# WEEK 6 – LECTURE 1

## COMPARING TWO SAMPLES

## (PART I)

# OBJECTIVES

- Review some hypothesis tests taught in elementary statistics courses, including
  1) Tests for population means
  2) Tests for population proportions
  3) Tests for population variances
  4) Tests for two population means
  5) Test for two population proportions

## Hypothesis Test for a Population Mean – Large Sample

- When the sample size is at least 30, the test statistic for a hypothesis test for a population mean is given by:

  + If σ is known: $$z = \frac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$$

  + If σ is unknown: $$z = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

  where $\bar{x}$ is the sample mean, $\mu$ is the presumed value of the population mean from the null hypothesis, $\sigma$ is the population standard deviation, $s$ is the sample standard deviation, and $n$ is the sample size.
- The test statistic has the standard normal distribution.

| Alternative Hypothesis, $H_a$ | Type of Hypothesis Test |
|---|---|
| $<$ Value | Left-tailed test |
| $>$ Value | Right-tailed test |
| $\neq$ Value | Two-tailed test |

**Decision Rule for Rejection Regions**

Reject the null hypothesis, $H_0$, if the test statistic calculated from the sample data falls within the rejection region.

Reject the null hypothesis, $H_0$, if:

$z \leq -z_{\alpha}$ for a left-tailed test

$z \geq z_{\alpha}$ for a right-tailed test

$|z| \geq z_{\alpha/2}$ for a two-tailed test

# *p*-Values

- A ***p*-value** is the probability of obtaining a sample statistic as extreme or more extreme than the one observed in the data, when the null hypothesis, $H_0$, is assumed to be true.

**Calculate p-values**

- For a left-tailed test, *p*-value = $P(Z \leq z)$.
- For a right-tailed test, *p*-value = $P(Z \geq z)$.
- For a two-tailed test, *p*-value = $P(|Z| \geq |z|)$.

**Conclusions Using *p*-Values**

- If *p*-value ≤ $\alpha$, then *reject* the null hypothesis.
- If *p*-value > $\alpha$, then *fail to reject* the null hypothesis.

## Hypothesis Test for a Population Mean – Small Sample

- Assume the population is normally distributed. When the sample size is small (less than 30), the test statistic for a hypothesis test for a population mean is given by:

  \+ If σ is known:
  $$z = \frac{\bar{x} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$$

  \+ If σ is unknown:
  $$t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

  where $\bar{x}$ is the sample mean, $\mu$ is the presumed value of the population mean from the null hypothesis, $\sigma$ is the population standard deviation, $s$ is the sample standard deviation, and $n$ is the sample size.
- The test statistic $z$ (when σ is known) has the standard normal distribution.
- The test statistic $t$ (when σ is unknown) has Student's $t$-distribution with **n-1** degrees of freedom.

## Rejection Regions for Hypothesis Tests for Population Means
## (small sample, $\sigma$ known)

Reject the null hypothesis, $H_0$, if:

$z \le -z_\alpha$ for a left-tailed test

$z \ge z_\alpha$ for a right-tailed test

$|z| \ge z_{\alpha/2}$ for a two-tailed test

**Calculate p-values**

- For a left-tailed test, $p$-value = $P(Z \le z)$.
- For a right-tailed test, $p$-value = $P(Z \ge z)$.
- For a two-tailed test, $p$-value = $P(|Z| \ge |z|)$.

**Conclusions Using *p*-Values**

- If $p$-value $\le \alpha$, then *reject* the null hypothesis.
- If $p$-value $> \alpha$, then *fail to reject* the null hypothesis.

## Rejection Regions for Hypothesis Tests for Population Means
## (small sample, $\sigma$ Unknown)

Reject the null hypothesis, $H_0$, if:

$t \le -t_\alpha$ for a left-tailed test

$t \ge t_\alpha$ for a right-tailed test

$|t| \ge t_{\alpha/2}$ for a two-tailed test

# Critical z-Values for Rejection Regions

| Level of Confidence $c$ | Level of Significance $\alpha = 1 - c$ | One-Tailed Test $z_\alpha$ | Two-Tailed Test $\pm z_{\alpha/2}$ |
|---|---|---|---|
| 0.90 | 0.10 | 1.28 | ±1.645 |
| 0.95 | 0.05 | 1.645 | ±1.96 |
| 0.98 | 0.02 | 2.05 | ±2.33 |
| 0.99 | 0.01 | 2.33 | ±2.575 |

## Hypothesis Test for a Population Proportion – Large Sample

- When the sample taken is a simple random sample, the conditions for a binomial distribution are met, and the sample size is large enough to ensure that $np \geq 10$ and $n(1-p) \geq 10$, the test statistic for a hypothesis test for a population proportion is given by

$$z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

where $\hat{p}$ is the sample proportion, $p$ is the presumed value of the population proportion from the null hypothesis, and $n$ is the sample size.

