CS 579: Online Social Network Analysis

Network Measures

Reading: Chapter 3

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Why Do We Need Measures?

- Who are the central figures (influential individuals) in the network?
  - **Centrality**
- What interaction patterns are common in friends?
  - **Reciprocity and Transitivity**
  - **Balance and Status**
- Who are the like-minded users and how can we find these similar individuals?
  - **Similarity**
- To answer these and similar questions, one first needs to define measures for quantifying **centrality**, **level of interactions**, and **similarity**, among others.

Centrality

Centrality defines how important a node is within a network

Centrality in terms of those who you are connected to

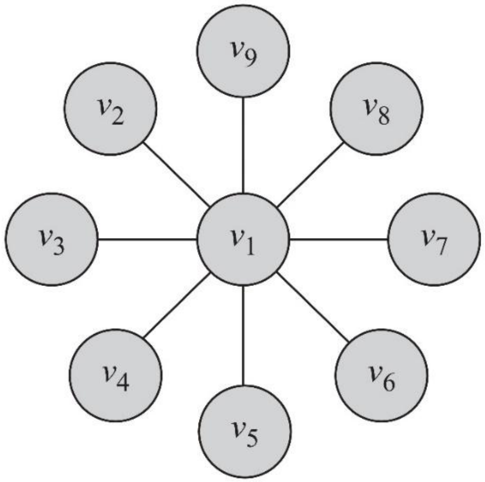
Degree Centrality

- **Degree centrality**: ranks nodes with more connections higher in terms of centrality

$$C_d(v_i) = d_i$$

- d_i is the degree (number of friends) for node v_i
  - i.e., the number of length-1 paths (can be generalized)

In this graph, degree centrality for node v_1 is d_1=8 and for all others is $d_j = 1, j \neq 1$



Degree Centrality in Directed Graphs

- In directed graphs, we can either use the in-degree, the out-degree, or the combination as the degree centrality value
- In practice, mostly in-degree is used.

$$
\begin{aligned}
C_d(v_i) &= d_i^{\text{in}} && \textit{(prestige)} \\
C_d(v_i) &= d_i^{\text{out}} && \textit{(gregariousness)} \\
C_d(v_i) &= d_i^{\text{in}} + d_i^{\text{out}}
\end{aligned}
$$

d_i^{out} is the number of outgoing links for node v_i

Normalized Degree Centrality

- Normalized by the maximum <u>possible</u> degree

$$C_d^{\text{norm}}(v_i) = \frac{d_i}{n-1}$$

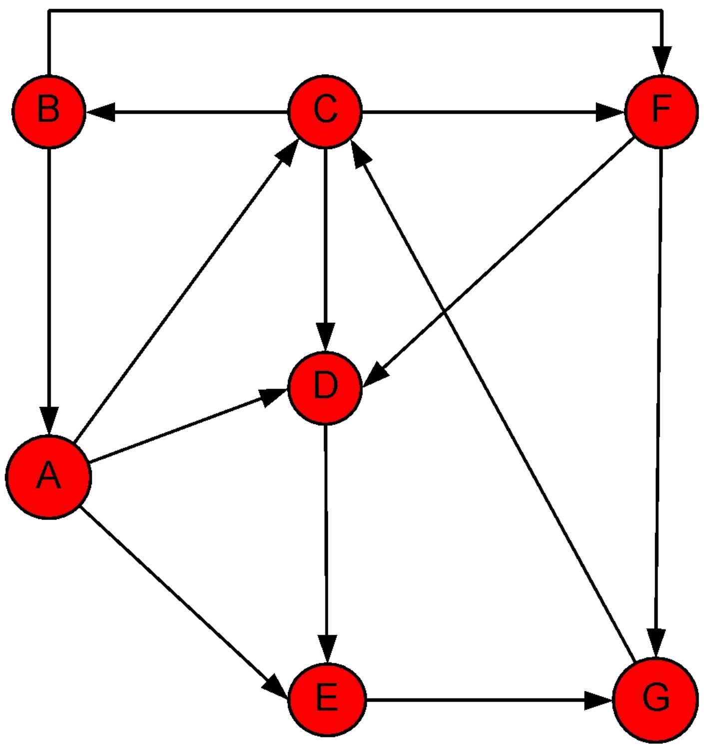
- Normalized by the maximum degree

$$C_d^{\max}(v_i) = \frac{d_i}{\max_j d_j}$$

- Normalized by the degree sum

$$C_d^{\text{sum}}(v_i) = \frac{d_i}{\sum_j d_j} = \frac{d_i}{2|E|} = \frac{d_i}{2m}$$

