FE535: Introduction to Financial Risk Management
Session 3

Anthony Diaco

Agenda

- Univariate Stochastic Processes
  - Review of GBM
  - Calibration
- Application to Portfolio Risk Management
  - Value-at-Risk
  - Stress-Testing

Review and Calibration of GBM

Geometric Brownian Motion

- The most common process to simulate stock prices is the Geometric Brownian Motion (GBM)
- In this case,

$$\Delta S_t = S_t \mu \Delta t + S_t \sigma \Delta Z_t \tag{1}$$

alternatively,

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \Delta Z_t \tag{2}$$

- Note that $\frac{\Delta S_t}{S_t}$ resembles the stock return between t and $t + \Delta t$.
- To see this,

$$\frac{\Delta S_t}{S_t} = \frac{S_{t+\Delta t} - S_t}{S_t} \approx log\left(\frac{S_{t+\Delta t}}{S_t}\right) = \Delta log(S_t) \tag{3}$$

- In fact, the solution to (1) or (2), requires the solution to the stochastic differential equation (SDE) $\Delta log(S_t)$

- This class doesn't require knowledge about SDEs, but it follows that the solution for the GBM is

$$log\left(\frac{S_{t+\Delta t}}{S_t}\right) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \Delta Z_t \tag{4}$$

which is the same as Equation (4.6) from the Textbook

- To simplify the notation, let ΔR_t denote the return of the stock over Δt

$$\Delta R_t = log\left(\frac{S_{t+\Delta t}}{S_t}\right) \tag{5}$$

- In fact, it follows that the ΔR_t is a general BM, such that

$$\Delta R_t \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right) \tag{6}$$

is an iid process

- This class doesn't require knowledge about SDEs, but it follows that the solution for the GBM is

$$log\left(\frac{S_{t+\Delta t}}{S_t}\right) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \Delta Z_t \tag{4}$$

  which is the same as Equation (4.6) from the Textbook
- To simplify the notation, let ΔR_t denote the return of the stock over Δt

$$\Delta R_t = log\left(\frac{S_{t+\Delta t}}{S_t}\right) \tag{5}$$

- In fact, it follows that the ΔR_t is a general BM, such that

$$\Delta R_t \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right) \tag{6}$$

  is an iid process
- Therefore, to simulate the price at time $t + \Delta t$, one needs
  1. the price at time t, S_t
  2. estimate μ and σ
  3. Finally, simulate ΔR_t, i.e. draw a random number from the normal distribution described in (6)
- In other words

$$S_{t+\Delta t} = S_t \times \exp(\Delta R_t) \tag{7}$$

Let's consider again the same example as before

Implementation of GBM Simulation

- Let $\Delta t = 1/252$
- The current price at time $t = 0$ is $S_0 = 100$
- Given μ and σ, draw a random number from (6) denoted by $\Delta R_{\frac{1}{252}}$
- The price next day is

$$S_{\frac{1}{252}} = S_0 \times \exp(\Delta R_{\frac{1}{252}}) \tag{8}$$

- To simulate the second day price, draw another random number from (6) denoted by $\Delta R_{\frac{2}{252}}$, such that

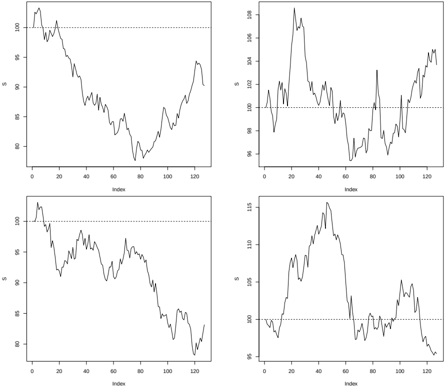
$$S_{\frac{2}{252}} = S_{\frac{1}{252}} \times \exp(\Delta R_{\frac{2}{252}}) \tag{9}$$

- To generalize it follows that the next d days price is given by

$$S_{\frac{d}{252}} = S_0 \prod_{i=1}^{d} \exp(\Delta R_{\frac{i}{252}}) = S_0 \times \exp\left(\sum_{i=1}^{d} \Delta R_{\frac{i}{252}}\right) \tag{10}$$

Let's demonstrate how to implement GBM

```
> S <- 100
> dt <- 1/252
> mu <- 0.1
> sig <- 0.2
> for(i in 1:126) {
+   dR <- rnorm(1,
+               dt*(mu - 0.5*sig^2),
+               sig*sqrt(dt) )
+   S_dt <- S[i]*exp(dR)
+   S <- c(S,S_dt)
+ }
> plot(S,type = "l")
> abline(h = S[1], lty = 2)
```



GBM Calibration

- The definition of the stock price process S_t is rather theoretical
- However, it denotes the price level at each time
- If we have historical data for D days, how can be bridge between theory and practice?
- This is commonly known as **calibration** (estimation)
  - Assume that the asset/stock price behaves with respect to a GBM
  - The only question is what parameters (μ,σ) would minimize the discrepancy between theory and practice?

