M.Sc. Simulation Sciences, Winter Semester 2016 / 2017

Numerical Methods for PDE

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Homework 4

Due: Tuesday, December 6, 2016 (by Midnight)

Summary

The homework assignment is computational. You will implement the Upwind scheme and the Lax-Wendroff scheme for the linear advection equation $u_t + au_x = 0$.

Problem Definition

In class, we defined the Upwind scheme, and the Lax-Wendroff scheme. Both schemes can be written, for i and n in a suitable index range, as

$$u_i^{n+1} = u_i^n - \frac{\nu}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{s}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \tag{1}$$

where $\nu = a\tau/h$. For $s = |\nu|$, respectively $s = \nu^2$, ones obtains the Upwind scheme, respectively the Lax-Wendroff scheme.

We consider problems that are periodic with respect to the interval $[0,1)$. To this end, we define a spatial grid $\mathcal{G}_h$, such that

$$\mathcal{G}_h := \{ih : \, i = 0, \ldots N-1; \, hN = 1\}. \tag{2}$$

and enforce periodic boundary conditions by replacing in (1):

- i with $N - |i|$, $(i < 0)$
- i with $i - N$, $(i > N-1)$

Test 1: Smooth Solution

a) Use initial conditions

$$u_0(x) = \sin(2\pi x), \tag{3}$$

and iterate, using $\nu = 0.9$ and $a = 1$, for $n = 0, 1, \ldots, M$, where $M = int(N/\nu)$. (Here, int means the next lower integer number. This will iterate approximately to $T = 1$). Measure the error at the final time $T = M\tau$ for grid sizes $N = 10, 100, 1000$, for both schemes. Recall that, for $a = 1$, the exact solution is given by

$$u(x,t) = u_0(x-t) = \sin(2\pi(x-t)).$$

Use the maximum norm, i.e. measure

$$||\mathbf{e}||_\infty = \max_{0 \le i \le N-1} |u_i^M - u(x_i, T)|$$

(Be sure to evaluate the exact solution at $T = M\tau$, *not* $T = 1$!) and plot the norm of the error against N on a logarithmic scale (i.e. logarithmic in *both* axes) for both schemes.

b) Recall that we derived the *modified equation* for the Upwind Scheme as

$$u_t + a u_x = \mu u_{xx}, \qquad \mu = \frac{|a|h}{2}(1 - |\nu|), \tag{4}$$

The numerical solution produced by the upwind scheme is, in fact, second order accurate with respect to the modified equation. The exact solution to (4) is given by

$$u(x,t) = \sin(2\pi(x-t))e^{-\mu 4\pi^2 t}.$$

Measure the error also with respect to this solution. (You should observe, choosing discretization parameters as above, second order accuracy in mesh refinement.)

Test 2: Discontinuous Solution

Use initial conditions

$$u_0(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ 0 & 0.5 \le x < 1 \end{cases}, \tag{5}$$

with assumed periodic extension beyond the interval $[0,1)$. Use otherwise the same setup as in the previous test. Plot the numerical solution obtained at $n = M$ against x, and overplot with $u_0(x - at)$.