FE535: Introduction to Financial Risk Management
Session 8

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Agenda

- Overview
- Managing Non-Linear Risk
  - Delta-Hedging
  - Options Price Sensitivity - Greeks

Managing Non-Linear Risk

- Last week focused on linear risk models, i.e. hedging using contracts such as forwards/futures whose values are linearly related to the underlying risk factors
- Market losses can be ascribed to the combination of two factors: exposure and adverse movements in the risk factor.
  - Thus a large loss could occur because of the risk factor, which is bad luck.

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- Market losses can be ascribed to the combination of two factors: exposure and adverse movements in the risk factor.
  - Thus a large loss could occur because of the risk factor, which is bad luck.
- Too often, however, losses occur because the exposure profile is similar to a short option position.
  - This is less forgivable, because exposure is under the control of the portfolio manager.
- Nonlinear risk models are much more complex.
  - option values can have sharply asymmetrical distributions

Niederhoffer's Case Study[1]

- A well-established hedge fund ran by Victor Niederhoffer
  - a star on Wall Street
  - his fund was wiped out in November 1997
- What happened?

[1]See Box 14.1 from this **chapter**.

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- What happened?
- Victor wrote (sold) "naked" options on the S&P 500 index
- His strategy was the following
  - collect many put option premiums for a small price
  - the chances of losses were small

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- Nonetheless, his main assumption was that the market won't drop more than 5% percent in a day
- During the Asian market crisis, the S&P 500 dropped more than 7% in a single day
- To meet margin calls, Victor had to liquidate his position in a fire-sale

[1] See Box 14.1 from this **chapter**.

- Let's take a closer look at what happened
- If R_d denotes the return on the S&P 500 in a single day d and $R_d \sim N(\mu_d, \sigma_d^2)$ then

$$\mathbb{P}\left(R_d < -0.05\right) = \mathbb{P}\left(\frac{-0.05 - \mu_d}{\sigma_d}\right) \tag{1}$$

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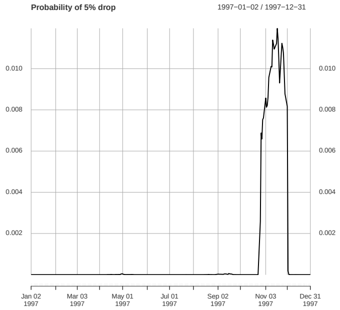
- The probability should be small as long as σ_d (μ_d) is small (large) enough
- However, in October 1997, the market exhibited a sudden increase in volatility due to worries about possible spillovers from Asian Financial Crisis
- For instance, compared to Sep, 1997, the S&P 500 volatility more than doubled in Oct, 1997
    - which, as a result, significantly increased the probability



How Traders Manager their Risk

- The trading function within a financial institution is referred to as the **front office**
- The part of the financial institution that is concerned with the overall level of the risks being taken, capital adequacy, and regulatory compliance is referred to as the **middle office**
- The record keeping function is referred to as the **back office**.

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- The record keeping function is referred to as the **back office**.
- There are two levels within a financial institution at which trading risks are managed
  1. The front office hedges risks by ensuring that exposures to individual market variables are not too great.
  2. The middle office aggregates the exposures of all traders to determine whether the total risk is acceptable.

Delta Hedging

- Delta of a portfolio is the partial derivative of a portfolio with respect to the price of the underlying asset

$$\Delta_V = \frac{\Delta V}{\Delta S} \tag{2}$$

- Suppose that a \$0.1 increase in the price of gold leads to the gold portfolio decreasing in value by \$100, then the delta of the portfolio is

$$\Delta_V = \frac{-100}{0.1} = -1000 \tag{3}$$

- The portfolio could be hedged against short-term changes in the price of gold by buying 1000 ounces of gold. This is known as making the portfolio **delta neutral**

Delta Hedging - Non-Linear Products

- When the price of a portfolio is linearly dependent on the price of an underlying asset a "hedge and forget" strategy can be used
- Non-linear products, however, require the hedge to be rebalanced to preserve delta neutrality
- Options and other more complex derivatives dependent on the price of an underlying asset are nonlinear products.
- The relationship between the value of the product and the underlying asset price at any given time is nonlinear.

