Fuzzy Relations, Rules and Inferences

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Fuzzy Relations

Crisp relations

To understand the fuzzy relations, it is better to discuss first crisp relation.

Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$

Note :

(1) $A \times B \neq B \times A$

(2) $|A \times B| = |A| \times |B|$

(3)$A \times B$ provides a mapping from $a \in A$ to $b \in B$.

The mapping so mentioned is called a relation.

Crisp relations

Example 1:

Consider the two crisp sets A and B as given below. A = {1, 2, 3, 4}
B = {3, 5, 7}.

Then, $A \times B = \{(1,3),(1,5),(1,7),(2,3),(2,5),(2,7),(3,3),(3,5),(3,7),(4,3),(4,5),(4,7)\}$

Let us define a relation R as $R = \{(a,b) | b = a+1, (a,b) \in A \times B\}$

Then, $R = \{(2,3),(4,5)\}$ in this case.

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c c} & \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

Union:

$$R(x, y) \cup S(x, y) = max(R(x, y), S(x, y));$$

Intersection:

$$R(x, y) \cap S(x, y) = min(R(x, y), S(x, y));$$

Complement:

$$\overline{R(x, y)} = 1 - R(x, y)$$

Example: Operations on crisp relations

Example:
Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Find the following:

1. $R \cup S$
2. $R \cap S$
3. $\overline{R}$

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y,Z. Then $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x, z)|(x, y) \in R \text{ and } (y, z) \in S \text{ and } \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S, the max-min composition is defined as $T = R \circ S$;

$$T(x, z) = max\{min\{R(x, y), S(y, z) \text{ and } \forall y \in Y\}\}$$

Example:

Given

$X = \{1,3,5\}; Y = \{1,3,5\}; R = \{(x,y)|y = x+2\}; S = \{(x,y)|x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have

$R = \{(1,3),(3,5)\}$

$S = \{(1,3),(1,5),(3,5)\}$

R= $\begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ and S= $\begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

Using max-min composition $R \circ S$= $\begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X_1, X_2, ..., X_n$
- Here, n-tuples $(x_1, x_2, ..., x_n)$ may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Example:

X = { typhoid, viral, cold } and Y = { running nose, high temp, shivering }

The fuzzy relation R is defined as

$$
\begin{array}{cc}
 & \begin{array}{ccc} \textit{runningnose} & \textit{hightemperature} & \textit{shivering} \end{array} \\
\begin{array}{c} \textit{typhoid} \\ \textit{viral} \\ \textit{cold} \end{array} &
\begin{bmatrix} 0.1 & 0.9 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{bmatrix}
\end{array}
$$

Fuzzy Cartesian product

Suppose

A is a fuzzy set on the universe of discourse X with $\mu_A(x) | x \in X$

B is a fuzzy set on the universe of discourse Y with $\mu_B(y) | y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given by $\mu_R(x, y) = \mu_{A \times B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$

Example :

$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$and $B = \{(b_1, 0.5), (b_2, 0.6)\}$

$$R = A \times B = \begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

Union:

$$\mu_{R \cup S}(a, b) = max\{\mu_R(a, b), \mu_S(a, b)\}$$

Intersection:

$$\mu_{R \cap S}(a, b) = min\{\mu_R(a, b), \mu_S(a, b)\}$$

Complement:

$$\mu_{\overline{R}}(a, b) = 1 - \mu_R(a, b)$$

Composition

$$T = R \circ S$$
$$\mu_{R \circ S} = max_{y \in Y}\{min(\mu_R(x, y), \mu_S(y, z))\}$$

Operations on Fuzzy relations: Examples

Example:

$X = (x_1, x_2, x_3)$; $Y = (y_1, y_2)$; $Z = (z_1, z_2, z_3)$;

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cc} y_1 & y_2 \\ \left[\begin{array}{cc} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{array}\right] \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{ccc} z_1 & z_2 & z_3 \\ \left[\begin{array}{ccc} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{array}\right] \end{array}$$

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{ccc} z_1 & z_2 & z_3 \\ \left[\begin{array}{ccc} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{array}\right] \end{array}$$

$\mu_{R \circ S}(x_1, y_1) = max\{min(x_1, y_1), min(y_1, z_1), min(x_1, y_2), min(y_2, z_1)\}$
$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = 0.5$ and so on.

