

# Algebraic Geometry Exercises

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## 1 Hartshorne – varieties

### 1.1 Affine varieties

**Exercise 1.** (a) Let  $Y$  be the plane curve  $y = x^2$  (i.e.,  $Y$  is the zero set of the polynomial  $f = y - x^2$ ). Show that  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ .

(b) Let  $Z$  be the plane curve  $xy = 1$ . Show that  $A(Z)$  is not isomorphic to a polynomial ring in one variable over  $k$ .

*Solution.* (a)  $y - x^2$  is irreducible, so  $I(Y) = (y - x^2)$ . Therefore  $A(Y) = k[x, y]/(y - x^2) = k[x]$ .

(b) Similarly,  $A(Z) = k[x, y]/(xy - 1) = k[x, x^{-1}]$ . Suppose there existed an isomorphism  $\phi : k[x, x^{-1}] \rightarrow k[t]$ .  $x, x^{-1}$  and all non-zero elements of  $k$  are units in  $k[x, x^{-1}]$ . Therefore, their images under  $\phi$  must be units. However,  $k[t]^\times = k \setminus \{0\}$ , so there is no element of  $k[x, x^{-1}]$  which maps to  $t$ , since the elements of  $k[x, x^{-1}]$  are polynomials in  $x$  and  $x^{-1}$  and  $\phi$  is a ring homomorphism. This contradicts the injectivity of  $\phi$ . ■