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- The relationship between the value of the product and the underlying asset price at any given time is nonlinear.
- This non-linearity makes them more difficult to hedge for two reasons:
  1. Making a nonlinear portfolio delta neutral only protects against small movements in the price of the underlying asset
  2. We are not in a hedge-and-forget situation. The hedge needs to be changed frequently. This is known as dynamic hedging.

- Consider as an example a trader who sells 100,000 European call options on a non-dividend-paying stock when
  1. Stock price is \$49, i.e. $S = 49$
  2. Exercise price is \$50, i.e. $K = 50$
  3. Interest rate is 5%, i.e. $r = 0.05$
  4. Stock volatility is 20% per annum, i.e. $\sigma = 0.2$
  5. Time to expiration is 20 weeks, i.e. $\tau = 20/52 = 0.3846$

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- Suppose that the amount received for the options is \$300,000 and that the trader has no other positions dependent on the stock.
- According to the BSM, the value of an option to buy one share of the stock is

$$C_S = f(S, K, r, \sigma, \tau) = \$2.4 \tag{4}$$

- At the same time, the delta of the option is

$$\Delta_C = \frac{C_{S+\Delta S} - C_S}{\Delta S} = \frac{f(S + \Delta S, K, r, \sigma, \tau) - f(S, K, r, \sigma, \tau)}{\Delta S} = \$0.522 \tag{5}$$

with small $\Delta S = 10^{-6}$

- The Δ_C from (5) is a numerical example of deriving the call option price with respect to S
- In other words, it approximates

$$\Delta_C = \frac{\partial C}{\partial S} \tag{6}$$

According to the BSM, it follows that

$$\Delta_C = \frac{\partial C}{\partial S} = \Phi(d_1) \tag{7}$$

where

$$\begin{aligned} d_1 =& d_2 + \sigma\sqrt{\tau} = \frac{\log\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \\ d_2 =& \frac{\log\left(\frac{S_t}{K}\right) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

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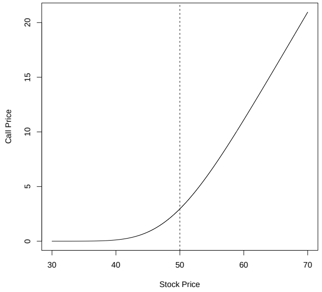
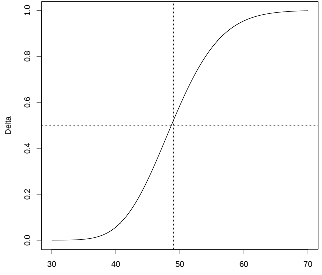
- In the previous example, we have $d_2 = -0.07$ and $d_1 = 0.054$
- Hence, $\Delta_C = \Phi(0.054) = \texttt{pnorm(0.054)} = 0.522$

```
> bs_f <- function(S,K,r,sig,tau) {
+   d2 <- (log(S/K) + (r-0.5*sig^2)*tau)/(sig*sqrt(tau))
+   d1 <- d2 + sig*sqrt(tau)
+   call_p <- S*pnorm(d1) - exp(-r*tau)*K*pnorm(d2)
+   return(call_p)
+ }
> S <- 49
> K <- 50
> r <- 0.05
> sig <- 0.2
> tau <- 20/52
> # define function as price alone
> C_S <- function(x) bs_f(x,K,r,sig,tau)
> C_S(S)
[1] 2.400527
> dS <- 10^-5
> Delta_C <- (C_S(S+dS) - C_S(S))/dS
> round(Delta_C,3)
[1] 0.522
> # call for different S
> S_seq <- seq(30,70,length = 10^5)
> P_seq <- sapply(S_seq,  function(p)  bs_f(p,K,r,sig,tau) )
> ds <- data.frame(Stock = S_seq, Call = P_seq)
> plot(Call ~ Stock,data = ds,type = "l",
+ ylab = "Call Price", xlab =  "Stock Price")
> abline(v = K,lty = 2)
> # compute the delta
> dS <- S_seq[2] - S_seq[1]
> Delta <- c(NA,(ds[-1,"Call"]-ds[-nrow(ds),"Call"])/dS)
> ds <- data.frame(ds,Delta)
> plot(Delta ~ Stock,data = ds,type = "l",
+ ylab = "Delta", xlab =  "Stock Price")
> abline(v = S,lty = 2)
> abline(h = 0.5,lty = 2)
```





