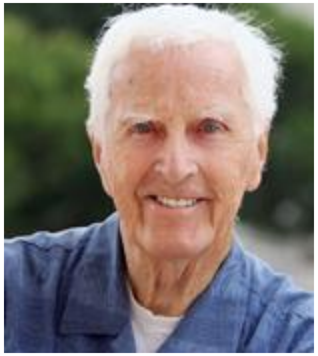
Centrality in terms of how you connect others
(information broker)

Betweenness Centrality

Another way of looking at centrality is by considering how important nodes are in connecting other nodes



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$$C_b(v_i) = \sum_{s \neq t \neq v_i} \frac{\sigma_{st}(v_i)}{\sigma_{st}}$$

σ_{st} The number of shortest paths from vertex s to t – *a.k.a.* information pathways

$\sigma_{st}(v_i)$ The number of **shortest paths** from s to t that pass through v_i

Normalizing Betweenness Centrality

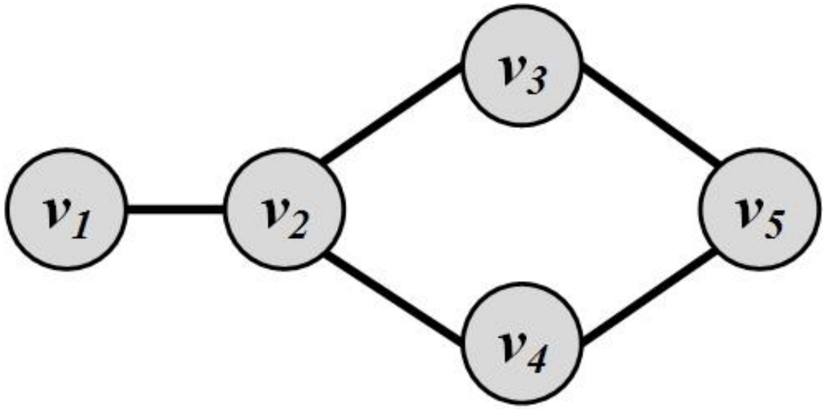
- In the best case, node v_i is on all shortest paths from s to t, hence, $\frac{\sigma_{st}(v_i)}{\sigma_{st}} = 1$

$$
\begin{aligned}
C_b(v_i) &= \sum_{s \neq t \neq v_i} \frac{\sigma_{st}(v_i)}{\sigma_{st}} \\
&= \sum_{s \neq t \neq v_i} 1 = 2\binom{n-1}{2} = (n-1)(n-2)
\end{aligned}
$$

Therefore, the maximum value is $(n-1)(n-2)$

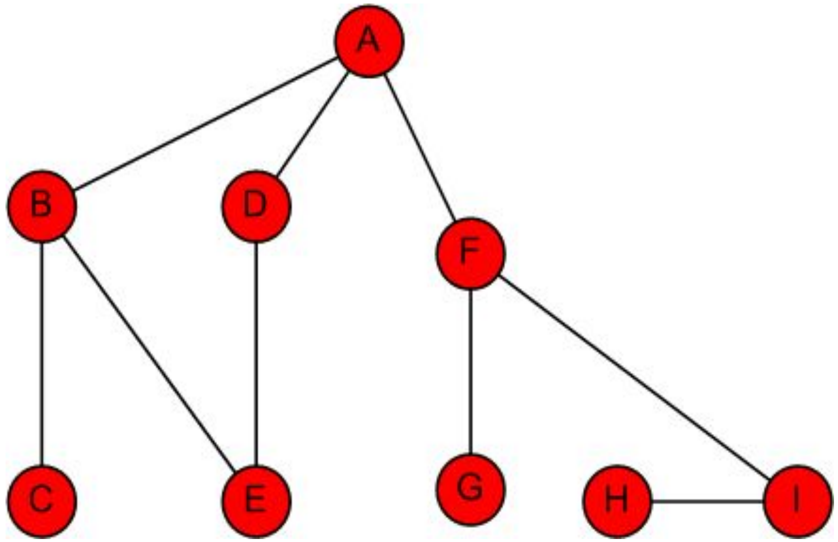
Betweenness centrality: $C_b^{\mathrm{norm}}(v_i) = \frac{C_b(v_i)}{2\binom{n-1}{2}}$

