FE535: Introduction to Financial Risk Management
Session 6

Anthony Diaco

Agenda

- Managing Risk
  - Linear Risk (forward/futures contract)
- Optimal Hedging
- Application
  - Equity - Beta Hedging
  - Bonds - Duration Hedging

Intro

Intro to Risk Hedging

- Risk that has been measured (quantified) can be managed.
  - we will focus on market risks
- An important aspect of managing risk is hedging
  - taking positions that lower the risk profile of the portfolio.
- Techniques for hedging have been developed in the futures market
  - farmers hedge output price

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Objective of Risk Hedging

- The objective is to find the optimal position in a futures contract that minimizes the volatility of the total portfolio.
- This portfolio consists of two positions, a fixed inventory exposed to a risk factor and a hedging instrument.

Linear and Non-Linear Risks

Linear Risk

- The value of the hedging instrument is linearly related to the underlying risk factor.
- This involves futures, forwards, and swaps.
- Because linear combinations of normal random variables are also normally distributed, linear hedging maintains normal distributions
  - which considerably simplifies the risk analysis.

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Non-Linear Risk

- Nonlinear risk models, however, are much more complex.
- This involves mainly options
  - which values can have sharply asymmetrical distributions.
- This requires an option pricing model (e.g. Black-Scholes) and its sensitivity to risk factors
  - Greeks

The Case of FX

Foreign Exchange Rates I

Quotes

- Foreign exchange (FX) rates are quoted in terms of a **base** currency and a **quote** currency
- Currency pairs are typically indicated as XXXYYY or XXX/YYY
  - XXX as the base currency
  - YYY as the quote currency
- The FX rate shows how much of the quote currency is needed to buy one unit of the base currency.
  - EURUSD (EUR/USD) quote of 1.18 indicates that $1.18 are needed to buy one euro (€1).

- In 2016, 88% of the trading was between the U.S. dollar (USD) and another currency.
- Like other markets, foreign-exchange markets are governed by the forces of supply and demand

Foreign Exchange Rates II

- Spot exchange rates are typically quoted with four decimal places
- The bid-ask spread faced by corporations when they trade large amounts of a currency is quite small.
  - On April 1, 2020, for example, EUR/USD was quoted as bid 1.0937 and 1.0939.
- However, when traveling such small currency exchanges exhibit much larger bid-ask spreads)

- Forward exchange rates are quoted with the same base currency as spot exchange rates
- Usually, forward FX rates are shown in basis points and added to the spot quote
- For instance, on April 1, 2020, the 1-year forward rate for EUR/USD was quoted at 118.9500 (mid-price)

Foreign Exchange Rates III

- Let's determine the forward rate using the previous info:

$$F_0(1) = S_0 + Q_0(1) \tag{1}$$

$F_0(1)$ is the 1-year forward rate
S_0 is the spot rate
$Q_0(1)$ is the quoted 1-year change

- Given the above info, we have

$$F_0(1) = 1.0938 + \frac{118.9500}{100^2} = 1.10562 \tag{2}$$

- Doing some manipulation, we can see that

$$F_0(1) = S_0 + Q_0(1) = S_0(1 + a) \approx S_0 e^a \tag{3}$$

with a denoting the appreciation (depreciation) rate of the EURUSD, such that

$$\log\left(\frac{F_0(1)}{S_0}\right) = a \tag{4}$$

Foreign Exchange Rates IV

- Suppose a U.S. trader starts with 1 EUR and wants to end up with USD in T years.
- The trader can choose one of the following options
  1. Lend the amount at r_f (foreign) rate and long $F_t(T)$ forward contract
  2. Exchange on the spot for S_t and lend at the r (domestic) rate
- Since both lead to the same economic outcome, then the price of which should be the same

$$e^{r_f(T-t)}F_t(T) = S_t e^{r(T-t)} \tag{5}$$

which leads to the interest rate parity (IRP)