- For out-of-the-money, the data from the previous code as the following:

S	C_S	Δ_C
30.0000	0.0000	
30.0004	0.0000	0.0000
30.0008	0.0000	0.0000
30.0012	0.0000	0.0000

- For at-the-money

S	C_S	Δ_C
49.9994	2.9542	0.5859
49.9998	2.9544	0.5859
50.0002	2.9546	0.5859
50.0006	2.9549	0.5859

- For the in-the-money

S	C_S	Δ_C
69.9988	20.9555	0.9983
69.9992	20.9559	0.9983
69.9996	20.9563	0.9983
70.0000	20.9567	0.9983

- Because the trader is short 100,000 options, i.e. $N = 100,000$, the value of the trader's portfolio is

$$V_S = -N \times C_S = -100,000 \times \$2.4 = -\$240,000 \tag{8}$$

- The delta of the portfolio is

$$\Delta_V = -\,N \times \Delta_C = -100,000 \times 0.522 = -52,200 \tag{9}$$

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- The trader can feel pleased that the options have been sold for $60,000 more than their theoretical value
- At the same time, she is faced with the problem of hedging the risk in the portfolio.
- Immediately after the trade, the trader's portfolio can be made delta neutral by buying 52,200 shares of the underlying stock.
- If there is a small decrease (increase) in the stock price, the gain (loss) to the trader of the short option position should be offset by the loss (gain) on the shares.

Delta Hedging - Rebalancing I

Hedged Portfolio Delta

- Let N_S denote the number of shares in the portfolio

$$V = N_S \times S + V_S = N_S \times S - N \times C_s \tag{10}$$

such that

$$\Delta V = N_S \times \Delta_S - N \times \Delta_C \tag{11}$$

- Since $\Delta_S = 1$ (why?), then $\Delta V = 0$ holds true when

$$N_S = N \times \Delta_C \rightarrow N_S = -\Delta_V \tag{12}$$

- Since the portfolio position has $\Delta_V = -52,200$, then the trader needs to buy 52,200 shares to achieve a delta-neutral portfolio
- As soon as the option is written, in order to purchase 52,200, the trader needs to borrow

$$49 \times 52,200 = \$2,557,800 \tag{13}$$

Delta Hedging - Rebalancing II

- Now the trader's portfolio consists of 100,000 options and 52,200 shares.
  - Options are liability
  - Shares are assets
- If the share price goes up, the trader faces further liability, however, the increase would be offset by assets
  - hence, delta-neutral
- Nonetheless, the value of delta depends on the location of the option. The above hedge is effective, if the delta of option were constant
- In the following trading day, the trader faces a similar challenge

- Consider the scenario in which the option is exercised at maturity

Week	S	C	Δ_C	Shares Needed	Shares Purchased	Cost of Shares Purchased	Cumulative Cash Outflow	Interest Cost
0	49.00	2.40	0.522	52200	52200	2557800	2557800	2459

- Consider the scenario in which the option is exercised at maturity

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0	49.00	2.40	0.522	52200	52200	2557800	2557800	2459
1	48.12	1.89	0.458	45800	-6400	-307968	2252291	2166