Degree Centrality (Directed Graph) Example



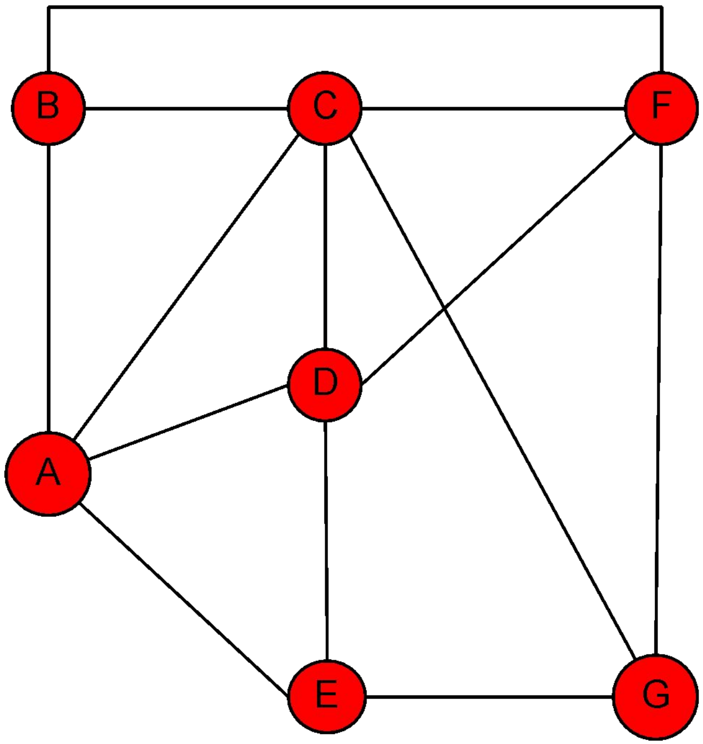
Calculate the outdegree centrality of nodes

Node	In-Degree	Out-Degree	Centrality	Rank
A	1	3	1/2	**1**
B	1	2	1/3	**3**
C	2	3	1/2	**1**
D	3	1	1/6	**5**
E	2	1	1/6	**5**
F	2	2	1/3	**3**
G	2	1	1/6	**5**

Normalized by the maximum possible degree

$$C_d^{\text{norm}}(v_i) = \frac{d_i}{n-1}$$

Degree Centrality (undirected Graph) Example



Calculate the degree centrality of nodes

Node	Degree	Centrality	Rank
A	4	2/3	**2**
B	3	1/2	**5**
C	5	5/6	**1**
D	4	2/3	**2**
E	3	1/2	**5**
F	4	2/3	**2**
G	3	1/2	**5**

Normalized by the maximum possible degree

$$C_d^{\text{norm}}(v_i) = \frac{d_i}{n-1}$$

Eigenvector Centrality

- Having more friends does not by itself guarantee that someone is more important
  - Having more **important friends** provides a stronger signal

Phillip Bonacich

- Eigenvector centrality generalizes degree centrality by incorporating the importance of the neighbors (undirected)

- For directed graphs, we can use incoming or outgoing edges

Formulation

- Let's assume the eigenvector centrality of a node is $c_e(v_i)$ (unknown)
- We would like $c_e(v_i)$ to be higher when important neighbors (node v_j with higher $c_e(v_j)$) point to us
  - Incoming or outgoing neighbors?
  - For incoming neighbors $A_{j,i} = 1$
- We can assume that v_i's centrality is the summation of its neighbors' centralities

$$c_e(v_i) = \sum_{j=1}^{n} A_{j,i} c_e(v_j)$$

- Is this summation bounded?
  - We have to normalize!