GBM Calibration Cont.

- In terms of trading days, let's consider the asset prices at $d/252$ and $(d+1)/252$
- Regardless of the model (motion), what does it mean to have

$$\frac{\Delta S_{\frac{d}{252}}}{S_{\frac{d}{252}}} = \frac{S_{\frac{d+1}{252}} - S_{\frac{d}{252}}}{S_{\frac{d}{252}}} = ? \tag{11}$$

GBM Calibration Cont.

- In terms of trading days, let's consider the asset prices at $d/252$ and $(d+1)/252$
- Regardless of the model (motion), what does it mean to have

$$\frac{\Delta S_{\frac{d}{252}}}{S_{\frac{d}{252}}} = \frac{S_{\frac{d+1}{252}} - S_{\frac{d}{252}}}{S_{\frac{d}{252}}} = ? \tag{11}$$

- It states the change in the price value with respect to the previous level

$$\frac{\Delta S_{\frac{d}{252}}}{S_{\frac{d}{252}}} = \Delta \log S_{\frac{d}{252}} = \log(S_{\frac{d+1}{252}}) - \log(S_{\frac{d}{252}}) = \Delta R_{\frac{d}{252}} \tag{12}$$

- with $\Delta R_{d/252}$ denoting the return (relative change) of the stock price
- In fact, the above result is consistent with Equation (7), such that

$$\log(S_{\frac{d+1}{252}}) - \log(S_{\frac{d}{252}}) = \Delta R_{\frac{d}{252}} \to S_{\frac{d+1}{252}} = S_{\frac{d}{252}} \exp(\Delta R_{\frac{d}{252}}) \tag{13}$$

GBM Calibration Cont.

- Let's consider the following example with real data for the SPY ETF:

Date	d	$S_{d/252}$	$\Delta R_{\frac{d}{252}}$ (in %)
2019-08-30	1	292.45	
2019-09-03	2	290.74	-0.59
2019-09-04	3	294.04	1.13
2019-09-05	4	297.82	1.28
2019-09-06	5	298.05	0.08
2019-09-09	6	298.20	0.05

- The above so far, makes no assumptions about the distribution of the returns (nor the motion of the asset price)
- The question remains, however, how do we link this to theory?

GBM Calibration Cont.

- The solution to the GMB states that the daily return on the asset follows a normal distribution (recall Equation (6))

$$\Delta R_{\frac{d}{252}} \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) \times \frac{1}{252}, \sigma^2 \times \frac{1}{252}\right) \tag{14}$$

- This means that we know the statistical properties of this stochastic component:

$$\mathbb{E}\left[\Delta R_{\frac{d}{252}}\right] = \left(\mu - \frac{\sigma^2}{2}\right) \times \frac{1}{252} \tag{15}$$

$$\mathbb{V}\left[\Delta R_{\frac{d}{252}}\right] = \sigma^2 \times \frac{1}{252} \tag{16}$$

- On the left hand side (LHS), we have the expectation and variance equation (data related)
- On the right hand side (RHS), we have the parameters μ and σ (theory)
- Hence, if were able to evaluate the LHS, **then the parameters can calibrated by solving two equations with two unknowns**

GBM Calibration Cont.

- Recall that the expectation corresponds to the population first moment
  - where the sample mean (average) corresponds to the sample first moment, denoted by m
- Also, recall that the variance corresponds to the population second central moment
  - whose estimate is given by the sample variance, s^2
- This means that

$$\widehat{\mathbb{E}\left[\Delta R_{\frac{d}{252}}\right]} = m \tag{17}$$

$$\widehat{\mathbb{V}\left[\Delta R_{\frac{d}{252}}\right]} = s^2 \tag{18}$$

where the $\hat{\theta}$ denotes the sample estimate of moment (parameter) θ

GBM Calibration Cont.

Finally, "putting hats" on Equation (15) and (16), while reconciling with Equations (17) and (18), indicates that

$$m = \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2}\right) \times \frac{1}{252} \tag{19}$$

$$s^2 = \hat{\sigma}^2 \times \frac{1}{252} \tag{20}$$

such that μ and σ can be estimated using sample moment (real data)

GBM Calibration Cont.

