Midterm-1 574

1b) True

If the derivative of a function $f: R \to R$ is always positive, it means the function is continuously increasing which means for every y ~~for~~ in $f^{-1}(y)$ there is a unique $x \Rightarrow$ Hence inverse function exists

Also, $f(x)$ is continuous and $f'(x)$ is never 0 (since it's always positive) $\Rightarrow$ inverse function is differentiable

Hence, inverse function exists and is differentiable.

2b) If $A = A^{-1}$ ~~(orthogonal matrix)~~

i) This does NOT mean $A = A^T$ since the inverse property and the symmetry property are independent of each other

Example :- Let $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

This satisfies $A \cdot A = I$

$A \times A \Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow A = A^{-1}$

However, $A^T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \neq A$

Hence, these are two different properties not dependent on each other.

ii) $A^2 = I$

$\det A \cdot \det A = \det(I)$

$(\det A)^2 = 1$

$\det A = \pm 1$

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iii) A^{-1} exists means that the matrix is a square matrix

Also, $A = A^{-1}$

This shows that A has a unique solution which means that A has a pivot in every column

iv) Given,

$$Ax = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

To prove, solution to the above is the first ~~xo~~ column of A.

$$A^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = x$$

$$\Rightarrow [x] = A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (\text{since } A = A^{-1})$$

$\Rightarrow$ This shows solution is first column of A

we can also, take an example matrix (3×3)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\cancel{[x] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

$$A^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \cancel{A} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A^{-1}}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \rightarrow \text{first column of } A$$

2d) Given,

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

det A =

using 1^{st} row $\Rightarrow$ $1\left(\det\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\right) + 1\left(\det\begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix}\right)$

using second column,

$$\det A = -1\left(\det\begin{pmatrix}1&1\\0&0\end{pmatrix}\right) + 1\left(\det\begin{pmatrix}1&1\\2&1\end{pmatrix}\right)$$

$$= -\det\begin{pmatrix}1&1\\0&0\end{pmatrix} + \det\begin{pmatrix}1&1\\2&1\end{pmatrix}$$

using third row,

$$\det A = +1\left(\det\begin{pmatrix}1&1\\2&1\end{pmatrix}\right) = \det\begin{pmatrix}1&1\\2&1\end{pmatrix}$$

2e) Given $d = \langle 1,1,2\rangle$ is the direction vector

The line passes through $(1,0,0)$

$L = \langle 1,0,0\rangle + t\langle 1,1,2\rangle$

$= \langle 1+t, t, 2t\rangle$

$\hookrightarrow$ t can be anything

✓ $S(t) = \langle 1+t, t, 2t\rangle$ satisfies the condition

✓ $x(t) = 1+t^3,\ y(t) = t^3,\ z(t) = 2t^3$ is in the form

$\langle x,y,z\rangle\ \cancel{z(t)} = \langle 1+k, k, 2k\rangle$ Satisfies the condition

✗ $r(t) = \langle 1+\cos t, \cos t, 2\cos t\rangle$

While this passes through $(1,0,0)$ and has the direction $\langle 1,1,2\rangle$, $0\leq \cos t \leq 1$ limits the line between $(1,0,0)$ and $(2,1,2)$ hence it is a line segment but not the line $\langle 1+t, t, 2t\rangle$

✗ $x+1 = y = \frac{z}{2}$ is not the equation since $(1,0,0)$ is not a points on the above equation.