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# A modification of eigenvalue localization for stochastic matrices

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ABSTRACT

A new eigenvalue localization for stochastic matrices is provided, and is used to estimate the moduli of the subdominant eigenvalue. Numerical examples are given to show that our results are better than those in Cvetković, Kostić and Peña (2011) [3].



## 1. Introduction

An entrywise nonnegative matrix $A = [a_{ij}] \in R^{n\times n}$ is called row stochastic (or simply stochastic) if all its row sums are 1, that is,

$$\sum_{j=1}^{n} a_{ij} = 1, \quad \text{for each } i \in N = \{1, 2, \ldots, n\}.$$

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Obviously, 1 is an eigenvalue of a stochastic matrix with a corresponding eigenvector $e = [1, 1, \ldots, 1]^T$. From the Perron–Frobenius Theorem [1], for any eigenvalue $\lambda$ of $A$, that is, $\lambda \in \sigma(A)$, we have $|\lambda| \leq 1$, so that in fact 1 is a dominant eigenvalue for $A$ [3]. Here we call $\lambda$ a subdominant eigenvalue of a stochastic matrix $A$ if $1 > |\lambda| > |\eta|$ for all eigenvalues $\eta$ different from 1 and $\lambda$ [3,5,6].

Stochastic matrices and eigenvalue localization of stochastic matrices play key roles in many application fields such as Computer Aided Geometric Design [9], Birth–Death Processes [2,4,7,8], and Markov chain [10].

In [3], L.J. Cvetković et al. presented a region including all eigenvalues of a stochastic matrix $A$ different from 1 by refining the Geršgorin circle [11] of $A$.

**Theorem 1.** *(See [3, Theorem 3.4].) Let $A = [a_{ij}] \in R^{n\times n}$ be a stochastic matrix, and let $s_i$ be the minimal element among the off-diagonal entries of the $i$-th column of $A$, that is, $s_i = \min_{j\neq i} a_{ji}$. Taking $\gamma(A) = \max_{i\in N}(a_{ii} - s_i)$, then for any $\lambda \in \sigma(A)\backslash\{1\}$,*

$$\big|\lambda - \gamma(A)\big| < r(A) = 1 - trace(A) + (n-1)\gamma(A).$$

Although Theorem 1 provides a circle with the center $\gamma(A)$ and radius equal to $1 - trace(A) + (n-1)\gamma(A)$ to localize the eigenvalue $\lambda$, it is not effective in some cases. Consider the following class of matrices

$$SM_0 = \big\{A \in R^{n\times n} : A \text{ is stochastic, and } a_{ii} = s_i = 0, \text{ for each } i \in N\big\}.$$

Then for any $A \in SM_0$, $trace(A) = 0$ and $\gamma(A) = 0$. Hence by Theorem 1, we have $|\lambda| < 1$ for $\lambda \in \sigma(A)\backslash\{1\}$. This is trivial. It is very interesting how to conquer this drawback.

Note that if $A$ is a stochastic matrix, then $A^m$ is also stochastic for any positive integer $m$. Therefore, we can apply Theorem 1 to $A^m$ and obtain

$$\big|\lambda^m - \gamma\big(A^m\big)\big| \leq r\big(A^m\big) = 1 - trace\big(A^m\big) + (n-1)\gamma\big(A^m\big). \tag{1}$$

When $A$ is a positive stochastic matrix, L.J. Cvetković et al. [3] proved that the sequence $\{\gamma(A^m)\}$ converges to 0, and the radii of the corresponding circles $r(A^m) = 1 - trace(A^m) + (n-1)\gamma(A^m)$ also tend to 0.

