MA540 Homework 5 19/10/23.

Q 5.7 $f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

Given $E[X] = \frac{3}{5}$

$$E(X) = \int_{-a}^{a} x \cdot f(x)\, dx. \quad \text{(formula of Expectation)}$$

$$\frac{3}{5} = \int_0^1 x\,(a + bx^2)\, dx$$

$$\frac{3}{5} = \int_0^1 ax + bx^3\, dx.$$

$$\frac{3}{5} = \left[\frac{ax^2}{2} + \frac{bx^4}{4}\right]_0^1$$

$$\frac{3}{5} = \left[\left(\frac{a}{2} + \frac{b}{4}\right) - 0\right] \quad \text{— (1).}$$

According to question, the density of X is given as

$$\int_{-a}^{a} f(x)\, dx = 1$$

$$\therefore \int_0^1 a + bx^2 = 1 \quad \Rightarrow \quad \left[ax + \frac{bx^3}{3}\right]_0^1$$

$$= \left[a + \frac{b}{3}\right] = 1 \quad \text{— (2).}$$

$$a = 1 - \frac{b}{3}$$

Substitute $a = 1 - \frac{b}{3}$ in eqⁿ (1).

$$\frac{3}{5} = \left(\frac{3-b}{6} + \frac{b}{4}\right)$$

$$\frac{3}{5} = \frac{12-4b+6b}{24} \Rightarrow \frac{6+b}{12}.$$

$$\frac{3b}{5} - 6 = b \Rightarrow \boxed{\frac{6}{5} = b.}$$

Substitute $b = \frac{6}{5}$ in eq^n 2.

