Homework 3

Problem 1

Consider the problem

$$\begin{aligned} \text{minimize} \;\; & f(x) \\ \text{s.t.} \;\; & x \in \mathcal{X} \end{aligned}$$

where $\mathcal{X} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 1\}$ and $f(x) = x_2$.

a. Find all point(s) satisfying a first order necessary optimality condition. Recall we studied two: FD-FONC and FONC.

b. Which of the point(s) in part a satisfy a second order necessary optimality condition?

c. Which of the point(s) in part a are local minimizers?

Problem 2

Consider the problem

$$\begin{aligned} \text{minimize} \;\; & f(x) = -x_2^2 \\ \text{s.t.} \;\; & |x_2| \leq x_1^2 \\ & x_1 \geq 0 \end{aligned}$$

where $x_1, x_2 \in \mathbb{R}$.

a. Does the point $(x_1, x_2)^\top = (0, 0)$ satisfy the FD-FONC? b. Is the point $(x_1, x_2)^\top = (0, 0)$ a local minimizer, a strict local minimizer, a local maximizer, a strict local maximizer, or none of the above?

Problem 3

Consider the problem

$$\begin{aligned} \text{minimize} \;\; & f(x) \\ \text{s.t.} \;\; & x \in \mathcal{X} \end{aligned}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x) = 3x_1$ with $x = (x_1, x_2)^\top$, and $\mathcal{X} = \{x = (x_1, x_2)^\top : x_1 + x_2^2 \geq 2\}$. Answer each of the following questions, showing complete justification.

a. Does the point $x^* = (2, 0)^\top$ satisfy the FD-FONC?

b. Does the point $x^* = (2, 0)^\top$ satisfy the FD-SONC?

c. Is the point $x^* = (2, 0)^\top$ a local minimizer?

Hint: Draw a picture with the constraint set and level sets of f.

Problem 4

Let $f : \mathbb{R}^2 \to \mathbb{R}$. Consider the problem

$$\begin{aligned} \text{minimize} \;\; & f(x) \\ \text{s.t.} \;\; & x_1, x_2 \geq 0 \end{aligned}$$

where $x = (x_1, x_2)^\top$. Suppose that $\nabla f(0) \neq 0$, and

$$\frac{\partial f}{\partial x_1}(0) \leq 0, \quad \frac{\partial f}{\partial x_2}(0) \leq 0$$

Show that 0 cannot be a minimizer for this problem.

Problem 5

Let $\mathcal{X} = \{x \in \mathbb{R}^n : Ax = b\}$. Show that $d \in \mathbb{R}^n$ is a feasible direction at $x \in \mathcal{X}$ iff $Ad = 0$.

Problem 6

Consider the problem

$$\begin{aligned} \text{maximize} \;\; & c_1 x_1 + c_2 x_2 \\ \text{s.t.} \;\; & x_1 + x_2 \leq 1 \, . \\ & x_1, x_2 \geq 0 \end{aligned}$$

where c_1 and c_2 are constants such that $c_1 > c_2 \geq 0$. This is a linear programming problem. Assuming that the problem has an optimal feasible solution, use the FD-FONC to show that the unique optimal feasible solution x^* is $(1, 0)^\top$.

Problem 7

Let $c \in \mathbb{R}^n, c \neq 0$, and consider the problem of minimizing the function $f(x) = c^\top x$ over a set $\mathcal{X} \subseteq \mathbb{R}^n$. Show that we cannot have a solution in the interior of $\mathcal{X}$.