* Homework 1

1

→ To find tangent line at point $(1,1)$ to curve $x^2y + xy^2 = 2e^{x-y}$

- To find tangent line we must find slope at that point of the curve; i.e., differentiating the curve

$\therefore\ x^2y + xy^2 = 2e^{(x-y)}$

differentiating

$$\frac{dy}{dx}\left[x^2y + xy^2\right] = \frac{dy}{dx}\left[2e^{(x-y)}\right]$$

$$\therefore\ \underset{1}{\frac{dy}{dx}[x^2y]} + \underset{2}{\frac{dy}{dx}[xy^2]} = \underset{3}{\frac{dy}{dx}\left[2e^{(x-y)}\right]}$$

Solving part 1 by product Rule method

$$y\frac{dy}{dx}(x^2) + x^2\frac{dy}{dx}$$

$$2xy + x^2\frac{dy}{dx}$$

Solving part 2 by Product Rule method

$$x\,2y\frac{dy}{dx} + y^2$$

$$2xy\frac{dy}{dx} + y^2$$

Solving 3 part

$$\cancel{2e^{(x-y)}}\ \frac{dy}{\cancel{dx}}\quad 2e^{(x-y)}\left(1-\frac{dy}{dx}\right)$$

$$\therefore\ 2xy + x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} + y^2 = 2\cdot e^{(x-y)}\cdot\left(1-\frac{dy}{dx}\right)$$

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} + 2xy + y^2 = 2e^{(x-y)} - 2e^{(x-y)}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(x^2 + 2xy + 2e^{(x-y)}\right) = 2e^{(x-y)} - 2xy - y^2$$

$$\therefore\ \frac{dy}{dx} = \frac{2e^{(x-y)} - 2xy - y^2}{x^2 + 2xy + 2e^{(x-y)}}$$

Now substituting (x, y) as $(1, 1)$

$$\therefore\ \frac{dy}{dx} = \frac{2e^{(1-1)} - 2(1)(1) - (1)^2}{(1)^2 + 2(1)(1) + 2e^{(1-1)}}$$

$$\frac{dy}{dx} = \frac{\cancel{2} - \cancel{2} - 1}{1+2+2} = \frac{-1}{5}$$

Now, to find tangent at that point we will use this derivative

$$\therefore\ \frac{-1}{5} = \frac{(y-1)}{(x-1)}$$

$$\Rightarrow\ -x+1 = 5y-5$$

$$\frac{6-x}{5} = y$$

$\therefore$ Tangent equation $\boxed{y = \frac{6-x}{5}}$

$\frac{B}{\sqrt{LB}} \times \frac{\sqrt{LB}}{\sqrt{LB}}$ $B\frac{\sqrt{LB}}{LB}$ $\frac{\sqrt{LB}}{L}$

(2)

→ Given depth = $\sqrt{\text{length} \times \text{breath/width}}$

volumn = length × breath × height

To find the rate of change of the length of the prism with respect to its width / breath

That is $\frac{dl}{dw}$

$D = \sqrt{l \times w}$

$V = l \times b \times D$ also $V = 2(lb + lD + b \times D)$

$\therefore L \times B \times D = 2LB + 2LD + 2BD$

Now puting $D = \sqrt{LB}$

$\therefore LB\sqrt{LB} = 2LB + 2L(\sqrt{LB}) + 2B(\sqrt{LB})$

÷ by $2\sqrt{LB}$

$\frac{LB}{2} = \frac{2B}{\sqrt{LB}} + L + B$

$\frac{LB}{2} = \sqrt{LB} + L + B$

Now differentiating $\frac{dL}{dB}$

$\frac{1}{2}\left[B \cdot \frac{dL}{dB} + L(1)\right] = \frac{1}{2\sqrt{LB}}\left[L + B\frac{dL}{dB}\right] + 1 + \frac{dL}{dB}$

$\therefore B \cdot \frac{dL}{dB} + L = \frac{L}{\sqrt{LB}} + \frac{B}{\sqrt{LB}}\frac{dL}{dB} + 2 + \frac{2dL}{dB}$

$\cancel{B\frac{dL}{dB} - \frac{B}{\sqrt{LB}}\frac{dL}{dB} - \frac{2dL}{dB}}$

$\cancel{B}\,\frac{\sqrt{LB}}{L}$ $\qquad$ $\frac{\cancel{L}\sqrt{LB}}{\cancel{L}B}$

