FE535: Introduction to Financial Risk Management Session 4

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Agenda

- Bond Fundamentals
  - Compounding and Discounting (Time Value of Money)
  - Fixed Coupon Bonds
- Interest Rate Risk
  - Duration
  - Convexity
- Introduction to Bond Portfolio Management

Asset Pricing - The Basic Case

- Risk management starts with asset pricing
- The simplest asset to price is a fixed income security, such as a zero-coupon bond
- Nonetheless, to price this, we need to think about the time value of money
    - This requires discounting and compounding interest
- In this case, the price is determined by one factor, interest rate

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  - This requires discounting and compounding interest
- In this case, the price is determined by one factor, interest rate
- After evaluating the asset, risk management tries to investigate the impact of the factor on the price
  - What is the impact of change of interest on the price of the bond?
  - As a result, what is the associated loss?

What is a Bond?

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- When you buy a bond, you lend your money to the issuer, such as corporation or government
- The borrowers routinely issue bonds to raise capital ranging between a few days up to 40 years
- The distinguishing character of a bond is that the issuer (government) enters a legal agreement to compensate the lender (you) through
  - periodic interest payments in the form of a *coupon*
  - repay the original sum (*face value* or *par value*) at end of the period (*maturity*)

Discounting, Present Value, and Future Value

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- The fair price should take into account
  - the likelihood of the payment (riskiness)
  - the time value of money
- Previously, we mainly focused on the former since it was an immediate game example
- Nonetheless, one should also consider the time-value of money when anticipating future cash flows

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- Nonetheless, one should also consider the time-value of money when anticipating future cash flows

- Assuming discrete time cash flows over T future periods, the fair price, P_0, should be

$$P_0 = \sum_{t=1}^{T} CF_{s,t} \times \mathbb{P}(s_t) \times \frac{1}{(1+y)^t} \tag{1}$$

  where y denotes a fixed rate of return (*yield*)

- If the security pays a CF_t with 100% certainty at each t, then (1) becomes

$$P_0 = \sum_{t=1}^{T} \frac{CF_t}{(1+y)^t} \tag{2}$$

Simple Case

- The simplest case is to consider a zero-coupon bond guaranteed by the U.S. government
  - A bond purchased today at $t = 0$ for a price of P_0
  - In T periods, it pays back the **face value**, CF_T, with 100% certainty (why?)
  - The one period yield on the bond is constant and equal to y
- By discounting the future cash flows at the rate of y, the fair price of the bond is equal to

$$P_0 = \frac{CF_T}{(1+y)^T} \tag{3}$$

- The main pricing Equation (1) assumes that discounting is constant over time $y_t = y, \forall t = 1, ..., T$

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- Since the future cash flow is 100% certain, y denotes the rate of return that an investor reaps on an investment today for T periods
- Put differently,

$$CF_T = P_0 \times (1+y)^T \tag{4}$$

or

$$y = \sqrt[T]{\frac{CF_T}{P_0}} - 1 \tag{5}$$

- When compared with other assets, it is easier to evaluate different assets using rates than prices
  - For instance, how much return the equity market gives over a T-bond?
- If T refers to units of years, then y denotes an annual rate
- The rate y is also known as
  - The Effective Annual Rate (EAR)
  - Internal Rate of Return (IRR)

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- Compounding could also take place on a more frequent basis - for instance, semi-annual
- If we consider a semi-annual rate of $y(2)$, then we have $2T$ compounding periods (i.e. $2T$ half years)

$$P_0 = \frac{CF_T}{(1 + \frac{y(2)}{2})^{2T}} \tag{6}$$

- From (3) and (6), it follows that

$$\left(1 + \frac{y(2)}{2}\right)^2 = 1 + y \tag{7}$$

- We can generalize the result from (7) to d increments over the year, i.e.

$$\left(1+\frac{y(d)}{d}\right)^{d}=1+y \tag{8}$$

- In fact, if we think about continuous compounding, i.e. $d \to \infty$, then it follows that

$$\lim_{d\to\infty}\left(1+\frac{y(d)}{d}\right)^{d}\to e^{y(c)} \tag{9}$$

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- Put differently, consider an asset that pays a continuous annual rate of t over time, which we denote by B_t
- If this asset obeys to GBM, we know that

$$\frac{\partial B_t}{B_t} = rdt + \sigma dZ_t \tag{10}$$

- However, for a risk-less asset, we have $\sigma = 0$, i.e.

