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Random variables

numeric outcomes from a random experiment.

Cumulative distribution function (cdf).

$$F(x) = P\{X \le x\}.$$

Probability mass function (pmf), discrete and countable

$$p(x) = P\{X = x\} \qquad \sum_{i=1}^{A} p(x_i) = 1.$$

Probability density function (pdf)

$$P\{X \in C\} = \int_C f_X(x)\,dx.$$

Expectation:

$$E[X] = \sum_i x_i \cdot P\{X = x_i\} \qquad E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx.$$

$$E[aX + b] = a \cdot E[X] + b, \qquad E[X_1 + X_2] = E[X_1] + E[X_2].$$

Variance:

$$Var[X] = E[(X-\mu)^2] = E[X^2] - (E[X])^2.$$

$$Var(aX + b) = a^2 \cdot Var[X].$$

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2).$$

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Common probability distributions (pdfs).

discrete

- $\boxed{B}$: Binomial (Bernoulli + Sampling)
- E : Exponential
- G : Gaussian
- C : Cauchy
- U : Uniform
- $\boxed{P}$: Poisson

Sampling

Outcomes	w/ replacement	w/o replacement
2	Binomial neg. binomial	Hypergeom.
≥3	multinomial	multivariate hypergeom.

Binomial:

n independent Bernoulli trials: success vs not success.

Let X be the number of successes in n trials. with success probability p.

$$P\{X = i\} = \binom{n}{i} p^i \cdot (1-p)^{n-i} \quad \text{for } i = 0, 1, \ldots, n.$$

$$\Downarrow$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \quad \longleftarrow \text{ \# of ways to choose } i \text{ elements from } n.$$

notation, $b(n, p)$.

$$E[X] = E[\underset{\substack{\uparrow \\ \text{indiv. bernoulli}}}{Y_1} + Y_2 + \cdots + Y_n] = n \cdot p.$$

$$Var[X] = n \cdot p(1-p).$$

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Geometric random variable

Indendent Bernoulli trials with success probability p. X is the number of trials until the first success, then

$$P[X=n] = \underbrace{p}_{\text{1-success}}\underbrace{(1-p)^{n-1}}_{(n-1)\text{ fail.}} \qquad \text{for } n \geq 1.$$

$$E[X] = \sum_{n=1}^{\infty} n \cdot p(1-p)^{n-1} = \frac{1}{p}.$$

$$Var[X] = \frac{1-p}{p^2}.$$

Issue spot: "until"

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Negative binomial:

Let X be the number of trials to reach r successes where each trial is a Bernoulli experiment.

$$P[X=n] = \binom{n-1}{r-1} p^r (1-p)^{n-r} \quad \text{for } n \geq r.$$

$$E[X] = \frac{r}{p}.$$

$$Var[X] = \frac{r(1-p)}{p^2}.$$

Hypergeometric:

Consider an urn w/ $N + m$ balls. Choose n balls randomly.

$$\underset{\substack{\text{light}\\\text{(success)}}}{N} + \underset{\substack{\text{dark}\\\text{(failure)}}}{m}$$

Let X be the number of light balls.

$$P\{X=i\} = \frac{\binom{N}{i}\binom{M}{n-i}}{\binom{N+M}{n}}$$

← # distinct subsets w/ n elements from $N+M$ elements

$$E[X] = \frac{n \cdot N}{N+M}.$$

$$Var[X] = \frac{n \cdot NM}{(N+m)^2} \cdot \left(1 - \frac{n-1}{N+m-1}\right).$$

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Poisson:

Approximate # of successes in large number of trials:

for p small, $p \ll 1$ and $n \cdot p \approx$ constant

$$P[X=i] = \frac{e^{-\lambda}\cdot\lambda^{K}}{K!}$$

$$P\{X=i\} = \frac{n!}{(n-i)!\,i!}\cdot p^{i}\cdot(1-p)^{n-i}$$

$$\cdots \quad \cdots$$

$$= \frac{n(n-1)\cdots(n-i+1)}{n^{i}}\cdot\frac{\lambda^{i}}{i!}\cdot\frac{(1-\lambda/n)^{n}}{(1-\lambda/n)^{i}}$$

for large n: $(1-\lambda/n)^{n} \approx e^{-\lambda}$.