    λ: some fixed constant

$$c_e(v_i) = \frac{1}{\lambda} \sum_{j=1}^{n} A_{j,i} c_e(v_j)$$

Eigenvector Centrality (Matrix Formulation)

- Let $\mathbf{C}_e = (C_e(v_1), C_e(v_2), \ldots, C_e(v_n))^T$

$$\rightarrow \quad \lambda \mathbf{C}_e = A^T \mathbf{C}_e$$

- This means that $\boldsymbol{C_e}$ is an eigenvector of adjacency matrix A^T (or A when undirected) and λ is the corresponding eigenvalue

- Which eigenvalue-eigenvector pair to choose?
  – We prefer centrality values to be positive for convenient comparison

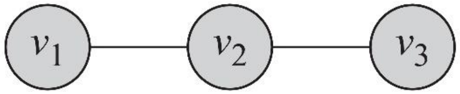
Eigenvector Centrality, cont.

Theorem 1 (Perron-Frobenius Theorem). *Let $A \in \mathbb{R}^{n \times n}$ represent the adjacency matrix for a [strongly] connected graph or $A : A_{i,j} > 0$ (i.e. a positive n by n matrix). There exists a positive real number (Perron-Frobenius eigenvalue) $\lambda_{\max}$, such that $\lambda_{\max}$ is an eigenvalue of A and any other eigenvalue of A is strictly smaller than $\lambda_{\max}$. Furthermore, there exists a corresponding eigenvector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ of A with eigenvalue $\lambda_{\max}$ such that $\forall v_i > 0$.*

So, to compute eigenvector centrality of A,

1. We compute the eigenvalues of A
2. Select the largest eigenvalue λ
3. The corresponding eigenvector of λ is $\mathbf{C_e}$.
4. Based on the Perron-Frobenius theorem, all the components of $\mathbf{C_e}$ will be positive
5. The components of $\mathbf{C_e}$ are the eigenvector centralities for the graph.

Eigenvector Centrality: Example 1



$$\lambda \mathbf{C}_e = A\mathbf{C}_e \qquad (A - \lambda I)\mathbf{C}_e = 0 \qquad \mathbf{C}_e = [u_1\ u_2\ u_3]^T$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0-\lambda & 1 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$(-\lambda)(\lambda^2 - 1) - 1(-\lambda) = 2\lambda - \lambda^3 = \lambda(2 - \lambda^2) = 0$$

Eigenvalues are

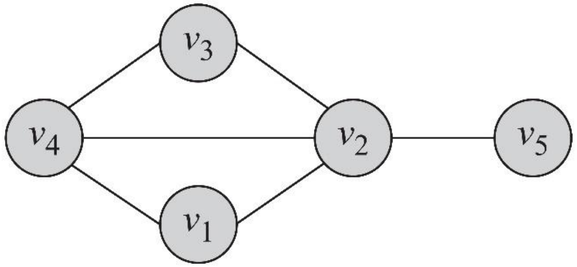
$$(-\sqrt{2}, 0, \boxed{+\sqrt{2}})$$

Largest Eigenvalue

Corresponding eigenvector (assuming $\mathbf{C_e}$ has norm 1)

$$\begin{bmatrix} 0-\sqrt{2} & 1 & 0 \\ 1 & 0-\sqrt{2} & 1 \\ 0 & 1 & 0-\sqrt{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{C}_e = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{bmatrix}$$

Eigenvector Centrality: Example 2



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{\lambda} = (2.68, \ -1.74, -1.27, \ 0.33, 0.00)$$

Eigenvalues

$$\lambda_{max} = 2.68 \Rightarrow C_e = \begin{bmatrix} 0.4119 \\ 0.5825 \\ 0.4119 \\ 0.5237 \\ 0.2169 \end{bmatrix}$$

