Homework 1

Important submission instructions:

- All explanations and answers must be clearly and neatly written. Explain each step in your solution. Your solutions should make very clear to the instructor that you understand all of the steps and the logic behind the steps.
- You are allowed to discuss the homework problems with other students (in particular via Canvas discussion board). However, you are not allowed to copy solutions from other students or other sources including AI chatbots. You should list at the end of the problem set the sources you consulted and people you worked with on this assignment including AI prompts.
- The final document should be saved and submitted as a single .pdf file, and please be sure all problem solutions are presented starting from the first to the last (that is, the first solution must correspond to problem 1 and the second to problem 2 and so on).
- Typed submissions (for example in LaTeX) will be positively considered in the grade. Overleaf is an easy avenue to start learning LaTeX. See the tutorials at https://www.overleaf.com/learn
- Honor code applies fully. Your submission should reflect your own understanding of the material. It is prohibited to post the following problems on any website/forum or any other online means.

Problem 1

Let $L \in \mathbb{R}^{n\times m}$ and $A = LL^\top$.

a Prove that A is positive semidefinite.

b Prove that A is positive definite iff L has full row rank.

Problem 2

Show that:

a If Q is positive definite, then the diagonal elements are positive.

b Let Q be a symmetric matrix. If there exist positive and negative elements in the diagonal, then Q is indefinite.

Problem 3

i) Write a program in MATLAB/Octave/Scilab or Python (any other language please contact me) that randomly generates a positive definite matrix.

ii) In order to check your previous code, write a function that implements Sylvester's criterion and use this function to check every matrix created in i).

Problem 4

Consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

When $a+b = c+d$, show that the vector $(1,1)^\top$ is an eigenvector of A and calculate both eigenvalues of A in terms of a, b, c, d.

Problem 5

Compute the gradient and the Hessian of:

a $f(x) = e^{\|x\|_2^2}$, with $x \in \mathbb{R}^n$ and $\|x\|_2^2 = \sum_j x_j^2$.

b $g(x) = \prod_{j=1}^n x_j$.

Problem 6

Compute

$$\lim_{(x,y)\to(0,0)} \frac{xy - \sin(x)\sin(y)}{x^2 + y^2}$$

using Taylor's Theorem.

Problem 7

Develop a quadratic approximation for the following functions:

(a) $g : \mathbb{R}^3 \to \mathbb{R}$ defined as $g(x,y,z) = e^{xe^y} + e^{ze^y} + y^2(1 + x + z)$ around the point $(0,1,0)$.

(b) $h : \mathbb{R}^n \to \mathbb{R}$ defined as $h = \langle x, Ax\rangle + e^{\langle c,x\rangle}$ around the point $(1, \dots, 1)$; here A is an $n \times n$ symmetric matrix and $c \in \mathbb{R}^n$ is a given vector.

Problem 8

For each of the following sets:

$$\mathcal{A} = \bigcup_{n \geq 1} \left[\frac{1}{n}, \frac{n}{n+1} \right],$$
$$\mathcal{B} = \{ [x_1, x_2, x_3]^\top \in \mathbb{R}^3 : \ \max\{ \ |x_1|, |x_2|, |x_3| \} < 1 \},$$
$$\mathcal{C} = \{ [x_1, x_2]^\top \in \mathbb{R}^2 : \ 0 \leq x_1 < x_2 \},$$
$$\mathcal{D} = \{ [x_1, x_2, x_3]^\top \in \mathbb{R}^3 : \ |x_1| + |x_2| + |x_3| = 1, \ x_1, x_2, x_3 \geq 0 \},$$

answer the following

(a) Is the set closed, open, or neither?

(b) Describe the interior and the closure of the set.

(c) Is the set bounded and if so, provide a bound.