$$F_t(T) = S_t e^{(r-r_f)\tau} \tag{6}$$

where $\tau = T - t$

- According to (6), we discern that a from (4) corresponds to

$$a = (r - r_f) \times 1 = r - r_f \tag{7}$$

which approximately 1.10% according to the previous example

Foreign Exchange Rates - Back to GBM

- Let S_t denote the spot exchange rate between foreign (base) and domestic (base) currencies, e.g. (EUR/USD)
- If the spot rate behaves according to geometric Brownian motion, then we have

$$S_T = S_t \times \exp\left((\theta - \frac{\sigma^2}{2})\tau + \sigma Z_\tau \right) \tag{8}$$

with

1. σ is the volatility of the exchange rate
2. θ is the drift component, i.e. expected return (growth)
3. $\tau = T - t$ time to maturity
4. Z_τ $N(0, \tau)$ is a standard Brownian motion

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- Under no-arbitrage pricing (risk-neutral), it holds true that

$$\theta = r - r_f \tag{9}$$

- In other words, the expected future spot rate in this case corresponds to the IRP:

$$\mathbb{E}_t[S_T] = S_t \times e^{(r - r_f)\tau} \tag{10}$$

with r and r_f, respectively, denote the domestic and foreign interest rates

Unitary Hedge

Unitary Hedging

- Consider the situation of a U.S. exporter who has been promised a payment of 125 million Japanese yen in seven months.
- This defines the underlying position, which can be viewed as an anticipated inventory.
  - Downside - the Yen depreciates, i.e. weakens YEN/USD $\downarrow$, Yen buys less dollars
  - Upside - the Yen appreciates, i.e. strengthens YEN/USD $\uparrow$, which translates to more dollars
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  - enter a seven-month forward contract over-the-counter (OTC).
- Assume for this illustration that this OTC contract is not convenient (why?)
- Instead, the exporter decides to turn to an exchange-traded futures contract, which can be bought or sold on exchange.
- What would be the hedge strategy then?

Unitary Hedging - Futures Hedging

- The Chicago Mercantile Exchange (CME Group) lists yen contracts with face amount of ¥12.5 millions that expire in nine months.
- A possible hedge is to do the following:
  1. Take a short position in 10 contracts, with the face amount of ¥12.5 millions
  2. In sevens months, take the opposite position.
- The hedge idea is to get paid from futures contract in case the Yen weakens
  - drop in the amount of dollars paid

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- The hedge idea is to get paid from futures contract in case the Yen weakens
  - drop in the amount of dollars paid
- By taking a short position over the 7 months, you would
  - gain from the futures contract
  - mitigate losses from weakened Yen
- Because the amount sold is the same as the underlying, this is called a **unitary hedge**.

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  - $/¥ goes from ¥125 to ¥150
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- The table below describes the initial and final conditions for the contract.

Item	Initial Time	Exit Time	Gain or Loss
Market Data:			
Maturity (months)	9	2	
U.S. interest rate	6.00%	6.00%	
Yen interest rate	5.00%	2.00%	
Spot (\$/¥)	¥125	¥150	
Spot (¥/\$)	$\frac{1}{125} = 0.008$	$\frac{1}{150} = 0.006667$	-\$166,667
Futures (¥/\$)	0.00806	0.006711	\$168,621
Basis (¥/\$)	-0.000060	-0.000045	\$1,954

- At each date, the futures price is determined by interest rate parity

$$F_t(T) = S_t \times e^{\tau(r-r_f)} \tag{11}$$

i.e., $0.00806 = 0.008 \times e^{(0.06-0.05)\times 9/12}$

- Without any hedging, the exporter loss is $12.5 \times 10^7 \times (1/125 - 1/150 = -\$166,667$
- If one were to long 10 contracts, then the payoff is $12.5 \times 10^7 \times (1/124.07 - 1/149 = -\$168,621$
- However, since the exporter took a short position in the futures, he ends up with a positive payoff of \$168,621.
- Taken altogether, the exporter reaped a profit of \$168,621 + (-\$166,667) = \$1,954

