PAGE NO.:
DATE: / /

HW-5:

Q1. A) $S = \{(x,y,z) \in R^3 \mid x = y\}$

- The zero vector $(0,0,0)$ is in subset because $0=0$.
- It is closed under vector addition: If (a,a,z) and (b,b,z) are in S, then their sum $(a+b, a+b, z)$ is also in S

- It is closed under scalar multiplication: If (x, x, z) is in S, then for any scalar c, (cx, cx, cz) is also in S.

$\therefore$ S is vector subspace of R^3

B) $T = \{(x,y,z) \in R^3 \mid y = 1\}$

- The zero vector $(0,1,0)$ is in subset.
- It is closed under vector addition: If $(a,1,z)$, $(b,1,w)$ are in T, then their sum $(a+b, 1, z+w)$ is also in T.
- It is closed under scalar multiplication: If $(x,1,z)$ is in T, then for any scalar c, $(cx, 1, cz)$ is also in T.

$\therefore$ T is vector subspace of R^3.

c) $U = \{(x,y,z) \in R^3 \mid xyz = 0\}$

- The zero vector $(0,0,0)$ in this subset.
- It is closed under vector addition: If (a,b,w) and (c,d,z) are in U, then their sum $(a+c, b+d, w+z)$ is also in U, because

$(a+c)(b+d)(w+z) = (abw + cdz + adw + bcz) = 0.$

- It is closed under scalar multiplication:
If (x,y,z) is in U, because, then for any scalar c, (cx, cy, cz) is also in U, because $(cx)(cy)(cz)$ $= c^3xyz = 0.$
$\therefore$ U is vector subspace of R^3

D) $V = \{(x,y,z) \in R^3 \mid x+y+z=0\}$

- The zero vector $(0,0,0)$ is in this subset because $0+0+0=0$.

- It is closed under vector addition: If (a,b,c) and (d,e,f) are in V, then their sum $(a+d, b+e, c+f)$ is also in V because $(a+d)+(b+e)+(c+f) =$ $(a+b+c) + (d+e+f) = 0.$

- It is closed under scalar multiplication:
If (x,y,z) is in V, then for any scalar c, (cx, cy, cz) is also in V, because ~~(a+d)+(b+e)+(c+f)~~
$(cx) + (cy) + (cz) = c(x+y+z) = c(0) = 0.$
$\therefore$ V is vector subspace of R^3.

Q2. $A = \begin{pmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & \alpha & 2 & 2 \\ 0 & 0 & 0 & \beta & 2 \end{pmatrix}$

- First row is linearly independent

- For 2^{nd} row to be linearly independent, α should be non-zero because it contains non-zero entries.

- For ~~3rd~~ 3^{rd} row to be linearly independent, β should be non-zero because it contains nonzero entries.

So, to have a rank of 2, α & β should both be non-zero. $\alpha \neq 0, \beta \neq 0$.

Q3 A) Matrix A :

Matrix A rotates clockwise by 30 degrees.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

For 30-degree clockwise rotation θ is 30°

$$A = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix}$$

- Column space of A has dimension 2 since both columns are linearly independent.
- Null space of A is 0.
- Row space of A has dimension 2 as its spanned by two rows.
- left null space of A is $\{0\}$.

B) Matrix B

- Matrix B projects elements of the plane onto the line $x+y=0$. To we can write B as the projection matrix onto the line. let v be the unit vector along the line $1/\sqrt{2}, -1/\sqrt{2}$

$$B = vv^T = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

- Column space of B is line $x+y=0$, so its dimension 1
- Null space of B is orthogonal complement of line which is all vector $\perp^{r}$ to line. Dimension = 1

- Row space of B is line $x+y=0$. So dimension
- Left null space of B is orthogonal complement of row space, which is set of all vectors perpendicular to line. It has dimension 1.

3) Matrix C.

