Homework - 3

1) a)

$$I = \int_{1}^{3}\int_{0}^{1} \frac{2xy}{x^2+1}\, dx\, dy$$

$$= \int_{x=1}^{3}\int_{y=0}^{1} \frac{2x}{x^2+1}\, y\, dy\, dx$$

$$= \int_{1}^{3} \left| \frac{2x}{x^2+1} \cdot \frac{y^2}{2} \right|_{0}^{1} dx$$

$$= \int_{1}^{3} \left(\frac{2x}{x^2+1} \cdot \frac{1}{2} - 0 \right) dx$$

$$= \frac{1}{2} \int_{1}^{3} \frac{2x}{x^2+1}\, dx$$

Let $u = x^2+1$,

$$\frac{du}{dx} = 2x \implies 2x\,dx = du$$

$$\implies I = \frac{1}{2} \int_{x=1}^{3} \frac{1}{u}\, du$$

$$= \frac{1}{2} \left| \ln u \right|_{x=1}^{3} \quad ; \quad \text{substitute } u = x^2+1$$

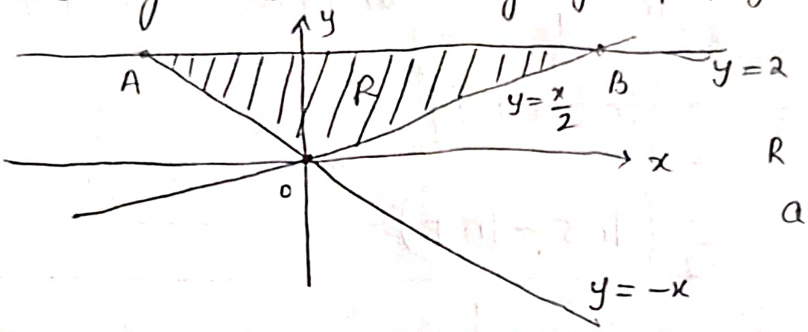
$$= \frac{1}{2} \left| \ln(x^2+1) \right|_{1}^{3}$$

$$= \frac{1}{2}(\ln 10 - \ln 2)$$

$$= \frac{1}{2} \ln 5$$

B) $$I = \iint_R \frac{1}{y^2+1}\, dA$$

R is the region bounded by $y = x/2$, $y = -x$, $y = 2$



R is the shaded area.

~~Meeting points A, B are~~ From equations, $x = -y$, $x = 2y$

~~A:~~ From diagram,

$0 \leq y \leq 2$

$-y \leq x \leq 2y$

$$I = \int_0^2 \left(\int_{-y}^{2y} \frac{1}{1+y^2}\, dx \right) dy$$

$$= \int_0^2 \left| \frac{x}{1+y^2} \right|_{-y}^{2y} dy$$

$$= \int_0^2 \left(\frac{2y}{1+y^2} - \left(\frac{-y}{1+y^2} \right) \right) dy$$

$$= \int_0^2 \frac{3y}{1+y^2}\, dy$$

$$= \frac{3}{2} \int_0^2 \frac{2y}{1+y^2}\, dy$$

Let $u = 1+y^2$

$$\frac{du}{dy} = 2y \Rightarrow du = 2y\,dy$$

$$I = \frac{3}{2}\int_{y=0}^{2} \frac{1}{u}\,du$$

$$= \frac{3}{2}\left|\ln u\right|_{y=0}^{2} \quad \text{Substitute } u = 1+y^2$$

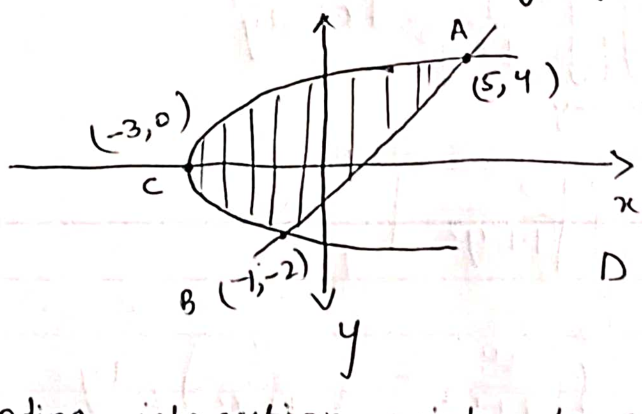
$$= \frac{3}{2}\left|\ln(1+y^2)\right|_{0}^{2}$$