(2) $$B\cdot\frac{dL}{dB} - \frac{B}{\sqrt{LB}}\frac{dL}{dB} - 2\frac{dL}{dB} = \frac{L}{\sqrt{LB}} + 2$$

$$\frac{dL}{dB}\left[B - \frac{\sqrt{LB}}{L} - 2\right] = \frac{\sqrt{LB}}{B} + 2$$

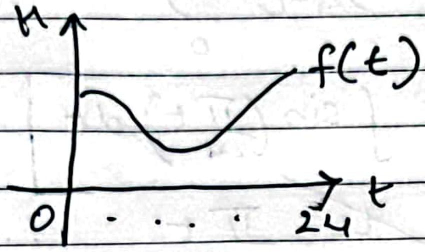
$$\frac{dL}{dB} = \frac{\sqrt{LB}+2B/B}{LB-\sqrt{LB}-2L/L}$$

FOR EDUCATIONAL USE

③ → Input variable $t \in [0, 24]$
$H(t)$ measures temperature in Fahrenheit at time t

Taking graph as example

value of $H = f(t)$



① Asked to find the average temperature over the course of the day

∴ using integration to find average

$$\text{average Temp} = \frac{1}{24-0}\int_0^{24} f(t)\,dt$$

Ⓐ Now, we are asked to find average temperature for the data set $D = \{(t_0, H_0) \ldots (t_N, H_N)\}$
& intervals are of 15 minutes
∴ Our Range will be from $[0, 96]$
here $N = 96$

So, Now the average temperature will be

$$\text{average Temp} = \frac{1}{96-0}\int_0^{96} f(t)\,dt$$

(c) Now, we are given the function $H(t) = 32 + 18\sin\left(\frac{\pi}{24}t\right)$

So, simply applying integration to get avg temp

$\therefore$ Average temperature $= \frac{1}{24-0}\int_0^{24} 32 + 18\sin\left(\frac{\pi}{24}t\right)dt$

$\therefore \frac{1}{24}\left[32t + 18\int \sin\left(\frac{\pi}{24}t\right)dt\right]$

$\therefore$ substituting $u = \frac{\pi}{24}t$ $\qquad \therefore du = \frac{\pi\, dt}{24}$

$\therefore \int \frac{24}{\pi}\sin(u)\,du$ $\qquad dt = \frac{24\,du}{\pi}$

$\therefore -\frac{24}{\pi}\cos u$

$= -\frac{24}{\pi}\cos\left(\frac{\pi t}{24}\right)$

$\therefore \frac{1}{24}\left[32t + 18\left[-\frac{24}{\pi}\cos\left(\frac{\pi t}{24}\right)\right]\right]_0^{24}$

$\therefore \left[\frac{2}{3}t - \frac{18}{\pi}\cos\left(\pi\frac{t}{24}\right)\right]_0^{24}$

for $t=0 \Rightarrow \left[0 - \frac{18}{\pi}\cos(0)\right] \Rightarrow -\frac{18}{\pi}$

for $t=24 \Rightarrow \left[16 - \frac{18}{\pi}(-1)\right] \Rightarrow 16 + \frac{18}{\pi}$

$\therefore t(24) - t(0) = 16 + \frac{18}{\pi} + \frac{18}{\pi} = 16 + \frac{36}{\pi}$

$\therefore$ $\boxed{\text{Average Temperature} = 16 + \frac{36}{\pi} \; + C}$

FOR EDUCATIONAL USE

(3) (B) $\left[\int_0^{24} (32 + 18\sin(\pi/24\, t)\,dt\right] \frac{1}{24}$

$\therefore \left[32t + 18 - \cos(\pi/24\, t)\right]_0^{24} \frac{1}{24}$

= ~~32t~~ putting upper & lower limit

$$\boxed{32 + \frac{36}{\pi}}$$

(B) function is evenly spaced $f: R \to R$ in interval $[a, b]$

As we know

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \sum \frac{f\,\Delta x}{b-a}$$

