

# Algebraic Geometry Exercises

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## 1 Hartshorne – varieties

### 1.1 Affine varieties

**Exercise 1.** (a) Let  $Y$  be the plane curve  $y = x^2$  (i.e.,  $Y$  is the zero set of the polynomial  $f = y - x^2$ ). Show that  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ .

(b) Let  $Z$  be the plane curve  $xy = 1$ . Show that  $A(Z)$  is not isomorphic to a polynomial ring in one variable over  $k$ .

*Solution.* (a)  $y - x^2$  is irreducible, so  $I(Y) = (y - x^2)$ . Therefore  $A(Y) = k[x, y]/(y - x^2) = k[x]$ .

(b) Similarly,  $A(Z) = k[x, y]/(xy - 1) = k[x, x^{-1}]$ . Suppose there existed an isomorphism  $\phi : k[x, x^{-1}] \rightarrow k[t]$ .  $x, x^{-1}$  and all non-zero elements of  $k$  are units in  $k[x, x^{-1}]$ . Therefore, their images under  $\phi$  must be units. However,  $k[t]^\times = k \setminus \{0\}$ , so there is no element of  $k[x, x^{-1}]$  which maps to  $t$ , since the elements of  $k[x, x^{-1}]$  are polynomials in  $x$  and  $x^{-1}$  and  $\phi$  is a ring homomorphism. This contradicts the injectivity of  $\phi$ . ■

**Exercise 2.** *The twisted cubic curve.* Let  $Y \subseteq \mathbb{A}^3$  be the set  $\{(t, t^2, t^3) : t \in k\}$ . Show that  $Y$  is an affine variety of dimension 1. Find generators for the ideal  $I(Y)$ . Show that  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ . We say that  $Y$  is given by the *parametric representation*  $x = t, y = t^2, z = t^3$ .

*Solution.* It is easy to verify that  $Y = Z(y - x^2, z - x^3)$ . Then  $A/(y - x^2, z - x^3) = k[x]$ , which is a principal ideal domain, so in particular an integral domain, meaning  $(y - x^2, z - x^3)$  is prime, so  $Y$  is an affine variety. Also, this tells us that  $I(Y) = (y - x^2, z - x^3)$  since all prime ideals are radical, so  $y - x^2$  and  $z - x^3$

are generators for  $I(Y)$ . We already saw that  $A(Y) = A/(y - x^2, z - x^3) = k[x]$ . To see that  $Y$  has dimension 1, observe that  $\dim A(Y) = \text{trdeg}_k k(x) = 1$ . ■

**Exercise 3.** Let  $Y$  be the algebraic set in  $\mathbb{A}^3$  defined by the two polynomials  $x^2 - yz$  and  $xz - x$ . Show that  $Y$  is a union of three irreducible components. Describe them and find their prime ideals.

*Solution.*

$$\begin{aligned} Y &= Z(x^2 - yz, xz - x) \\ &= Z(x^2 - yz) \cap (Z(x) \cup Z(z - 1)) \\ &= (Z(x^2 - yz) \cap Z(x)) \cup (Z(x^2 - yz) \cap Z(z - 1)) \\ &= ((Z(y) \cup Z(z)) \cap Z(x)) \cup (Z(x^2 - y) \cap Z(z - 1)) \\ &= (Z(y) \cap Z(x)) \cup (Z(z) \cap Z(x)) \cup (Z(x^2 - y) \cap Z(z - 1)) \\ &= Z(x, y) \cup Z(x, z) \cup Z(x^2 - y, z - 1). \end{aligned}$$

These are irreducible because  $(x, y)$ ,  $(x, z)$  and  $(x^2 - y, z - 1)$  are prime ideals of  $k[x, y, z]$ . ■

**Exercise 4.** If we identify  $\mathbb{A}^2$  with  $\mathbb{A}^1 \times \mathbb{A}^1$  in the natural way, show that the Zariski topology on  $\mathbb{A}^2$  is not the product topology of the Zariski topologies on the two copies of  $\mathbb{A}^1$ .