Finally, "putting hats" on Equation (15) and (16), while reconciling with Equations (17) and (18), indicates that

$$m = \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2}\right) \times \frac{1}{252} \tag{19}$$

$$s^2 = \hat{\sigma}^2 \times \frac{1}{252} \tag{20}$$

such that μ and σ can be estimated using sample moment (real data)

Proposed Approach for Daily Data

1. Given the portfolio realized daily returns
   1. compute mean daily return denoted by m
   2. compute daily standard deviation (volatility) denoted by s
2. if we scale s by $\sqrt{252}$, then we have

$$\hat{\sigma} = \sqrt{252} \times s \tag{21}$$

3. For the μ_p, we need to scale and adjust for the $\sigma_p^2/2$ component, i.e.

$$\hat{\mu} = 252 \times m + \frac{\hat{\sigma}^2}{2} \tag{22}$$

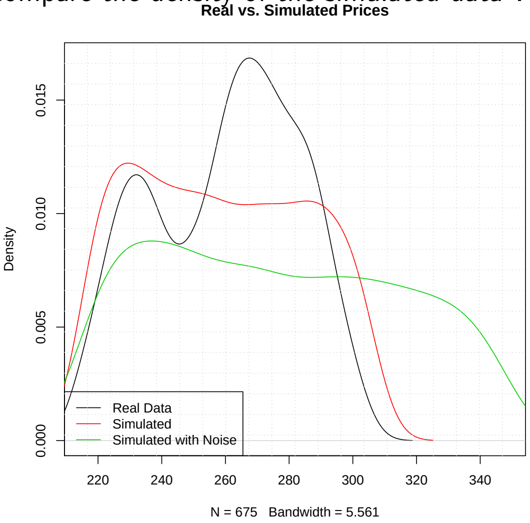
GBM Calibration Cont.

- Let's use the same SPY data dating back to 2017
- According to the daily returns, we have

	Values Reported in Percentage (%)
Daily Mean Return: m	0.05
Daily Volatility: s	0.83
Annual Mean Return: $252 \times m$	12.53
Annual Volatility: $\hat{\sigma} = \sqrt{252} \times s$	13.24
Calibration	
$\hat{\sigma} = \sqrt{252} \times s$	13.24
$\hat{\mu} = 252 \times m + \hat{\sigma}^2/2$	13.40

GBM Calibration Cont.

- Let's compare the density of the simulated data versus the true ones



Application to Portfolio Risk Management

Application to Portfolio Risk Management

- Let's denote today's portfolio value by F_0
- We are interested in evaluating the future portfolio performance at t, i.e. F_t which is unknown
- MC simulations allow us to generate different scenarios as a robustness check, for instance
  - With 95% confidence, what's the maximum loss on the portfolio?
  - What happens to the underlying portfolio if the market premium (volatility) drops (increases) by x%?

Application to Portfolio Risk Management

- Let's denote today's portfolio value by F_0
- We are interested in evaluating the future portfolio performance at t, i.e. F_t which is unknown

- MC simulations allow us to generate different scenarios as a robustness check, for instance
  - With 95% confidence, what's the maximum loss on the portfolio?
  - What happens to the underlying portfolio if the market premium (volatility) drops (increases) by x%?

- Assuming the portfolio obeys to GBM as in (7), then its future price can be described as

$$F_t = F_0 \times \exp\left((\mu_p - \frac{\sigma_p^2}{2})t + \sigma_p Z_t \right) \tag{23}$$

  where μ_p and σ_p denote the portfolio mean return and volatility, respectively
- Hence, if one knows μ_p and σ_p, then the future price is determined by the stochastic component Z_t

Value-at-Risk

- According to Equation (2.42) from the textbook, the c% Value-at-Risk (VaR) is given by

$$VaR(F_t, c) = \bar{F}_t - Q(F_t, c) \tag{24}$$

with

1. $\bar{F}_t$ denoting the expected value of portfolio at time t, i.e. $\bar{F}_t = \mathbb{E}[F_t]$
2. $Q(F_t, c)$ is the c percentile of the F_t, such that $1 - c$ denotes the level of confidence

Value-at-Risk Simplified

- If the daily portfolio values follow an iid normal distribution, i.e. $F_d \sim N(\mu, \sigma)$ $\forall d = 1, ..., T$, then we know that

$$\mathbb{V}[\sum_{d=1}^{D} F_d] = D \times \sigma^2 \tag{25}$$

- This is the result of an iid assumption for time aggregation (see Section 5.1.2 of the Jorion)