Now, $\Delta x = \frac{b-a}{n}$

If $n \to \infty$; then it will make Δx very small

Sundaram FOR EDUCATIONAL USE

(4) We have to calculate the Variance

$$V(x) = \int_{-\infty}^{\infty} (x-\lambda^{-1})^2 \, p(x)\, dx$$

& $p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Now when Integrating $V(x)$, we have to split it in interval $(-\infty, 0)$ & $[0, +\infty)$

$$\therefore \int_{-\infty}^{0} (x-\lambda^{-1})^2 p(x)\,dx + \int_{0}^{\infty} (x-\lambda^{-1})^2 p(x)\,dx$$

Now, $p(x) = 0$ when $x < 0$ $\therefore$ first side is 0

$$\therefore \int_{0}^{\infty} (x-\lambda^{-1})^2 \cdot (\lambda e^{-\lambda x})$$

$$\int_{0}^{\infty} (x^2 + \lambda^{-2} - 2x\lambda^{-1})(\lambda e^{-\lambda x})$$

$$\int_{0}^{\infty} x^2 \cdot \lambda e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda} - 2x e^{-\lambda x}$$

Now, we have 3 integration to solve

$$\underset{1}{\int x^2 \cdot \lambda e^{-\lambda x}} + \underset{2}{\int \frac{e^{-\lambda x}}{\lambda}} - \underset{3}{\int 2x e^{-\lambda x}}$$

① $\int x^2 \lambda e^{-\lambda x} dx \rightarrow \lambda \int x^2 e^{-\lambda x} dx$

Integrating by parts

$\lambda \left[x^2 \int e^{-\lambda x} - \int \left[\frac{d}{dx} x^2 \int e^{-\lambda x} dx \right] dx \right]$

$\lambda \left[x^2 (\cancel{\mp} e^{-\lambda x}) + \int 2x (\lambda e^{-\lambda x}) dx \right.$

$\lambda \left[-\lambda x^2 e^{-\lambda x} + 2\lambda \int x \cancel{\lambda} (e^{-\lambda x}) dx \right]$

Again Inte. by parts

$x \int e^{-\lambda x} - \int (1 \cdot \int e^{-\lambda x} dx) dx$

$-x \lambda e^{-\lambda x} - \int \cancel{e} - \lambda e^{-\lambda x} dx$

$-x \lambda e^{-\lambda x} \cancel{+} \lambda^2 e^{-\lambda x}$

$\therefore \lambda \left[-\lambda x^2 e^{-\lambda x} + 2 \left[-x \lambda e^{-\lambda x} - \lambda^2 e^{-\lambda x} \right] \right.$

$-\lambda^2 x^2 e^{-\lambda x} \cancel{+} - 2x \lambda e^{-\lambda x} - 2\lambda^2 e^{-\lambda x}$

② $\int \frac{e^{-\lambda x}}{\lambda} dx = -\frac{\cancel{\lambda} e^{-\lambda x}}{\cancel{\lambda}} = -e^{-\lambda x}$

③ $\int 2x e^{-\lambda x} dx$

$2 \left[x \int e^{-\lambda x} - \int (1 \cdot \int e^{-\lambda x} dx) dx \right]$

$= 2 \left[-x \lambda e^{-\lambda x} - \lambda^2 e^{-\lambda x} \right]$

$\therefore \; -\lambda^2 x^2 e^{-\lambda x} - \cancel{2x\lambda e^{-\lambda x}} - \cancel{2\lambda^2 e^{-\lambda x}} - e^{-\lambda x} \qquad \left(-\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

$- \cancel{e^{-\lambda x}} + \cancel{2x\lambda e^{-\lambda x}} + \cancel{2\lambda^2 e^{-\lambda x}}$

$\therefore \; -\lambda^2 x^2 e^{-\lambda x} - e^{-\lambda x}$

$\therefore \; e^{-\lambda x}\left(-\lambda^2 x^2\right)$

$\therefore$ Value of Variance

$\left[e^{-\lambda x}\left(-\lambda^2 x^2\right)\right]_{-\lambda^{-1}}^{\infty}$

limit of integration is $x \neq 0 \to \infty$, but parts; $-\lambda^{-1} \to \infty$

since; $e^{\infty} = 0 \to$ for $(\infty) = 0$

for $(-\lambda^{-1}) = e^{+\cancel{\lambda}\times+\cancel{\lambda^{-1}}} \cdot \left(-\lambda^2 (-\lambda^{-1})^2\right)$

$= e \cdot \left(-\cancel{\lambda^2} \times \frac{1}{\cancel{\lambda^2}}\right)$

$= -e$