E(R_Z)}{SD(R_X) * SD(R_Z)} = \frac{-5 * 10^{-6} - 0.001 * 0.001}{0.003 * 0.002} = -1$$

Calculating Correlation Between R_Y and R_Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
2	-0.002	-0.001	0.003	0.5

E(R)	0.001	0.001	0.001
SD(R)	0.003	0.002	0.002

R_Y and R_Z are also "perfectly anti-correlated"

- Expected value of the product of R_Y and R_Z

$$E(R_Y*R_Z) = 0.5*(0.003)*(-0.001)+0.5*(-0.001)*(0.003)$$
$$= -5*3*10^{-7}-5*3*10^{-7} = -3*10^{-6} = -0.000003$$

- Correlation between R_Y and R_Z

$$Corr(R_Y*R_Z) = \frac{E(R_Y*R_Z)-E(R_Y)*E(R_Z)}{SD(R_Y)*SD(R_Z)} = \frac{-3*10^{-6}-0.001*0.001}{0.002*0.002} = -1$$

Positive and Negative Correlation Values

- Perfect correlation and perfect anti-correlation are extreme cases of positive and negative correlation
- Correlation values always fall in the interval between -1 and 1
- In general, combining negatively correlated assets in a portfolio leads to a reduction in the standard deviation of the portfolio's return

Side-by-Side Comparison: X and Y vs. X and Z

Scenario	Return on Stock X	Return on Stock Y	Return on Stock Z	Probability
1	0.004	0.003	-0.001	0.5
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E(R)	0.001	0.001	0.001
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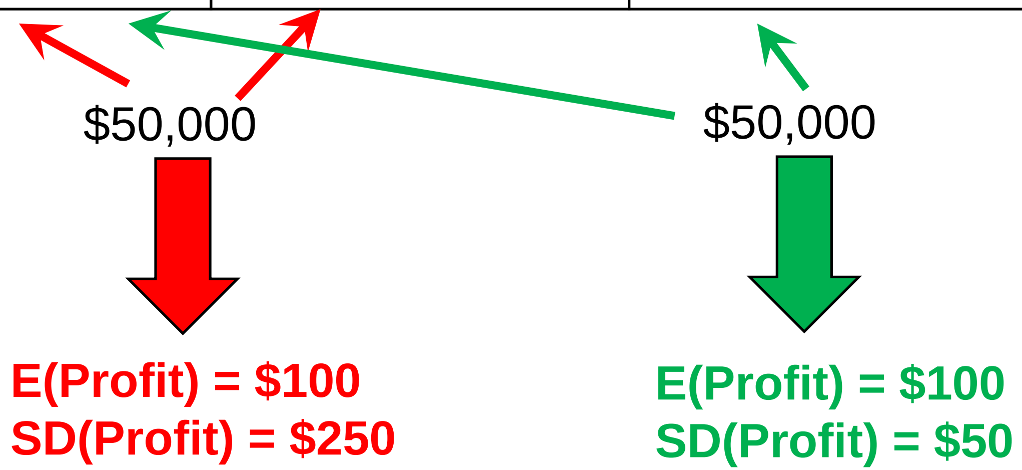

- In our example, combining stocks with perfectly anti-correlated returns (X and Z) resulted lower risk as compared to combining stocks with perfectly correlated returns (X and Y)