- The test statistic has the standard normal distribution.

## Conclusions Using *p*-Values

- If *p*-value ≤ $\alpha$, then *reject* the null hypothesis.
- If *p*-value > $\alpha$, then *fail to reject* the null hypothesis.

**Hypothesis Test for a Population Variance (or Standard Deviation)**

- When the sample taken is a simple random sample and the population distribution is approximately normal, the test statistic for a hypothesis test for a population variance (or standard deviation) is given by

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

where $s^2$ is the sample variance, $\sigma^2$ is the presumed value of the population variance from the null hypothesis, and *n* is the sample size.

- The test statistic has the chi-square distribution with $df = n - 1$ degrees of freedom.

## Rejection Regions for Hypothesis Tests for Population Variance

Reject $H_0$ if:

$\chi^2 \leq \chi^2_{n-1}(1-\alpha)$ for a left-tailed test

$\chi^2 \geq \chi^2_{n-1}(\alpha)$ for a right-tailed test

$\chi^2 \leq \chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)$ or $\chi^2 \geq \chi^2_{n-1}\left(\frac{\alpha}{2}\right)$ for a two-tailed test

# Practice Problem 1

A cosmetic company fills its best-selling 8-ounce jars of facial cream by an automatic dispensing machine. The machine is set to dispense a mean of 8.1 ounces per jar. Uncontrollable factors in the process can shift the mean away from 8.1 and cause either underfill or overfill, both of which are undesirable. In such a case the dispensing machine is stopped recalibrated. Regardless of the mean amount dispensed, the standard deviation of the amount dispensed always has value 0.22 ounce. A quality control engineer routinely select 30 jars from the assembly line to check the amount filled. On one occasion, the sample mean is 8.2 ounces. Determine if there is sufficient evidence in the sample to indicate, at the 1% level of significance, that the machine should be recalibrated.

# Practice Problem 2

The total score in a professional basketball game is the sum of the scores of the two teams. An expert commentator claims that the average total score for NBA games is 202.5. A fan suspect that this is an overstatement and that the actual average is less than 202.5. He selects a random sample of 85 games and obtains a mean total score of 199.2 with standard deviation 19.63. Determine, at the 5% level of significance, whether there is sufficient evidence in the sample to reject the expert commentator's claim.

# Practice Problem 3

The price of a popular tennis racket at a national chain store is $179. Portia bought five of the same racket at an online auction site for the following prices (in $):

155 179 175 175 161

Assuming that the auction prices of rackets are normally distributed, determine whether there is sufficient evidence in the sample, at the 5% level of significance, to conclude that the average price of the racket is less than $179 if purchased at an online auction. Assume that the standard deviation of prices for rackets purchased at an online auction is $10.

# Practice Problem 4

A locally owned, independent department store has chosen its marketing strategies for many years under the assumption that the mean amount spent by each shopper in the store is no more than $100.00. A newly hired store manager claims that the current mean is higher and wants to change the marketing scheme accordingly. A group of 27 shoppers is chosen at random and found to have spent a mean of $104.93 with a standard deviation of $9.07. Assume that the population distribution of amounts spent is approximately normal, and test the store manager's claim at the 0.05 level of significance.

# Practice Problem 5

According to a report published by the Mathematical Association of America, 1% of bachelor's degrees in the United States are awarded in the fields of mathematics and statistics. One recent mathematics graduate believes that this number is incorrect. She has no indication if the actual percentage is more or less than 1%. To perform a hypothesis test, she uses a simple random sample of 12,317 recent college graduates for her sample. According to the graduation programs from the colleges, she notes that bachelor's degrees in mathematics or statistics were awarded to 148 graduates in her sample. Does this evidence support this graduate's belief that the percentage of bachelor's degrees awarded in the fields of mathematics and statistics is not 1%? Use a 0.10 level of significance.