$$a = 1 - \frac{b}{3}. \Rightarrow a = \frac{3-b}{3}.$$

$$a = \frac{3 - 6/5}{3} \Rightarrow a = \frac{15-6}{15}.$$

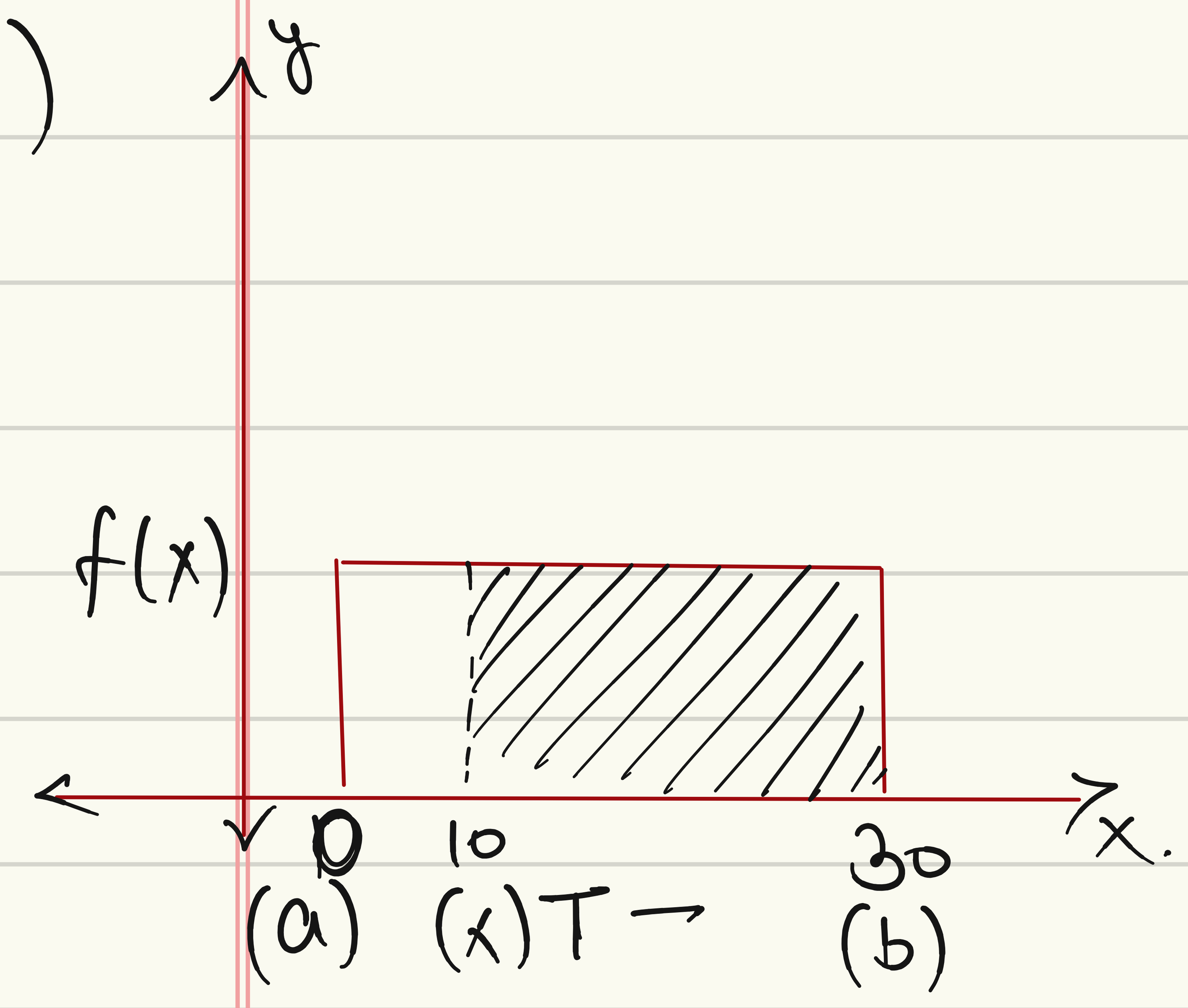
$$\boxed{a = \frac{3}{5}}$$

$$\therefore f(x) = \begin{cases} \frac{3}{5} + \frac{6}{5}x^2 & 0 \le x \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

Q5.13. Given that,

10 → 10:30 uniform distribution.

(A)



for uniform distribution, we can calculate the area of rectangle.

