M.Sc. Simulation Sciences, Winter Semester 2016 / 2017

Numerical Methods for PDE

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Homework 2

Due: Tuesday, November 22, 2016 (by midnight)

Consider the linear heat equation with homogeneous boundary conditions,

$$
\begin{aligned}
u_t &= a u_{xx}, \qquad a > 0, \qquad && (x,t) \in (0,1) \times (0,T] && (1)\\
u(0,t) &= 0, && t \in [0,T] && (2)\\
u(1,t) &= 0, && t \in [0,T] && (3)\\
u(x,0) &= \sin(\pi x), && x \in [0,1] && (4)
\end{aligned}
$$

We solve this via finite difference approximation. Use a spatial grid with constant mesh spacing,

$$\mathcal{G}_h := \{ih : i = 0, \dots N;\ hN = 1\}. \tag{5}$$

Furthermore, use discrete time instances $t^n = n\tau$, for $n = 1, 2, \dots$, and $t \le T = 0.2$. (We choose τ below.)

Implement the Crank Nicolson Method, which is defined for $i = 1, \dots, N-1$ and $n = 0, 1, 2, \dots$

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{a}{2}\left(\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} + \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{h^2}\right), \tag{6}$$

where values for $i = 0, N$, and $n = 0$, are defined by the boundary conditions (2) (3), and the initial conditions (4), respectively.

- For simplicity, choose $a = 1$. Define $\lambda := \frac{a\tau}{h^2} = \frac{\tau}{h^2}$.
- Perform computations with $N = 20$ and $N = 200$. Choose $\tau = h = 1/N$. (Note that this implies $\lambda = N$, i.e λ changes with the mesh!)
- The exact solution is $u(x,t) = \sin(\pi x)e^{-\pi^2 t}$. Report the error

  $$\max_{0 \le i \le N} |u_i^n - u(ih, n\tau)| \tag{7}$$

  for $n\tau = T = 0.2$, and both values of N.