Fuzzy relation : An example

Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants
$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, then R can be stated as

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : An example

Also, consider T be the another relation on $D \times S$, which is given by

$$S = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition.**

Hint: Find $R \circ T$, and verify that

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy relation : Another example

Let, $R = x$ is relevant to y

and $S = y$ is relevant to z

be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively, where $X = \{1, 2, 3\}$,$Y = \{\alpha, \beta, \gamma, \delta\}$ and $Z = \{a, b\}$.
Assume that R and S can be expressed with the following relation matrices :

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \end{array} \text{ and}$$

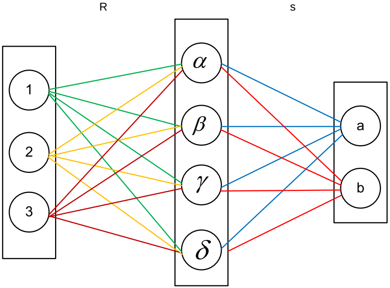
$$S = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : Another example

Now, we want to find $R \circ S$, which can be interpreted as a derived fuzzy relation x **is relevant to** z.

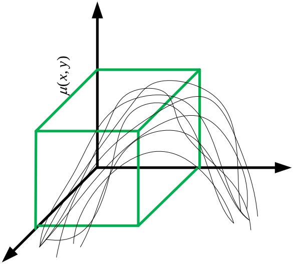
Suppose, we are only interested in the degree of relevance between $2 \in X$ and $a \in Z$. Then, using max-min composition,

$$\mu_{R \circ S}(2, a) = max\{(0.4 \wedge 0.9), (0.2 \wedge 0.2), (0.8 \wedge 0.5), (0.9 \wedge 0.7)\}$$
$$= max\{0.4, 0.2, 0.5, 0.7\} = 0.7$$



2D Membership functions : Binary fuzzy relations

(Binary) fuzzy relations are fuzzy sets $A \times B$ which map each element in $A \times B$ to a membership grade between 0 and 1 (both inclusive). Note that a membership function of a binary fuzzy relation can be depicted with a 3D plot.



Important: Binary fuzzy relations are fuzzy sets with two dimensional MFs and so on.

2D membership function : An example

Let, $X = R^+ = y$ (the positive real line)
and $R = X \times Y$ = "y is much greater than x"

The membership function of $\mu_R(x, y)$ is defined as

$$\mu_R(x, y) = \begin{cases} \frac{(y-x)}{4} & if \quad y > x \\ 0 & if \quad y \leq x \end{cases}$$

Suppose, $X = \{3, 4, 5\}$ and $Y = \{3, 4, 5, 6, 7\}$, then

$$R = \begin{array}{c c} & \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0.25 & 0.5 & 0.75 & 1.0 \\ 0 & 0 & 0.25 & 0.5 & 0.75 \\ 0 & 0 & 0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

Problems to ponder:

How you can derive the following?

If x is A or y is B then z is C;

Given that

1. R_1: If x is A then z is c $[R_1 \in A \times C]$
2. R_2: If y is B then z is C $[R_2 \in B \times C]$

- **Hint:**
  - You have given two relations R_1 and R_2.
  - Then, the required can be derived using the union operation of R_1 and R_2

Fuzzy Propositions

Two-valued logic vs. Multi-valued logic

- The basic assumption upon which crisp logic is based - that every proposition is either TRUE or FALSE.
- The classical two-valued logic can be extended to multi-valued logic.
- As an example, three valued logic to denote true(1), false(0) and indeterminacy ($\frac{1}{2}$).

Different operations with three-valued logic can be extended as shown in the following truth table:

a	b	$\wedge$	$\vee$	$\neg$a	$\Longrightarrow$	$=$
0	0	0	0	1	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	0	1	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1

Fuzzy connectives used in the above table are:

AND ($\wedge$), OR ($\vee$), NOT ($\neg$), IMPLICATION ($\Longrightarrow$) and EQUAL ($=$).