$\therefore A = l \times b. = P(X > 10).$

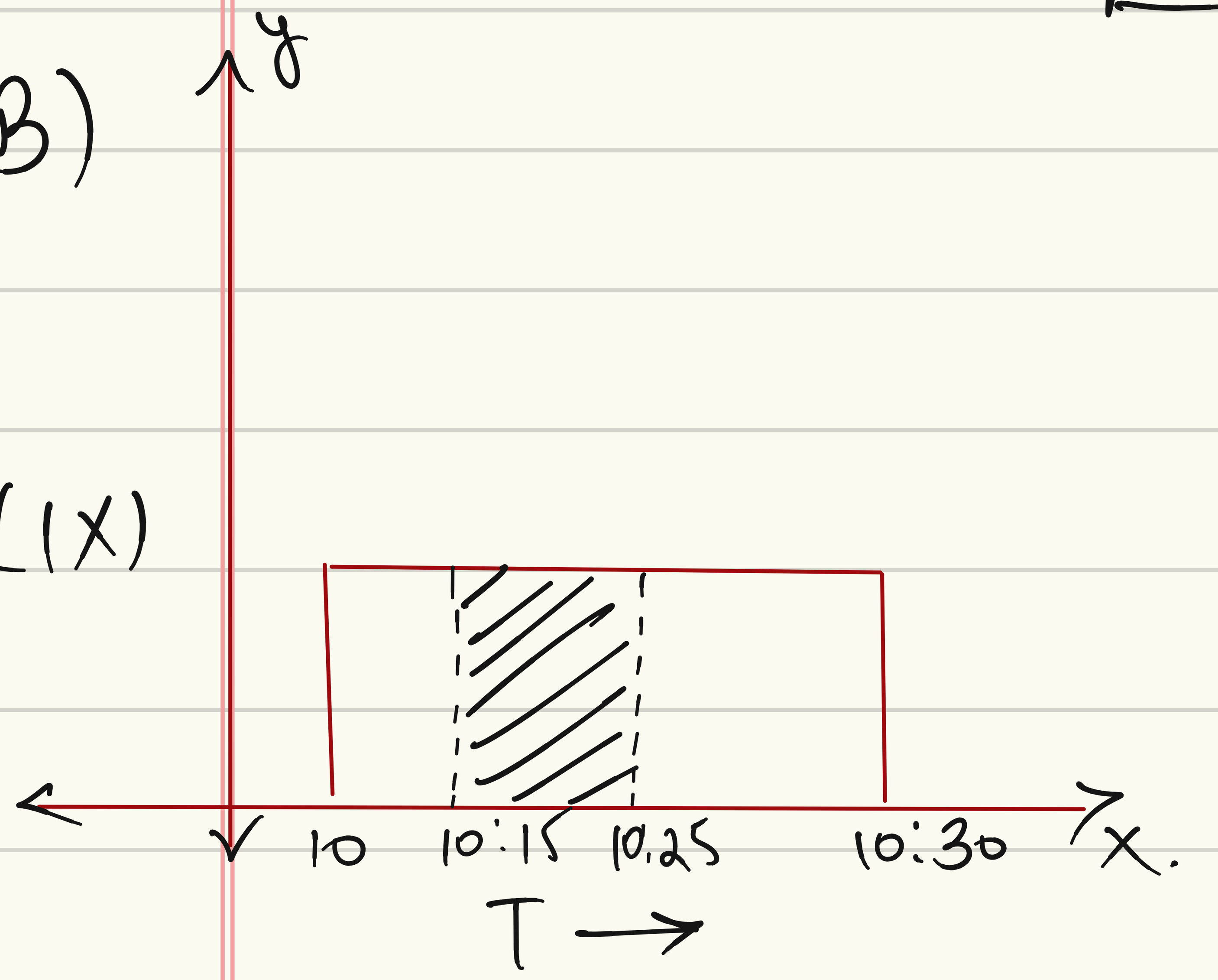
$A = 20 \times f(x).$

$f(x) = \frac{1}{b-a}. \Rightarrow \frac{1}{30}.$

$A = 20 \times \frac{1}{30}.$

$$\boxed{\therefore p(x > 10) = \frac{2}{3}}$$

(B)



let a be the event that you have to wait additional 10 mins & b be an event that bus has not arrived till 10:15 pm.