Betweenness Centrality: Example 1



$$C_b(v_2) = 2 \times (\underbrace{(1/1)}_{s=v_1,t=v_3} + \underbrace{(1/1)}_{s=v_1,t=v_4} + \underbrace{(2/2)}_{s=v_1,t=v_5} + \underbrace{(1/2)}_{s=v_3,t=v_4} + \underbrace{0}_{s=v_3,t=v_5} + \underbrace{0}_{s=v_4,t=v_5})$$
$$= 2 \times 3.5 = 7,$$
$$C_b(v_3) = 2 \times (\underbrace{0}_{s=v_1,t=v_2} + \underbrace{0}_{s=v_1,t=v_4} + \underbrace{(1/2)}_{s=v_1,t=v_5} + \underbrace{0}_{s=v_2,t=v_4} + \underbrace{(1/2)}_{s=v_2,t=v_5} + \underbrace{0}_{s=v_4,t=v_5})$$
$$= 2 \times 1.0 = 2,$$
$$C_b(v_4) = C_b(v_3) = 2 \times 1.0 = 2,$$
$$C_b(v_5) = 2 \times (\underbrace{0}_{s=v_1,t=v_2} + \underbrace{0}_{s=v_1,t=v_3} + \underbrace{0}_{s=v_1,t=v_4} + \underbrace{0}_{s=v_2,t=v_3} + \underbrace{0}_{s=v_2,t=v_4} + \underbrace{(1/2)}_{s=v_3,t=v_4})$$
$$= 2 \times 0.5 = 1,$$

Betweenness Centrality: Example 2

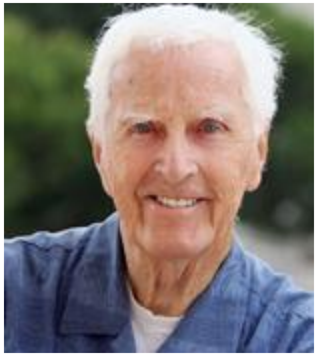


Node	Betweenness Centrality	Rank
A	16 + 1/2 + 1/2	**1**
B	7+5/2	**3**
C	0	**7**
D	5/2	**5**
E	1/2 + 1/2	**6**
F	15 + 2	**1**
G	0	**7**
H	0	**7**
I	7	**4**