*Solution.* The complement of  $Z(y - x)$  is open in  $\mathbb{A}^2$  by definition, but can be seen not to be open in  $\mathbb{A}^1 \times \mathbb{A}^1$ . The topology on  $\mathbb{A}^1 \times \mathbb{A}^1$  is generated by sets of the form  $U \times V$ , where  $U$  and  $V$  are open subsets of  $\mathbb{A}^1$ . An open subset of  $\mathbb{A}^1$  is either empty or the complement of a finite set of points. However,  $Z(y - x)$  is infinite, so its complement cannot be written as a union of sets of the form  $U \times V$ . ■

**Exercise 5.** Show that a  $k$ -algebra  $B$  is isomorphic to the affine coordinate ring of some algebraic set in  $\mathbb{A}^n$ , for some  $n$ , if and only if  $B$  is a finitely generated  $k$ -algebra with no nilpotent elements.

*Solution.* We already know that for all algebraic sets  $Y \subseteq \mathbb{A}^n$ , the coordinate ring  $A(Y)$  is finitely generated. We will show that  $A(Y)$  has no nilpotent elements. Suppose  $f \in A(Y)$  is nilpotent. We can think of  $f$  as a polynomial such that  $f^r \in I(Y)$  for some positive integer  $r$ . Since  $Y$  is algebraic,  $I(Y)$  is radical, so  $f \in I(Y)$ , meaning  $f$  is the zero element of  $A(Y)$ .

Now we show that if  $B$  is a finitely generated  $k$ -algebra with no nilpotent elements then it is isomorphic to a coordinate ring.  $B$  is finitely generated, say by  $n$  elements, so there is a surjective homomorphism  $\phi : A \rightarrow B$ . It is surjective, so by the isomorphism theorem  $A/\ker \phi \cong B$ . Let  $Y = Z(\ker \phi)$ .  $\phi$  is a homomorphism, so if  $f^r \in \ker \phi$ , then that means  $\phi(f^r) = \phi(f)^r = 0$ ,

implying  $f \in \ker \phi$ , since  $B$  has no nilpotent elements. In other words  $\ker \phi$  is a radical ideal. Therefore  $A(Y) = A/I(Y) = A/\ker \phi \cong B$ . ■

**Exercise 6.** Any non-empty open subset of an irreducible topological space is dense and irreducible. If  $Y$  is subset of a topological space  $X$ , which is irreducible in its induced topology, then the closure  $\bar{Y}$  is also irreducible.

*Solution.* Let  $Z$  be an irreducible topological space and  $U \subseteq Z$  be non-empty and open. Let  $V = Z \setminus U$  be the complement of  $U$ .  $V$  is closed. We can write  $Z = \bar{U} \cup V$ , a union of closed sets.  $Z$  is irreducible, so either  $\bar{U} = Z$  or  $V = Z$ .  $U$  was non-empty, so we cannot have  $V = Z$ . Therefore  $\bar{U} = Z$ . Suppose, for contradiction, that  $U$  is not irreducible. Then we could write  $U = U_1 \cup U_2$ , where  $U_1$  and  $U_2$  are closed proper subsets of  $U$ . There exist closed sets  $W_1$  and  $W_2$  such that  $U_1 = U \cap W_1$  and  $U_2 = U \cap W_2$ . Then we can write  $Z = (W_1 \cup W_2) \cup V$ . We know  $Z$  is irreducible and  $Z \neq V$ , so  $Z = W_1 \cup W_2$ . Again by irreducibility, we have, without loss of generality, that  $Z = W_1$ . Then  $U_1 = U \cap Z = U$ , which contradicts  $U_1$  being a proper subset.

Suppose, for contradiction, that  $\bar{Y}$  is not irreducible. Then we can write  $\bar{Y} = Y_1 \cup Y_2$ , where  $Y_1$  and  $Y_2$  are closed proper subsets of  $\bar{Y}$ . There exist closed sets  $V_1$  and  $V_2$  such that  $Y_1 = \bar{Y} \cap V_1$  and  $Y_2 = \bar{Y} \cap V_2$ . Then we can write  $Y = (\bar{Y} \cap V_1 \cap Y) \cup (\bar{Y} \cap V_2 \cap Y)$ . Since  $Y$  is irreducible,  $Y = \bar{Y} \cap V_1 \cap Y$  without loss of generality. Therefore  $Y \subseteq \bar{Y} \cap V_1 \subseteq \bar{Y}$ . Since  $\bar{Y}$  is the minimal closed set containing  $Y$ , we have  $\bar{Y} \cap V_1 = \bar{Y}$ . Therefore  $Y_1 = \bar{Y}$ , which is a contradiction. ■

**Exercise 7.** (a) Show that the following conditions are equivalent for a topological space  $X$ : (i)  $X$  is Noetherian; (ii) every non-empty family of closed subsets has a minimal element; (iii)  $X$  satisfies the ascending chain condition for open sets; (iv) every non-empty family of open sets has a maximal element.