Unitary Hedging - Basis Risk

- The previous example shows that futures hedging can be quite effective
- However, there is also potential loss if the Yen didn't weaken.
  - This results in what is known as **basis risk**
- Define Q as the amount of yen transacted and S and F as the spot and futures rates
- To simplify, index each 1 for the the initial time and by 2 for the exit time.
- The profit and loss (P&L) on the unhedged transaction is

$$QS_2 - QS_1 = Q \times [S_2 - S_1] \tag{12}$$

- With unitary hedging, the P&L is

$$Q[S_2 - S_1] - Q[F_2 - F_1] = Q[S_2 - F_2] - Q[S_1 - F_1] = Q[b_2 - b_1] \quad (13)$$

where $b_t = S_t - F_t$ is the *basis*

- Hence, the P&L from the hedged position depends on $b_2 > b_1$

  - The effect of hedging is to transform price risk into basis risk.
  - A short hedge position is said to be long the basis, since it benefits from an increase in the basis.

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- In the previous example, we have

Item	Initial Time	Exit Time	Gain or Loss
Contract Data:			
Spot (¥/\$)	S_1	S_2	$Q[S_2 - S_1]$
Futures (¥/\$)	F_1	F_2	$-Q[F_2 - F_1]$
Basis (¥/\$)	$b_1 = S_1 - F_1$	$b_2 = S_2 - F_2$	$Q[b_2 - b_1]$

- Obviously, in the original example, we have

$$b_2 = -0.000045 > b_1 = -0.000060 \quad (14)$$

- Now consider the case in which the Yen appreciates, i.e. strengthens, at exit with $S_2 = 1/100$ while holding everything else equal

Item	Initial Time	Exit Time	Gain or Loss
Contract Data:			
Spot (¥/\$)	$\frac{1}{125} = 0.008$	$\frac{1}{100} = 0.01$	\$250,000
Futures (¥/\$)	$\frac{1}{124.07} = 0.00806$	$\frac{1}{99.33} = 0.0100667$	-\$250,833
Basis (¥/\$)	-0.00006	-0.00007	-\$833

- In this case, $b_2 - b_1 < 0$, by longing the basis, the hedge results in a loss

Basis Risk - Summary

- There are two main factors causing the basis risk
  1. The underlying assets
  2. Difference in maturity
- Basis risk is lowest when the underlying position and the futures correspond to the same asset
  - Unlike the case of cross-hedging
- Basis risk remains because of differing maturities
  - As we have seen in the yen hedging example, the maturity of the futures contract is nine months rather than seven months.
  - As a result, the liquidation price of the futures is uncertain
  - If the exit were 9 months same as the maturity, then $b_2 = 0 > b_1$

FRM Exam 2000

Under which scenario is basis risk likely to exist?

1. A hedge (which was initially matched to the maturity of the underlying) is lifted before expiration.
2. The correlation of the underlying and the hedge vehicle is less than one and their volatilities are unequal.
3. The underlying instrument and the hedge vehicle are dissimilar.
4. All of the above are correct.

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- Answers B and C imply dissimilarity between the asset and the futures contract
- Answer A denotes mismatch in maturity. Even if the futures and the underlying were similar, this results in a basis risk

Optimal Hedge

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- The previous discussion gave an example of a unit hedge, where the quantities transacted are identical in the two markets.
- In general, we need to decide how much of the hedging instrument to transact.

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Bond Portfolio Manager Example

- Consider a situation where a portfolio manager has an inventory of carefully selected corporate bonds that should do better than their benchmark.
- The manager wants to guard against interest rate increases, however, over the next three months.
- In this situation, it would be too costly to sell the entire portfolio only to buy it back later.
- Instead, the manager can implement a temporary hedge using derivative contracts, for instance T-bond futures.