Value-at-Risk Simplified

- If the daily portfolio values follow an iid normal distribution, i.e. $F_d \sim N(\mu, \sigma)$ $\forall d = 1, ..., T$, then we know that

$$\mathbb{V}[\sum_{d=1}^{D} F_d] = D \times \sigma^2 \tag{25}$$

- This is the result of an iid assumption for time aggregation (see Section 5.1.2 of the Jorion)
- In terms of VaR, under normal distribution, it follows that

$$VaR(F_d, c) = \mathbb{E}[F_d] - Q(F_d, c) = \mu - Q(F_d, c) \tag{26}$$

- Under normal distribution the c quantile of F_d

$$Q(F_d, c) = \mu + \sigma Z_c \tag{27}$$

such that

$$VaR(F_d, c) = \mu - [\mu + \sigma Z_c] = -\sigma Z_c = \sigma Z_{1-c} \tag{28}$$

where Z_{1-c} is the $1 - c$ percentile of the standard normal distribution, i.e.

$$\mathbb{P}(Z < Z_{1-c}) = 1 - c \text{ with } Z \sim N(0, 1) \tag{29}$$

Value-at-Risk Simplified Cont.

- Equation (28) indicates that the daily VaR at the $1-c$ level of confidence is mainly determined by the assessment of the daily volatility
- Obviously, a daily monitoring of the VaR requires a daily monitoring of the volatility
- If returns were iid, then the D multiple periods VaR is given by

$$VaR\left(\sum_{d=1}^{D} F_d, c\right) = \sqrt{D} \times \sigma \times Z_{1-c} \tag{30}$$

- The above results are relevant if returns were normal and iid.
- However, in practice, distributions do deviate from normality or iid, making the analytical solution for VaR complicated
- Hence, it is common to use MC to compute the VaR of the portfolio

Value-at-Risk using MC

Value at Risk (VaR) using MC

To compute the portfolio VaR using MC, one can do the following steps:

1. Simulate $Z_t(n) \sim N(0, t)$ for $n = 1, ..., N$
2. For each $Z_t(n)$, compute the corresponding value of $F_t(n)$
3. Given the distribution of $F_t(n)$,
   - take the average of $F_t(n)$ for $n = 1, ..., N$, denoted by $\bar{F}_t$
   - compute the c percentile, denoted by $Q(F_t, c)$
4. Finally, we have

$$VaR(c) = \bar{F}_t - Q(F_t, c) \tag{31}$$

Example using Portfolio 1 and Portfolio 2

- Recall Portfolios 1 and 2 from Session 1?
- Using back-testing, we had:

	Portfolio 1	Portfolio 2
$252 \times m$	0.07	0.15
$\hat{\sigma}_p = \sqrt{252} \times s$	0.32	0.31

Example using Portfolio 1 and Portfolio 2

- Recall Portfolios 1 and 2 from Session 1?
- Using back-testing, we had:

	Portfolio 1	Portfolio 2
$252 \times m$	0.07	0.15
$\hat{\sigma}_p = \sqrt{252} \times s$	0.32	0.31
$\hat{\mu}_p = 252 \times m + \hat{\sigma}_p^2/2$	0.12	0.20

Example using Portfolio 1 and Portfolio 2

- Recall Portfolios 1 and 2 from Session 1?
- Using back-testing, we had:

	Portfolio 1	Portfolio 2
$252 \times m$	0.07	0.15
$\hat{\sigma}_p = \sqrt{252} \times s$	0.32	0.31
$\hat{\mu}_p = 252 \times m + \hat{\sigma}_p^2/2$	0.12	0.20

- Assume that the current value of each portfolio is \$100K
- As a risk manager, you need to assess the downside risk of each portfolio
- To quantify the above, you are asked to compute the VaR for each portfolio

```r
> S <- 100
> sig1 <- 0.32
> mu1 <- 0.07 + (sig1^2)/2
> sig2 <- 0.31
> mu2 <- 0.15 + (sig2^2)/2
> gbm_path <- function(N,mu,sig,T_end) {
+   R_t <-  rnorm(N,T_end*(mu - 0.5*sig^2),sig*sqrt(T_end) )
+   S_t <- S*exp(R_t)
+   return(S_t)
+ }
> sim1 <- gbm_path(1000,mu1,sig1,1)
> sim2 <- gbm_path(1000,mu2,sig2,1)
> F_bar_1 <-  mean(sim1)
> F_bar_2 <-  mean(sim2)
> var1_0.05 <- F_bar_1 - quantile(sim1,0.05)
> var2_0.05 <- F_bar_2 - quantile(sim2,0.05)
> var1_0.01 <- F_bar_1 - quantile(sim1,0.01)
> var2_0.01 <- F_bar_2 - quantile(sim2,0.01)
```