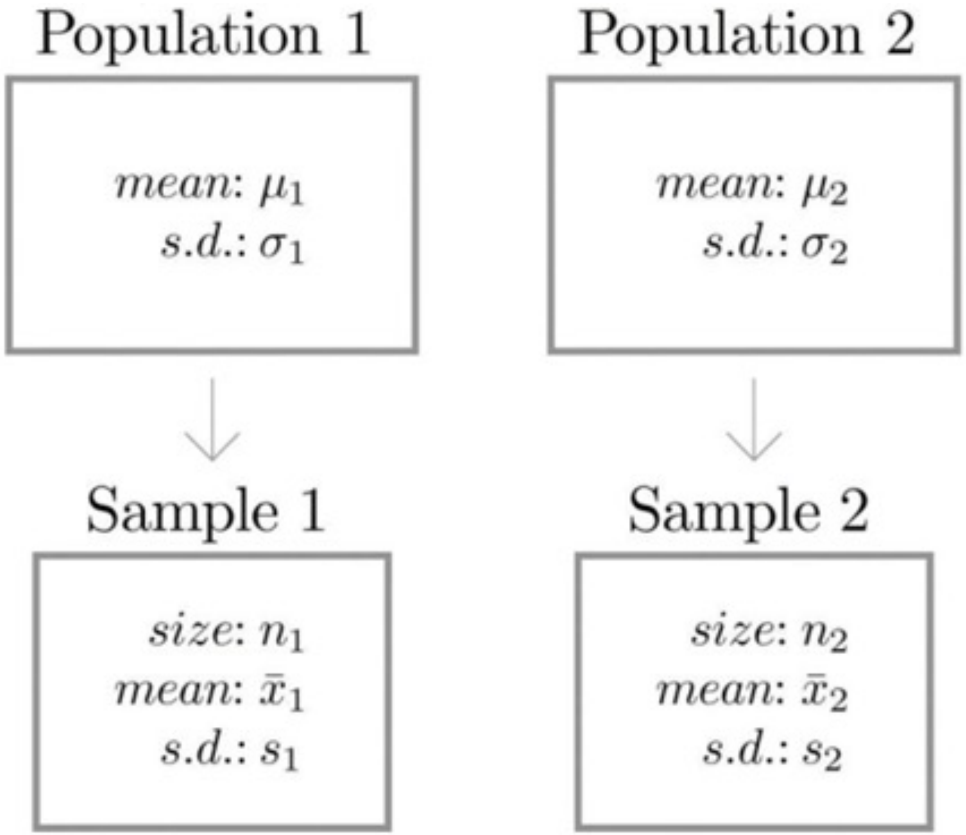
# Practice Problem 6

A manufacturer of golf balls requires that the weights of its golf balls have a standard deviation that does not exceed 0.08 ounces. One of the quality control inspectors says that the machines need to be recalibrated because he believes the standard deviation of the weights of the golf balls is more than 0.08 ounces. To test the machines, he selects a simple random sample of 30 golf balls off the assembly line and finds that they have a mean weights of 1.6200 ounces and a standard deviation of 0.0804 ounces. Does this evidence support the need to recalibrate the machines, at the 0.05 level of significance? Assume that the weights of the golf balls are normally distributed.

# Comparing Two Population Means

Two samples are **independent** if the data from the first sample are not connected to the data from the second sample.

Two samples are **dependent** if the data from both samples are systematically connected, or ***paired***.



## Hypothesis Test for Two Population Means - Independent Samples, σ(s) known

Assume that both sample sizes are at least 30 or both population distributions are approximately normal.

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

- $(\mu_1 - \mu_2)$ is the presumed value of the difference between the two population means from the null hypothesis.
- The test statistic has the standard normal distribution.

### Calculate p-values

- For a left-tailed test, $p$-value = $P(Z \le z)$.
- For a right-tailed test, $p$-value = $P(Z \ge z)$.
- For a two-tailed test, $p$-value = $P(|Z| \ge |z|)$.