Optimal Hedge Ratio

- Let ΔS denote the change in the dollar value of the inventory
- Let ΔF denote the change in the dollar value of the futures contract
- If the manager longs N futures contracts, the change in the value of the portfolio is
$$\Delta V = \Delta S + N \times \Delta F \tag{15}$$
- The question is what is the optimal N?
- From risk perspectives, an optimal hedge should to minimize the volatility of such portfolio

- Let $\sigma^2_{\Delta V}$ denote the portfolio volatility, where

$$\sigma^2_{\Delta V} = \sigma^2_{\Delta S} + N^2\sigma^2_{\Delta F} + 2N\sigma_{S,F} \tag{16}$$

with $\sigma_{S,F} = cov(\Delta S, \Delta F)$ denoting the covariance in the change of value between the underlying and the contracts

- Hence, the optimal hedge is given by $\partial\sigma^2_{\Delta V}/\partial N = 0$, i.e.

$$2N\sigma^2_{\Delta F} + 2\sigma_{\Delta S \Delta F} = 0$$
$$N^* = -\frac{\sigma(\Delta S, \Delta F)}{\sigma^2_{\Delta F}} = -\beta_{S,F}$$

where $\beta_{S,F}$ denotes the slope derived by regressing ΔS (dependent variable) on ΔF (independent variable)

- Hence, optimal hedge ratio takes an opposite position of the covariance/correlation sign

Optimal Hedge Ratio - Hedge Effectiveness

- Hedge effectiveness is determined by the decrease in the portfolio volatility
- Given the optimal hedge, N^*, the portfolio volatility is, hence:

$$\begin{aligned} \sigma^2_{\Delta V}(N^*) =& \sigma^2_{\Delta S} + (N^*)^2 \sigma^2_{\Delta F} + 2N^* \sigma_{S,F} && (17) \\ =& \sigma^2_{\Delta S} - \frac{\sigma^2_{S,F}}{\sigma^2_{\Delta F}} && (18) \end{aligned}$$

- Relative to the unhedged position, effectiveness is measured by the degree to which the hedged position reduces the volatility the volatility by

$$R^2 = \frac{\sigma^2_{\Delta S} - \sigma^2_{\Delta V}(N^*)}{\sigma^2_{\Delta S}} \in (0, 1) \tag{19}$$

  with $R^2 = 1$ implying maximum effectiveness and $R^2 = 0$ zero effectiveness.
- Put differently, Equation (19) implies that

$$\sigma^2_{\Delta V}(N^*) = (1 - R^2)\sigma^2_{\Delta S} \tag{20}$$

Optimal Hedge Ratio - Standardization

- In practice, there is often confusion about the definition of the portfolio value and unit prices
- Here S consists of the number of units (shares, bonds, bushels, gallons) times the unit price (stock price, bond price, wheat price, fuel price).
- It is easier to deal with unit prices and to express volatilities in terms of rates of changes in unit prices, which are unitless.

- Define Q as the quantity of the underlying with price s, such that $S = Q \times s$
- The notional amount of one futures is $F = Q_f \times f$
- The optimal number of contracts needed is

$$N^* = -\beta_{s,f} \frac{Q \times s}{Q_f \times f} \tag{21}$$

- To estimate $\beta_{s,f}$, one needs a price time series of the underlying and futures, i.e. s_t and f_t
- Given each, compute the rate of change for each time series, i.e.

$$r_t^s = \frac{s_t - s_{t-1}}{s_{t-1}} \text{ and } r_t^f = \frac{f_t - f_{t-1}}{f_{t-1}} \tag{22}$$

- Then $\beta_{s,f}$ is estimated by regressing r_t^s on r_t^f, i.e.

$$r_t^s = \alpha + \beta_{s,f} r_t^f + \epsilon_t \tag{23}$$

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JPY USD Example

In the JPY/USD example we had

- $Q = 125,000,000$ and $s_0 = \$1/125$
- Since the notional for one JPY/USD contract is ¥12,500,000, then $Q_f = 12,500,000$ and $f_0 = 1/124.07$
- For simplicity, assume that $\beta_{s,f} = 1$, then

$$N^* = -1 \times \frac{125 \times 10^6 \times \frac{1}{125}}{125 \times 10^5 \times \frac{1}{124.07}} \approx -10 \tag{24}$$