Katz Centrality

- A major problem with eigenvector centrality arises when it deals with directed graphs
- Centrality only passes over *outgoing* edges and in special cases such as when a node is in a directed acyclic graph the centrality can become zero

Elihu Katz

- To resolve this problem we add bias term β to the centrality values for all nodes

Eigenvector Centrality

$$C_{\text{Katz}}(v_i) = \boxed{\alpha \sum_{j=1}^{n} A_{j,i} C_{\text{Katz}}(v_j)} + \beta$$

Katz Centrality, cont.

$$C_{\text{Katz}}(v_i) = \alpha \sum_{j=1}^{\mathrm{n}} A_{j,i} C_{\text{Katz}}(v_j) + \beta$$

Controlling term

Bias term

Rewriting equation in a vector form

$$\mathbf{C}_{\text{Katz}} = \alpha A^T \mathbf{C}_{\text{Katz}} + \beta \mathbf{1}$$

vector of all 1's

Katz centrality: $\mathbf{C}_{\text{Katz}} = \beta(\mathbf{I} - \alpha A^T)^{-1} \cdot \mathbf{1}$

Katz Centrality, cont.

- $$C_{\text{Katz}}(v_i) = \alpha \sum_{j=1}^{\mathrm{n}} A_{j,i} C_{\text{Katz}}(v_j) + \beta$$

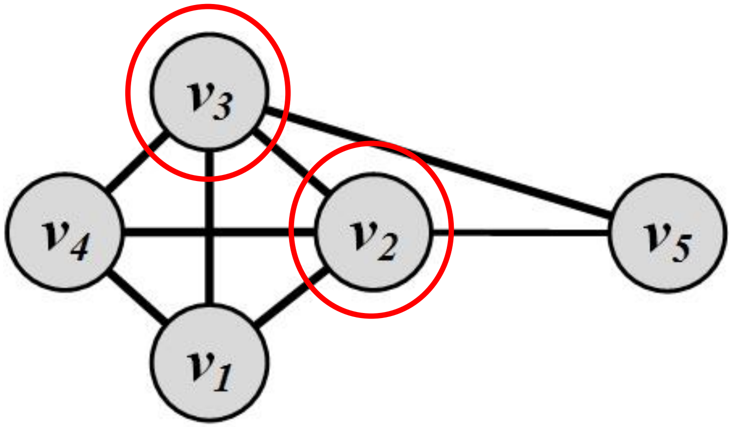
- When α=0, the eigenvector centrality is removed and all nodes get the same centrality value β
  - As α gets larger the effect of β is reduced

- For the matrix $(I - \alpha A^T)$ to be invertible, we must have
  - $det(I - \alpha A^T) \neq 0$
  - By rearranging we get $det(\mathrm{A}^{\mathrm{T}} - \alpha^{-1} I) = 0$
  - This is basically the characteristic equation,
  - The characteristic equation **first** becomes zero when the largest eigenvalue equals α^{-1}

The largest eigenvalue is easier to compute (power method)

In practice we select $\alpha < 1/\lambda$, where λ is the largest eigenvalue of A^T

Katz Centrality Example



$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = A^T$$

- The Eigenvalues are -1.68, -1.0, -1.0, 0.35, $\boxed{3.32}$
- We assume α=0.25 < $\boxed{1/3.32}$ and $\beta = 0.2$

$$\mathbf{C}_{Katz} = \beta(\mathbf{I} - \alpha A^T)^{-1} \cdot \mathbf{1} = \begin{bmatrix} 1.14 \\ 1.31 \\ 1.31 \\ 1.14 \\ 0.85 \end{bmatrix}$$

Most important nodes!

