CS512-S22-jain-akshay

[Assignment 0 (Solutions)]

A. $a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $c = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

1.) $2a - b = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 2-4 \\ 4-5 \\ 6-6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

2) $\hat{a}$, a unit vector in the direction of a

$|\vec{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$\hat{a} = \dfrac{a}{|\vec{a}|} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$

3) $\|a\|$ and the angle of a relative to the positive x axis

- $\|a\| = \sqrt{1^2+2^2+3^2} = \boxed{\sqrt{14}}$
- 3x1 matrix for +ve X-axis (a_x) $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where $|\vec{a_x}| = 1$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{a_x}}{|\vec{a}| \cdot |\vec{a_x}|}$

$\vec{a} \cdot \vec{a_x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 + 0 + 0 = 1$

$\cos\theta = \dfrac{1}{\sqrt{14} \cdot 1} = 1/\sqrt{14}$ or $\boxed{\theta = \cos^{-1}(1/\sqrt{14})}$

4) The dir^n cosines of a

$\cos a = \frac{a_x}{\|a\|} = \frac{1}{\sqrt{14}}$, $\cos b = \frac{a_y}{\|a\|} = \frac{2}{\sqrt{14}}$, $\cos c = \frac{a_z}{\|a\|} = 3/\sqrt{14}$

where $\cos a$, $\cos b$ & $\cos c$ are dir^n cosines of a

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5) the angle b/w a & b

$a \cdot b = \|a\| \cdot \|b\| \cos\theta$

$\cos\theta = \frac{a \cdot b}{\|a\| \cdot \|b\|} \Rightarrow \frac{32}{\sqrt{14}\sqrt{77}} = \frac{32.0}{32.8} = 1$

where; $a \cdot b = 32$

$\|a\| = \sqrt{14}$

$\|b\| = \sqrt{4^2+5^2+6^2}$

$= \sqrt{16+25+36} = \sqrt{77}$

$\therefore \boxed{\theta = 0 \text{ degrees}}$

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6) $a \cdot b$ & $b \cdot a$

$a \cdot b = 32$

$b \cdot a = 32$

7) $a \cdot b$ by using angle b/w a & b $(\theta) = (0 \text{ degrees})$

$\cos\theta = \frac{a \cdot b}{\|A\|\|B\|} = 1$

$\boxed{a \cdot b = \sqrt{14}\sqrt{77} = 32}$

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8) Scaler Projection of b onto $\hat{a}$

S. Projection $= \frac{b \cdot a}{\|a\|} = \frac{32}{\sqrt{14}}$

or $\left(\frac{b \cdot a}{\|a\|}\right)$

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9) a vector which is perpendicular to 'a'

Let's take an arbitory vector V ⊥ to 'a' such that $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

So; $a \cdot V = 0$ $\quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ or $\boxed{x + 2y + 3z = 0}$

for $\boxed{y = 0}$, $\boxed{z = 1}$ $\quad x = -3z$ $\quad \boxed{x = -3}$

So; $\boxed{V \text{ can be } = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}$

perpendicular vector to a

10) $a \times b$ & $b \times a$

Cross Product Using Matrix Notation

So. $a = \hat{i} + 2\hat{j} + 3\hat{k}$ & $b = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$$\underline{a \times b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \hat{i}(12-15) - \hat{j}(6-12) + \hat{k}(5-8)$$

$$= \boxed{3\hat{i} + 6\hat{j} - 3\hat{k}}$$

&

$$\underline{b \times a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i}(15-12) - \hat{j}(12-6) + \hat{k}(8-5)$$

$$= \boxed{3\hat{i} - 6\hat{j} + 3\hat{k}}$$

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11) a vector which is $\perp$ to both a & b

let's take 'v' a vector $\perp$ to a & b

then we can write '$\underline{v}$' as $a \times b$

Hence, $\boxed{v = 3\hat{i} + 6\hat{j} - 3\hat{k}}$

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12) The linear dependency b/w a, b, c

to find linear dependency let's find determinant of the matrix formed using a, b, c

$$M = \begin{vmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 16 & 3 \end{vmatrix}$$

$$\det(M) = 1(15-6) - 4(6-3) - 1(12-15)$$

$$= 9 - 4(3) + 3$$

$$= 0$$

Hence, (a, b & c vectors are linearly dependent)

