M.Sc. Simulation Sciences, Winter Semester 2016 / 2017

Numerical Methods for PDE

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Homework 5

Due: Tuesday, December 20, 2016 (by Midnight)

Problem 1

Consider the periodic problem

$$
\begin{aligned}
u_t + (f(u))_x &= 0, && (x,t) \in [0,1) \times (0,T] \\
u(x,0) &= 1 + \frac{1}{2}\sin(2\pi x), && x \in [0,1)
\end{aligned}
$$

for $f(u) = \frac{1}{2}u^2$, and *periodic* boundary conditions, i.e. $u(x,t) = u(x-1,t)$ for all $x \in \mathbb{R},\ t > 0$.

Solve this problem numerically using the finite volume scheme discussed in class, i.e. set for $n = 0, 1, \ldots$, and all $i = 0, \ldots N-1$,

$$
\frac{u_i^{n+1} - u_i^n}{\tau} = -\frac{1}{h}\left(\hat{f}^n_{i+\frac{1}{2}} - \hat{f}^n_{i-\frac{1}{2}}\right), \tag{1}
$$

where

$$
\hat{f}^n_{i+\frac{1}{2}} := \frac{1}{2}\left(f(u^n_{i+1}) + f(u^n_i)\right) - \frac{\alpha^n_{i+\frac{1}{2}}}{2}\left(u^n_{i+1} - u^n_i\right). \tag{2}
$$

and N corresponds to the grid

$$
\mathcal{G}_h := \{ih :\ i = 0, \ldots N-1,\ hN = 1\}.
$$

(We enforce periodic boundary conditions, just like in the previous homework.)

Use meshes with $N = 100$ and $N = 1000$ points, and both

- $\alpha^n_{i+\frac{1}{2}} = |\overline{a}^n_{i+\frac{1}{2}}|$,
- $\alpha^n_{i+\frac{1}{2}} = 2|\overline{a}^n_{i+\frac{1}{2}}|$,

where

$$
\overline{a}^n_{i+\frac{1}{2}} := \begin{cases} \frac{f(u^n_{i+1}) - f(u^n_i)}{u^n_{i+1} - u^n_i} & u^n_{i+1} \neq u^n_i, \\ f'(u^n_i) & u^n_{i+1} = u^n_i. \end{cases} \tag{3}
$$

(For the case distinction you can test if $|u^n_{i+1} - u^n_i| < \epsilon$, where ϵ is a small tolerance, somewhat larger than machine zero.) Compute the permissible time step $\tau = \tau^n$ at *each* time t^n using the stability criterion

$$
\tau^n \leq \min_i \left\{ \frac{h}{c^n_{i,i-1} + c^n_{i,i+1}} \right\}, \tag{4}
$$

where the coefficients $c_{i,m}^n = c_{i,m}(u_h^n)$ are those of the canonical form discussed in class.

Note that the time step will be different at each time instance t^n, because the coefficients depend on the solution. Consequently, we cannot tell a priori how many time steps we need to reach a particular time T. Just stop the iteration when $t^n \geq T = 1$ for the first time. Plot the solution at the final time against x for both meshes, and both choices of α.

Problem 2

Repeat Problem 1, but this time using the initial conditions

$$\phi(x) = \begin{cases} 0, & 0 \leq x \leq 0.1 \\ 1, & 0.1 < x < 0.3 \\ 0, & 0.3 \leq x \leq 1 \end{cases}, \tag{5}$$