PageRank

- Problem with Katz Centrality:
  - In directed graphs, once a node becomes an authority (high centrality), it passes **all** its centrality along **all** of its out-links

- This is less desirable since not everyone known by a well-known person is well-known

- **Solution?**
  - We can divide the value of passed centrality by the number of outgoing links, i.e., out-degree of that node
  - Each connected neighbor gets a fraction of the source node's centrality

PageRank, cont.

$$C_p(v_i) = \alpha \sum_{j=1}^{n} A_{j,i} \frac{C_p(v_j)}{d_j^{\text{out}}} + \beta$$

What if the degree is zero?

$$\begin{cases} d_j^{out} > 0 \\ D = diag(d_1^{\text{out}}, d_2^{\text{out}}, \ldots, d_n^{\text{out}}) \end{cases}$$

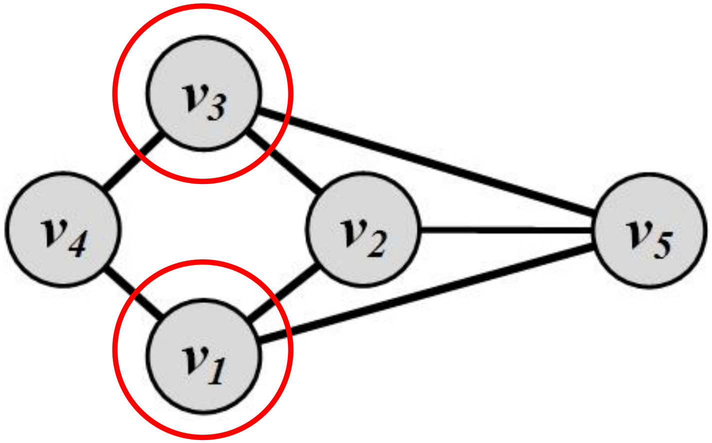
$$\mathbf{C}_p = \alpha A^T D^{-1} \mathbf{C}_p + \beta \mathbf{1}$$

$$\mathbf{C}_p = \beta (\mathbf{I} - \alpha A^T D^{-1})^{-1} \cdot \mathbf{1}$$

Similar to Katz Centrality, in practice, $\boldsymbol{\alpha} < \mathbf{1}/\boldsymbol{\lambda}$, where λ is the largest eigenvalue of $A^T D^{-1}$. In undirected graphs, the largest eigenvalue of $A^T D^{-1}$ is $\boldsymbol{\lambda} = 1$; therefore, $\boldsymbol{\alpha} < \mathbf{1}$.

PageRank Example

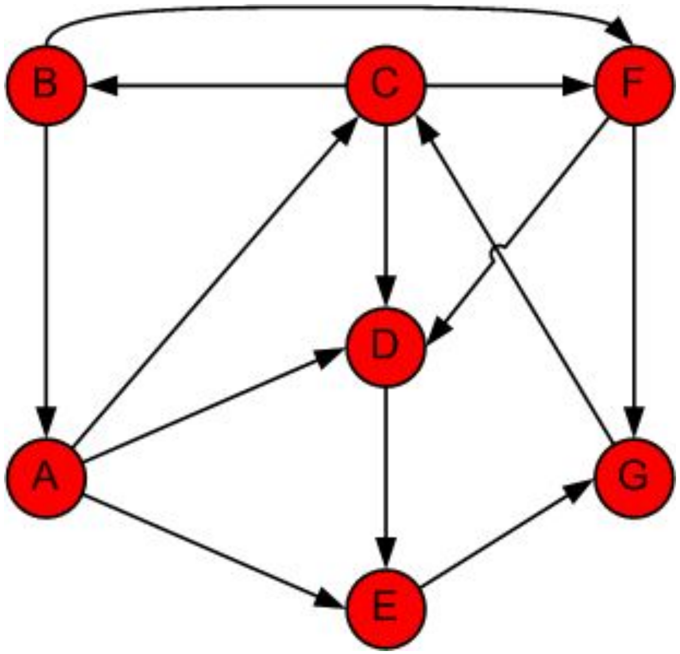
- We assume α=0.95 < 1 and and $\beta = 0.1$