Short Vs. Long

- The sign of optimal hedge from Equation (21) depends on whether the investor is hedging the input or output.
- In the case of hedging the output, i.e. the future sale of 125 Yens, the trader needs to mitigate the downside and, hence, goes short
- In the short position we have

$$N^* = -\beta_{s,f} \frac{Q \times s}{Q_f \times f} \tag{25}$$

- On the other hand, if the traded is hedging input, such as cost of inventory, then the traded goes long.
- In the long position, we have

$$N^* = \beta_{s,f} \frac{Q \times s}{Q_f \times f} \tag{26}$$

Airline Example

- An airline knows that it will need to purchase 10,000 metric tons of jet fuel in three months. It wants some protection against an upturn in prices using futures contracts.
- As there is no futures contract on jet fuel, the company can refer to heating oil instead
- The company can hedge using heating oil futures contracts traded on NYME and the notional for one contract is 42,000 gallons.
- The current price of jet fuel is \$277/metric ton. The futures price of heating oil is \$0.6903/gallon.
- The standard deviation of the rate of change in jet fuel prices over three months is 21.17%, that of futures is 18.59%, and the correlation is 0.8243.

Compute

1. The notional and standard deviation of the unhedged fuel cost in dollars
2. The optimal number of futures contract to buy/sell, rounded to the closest integer
3. The standard deviation of the hedged fuel cost in dollars

1. Note that
   - the position has $Q = 10,000$ metric tons with $s = \$277$ per metric ton, hence the notional amount is $S = Q \times s = \$2,770,000$.
   - The cost refers to the change in the notional amount S, i.e. ΔS. This is approximated by $\sigma_{\Delta S}$:

$$\sigma^2_{\Delta S} = \mathbb{V}(\Delta S) = \mathbb{V}(\Delta(sQ)) = \mathbb{V}(\Delta s)Q^2 = \mathbb{V}\left(\frac{\Delta s}{s}\right) s^2 \times Q^2 \tag{27}$$

   - The question provides us s and Q along with $\mathbb{V}\left(\frac{\Delta s}{s}\right) = 0.2117^2$
   - Therefore, the answer is

$$\sigma_{\Delta S} = \sqrt{\mathbb{V}\left(\frac{\Delta s}{s}\right)} \times s \times Q = 0.2117 \times \$2,770,000 = \$586,409 \tag{28}$$

   - For reference, that of the futures contract is

$$\sigma_{\Delta F} = \sqrt{\mathbb{V}\left(\frac{\Delta f}{f}\right)} \times f \times Q_f = 0.1859 \times \$0.6903 \times 42,000 = \$5,389.72 \tag{29}$$

2. To find optimal number of contracts, we need to compute $\beta_{s,f}$, which is decomposed into

$$\beta_{s,f} = \frac{Cov\left(\frac{\Delta s}{s}, \frac{\Delta f}{f}\right)}{\mathbb{V}\left(\frac{\Delta f}{f}\right)} = \frac{\rho_{s,f}\sqrt{\mathbb{V}\left(\frac{\Delta s}{s}\right)}\sqrt{\mathbb{V}\left(\frac{\Delta f}{f}\right)}}{\mathbb{V}\left(\frac{\Delta f}{f}\right)} = \rho_{s,f}\frac{\sqrt{\mathbb{V}\left(\frac{\Delta s}{s}\right)}}{\sqrt{\mathbb{V}\left(\frac{\Delta f}{f}\right)}} \tag{30}$$

The question provides us the correlation between the rates of change in prices, which is 0.8243, along with the volatility for each, such that

$$\beta_{s,f} = 0.8243 \times \frac{0.2117}{0.1859} = 0.9387 \tag{31}$$

Since the risk manager is hedging input, then it the optimal number of futures it needs to short/long is

$$N^* = \beta_{s,f}\frac{Q \times s}{Q_f \times f} = 0.9387\frac{10,000 \times \$277}{42,000 \times \$0.6903} = 89.7 \approx 90 \tag{32}$$

Clearly, since the correlation (hence the beta) is positive, then a hedge position should go long, i.e. gaining a profit from an increase in the price.

