Homework 9

1 Let $f(t) = t\sin(t^3)$

A Find the Taylor Series of $f(t)$ centered at $t=0$

B Use the degree-16 Taylor polynomial of $f(t)$ to approximate the value of $\int_0^1 f(t)dt$

C Use your answer to part A to explain why $\lim_{t \to 0} \frac{f(t)}{t^2} = 0$

Sol given $f(t) = t\sin(t^3)$

a Taylor Series centred at 0

$$= f(0) + f'(0)(t-0) + \frac{f''(0)(t-0)^2}{2!} + \frac{f'''}{3!}(t-0)^3$$

$$\sin(t^3) = t\left(t^3 - \frac{(t^3)}{3!} + \frac{(t^5)^5}{5!} + \dots\right)$$

$$F(t) = t\sum_{n=0}^{\infty} \frac{(-1)^n (t^3)^{2n+1}}{(2n+1)!}$$

$$F(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n+4}}{(2n+1)!}$$

$$F(t) = t^4 - \frac{t^{10}}{6} + \frac{t^{16}}{120} + \dots$$

Taylor Series → $F(t) = t^4 - \frac{t^{10}}{6} + \frac{t^{16}}{120} + \dots$

b
$$\int_0^1 F(t)dt = \int_0^1 \left[t^4 - \frac{t^{10}}{6} + \frac{t^{16}}{120} + \dots\right] dt$$

$$= \left[\frac{t^5}{5} - \frac{1}{6}\frac{t^{11}}{11} + \frac{1}{120}\frac{t^{17}}{17}\right]_0^1$$

$$= 0.1853$$

$$\int_0^1 F(t)dt = 0.1853$$

C
$$\lim_{t \to 0} \frac{F(t)}{t^2} = \lim_{t \to 0} \frac{1}{t^2}\left[t^4 - \frac{t^{10}}{3!} + \frac{t^{16}}{5!} \dots\right]$$

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$$= \lim_{t \to 0} \left[t^2 - \frac{t^8}{3!} + \frac{t^{14}}{5!} + \cdots \right]$$

$$= \left[0^2 - \frac{0^8}{3!} + \frac{0^{14}}{5!} + \cdots \right]$$

$$\lim_{t \to 0} \frac{F(t)}{t^2} = 0$$

2 Suppose you decide to build a calculator that will compute values of e^x. Having learned about Taylor Series, and remembering that computers are good at addition and multiplication, you decide to use a Taylor polynomial to do the job. The display of your calculator will show six digits after the decimal point.

A If you would like your calculator to display accurate values of e^x whenever $-1 \le x \le 1$, then what is the lowest-degree Taylor polynomial you can use?

B What is the lowest-degree Taylor polynomial you can use if you would like your calculator to display accurate values of e^x whenever $-2 \le x \le 2$?

C The following thought occurs to you: To compute a value e^N, where N is some large number, why not program your calculator to use a Taylor polynomial centered at a number that is close to N? Is this a practical strategy? Explain

Sol Since the calculator can only show six digits after the decimal, we must have an accuracy of six digits.

The Taylor polynomial for e^x is given by.

$$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!}$$

$$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362800} + \frac{x^{10}}{362880}$$

a The limits are $-1 \le x \le 1$

The value of e^x at -1 is $e^{-1} = 0.367879$

e^x at 1 is $e^1 = 2.718281$

Taylor's polynomial at -1,

$$P(-1) = 1 + (-1) + \frac{(-1)^2}{2} + \frac{(-1)^3}{6} + \frac{(-1)^4}{24} + \frac{(-1)^5}{120} + \frac{(-1)^6}{720} + \frac{(-1)^7}{5040} + \frac{(-1)^8}{40320} + \frac{(-1)^9}{362880}$$