Fuzzy connectives defined for such a three-valued logic better can be stated as follows:

Symbol	Connective	Usage	Definition
$\neg$	NOT	$\neg$P	$1 - T(P)$
$\vee$	OR	$P \vee Q$	max{T(P), T(Q) }
$\wedge$	AND	$P \wedge Q$	min{ T(P),T(Q) }
$\Longrightarrow$	IMPLICATION	$(P \Longrightarrow Q)$ or $(\neg P \vee Q)$	max{(1 - T(P)), T(Q) }
$=$	EQUALITY	$(P = Q)$ or $[(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)]$	$1 - \lvert T(P) - T(Q) \rvert$

Example 1:

P : Ram is honest

1. T(P) = 0.0 : Absolutely false
2. T(P) = 0.2 : Partially false
3. T(P) = 0.4 : May be false or not false
4. T(P) = 0.6 : May be true or not true
5. T(P) = 0.8 : Partially true
6. T(P) = 1.0 : Absolutely true.

Example 2 :Fuzzy proposition

P : Mary is efficient ; T(P) = 0.8;

Q : Ram is efficient ; T(Q) = 0.6

1. **Mary is not efficient.**

   $T(\neg P) = 1 - T(P) = 0.2$

2. **Mary is efficient and so is Ram.**

   $T(P \wedge Q) = min\{T(P), T(Q)\} = 0.6$

3. **Either Mary or Ram is efficient**

   $T(P \vee Q) = max T(P), T(Q) = 0.8$

4. **If Mary is efficient then so is Ram**

   $T(P \Longrightarrow Q) = max\{1 - T(P), T(Q)\} = 0.6$

Fuzzy proposition vs. Crisp proposition

- The fundamental difference between crisp (classical) proposition and fuzzy propositions is in the range of their truth values.
- While each classical proposition is required to be either true or false, the truth or falsity of fuzzy proposition is a matter of degree.
- The degree of truth of each fuzzy proposition is expressed by a value in the interval [0,1] both inclusive.

Canonical representation of Fuzzy proposition

- Suppose, X is a universe of discourse of five persons. Intelligent of $x \in X$ is a fuzzy set as defined below.

  Intelligent: $\{(x_1, 0.3), (x_2, 0.4), (x_3, 0.1), (x_4, 0.6), (x_5, 0.9)\}$

- We define a fuzzy proposition as follows:

  P : x is intelligent

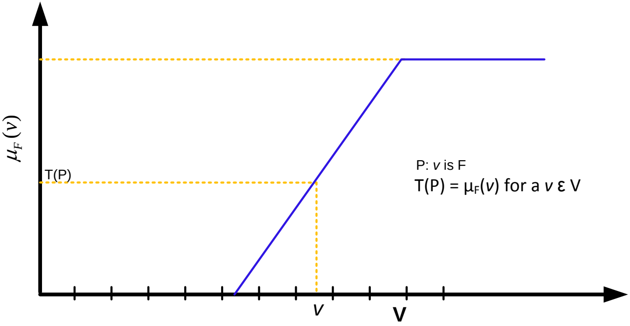
- The canonical form of fuzzy proposition of this type, P is expressed by the sentence P : v is F .

- Predicate in terms of fuzzy set.

  P : v is F ; where v is an element that takes values v from some universal set V and F is a fuzzy set on V that represents a fuzzy predicate.

- In other words, given, a particular element v, this element belongs to F with membership grade $\mu_F(v)$.

Graphical interpretation of fuzzy proposition



- For a given value v of variable V in proposition P, T(P) denotes the degree of truth of proposition P.

Fuzzy Implications

Fuzzy rule

- A fuzzy implication (also known as fuzzy If-Then rule, fuzzy rule, or fuzzy conditional statement) assumes the form :

  If x is A then y is B

  where, A and B are two linguistic variables defined by fuzzy sets A and B on the universe of discourses X and Y, respectively.

- Often, x **is** A is called the antecedent or premise, while y **is** B is called the consequence or conclusion.