**Theorem 2.** *(See [3, Theorem 3.5].) Let $A = [a_{ij}] \in R^{n\times n}$ be a positive stochastic matrix. Then*

$$\lim_{m\to\infty} \gamma\big(A^m\big) = 0$$

*and*

$$\lim_{m\to\infty} r\big(A^m\big) = 0.$$

Let $v(A^m) = (r_m(A) + |\gamma(A)|)^{\frac{1}{m}}$ and $d(A^m) = 1 - v(A^m)$. Then from inequality (1), we know that $v(A^m)$ is an upper bound for the moduli of subdominant eigenvalues of a stochastic matrix $A$, and that $d(A^m)$ represents the distance of the dominant eigenvalue 1 and the localization set $z \in C : |z^m - \gamma(A^m)| \leq r(A^m)$. Therefore, we can use $v(A^m)$ and $d(A^m)$ to estimate the subdominant eigenvalue and the spectral gap of $A$, respectively.

Also in [3], L.J. Cvetković et al. compared the upper bound $v(A^m)$ for the subdominant eigenvalue of stochastic matrices with those in [5,6], and give some numerical examples to show the upper bound $v(A^m)$ is better than those in [5,6].

In this paper, we also consider the eigenvalue localization for stochastic matrices, and give a new region including all eigenvalues different from 1. Furthermore, for a primitive stochastic matrix, especially, a positive stochastic matrix, we give a simple algorithm to estimate the spectral gap and the bound for the moduli of subdominant eigenvalues of the primitive stochastic matrix. Numerical examples are given to show our results are better than those in [3,5].

## 2. A modification of eigenvalue localization for stochastic matrices

Here $A = [a_{ij}] \in R^{n\times n}$ is called a matrix with constant row (column) sum $\lambda$ if $Ae = \lambda e$ ($A^T e = \lambda e$, respectively). In particular, when $\lambda = 1$, it is stochastic. Therefore, we first study eigenvalue localization of a matrix with constant row sum $\lambda$. Obviously, $\lambda$ is an eigenvalue of $A$. So we first give some results to localize the eigenvalues of $A$ different from $\lambda$.

**Proposition 1.** *(See [3, Proposition 2.1].) Let $A = [a_{ij}] \in R^{n\times n}$ such that $A^T e = \lambda e$, and let $\mu \in \sigma(A)\backslash\{\lambda\}$. Then for any real number $d_i$, $i \in N$, we have $-\mu \in \sigma(C)$, where*

$$C = diag(d_1, d_2, \ldots, d_n)ee^T - A.$$

By applying this result to $A^T$, we have

**Proposition 2.** *(See [3, Remark 1].) Let $A = [a_{ij}] \in R^{n\times n}$ such that $Ae = \lambda e$, and let $\mu \in \sigma(A)\backslash\{\lambda\}$. Then for any real number $d_i$, $i \in N$, we have $-\mu \in \sigma(B)$, where*

$$B = diag(d_1, d_2, \ldots, d_n)ee^T - A^T. \tag{2}$$

As is shown in [3], the best choice of $d_i$ in Proposition 1 would refine the Geršgorin circle [11] of $C$, which is a so-called modification of the Geršgorin circle of $A$. In particular, for a nonnegative matrix $A = [a_{ij}] \in R^{n\times n}$ we present the following approach:

- the choice $d_i = a_{ii} + \max_{i\in N}\{S_i - a_{ii}\}$, where

$$S_i = \max_{j\neq i} a_{ji}, \quad i \in N. \tag{3}$$

We next provide a new region including all eigenvalues different from 1 of a stochastic matrix. Before that three lemmas are given.

**Lemma 3.** *Let $A = [a_{ij}] \in R^{n\times n}$ be nonnegative such that $Ae = \lambda e$, let $B = [b_{ij}]$ be the matrix given in (2) with $d_i = S_i$ for each $i \in N$, and let $\mu \in \sigma(A)\backslash\{\lambda\}$. Then $-\mu \in \sigma(B)$, and for any $i, j \in N$*

$$R_i(B) = (n-1)S_i - C_i(A), \qquad C_j(B) = \sum_{k\neq j} S_k - R_j(A)$$

*where $R_i(B) = \sum_{j\neq i} |b_{ij}|$ and $C_j(B) = R_i(B^T)$.*

**Proof.** By Proposition 2, $-\mu \in \sigma(B)$ holds obviously. Note that $b_{ij} = S_i - a_{ji} \geq 0$ for $i, j \in N$. Hence, for any $i, j \in N$

$$R_i(B) = \sum_{j\neq i}(S_i - a_{ji}) = (n-1)S_i - \sum_{j\neq i} a_{ji} = (n-1)S_i - C_i(A)$$

and

$$C_j(B) = \sum_{k\neq j}(S_k - a_{jk}) = \sum_{k\neq j} S_k - \sum_{k\neq j} a_{jk} = \sum_{k\neq j} S_k - R_j(A). \qquad \square$$