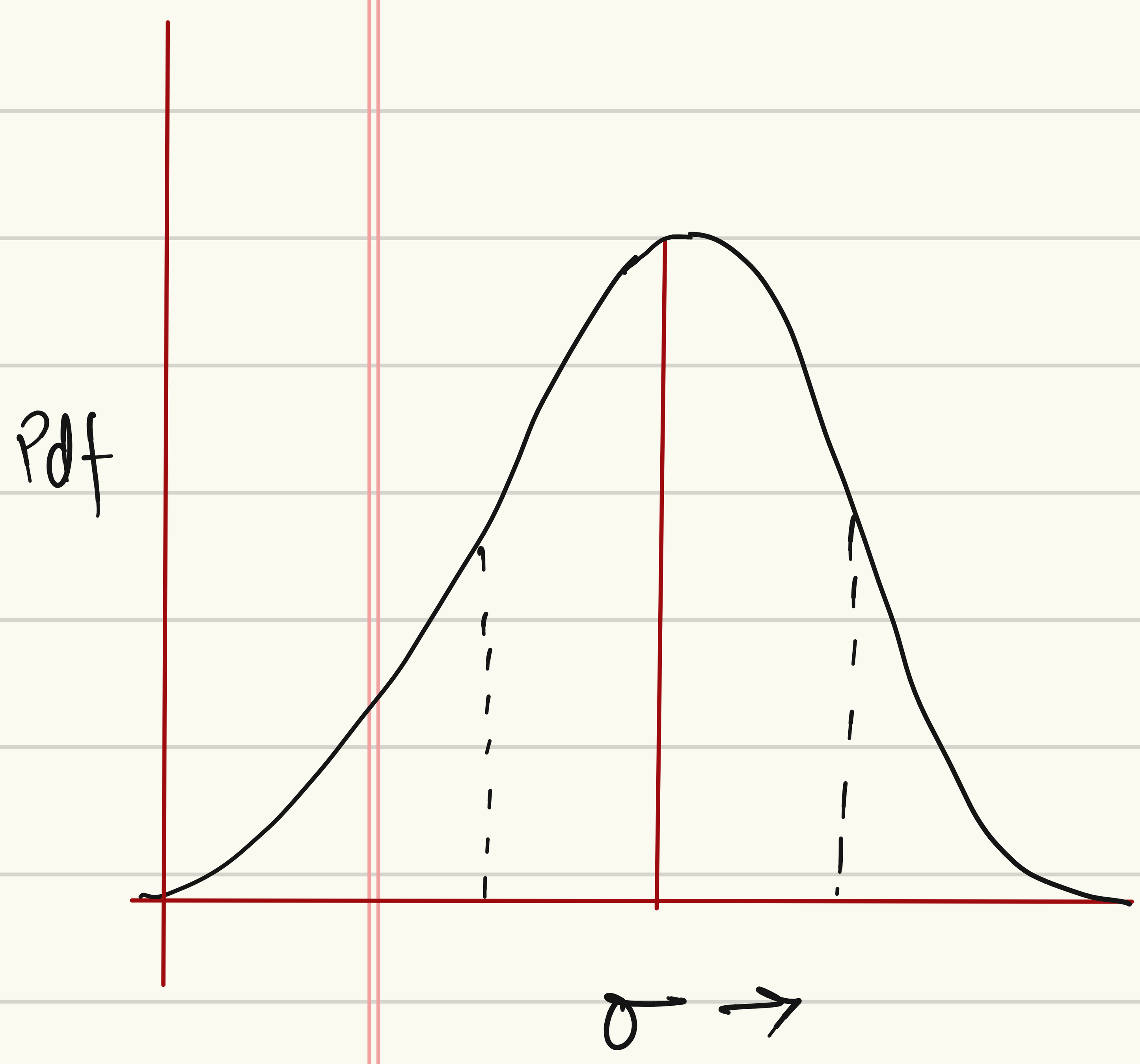
then $p(A|B) = p(X < 25) - p(X < 15).$

$P(A|B) = 25 \times \frac{1}{30} - 15 \times \frac{1}{30}.$

$P(A|B) = \frac{1}{30}\left[25 - 15\right] \Rightarrow \frac{10}{30}.$

$$\boxed{P(A|B) = \frac{1}{3}}$$

Q5.16 Given that,
normal distribution with $\mu = 40$, $\sigma = 4$



let $z = \frac{x - \mu}{\sigma}$.

$z = \frac{50 - 40}{4}$.

$z = \underline{\underline{2.5}}$.

standard normal variable.

$p(Z > 2.5) = 0.0062$ (using standard normal tables)

Probability that it will take more than 10 years is

1 − Probability that there is no rainfall of over 50 inches in next 10 years.

$= (1 - 0.0062)^{10}$.

$= \underline{\underline{0.9397}}$.

we have assumed that all years have independent rainfall.

Q5.21. Given that,

$\mu = 71$, $\sigma^2 = 6.25$.

converting foot into inches, we get.

$6'2'' = 74$ inches

Probability of 25 year old man over 74 inches is

$$P(X \geq 74) = 1 - p(X < 74)$$
$$= 1 - p(Z < 1.2)$$
$$= 1 - 0.8849$$
$$= 0.1151$$

$$Z = \frac{x - \mu}{\sigma}$$
$$Z = \frac{74 - 71}{\sqrt{6.25}}$$
$$Z = 1.2$$

$\therefore$ 11.51 % of 25 year old men will have height > 6'2''

