HW 2

I. A. $\frac{d}{dx}\int_{\cos x}^{\sin x} \sqrt{1-t^2}\,dt$

Using Leibniz rule of differentiation:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = \frac{d}{dx}v(x)\cdot f(v(x)) - \frac{d}{dx}u(x)\cdot f(u(x))\ ; \quad v(x) = \sin x \quad u(x) = \cos x \quad f(t) = \sqrt{1-t^2}$$

where: $\frac{d}{dx}v(x) = \frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}u(x) = \frac{d}{dx}\cos x = -\sin x$

back to the equation:

$$= \cos x \cdot \left(\sqrt{1-\cos^2 x}\right) + \sin x \cdot \sqrt{1-\sin^2 x}$$

B. $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$

$u = x^2$, and $du = 2x\,dx$

and $s = 4-u$ and $ds = -du$

$$= -\frac{1}{2}\int \frac{4-s}{\sqrt{s}}\,ds = \frac{1}{2}\int \sqrt{s}\,ds - 2\int \frac{1}{\sqrt{s}}\,ds = \frac{s^{\frac{3}{2}}}{3} - 4\sqrt{s}$$

sub back to u:

$$\frac{1}{3}(4-u)^{3/2} - 4\sqrt{4-u}$$

sub back to $u = x^2$

$$= \frac{1}{3}(4-x^2)^{3/2} - 4\cdot\sqrt{4-x^2}$$

$$= -\frac{1}{3}\sqrt{4x-x^2}\,(x^2+8)$$

C. $\int x^2 \arctan x \, dx$

$= \int x^2 \cdot \tan^{-1}(x) \, dx$

Integrate by parts, $\int f \, dg = f \cdot g - \int g \, df$

$f = \tan^{-1} x$, $dg = x^2 \, dx$

$df = \frac{1}{x^2+1} \, dx$, $g = \frac{x^3}{3}$

We get:

$= \frac{1}{3} x^3 \tan^{-1}(x) - \int \frac{x^3}{3(x^2+1)} \, dx$

$= \frac{1}{3} x^3 \tan^{-1}(x) - \frac{1}{3} \int \frac{x^3}{x^2+1} \, dx$

$u = x^2$, $du = 2x \, dx$ (u-sub):

$= \frac{1}{3} x^3 \tan^{-1}(x) - \frac{1}{6} \int \frac{u}{u+1} \, du$

sub in $s = u+1$ and $ds = du$

$= \frac{1}{3} \tan^{-1} x \cdot x^3 + \frac{1}{6} \int \frac{1}{s} \, ds - \frac{1}{6} \cdot \int 1 \, du$

$= \frac{1}{3} \tan^{-1} x + \frac{\log s}{6} - \frac{1}{6} \cdot \int 1 \, du$

Back to u:

$= \frac{\log s}{6} - \frac{u}{6} + \frac{1}{3} x^3 \tan^{-1} x$

$= \frac{1}{6} \left(2x^3 \tan^{-1}(x) - x^2 + \log(x^2+1) \right)$

2. Given:

$\vec{r}_1(t) = \langle 6 + t + 0.5t^2, t^2 + 2t, 5t - 2t^2 \rangle$

$\vec{r}_2(t) = \langle 7t - 0.5t^2, 1 + 0.5t^2 - t, t^2 - 9t \rangle$

Velocity vectors

$\vec{v}_1(t) = \frac{d}{dt}\vec{r}_1(t) = \langle 1+t, 2t+2, 5-4t \rangle$

$\vec{v}_2(t) = \frac{d}{dt}\vec{r}_2(t) = \langle 7-t, t-1, 2t-9 \rangle$

When the velocities equal:

$\vec{v}_1(t) = \vec{v}_2(t)$

$$\left\langle \begin{array}{l} 1+t_1 = 7-t_2 \\ 2t_1 + 2 = t_2 - 1 \\ 5 - 4t_1 = 2t_2 - 9 \end{array} \right\rangle$$

Solve the system of equations:

$t_1 = 1$
$t_2 = 5$

Plug into velocity functions:

$\vec{v}_1(t_1) = \vec{v}_2(t_2) = \langle 2, 4, 1 \rangle$

3. A

$(-1,0,0)$ and $(1,0,0)$

Mid-point:

$(0,0,0)$

Let point A be (x,y,z); distance between A and $(-1,0,0)$:

$s^2 = (x-(-1))^2 + y^2 + z^2$ 1

Distance between A and (1,0,0)

$s^2 = (x+1)^2 + y^2 + z^2$ (2)

(1) - (2):

$4x = 0$; $x = 0$

We get for point A and B $z=0$:

$y^2 = \frac{s^2-1}{4}$

$A = (0, \pm\sqrt{\frac{s^2-1}{4}}, 0)$

Makes B with the following coordinates:

$(-\sqrt{\frac{s^2-1}{4}}, 0, 0)$

B. Solving the equations, we get:

$v = -\frac{\sqrt{2}}{4}$

Sub this value we get:

$u = \frac{1}{2}$

Centroid of the tetrahedron is:

$O = (-\sqrt{\frac{s^2-1}{16}}, \frac{\sqrt{s^2-1}}{16}, \frac{s \cdot \sqrt{\frac{2}{3}}}{4}$

C. Let points P and Q be two adjacent vertides of the tetrahedron, and R and S be the other two vertices:

$\cos PQR = -\frac{1}{3}$, $\cos PQS = -\frac{1}{3}$, $\cos PPS = -\frac{1}{3}$, $\cos QRS = -\frac{1}{3}$

Let G be the centroid of the tetrahedron:

Use the law of cosines

$$\cos GOP = \frac{OG^2 + OP^2 - GP^2}{2OGOP} = -\frac{s^2-1}{4s\sqrt{3}}$$

$108^\circ \rightarrow \frac{\pi}{3}$ in radian.

4. Given that:

Domain of $f\ (D_f) = (-\infty, 1)$
" $g\ (D_g) = [0, \pi]$
$h\ (D_h) = (-1, 1)$

$F(x,y) = (D_f \cap D_g) \times D_h$

$= \{(-\infty, 1) \cap [0, \pi]\} \times (-1, 1)$

$= [0, 1) \times (-1, 1)$

5A. f_x is positive at C and as the contour increases the move along $+x$-axis

decreases as moving along $-x$-axis

making it negative at A and B.

C is the highest value.

B. f_y is positive at B increases as we move along $+y$ axis, the contour value

increases at A. Contour value $=0$ at B. Contour value decreases at C

$\therefore$ Highest at B

C. The red curve corresponds to 2 different values

$\hookrightarrow$ Not a level curve