Algebraic Geometry Exercises

Akshay Popat

January 11, 2026

Contents

1 Hartshorne – varieties

1.1 Affine varieties

Exercise 1. (a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

Solution. (a) $y - x^2$ is irreducible, so $I(Y) = (y - x^2)$. Therefore $A(Y) = k[x, y]/(y - x^2) = k[x]$.

(b) Similarly, $A(Z) = k[x, y]/(xy - 1) = k[x, x^{-1}]$. Suppose there existed an isomorphism $\phi : k[x, x^{-1}] \to k[t]$. x, x^{-1} and all non-zero elements of k are units in $k[x, x^{-1}]$. Therefore, their images under ϕ must be units. However, $k[t]^\times = k \setminus \{0\}$, so there is no element of $k[x, x^{-1}]$ which maps to t, since the elements of $k[x, x^{-1}]$ are polynomials in x and x^{-1} and ϕ is a ring homomorphism. This contradicts the injectivity of ϕ.

■

Exercise 2. *The twisted cubic curve.* Let $Y \subseteq \mathbb{A}^3$ be the set $\{(t, t^2, t^3) : t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t$, $y = t^2$, $z = t^3$.

Solution. It is easy to verify that $Y = Z(y-x^2, z-x^3)$. Then $A/(y-x^2, z-x^3) = k[x]$, which is a principal ideal domain, so in particular an integral domain, meaning $(y - x^2, z - x^3)$ is prime, so Y is an affine variety. Also, this tells us that $I(Y) = (y-x^2, z-x^2)$ since all prime ideals are radical, so $y-x^2$ and $z-x^2$

are generators for $I(Y)$. We already saw that $A(Y) = A/(y-x^2, z-x^3) = k[x]$. To see that Y has dimension 1, observe that $\dim A(Y) = \operatorname{trdeg}_k k(x) = 1$. ■

Exercise 3. Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

Solution.

$$\begin{aligned}
Y &= Z(x^2 - yz, xz - x) \\
&= Z(x^2 - yz) \cap (Z(x) \cup Z(z-1)) \\
&= (Z(x^2 - yz) \cap Z(x)) \cup (Z(x^2 - yz) \cap Z(z-1)) \\
&= ((Z(y) \cup Z(z)) \cap Z(x)) \cup (Z(x^2 - y) \cap Z(z-1)) \\
&= (Z(y) \cap Z(x)) \cup (Z(z) \cap Z(x)) \cup (Z(x^2 - y) \cap Z(z-1)) \\
&= Z(x, y) \cup Z(x, z) \cup Z(x^2 - y, z - 1).
\end{aligned}$$

These are irreducible because (x, y), (x, z) and $(x^2 - y, z - 1)$ are prime ideals of $k[x, y, z]$. ■

Exercise 4. If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Solution. The complement of $Z(y - x)$ is open in $\mathbb{A}^2$ by definition, but can be seen not to be open in $\mathbb{A}^1 \times \mathbb{A}^1$. The topology on $\mathbb{A}^1 \times \mathbb{A}^1$ is generated by sets of the form $U \times V$, where U and V are open subsets of $\mathbb{A}^1$. An open subset of $\mathbb{A}^1$ is either empty or the complement of a finite set of points. However, $Z(y - x)$ is infinite, so its complement cannot be written as a union of sets of the form $U \times V$. ■

Exercise 5. Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

Solution. We already know that for all algebraic sets $Y \subseteq \mathbb{A}^n$, the coordinate ring $A(Y)$ is finitely generated. We will show that $A(Y)$ has no nilpotent elements. Suppose $f \in A(Y)$ is nilpotent. We can think of f as a polynomial such that $f^r \in I(Y)$ for some positive integer r. Since Y is algebraic, $I(Y)$ is radical, so $f \in I(Y)$, meaning f is the zero element of $A(Y)$.

Now we show that if B is a finitely generated k-algebra with no nilpotent elements then it is isomorphic to a coordinate ring. B is finitely generated, say by n elements, so there is a surjective homomorphism $\phi : A \to B$. It is surjective, so by the isomorphism theorem $A/\ker\phi \cong B$. Let $Y = Z(\ker\phi)$. ϕ is a homomorphism, so if $f^r \in \ker\phi$, then that means $\phi(f^r) = \phi(f)^r = 0$, implying $f \in \ker\phi$, since B has no nilpotent elements. In other words $\ker\phi$ is a radical ideal. Therefore $A(Y) = A/I(Y) = A/\ker\phi \cong B$. ∎

Exercise 6. Any non-empty open subset of an irreducible topological space is dense and irreducible. If Y is subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Solution. Let Z be an irreducible topological space and $U \subseteq Z$ be non-empty and open. Let $V = Z \setminus U$ be the complement of U. V is closed. We can write $Z = \overline{U} \cup V$, a union of closed sets. Z is irreducible, so either $\overline{U} = Z$ or $V = Z$. U was non-empty, so we cannot have $V = Z$. Therefore $\overline{U} = Z$. Suppose, for contradiction, that U is not irreducible. Then we could write $U = U_1 \cup U_2$, where U_1 and U_2 are closed proper subsets of U. There exist closed sets W_1 and W_2 such that $U_1 = U \cap W_1$ and $U_2 = U \cap W_2$. Then we can write $Z = (W_1 \cup W_2) \cup V$. We know Z is irreducible and $Z \neq V$, so $Z = W_1 \cup W_2$. Again by irreducibility, we have, without loss of generality, that $Z = W_1$. Then $U_1 = U \cap Z = U$, which contradicts U_1 being a proper subset.