$P(-1) = 0.367879$

Taylor's polynomial at 1,

$$P(1) = 1 + 1 + \frac{1^2}{2} + \frac{1^3}{6} + \frac{1^4}{24} + \frac{1^5}{120} + \frac{1^6}{720} + \frac{1^7}{5040} + \frac{1^8}{40320} + \frac{1^9}{362830}$$

$P(1) = 2.718281$

Therefore we need to use 9^{th} degree polynomial

b $-2 \le x \le 2$

e^x at $+2 = e^2 = 7.369056$

$$P(2) = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!} + \frac{2^8}{8!} + \frac{2^9}{9!} + \frac{2^{10}}{10!} + \frac{2^{11}}{11!} + \frac{2^{12}}{12!} + \frac{2^{13}}{13!} + \frac{2^{14}}{14!}$$

$P(2) = 7.389056$

Therefore, we need 14^{th} degree polynomial

c If we program the calculator to be centered at a number that is close to N, there will be a bias towards it. There will be a weighted shift towards the new centre. Also, calculating other Taylor approximation for different functions would be very hard and would require reprogramming of the calculator. So, it is not a practical solution.

3 Let $f(x,y) = (x^2+y)e^{y/2}$

A The function f has exactly one critical point c. Find it

B Find the 2nd Taylor polynomial P_2 of f centered at $x=C$

C Use the Hessian of f to classify c as a local maximum, local minimum or Saddle point.

D is c a critical point of P_2 as well? If so, classify it. What can you conclude?

Sol Given $f(x,y) = (x^2+y)e^{y/2}$

A To find critical point, differentiate w.r.t x, y partially respectively.

$$f_x = \frac{\partial f}{\partial x} = (2x)e^{y/2}$$

$$f_y = \frac{\partial f}{\partial y} = e^{y/2} + (x^2+y)e^{y/2}\left(\frac{1}{2}\right)$$

$$f_x = f_y = 0$$

$$2xe^{y/2} = 0 \quad \text{and} \quad \left(\frac{x^2+y}{2}+1\right)e^{y/2} = 0$$

for $x=0$

$$\frac{x^2+y}{2}+1=0 \rightarrow \frac{y}{2}+1=0$$

$$y=-2$$

The critical point is $(0,-2)$

B Second degree Taylor polynomial at a point c,

$$f(x,y) = f(c) + f_x(c)(x-0) + f_y(c)(y+2) + \frac{f_{xx}(c)}{2}(x-0)^2 +$$

$$f_{xy}(c)(x-0)(y+2) + \frac{f_{yy}}{2}(c)(y+2)^2$$

$f_{xx} = 2e^{y/2}$, $f_{yy} = e^{y/2} + e^{y/2} + \frac{(x^2+y)}{4}e^{y/2}$, $f_{xy} = xe^{y/2}$

At $c=(0,-2)$

$$f_{xx} = 2e^{-1}$$

$$f_{xy} = 0$$

$$f_{yy} = e^{-1} - e^{-1}/2 = e^{-1}/2$$

Hence $P_2(x, y) = P_2(0, -2)$

$$\rightarrow -2e^{-1} + \frac{2e^{-1}x^2}{2} + 0 + \frac{e^{-1}}{4}(y+2)^2$$

$$P_2(0, -2) \rightarrow -2e^{-1} + e^{-1}x^2 + \frac{e^{-1}}{4}(y+2)^2$$

C Hessian matrix for f is given as,

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{y/2} & xe^{y/2} \\ xe^{y/2} & \left(\frac{x^2+y}{4} + 1\right)e^{y/2} \end{bmatrix}$$

at $(x, y) = C$, $H = \begin{bmatrix} 2e^{-1} & 0 \\ 0 & e^{-1}/2 \end{bmatrix}$

$$|H| = (\cancel{2}e^{-1})\left(\frac{e^{-1}}{\cancel{2}}\right)$$

$$|H| = e^{-2} > 0$$

C is a point of local minimum

D Yes, it is a critical point for P_2 also

since $P_x(x, y) = 0$ and $P_y(c) = 0$.