$$p(X > 77) = 1 - p(X < 77)$$
$$= 1 - p(Z < 2.4)$$
$$= 1 - 0.9918$$
$$= 0.0082$$

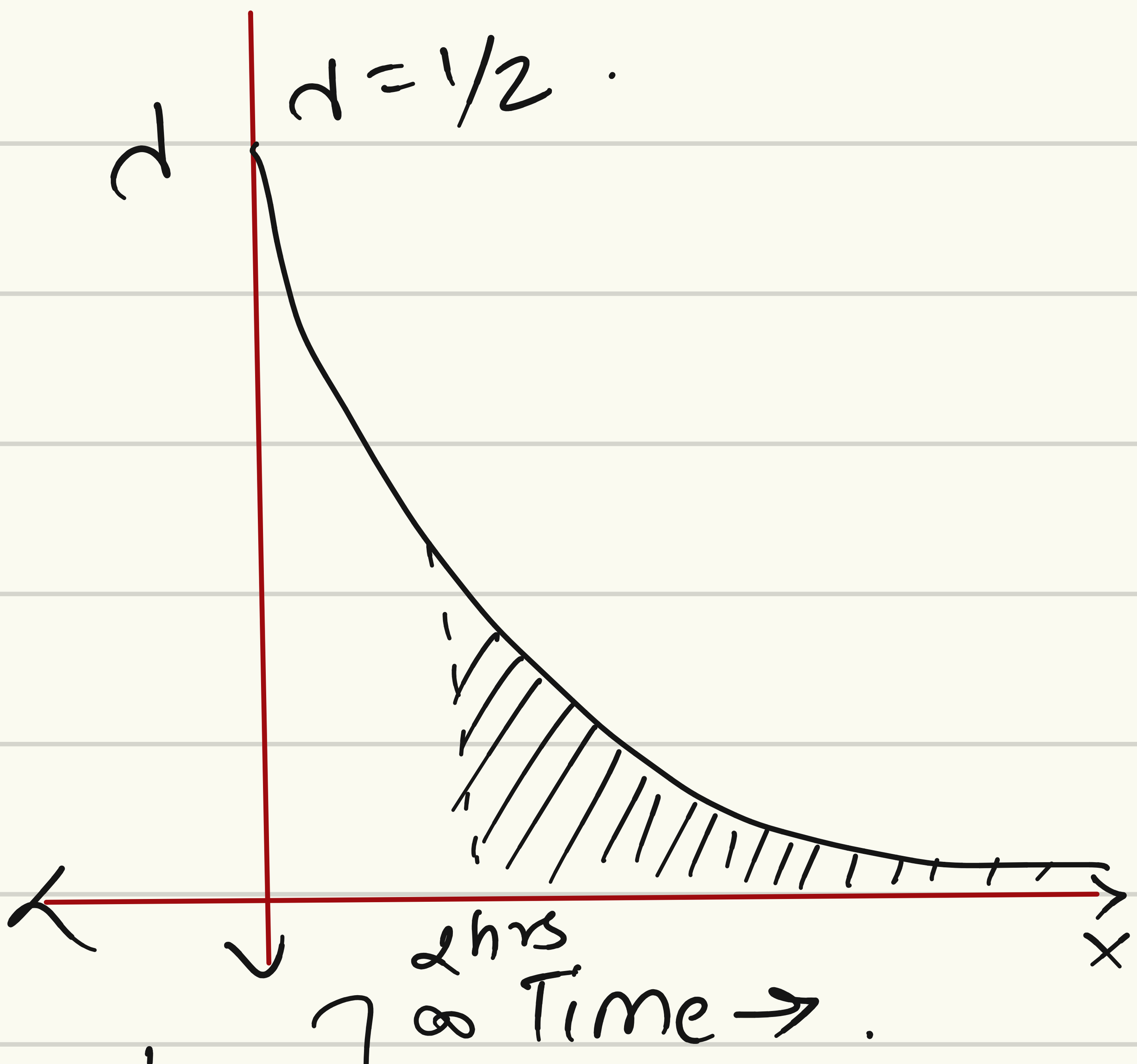
Thus, 0.82 % of men in club are > 6'5''

Q. 5.32 Given that,

Exponential distribution with $\lambda = \frac{1}{2}$

a) $P(X>2)$

$$= \int_0^\infty \lambda e^{-\lambda x}\, dx \leftarrow \text{exp dist.}$$



$$\therefore P(X>2) = \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x}\, dx$$

$$= \frac{1}{2}\left[\frac{-e^{-\frac{1}{2}x}}{\cancel{\frac{1}{2}}}\right]_2^\infty dx. \quad = \left[-e^{-\frac{1}{2}x}\right]_2^\infty$$