**Lemma 4.** *Let $A = [a_{ij}] \in R^{n\times n}$ be a stochastic matrix, and $S_i \geq a_{ii}$, $i \in N$. Taking $\tilde{\gamma}(A) = \max_{i\in N}\{S_i - a_{ii}\}$, if $\mu \in \sigma(A)\backslash\{1\}$, then*

$$\left|\mu + \tilde{\gamma}(A)\right| \leq trace(A) + (n-1)\tilde{\gamma}(A) - 1. \tag{4}$$

**Proof.** Let $d_i = a_{ii} + \tilde{\gamma}(A)$, $i \in N$, and

$$B = diag(d_1, d_2, \ldots, d_n)ee^T - A^T.$$

Note that for $i \in N$ and $j \neq i$,

$$b_{ij} = d_i - a_{ji} = a_{ii} + \tilde{\gamma}(A) - a_{ji} \geq a_{ii} + \tilde{\gamma}(A) - S_i = \tilde{\gamma}(A) - (S_i - a_{ii}) \geq 0, \tag{5}$$

and

$$b_{ii} = d_i - a_{ii} = a_{ii} + \tilde{\gamma}(A) - a_{ii} = \tilde{\gamma}(A). \tag{6}$$

From (5), we get that for each $i \in N$,

$$C_i(B) = \sum_{k\neq i} d_k - R_i(A) = \sum_{k\neq i} d_k - (1 - a_{ii}) = \sum_{k\in N} d_k - 1 + (a_{ii} - d_i).$$

Note that $d_i = a_{ii} + \tilde{\gamma}(A)$, then

$$C_i(B) = \sum_{k\in N} d_k - 1 - \tilde{\gamma}(A) = \sum_{k\in N}\big(a_{kk} + \tilde{\gamma}(A)\big) - 1 - \tilde{\gamma}(A)$$
$$= trace(A) + (n-1)\tilde{\gamma}(A) - 1.$$

Moreover, by (6) and Geršgorin circle theorem [11], we have all eigenvalues of $B$ are contained in the region $\Gamma(B)$, where

$$\Gamma(B) = \big\{z \in C : \big|z - \tilde{\gamma}(A)\big| \le trace(A) + (n-1)\tilde{\gamma}(A) - 1\big\}.$$

From Proposition 2, the conclusion follows. □

Lemma 4 provides one circle with center in $\tilde{\gamma}(A)$ and radius equal to $trace(A) + (n-1)\tilde{\gamma}(A) - 1$ to localize all eigenvalues different from 1 of a stochastic matrix $A = [a_{ij}] \in R^{n\times n}$ with the restriction $S_i \ge a_{ii}$, $i \in N$. In fact, combining with the following lemma, we can prove that the same result holds for all stochastic matrices.

**Lemma 5.** *Let $A = [a_{ij}] \in R^{n\times n}$ be a stochastic matrix, and let $\Delta = \{i \in N : a_{ii} > S_i\}$. If $\Delta \neq \emptyset$, then*

$$A_\delta = \frac{1}{1 + (n-1)\delta}\big(A + \delta\big(ee^T - I\big)\big),$$

*where $\delta = -\min_{i\in N}(S_i - a_{ii})$, is a stochastic matrix. Furthermore, for any $\mu \in \sigma(A)\backslash\{1\}$, $\frac{\mu-\delta}{1+(n-1)\delta} \in \sigma(A_\delta)$.*

**Proof.** Let $A_\delta = [\tilde{a}_{ij}]$. Then

$$\tilde{a}_{ij} = \begin{cases} \frac{a_{ii}}{1+(n-1)\delta}, & j = i, \\ \frac{a_{ij}+\delta}{1+(n-1)\delta}, & j \neq i. \end{cases}$$