$$= \frac{3}{2}(\ln 5 - \ln 1)$$

$$\boxed{I = \frac{3}{2}.\ln 5}$$

2) Given,

$I = \iint_D xy\,dA$, D is the region enclosed by $y^2 = 2x+6$, $y = x-1$



At $y = 0$, $x = -3$

$\Rightarrow C = (-3, 0)$

D is the shaded region

Finding intersection points to plot diagram

$$(x-1)^2 = 2x+6$$

$$x^2 - 2x + 1 = 2x + 6$$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0 \Rightarrow x = 5,\ x = -1$$

$$\Rightarrow y = 4,\ y = -2$$

$A: (5,4)$; $B: (-1,-2)$

From diagram,

$$-3 \leq x \leq 5$$

$$x-1 \leq y \leq \sqrt{2x+6}$$

$$\Rightarrow \quad I = \int_{-3}^{5} \left(\int_{x-1}^{\sqrt{2x+6}} xy \, dy \right) dx$$

$$= \int_{-3}^{5} \left| \frac{xy^2}{2} \right|_{x-1}^{\sqrt{2x+6}} dx$$

$$= \frac{1}{2} \int_{-3}^{5} \left(x(2x+6) - x(x-1)^2 \right) dx$$

$$= \frac{1}{2} \int_{-3}^{5} \left(2x^2 + 6x - x(x^2+1-2x) \right) dx$$

$$= \frac{1}{2} \int_{-3}^{5} (2x^2 + 6x - x^3 - x + 2x^2) \, dx$$

$$= \frac{1}{2} \int_{-3}^{5} (4x^2 + 5x - x^3) \, dx$$

$$= \frac{1}{2} \left| \frac{4x^3}{3} + \frac{5x^2}{2} - \frac{x^4}{4} \right|_{-3}^{5}$$

$$= \frac{1}{2} \left[\frac{4 * 5^3}{3} + \frac{5 * 5^2}{2} - \frac{5^4}{4} + \frac{4 * 3^3}{3} - \frac{5 * 3^2}{2} + \frac{3^4}{4} \right]$$

$$= \frac{1}{2} (106.66)$$

$$\boxed{I = 53.33}$$

3) $$I = \int_0^1 \int_{2\sqrt{y}}^{2} 12\sqrt{x^3+1}\, dx\, dy$$

Describe the region of integration

$\Rightarrow \quad 0 \leq y \leq 1$

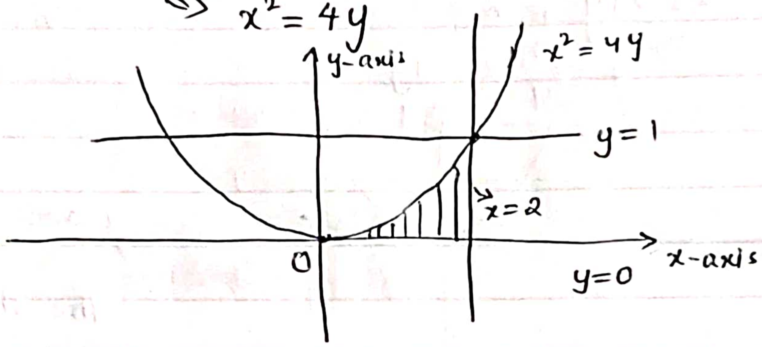
$2\sqrt{y} \leq x \leq 2$

This is the region bounded by

$y = 0, \quad y = 1,$

$x = 2\sqrt{y}, \quad x = 2$

$\Rightarrow \quad x^2 = 4y$



The shaded region is the region of integration

(To plot the graph, at $y = 1$; $x = 2$)

