Homework - 10.

Q1 let y be i.i.d uniform $(0, \theta_1)$
let z be i.i.d uniform (θ_1, θ_2)

pdf of y will be.

$$f(y) = \frac{1}{\theta_1 - 0} \Rightarrow \frac{1}{\theta_1}$$

$$E(y) = \int_{-\infty}^{\infty} y \cdot f(y)\, dy$$

$$= \int_{0}^{\theta_1} y \cdot \frac{1}{\theta_1}\, dy. \Rightarrow \frac{1}{2\theta_1}\left[y^2\right]_0^{\theta_1}$$

$$E(y) = \frac{\theta_1^{\cancel{2}}}{2\cancel{\theta_1}} = \frac{\theta_1}{2}$$

According to the property, mean is the first moment of the distribution.

$$E(y) = \bar{y}$$

$$\therefore \bar{y} = \frac{\hat{\theta_1}}{2}.$$

$$\boxed{2\bar{y} = \hat{\theta_1}} \leftarrow$$ 1st moment estimator for θ_1

pdf of z.

$$f(z) = \frac{1}{\theta_2 - \theta_1}$$

$$E(z) = \int_{-\infty}^{\infty} z \cdot f(z)\, dz$$

$$= \int_{\theta_1}^{\theta_2} z \cdot \left(\frac{1}{\theta_2 - \theta_1}\right) dz.$$

$$= \frac{1}{\theta_2 - \theta_1} \left[\frac{z^2}{2} \right]_{\theta_1}^{\theta_2} \qquad = \frac{1}{2(\theta_2 - \theta_1)} \left[\theta_2^2 - \theta_1^2\right]$$

$$= \frac{\cancel{(\theta_2 - \theta_1)}\ (\theta_2 + \theta_1)}{2\ \cancel{(\theta_2 - \theta_1)}}$$

$$= \frac{\theta_2 + \theta_1}{2}$$

since $\theta_1 = 2\bar{y}$

$$E(z) = \frac{\theta_2 + 2\bar{y}}{2}$$

$$\because \quad \bar{z} = E(z)$$

$$2\bar{z} = \hat{\theta}_2 + 2\bar{y}$$

$$\hat{\theta}_2 = 2\bar{z} - 2\bar{y}$$

$$\boxed{\hat{\theta}_2 = 2(\bar{z} - \bar{y})} \leftarrow$$

1st moment estimator for θ_2

Q2. Rayleigh distribution

$$f(x) = \left(\frac{x}{\sigma^2}\right) e^{\left(\frac{-x^2}{2\sigma^2}\right)} \qquad \text{find MLE of } \sigma^2$$

Taking product on both sides.

$$\Pi f(x) = \Pi \left[\left(\frac{x}{\sigma^2}\right) e^{\left(\frac{-x^2}{2\sigma^2}\right)}\right]$$

$$= \frac{\Sigma x}{(\sigma^2)^n} \, e^{\frac{-1}{2\sigma^2} \Sigma x^2}$$

Taking log on both the sides

$$\ln(L) = \ln\left(\frac{\Sigma x}{(\sigma^2)^n} \cdot e^{\frac{-1}{2\sigma^2} \Sigma x^2}\right)$$

$$= \ln\left(\frac{\Sigma x}{(\sigma^2)^n}\right) + \cancel{\ln}\left(\cancel{e}^{\frac{-1}{2\sigma^2} \cdot \Sigma x^2}\right)$$

$$= \ln(\Sigma x) - \ln((\sigma^2)^n) + \left(\frac{-1}{2\sigma^2} \cdot \Sigma x^2\right)$$

Taking partial derivative w.r.t σ^2

$$= \frac{\partial \ln(L)}{\partial \sigma^2} = \frac{\partial}{\partial \sigma^2}\left(\ln(\Sigma x) - n \ln \sigma^2 + \left(\frac{-1}{2\sigma^2} \cdot \Sigma x^2\right)\right)$$

$$= 0 - n \times \frac{1}{\sigma^2} + \frac{\Sigma x^2}{2}\left(\frac{1}{\sigma^4}\right)$$

let $\frac{\partial}{\partial \sigma^2} \ln(L) = 0.$

$$\therefore \quad 0 = -\frac{n}{\sigma^2} + \frac{\Sigma x^2}{2\sigma^4}$$

$$\frac{n}{\cancel{\sigma^2}} = \frac{\Sigma x^2}{2\sigma^{\cancel{4}}}$$

$$\boxed{\hat{\sigma}^2 = \frac{\Sigma x^2}{2n}}$$