Centrality in terms of how fast you can reach others

Closeness Centrality

- The intuition is that influential/central nodes can quickly reach other nodes



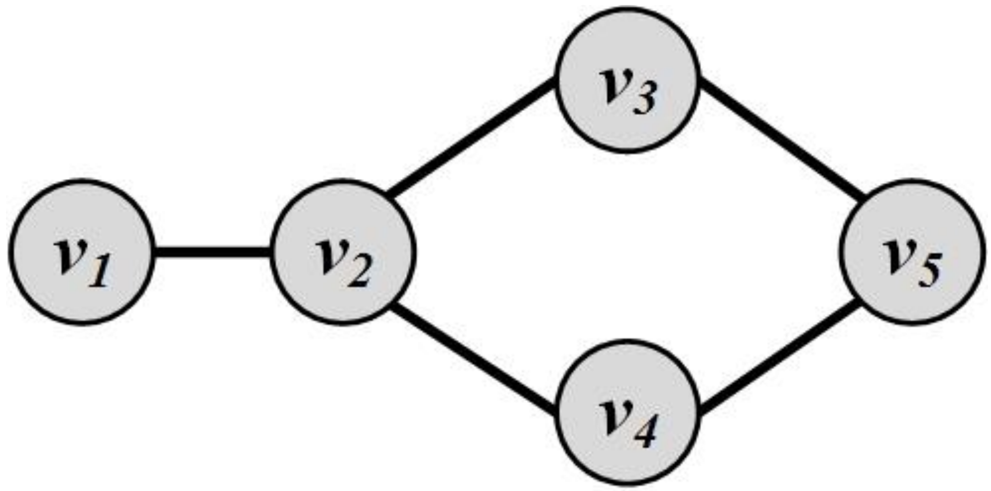
Linton Freeman

- These nodes should have a smaller average shortest path length to others

Closeness centrality: $C_c(v_i) = \frac{1}{\bar{l}_{v_i}}$

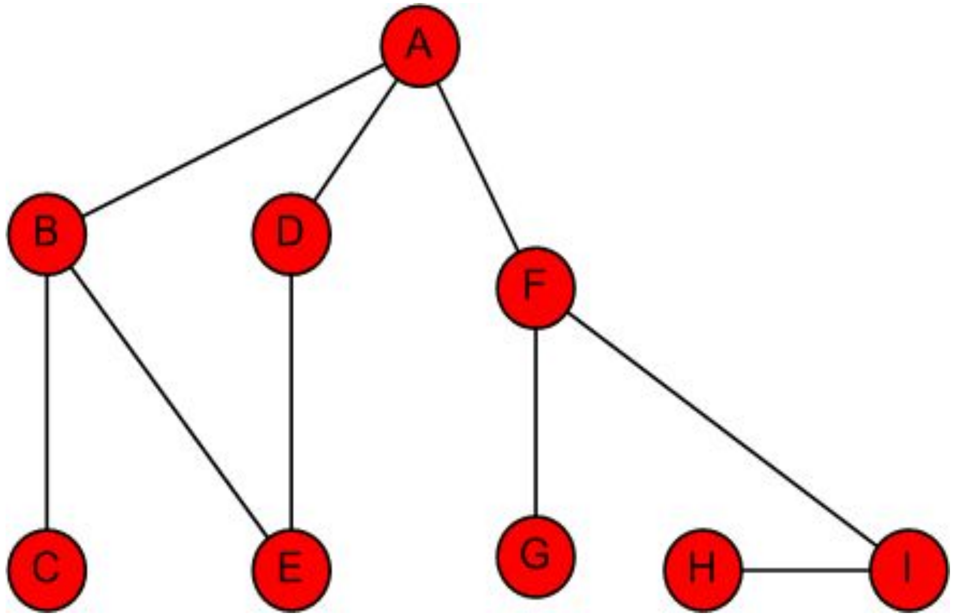
$$\bar{l}_{v_i} = \frac{1}{n-1} \sum_{v_j \neq v_i} l_{i,j}$$

Closeness Centrality: Example 1



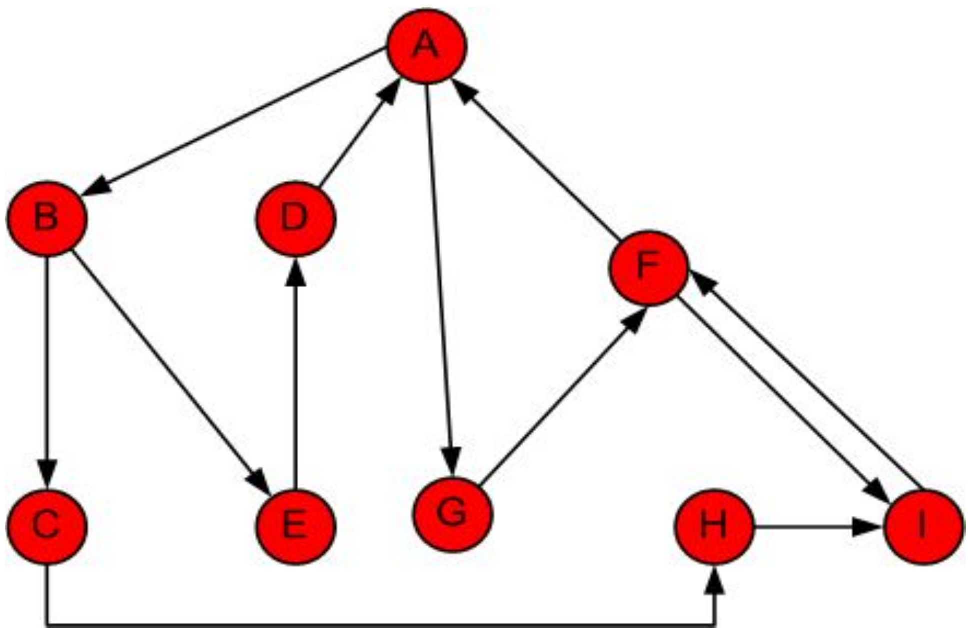
$$
\begin{aligned}
C_c(v_1) &= 1 \;/\; (\;(1+2+2+3)/4\;) = 0.5, \\
C_c(v_2) &= 1 \;/\; (\;(1+1+1+2)/4\;) = 0.8, \\
C_c(v_3) = \quad C_b(v_4) &= 1 \;/\; (\;(1+1+2+2)/4\;) = 0.66, \\
C_c(v_5) &= 1 \;/\; (\;(1+1+2+3)/4\;) = 0.57.
\end{aligned}
$$