13) a^Tb & ab^T

$$a^Tb = \underset{1\times3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} = 4 + 10 + 18$$

$$= \boxed{32}$$

$$ab^T = \underset{3\times1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \underset{1\times3}{\begin{bmatrix} 4 & 5 & 6 \end{bmatrix}}$$

So, ab^T will be 3×3 matrix

$$ab^T = \begin{bmatrix} 4 & 5 & 6 \\ 4\times2 & 5\times2 & 6\times2 \\ 4\times3 & 5\times3 & 6\times3 \end{bmatrix}$$

$$\boxed{ab^T = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}}$$

B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

$$c = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix} \qquad d = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

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1) $2A - B$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}}$$

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2) AB & BA

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4+9 & - & - \\ 4-4\ 9 & - & - \\ 0+10-3 & - & - \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$$

Similarly,

$$BA = \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$$

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3) $(AB)^T$ & $B^T A^T$

from Answer 2

$$AB = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

As $(AB)^T = B^T A^T$

$$B^T A^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

4). $|A|$ & $|C|$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} = 1(2-15) - 2(-4) + 3(20) = 55$$

$$|C| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{vmatrix} = 0$$

5) matrix (A, B, or C) in which the row vectors form an orthogonal set.

* B is the only matrix in which row vectors form an orthogonal set as the dot product of the rows in B is $\boxed{0}$.

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

Row 1 . Row 2 = 2 + 2 - 4 = 0
Row 2 . Row 3 = 6 - 2 - 4 = 0
Row 3 . Row 1 = 3 - 4 + 1 = 0

6) A^{-1} & B^{-1}

$AA^{-1} = I$ ↳ Identity Matrix

$$\therefore A^{-1} = \begin{bmatrix} -13/55 & 17/55 & 12/55 \\ 4/55 & -1/55 & 9/55 \\ 20/55 & -5/55 & -10/55 \end{bmatrix}$$

Similarly,

$$B^{-1} = \begin{bmatrix} 7/42 & 4/42 & 9/42 \\ 14/42 & 2/42 & -6/42 \\ 7/42 & -8/42 & 3/42 \end{bmatrix}$$

7) C^{-1}

Determinant of matrix C is 0

Hence, C has no inverse.

8) product Ad

$$Ad = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$$

9) the scaler projection of rows of A onto vector d with normalizing d

So, $$\hat{d} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

Scaler Projection for Row 1 $= \vec{Row1} \cdot \hat{d}$

$$= 1/\sqrt{14} + 4/\sqrt{14} + 9/\sqrt{14} = \sqrt{14}$$

" " " Row 2 $= \vec{Row2} \cdot \hat{d}$

$$= 4/\sqrt{14} - 4/\sqrt{14} + 9/\sqrt{14} = 9/\sqrt{14}$$

" " " Row 3 $= \vec{Row3} \cdot \hat{d}$

$$= 0 + \frac{5 \times 2}{\sqrt{14}} - 3/\sqrt{14} = 7/\sqrt{14}$$

10) Vector projection " — with d

$$Proj_d(Row1) = \frac{\vec{Row1} \cdot \vec{d}}{|d|^2} \vec{d} = \frac{14}{(\sqrt{14})^2} \langle 1, 2, 3 \rangle$$

$$Proj_d(Row2) = \frac{\vec{Row2} \cdot \vec{d}}{|d|^2} \vec{d} = \frac{9}{(\sqrt{14})^2} \langle 1, 2, 3 \rangle$$

$$\text{Proj}_{\vec{d}}(\text{Row3}) = \frac{\vec{\text{Row3}} \cdot \vec{d}}{|\vec{d}|^2} \vec{d}$$

$$= \frac{7}{(\sqrt{14})^2} \langle 1, 2, 3 \rangle$$

11) The linear combinations of a using elements of d

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & 1 \end{bmatrix} \qquad \vec{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 9 \\ 3 \end{bmatrix}$$