### Conclusions Using *p*-Values

- If $p$-value $\le \alpha$, then *reject* the null hypothesis.
- If $p$-value $> \alpha$, then *fail to reject* the null hypothesis.

**Hypothesis Test for Two Population Means – Independent Samples, σ(s) unknown, unequal**

Assume both sample sizes are at least 30 or both populations are approximately normal, and the population standard deviations are unequal.

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- $(\mu_1 - \mu_2)$ is the presumed value of the difference between the two population means from the null hypothesis.
- The number of degrees of freedom for the t-distribution of the test statistic is the smaller of the values $n_1 - 1$ and $n_2 - 1$.

**Rejection Regions for Hypothesis Tests for Population Means (small samples)**

Reject the null hypothesis, $H_0$, if:

$t \leq -t_\alpha$ for a left-tailed test

$t \geq t_\alpha$ for a right-tailed test

$|t| \geq t_{\alpha/2}$ for a two-tailed test

## Hypothesis Test for Two Population Means – Independent Samples, σ(s) unknown, equal

Assume the samples are independent, both sample sizes are at least 30 or the populations are approximately normal, and the population standard deviations are equal.

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \qquad \text{where} \qquad s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

- The number of degrees of freedom is $df = n_1 + n_2 - 2$.

## Rejection Regions for Hypothesis Tests for Population Means (small samples)

Reject the null hypothesis, $H_0$, if:

$t \le -t_\alpha$ for a left-tailed test

$t \ge t_\alpha$ for a right-tailed test

$|t| \ge t_{\alpha/2}$ for a two-tailed test

# Practice Problem 7

Two universities in the same state are bitter rivals. Each university believes that its students are more physically fit than the students at the other university. To test the claim that there is a difference in the average fitness levels of students at the two universities, 36 randomly selected students at the first university were surveyed and exercised for a mean of 2.9 hours per week. A random sample of 38 students at the second university was also surveyed, and exercised for a mean of 2.7 hours per week. Assume that the population standard deviation for hours of exercise at the first university is known to be 1.1 hours per week and the population standard deviation for the second university is known to be 1.0 hour per week. Use a 0.05 level of significance to perform a hypothesis test to determine if there is a difference in the average fitness levels of students at the two universities.

# Practice Problem 8

| Sample 1 (genetically modified) | Sample 2 (regular) |
|---|---|
| 20 | 21 |
| 23 | 21 |
| 27 | 22 |
| 25 | 18 |
| 25 | 20 |
| 25 | 20 |
| 27 | 18 |
| 23 | 25 |
| 24 | 23 |
| 22 | 20 |

A generic engineering company claims that it has developed a genetically modified tomato plant that yield on average more tomatoes than other varieties. A farmer wants to test the claim on a small scale before committing to a full-scale planting. Ten genetically modified tomato plants are grown from seeds along with ten other tomato plants. At the season's end, the resulting yields in pounds are recorded as below.
Test, at the 1% level of significance, whether the data provide sufficient evidence to conclude that the mean yield of genetically modified variety if greater than that for the standard variety. Assume all populations are normally distributed and they have equal standard deviation.

## Paired Difference

- When two dependent samples consist of paired data, the paired difference for any pair of data values is given by

$$d_i = y_i - x_i$$

where $x_2$ is a data value from the second sample and

$x_1$ is the data value from the first sample that is paired with $x_2$.

- When two dependent samples consist of paired data, the mean of the paired differences for the sample data is given by

$$\bar{d} = \frac{\sum d_i}{n}$$

where $d_i$ is the paired difference for the $i^{th}$ pair of data values and $n$ is the number of paired differences in the sample data.

## Sample Standard Deviation of Paired Differences

- When two dependent samples consist of paired data, the sample standard deviation of the paired differences for the sample data is given by

$$s_d = \sqrt{\frac{\sum(d_i - \bar{d})^2}{n-1}}$$

## Example 1: Calculating Paired Differences and the Mean of Paired Differences

The amounts of home utility bills are given for two consecutive months for 5 different homes in a neighborhood. For each home in the sample, we can pair the billing amounts for March and April. Given the following data, calculate the paired differences and the mean of the paired differences for the five homes.