$$\frac{\partial B_t}{B_t} = rdt \tag{11}$$

- If we know B_0, then the future prices is determined by the solution to the ordinary differential equation (ODE) from (11), such that

$$B_t = B_0 e^{r\tau} \tag{12}$$

Example 1 - Exam Question

You have \$1 million to invest for one year in a certified deposit account. You have 4 options among which you need to choose the one that returns the highest EAR:

1. monthly compounding, i.e. $y(12) = 7.82\%$
2. quarterly compounding, i.e. $y(4) = 8.00\%$
3. semi-annually compounding, i.e. $y(2) = 8.05\%$
4. continuous compounding, i.e. $y(c) = 7.95\%$

Example 1 - Exam Question (solution)

To answer this question, we need to compute the EAR for each alternative, i.e. find the corresponding y from Equation (8)

1. For the first one, we have $d = 12$ and $y(12) = 7.82$, such that
$$\left(1 + \frac{7.82/100}{12}\right)^{12} = 1.0811 \Rightarrow y = 8.11\% \tag{13}$$
2. Solving the same for alternative 2, we get $y = 8.24\%$
3. For alternative 3, we have $y = 8.21\%$
4. Finally, for the continuous compounding alternative, it follows that $e^{7.95/100} = 1.0827$, i.e. $y = 8.27\%$

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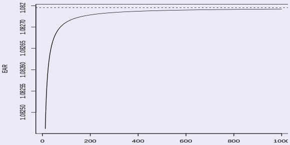
Note: in fact, if one computes the EAR for a large d for the continuous compounding alternative, the answer should converge to 1.0827. To see this, consider the following

```
> y <- function(y_d,d) (1+((y_d/100)/d))^d
> d_seq <- 12:1000
> y_d <- sapply(d_seq, function(d) y(7.95,d))
> plot(y_d~d_seq, type = "l", ylab = "EAR",xlab = "d")
> abline(h = exp(7.95/100),lty = 2)
```



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In this case, the annual rate is $y = 8\%$, where today's price is $P_0 = 1$. The future cash flow is $CF_T = 2$. Hence, we need to find the value T for which Equation (4) holds, i.e.

$$1 \times (1.08^T) = 2 \tag{14}$$

In other words,

$$1.08^T = \frac{2}{1} \tag{15}$$

$$\log\left(1.08^T\right) = \log\left(\frac{2}{1}\right) \tag{16}$$

$$T \times \log(1.08) = \log(2) \tag{17}$$

$$T = \frac{\log(2)}{\log(1.08)} \approx 9 \tag{18}$$

$$\tag{19}$$

- How would your answer change if the saving account in Example 2 would use compounding with
  1. semi-annual with $y(2) = 8\%$
  2. monthly-annual with $y(12) = 8\%$
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$$\left(\left(1 + \frac{y(d)}{d}\right)^{d}\right)^{T} = 2 \Rightarrow T = \frac{\log(2)}{d \times \log\left(1 + \frac{0.08}{d}\right)} \tag{20}$$

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d	1.00	3.00	6.00	12.00	50.00	100.00
T	9.01	8.78	8.72	8.69	8.67	8.67

Fixed-Coupon Bond

- Bonds usually pay fixed coupons on a semi-annual basis
  - the case for U.S. Treasury and corporate bonds
- The face value of the bond is standardized to $F = 100$, which is known as the par value
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- The face value of the bond is standardized to $F = 100$, which is known as the par value
- The coupon is written in percentage, such that c denotes a CF of $c \times F$
- According to (2), the price of a fixed annual coupon bond is

$$P_0 = \frac{c \times F}{(1+y)} + \frac{c \times F}{(1+y)^2} + \dots \frac{c \times F}{(1+y)^{T-1}} + \frac{(1+c) \times F}{(1+y)^T} \tag{21}$$

which can simplified to

$$P_0 = c \times F \sum_{t=1}^{T} \frac{1}{(1+y)^t} + \frac{F}{(1+y)^T} \tag{22}$$

and, hence, to

$$P_0 = \frac{c}{y} \times F \left[1 - \frac{1}{(1+y)^T}\right] + \frac{F}{(1+y)^T} \tag{23}$$

At Par

A bond is called selling at par, if the current price is equal to the face value.