	Portfolio 1	Portfolio 2
$\bar{F}_1$	113.39	123.05
$VaR(0.05)$	47.67	53.18
$VaR(0.01)$	61.62	66.68

Stress Testing

- As a robustness check, you need to evaluate the downside risk under stressed scenarios
- Using a one risk factor model (CAPM), for instance, we can assess the portfolio sensitivity to market volatility
- Recall that under CAPM, it follows that

$$\hat{\sigma}_p = \hat{\beta}_p \times \hat{\sigma}_M \tag{32}$$

- All else equal, an increase of 10% in the market volatility would increase the portfolio volatility by $0.1 \times \hat{\beta}_p \times \hat{\sigma}_M$
  - hence, the larger the beta is the greater the market risk is

Stress Testing

- As a robustness check, you need to evaluate the downside risk under stressed scenarios
- Using a one risk factor model (CAPM), for instance, we can assess the portfolio sensitivity to market volatility
- Recall that under CAPM, it follows that

$$\hat{\sigma}_p = \hat{\beta}_p \times \hat{\sigma}_M \tag{32}$$

- All else equal, an increase of 10% in the market volatility would increase the portfolio volatility by $0.1 \times \hat{\beta}_p \times \hat{\sigma}_M$
  - hence, the larger the beta is the greater the market risk is
- Recall slide 40 from Session 1, Portfolios 1 and 2 have betas of 1.00 and 0.91, respectively.
- According to the SPY volatility, we had $\hat{\sigma}_M = 0.32$. For an increase of 10% in market volatility we get

	Portfolio 1	Portfolio 2
$\hat{\sigma}_p$	0.35	0.34
$\bar{F}_t$	112.26	122.39
$VaR(0.05)$	52.48	56.31
$VaR(0.01)$	66.75	70.08

Summary

Today we...

- talked about random number generators and their application in financial risk management
- showed how MC methods can be useful to provide numerical solutions
- demonstrated how to simulate stock prices
- illustrated the application of MC portfolio management
  - downside risk (VaR)
  - stress-testing - sensitivity to market risk

Teamwork

Teamwork

1. Download prices for your the SPY ETF, dating back to 2017-01-01
2. Compute the daily returns over this sample period
3. Assume that the price obeys to the GBM, calibrate the model
4. After calibration, simulate the future price over a one year period - create a couple of plots
5. What is the $VaR(0.05)$?

Appendix

```
> library(quantmod) # CODE FOR THE DENSITY PLOT <-----------------
> library(lubridate)
>
> P <- get(getSymbols("SPY"))
> P <- P$SPY.Adjusted
> P <- P[year(P) >= 2017,]
> R <- na.omit(log(P/lag(P)))
>
> # daily estimates
> m <- mean(R)
> s <- sd(R)
>
> # calibrate
> sig_hat <- s*sqrt(252)
> mu_hat <- m*252 + (sig_hat^2)/2
> gbm_path_single <- function(S,mu,sig,Days) {
+   R_t <-  rnorm(Days,(1/252)*(mu - 0.5*sig^2),sig*sqrt(1/252) )
+   S_t <- S*exp(cumsum(R_t))
+   return(S_t)
+ }
>
> S0 <- as.numeric(first(P))
> Days <- length(P) - 1
>
> S_mat <- sapply(1:10^3,function(i) gbm_path_single(S0,mu_hat,sig_hat,Days) )
> S_sim1 <- apply(S_mat,1,mean)
> noise <- 0.05
> S_mat2 <- sapply(1:10^3,function(i) gbm_path_single(S0,mu_hat + noise,sig_hat + noise,Days) )
> S_sim2 <- apply(S_mat2,1,mean)
>
> plot(density(P), xlim = range(c(as.numeric(P),S_sim1,S_sim2)), main = "Real vs. Simulated Prices" )
> lines(density(S_sim1),col = 2)
> lines(density(S_sim2),col = 3)
> legend("bottom",
c("Real Data","Simulated","Simulated with Noise")
, col = 1:3, lty = 1)
> grid(20)
```