we can change the order of integration,

$0 \leq x \leq 2$

From diagram, $0 \leq y \leq x^2/4$

$$\Rightarrow \quad I = \int_0^2 \int_0^{x^2/4} 12\sqrt{x^3+1}\, dx\, dy$$

$$= \int_0^2 \left(\int_0^{x^2/4} 12\sqrt{x^3+1}\, dy \right) dx$$

$$= \int_0^2 \left| 12y\sqrt{x^3+1} \right|_0^{x^2/4} dx$$

$$= \int_0^2 3x^2\sqrt{x^3+1}\; dx$$

Let $u = x^3+1$

$\Rightarrow du = 3x^2\,dx$

So, $$I = \int_{x=0}^{x=2} \sqrt{u}\, du$$

$$= \left| \frac{2}{3}u^{3/2} \right|_{x=0}^{x=2} \quad ; \text{ Substitute } u = x^3+1$$

$$= \frac{2}{3}\left|(x^3+1)^{3/2}\right|_0^2$$

$$= \frac{2}{3}\left(9^{3/2} - 1\right)$$

$$= \frac{2}{3}(26)$$

$$\boxed{I = \frac{52}{3}}$$

4) A) $I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2} \, dy \, dx$

we can solve this easily by using polar form

$$I = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta) \, r \, dr \, d\theta$$

$$x^2+y^2 = a^2 \Rightarrow r_2 = a$$

$$\cancel{x^2+y^2 = 0} \Rightarrow r_1 = 0$$

$x_2 = a\cos\theta_2$ $\quad\quad$ $x_1 = a\cos\theta_1$

$a = a\cos\theta_2$ $\quad\quad$ $-a = a\cos\theta_1$

$\theta_2 = 2\pi$ $\quad\quad$ $\theta_1 = \pi$

$$I = \int_{\pi}^{2\pi} \int_{0}^{a} (a^2)^{3/2} \, r \, dr \, d\theta$$

$$= \int_{\pi}^{2\pi} \int_{0}^{a} a^3 \, r \, dr \, d\theta$$

$$= a^3 \int_{\pi}^{2\pi} \left| \frac{r^2}{2} \right|_{0}^{a} d\theta$$

$$= a^3 \int_{\pi}^{2\pi} \frac{a^2}{2} \, d\theta$$

$$= \frac{a^5}{2} \left| \theta \right|_{\pi}^{2\pi} = \frac{a^5}{2} \left| 2\pi - \pi \right|$$

$$\Rightarrow \boxed{I = \frac{\pi a^5}{2}}$$

B) $$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

using polar form,

$$I = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)\, r\, dr\, d\theta$$

$$y = \sqrt{2x-x^2} \Rightarrow x^2-2x+y^2 = 0$$

$$x^2-2x+1+y^2 = 1$$

$$(x-1)^2+y^2 = 1$$

$r_2 = 1$, $\quad x = 1+\cos\theta$, $y = \sin\theta$

$r_1 = 0$

$1+\cos\theta_1 = 0$	$1+\cos\theta_2 = 2$
$\cos\theta_1 = -1$	$\cos\theta_2 = 1$
$\theta_1 = \pi$	$\theta_2 = 2\pi$

$$I = \int_{\pi}^{2\pi} \int_0^1 \sqrt{(1+\cos\theta)^2+\sin^2\theta} \cdot r\, dr\, d\theta$$

$$= \int_{\pi}^{2\pi} \int_0^1 \sqrt{1+\cos^2\theta+\sin^2\theta+2\cos\theta}\ r\, dr\, d\theta$$

$$= \int_{\pi}^{2\pi} \int_0^1 \sqrt{2(1+\cos\theta)}\ r\, dr\, d\theta$$

$$= \int_{\pi}^{2\pi} \int_0^1 \sqrt{4\cos^2\theta/2}\ r\, dr\, d\theta$$

$$= \int_{\pi}^{2\pi} \int_0^1 2\cos\theta/2 \cdot r\, dr\, d\theta$$

$$= \int_{\pi}^{2\pi} \left| 2\cos\theta/2 \cdot \frac{r^2}{2} \right|_0^1 d\theta$$