- A special case for a fixed-coupon bond is when the yield is equal to the coupon, $c = y$
- If $c = y$, then it follows from (23) that

$$P_0 = F\left[1 - \frac{1}{(1+y)^T} + \frac{1}{(1+y)^T}\right] = F \tag{24}$$

- For instance, if $c > y$, then bond investors are willing to pay a premium, i.e. $P_0 > F$
- On the other hand, if $c < y$, then the bond should be sell than par, $P_0 < F$, to encourage investors

Example 3 - Sensitivity to Yield

Consider a bond that pays 100 in 10 years with 6% annual coupon. What is the price of the bond if $y = 6\%$, $y = 7\%$, $y = 5\%$?

- If $c = y$, then the price should be equal to the face value, $P_0 = 100$
- What about $y = 7\%$? According to (23), the price is

$$P_0 = \frac{0.06}{0.07} \times 100 \left[1 - \frac{1}{(1+0.07)^{10}}\right] + \frac{100}{(1+0.07)^{10}} = 92.98 \tag{25}$$

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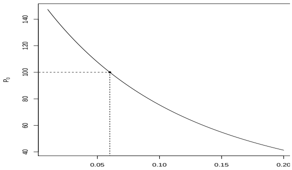
```
> P <- function(y,c,FV,T_end)  (c/y)*FV*(1 - 1/(1+y)^T_end) + FV/(1+y)^T_end
> P_y <- function(y) P(y,0.06,100,10)
> plot(P_y,0.01,0.2,ylab = expression(P[0]),xlab = "y" )
> points(0.06,P_y(0.06), pch = 20)
> segments(0.06,0,0.06,P_y(0.06), lty =2)
> segments(0,P_y(0.06),0.06,P_y(0.06), lty =2)
```



Example 4 - FRM Exam 2009 Question

A five year corporate bond is paying an annual coupon of 8% is sold a price reflecting a yield to maturity of 6%. One year passes and the interest rate remains unchanged. Assuming a flat term structure and holding all other factors constant, the bond's price during this period will have

1. Increased
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- To answer this we need to consider two things
  - the yield relative to coupon - is the bond selling at, above, or below par?
  - what happens to the price of the bond as the time to maturity shortens?

Case Study - Long-Term Capital Management's Big Loss

- Long-Term Capital Management (LTCM), was a hedge fund formed in the mid-1990s
- The fund's strategy was known as a convergence arbitrage
  - Find bonds by the same issuer with same payoffs
  - However, one was more liquid than the other
  - Buy a discount bond (underpriced) X
  - Short a premium bond (overpriced) Y

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  - However, one was more liquid than the other
  - Buy a discount bond (underpriced) X
  - Short a premium bond (overpriced) Y
- The main assumption behind the above is that the prices of each will eventually converge to par
  - i.e. $Y - X \to 0$
- If interest rate would increase, then both prices would change in the same fashion

- In Aug 1998, however, Russia defaulted on its debt and investors valued more safe and liquid assets
  - a phenomenon known as *Flight to Quality* during market panics
- This created more (less) demand for liquid (illiquid) assets
  - The price of Y went up, while X went down
  - The spread, hence, $Y - X \nrightarrow 0$ started to diverge rather converge
- Given this divergence, LTCM had to liquidate its position at large losses
- These losses were mainly amplified by high leverage of the fund
  - LTCM held huge positions, totaling roughly 5% of the total global fixed-income market
  - Borrowed massive amounts of money to finance these leveraged trades
- Eventually the fund was bailed out with the help of the Federal Reserve
- Then its creditors took over, and a systematic meltdown of the market was prevented

Interest Rate Risk

- In Example 3, we illustrated how sensitive the bond price is to the yield
- In particular, bond holders are concerned with a number of risk factors
    - interest rate risk: change in interest rate
    - credit risk: credit worthiness of issuers
    - liquidity risk: bonds tend to be less liquid that stocks
- Nonetheless, **all bonds subject to interest rate risk**
- For instance, if you are holding a \$1 million portfolio of T-bonds, what would happen if the interest rate go up/down?