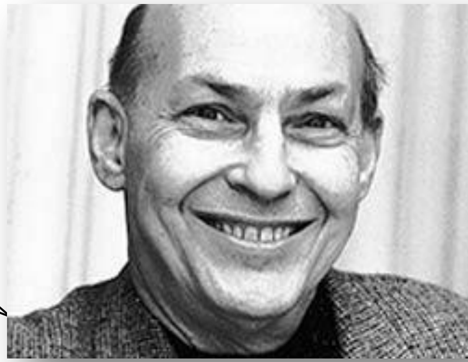
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}_p = \beta(\mathbf{I} - \alpha A^T D^{-1})^{-1} \cdot \mathbf{1} = \begin{bmatrix} 2.14 \\ 2.13 \\ 2.14 \\ 1.45 \\ 2.13 \end{bmatrix}$$

PageRank Example – Alternative Approach [Markov Chains]



"You don't understand anything until you learn it more than one way"



Marvin Minsky (1927-2016)

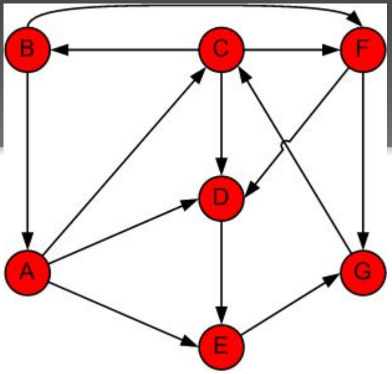
Using Power Method

α=1 and β =0?

$$C_p(v_i) = \alpha \sum_{j=1}^{n} A_{j,i} \frac{C_p(v_j)}{d_j^{\text{out}}} + \beta$$

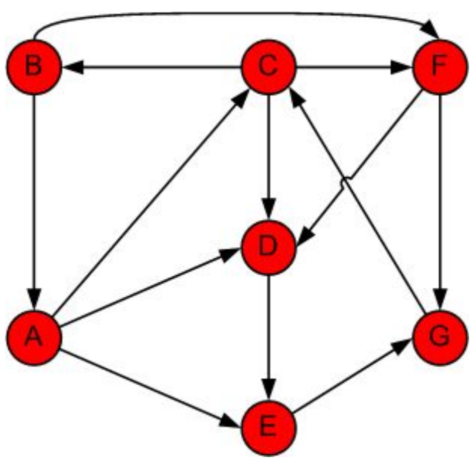
Step	A	B	C	D	E	F	G
0	1/7	1/7	1/7	1/7	1/7	1/7	1/7
1	B/2	C/3	A/3 + G	A/3 + C/3 + F/2	A/3 + D	C/3 + B/2	F/2 + E
	0.071	0.048	0.190	0.167	0.190	0.119	0.214

PageRank: Example



Step	A	B	C	D	E	F	G	Sum
1	0.143	0.143	0.143	0.143	0.143	0.143	0.143	1.000
2	0.071	0.048	0.190	0.167	0.190	0.119	0.214	1.000
3	0.024	0.063	0.238	0.147	0.190	0.087	0.250	1.000
4	0.032	0.079	0.258	0.131	0.155	0.111	0.234	1.000
5	0.040	0.086	0.245	0.152	0.142	0.126	0.210	1.000
6	0.043	0.082	0.224	0.158	0.165	0.125	0.204	1.000
7	0.041	0.075	0.219	0.151	0.172	0.115	0.228	1.000
8	0.037	0.073	0.241	0.144	0.165	0.110	0.230	1.000
9	0.036	0.080	0.242	0.148	0.157	0.117	0.220	1.000
10	0.040	0.081	0.232	0.151	0.160	0.121	0.215	1.000
11	0.040	0.077	0.228	0.151	0.165	0.118	0.220	1.000
12	0.039	0.076	0.234	0.148	0.165	0.115	0.223	1.000
13	0.038	0.078	0.236	0.148	0.161	0.116	0.222	1.000
14	0.039	0.079	0.235	0.149	0.161	0.118	0.219	1.000
15	0.039	0.078	0.232	0.150	0.162	0.118	0.220	1.000
Rank	**7**	**6**	**1**	**4**	**3**	**5**	**2**	

Effect of PageRank



Node	Rank
A	**7**
B	**6**
C	**1**
D	**4**
E	**3**
F	**5**
G	**2**