MA 574
Fall 2023
Exam 1
10/24/2023
Time Limit: 85 Minutes

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- This exam contains 7 pages (including this cover page) and 4 questions. Make sure you have every page!
- Make sure to show as much work as you can. There is no such thing as explaining too much! You also increase your chances for getting partial credit by including more explanation.
- There will be no calculators allowed during the exam.
- There will be no collaboration permitted during the exam. Any questions must be presented to me directly.

Question	Points	Score
1	23	
2	15	
3	20	
4	20	
Total:	67	

1. **(12 points)** Determine whether each of the following statements are True or False. **You must include a short explanation for each answer.**

   (a) **(3 points)** True or False: If the **coefficient matrix** of a linear system has at least one row of zeros, then the system must either have no solution or infinitely many solutions.

FALSE This is not always true as it is possible for a system with a zero row to have a unique solution.

   (b) **(3 points)** True or False: If $f : \mathbb{R} \to \mathbb{R}$ is invertible and differentiable, and its inverse $f^{-1}(x)$ is **NOT** differentiable at $x = x_0$, then the tangent line to the curve $y = f(x)$ at the point $x = f^{-1}(x_0)$ is given by $y = c$ for some constant c.

TRUE. If a function shows severe bend, or cusp, at point then tangent turns into vertical line & inverse is not differentiable

   (c) **(3 points)** True or False: The line $\mathbf{r}(t) = \langle 1 - t, 3t, 2 + 2t \rangle$ intersects the plane $2x - 2y + 4z = 10$ at a single point.

FALSE

$1 - t = 2$ $\quad t = -1$

$3t = -2$ $\quad t = -2/3$

$2 + 2t = 4$ $\quad 2t = 2,\ t = 1$

Hence more than one point.

   (d) **(3 points)** True or False: If $f(x, y)$ is a strictly positive function, where each level curve $f(x, y) = c$ is a circle of radius $r = -\ln(c)$ centered at a fixed point (x_0, y_0), then the partial derivatives of f are always negative.

FALSE

Partial derivative can be positive, as $r = -\ln(c)$ which is to be positive.

& $f(x, y)$ is a strictly positive function.

2. (15 points) Select all correct answer(s) in each of the following problems.

(a) (3 points) The trajectory of the vector-valued function $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2) \rangle$ most closely resembles what?

- A circle
- A parabola
- A graph of the tangent function
- A line

x comp : sin t^2
y comp : cos t^2
Both periodic functions & squared inside the argument the trajectory resembles that of a circle.

(b) (3 points) Let A be an invertible 3×3 matrix, and assume that $A^{-1} = A^T$. Select all (if any) of the following statements that are also true.

- $A = A^{-1}$
- Any row echelon form of A has a pivot in every column.
- The determinant of A is equal to either 1 or -1
- The unique solution to $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the first row of A

since matrix is invertible as ∴ Identity matrix as row-echelon.

This is resembling the 1^{st} row. as both other rows are zero.

$\det|A^{-1}| = \det|A^T|$

$\det|A^{-1}| = \frac{1}{\det|A|}$

$\therefore \det|A^T| = \frac{1}{\det|A|}$

(c) (3 points) Let R denote the bounded region between the curves $y = \sqrt{x}$ and $y = \frac{x}{3}$. If $f(x, y)$ is some continuous function, which of the following integrals compute $\iint_R f(x, y)dA$?

- $\int_0^9 \int_{\frac{x}{3}}^{\sqrt{x}} f \, dy \, dx$
- $\int_0^9 \int_{\sqrt{x}}^{\frac{x}{3}} f \, dy \, dx$
- $\int_0^3 \int_{3y}^{y^2} f \, dx \, dy$
- $\int_0^3 \int_{y^2}^{3y} f \, dx \, dy$

T1 $f(x,y)$ $0 < x < 9$
$\frac{x}{3} < y < x$

T2 $f(x,y)$ $0 < y < 3$
$y^2 < x < 3y$

(d) (3 points) Let $A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix}$. Which of the following correctly compute $\det(A)$?