$$\frac{n(n-1)\cdots(n-i+1)}{n^{i}} \approx 1. \qquad (1-\lambda/n)^{i} \approx 1.$$

$$\therefore \longrightarrow P\{X_i\} = e^{-\lambda}\frac{\lambda^{i}}{i!}.$$

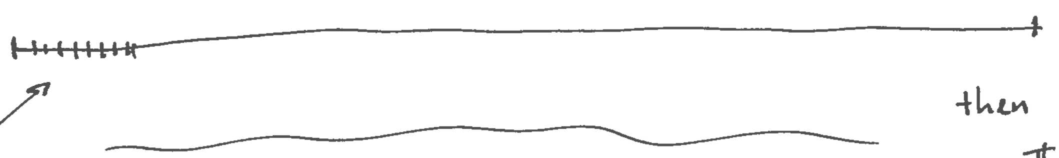
Using binomial result: $E[X] = np$ $\qquad Var[X] = n\cdot p(1-p) \approx np$

since $1-p \approx 0$.

$$\therefore E[X] = Var[X] = \lambda.$$

Useful for arrival counting:

Ex: # cars entering freeway per hour.



subdivide so $P(car) \ll 1$.

1 hour

$\lambda = n\cdot p$ is average # of entry per hour.

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Generating discrete random variables:

To generate arbitrary discrete random variable w/ p.m.f.

$$P\{X=x_j\} = P_j \qquad j=0,1,\ldots \text{ w/ } \sum_j P_j = 1.$$

Generate a random # $\sim u[0,1]$. Then

$$X = \begin{cases} x_0 & u < P_0 \\ x_1 & P_1 \le u < P_0 + P_1 \\ \ldots & \\ x_j & \sum_{i=0}^{j-1} P_i \le u < \sum_{i=0}^{j} P_i \end{cases}$$

Alg:

1. Generate $x \sim u[0,1]$.
2. if $u < P_0$ $\quad X = x_0$, Stop
3. if $u < P_0 + P_1$ $\quad X = X_1$, Stop

...

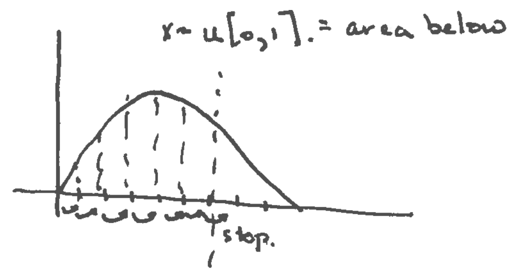
The above is called the "discrete inverse transform method" to generate r.v. X.

Generating Poisson random samples:

recall $P\{x=i\} = \dfrac{e^{-\lambda}\lambda^i}{i!}$. $\qquad$ Can show: $P_{i+1} = \dfrac{\lambda}{i+1} \cdot P_i$. $i \ge 0$.

Using the recursion method above:

1. Generate $x \sim u[0,1]$.
2. $i=0$, $p = e^{-\lambda}$, $F=p$.
3. if $u < F$; $x = i$, Stop
4. $p = \lambda p/(i+1)$, $F = F+P$, $i = i+1$
5. Goto 3.



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Acceptance - Rejection Sampling.

Generate samples from an arbitrary density: $\{p_j ; j \geq 0\}$.

But use samples from an "auxiliary" distribution $\{q_j ; j \geq 0\}$ that dominates p_j. Specifically: $p_j \leq c \cdot q_j$ for all j where c is a constant.

Alg: 1. ~~Generate~~ $Y \sim q_j$

2. Generate $U \sim U[0,1]$.

3. if $U < p_Y / c \cdot q_Y$ set $X = Y$, stop. Else step 1.

Thrm: The accept-reject algorithm generates a random variable X such that $P[X=j] = p_j \quad j = 0, 1 \ldots$

and the number of iterations to generate X is a geometric r.v. w/ mean c.

Pf: $$P[Y=j, \text{it is accepted}] = \overbrace{P[Y=j] \cdot P[\text{accept} \mid Y=j]}^{\text{thrm total probability.}}$$
$$= q_j \cdot \frac{p_j}{c \cdot q_j}$$
$$= \frac{p_j}{c}.$$

$$P[\text{accept}] = \sum_j \frac{p_j}{c} = \frac{1}{c} \cdot \sum p_j = \frac{1}{c}.$$

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$\therefore$ each iteration is an independent Bernoulli trial w/ $p[\text{success}] = \frac{1}{c}$.

$\therefore$ geometric random variable w/ mean c.

$$P\{X=j\} = \sum_n P\{j \text{ accepted on iteration } n\}.$$

$$= \sum \left(1-\frac{1}{c}\right)^{n-1} \frac{P_j}{c}.$$

$$= P_j$$