MA 574 - HW 9

① Let $f(t) = t(\sin(t^3))$

ⓐ We know that for $\sin x$ we already have a Taylor series, i.e.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \Rightarrow \text{centered at } x=0$$

so, similarily for $\sin(t^3)$, we can replace

$$f(t) = t\left[t^3 - \frac{(t^3)^3}{3!} + \frac{(t^3)^5}{5!} - \cdots\right]$$

$$f(t) = t\sum_{n=0}^{\infty} \frac{(-1)^n (t^3)^{2n+1}}{(2n+1)!}$$

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n+4}}{(2n+1)!}$$

$$\boxed{f(t) = t^4 - \frac{t^{10}}{6} + \frac{t^{16}}{120} - \cdots}$$

ⓑ $$\int_0^1 f(t)\,dt = \int_0^1 \left(t^4 - \frac{t^{10}}{6} + \frac{t^{16}}{120} - \cdots\right) dt$$

Considering the series up until 16 degree term as mentioned in the question

$$= \left[\frac{t^5}{5} - \frac{t^{11}}{66} + \frac{t^{17}}{17 \times 120}\right]_0^1 = \frac{1}{5} - \frac{1}{66} + \frac{1}{2040} \Rightarrow 0.1853$$

ⓒ $$\lim_{t\to 0} \frac{f(t)}{t^2} = \lim_{t\to 0} \frac{1}{t^2}\left[t^4 - \frac{t^{10}}{3!} + \frac{t^{16}}{5!} - \cdots\right]$$

$$= \lim_{t\to 0}\left[t^2 - \frac{t^8}{3!} + \frac{t^{14}}{5!} - \cdots\right] \Rightarrow 0$$

(2) Usually a calculator shows six-digit decimal pt, so we must have an accuracy of upto six digits.

Then,

Let us write the Taylor Polynomial for e^x then we can solve each subpart,

$$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \cdots$$

(a) The value of e^x at -1 is $e^{-1} = 0.367879$ and the value of e^x at 1 is $e = 2.718281$.

$$P(-1) = 1 - 1 + \frac{(-1)^2}{2} + \frac{(-1)^3}{6} + \frac{(-1)^4}{24} + \frac{(-1)^5}{120} + \frac{(-1)^6}{720} + \frac{(-1)^7}{5040} + \frac{(-1)^8}{40320} + \frac{(-1)^9}{362880}$$

$$\Rightarrow 0.367879$$

So, to obtain the value of e^{-1} exactly equal to the Taylor approximation, we have to use 9^{th} degree taylor polynomial.

Now, e^1

$$P(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320} + \frac{1}{362880}$$

$$\Rightarrow 2.718281$$

Hence, the answer is we have to use 9^{th} degree Taylor Polynomial.

(b) We know that $e^2 = 7.389056$

$$P(2) = 1 + 2 + \frac{(2)^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots$$

$$= 1 + 2 + \frac{4}{2} + \frac{8}{6} + \frac{16}{24} + \frac{32}{120} + \frac{64}{720} + \frac{128}{5040} + \frac{256}{40320}$$

$$+ \frac{512}{362880} + \frac{1024}{3628800} + \frac{2048}{39916800} + \frac{4096}{479001600} + \frac{8192}{6227020800}$$

$$+ \frac{16384}{87178291200} \Rightarrow 7.389056$$

Hence, we require a 14^{th} degree Taylor Polynomial.

(c) If the calculator programmed at a center value then it calculates only for that value. We normally take center at 0, so that we can compute different approximations but if we take a particular value for the derivatives will result as per the center value and we cannot compute other Taylor approximations using the same calculator unless it is reprogrammed.

(3) $f(x,y) = (x^2+y)e^{y/2}$

(A) For critical point, differentiate the above eqn w.r.t to x, y partially respectively and find the gradient vector.

$\nabla f = \langle f_x, f_y \rangle$

$f_x = 2xe^{y/2}$, $f_y = \frac{\partial}{\partial y}(x^2e^{y/2} + y.e^{y/2})$

$$= \frac{x^2}{2}e^{y/2} + e^{y/2} + \frac{y}{2}e^{y/2}$$

$$\Rightarrow e^{y/2}\left(1 + \frac{1}{2}(x^2+y)\right)$$

For critical point, $\nabla f = 0$

or $f_x = f_y = 0$, then

$f_x = 0$ & $\left(\frac{x^2+y}{2} + 1\right)e^{y/2} = 0$

$2xe^{y/2} = 0$

$\Rightarrow x = 0$, as $e^{y/2} \neq 0$. $\frac{y}{2} + 1 = 0 \Rightarrow y = -2$