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1	48.12	1.89	0.458	45800	-6400	-307968	2252291	2166
2	47.37	1.49	0.400	40000	-5800	-274746	1979711	1904
3	50.25	2.84	0.596	59600	19600	984900	2966515	2852
4	51.75	3.71	0.693	69300	9700	501975	3471342	3338
5	53.12	4.62	0.774	77400	8100	430272	3904952	3755
6	53.00	4.44	0.771	77100	-300	-15900	3892807	3743
7	51.87	3.50	0.706	70600	-6500	-337155	3559395	3422
8	51.38	3.06	0.674	67400	-3200	-164416	3398401	3268
9	53.00	4.15	0.787	78700	11300	598900	4000569	3847
10	49.88	1.92	0.550	55000	-23700	-1182156	2822260	2714
11	48.50	1.15	0.413	41300	-13700	-664450	2160523	2077
12	49.88	1.69	0.543	54300	13000	648440	2811041	2703
13	50.37	1.84	0.591	59100	4800	241776	3055520	2938
14	52.13	2.91	0.768	76800	17700	922701	3981159	3828
15	51.88	2.59	0.759	75900	-900	-46692	3938295	3787
16	52.87	3.27	0.865	86500	10600	560422	4502504	4329
17	54.87	5.04	0.978	97800	11300	620031	5126864	4930
18	54.62	4.72	0.990	99000	1200	65544	5197338	4997
19	55.87	5.92	1.000	100000	1000	55870	5258205	5056
20	57.25	7.25	1.000	100000	0	0	**5263261**	5061

- The previous example is from Chapter 8 from Hull's textbook, (see Table 8.2)
- At maturity, it costs the trader \$5,263.3K to perform the hedge
    - Paying interest and principle
    - Buy-Sell stocks
- Since the option is exercised, the trader delivers 100,000 shares at \$50, i.e. \$5,000K
- At time 0, the trader received \$300K for writing options
- At maturity, the P&L of the trader is

$$\$5,000K + \$300K \times \exp^{(0.05 \times \frac{20}{52})} - \$5,263.3K = \$42.6K \tag{14}$$

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- A profit \$42.6K seems less appealing. Nonetheless, the trader can rest assure that she is incurring huge losses
- Without hedging, i.e. writing a naked option, the trader needs to deliver 100,000 shares
  - Since the market price is \$57.25, the trader would have absorbed a loss of \$7.25 per option

- To summarize we have

Cash Inflow from Selling Calls	\$306K
Cash Inflow from Delivering Stock	\$5,000K
Cash Outflow	-\$5,263.3K
Final P&L	\$42.6K

- Note that net cash-flow cost from one option is equal to

$$\frac{\$5,000K - \$5,263.3K}{100K} = -\$2.63 \tag{15}$$

- Without taking into account the initial premium paid on these premiums, the cost of selling these options per contract is \$2.63
- Hence, if the trader were able to sell any contract above that price, she is able to reap a profit
- Nonetheless, note that this cost will be random depending on the final price

- Consider another scenario in which the option expires at maturity

Week	S	C	Δ_C	Shares Needed	Shares Purchased	Cost of Shares Purchased	Cumulative Cash Outflow	Interest Cost
0	49.00	2.40	0.522	52200	52200	2557800	2557800	2459
1	49.75	2.72	0.568	56800	4600	228850	2789109	2682

