MA 540 Homework 8

29/11/23

Q 7.64.

Given that.

There are two types of bulb : 1 & 2

mean of b1 $= \mu_1$

mean of b2 $= \mu_2$.

SD of b1 $= \sigma_1$

SD of b2 $= \sigma_2$.

$$P_X(x) = \begin{cases} \mu_1 & (p) \\ \mu_2 & (1-p) \end{cases}$$

let x denote the lifetime of the bulb.

a) $E(X)$

→ $$E(X) = \sum_{X=1}^{2} x \cdot P_X(x)$$

$$E(X) = \mu_1 (p) + \mu_2 (1-p)$$ [Expected lifetime of the bulb]

b) $var(X)$

→ $$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{X=1}^{2} x^2 \cdot P_X(x)$$

$$= (\mu_1)^2 \cdot p + (\mu_2)^2 \cdot (1-p)$$

$$Var(X) = \mu_1^2 \cdot p + \mu_2^2 \cdot (1-p) - \left(\mu_1 (p) + \mu_2 (1-p)\right)^2$$

Q 7.65 Given that,

Poisson distribution : Good year mean = 3 (λ_1)
Bad year mean = 5 (λ_2)

P(Good year) = 0.4
P(Bad year) = 0.6

$E(x) = p_1 E(x|\text{good}) + p_2 E(x|\text{Bad})$

$= p_1\lambda_1 + p_2\lambda_2$ $\qquad \because$ mean, variance $\}\lambda$

$= (0.4)3 + (0.6)5$

$= 1.2 + 3$

$E(X) = 4.2$

$\therefore$ Expected num of storms next year is 4.2

$\text{var}(X) = E[\text{var}(X|Y) + \text{var}(E(X|Y))]$ (Total law of variance)

$= (p_1\lambda_1 + p_2\lambda_2) + [(p_1(\lambda_1 - E(\lambda))^2 + p_2(\lambda_2 - E(\lambda))^2]$

$= 4.2 + (0.41 \times 0.44 + 0.60 \times 0.64)$

$= 4.2 + (0.576 + 0.384)$

$\text{var}(X) = \underline{\underline{5.16.}}$

Q. 7.9 Given that,

$\mu = 40$
$\sigma = 6.$
$\rho = 0.6$

let x & y denote the successive weekly sales at a company.

(a) $P(x+y > 90)$

we know that, if the joint distribution follows bivariate normal distribution, then the variables individually follow univariate normal distribution.

So, $x \sim N(40, 6^2)$, $y \sim N(40, 6^2)$

$x+y \sim N(80, 115.2)$

$$E(x+y) = E(x) + E(y)$$
$$= 40 + 40$$
$$= 80$$

$$\text{cov}(x,y) = \rho \sigma_x \sigma_y.$$

$$\text{var}(x+y) = \text{var}(x) + \text{var}(y) + 2\,\text{cov}(x,y)$$
$$= 6^2 + 6^2 + 2(0.6) \times 6 \times 6$$
$$= \underline{115.2}$$

$$\therefore p(x+y > 90)$$
$$= P\left(\frac{x+y-80}{\sqrt{115.2}} > \frac{90-80}{\sqrt{115.2}}\right)$$
$$= P(z > 0.932)$$
$$= 1 - 0.824$$
$$= \underline{0.176.}$$

(c) If the correlation were 0.2 instead of 0.6 then the var $(x+y)$ would have changed.
thus $p(x+y)$ would have also changed.
$p(x+y)$ would have decreased to 0.2.

(d) $p = 0.2$

$\text{var}(x+y) = 86.4$

$x+y \sim N(80, 86.4)$

$$\text{var}(x+y) = \text{var}(x) + \text{var}(y) + 2\text{cov}(x,y)$$
$$= 6^2 + 6^2 + 2(0.2 \times 6 \times 6)$$
$$= 86.4$$

$$\therefore P(x+y > 90)$$
$$= P\left(\frac{x+y-80}{\sqrt{86.4}} > \frac{90-80}{\sqrt{86.4}}\right)$$
$$= P(Z > 1.07)$$
$$= \underline{\underline{0.141}}.$$

Theoretical.

Q.7.40 $\quad var(x) = E(x^2) - (E(x))^2$

The pmf of geometric RV is given by

$$P(x=k) = p(1-p)^{k-1}$$

$$E(x) = \frac{1}{p}$$

$$E(x^2) = \sum_{k=1}^{\infty} [k^2 \cdot p(X=k)]$$

$$E(x^2) = \sum_{k=1}^{\infty} [k^2 \cdot p(1-p)^{k-1}]$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$$E(x^2) = p\left(\frac{2-p}{(1-p)^2}\right)$$

$$var(x) = \frac{1}{p} - \left(p\left(\frac{2-p}{(1-p)^2}\right)\right)$$

$$\boxed{var(x) = \left(\frac{1-p}{p^2}\right)}$$

Q 7.51 Since $x_1, x_2, x_3 \ldots x_n$ are identically distributed exponential RV each having mean $1/\lambda$

The MGF of x_i is

$$M_{x_i}(t) = E(e^{tx_i}) = \left(1 - \frac{t}{\lambda}\right)^{-1} \text{ for } i = 1, 2, 3 \ldots n$$

The MGF of $y = \sum_{i=1}^{n} x_i$ is

$$M_y(t) = E(e^{ty})$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-n}$$

Sum of independent & identically distributed exponential RV follows gamma distribution

The pmf of gamma RV $= f_Y(y) = \dfrac{\lambda^n y^{n-1} e^{-\lambda y}}{\Gamma(n)}$ for $y > 0$.