Hence, critical pt is $(0,-2)$

(B) To find 2nd Taylor Polynomial centered at $x=c$

We know that Taylor series for multivariate is given by

~~$f(x,y) \approx f(c) + f_x(c)(x-c) + f_{xx}$~~

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) +$$
$$\frac{1}{2!}\left(f_{xx}(a,b)(x-a)^2 + f_{yy}(a,b)(y-b)^2 + 2f_{xy}(a,b)(x-a)(y-b)\right) + \ldots$$

Hence, we need to find at $(0,-2)$

$$f(0,-2) = (0^2 + (-2))e^{-2/2} \Rightarrow -2 \times e^{-1} \Rightarrow -0.7358$$

$$f_{xx} = 2e^{y/2} \xrightarrow{(0,-2)} 2 \cdot e^{-2/2} = 0.7358$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\left(\frac{x^2+y}{2}+1\right)e^{y/2}\right) = \frac{1}{2}e^{y/2} + \frac{1}{2}e^{y/2}\left(\frac{x^2+y}{2}+1\right) \Rightarrow \frac{(x^2+y)}{4}e^{y/2} + e^{y/2}$$

$$\xrightarrow{(0,-2)} \frac{(0^2+(-2))}{4}e^{-1} + e^{-1} \Rightarrow -\frac{1}{2}e^{-1} + e^{-1} \Rightarrow \frac{e^{-1}}{2} \Rightarrow 0.1839$$

$$f_{xy} = xe^{y/2} \xrightarrow{(0,-2)} 0 \cdot e^{-1} \Rightarrow \cancel{0.3679} \Rightarrow 0$$

$$f_x = 2xe^{y/2} \xrightarrow{(0,-2)} 2(0) \cdot e^{-1} \Rightarrow 0$$

$$f_y = \left(\frac{x^2+y}{2}+1\right)e^{y/2} \xrightarrow{(0,-2)} \left(\frac{0^2+(-2)}{2}+1\right)e^{-1} = 0$$

Putting values in the function,

$$f(x,y) = (-0.7358) + 0(x-0) + 0(y+2)$$
$$+ \frac{1}{2}\left(0.7358(x-0)^2 + 0.1839(y+2)^2 + 2(0)(x-0)(y+2)\right)$$

$$\boxed{f(x,y) \Rightarrow (-0.7358) + 0.3679x^2 + 0.09195(y+2)^2 + \ldots}$$

(C) Hessian matrix,

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2e^{y/2} & xe^{y/2} \\ xe^{y/2} & \left(\frac{x^2+y}{4}+1\right)e^{y/2} \end{bmatrix}$$

At critical point $(0,-2)$,

$$H = \begin{bmatrix} 2e^{-1} & 0 \\ 0 & \frac{e^{-1}}{2} \end{bmatrix}, \quad |H| = \det(H) \Rightarrow e^{-2} > 0$$

always greater than 0

and $f_{xx}(0,-2) = 2e^{-1} > 0$

$f_{yy}(0,-2) = e^{-1}/2 > 0$

Hence, c is a point of local minimum.

(D) p_2 will be

$f_{xx} = 2e^{y/2}$

$f_{yy} = \left(\frac{x^2+y}{4}+1\right)e^{y/2}$

$f_{xxx} = 0$, $f_{yyy} = \frac{1}{4}e^{y/2} + \frac{1}{2}e^{y/2}\left(\frac{x^2+y}{4}+1\right)$

$= \left(\frac{x^2+y}{8} + \frac{3}{4}\right)e^{y/2}$

$f_{yyy}(0,-2) = \left(\frac{0^2+(-2)}{8} + \frac{3}{4}\right)e^{-1} = \left(\frac{-1}{4}+\frac{3}{4}\right)e^{-1} = \frac{e^{-1}}{2}$

As, $f_{yyy} \neq 0$. Hence, it is not a critical point for p_2.

MA_574_Python_Exercise_10

December 1, 2023

[1]:

```
import numpy as np
import matplotlib.pyplot as plt

plt.style.use('seaborn-poster')
```

```
C:\Users\sid23\AppData\Local\Temp\ipykernel_4844\1421859367.py:4:
MatplotlibDeprecationWarning: The seaborn styles shipped by Matplotlib are
deprecated since 3.6, as they no longer correspond to the styles shipped by
seaborn. However, they will remain available as 'seaborn-v0_8-<style>'.
Alternatively, directly use the seaborn API instead.
  plt.style.use('seaborn-poster')
```

TD - 1 :

Recall that the Taylor series of $\sin x$ centered at $x = 0$ is

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

Implement this equation (*in one line*) below. [Hint : Use np.math.factorial() for the factorial function.]

