M.Sc. Simulation Sciences, Winter Semester 2016 / 2017

Numerical Methods for PDE

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Homework 7

Due: Tuesday, Jan 31, 2017 (by Midnight)

Problem Statement

Consider the one-dimensional Poisson-equation

$$\begin{aligned} -u'' &= f, \qquad x \in (0,1) \\ u(0) = u(1) &= 0, \end{aligned}$$

where $f(x) = \pi^2 \sin(\pi x)$. Note that the solution to this problem is given by $u(x) = \sin(\pi x)$. Use the Ritz-Galerkin Method to solve the proper variational form of this equation, as discussed in class.

Construct a grid $0 = x_0 < x_1 < \cdots < x_N = 1$, where

$$x_i = \frac{e^{\alpha \frac{i}{N}} - 1}{e^{\alpha} - 1}, \qquad \alpha > 0. \tag{1}$$

Thus $[0,1] = \bigcup_{i=1}^{N} I_i$ where $I_i = [x_i, x_{i-1}]$. Construct the Ritz-Galerkin method using the finite dimensional space

$$S_h = \{v \in C^0[0,1] : \; v|_{I_i} \in \mathcal{P}^1(I_i), i = 1, \dots N\,, \; v(0) = v(1) = 0\}. \tag{2}$$

and a nodal basis $\{\phi_k\}_{k=1,\dots N-1}$ (i.e. a basis satisfying $\phi_k(x_i) = \delta_{ik}$ for $k = 1, \dots N-1$ and $i = 0, \dots, N$).

- Use $\alpha = 10$, and meshes of the size $N = 100$ and $N = 1,000$
- It is recommended that you implement the integral for the right-hand side with a trapezoidal numerical integration rule, i.e. the integral over the interval $[x_{i-1}, x_i]$ is approximated as

$$\int_{x_{i-1}}^{x_i} g(x)dx = (x_i - x_{i-1})\frac{1}{2}\left(g(x_i) + g(x_{i-1})\right) \tag{3}$$

- Compute the error $\|u_h - u\|_{0,\Omega}$ for both meshes, and plot against $h = max_{1 \le i \le N} |I_i|$ on a log-log scale. (again you can use the trapezoidal rule for the integrals.)