Closeness Centrality: Example 2 (Undirected)



Node	A	B	C	D	E	F	G	H	I	D_Avg	Closeness Centrality	Rank
A	0	1	2	1	2	1	2	3	2	1.750	0.571	**1**
B	1	0	1	2	1	2	3	4	3	2.125	0.471	**3**
C	2	1	0	3	2	3	4	5	4	3.000	0.333	**8**
D	1	2	3	0	1	2	3	4	3	2.375	0.421	**4**
E	2	1	2	1	0	3	4	5	4	2.750	0.364	**7**
F	1	2	3	2	3	0	1	2	1	1.875	0.533	**2**
G	2	3	4	3	4	1	0	3	2	2.750	0.364	**7**
H	3	4	5	4	5	2	3	0	1	3.375	0.296	**9**
I	2	3	4	3	4	1	2	1	0	2.500	0.400	**5**

Closeness Centrality: Example 3 (Directed)



Node	A	B	C	D	E	F	G	H	I	D_Avg	Closeness Centrality	Rank
A	0	1	2	3	2	2	1	3	3	2.125	0.471	**1**
B	3	0	1	2	1	4	4	2	3	2.500	0.400	**2**
C	4	5	0	7	6	3	5	1	2	4.125	0.242	**9**
D	1	2	3	0	3	3	2	4	5	2.875	0.348	**3**
E	2	3	4	1	0	4	3	5	5	3.375	0.296	**6**
F	1	2	3	4	3	0	2	4	4	2.875	0.348	**4**
G	2	3	4	5	4	1	0	5	2	3.250	0.308	**5**
H	4	4	5	6	5	2	4	0	1	3.875	0.258	**8**
I	2	3	4	5	4	1	4	5	0	3.500	0.286	**7**

Centrality for a group of nodes

Group Centrality

- All centrality measures defined so far measure centrality for a single node. These measures can be generalized for a group of nodes.

- A simple approach is to replace all nodes in a group with a super node
  - The group structure is disregarded.

- Let S denote the set of nodes in the group and $V - S$ the set of outsiders

Group Centrality

I. Group Degree Centrality

$$C_d^{\text{group}}(S) = |\{v_i \in V - S | v_i \text{ is connected to } v_j \in S\}|$$

- **Normalization**: divide by $|V - S|$

II. Group Betweenness Centrality

$$C_b^{\text{group}}(S) = \sum_{s \neq t, s \notin S, t \notin S} \frac{\sigma_{st}(S)}{\sigma_{st}}$$

- **Normalization**: divide by $2\binom{|V-S|}{2}$

III. Group Closeness Centrality

$$C_c^{\text{group}}(S) = \frac{1}{\bar{l}_S^{\text{group}}}$$

– It is the average distance from non-members to the group

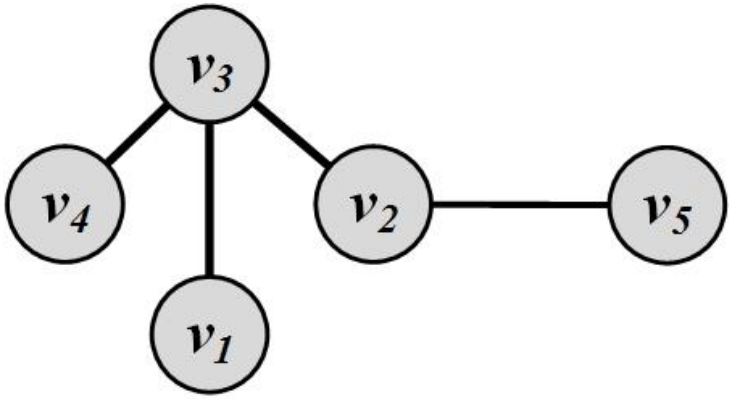
$$\bar{l}_S^{\text{group}} = \frac{1}{|V-S|} \sum_{v_j \notin S} l_{S,v_j}$$

$$l_{S,v_j} = \min_{v_i \in S} l_{v_i,v_j}$$

- One can also utilize the *maximum distance* or the *average distance*

Group Centrality Example

- Consider $S = \{v_2, v_3\}$



- Group degree centrality = **3**
- Group betweenness centrality = **6**
- Group closeness centrality = **1**