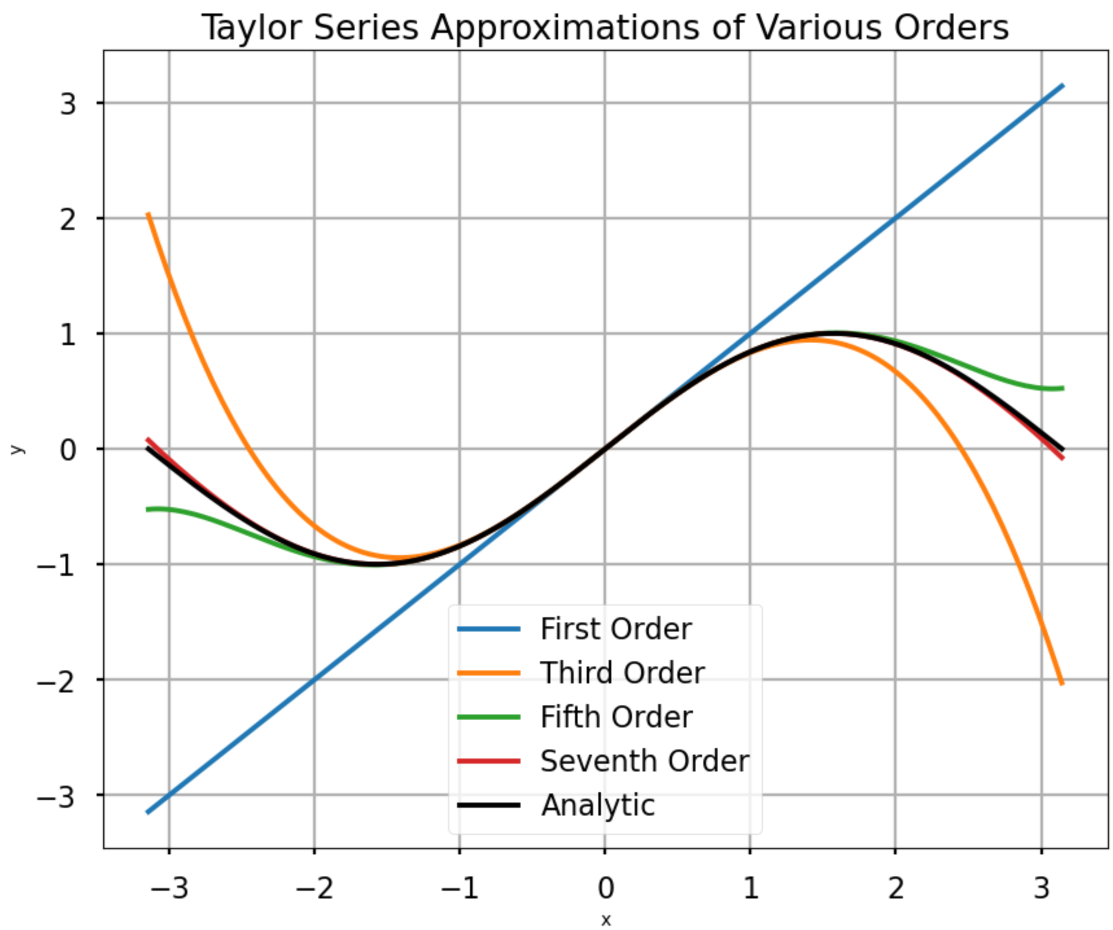
[4]:

```
x = np.linspace(-np.pi, np.pi, 200)
y = np.zeros(len(x))

labels = ['First Order', 'Third Order', 'Fifth Order', 'Seventh Order']

plt.figure(figsize = (10,8))
for n, label in zip(range(4), labels):
    y = y + ((-1) ** n) * (x ** (2 * n + 1)) / np.math.factorial(2 * n + 1)
    plt.plot(x,y, label = label)

plt.plot(x, np.sin(x), 'k', label = 'Analytic')
plt.grid()
plt.title('Taylor Series Approximations of Various Orders')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()
```



Now test the value of the Taylor approximation of $\sin x$ at $x = 0$ and $x = \frac{\pi}{2}$:

[8]:
```
x = np.pi/2
y = 0

for n in range(4):
    y = y + (((-1) ** n) * (x ** (2 * n + 1)) / np.math.factorial(2 * n + 1))

print(y)
```

```
0.9998431013994987
```

[9]:
```
x = 0
y = 0

for n in range(4):
    y = y + (((-1) ** n) * (x ** (2 * n + 1)) / np.math.factorial(2 * n + 1))

print(y)
```

```
0.0
```

[ ]:

```

```