Fuzzy implication : Example 1

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High
- The fuzzy implication is denoted as $R : A \rightarrow B$
- In essence, it represents a binary fuzzy relation R on the (Cartesian) product of $A \times B$

Fuzzy implication : Example 2

- Suppose, P and T are two universes of discourses representing pressure and temperature, respectively as follows.
- $P = \{ 1,2,3,4\}$ and $T = \{ 10, 15, 20, 25, 30, 35, 40, 45, 50 \}$
- Let the linguistic variable High temperature and Low pressure are given as
- $T_{HIGH} =$
  $\{(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)\}$
- $P_{LOW} = (1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)$

Fuzzy implications : Example 2

- Then the fuzzy implication **If temperature is High then pressure is Low** can be defined as

  $R : T_{HIGH} \rightarrow P_{LOW}$

$$
\text{where, R} = \begin{array}{c} \\ 20 \\ 25 \\ 30 \\ 35 \\ 40 \\ 45 \\ 50 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.7 & 0.7 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}
$$

Note : If temperature is 40 then what about low pressure?

Interpretation of fuzzy rules

In general, there are two ways to interpret the fuzzy rule $A \rightarrow B$ as

- A coupled with B
- A entails B

Interpretation as *A* coupled with *B*

$R : A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x) * \mu_B(y)|_{(x,y)}$; where $*$ is called a T-norm operator.

T-norm operator

The most frequently used T-norm operators are:

Minimum : $T_{min}(a, b) = min(a, b) = a \wedge b$

Algebric product : $T_{ap}(a, b) = ab$

Bounded product : $T_{bp}(a, b) = 0 \vee (a + b - 1)$

Drastic product : T_{dp} = $\begin{cases} a & if \quad b = 1 \\ b & if \quad a = 1 \\ 0 & if \quad a, b < 1 \end{cases}$

Here, a = $\mu_A(x)$ and b = $\mu_B(y)$. T_* is called the function of T-norm operator.

Interpretation as A coupled with B

Based on the T-norm operator as defined above, we can automatically define the fuzzy rule $R : A \rightarrow B$ as a fuzzy set with two-dimentional MF:

$\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b)$ with a=$\mu_A(x)$, b=$\mu_B(y)$, and f is the fuzzy implication function.

Interpretation as A coupled with B

In the following, few implications of $R : A \rightarrow B$

Min operator:

$R_m = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y)|_{(x,y)}$ or $f_{min}(a, b) = a \wedge b$
[Mamdani rule]

Algebric product operator

$R_{ap} = A \times B = \int_{X \times Y} \mu_A(x).\mu_B(y)|_{(x,y)}$ or $f_{ap}(a, b) = ab$
[Larsen rule]

Product Operators

Bounded product operator

$R_{bp} = A \times B = \int_{X \times Y} \mu_A(x) \odot \mu_B(y)|_{(x,y)} =$
$\int_{X \times Y} 0 \vee (\mu_A(x) + \mu_B(y) - 1)|_{(x,y)}$

or $f_{bp} = 0 \vee (a + b - 1)$

Drastic product operator

$R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \hat{\bullet} \mu_B(y)|_{(x,y)}$

or $f_{dp}(a, b) = \begin{cases} a & if \quad b = 1 \\ b & if \quad a = 1 \\ 0 & if \quad otherwise \end{cases}$

Interpretation of A entails B

There are three main ways to interpret such implication:

Material implication :

$R : A \rightarrow B = \bar{A} \cup B$

Propositional calculus :

$R : A \rightarrow B = \bar{A} \cup (A \cap B)$

Extended propositional calculus :

$R : A \rightarrow B = (\bar{A} \cap \bar{B}) \cup B$

Interpretation of A entails B

With the above mentioned implications, there are a number of fuzzy implication functions that are popularly followed in fuzzy rule-based system.

Zadeh's arithmetic rule :

$R_{za} = \bar{A} \cup B = \int_{X \times Y} 1 \wedge (1 - \mu_A(x) + \mu_B(y))|_{(x,y)}$
or
$f_{za}(a, b) = 1 \wedge (1 - a + b)$

Zadeh's max-min rule :

$R_{mm} = \bar{A} \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))|_{(x,y)}$
or
$f_{mm}(a, b) = (1 - a) \vee (a \wedge b)$

Interpretation of A entails B

Boolean fuzzy rule

$R_{bf} = \bar{A} \cup B = \int_{X \times Y} (1 - \mu_A(x)) \vee \mu_B(x)|_{(x,y)}$
or
$f_{bf}(a, b) = (1 - a) \vee b;$

Goguen's fuzzy rule:

$R_{gf} = \int_{X \times Y} \mu_A(x) * \mu_B(y)|_{(x,y)}$ where $a * b = \begin{cases} 1 & if \quad a \leq b \\ \frac{b}{a} & if \quad a > b \end{cases}$

Example 3: Zadeh's Max-Min rule

If x is A then y is B with the implication of Zadeh's max-min rule can be written equivalently as :

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y)$$

Here, Y is the universe of discourse with membership values for all $y \in Y$ is 1, that is , $\mu_Y(y) = 1 \forall y \in Y$.