- (b) A Noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.
- (c) Any subset of a Noetherian topological space must be Noetherian in its induced topology.
- (d) A Noetherian space which is also Hausdorff must be a finite set with the discrete topology.

*Solution.* (a) (i)  $\Rightarrow$  (ii). If there were a non-empty family without a minimal element, then we could construct a descending chain that does not stabilise.

(ii)  $\Rightarrow$  (i). Given a descending chain  $(Y_i)_{i=1}^{\infty}$ , we know there is a minimal element, say  $Y_r$ . Then for all  $s > r$ , we know  $Y_s \subseteq Y_r$ , but  $Y_r$  is minimal, so  $Y_s = Y_r$ .

Similarly (iii)  $\Leftrightarrow$  (iv).

(i) $\Leftrightarrow$ (iii). Take complements.

- (b) Let  $X$  be a Noetherian topological space and let  $\{U_\alpha\}$  be an open cover of  $X$ . Suppose  $\{U_\alpha\}$  does not have a finite subcover. Choose a set  $U_1$  from the cover.  $\{U_1\}$  is not a cover, so there exists a set  $U_2 \not\subseteq U_1$ .  $\{U_1, U_2\}$  is not a cover, so there exists a set  $U_3 \not\subseteq U_1 \cup U_2$ . Continuing to choose sets in this way, we obtain a countable subcollection such that, by construction,

$$X \setminus U_1 \supsetneq X \setminus (U_1 \cup U_2) \supsetneq \cdots$$

is a non-stabilising descending chain of closed subsets of  $X$ , a contradiction.

- (c) Let  $X$  be a Noetherian topological space,  $Y \subseteq X$  a subset and  $V_1 \supseteq V_2 \supseteq \cdots$  a descending chain of closed subsets of  $Y$ . Then, for all  $i$ , there exist closed sets  $W_i$  such that  $V_i = Y \cap W_i$ . Since  $X$  is Noetherian,  $W_1 \supseteq W_1 \cap W_2 \supseteq \cdots$  stabilises, so

$$Y \cap W_1 \supseteq Y \cap (W_1 \cap W_2) \supseteq \cdots$$

also stabilises. Notice that for all  $i$ ,

$$\begin{aligned} Y \cap (W_1 \cap \cdots \cap W_i) &= (Y \cap W_1) \cap \cdots \cap (Y \cap W_i) \\ &= V_1 \cap \cdots \cap V_i \\ &= V_i, \end{aligned}$$

so  $V_1 \supseteq V_2 \supseteq \cdots$  stabilises.

- (d) Let  $X$  be a Noetherian Hausdorff space. We can write  $X$  as a union of closed irreducible subsets  $X = Y_1 \cup \cdots \cup Y_n$ .  $X$  is Hausdorff so all  $Y_i$  are Hausdorff. Given two distinct points  $x, y \in Y_i$  there are disjoint open neighbourhoods  $U_x$  of  $x$  and  $U_y$  of  $y$ . Then  $(Y_i \setminus U_x) \cup (Y_i \setminus U_y) = Y_i$ , which contradicts irreducibility of  $Y_i$ , so there are not two distinct points in  $Y_i$ . The empty set is not irreducible so  $Y_i$  is a singleton. This holds for all  $i$ , so  $X$  is finite, and a finite Hausdorff space is discrete. ■

**Exercise 8.** Let  $Y$  be an affine variety of dimension  $r$  in  $\mathbb{A}^n$ . Let  $H$  be a hypersurface in  $\mathbb{A}^n$ , and assume that  $Y \not\subseteq H$ . Then every irreducible component of  $Y \cap H$  has dimension  $r - 1$ .