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1	49.75	2.72	0.568	56800	4600	228850	2789109	2682
2	52.00	4.07	0.705	70500	13700	712400	3504191	3369
3	50.00	2.69	0.579	57900	-12600	-630000	2877561	2767
4	48.38	1.76	0.459	45900	-12000	-580560	2299768	2211
5	48.25	1.61	0.443	44300	-1600	-77200	2224779	2139
6	48.75	1.75	0.475	47500	3200	156000	2382918	2291
7	49.63	2.10	0.540	54000	6500	322595	2707804	2604
8	48.25	1.34	0.420	42000	-12000	-579000	2131408	2049
9	48.25	1.25	0.411	41100	-900	-43425	2090032	2010
10	51.12	2.67	0.658	65800	24700	1262664	3354706	3226
11	51.50	2.82	0.692	69200	3400	175100	3533032	3397
12	49.88	1.69	0.543	54300	-14900	-743212	2793217	2686
13	49.88	1.57	0.538	53800	-500	-24940	2770963	2664
14	48.75	0.91	0.400	40000	-13800	-672750	2100877	2020
15	47.50	0.40	0.236	23600	-16400	-779000	1323897	1273
16	48.00	0.41	0.262	26200	2600	124800	1449970	1394
17	46.25	0.06	0.062	6200	-20000	-925000	526364	506
18	48.13	0.19	0.183	18300	12100	582373	1109243	1067
19	46.63	0.00	0.007	700	-17600	-820688	289622	278
20	48.12	0.00	0.000	0	-700	-33684	**256217**	246

- Again, the previous table was taken from Hull's textbook (see Table 8.3)
- In the second case, the stock price at maturity is $S < K$, hence, the option is not exercised

- A standard hedge would be holding 100,000 shares at the beginning
- Nonetheless, using dynamic hedging such position changes over time
- Using hedging on a dynamic basis,

Cash Inflow from Selling Calls	\$306K
Cash Inflow from Delivering Stock	\$0.00
Cash Outflow	-\$256K
Final P&L	\$50K

- The above table states that the average cost is $\$256K/100,000 = \$2.56$
  - which is approximately the price of the call option

Delta Hedging - Relation to Black-Scholes

- The key economic concept behind the BSM, is that the investor can create a risk-free portfolio using options
- In other words, use Δ_C stock shares to create a risk-free portfolio

$$V = P - \Delta_C \times S \tag{16}$$

- This implies that

$$dV = rVdt \tag{17}$$

- In the BSM class, we covered the statistical interpretation of the model
- The economic solution is the same, however, is derived by solving a PDE
- Nevertheless, the main take is that option's price, C, should be reflected by the average cost needed for hedging

Delta Hedging - Relation to Black-Scholes II

- The costs of hedging the option in the previous example are close to, but not exactly the same as that of, the BSM

$$(263,261 \times e^{-0.05\times\frac{20}{52}})/100,000 = 2.58$$
$$(256,217 \times e^{-0.05\times\frac{20}{52}})/100,000 = 2.51$$

- If the hedging scheme worked perfectly, the cost of hedging would, after discounting, be exactly equal to the BSM price for every simulated stock price path.
- The reason for the variation in the cost of delta hedging is that the hedge is rebalanced only once a week.
- As rebalancing takes place more frequently, the variation in the cost of hedging is reduced.
- Of course, these examples are idealized in that they assume the model underlying the BSM is exactly correct and there are no transactions costs.

Delta Hedging - Multiple Simulations

- Let's repeat the same example as before but using 1000 random price paths
- At each point, we incur cost of hedging
- Nonetheless, since the prices are simulated from GBM, we should get a consistent result

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Cost/Contract	0.78	2.13	2.41	2.41	2.66	4.61