Since $\Delta \neq \emptyset$, we have that $\delta > 0$, and for any $i, j \in n$, $\tilde{a}_{ij} \ge 0$. Hence, $A_\delta$ is nonnegative. Note that

$$A_\delta e = \frac{1}{1 + (n-1)\delta}\big(A + \delta\big(ee^T - I\big)\big)e = \frac{1}{1 + (n-1)\delta}\big(e + (n-1)\delta e\big) = e.$$

Therefore, $A_\delta$ is stochastic.

Furthermore, let $\mu \neq 1$, and suppose that $x$ is an eigenvector of $A^T$ corresponding to $\mu$. Similar to the proof of Proposition 1, we have $x^T e = 0 = e^T x$. Then

$$A_\delta x = \frac{1}{1 + (n-1)\delta}\big(A + \delta\big(ee^T - I\big)\big)x = \frac{1}{1 + (n-1)\delta}(\mu - \delta)x,$$

that is, $\frac{\mu-\delta}{1+(n-1)\delta} \in \sigma(A_\delta)$. □

We now give the main result in this paper.

**Theorem 6.** *Let $A = [a_{ij}] \in R^{n\times n}$ be a stochastic matrix. Taking $\tilde{\gamma}(A) = \max_{i\in N}\{S_i - a_{ii}\}$, if $\mu \in \sigma(A)\backslash\{1\}$, then*

$$\left|\mu + \tilde{\gamma}(A)\right| \leq trace(A) + (n-1)\tilde{\gamma}(A) - 1. \tag{7}$$

**Proof.** Without lose of generality, assume that $\Delta \neq \emptyset$. In fact, if $\Delta = \emptyset$, then by Lemma 4 we can prove that inequality (7) holds.

Let

$$A_\delta = \frac{1}{1+(n-1)\delta}\left(A + \delta\left(ee^T - I\right)\right),$$

where $\delta = -\min_{i\in N}(S_i - a_{ii})$. From Lemma 5, $A_\delta$ is stochastic. Note that for each $j \in N$,

$$\begin{aligned} S_j(A_\delta) &= \frac{S_j + \delta}{1+(n-1)\delta} = \frac{S_j - \min_{i\in N}(S_i - a_{ii})}{1+(n-1)\delta} \\ &\geq \frac{S_j - (S_j - a_{jj})}{1+(n-1)\delta} = \frac{a_{jj}}{1+(n-1)\delta} = \tilde{a}_{jj}. \end{aligned}$$

By Lemma 5, we know that if $\mu \in \sigma(A)\backslash\{1\}$, then $\frac{\mu-\delta}{1+(n-1)\delta} \in \sigma(A_\delta)$, and apply Lemma 4 to the matrix $A_\delta$ which leads to

$$\left|\frac{\mu-\delta}{1+(n-1)\delta} + \tilde{\gamma}(A_\delta)\right| \leq trace(A_\delta) + (n-1)\tilde{\gamma}(A_\delta) - 1.$$

Since

$$\tilde{\gamma}(A_\delta) = \max_{i\in N}\left(S_i(A_\delta) - \tilde{a}_{ii}\right) = \max_{i\in N}\left(\frac{S_i+\delta}{1+(n-1)\delta} - \frac{a_{ii}}{1+(n-1)\delta}\right) = \frac{\delta + \tilde{\gamma}(A)}{1+(n-1)\delta},$$

we have

$$\left|\frac{\mu-\delta}{1+(n-1)\delta} + \frac{\delta+\tilde{\gamma}(A)}{1+(n-1)\delta}\right| \leq \frac{trace(A)}{1+(n-1)\delta} + (n-1)\frac{\delta+\tilde{\gamma}(A)}{1+(n-1)\delta} - 1,$$

equivalently,

$$\left|\mu + \tilde{\gamma}(A)\right| \leq trace(A) + (n-1)\tilde{\gamma}(A) - 1.$$

The proof is completed. □

**Remark 1.** (I) The bound provided by Theorem 6 is sharp, which is illustrated by the following example. Take the matrix

$$A = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}.$$

Then $\tilde{\gamma}(A) = 0$ and the region in Theorem 6 becomes $|\mu| \leq 0$, that is, the eigenvalues of $A$ different from 1 are 0.

