Algebraic Geometry Exercises

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1 Hartshorne – varieties

1.1 Affine varieties

Exercise 1. (a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

Solution. (a) $y - x^2$ is irreducible, so $I(Y) = (y - x^2)$. Therefore $A(Y) = k[x, y]/(y - x^2) = k[x]$.

(b) Similarly, $A(Z) = k[x, y]/(xy - 1) = k[x, x^{-1}]$. Suppose there existed an isomorphism $\phi : k[x, x^{-1}] \to k[t]$. x, x^{-1} and all non-zero elements of k are units in $k[x, x^{-1}]$. Therefore, their images under ϕ must be units. However, $k[t]^\times = k \setminus \{0\}$, so there is no element of $k[x, x^{-1}]$ which maps to t, since the elements of $k[x, x^{-1}]$ are polynomials in x and x^{-1} and ϕ is a ring homomorphism. This contradicts the injectivity of ϕ.

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Exercise 2. *The twisted cubic curve.* Let $Y \subseteq \mathbb{A}^3$ be the set $\{(t, t^2, t^3) : t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t$, $y = t^2$, $z = t^3$.

Solution. It is easy to verify that $Y = Z(y - x^2, z - x^3)$. Then $A/(y - x^2, z - x^3) = k[x]$, which is a principal ideal domain, so in particular an integral domain, meaning $(y - x^2, z - x^3)$ is prime, so Y is an affine variety. Also, this tells us that $I(Y) = (y - x^2, z - x^2)$ since all prime ideals are radical, so $y - x^2$ and $z - x^2$

are generators for $I(Y)$. We already saw that $A(Y) = A/(y - x^2, z - x^3) = k[x]$. To see that Y has dimension 1, observe that $\dim A(Y) = \operatorname{trdeg}_k k(x) = 1$. ■