Price Sensitivity

- We saw in Equation (23) that there is an inverse relation between P_0 and y
- In particular, the price of P_0 can be described as a function of y, all else equal

$$P_0 = f(y) \tag{26}$$

- One can investigate the sensitivity of the price given yield, y_0, i.e. what's the price of the bond if the yield changes from y_0 to y_1, where $y_1 = y_0 + \Delta y$
- Using Taylor Expansion, it can be shown for a small Δy that

$$P_1 = P_0 + f'(y_0) \times (\Delta y) + \frac{1}{2} f''(y_0) \times (\Delta y)^2 \tag{27}$$

Price Sensitivity - Duration

- The first order change in the bond price is known as **duration**
- In particular, the first derivative is described by

$$f'(y_0) = -DD = -D^* P_0 = -\frac{D}{1+y} P_0 \tag{28}$$

where

- DD is the **dollar duration**
- D^* is called the **modified duration**
- D is the **Macaulay duration**

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- According first order approximation (27), we get

$$P_1 = P_0 - \Delta y \times D^* \times P_0 \Rightarrow \frac{P_1 - P_0}{P_0} = -\Delta y \times D^* \tag{29}$$

Calculating Duration

- The Macaulay duration is the more intuitive one, which can be computed as

$$D = \sum_{t=1}^{T} w_t t \tag{30}$$

where

$$w_t = \frac{CF_t/(1+y)^t}{\sum_{t=1}^{T} CF_t/(1+y)^t} = \frac{CF_t/(1+y)^t}{P_0} \tag{31}$$

- The above computation can be generalized to any debt instrument
- Hence, if we know D, then we know $D^* = D/(1+y)$ and $DD = D^* P_0$
  - and hence the sensitivity of the debt instrument to y, i.e. $f'(y_0) = -DD$

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- Taking the derivative w.r.t y, we have

$$f'(y_0) = \frac{\partial P_0}{\partial y} = 100 \times -T \times (1+y)^{-T-1} \tag{33}$$

$$= -100 \times (1+y)^{-T} \times \frac{T}{1+y} = -P_0 \times \frac{T}{1+y} \tag{34}$$

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- According to (28), it is clear that the Macaulay duration is T

- Alternatively, according to (30), we see that there is one payment at time T, such that $w_T = 1$, and, therefore, the Macaulay duration is T

Duration Summary

- For bonds with coupons duration should be less than T, since the bonds holder receives payment before maturity
  - the higher the coupon the lower the duration is
- Moreover, the higher the yield is the lower duration is
  - higher yield means lower price paid for the bond
  - lower price means less time takes to recover the down payment
  - less weights assigned to the distant future - recall Equation (31)
- The longer the maturity the higher the duration is
  - regardless of the coupons, the face value is the major cash flow
  - the later it is received the higher the duration is

EXAMPLE 6.13: FRM EXAM 2000 - QUESTION 106

- How would you rank the following from shortest to longest duration?

Bond Number	Maturity	Coupon Rate	Frequency	Yield
1	10	6.00%	1	6.00%
2	10	6.00%	2	6.00%
3	10	0.00%	1	6.00%
4	10	6.00%	1	5.00%
5	9	6.00%	1	6.00%

1. 5-2-1-4-3
2. 1-2-3-4-5
3. 5-4-3-1-2
4. 2-4-5-1-3

Price Sensitivity - Convexity

- **Convexity** refers to the second order change in the bond price with respect to yield, y
- In particular, the second derivative can be described as

$$f''(y_0) = C \times P_0 \tag{35}$$

- In Taylor's expansion (27), it follows that

$$P_1 = P_0 - \Delta y \times D^* \times P_0 + (\Delta y)^2 \times \frac{C \times P_0}{2} \tag{36}$$

$$\frac{P_1 - P_0}{P_0} = -\Delta y \times D^* + (\Delta y)^2 \times \frac{C}{2} \tag{37}$$

- In economic intuition, C is given by

$$C = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times w_t \tag{38}$$

where w_t is given by (31)

- Like duration, convexity computes the weighted-average of the squared time periods of cash flows

Approximating Price Change using Duration and Convexity

- If you know the duration and convexity of a bond, then you can approximate the change to its price if the yield goes up by Δy
- According to (27), Equation (36) indicates that

$$P_1 = P_0 \left[1 - \frac{D}{1+y} \times \Delta y + \frac{1}{2} \times C \times (\Delta y)^2\right] \tag{39}$$

where D and C follow from (30) and (38), respectively

- One can see that the first order change is always negative
  - negative relation between price and yield
- Convexity serves as a correction to provide a better approximation price sensitivity
  - which captures the non-linearity in the price change

Basis Points (bps)

In the bond market, it is common talk in **basis points** (bps)

- 1% is equal to 100 bps - or x bps are equal to $x/100^2$

Example 5 - Exam Question

A portfolio manager has a bond position worth $100 million. The position has a modified duration of 8 years and convexity of 150 years. By how much the position would change if interest rates increase by 25bps?

