25/02/24

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INSTITUTE OF TECHNOLOGY
1870

MA 541-A – Spring 2024

HOMEWORK 5

(Due date: Thursday, 02/25/2024, at 11:00 pm)

Show ALL WORK to get full credit.

(Write the pledge on top of your work and sign under it.)

I pledge my honor that I have abided by the Stevens honor system

Akshay.

Problem 1:

Let $X_1, X_2, \dots, X_n$ be a random sample from a Poisson distribution. Find the likelihood ratio for testing $H_0: \lambda = \lambda_0$ versus $H_1: \lambda = \lambda_1$, where $\lambda_1 > \lambda_0$. Use the fact that the sum of independent Poisson random variables follows a Poisson distribution to explain how to determine a rejection region for a test at level α.

Problem 2:

Currently, 20% of potential customers buy soap of brand A. To increase sales, the company will conduct an extensive advertising campaign. At the end of the campaign, a sample of 400 potential customers will be interviewed to determine whether the campaign was successful.

a) State H_0 and H_1 in terms of p, the probability that a customer prefers soap brand A.

b) The company decides to conclude that the advertising campaign was a success if at least 92 of the 400 customers interviewed prefer brand A. Find α. (Use the normal approximation to the binomial distribution to evaluate the desired probability.)

Problem 3:

Let X be a binomial random variable with n trials and probability p of success. What is the generalized likelihood ratio for testing $H_0: p = 0.5$ versus $H_1: p \neq 0.5$?

Problem 4:

A coin is thrown independently 10 times to test the hypothesis that the probability of heads is 0.5 versus the alternative that the probability is not 0.5. The test rejects if either 0 or 10 heads are observed. What is the significance level of the test?

Problem 5:

The intensity of light reflected by an object is measured. Suppose there are two types of possible objects, A and B. If the object is of type A, the measurement is normally distributed with mean 100 and standard deviation 25; if it is of type B, the measurement is normally distributed with mean 125 and standard deviation 25. A single measurement is taken with the value X = 120. Consider the test H_0: Item is of type A versus H_a: Item is of type B.

a) Calculate the likelihood ratio statistic of this test.

b) If the prior probabilities of A and B are equal (1/2 each), what is the posterior probability that the item is of type B?

c) Suppose that a decision rule has been formulated that declares the object to be of type B if $X > 125$. What is the significance level of this test?

d) What is the power of this test? (*Hint:* Compute the power using its definition.)

e) What is the p-value when $X = 120$? (*Hint:* Values as extreme or more extreme than the observed value are $X \geq 120$.)

Bonus Question:

Under H_0, a random variable has the cumulative distribution function $F_0(x) = x^2$, $0 \leq x \leq 1$; and under H_1, it has the cumulative distribution function $F_1(x) = x^3$, $0 \leq x \leq 1$.

a) If the two hypotheses have equal prior probability, for what values of x is the posterior probability of H_0 greater than that of H_1?

b) What is the form of the likelihood ratio test of H_0 versus H_1?

c) What is the rejection region of a level α test?

d) What is the power of the test?

Problem 1

Given that,

$$H_0 : \lambda = \lambda_0 \qquad H_1 : \lambda = \lambda_1$$

where $\lambda_1 > \lambda_0$

→ H_0 & H_1 are simple hypothesis and that the test rejects the H_0 whenever the likelihood ratio is less than c has significance level α

$$\text{likelihood ratio} = \frac{L(\text{Data} \mid H_0)}{L(\text{Data} \mid H_1)}$$

$$\because \; P(X = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!} \leftarrow \text{Poisson dist}$$

$$\therefore \; \text{likelihood ratio} = \frac{\prod_{i=1}^{n} \left(\frac{\lambda_0^{x} \cdot e^{-\lambda_0}}{x!} \right)}{\prod_{i=1}^{n} \left(\frac{\lambda_1^{x} \cdot e^{-\lambda_1}}{x!} \right)} = \frac{\frac{\lambda_0^{\Sigma x_i} \cdot e^{-n\lambda_0}}{\cancel{\prod_{i=1}^{n} x_i}}}{\frac{\lambda_1^{\Sigma x_i} \cdot e^{-n \cdot \lambda_1}}{\cancel{\prod_{i=1}^{n} x_i}}}$$

$$= \frac{\lambda_0^{\Sigma x_i} \cdot e^{-n\lambda_0}}{\lambda_1^{\Sigma x_i} \cdot e^{-n \cdot \lambda_1}} = \left(\frac{\lambda_0}{\lambda_1} \right)^{\Sigma x_i} \cdot e^{-n(\lambda_0 + \lambda_1)}$$

$$RR = \left\{ X : \left(\frac{\lambda_0}{\lambda_1} \right)^{\Sigma x_i} \cdot e^{-n(\lambda_0 - \lambda_1)} < c \right\}$$

$$RR = \left\{ X : \Sigma x_i \ln\left(\frac{\lambda_0}{\lambda_1} \right) < \ln(c) + n(\lambda_0 - \lambda_1) \right\}$$

$$RR = \left\{ X : \Sigma x_i > \underbrace{\frac{\ln(c) + n(\lambda_0 - \lambda_1)}{\ln\left(\frac{\lambda_0}{\lambda_1} \right)}}_{c_1} \right\} \quad \because \ln\left(\frac{\lambda_0}{\lambda_1} \right) > 0$$

$\therefore$ Reject H_0 if

$$x_1 + x_2 + \dots x_n > \frac{\ln(c) + n(\lambda_0 - \lambda_1)}{\ln\left(\frac{\lambda_0}{\lambda_1} \right)}$$

Problem 2.