(II) In [3], a region including all eigenvalues different from 1 of a stochastic matrix is given; also see Theorem 1. As is shown in Section 1, it is not always effective because $s_i$ for all $i \in N$ may be zero, which weaken the function to refine the eigenvalue localization. Take the following matrix

$$A = \begin{bmatrix} 0.0781 & 0.1563 & 0.1406 & 0.5156 & 0.1094 \\ 0.0833 & 0.1302 & 0.1250 & 0.5208 & 0.1406 \\ 0.0729 & 0.2187 & 0.1146 & 0.4062 & 0.1875 \\ 0 & 0 & 0 & 1 & 0 \\ 0.0625 & 0.1458 & 0.1458 & 0.5625 & 0.0833 \end{bmatrix}.$$

By Theorem 1, for any $\mu \in \sigma(A)\backslash\{1\}$, we have

$$|\mu - 0.5938| \leq 1.9688.$$

By Theorem 6, we have

$$|\mu + 0.1042| \leq 0.8229.$$

These regions are shown in Fig. 1. It is easy to see that the region in Theorem 6 localizes all eigenvalues different from 1 of $A$ more precisely than that in Theorem 1. Therefore, our result is better than that in [3] in some cases.

**Proposition 3.** *Let $A = [a_{ij}] \in R^{n\times n}$ be a stochastic matrix, and let $\mu_1 = 1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $A$. Then*

(I) *the mean of the eigenvalues $\mu_2, \ldots, \mu_n$ is more than or equal to $-\max_{i\in N}\{S_i - a_{ii}\}$, i.e.,*

$$\frac{\sum_{i=2}^{n} \mu_i}{n-1} \geq -\max_{i\in N}\{S_i - a_{ii}\};$$

(II) *if $a_{ii} = S_i$ holds for any $i \in N$, then the mean of the eigenvalues $\mu_2, \ldots, \mu_n$ is nonnegative.*

**Proof.** (I) By Theorem 6, we have

$$0 \leq trace(A) + (n-1)\tilde{\gamma}(A) - 1 = trace(A) + (n-1)\max_{i\in N}\{S_i - a_{ii}\} - 1,$$

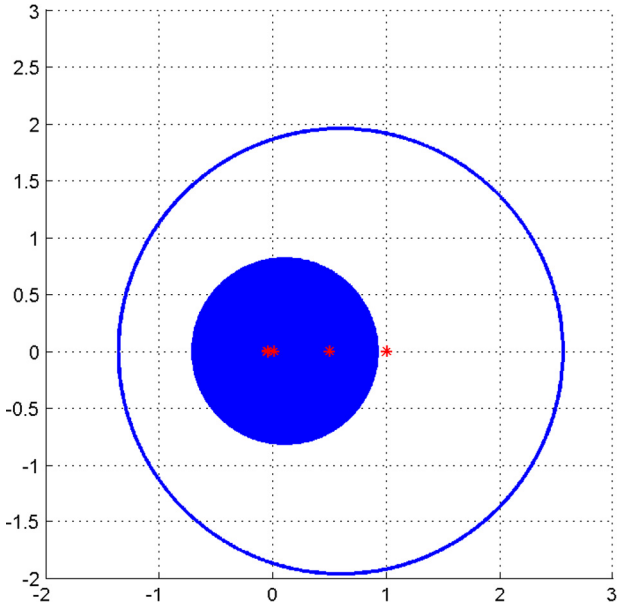


**Fig. 1.** The region $|\mu-0.5938| \leq 1.9688$ is represented by the thick boundary. The region $|\mu+0.1042| \leq 0.8229$ is filled and the exact eigenvalues of $A$ are plotted with asterisks.

which implies that

$$\max_{i\in N}\{S_i - a_{ii}\} \geq \frac{1 - trace(A)}{n-1} = -\frac{\sum_{i=2}^{n}\mu_i}{n-1}.$$

Hence, $\frac{\sum_{i=2}^{n}\mu_i}{n-1} \geq -\max_{i\in N}\{S_i - a_{ii}\}$.

