Homework 2

Important submission instructions:

- All explanations and answers must be clearly and neatly written. Explain each step in your solution. Your solutions should make very clear to the instructor that you understand all of the steps and the logic behind the steps.
- You are allowed to discuss the homework problems with other students (in particular via Canvas discussion board). However, you are not allowed to copy solutions from other students or other sources including AI chatbots. You should list at the end of the problem set the sources you consulted and people you worked with on this assignment including AI prompts.
- The final document should be saved and submitted as a single .pdf file, and please be sure all problem solutions are presented starting from the first to the last (that is, the first solution must correspond to problem 1 and the second to problem 2 and so on).
- Typed submissions (for example in LaTeX) will be positively considered in the grade. Overleaf is an easy avenue to start learning LaTeX. See the tutorials at https://www.overleaf.com/learn
- Honor code applies fully. Your submission should reflect your own understanding of the material. It is prohibited to post the following problems on any website/forum or any other online means.

Problem 1

Prove: Let $Q \in \mathbb{R}^{n\times n}$ symmetric. Then,

$$\lambda_1 x^\top x \leq x^\top Q x \leq \lambda_n x^\top x, \quad \forall x \in \mathbb{R}^n,$$

where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ its evas.

Problem 2

Let $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ symmetric. Prove that

$$Q, R \text{ are psd} \quad \Leftrightarrow \quad \begin{pmatrix} Q & 0 \\ 0^\top & R \end{pmatrix} \text{ is psd},$$

where 0 denote the zero matrix of dimension $n \times m$.

Problem 3

Find all stationary points of the following $\mathbb{R}^2$ functions and classify them (only local):

(a) $f(x,y) = 1 - y^2 - x^4$.

(b) $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

(c) $f(x,y) = (ax^2 + by^2)e^{-x^2+y^2}$, where $a, b \in \mathbb{R}$.

Problem 4

Let h be a real function with continuous and positive second derivative such that $h'(0) = 0$. Let the following function be defined as $f(x,y) := h(x+y)$

(a) Determine the stationary points of f.

(b) Classify them (only local). Hint: note h has a minimum at $x = 0$.

Problem 5

Let $b \neq 0 \in \mathbb{R}^n$. Show that the maximum of $f(x) = b^\top x$ over $X = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is attained at $x^* = \frac{b}{\|b\|}$ and $f(x^*) = \|b\|$. Hint: Use Cauchy-Schwartz inequality to bound f. This problem is a typical example of how to optimize using an inequality.

Problem 6

Show if the following functions are coercive or not.

(a) $f(x,y) = 4x^2 + 2xy + 2y^2$.

(b) $f(x,y) = 2x^2 - 8xy + y^2$.

(c) $f(x,y,z) = x^3 + y^3 + z^3$.

(d) $f(x,y) = x^2 - 2xy^2 + y^4$.

Problem 7

Under what conditions (if any) the following functions are coercive?

(a) $f(x) = a^\top x$, where $a, x \in \mathbb{R}^n$, and $n > 1$.

(b) $f(x) = a^\top x + \delta\|x\|^2$, where $a, x \in \mathbb{R}^n$ and $\delta > 0$.

(c) $f(x) = \|Ax - b\|^2$, where $x, b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$.

Problem 8

Find all stationary points and classify them (local and global).

(a) $f(x, y) = (4x^2 - y)^2$.

(b) $f(x, y) = 2x^2 + 3y^2 - 2xy + 2x - 3y$.

(c) $f(x, y, z) = x^4 - 2x^2 + y^2 + 2yz + 2z^2$.

Problem 9

Consider the problem of minimizing the function

$$f(x, y) = a(x^2 + y^2) + bxy - (x + y) + c$$

over $\mathbb{R}^2$, where $a, b, c \in \mathbb{R}$. Find all values of a, b and c such that the problem has a unique optimal solution.