- $\begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -1 \\ 1 & 0 \end{vmatrix}$
- ✓ $2\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$ $2[0] + [2-1] = 1$
- ✓ $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$ $[2-1] = 1$
- ✓ $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix}$ $1[-1] - 2[-1]$

(e) Which (if any) among the following expressions is a representation of the line in $\mathbb{R}^3$ that has direction vector $\mathbf{v} = \langle -2, 1, 5 \rangle$ and passes through the point $(0, 3, -2)$?

- $\mathbf{s}(t) = \langle -2, 1 + 3t, 5 - 2t \rangle$
- ✓ $\mathbf{r}(t) = \langle -2t, 3 + t, -2 + 5t \rangle$
- ✓ $x(t) = -2\sqrt[3]{t}, 3 + \sqrt[3]{t}, -2 + 5\sqrt[3]{t}$
- $-2x = 3 + y = -2 + 5z$

(ii) $\langle -2t, 3+t, -2+5t \rangle$

$rt = r \cdot t \cdot v.$

$= (0, 3, -2) + (-2, 1, 5)$

$= (-2t, 3+t, -2+5t \rangle.$

(iii) $xt = -2\sqrt[3]{t}, 3+\sqrt[3]{t}, -2+5\sqrt[3]{t}$

Instead of t, here $\sqrt[3]{t}$ has bee taken to represent line $\mathbb{R}^3$.

3. (20 points) Throughout this problem, let $f(x, y) = \ln(x + \ln(y))$.

(a) (5 points) What is the domain of f?

For domain of $F(x,y)$; $\left.\begin{matrix} x+\log y > 0 \\ y > 0 \end{matrix}\right\} \rightarrow$ Since log is defined for positive real numbers.

(b) (5 points) Compute $\dfrac{\partial f}{\partial y}$

$\dfrac{df}{dy} = \dfrac{d}{dx}\left[\ln(x+\ln(y))\right] = \dfrac{1}{(x+\ln(y))} \cdot \dfrac{d}{dy}\left(\ln(y)+x\right) = \dfrac{1}{[x+\ln(y)]} \cdot \dfrac{d}{dy}\left[\ln(y)\right] + \dfrac{d}{dy}(x)\Big] = \dfrac{1}{y} \cdot \dfrac{1}{[x+\ln(y)]}$

$\therefore \boxed{\dfrac{df}{dy} = \dfrac{1}{y(x+\ln(y))}}$

(c) (5 points) **Using implicit differentiation**, compute the slope of the tangent line to the level curve $f(x, y) = 1$ at the point $(e, 1)$.

$F(x,y) = 1$.

differentiating w.r.t x we get

$\therefore \dfrac{d}{dx}[F(x,y)] = 0$.

$\therefore \dfrac{1}{x+\log(y)} \cdot \left[1 + \dfrac{1}{y}\dfrac{dy}{dx}\right] = 0$.

Now substituting the point $(e,1)$, we get.

$\dfrac{1}{e+\log(1)} \cdot \left[1 + \dfrac{dy}{dx}\right] = 0$.

$\therefore \dfrac{1}{e}\left[1 + \dfrac{dy}{dx}\right] = 0$. $\therefore$

$\therefore 1 + \dfrac{dy}{dx} = 0$

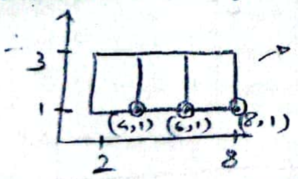
$\therefore \boxed{\dfrac{dy}{dx} = -1}$

$\therefore$ Slope of tangent line to level curve at point $(e,1)$ is -1,;

(d) (5 points) Approximate the volume under the surface defined by f, but above the rectangle $[2, 8] \times [1, 3]$, using Riemann sums. You should break the region into 3 smaller squares and use the lower right corners of each square. Your answer may be left as a sum. $\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\, \Delta A$.

Above rectangle $[2,8] \times [1,3]$

Break the region is 3 smaller squares.



