Lesson 2 Board-work

Plan for Lesson 2

- Using various rules for derivatives.
- Implicit differentiation.
- Inverse functions and their derivatives.
- Numerical Differentiation (Python File).

Q 1. Find the derivative of the function $f(x) = \left(\dfrac{x+5}{x-1}\right)^x$.

Q 2. The surge function $f(t) = \alpha t e^{-t}$, where $t \geq 0$ and $\alpha \geq 0$ is a constant, can be used to model a variety of phenomena, such as the amount of a drug in a patient's bloodstream after the drug is ingested, electrical surges, or the number of sales of an item following a successful advertising campaign. In each case, there is a quantity that initially "surges", then gradually decays.

A. For which values of t does f increase ("surge")? For which values of t does f decay?

B. Can you find the maximum value of the surge function? If so, how, and what is it?

Q 3. Application to Data. Suppose you are presented with a large data set $S = \{(x_1,y_1),(x_2,y_2),\ldots,(x_N,y_N)\}$, where $0 < x_{n+1} - x_n < \Delta x$ for some (small) increment $\Delta x > 0$ when $1 \leq n < N$.

A. Given an index n such that $1 \leq n < N$, explain how the data set S can be used to estimate the value of a derivative at x_n.

B. Explain another way in which the data set can be used to estimate the value of a derivative at x_n. Can you think of a third or even fourth way?

C. Explain when and how the data set S can be used to estimate the value of a second derivative at x_n.

Python Resources available on Canvas for numerical differentiation.

Inverse functions: A discussion

Q 4. Find the inverse of the hyperbolic tangent function $g(x) = \operatorname{artanh}(x)$.

Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'(f^{-1}(x)) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x)) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Q 5. If a function $f(x)$ is invertible and differentiable for all x, does it follow from the inverse function theorem that $f^{-1}(x)$ is differentiable for all x? Explain.

Q 6. What is the derivative of the inverse of a linear function?

Q 7. Find and compare the derivatives of $\ln(x)$, $\ln(2x)$, and $\ln(3x)$ using the inverse function theorem. How can you make sense of your answers?

Q 8. Consider the logistic function $g(z) = \dfrac{1}{1+e^{-z}}$ to answer the following.

(a). Find the derivative of the logistic function. Prove that $g' = g(1-g)$.

(b). Find its inverse (called logit or log-odd function) and the derivative of the inverse. Discuss the domain and range of the logit function.

Q 9. For each of the following equations, use implicit differentiation to find $\frac{dy}{dx}$ (which you're free to denote y' if you prefer).

A. $x^2 - y^2 = 4$

B. $x^2 = y - 7$

C. $e^{x/y} = x$

D. $y^3 - \ln(x^2 y) = 1$

Q 10. A closed rectangular box with a square base has a side length of x cm and a height of y cm . Its surface area is 1000 cm^2.

A. Express the above information using a single equation.

B. What is the rate of change of the height of the box with respect to the length of its base when the height is 20 cm?

C. Why does it even make sense to talk about the rate of change of the height of the box with respect to the length of its base?

Survey 2 (Sept 8):

Q 11. Find the slope of the tangent line to the curve $2x^2 - y^3 = x^3y$ at $(x, y) = (1,1)$.

Indicate your level of understanding of the following by putting a number from 1 to 5.

- How comfortable you feel about finding implicit differentiation.
  [ ]

- How comfortably can you work with the inverse of a function.
  [ ]