- On average, we get 2.41, which closely approximates the true answer
- We also note that there are cases in which the hedging cost could be high, resulting in a negative P&L

```
> dt <- 1/52
> dS <- 10^-7
> P_sim <- function(S,K,r,sig,tau) {
+   R_t <-  rnorm(20,dt*(r - 0.5*sig^2),sig*sqrt(dt) )
+   S_t <- S*exp(cumsum(R_t))
+   return(c(S,S_t))
+ }
> hedge_cost <- function(S_seq) {
+   P_seq <- P_seq2 <-numeric()
+   for (i in 1:length(S_seq) ) {
+     p <- S_seq[i]
+     T_end <- tau - (i-1)*dt
+     P_seq <- c(P_seq,bs_f(p,K,r,sig,T_end) )
+     P_seq2 <- c(P_seq2,bs_f(p+dS,K,r,sig,T_end) )
+     }
+   Delta <- (P_seq2 - P_seq)/dS
+
+   ds <- data.frame(Week = 0:20, Stock_Price =  S_seq, C_Price = P_seq , Delta = Delta  )
+   ds$Shares_Needed <- round(ds$Delta,3)*10^5
+   ds$Shares_Purchased <-ds$Shares_Needed
+   ds$Shares_Purchased[2:nrow(ds)] <- ds$Shares_Purchased[2:nrow(ds)]  -  ds$Shares_Purchased[1:(nrow(ds)-1)]
+   ds$Cost_Shares <- ds$Shares_Purchased * ds$Stock_Price
+
+   ds$Cum_Outflow <- ds$Cost_Shares
+   ds$Interest <- ds$Cost_Shares[1]*r*dt
+
+   for(i in 2:nrow(ds)) {
+     ds$Cum_Outflow[i] <-   ds$Cum_Outflow[i-1] +  ds$Cost_Shares[i] + ds$Interest[i-1]
+     ds$Interest[i] <- ds$Cum_Outflow[i]*r*dt
+     }
+
+   hedge_cost <- ds$Cum_Outflow[nrow(ds)] - 100000*K*(ds$Stock_Price[nrow(ds)] > K)
+   hedge_cost <- exp(-r*tau)*hedge_cost/100000
+   return(hedge_cost)
+   }
```

Delta Hedging - Where the Cost Comes From?

- The previous examples demonstrate a case in which the trader is buying high and selling low

Transactions Costs

- Maintaining a delta-neutral position also incurs transactions costs
- Delta neutrality is more feasible for a large portfolio of derivatives dependent on a single asset
- The hedging transactions costs are absorbed by the profits on many different trades.
- There are economies of scale in trading derivatives.
  - It is not surprising that the derivatives market is dominated by a small number of large dealers.

Option Greeks - Call

- Under the BSM, recall that the call option is

$$C = S_t\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \tag{18}$$

- While the put price is given by Put-Call Parity, i.e.

$$P = C - \left[S_t - Ke^{-r\tau}\right] \tag{19}$$

- Given a pricing function, it is straightforward to investigate the price sensitivity to each underlying input

	Greek	Definition	Sign
Delta	Δ	$\frac{\partial C}{\partial S}$	?
Gamma	Γ	$\frac{\partial^2 C}{\partial S^2}$	?
Vega	$\mathcal{V}$	$\frac{\partial C}{\partial \sigma}$	?
Rho	ρ	$\frac{\partial C}{\partial r}$	?
Theta	Θ	$\frac{\partial C}{\partial t}$	?

Option Greeks - Call

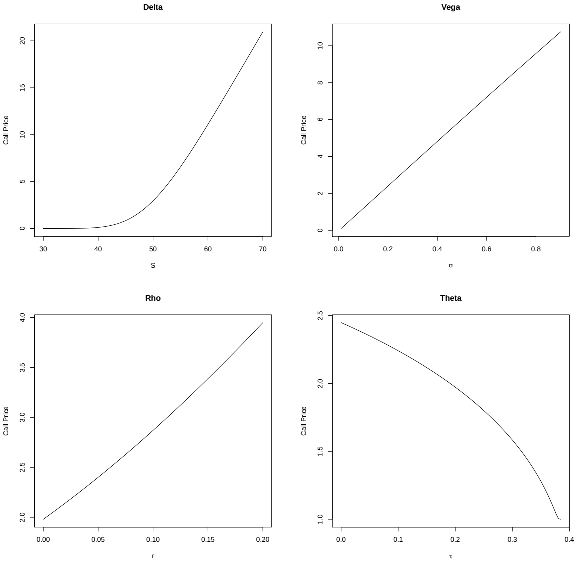


Table: Call Option Greeks

	Greek	Definition	Sign
Delta	Δ	$\frac{\partial C}{\partial S}$	$+$
Gamma	Γ	$\frac{\partial^2 C}{\partial S^2}$	$+$
Vega	$\mathcal{V}$	$\frac{\partial C}{\partial \sigma}$	$+$
Rho	ρ	$\frac{\partial C}{\partial r}$	$+$
Theta	Θ	$\frac{\partial C}{\partial t}$	-