| Utility Bills for Homes | | Paired Differences $d_i$ |
|---|---|---|
| March | April | |
| $119.75 | $121.06 | 1.31 |
| $68.43 | $79.04 | 10.61 |
| $202.39 | $189.55 | −12.84 |
| $47.88 | $49.64 | 1.76 |
| $66.01 | $68.52 | 2.51 |

$$\bar{d} = \frac{\sum d_i}{n}$$

$$= \frac{1.31 + 10.61 - 12.84 + 1.76 + 2.51}{5}$$

$$= 0.67$$

$$s_d = \sqrt{\frac{\sum (d_i - \bar{d})^2}{n-1}}$$

$$= \sqrt{\frac{(1.31-0.67)^2 + (10.61-0.67)^2 + (-12.84-0.67)^2 + (1.76-0.67)^2 + (2.51-0.67)^2}{5-1}}$$

## Hypothesis Test for the Mean of Two Populations - Paired Samples

$$t = \frac{\bar{d} - \mu_d}{\left(\frac{s_d}{\sqrt{n}}\right)}$$

where $\mu_d$ is the presumed value of the difference of the population means from the null hypothesis.

- The test statistic has Student's t-distribution with $n$ – 1 degrees of freedom.
- The population of differences must be normally distributed.

### Rejection Regions for Hypothesis Test for the Mean of Two Populations - Paired Samples

Reject the null hypothesis, $H_0$, if:

$t \le -t_\alpha$ for a left-tailed test

$t \ge t_\alpha$ for a right-tailed test

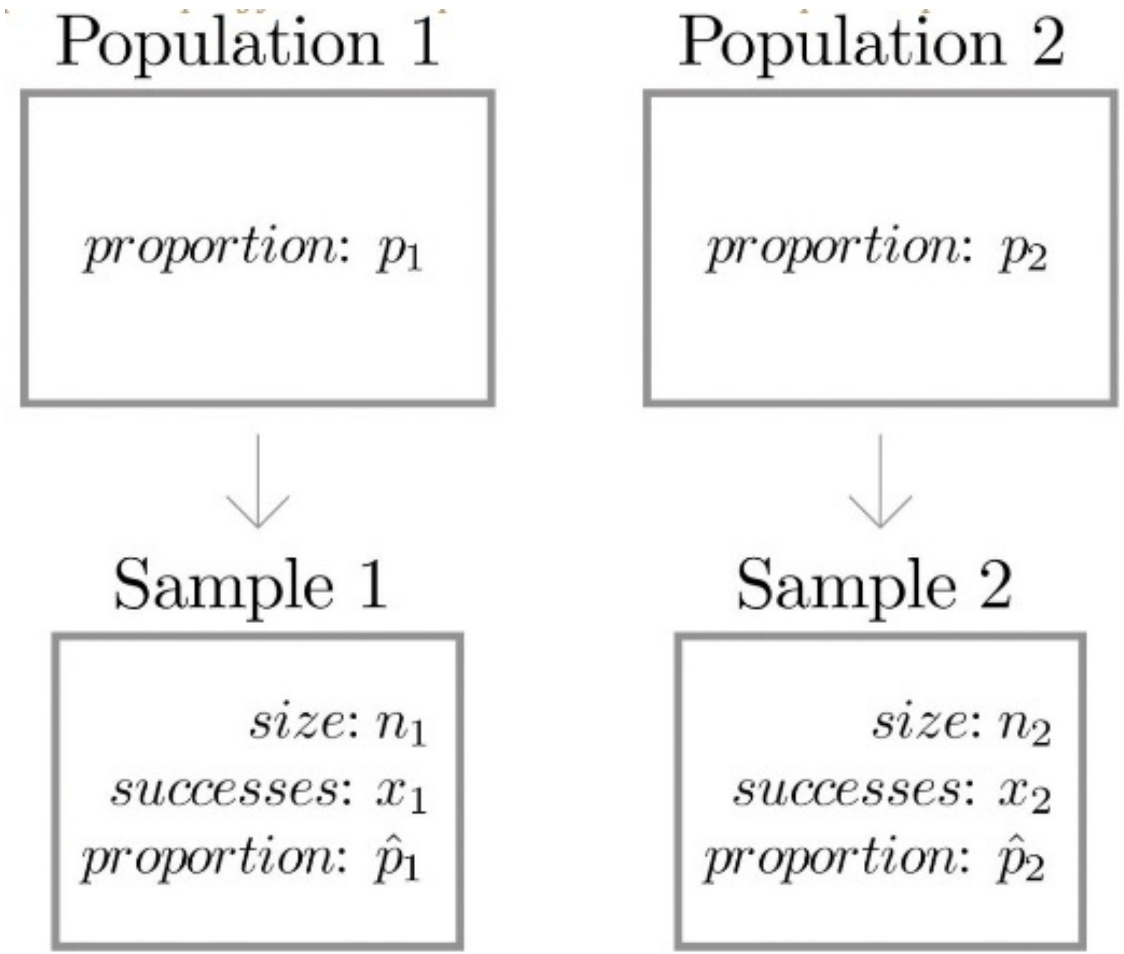
$|t| \ge t_{\alpha/2}$ for a two-tailed test