(II) It follows from (I) because $\max_{i\in N}\{S_i - a_{ii}\} = 0$. □

## 3. Estimating the spectral gap and the moduli of the subdominant eigenvalue of a primitive stochastic matrix

Note that if $A$ is stochastic, then $A^m$ is also stochastic for any positive integer $m$. So we can apply Theorem 6 to obtain

$$\left|\mu^m + \tilde{\gamma}(A^m)\right| \leq \tilde{r}_m(A), \tag{8}$$

where $\mu \in \sigma(A)\backslash\{1\}$, $\tilde{r}_m(A) = trace(A^m) + (n-1)\tilde{\gamma}(A^m) - 1$. Furthermore we can prove that when $m \to \infty$, the sequences $\{\tilde{\gamma}(A^m)\}$ and $\tilde{r}_m(A)$ all converge to 0. Since its proof is similar to that of Theorem 3.5 in [3], we omit it here.

**Proposition 4.** *Let $A = [a_{ij}] \in R^{n\times n}$ be a primitive stochastic matrix. Denote by $\tilde{\gamma}_m(A) = \tilde{\gamma}(A^m)$ for all positive integers $m$. Then*

$$\lim_{m\to\infty} \tilde{\gamma}_m(A) = 0,$$

*and for the radii* $\tilde{r}_m(A) = trace(A^m) + (n-1)\tilde{\gamma}_m(A) - 1$ *of the corresponding circles (which contain eigenvalues* $\mu^m$ *of* $A^m$ *different from 1),*

$$\lim_{m\to\infty} \tilde{r}_m(A) = 0.$$

Let

$$\tilde{d}_m(A) = 1 - \left(\tilde{r}_m(A) + \left|\tilde{\gamma}_m(A)\right|\right)^{\frac{1}{m}}. \tag{9}$$

Then $\tilde{d}_m(A)$ represents the distance of the dominant eigenvalue 1 and the localization set $\{z \in C : |z^m - \tilde{\gamma}_m(A)| \le \tilde{r}_m(A)\}$. Hence, by $\tilde{d}_m(A)$ we can estimate the spectral gap of the positive stochastic matrix $A$. By Proposition 4, we can give the following algorithm.

**Algorithm.** Given a primitive stochastic matrix $A = [a_{ij}] \in R^{n\times n}$ and a positive integer $T$. Let $t = 1$, then do the following:

(1) Set $m = 2^{t-1}$.
(2) For $i = 1, \ldots, n$, compute $S_i = \max_{j\neq i} a_{ji}$.
(3) Compute $\tilde{\gamma} = \max_{i\in N}\{S_i - a_{ii}\}$.
(4) Compute $\tilde{r} = trace(A^m) + (n-1)\tilde{\gamma} - 1$.
(5) Compute $\tilde{d} = 1 - (\tilde{r} + |\tilde{\gamma}|)^{\frac{1}{m}}$.
(6) Set $A = A * A$ and $t = t + 1$. If $t > T$, then output $\tilde{\gamma}$, $\tilde{r}$, $\tilde{d}$, stop, else, go to (1).

**Example 1.** Take the stochastic matrices $A_1$ and $A_2$ in [3], where

$$A_1 = \begin{bmatrix} 0.27 & 0.18 & 0.18 & 0.10 & 0.27 \\ 0.20 & 0.40 & 0.20 & 0 & 0.20 \\ 0.11 & 0.22 & 0.22 & 0.34 & 0.11 \\ 0.06 & 0.25 & 0.31 & 0.19 & 0.19 \\ 0.08 & 0.17 & 0 & 0.42 & 0.33 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

Using Matlab 7.1, we produce $\gamma_m(A_i)$, $r_m(A_i)$, $d_m(A_i)$, $i = 1, 2$ in [3], and $\tilde{\gamma}_m(A_i)$, $\tilde{r}_m(A_i)$, $\tilde{d}_m(A_i)$, $i = 1, 2$ for the first few iterations (see Table 1 and Table 2, respectively).