Suppose $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$

and $A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$ are two fuzzy sets.

We are to determine $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

Example 3: Zadeh's min-max rule:

The computation of $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$ is as follows:

$$A \times B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array} \text{ and}$$

$$\bar{A} \times Y = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 3: Zadeh's min-max rule:

Therefore,

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 3 :

$X = \{a, b, c, d\}$

$Y = \{1, 2, 3, 4\}$

Let, $A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$

Determine the implication relation :

If x is A then y is B

Here, $A \times B$ =
$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 3 :

and $\bar{A} \times Y =$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0.2 & 0.2 & 0.2 & 0.2 \\ c & 0.4 & 0.4 & 0.4 & 0.4 \\ d & 0 & 0 & 0 & 0 \end{array}$$

$R_{mm} = (A \times B) \cup (\bar{A} \times Y) =$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0.2 & 0.8 & 0.8 & 0.2 \\ c & 0.4 & 0.6 & 0.6 & 0.4 \\ d & 0.2 & 1.0 & 0.8 & 0 \end{array}$$

This R represents **If** x **is A then** y **is B**

Example 3 :

IF x is A THEN y is B ELSE y is C.

The relation R is equivalent to

$$R = (A \times B) \cup (\bar{A} \times C)$$

The membership function of R is given by

$\mu_R(x, y) = max[min\{\mu_A(x), \mu_B(y)\}, min\{\mu_{\bar{A}}(x), \mu_C(y)]$

Example 4:

$X = \{a, b, c, d\}$

$Y = \{1, 2, 3, 4\}$

$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$

$C = \{(1, 0), (2, 0.4), (3, 1.0), (4, 0.8)\}$

Determine the implication relation :

If x is A then y is B else y is C

Here, $A \times B$ = $$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

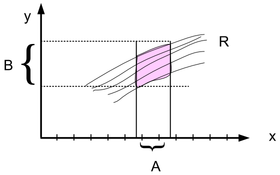
Example 4:

$$\text{and } \bar{A} \times C = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

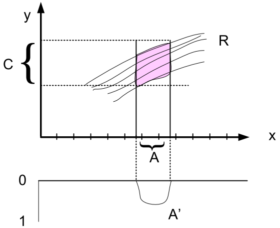
$$R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Interpretation of fuzzy implication

If x is A then y is B



If x is A then y is B else y is C



Fuzzy Inferences

Fuzzy inferences

Let's start with propositional logic. We know the following in propositional logic.

1. Modus Ponens : $P, P \implies Q, \qquad \Leftrightarrow Q$
2. Modus Tollens : $P \implies Q, \neg Q \qquad \Leftrightarrow, \neg P$
3. Chain rule : $P \implies Q, Q \implies R \qquad \Leftrightarrow, P \implies R$

An example from propositional logic

Given

1. $C \vee D$
2. $\sim H \implies (A \wedge \sim B)$
3. $C \vee D \implies \sim H$
4. $(A \wedge \sim B) \implies (R \vee S)$

From the above can we infer $R \vee S$?

Similar concept is also followed in fuzzy logic to infer a fuzzy rule from a set of given fuzzy rules (also called fuzzy rule base).