$\rightarrow$ consider lower right corner

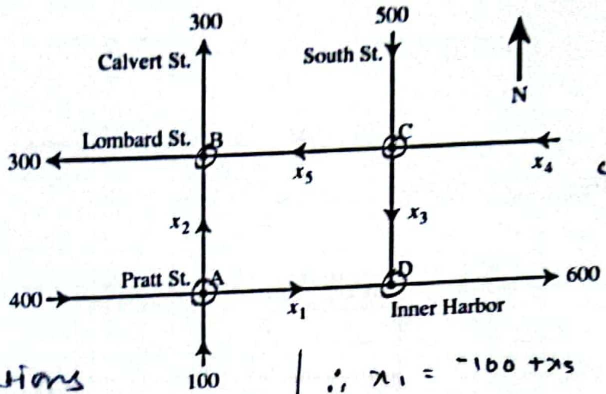
For finding volume under surface we can get the Reimann sum of height multiplied by Area.

$\therefore \Delta x = \dfrac{6}{3} = 2$, $\Delta y = \dfrac{2}{1} = 2$ $\therefore \Delta A = 2 \cdot 2 = 4$;

Volume $= F(x_1, y_1) \cdot \Delta A_1 + F(x_2, y_2) \cdot \Delta A_2 + F(x_3, y_3) \cdot \Delta A_3$

$= F(4,1) \cdot 4 + F(6,1) \cdot 4 + F(8,1) \cdot 4$

(c) (10 points) Pictured below is a network that displays the traffic flow of every street in a city (measured in cars/hour). Notice that every street pictured only has traffic flowing in one direction. Also notice that a number of the streets have undetermined traffic flows indicated by variables x_1, x_2, x_3, x_4, and x_5. Being that this is a "real world" network, you know that at every intersection the amount of traffic flow into the intersection is equal to the amount of traffic flow out. Moreover, you know that the total amount of traffic flowing into the network must equal the amount flowing out. Putting all of this together, write down a system of equations that models this situtation, and find all possible values of x_1, x_2, x_3, x_4 and x_5 giving solutions to the system.



At Intersections Traffic flow in = Traffic flow out. Using this into we can create a system of equation from the diagram.

$x_1 + x_2 = 500$ - (1)

$x_2 + x_5 = 600$ - (2)

$x_3 + x_5 = 500 + x_4$

$\therefore x_3 + x_5 - x_4 = 500$ - (3)

$x_1 + x_3 = 600$ - (4)

From the system of equations we can obtain a sol^n by solving $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \quad \therefore b = \begin{bmatrix} 500 \\ 600 \\ 500 \\ 600 \end{bmatrix}$$

$$\therefore \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 500 \\ 0 & 1 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & -1 & 1 & 500 \\ 1 & 0 & 1 & 0 & 0 & 600 \end{array}\right] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$\therefore R_4 \to R_4 - R_1$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 500 \\ 0 & 1 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & -1 & 1 & 500 \\ 0 & -1 & 1 & 0 & 0 & 100 \end{array}\right]$$

$R_4 \to R_4 + R_2$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 500 \\ 0 & 1 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & -1 & 1 & 500 \\ 0 & 0 & 1 & 0 & 0 & 700 \end{array}\right]$$

$R_4 \to R_4 - R_3$

$R_3 + R_4 \to R_3$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 500 \\ 0 & 1 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 1 & 700 \\ 0 & 0 & 0 & 1 & 0 & 200 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 1 & 700 \\ 0 & 0 & 0 & 1 & 0 & 200 \end{array}\right]$$

$\therefore x_1 - x_5 = -100$

$x_2 + x_5 = 600$

$x_3 + x_5 = 700$

$x_4 = 200$

$x_5 = x_5$

$\therefore x_5$ is a free variable

$\therefore x_1 = -100 + x_5$

$x_2 = 600 - x_5$

$x_3 = 700 - x_5$

$x_4 = 200$

$x_5 = x_5$.

$\therefore$ General Solution of system

$$X = \begin{bmatrix} -100 + x_5 \\ 600 - x_5 \\ 700 - x_5 \\ 200 \\ x_5 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} -100 \\ 600 \\ 700 \\ 200 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} x_5$$