**Table 1**
The values for $A_1$ when $m = 2^t$, $t = 0, 1, 2$.

| $m$ | $\gamma_m(A_1)$ | $\tilde{\gamma}_m(A_1)$ | $r_m(A_1)$ | $\tilde{r}_m(A_1)$ | $d_m(A_1)$ | $\tilde{d}_m(A_1)$ |
|---|---|---|---|---|---|---|
| 1 | 0.2300 | 0.2300 | 0.5100 | 1.3300 | 0.2600 | −0.5600 |
| 2 | 0.0624 | 0.0157 | 0.1301 | 0.1823 | 0.5613 | 0.5550 |
| 4 | 0.0038 | 0.0027 | 0.0159 | 0.0100 | 0.6255 | 0.6642 |

**Table 2**
The values for $A_2$ when $m = 2^t$, $t = 0, \ldots, 4$.

| $m$ | $\gamma_m(A_2)$ | $\tilde{\gamma}_m(A_2)$ | $r_m(A_2)$ | $\tilde{r}_m(A_2)$ | $d_m(A_2)$ | $\tilde{d}_m(A_2)$ |
|---|---|---|---|---|---|---|
| 1 | 1.0000 | 0.5000 | 3.7500 | 2.2500 | −3.7500 | −1.7500 |
| 2 | 0.7083 | 0.0000 | 1.8958 | 0.9375 | −0.6137 | 0.0318 |
| 4 | 0.4583 | 0.0143 | 1.3997 | 0.4909 | −0.1675 | 0.1569 |
| 8 | 0.1828 | 0.0108 | 0.5741 | 0.2005 | 0.0342 | 0.1766 |
| 16 | 0.0286 | 0.0018 | 0.0900 | 0.0316 | 0.1248 | 0.1914 |

Note that

$$\tilde{d}_4(A_1) = 0.6642 > d_4(A_1) = 0.6255$$

and

$$\tilde{d}_{16}(A_2) = 0.1914 > d_{16}(A_2) = 0.1248.$$

This shows that we estimate the gap between the dominant eigenvalue 1 and the cluster of all other eigenvalues for a primitive stochastic matrix by using $\tilde{d}_m(A)$ more precisely than by using $d_m(A)$ of [3] in some cases.

Let $\tilde{v}_m(A) = (\tilde{r}_m(A) + |\tilde{\gamma}_m(A)|)^{\frac{1}{m}}$. Then from inequalities (8) and (9), we have

$$\left|\mu_2(A)\right| \leq \tilde{v}_m(A) = 1 - \tilde{d}_m(A),$$

where $\mu_2(A)$ is a subdominant eigenvalue of a stochastic matrix $A$. Hence, we can use $\tilde{v}_m(A)$ to estimate the moduli of the subdominant eigenvalue of $A$. Next, we compare our obtained bound $\tilde{v}_m(A)$ with those in [3,5].

**Example 2.** Take the stochastic companion matrix

$$A_3 = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which is the matrix $C$ of Example 4.1 in [5] with $x_i = i/10$, $i \in \{1, 2, 3, 4\}$ (for details, see [3,5]). By computations, we have that $\mu_2(A_3) = 0.7513$, the upper bound (2.3) in [5] is 0.9, and $v_m(A_3)$, $\tilde{v}_m(A_3)$ are shown in Table 3.

**Table 3**
The values of $v_m(A_3)$ and $\tilde{v}_m(A_3)$.

| $t$ | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|
| $m = 2^{t-1}$ | 4 | 8 | 16 | 32 | 64 |
| $v_m(A_3)$ | 0.9505 | 0.8951 | 0.8310 | 0.7757 | 0.7682 |
| $\tilde{v}_m(A_3)$ | 0.8604 | 0.8395 | 0.8062 | 0.7705 | 0.7660 |

It is not difficult to see from this example that our bound performs better than that in [3,5].

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