- It shears the plane horizontally by factor of 0.5.

$$C = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}.$$

- Column space of C has dimension 2 since both columns are linearly independent.
- Null space is 0.
- Row space of C has dimension 2 as it is spanned by 2 rows.
- Left null space of C is 0.

4) ABC = A·B·C

$$X = \begin{pmatrix} \frac{\sqrt{3}}{2} & -1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}-1}{4} & \frac{-\sqrt{3}+1}{4} \\ \frac{-1+\sqrt{3}}{4} & \frac{1-\sqrt{3}}{4} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}-1}{4} & \frac{3\sqrt{3}-3}{8} \\ \frac{-1+\sqrt{3}}{4} & \frac{-3+3\sqrt{3}}{8} \end{pmatrix}$$

$\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$

$\frac{-\sqrt{3}}{4} + \frac{1}{4}$

$\frac{\sqrt{3}-1}{8} + \frac{\sqrt{3}-1}{4}$

$\frac{\sqrt{3}-1+2\sqrt{3}-2}{8}$

$3\sqrt{3}-3$

$\frac{-1+\sqrt{3}}{8} - \frac{1+\sqrt{3}}{4}$

(d) Column space of X is 2 as both columns are linearly independent.
- Null space is 0.
- Row space has dimension 2 because it is spanned by 2 rows.
- Left null space is 0.

Q4. $A = \begin{bmatrix} 5 & 14 & 38 & 47 & 10 \\ 1 & 3 & 8 & 10 & 2 \\ 1 & 2 & 6 & 7 & 3 \\ 4 & 8 & 24 & 28 & 12 \end{bmatrix}$

Swap R_1 and R_2

$$\begin{bmatrix} 1 & 3 & 8 & 10 & 2 \\ 5 & 14 & 38 & 47 & 10 \\ 1 & 2 & 6 & 7 & 3 \\ 4 & 8 & 24 & 28 & 12 \end{bmatrix}$$

$R_2 \to R_2 R_1(5)$ | $R_3 \to R_3 - R_1$ | $R_4 \to$ ~~$R_1(4)$~~ $R_4 - 4(R_1)$

$$\begin{bmatrix} 1 & 3 & 8 & 10 & 2 \\ 0 & -1 & -2 & -3 & 0 \\ 0 & -1 & -2 & -3 & 1 \\ 0 & -4 & -8 & -12 & 4 \end{bmatrix}$$

$R_2 \Leftrightarrow R_2(-1)$

$$\begin{bmatrix} 1 & 3 & 8 & 10 & 2 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & -3 & 1 \\ 0 & -4 & -8 & -12 & 4 \end{bmatrix}$$

$R_3 \to R_3 + R_2 \mid R_1 \to R_1 - R_2(3) \mid R_4 \to R$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3 \mid R_4 \to R_4 - 4R_3$

$$= \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Dimension of column space of matrix = 1.
(rank of matrix)

Basis of H is 5 - as all columns are linearly independent = $\{v_1, v_2, v_3, v_4, v_5\}$.

Q5. $A = \begin{pmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 3 & 9 & 1 & -7 & 23 & 8 \\ 1 & 3 & 1 & -3 & 9 & 2 \\ 1 & 3 & -1 & -1 & 5 & 4 \end{pmatrix}$

$R_2 \to R_2 - R_1(3) \mid R_3 \to R_3 - R_1 \mid R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ | $R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

① Column space -
- Dimension of column space is 2, rank of A.

② Null space.
- Null space is set of solution to equation $Ax = 0$.

$$x_1 + 3x_2 - 2x_4 + 7x_5 + 3x_6 = 0.$$
$$x_3 - x_4 + 2x_5 + x_6 = 0$$

Solving system will yield vectors from null space. It has dimension 6 - Rank (A)
$6 - 2 = \underline{\underline{4}}$.

③ Row space.
- Row space is subspace panned by non-zero rows.
Rank of A is 2 which number of non-zero rows.

④ Left null space = No. of Rows in A^T - Rank of A
$= 6 - 2$
$= \underline{\underline{4}}$.