Inferring procedures in Fuzzy logic

Two important inferring procedures are used in fuzzy systems :

- **Generalized Modus Ponens (GMP)**

  If x is A Then y is B

  x is A'

  ---

  y is B'

- **Generalized Modus Tollens (GMT)**

  If x is A Then y is B

  y is B'

  ---

  x is A'

Fuzzy inferring procedures

- Here, $A, B, A^{'}$ and $B^{'}$ are fuzzy sets.
- To compute the membership function $A^{'}$ and $B^{'}$ the max-min composition of fuzzy sets $B^{'}$ and $A^{'}$,respectively with $R(x, y)$ (which is the known implication relation) is to be used.
- Thus,

$$B^{'} = A^{'} \circ R(x, y) \qquad \mu_B(y) = max[min(\mu_{A'}(x), \mu_R(x, y))]$$

$$A^{'} = B^{'} \circ R(x, y) \qquad \mu_A(x) = max[min(\mu_{B'}(y), \mu_R(x, y))]$$

Generalized Modus Ponens

Generalized Modus Ponens (GMP)

$$P : \textbf{If } x \textbf{ is } A \textbf{ then } y \textbf{ is } B$$

Let us consider two sets of variables x and y be

$X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, respectively.

Also, let us consider the following.

$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$

$B = \{(y_1, 1), (y_2, 0.4)\}$

Then, given a fact expressed by the proposition x is $A^{'}$,
where $A^{'} = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$
derive a conclusion in the form y **is** $B^{'}$ (using generalized modus ponens (GMP)).

Example: Generalized Modus Ponens

If x is A Then y is B

x is A'

---

y is B'

We are to find $B' = A' \circ R(x, y)$ where $R(x, y) = max\{A \times B, \overline{A} \times Y\}$

$$A \times B = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.4 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} \text{ and } \overline{A} \times Y = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Note: For $A \times B$, $\mu_{A \times B}(x, y) = min(\mu_A x, \mu_B(y))$

Example: Generalized Modus Ponens

$$R(x,y) = (A \times B) \cup (\overline{A} \times y) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

Now, $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

Therefore, $B' = A' \circ R(x,y) =$

$$\begin{bmatrix} 0.6 & 0.9 & 0.7 \end{bmatrix} \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 \end{bmatrix}$$

Thus we derive that y is B' where $B' = \{(y_1, 0.9), (y_2, 0.5)\}$

Example: Generalized Modus Tollens

Generalized Modus Tollens (GMT)

P: If x is A Then y is B

Q: y is B'

---

x is A'

Example: Generalized Modus Tollens

- Let sets of variables x and y be $X = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2\}$, respectively.

- Assume that a proposition **If** x **is** A **Then** y **is** B given where $A = \{(x_1, 0.5), (x_2, 1.0), (x_3, 0.6)\}$ and $B = \{(y_1, 0.6), (y_2, 0.4)\}$

- Assume now that a fact expressed by a proposition y **is** B is given where $B^{'} = \{(y_1, 0.9), (y_2, 0.7)\}$.

- From the above, we are to conclude that x **is** $A^{'}$. That is, we are to determine $A^{'}$

Example: Generalized Modus Tollens

- We first calculate $R(x, y) = (A \times B) \cup (\overline{A} \times y)$

$$R(x,y) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

- Next, we calculate $A^{'} = B^{'} \circ R(x, y)$

$$A^{'} = \begin{bmatrix} 0.9 & 0.7 \end{bmatrix} \circ \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} = \begin{bmatrix} 0.5 & 0.9 & 0.6 \end{bmatrix}$$

- Hence, we calculate that x is $A^{'}$ where $A^{'} = [(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)]$

Practice

Apply the fuzzy GMP rule to deduce **Rotation is quite slow**

Given that :

- If temperature is High then rotation is Slow.
- temperature is Very High

Let,

$X = \{30, 40, 50, 60, 70, 80, 90, 100\}$ be the set of temperatures.

$Y = \{10, 20, 30, 40, 50, 60\}$ be the set of rotations per minute.

Practice

The fuzzy set High(H), Very High (VH), Slow(S) and Quite Slow (QS) are given below.

$H = \{(70, 1), (80, 1), (90, 0.3)\}$

$VH = \{(90, 0.9), (100, 1)\}$

$S = \{(30, 0.8), (40, 1.0), (50, 0.6)\}$

$QS = \{(10, 1), (20, 0.8)\}$

1. If temperature is High then the rotation is Slow.

$$R = (H \times S) \cup (\overline{H} \times Y)$$

2. temperature is Very High

Thus, to deduce "rotation is Quite Slow", we make use the composition rule $QS = VH \circ R(x, y)$

Any questions??