- The call price is a convex increasing function of the spot price
- Larger volatility implies larger swings, which makes the options more likely to get exercised
- From risk-neutral probability, r denotes potential growth in the stock
  - higher return $\rightarrow$ higher likelihood of exercise
- Longer maturity also implies higher likelihood of exercise

Option Greeks - Put

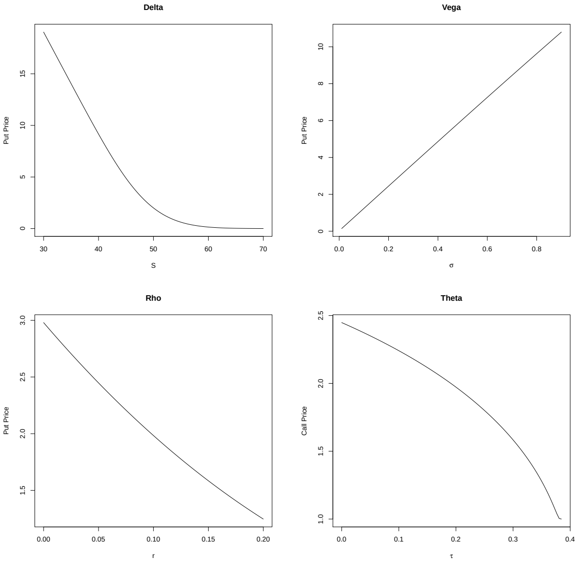


Table: Put Option Greeks

	Greek	Definition	Sign
Delta	Δ	$\frac{\partial P}{\partial S}$	-
Gamma	Γ	$\frac{\partial^2 P}{\partial S^2}$	$+$
Vega	$\mathcal{V}$	$\frac{\partial P}{\partial \sigma}$	$+$
Rho	ρ	$\frac{\partial P}{\partial r}$	-
Theta	Θ	$\frac{\partial P}{\partial t}$	-

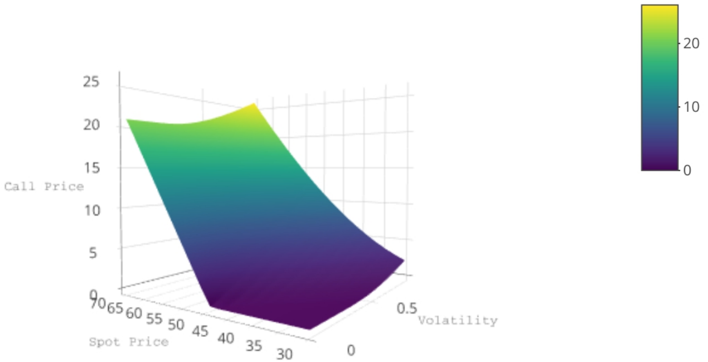
- The same logic for the call option holds true for the put
- A couple of comments are worth mentions
  - Note that $\Gamma > 0$ due to the convexity, while $\Delta < 0$
  - $\rho < 0$ as r increases, so does the stock price, reducing the likelihood of exercise

Vega Hedging

- Once a portfolio has been made delta neutral, the next stage is often to look at its gamma.
- The gamma of a portfolio is the rate of change of its delta with respect to the price of the underlying asset.
  - measures curvature
- Another important hedge statistic is vega.
- Gamma and vega can be changed by trading options on the underlying asset.
- In practice, derivatives traders usually rebalance their portfolios at least once a day to maintain delta neutrality.
- It is usually **not feasible** to maintain gamma and vega neutrality on a regular basis.
- Typically a trader monitors these measures. If they get too large, either corrective action is taken or trading is curtailed.