Given that,

\# of samples $(n) = 400$

a) $H_0 : p = 0.2$

$H_1 : p > 0.2$

b) we will reject H_0 if $P(X \geq 92)$

where x is the # of customer who prefer brand 1

$$\text{mean} = n \cdot p \Rightarrow 400 \times 0.2 \Rightarrow 80$$

$$SD = \sqrt{np(1-p)} \Rightarrow \sqrt{400 \times 0.2 (0.8)} = 8$$

Now Standardize the value of normal dist

$$Z = \frac{x - \mu}{\sigma} = \frac{92 - 80}{8} \Rightarrow \frac{3}{2} \Rightarrow 1.5$$

$\because$

$$\alpha = P(Z \geq 1.5)$$

$$\alpha = 1 - P(Z \leq 1.5)$$

$$= 1 - 0.9332$$

$$\alpha = 0.0668$$

problem 3.

Given that: $X \sim \text{Binomial}(n,p)$

$H_0: p = 0.5$ vs $H_1: p \neq 0.5$

$$\Lambda(x) = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)}$$

$$L(P) = \prod_{i=1}^{n} \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$= \prod_{i=1}^{n} {}^{n}C_{x_i} \cdot p^{\Sigma x_i} (1-p)^{n-\Sigma x_i}$$

$$\because \max_{\theta \in \Theta} L(\theta) = L(\hat{\theta}_{MLE})$$

$$\therefore L(\hat{\theta}_{MLE}) = \ln\left(\prod_{i=1}^{n} {}^{n}C_{x_i}\right) + \Sigma x_i . \ln(p) + (n - \Sigma x_i) . \ln(1-p)$$

Taking derivative w.r.t p and equating with 0

$$L(\hat{\theta}_{MLE}) = 0 + \frac{\Sigma x_i}{p} - \frac{n - \Sigma x_i}{(1-p)} = 0.$$

$$\frac{\Sigma x_i}{p} = \frac{n - \Sigma x_i}{(1-p)}$$

$$\Sigma x_i - \cancel{p\Sigma x_i} - np + \cancel{p\Sigma x_i} = 0.$$

$$\boxed{\hat{p} = \frac{\Sigma x_i}{n} = \bar{x}}$$

$$\therefore \Lambda(x) = \frac{L(P)}{L(\hat{P})} = \frac{\cancel{\prod_{i=1}^{n} {}^{n}C_{x_i}} \cdot p^{\Sigma x_i} (1-p)^{n-\Sigma x_i}}{\cancel{\prod_{i=1}^{n} {}^{n}C_{x_i}} \cdot \hat{p}^{\Sigma x_i} (1-\hat{p})^{n-\Sigma x_i}}$$

$$\Lambda(x) = \frac{L(P)}{L(\hat{P})} = \frac{0.5^{n}}{\hat{p}^{\Sigma x_i} (1-\hat{p})^{n-\Sigma x_i}}$$

where $\hat{p} = \frac{\Sigma x_i}{n}$

Problem 4.

Given that :

$$X \sim \text{Binomial}(n, p)$$

$H_0 : p = 0.5$ vs $H_1 : p \neq 0.5$

$$P(X = x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}.$$

we will reject the hypothesis if # of heads are 0 or 10.

$\therefore \alpha = P(\text{Reject } H_0 \mid H_0 \text{ true})$

we will calculate probability of 0 or 10 heads

$P(X=0) = {}^{10}C_0 \cdot (0.5)^0 (0.5)^{10} = \frac{\cancel{10!}}{\cancel{10!}} \cdot (0.5)^{10} \cdot 1 = 1/1024$

$P(X=10) = {}^{10}C_{10} \cdot (0.5)^{10} (0.5)^0 = \frac{\cancel{10!}}{\cancel{10!}} \cdot (0.5)^{10} \cdot 1 = 1/1024$

$\therefore \alpha = P(X=0) + P(X=10)$

$\alpha = \frac{\cancel{2}}{\cancel{1024}} \quad \frac{1}{512}$

$$\boxed{\therefore \alpha = \frac{1}{512}}$$

Problem 5 .

a) $\Lambda(x) = \dfrac{f_A(x)}{f_B(x)}$

let $x = 120$

$\Lambda(x) = \dfrac{f_A(120)}{f_B(120)}$

$$f_A(120) = \frac{1}{25\sqrt{2\pi}} e^{-\frac{(120-100)^2}{2 \cdot 25^2}}$$

$$f_B(120) = \frac{1}{25\sqrt{2\pi}} \cdot e^{-\frac{(120-125)^2}{2 \cdot 25^2}}$$

B) $p(\text{Type } B|x) = \dfrac{p(x|\text{Type } B) \cdot p(\text{Type } B)}{p(x)}$

$$p(x|\text{Type } B) = \frac{1}{25\sqrt{2\pi}} e^{-\frac{(120-125)^2}{2 \cdot 25^2}}.$$

$p(\text{Type } B) = 1/2$

$p(x) = p(x|\text{Type } A) \cdot p(\text{Type } A) + p(x|\text{Type } B) \cdot p(\text{Type } B)$

c) $\alpha = p(\text{reject } H_0 | H \text{ True})$

$= p(x > 125 \,|\, \text{Type } A)$

$\alpha = 1 - p(x \leq 125 \,|\, \text{Type } A)$

d) power = p(reject H_0 | H_0 is false)

$= p(X > 125 \mid \text{Type B})$

e) p value $= p(X \geq 120 \mid \text{Type A})$

$= 1 - p(X < 120 \mid \text{Type A})$