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- 1% is equal to 100 bps - or x bps are equal to $x/100^2$

Example 5 - Exam Question

A portfolio manager has a bond position worth \$100 million. The position has a modified duration of 8 years and convexity of 150 years. By how much the position would change if interest rates increase by 25bps?

Example 5 - Exam Question Solution

- First, note that the question provides info about D^*, which is $D/(1+y)$, hence $D/(1+y) = 8$ and $C = 150$
- Second, $\Delta y = 25/100^2$
- Third, according to (30) and (39), it follows that

$$P_1 = 100 \left[1 - 8 \times \frac{25}{100^2} + \frac{1}{2} \times 150 \times \left(\frac{25}{100^2} \right)^2 \right] = 98.05 \tag{40}$$

- Finally, the change in the position is $P_1 - P_0 = 98.05 - 100 = -1.95$

Portfolio Duration and Convexity

- Fixed income portfolios involve large number of securities
- It is more practical to assess the sensitivity of the portfolio rather than each asset
- For instance, consider the case where a bond fund is compared to T-bond with a duration of 5 years
- The manager may wish her portfolio duration to, let's say, 1 year
- If interest rates increase by 1% then the benchmark would suffer approximately 5%
- Whereas, the bond fund would only suffer 1%, hence outperforming the benchmark by 4%

- Since portfolio is a linear combination of bond prices, it holds true that

$$D_p^* = \sum_{i=1}^{N} D_i^* x_i \tag{41}$$

and

$$C_p = \sum_{i=1}^{N} C_i x_i \tag{42}$$

where

  - D_p^* (C_p) is the modified duration (convexity) of the portfolio
  - D_i^* (C_i) is the modified duration (convexity) of bond i, for $i = 1, .., N$
  - x_i is the weight allocated to bond i, for $i = 1, .., N$

$$x_i = \frac{n_i \times P_{i,0}}{\sum_{j=1}^{N} n_j \times P_{j,0}} \tag{43}$$

with n_i the number of bond i held, $P_{i,0}$ denoting the price of which today

- Note that the portfolio weights should sum to 1, i.e. $\sum_{i=1}^{N} x_i = 1$

Bond Portfolio Problem

- Put formally, let $\mathbf{x}$ denote the vector of weights allocated to bond 1 and 2, i.e.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{44}$$

- At the same time, define the 2×2 $\mathbf{A}$ matrix and the 2×1 column vector $\mathbf{b}$ as

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ D_1^* & D_2^* \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ D_p^* \end{bmatrix} \tag{45}$$

- Then, the portfolio weights must satisfy the following condition

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{46}$$

and, as a result:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{47}$$

Bond Portfolio Problem - Example

- Suppose you have two bonds trading at par one with duration of 3 years and the other with duration of 6 years
- In total, you need to invest \$100K between the two, but, at the same time you need to ensure the duration is no more than 5 years
- How many bonds you should buy from each?

Bond Portfolio Problem - Example

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- In total, you need to invest \$100K between the two, but, at the same time you need to ensure the duration is no more than 5 years
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- Using the former notation, we have

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \tag{48}$$

- Therefore, the weights should are given by

$$\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{3} \\ -1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \tag{49}$$

- Finally, we need to purchase $N_1 = \frac{\$100{,}000}{1000} \times \frac{1}{3} \approx 34$ and $N_2 = \frac{\$100{,}000}{1000} \times \frac{2}{3} \approx 66$

Summary

- Interest rate risk is the most common risk factor in bond valuations
- Duration and convexity provide first and second order sensitivity approximation to changes in bond prices
- Both provide risk metrics to measure portfolio risk exposure
  - Allowing bond portfolio managers to track/outperform a benchmark
- For those interested in further reading on bonds, I recommend The Bond Book by Annette Thau (see **link**)