# Practice Problem 9

Dr. James, a clinical psychologist, wishes to test the claim that there is a significant difference in a person's adult weight if he is raised by his father instead of his mother. Luckily, Dr. James knows of five sets of identical twin boys who were raised separately, one twin by the mother and one twin by the father, and who are willing to participate in a study to help her test her claim. Each twin is weighed and identified as having been raised by his mother or his father. The following table lists the results. Do these data support Dr. James' claim at the 0.01 level of significance?

| Weights of Twins (in Pounds) | | | | | |
|---|---|---|---|---|---|
| Twin Raised by Father | 143.67 | 235.91 | 156.34 | 187.21 | 129.81 |
| Twin Raised by Mother | 134.81 | 221.37 | 163.92 | 193.45 | 131.38 |

# Notations


Population 1
*proportion*: $p_1$
Population 2
*proportion*: $p_2$
Sample 1
*size*: $n_1$
*successes*: $x_1$
*proportion*: $\hat{p}_1$
Sample 2
*size*: $n_2$
*successes*: $x_2$
*proportion*: $\hat{p}_2$

## Hypothesis Test for the Difference Between Two Population Proportions

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$$

- The test statistic has the standard normal distribution.
- The samples must be independent and large, each of the intervals

$$\left[\hat{p}_1 - 3\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}}, \hat{p}_1 + 3\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}}\right] \quad \text{and} \quad \left[\hat{p}_2 - 3\sqrt{\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}, \hat{p}_2 + 3\sqrt{\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}\right]$$

must lie wholly within the interval [0, 1].

# Practice Problem 10

The mayor's chief of staff thinks that a local newspaper article has changed the community's opinion about the mayor. To test his theory, he finds a poll of the mayor's approval rating that was taken before the article came out and compares it to the mayor's approval rating after the article. Before the article ran in the paper, 480 out of a simple random sample of 1200 voters thought the mayor was trustworthy. After the article, 550 out of a simple random sample of 1180 voters thought he was trustworthy. Based on these data, use a 5% level of significance to test the chief of staff's claim that the mayor's approval rating has increased.

## Hypothesis Test for Two Population Variances

Assume the samples taken are independent, simple random samples and both population distributions are approximately normal.

$$F = \frac{s_1^2}{s_2^2}$$

- The test statistic has the $F$-distribution with $df_1 = n_1 - 1$ degrees of freedom for the numerator and $df_2 = n_2 - 1$ degrees of freedom for the denominator, where $n_1$ and $n_2$ are the two sample sizes.

## Rejection Regions for Hypothesis Test for for Two Population Variances

Reject the null hypothesis, $H_0$, if:

$F < F_{1-\alpha}$ for a left-tailed test

$F > F_{\alpha}$ for a right-tailed test

$F < F_{1-\alpha/2}$ or $F > F_{\alpha/2}$ for a left-tailed test

# Practice Problem 11

Suppose that a quality control inspector believes that the candy-making machines on Assembly Line A are not adjusted properly. She thinks that the variance in the sizes of the cady produced by Assembly Line A is greater than the variance in the sizes of candy produced by Assembly Line B. A random sample of 20 pieces of candy is taken from each assembly line and measured. The sizes of the candy from Assembly Line A have a sample variance of 1.45, while the sizes of the candy from Assembly Line B have a sample variance of 0.47. Using a 0.01 level of significance, perform a hypothesis test to test the quality control inspector's claim. Assume that both population are normally distributed.