12) Solution of x for $Bx = \vec{d}$

or $x = B^{-1} \vec{d}$

$$= \begin{bmatrix} 7/42 & 4/42 & 9/42 \\ 14/42 & 2/42 & -6/42 \\ 7/42 & -8/42 & -3/42 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(from Ans 6)

$$\boxed{x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

13) For $Cx = d$

or $\boxed{x = C^{-1} d}$

As we have seen from Ans 7 C^{-1} doesn't exit. Hence, no solution for 'x'

$\boxed{C}$ $D = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $E = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$,

$F = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

1) eigen values & corresponding eigenvector of D

sol^n $\det(D - \lambda I)$ or $|D - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0 \text{ or } (1-\lambda)(2-\lambda) - 6 = 0$$

or $\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$ So, $\lambda = 4, -1$ (Eigen Values)

~~For~~ $\boxed{\text{Eigen vectors}}$

for $\lambda = 4$

$$\begin{bmatrix} 1-4 & 2 \\ 3 & 2-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$-3x + 2y = 0$ & $3x = 2y$

if $y = 1$, $x = 2/3$

$\therefore$ Eigen vectors $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

for $\lambda = -1$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$2(x+y) = 0, \quad 3(x+y) = 0$

$\boxed{x = -y}$ Eigen vector (for $y = 1$) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2) Dotproduct b/w vectors of D

$$\begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2/3 + 1 = +1/3.$$

3) Dot product b/w eigen vectors of E

$$\det(E - \lambda I) = \begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$$

$\lambda = 6, 1$

Eigen vectors

$\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

dotproduct $= \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= 0$

4) property of eigenvectors of E & its reason

Sol^n As dot product of eigen vectors of E is Zero so one of the property is both the vectors are '⊥' to each other.

5) $Fx = 0 \Rightarrow$ trivial sol^n ?

Sol^n

$$|F| = \left|\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right| = 0$$

determinat

So, as the $det(F) = 0$
trivial Sol^n for $F(x) = 0$ is $\boxed{(0,0)}$

6) Non-trivial Sol^n for $Fx = 0$

Sol^n As $|F| = 0$

So, as $det(F) = 0$
non-trivial sol^n for $f(x) = 0$ is
$\boxed{x = -2y}$, $\boxed{3x = -2y}$

7) only sol^n of x to the eq^n
$Dx = 0$ & reason of Single Sol^n

Sol^n

As $|D| \neq 0$.
~~After multiplying~~ $\cancel{x+2y=0}$

On Solving D matrix we get
$x + 2y = 0$ & $3x = -2y$
Hence, the only possible sol^n of x & y
is $(0,0)$

D. $f(x) = x^2 + 3$, $g(x) = x^2$, $q(x,y) = x^2 + y^2$ find:

1) I^{st} & II^{nd} derivative of $f(x)$ with respect to x:

$f'(x) = 2x$, $f''(x) = 2$

2) The partial derivatives:

$\frac{\partial q}{\partial x} = 2x$, $\frac{\partial q}{\partial y} = 2y$

3) The gradient vector: $\nabla q(x,y)$

$\nabla q(x,y) = \langle \frac{\partial q}{\partial x}, \frac{\partial q}{\partial y} \rangle$

$= (2x, 2y)$

4) The derivative $\frac{\partial}{\partial x} f(g(x))$ with & without using the chain rule of derivatives:

$f(g(x)) = f(x^2) = (x^2)^2 + 3 = x^4 + 3$

$\frac{\partial}{\partial x} f(g(x)) = \boxed{4x^3}$