3. Finally, we need to compute the volatility of the hedged position
   - Recall Equation (20), where

$$\sigma^2_{\Delta V}(N^*) = \sigma^2_{\Delta S}(1 - R^2) \tag{33}$$

   - From part 1 of this question we know that $\sigma^2_{\Delta S} = 586,409$
   - Additionally, recall that $R^2 = \rho^2_{s,f}$, therefore $R^2 = 0.8243^2$
   - As a result,

$$\sigma_{\Delta V}(N^*) = \sigma_{\Delta S} \times \sqrt{(1 - R^2)} = 586,409 \times \sqrt{1 - 0.8243^2} = 331,997 \tag{34}$$

   - This implies the hedging is effective in reducing the original position's volatility by almost 43%, where $1 - \sqrt{1 - R^2} = 43\%$

Optimal Hedge - Liquidity Issues

- Although futures hedging can be successful at mitigating market risk, it can create other risks.
- Recall that futures contracts are marked to market daily.
  - Hence they can involve large cash inflows or outflows
- Cash outflows, in particular, needed to maintain the futures position can create liquidity problems
  - especially when they are not offset by cash inflows from the underlying position.

Application

Hedging Equity

- We now turn to equity hedging using stock index futures.
- From a single factor model, beta denotes a risk exposure of an asset to a systematic factor such as the market
- Recall that

$$R_i = \alpha + \beta_{iM} R_M + \epsilon_i \tag{35}$$

- Ignoring the residual and intercept, the relative change in the price of the asset is

$$\frac{\Delta S_i}{S_i} = R_i = \beta_{iM} R_M = \beta_{iM} \frac{\Delta M}{M} \tag{36}$$

- Assume that we have at our disposal a stock index futures contract, which has a beta of unity
  - perfectly tracks the market

- Consider a portfolio consists of \$$S$ in asset i and N stock index futures contracts
- The change in value of such portfolio is given by

$$\begin{aligned}\Delta V =& \Delta S_i + N\Delta F = \frac{\Delta S_i}{S_i} \times S_i + N\frac{\Delta F}{F} \times F \\ &= \beta_{iM}\frac{\Delta M}{M} \times S_i + N\frac{\Delta M}{M} \times F\end{aligned}$$

- Hence, a perfect hedge implies that $\Delta V = 0$, which holds true if

$$\begin{aligned}&\beta_{iM}\frac{\Delta M}{M} \times S_i + N\frac{\Delta M}{M} \times F = 0 \\ &\beta_{iM} \times S_i + N \times F = 0 \\ &N^* = -\beta_{iM} \times \frac{S_i}{F} = -\beta_{iM} \times \frac{Q \times s_i}{Q_f \times f}\end{aligned}$$

which is the same result we have from (21)

Hedging Equity - Example

A portfolio manager holds a stock portfolio worth \$10 million with a beta of 1.5 relative to the S&P 500. According to the CME Group, on Oct 29th 2018 the last traded price of a futures contract on the S&P 500 maturing in June 2019 is \$2627 (see **link**). Given this, compute the number of "e-mini" contracts to sell short for an optimal protection.