Vega Hedging II

Figure: Call Option Price as a function of spot price and volatility



- For an interactive visualization check this **link**

Vega and Delta Hedging

- Consider an option portfolio with a delta of 2,000 and vega of 60,000.[2]
- We need to make the portfolio both delta and vega neutral using:
  1. The underlying stock
  2. A traded option with delta 0.5 and vega 10.
- How would you achieve so?

[2] Example were taken from Prof. Liuren Wu's website (**link**)

Vega and Delta Hedging

- Consider an option portfolio with a delta of 2,000 and vega of 60,000.[2]
- We need to make the portfolio both delta and vega neutral using:
  1. The underlying stock
  2. A traded option with delta 0.5 and vega 10.
- How would you achieve so?

- It is easier to take care of the vega first using options
- After achieving so, one can achieve zero delta using the stock
  - Note that the stock has 1 delta and 0 vega
- Let N_1 and N_2 denote the number of options and shares, respectively
- The vega of the hedged position is given by

$$0 = 60,000 + N_1 \times 10 + N_2 \times 0 \rightarrow N_1 = -6,000 \tag{20}$$

- Whereas, its delta is given by

$$0 = 2,000 + (-6000) \times 0.5 + N_2 \times 1 \rightarrow N_2 = 1,000 \tag{21}$$

[2] Example were taken from Prof. Liuren Wu's website (**link**)

Vega and Delta Hedging - Generalization I

- Let Δ_O (Δ_S) denote the delta of the option (share), where Δ_P denoting the delta of the portfolio.
- Let N_1 and N_2 denotes the number of shares and options needed to long/short, respectively, in order hedge.
- A delta-neutral portfolio is achieved by

$$N_1 \times \Delta_S + N_2 \times \Delta_O + \Delta_P = 0 \tag{22}$$

- Let $\mathcal{V}$ refer to the vega, respectively
- It follows that the number of shares and options needed to achieve a zero-vega portfolio is

$$N_1 \times \mathcal{V}_S + N_2 \times \mathcal{V}_O + \mathcal{V}_p = 0 \tag{23}$$

Vega and Delta Hedging - Generalization II

- The information from the previous example tells us that

Notation	Value
Δ_S	1
Δ_O	0.5
Δ_p	2,000
$\mathcal{V}_S$	0
$\mathcal{V}_O$	10
$\mathcal{V}_p$	60,000
N_1	?
N_2	?

- We have two equations and need to find two unknowns, N_1 and N_2
- The problem can be presented as a system of linear equations in the follow manner:

$$\begin{bmatrix} \Delta_S & \Delta_O \\ \mathcal{V}_S & \mathcal{V}_O \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} -\Delta_p \\ -\mathcal{V}_p \end{bmatrix} \rightarrow \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \Delta_S & \Delta_O \\ \mathcal{V}_S & \mathcal{V}_O \end{bmatrix}^{-1} \begin{bmatrix} -\Delta_p \\ -\mathcal{V}_p \end{bmatrix} \tag{24}$$

Summary

- A trader working for a bank, who is responsible for all the trades involving a particular asset, monitors a number of Greek letters and ensures that they are kept within the limits specified by the bank.
- The delta, Δ, of a portfolio is the rate of change of its value with respect to the price of the underlying asset.
- Delta hedging involves creating a position with zero delta (sometimes referred to as a delta-neutral position).
- Because the delta of the underlying asset is 1.0, one way of hedging the portfolio is to take a position of $-\Delta$ in the underlying asset.
- For portfolios involving options and more complex derivatives, the position taken in the underlying asset has to be changed periodically.
  - This is known as rebalancing.