Homework 4

Important submission instructions:

- All explanations and answers must be clearly and neatly written. Explain each step in your solution. Your solutions should make very clear to the instructor that you understand all of the steps and the logic behind the steps.
- You are allowed to discuss the homework problems with other students (in particular via Canvas discussion board). However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- The final document should be saved and submitted as a single .pdf file, and please be sure all problem solutions are presented starting from the first to the last (that is, the first solution must correspond to problem 1 and the second to problem 2 and so on).
- Typed submissions (for example in LaTeX) will be positively considered in the grade. Overleaf is an easy avenue to start learning LaTeX. See the tutorials at https://www.overleaf.com/learn
- Very important: Honor code applies fully. You must submit your own work only. In particular, it is prohibited to post the following problems on any website/forum or any other virtual means (for example Chegg).

Problem 1

The objective of this exercise is to classify the stationary points locally and globally and observe the effect of the restrictions. Later in the course we will use a more systematic way by the Lagrangian and Karush-Kuhn-Tucker conditions.

a)

$$\begin{aligned} \text{minimize } & 4x^4 - x^3 - 4x^2 + 1 \\ \text{s.t. } & x \in [-1, 1] \,. \end{aligned}$$

b)

$$\begin{aligned} \text{minimize } & (x-a)^2 + 1 \\ \text{s.t. } & x \in [-1, 1] \,, \end{aligned}$$

where $a \in \mathbb{R}$.

c)

$$\begin{aligned} \text{minimize } & \|x-c\|^2 + 1 \\ \text{s.t. } & x \in [0, 1]^2 \,, \end{aligned}$$

where $x \in \mathbb{R}^2$, $c = (2, 1/2)^\top$ and $[0,1]^2 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1,\ \ 0 \le x_2 \le 1\}$.

d)

$$\begin{aligned} \text{minimize} \ \ & (x-2)^2 + y^2 - y \\ \text{s.t.} \ \ & x + y \le 4 \\ & x \ge 0,\ y \ge 0\,. \end{aligned}$$

Hint: Plotting the contour lines of the objetive function and the feasible region in each case may help to solve the problem.

Problem 2

Given the experimental data

```
y = 13.26 14.15 13.86 14.81 15.68 15.64 15.58 15.67 16.08 16.36 16.14 ..
16.68 16.00 15.66 16.05 16.22 16.17 16.03 16.69 15.83 16.21 16.13 ..
16.36 16.42 16.92 16.65 16.94 17.47 18.07 17.52];

x = 1.00 1.13 1.27 1.41 1.55 1.68 1.82 1.96 2.10 2.24 2.37 2.51 2.65 ..
2.79 2.93 3.06 3.20 3.34 3.48 3.62 3.75 3.89 4.03 ..
4.17 4.31 4.44 4.58 4.72 4.86 5.00];
```

1. Find the least-squares line that fit this data.
2. By inspecting the residuals, can you determine if the previous model is suitable?
3. Can you find a better model than 1.? Use residual plot to measure "betterness".

Problem 3

Rewrite each problem minimize$_x f(x)$ as a LS problem. Specify matrix A and vector b, and then solve the problem.

(a) $f(x) = (2x_1 - x_2 + 1)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

(b) $f(x) = (1 - x_1)^2 + \sum_{j=1}^{3}(x_j - x_{j+1})^2$

Problem 4

Consider the LS problem with

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and } b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

(a) Solve the LS problem.

(c) Write down the matrix that represents the orthogonal projection onto col(A).

Problem 5

Write the following function as a quadratic one ($x^\top Q x + 2b^\top x + c$).

(a) $f = (2x - y)^2 + (y - z)^2 + (z - 1)^2$

(b) $f = x^2 + 16xy + 4yz + y^2$

Problem 6

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $D \in \mathbb{R}^{p\times n}$, and $\lambda > 0$. Consider the regularized least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|^2 + \lambda\|Dx\|^2.$$

Show that the problem has a unique solution iff

$$\mathrm{null}(A) \cap \mathrm{null}(D) = \{0\},$$

where the the null space of a linear map T, denoted by $\mathrm{null}(T)$, is the set of vectors x such that $Tx = 0$. A synonym for null space is kernel.

Note that $\{0\}$ is *not* the emptyset.

Problem 7

Let $A \in \mathbb{R}^{m\times n}$, $m \le n$, $\mathrm{rank}(A) = m$, and $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Show that the problem

$$\begin{aligned} \text{minimize } & \|x - c\|^2 \\ s.t. \ & Ax = b \end{aligned}$$

has a unique solution and find the expression for x^*.

Problem 8

Find the MLE of θ in the following cases:

1. Consider the random variable X

x	0	1	2	3
$\mathbb{P}(X = x) = p(x;\theta)$	$\frac{2}{3}\theta$	$\frac{1}{3}\theta$	$\frac{2}{3}(1-\theta)$	$\frac{1}{3}(1-\theta)$

2. Consider a random variable with probability density function

$$f(x,\theta) = \begin{cases} \frac{(1+\theta x)}{2} & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$