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- Note that there are different sizes of stock baskets for the S&P 500. For instance, CME, offers a "big contract" and an "e-mini" contract
  - the big contract has a multiplier of \$250
  - the e-mini contract has a multiplier of \$50
- The notional of the futures contract is $2,627 \times 50 = \$131,350$
- The optimal hedge, therefore, is given by

$$N^* = -\beta \times \frac{S}{F} = -1.5 \times \frac{\$10,000,000}{\$131,350} = -76.13 \tag{37}$$

Another Example - Beta Target

The current value of a stock index is $1,457, and each futures contract on the index is for delivery of 250 times the index. A long-only equity portfolio with market value of $300,100,000 has a beta of 1.1. To reduce the portfolio beta to 0.75, how many futures contracts should you long/short?

Another Example - Beta Target

The current value of a stock index is \$1,457, and each futures contract on the index is for delivery of 250 times the index. A long-only equity portfolio with market value of \$300,100,000 has a beta of 1.1. To reduce the portfolio beta to 0.75, how many futures contracts should you long/short?

- Recall that N^* is the optimal hedge, in the sense of creating a 0-beta portfolio
- Instead, we need to find the number N, for which the beta of portfolio is 0.75
- First, by selling N^* contracts, we can reduce the portfolio beta to 0, i.e.

$$N^* = -1.1 \times \frac{300,100,000}{1,457 \times 250} = -906 \tag{38}$$

- If we sell $N = 0$ zero contracts, the portfolio beta is 1.1
- Hence, there is should be a point in between, i.e. $-906 < N < 0$, that reduces the portfolio beta to 0.75
- Hypothetically, if the portfolio beta were 0.75, then we need to sell 618 to bring its beta to zero
- Therefore, we need to sell $906 - 618 = 288$

Hedging Bonds

- Going back to the bond portfolio manager, what is the optimal hedge to guard against interest rates over three months period?
- Recall that according to first order Taylor's expansion, the price of the bond is given by

$$P_1 = P - PD^*\Delta y$$
$$\Delta P = -PD^*\Delta y$$

where D^* denotes the modified duration, i.e. $D^* = D/(1+y)$

- Let S denote the dollar value of the bonds portfolio, where F is the notional value of the corresponding futures contract
- Following the same notation as before, the change in the hedged position is given by

$$\Delta V = \Delta S + N\Delta F = -S \times D_S^* \times \Delta y + N \times (-F \times D_F^* \times \Delta y) \qquad (39)$$

which is equal to zero, when

$$N^* = -\frac{D_S^* S}{D_F^* F} \qquad (40)$$

Example - Hedging Interest Rate Risk

On June 2, a fund manager with USD 10 million invested in government bonds is concerned that interest rates will be highly volatile over the next three months. The manager decides to use the September Treasury bond futures contract to hedge the portfolio. The current futures price is USD 95.0625. Each contract is for the delivery of USD 100,000 face value of bonds. The duration of the manager's bond portfolio in three months will be 7.8 years. The cheapest-to-deliver (CTD) bond in the Treasury bond futures contract is expected to have a duration of 8.4 years at maturity of the contract. What position should the fund manager undertake to mitigate his interest rate risk exposure?

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- For the underlying, we have $S = \$10,000,000$ and $D_S^* = 7.8$
- For the futures, prices are quoted similar to bonds. In particular, note that the face value a \$95.0625 is \$100, hence the futures contract has 1000 bonds, as a result $F = 95062.5$. Additionally, we are given that $D_F^* = 8.4$
- Putting altogether, the optimal hedge is

$$N^* = -\frac{D_S^* S}{D_F^* F} = -\frac{7.8 \times \$10,000,000}{8.4 \times 95,062.5} = -97.7 \approx -98 \tag{41}$$

General Considerations

- The quality of the hedge depends on the size of the residual risk, idiosyncratic component
- For large portfolios, the approximation may be good because residual risk diversifies away
- In contrast, hedging an individual stock with stock index futures may give poor results.

- If the objective of hedging is to lower volatility, hedging will eliminate downside risk but also any upside potential.
- The objective of hedging is to lower risk, not to make profits, so this is a double-edged sword.
- Whether hedging is beneficial should be examined in the context of the trade-off between risk and return.