$$= \int_{\pi}^{2\pi} \cos \theta/_2 \, d\theta$$

$$= \left| 2 \sin \theta/_2 \right|_{\pi}^{2\pi}$$

$$= 2\left[\sin \pi - \sin \tfrac{\pi}{2}\right]$$

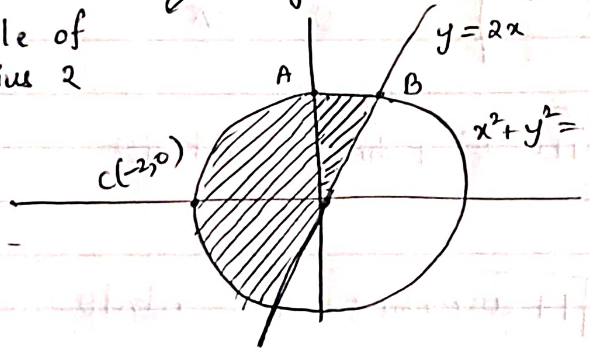
$$= 2(-1)$$

$$\boxed{I = -2}$$

5) A) R is the upper region bounded by

$x^2 + y^2 = 4$, $y = 2x$

Circle of radius 2



$y = \sqrt{4-x^2}$

The intersection points A, B are

A : $(0, 2)$

B : $x^2 + (2x)^2 = 4$

$x = \frac{2}{\sqrt{5}} \Rightarrow y = \frac{4}{\sqrt{5}}$ B: $\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$

The bounds are,

$-2 \leq x \leq 2/\sqrt{5}$

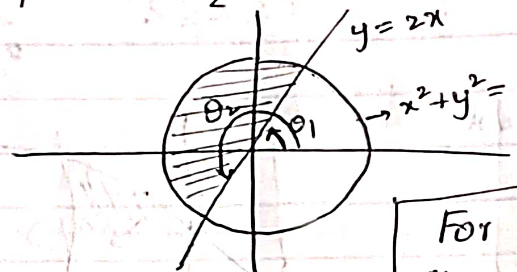
$2x \leq y \leq \sqrt{4-x^2}$

$$I = \iint_R (x+y)\,dA$$

$$\boxed{I = \int_{-2}^{2/\sqrt{5}} \int_{2x}^{\sqrt{4-x^2}} (x+y)\,dx\,dy}$$

B) From the diagram in 5 A,

$r_1 = 0$, $r_2 = 2$ (radius of $x^2+y^2=4$)



$x = r\cos\theta$
$y = r\sin\theta$

For $y = 2x$
Slope $m = 2 \Rightarrow \tan\theta_1 = 2$
$\theta_1 = \tan^{-1} 2$

$\Rightarrow \theta_2 = \pi + \theta_1$
$= \pi + \tan^{-1} 2$

$$I = \iint_R (x+y)\,dA$$

$$I = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)\, r\,dr\,d\theta$$

$$= \int_{\tan^{-1}2}^{\pi+\tan^{-1}2} \int_0^2 (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

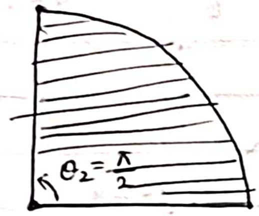
$$\Rightarrow \boxed{I = \int_{\tan^{-1}2}^{\pi+\tan^{-1}2} \int_0^2 r^2(\cos\theta+\sin\theta)\,dr\,d\theta}$$

6) Given a metal plate shaped like a quarter circle of radius 2m

Density, $d = 2x$

We know,

$$\text{mass, } M = \iint d \cdot dA$$



Using polar form,

$$M = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta) \cdot r dr \cdot d\theta$$

From diagram; $r_1 = 0$, $r_2 = 2$

$\theta_1 = 0$, $\theta_2 = \pi/2$

$$\Rightarrow M = \int_0^{\pi/2}\int_0^2 2r \cdot r dr d\theta = \int_0^{\pi/2}\int_0^2 2r^2 dr d\theta$$

$$= \int_0^{\pi/2} \left| 2\frac{r^3}{3} \right|_0^2 d\theta$$

$$= \int_0^{\pi/2} \frac{16}{3} d\theta$$

$$= \left| \frac{16}{3}\theta \right|_0^{\pi/2}$$

$$= \frac{16}{3} \cdot \frac{\pi}{2} - 0$$

$$\boxed{M = \frac{8\pi}{3} \text{ kgs}}$$