$$= \left[-e^{-\infty} + e^{-\frac{1}{\cancel{2}}\times\cancel{2}}\right]$$

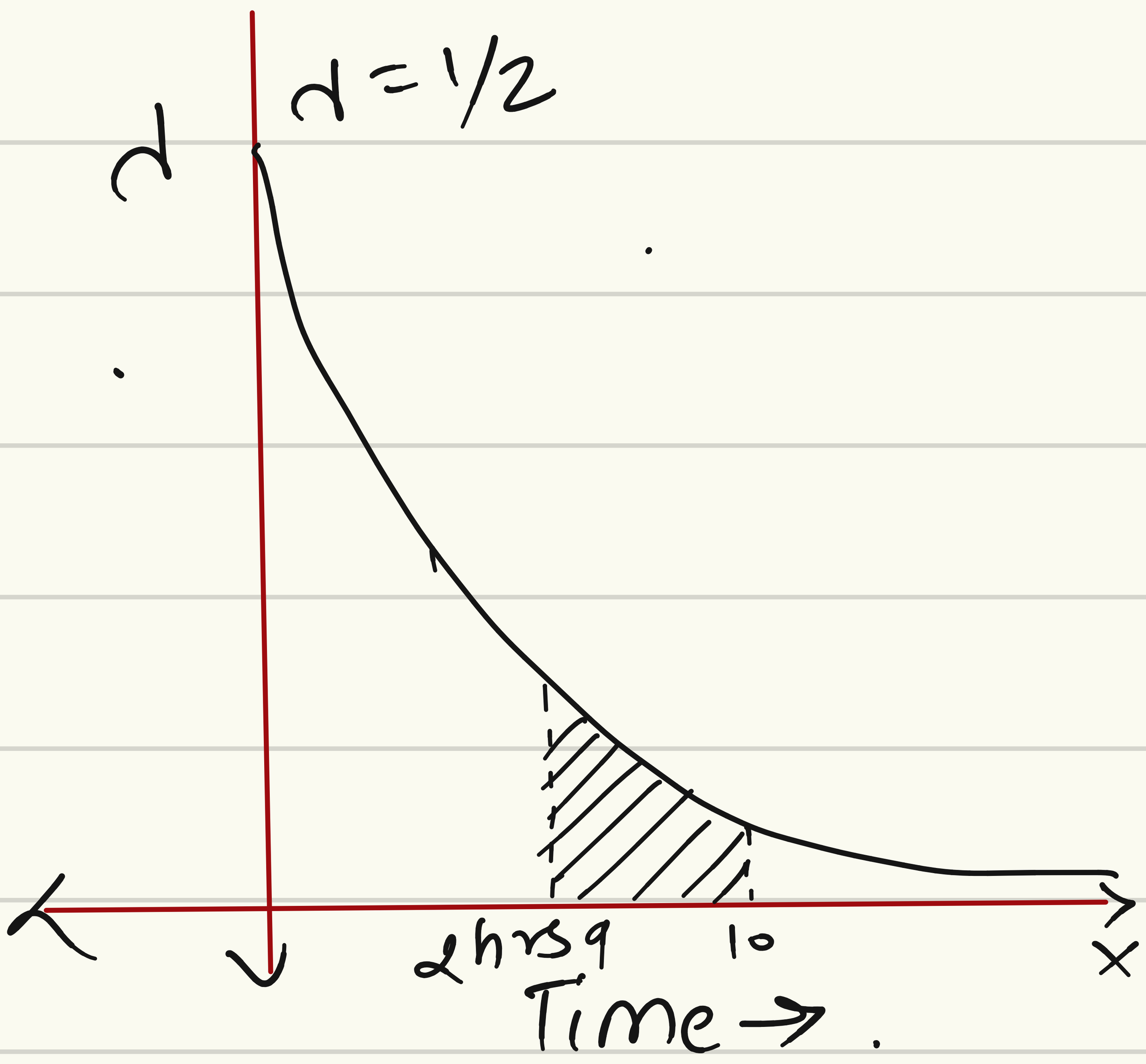
$$P(X>2) = e^{-1} \approx \underline{0.36787}.$$

(b)

$$P(\underset{(A)}{x>10} \mid \underset{(B)}{x>9}) = \frac{P(A\cap B)}{P(B)}.$$

$$= \frac{P(x\geq 10)}{P(x>9)}$$

$$= \underline{0.6065}$$



Q5.40. Given that,

uniform distribution over $(0,1)$

$$f(x) = \frac{1}{b-a} \Rightarrow \frac{1}{1}$$

now, let $Y = e^{x}$

$$\therefore G_Y(y) = P(Y \leq y).$$

$$= P(e^{x} \leq y)$$

$$= P(x \leq \log y).$$

$$= \int_{-\infty}^{\log y} 1 \cdot f(x)\, dx.$$

$$G_Y(y) = \log(y)$$

$\therefore$ pdf of y is

$$g_Y(y) = \frac{d}{dx} G(y) = \frac{d}{dy} \log(y)$$

$$= \frac{1}{y}, \quad 1 < y < 0.$$