Homework - 6

1. A. Matrix Norms

Frobenius Norm $\|A\|_F$: The Frobenius norm of a matrix is the square root of the sum of squares of its elements.

$$A = \begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 3 & 9 & 1 & -7 & 23 & 8 \\ 1 & 3 & 1 & -3 & 9 & 2 \\ 1 & 3 & -1 & -1 & 5 & 4 \end{bmatrix}$$

$$\|A\|_F \Rightarrow \sqrt{(1)^2+(3)^2+(0)^2+(-2)^2+(7)^2+(3)^2+(3)^2+(9)^2+(1)^2+(-7)^2 + (23)^2+(8)^2+(1)^2+(3)^2+(1)^2+(-3)^2+(9)^2+(2)^2+(1)^2+ (3)^2+(-1)^2+(-1)^2+(5)^2+(4)^2}$$

$$= \sqrt{957}$$

$$= 31.03224$$

Infinity Norm $\|A\|_\infty$: The infinity norm of a matrix is the maximum absolute row sum.

row 1 $\Rightarrow 1+3+0+2+7+3 = 16$

row 2 $\Rightarrow 3+9+1+7+23+8 = 51$

row 3 $\Rightarrow 1+3+1+3+9+2 = 19$

row 4 $\Rightarrow 1+3+1+1+5+4 = 14$

$\|A\|_\infty = 51$

one Norm $\|A\|_1$: The one norm of a matrix is the maximum absolute column sum.

$C_1 = 1+3+1+1 = 6$

$C_2 = 3+9+3+3 = 18$

$C_3 = 0+1+1+1 = 3$

$C_4 = 2+7+3+1 = 13$

$C_5 = 7+23+9+5 = 44$

$C_6 = 3+8+2+4 = 17$

$\|A\|_1 = 44$

B. Spectral Norm $\|A\|_2$: The spectral norm is the square root of the largest eigenvalue of A^TA or AA^T

$$A^T = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 3 & 9 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ -2 & -7 & -3 & -1 \\ 7 & 23 & 9 & 5 \\ 3 & 8 & 2 & 4 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 3 & 9 & 1 & -7 & 23 & 8 \\ 1 & 3 & 1 & -3 & 9 & 2 \\ 1 & 3 & -1 & -1 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 3 & 9 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ -2 & -7 & -3 & -1 \\ 7 & 23 & 9 & 5 \\ 3 & 8 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 91 & 174 & 13 & -27 \\ 174 & 504 & 31 & -63 \\ 13 & 31 & 19 & -13 \\ -27 & -63 & -13 & 86 \end{bmatrix}$$

$$\|A\|_2 = \sqrt{930}$$

$$= 30.4959$$

c. Cosine of the Angle between the First and Last row vectors: To find the cosine of the angle between two vectors, we can use the dot product formula and the norms of the vectors.

cosine of the Angle between the first and last row vectors

$= 0.95509$

2.

A. The orthogonal projection of a vector x onto a subspace U can be found using the formula

$$P_{x \text{ onto } U} = \frac{x \cdot u_1}{|u_1|^2} \cdot u_1 + \frac{x \cdot u_2}{|u_2|^2} \cdot u_2$$

$u_1 = (1, -2, 2, 0)^T$

$u_2 = (-1, 1, 1, -1)^T$

$x = (1, 2, 3, 4)^T$

$$= \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}, 0\right) + (0, 0, 0, 0)$$

$$= \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, 0\right)$$

B. The projection matrix P that performs the projection onto the subspace U can be calculated as

$$P = U\cdot(U^T\cdot U)^{-1}\cdot U^T$$

where U is a matrix whose columns are the basis vectors of U.

$$U = \begin{bmatrix} 0.314 & -0.4 & -0.0571 & 0.2285 \\ -0.4 & 0.6 & -0.2 & -0.2 \\ -0.057 & -0.2 & 0.8285 & -0.3142 \\ 0.228 & -0.2 & -0.3142 & 0.2571 \end{bmatrix}$$

C. The distance of a vector x from the subspace U can be found using the formula

$$\text{Distance} = \|x - (\text{projection of } x \text{ onto } U)\|$$

$$= \|(1,2,3,4)^T - (\tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{2}{3}, 0)\|$$

Distance of x from subspace U: 5.3851

3.

A. The projection matrix P that projects elements onto the line spanned by n can be found using

$$P = \frac{nn^T}{n^Tn}$$

$n = (1,2,3)$

$$n = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$P = \frac{1}{1^2+2^2+3^2}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

B. The rank of the projection matrix P is 1. This is because P is a rank-1 matrix, which means that its column space has dimension 1. In this case, P projects vectors onto a 1-D subspace, so its rank is 1.

C. To find the projection of b onto L using the projection matrix P, we can use the formula

projection of b onto $L = Pb$

$$b = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

$$P = \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$P_{b\,onto\,L} = \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 26 \\ 52 \\ 78 \end{bmatrix} = \begin{bmatrix} \frac{13}{7} \\ 2 \\ 6 \end{bmatrix}$$

D. The plane V that is orthogonal to n can be defined by the equation

$$n \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

now,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + 2y + 3z = 0$$

The equation that defines the plane V is $x + 2y + 3z = 0$

E. The projection matrix Q that projects elements onto the plane V can be found as the identity matrix minus the projection matrix P

$$Q = I - P$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q = I - P$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{14} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 13/14 & 2 & 3 \\ 2 & 5/7 & 6 \\ 3 & 6 & 5/14 \end{bmatrix}$$

F. The rank of Q is 2. This is because Q is the identity matrix minus a rank 1 matrix P. When we subtract a rank1 matrix from an identity matrix. The resulting matrix has full rank, which means that its column space has dimension 3. Since Q is projecting vectors onto a 2-D subspace, its rank is 2.

G. $P_{b \text{ onto } V} = Qb$

$$b = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 13/14 & 2 & 3 \\ 2 & 5/7 & 6 \\ 3 & 6 & 5/14 \end{bmatrix}$$

$$P_{b \text{ onto } V} = \begin{bmatrix} 13/14 & 2 & 3 \\ 2 & 5/7 & 6 \\ 3 & 6 & 5/14 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 429/14 \\ 39 \\ \frac{409}{7} \end{bmatrix}$$

4. b is not equal to the sum of the answers to C and G. The projection of b onto L and the Projection of b onto V are orthogonal to each other. when adding them, and taking a vector in one subspace and adding it to a vector in a different subspace and the result will not be equal to the original vector b.