Suppose, for contradiction, that $\overline{Y}$ is not irreducible. Then we can write $\overline{Y} = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed proper subsets of $\overline{Y}$. There exist closed sets V_1 and V_2 such that $Y_1 = \overline{Y} \cap V_1$ and $Y_2 = \overline{Y} \cap V_2$. Then we can write $Y = (\overline{Y} \cap V_1 \cap Y) \cup (\overline{Y} \cap V_2 \cap Y)$. Since Y is irreducible, $Y = \overline{Y} \cap V_1 \cap Y$ without loss of generality. Therefore $Y \subseteq \overline{Y} \cap V_1 \subseteq \overline{Y}$. Since $\overline{Y}$ is the minimal closed set containing Y, we have $\overline{Y} \cap V_1 = \overline{Y}$. Therefore $Y_1 = \overline{Y}$, which is a contradiction. ∎

Exercise 7. (a) Show that the following conditions are equivalent for a topological space X: (i) X is Noetherian; (ii) every non-empty family of closed subsets has a minimal element; (iii) X satisfies the ascending chain condition for open sets; (iv) every non-empty family of open sets has a maximal element.

(b) A Noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.

(c) Any subset of a Noetherian topological space must be Noetherian in its induced topology.

(d) A Noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Solution. (a) (i)⇒(ii). If there were a non-empty family without a minimal element, then we could construct a descending chain that does not stabilise.

(ii)⇒(i). Given a descending chain $(Y_i)_{i=1}^{\infty}$, we know there is a minimal element, say Y_r. Then for all $s > r$, we know $Y_s \subseteq Y_r$, but Y_r is minimal, so $Y_s = Y_r$.

Similarly (iii)⇔(iv).

(i)$\Leftrightarrow$(iii). Take complements.

(b) Let X be a Noetherian topological space and let $\{U_\alpha\}$ be an open cover of X. Suppose $\{U_\alpha\}$ does not have a finite subcover. Choose a set U_1 from the cover. $\{U_1\}$ is not a cover, so there exists a set $U_2 \not\subseteq U_1$. $\{U_1, U_2\}$ is not a cover, so there exists a set $U_3 \not\subseteq U_1 \cup U_2$. Continuing to choose sets in this way, we obtain a countable subcollection such that, by construction,

$$X \setminus U_1 \supsetneq X \setminus (U_1 \cup U_2) \supsetneq \cdots$$

is a non-stabilising descending chain of closed subsets of X, a contradiction.

(c) Let X be a Noetherian topological space, $Y \subseteq X$ a subset and $V_1 \supseteq V_2 \supseteq \cdots$ a descending chain of closed subsets of Y. Then, for all i, there exist closed sets W_i such that $V_i = Y \cap W_i$. Since X is Noetherian, $W_1 \supseteq W_1 \cap W_2 \supseteq \cdots$ stabilises, so

$$Y \cap W_1 \supseteq Y \cap (W_1 \cap W_2) \supseteq \cdots$$

also stabilises. Notice that for all i,

$$\begin{aligned} Y \cap (W_1 \cap \cdots \cap W_i) &= (Y \cap W_1) \cap \cdots \cap (Y \cap W_i) \\ &= V_1 \cap \cdots \cap V_i \\ &= V_i, \end{aligned}$$

so $V_1 \supseteq V_2 \supseteq \cdots$ stabilises.

(d) Let X be a Noetherian Hausdorff space. We can write X as a union of closed irreducible subsets $X = Y_1 \cup \cdots \cup Y_n$. X is Hausdorff so all Y_i are Hausdorff. Given two distinct points $x, y \in Y_i$ there are disjoint open neighbourhoods U_x of x and U_y of y. Then $(Y_i \setminus U_x) \cup (Y_i \setminus U_y) = Y_i$, which contradicts irreducibility of Y_i, so there are not two distinct points in Y_i. The empty set is not irreducible so Y_i is a singleton. This holds for all i, so X is finite, and a finite Hausdorff space is discrete.

■

Exercise 8. Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

Solution. Write $H = Z(f)$ for some irreducible non-constant polynomial $f \in A$. If $Y \cap Z(f) = \varnothing$, then the statement is vacuously true. Assume $Y \cap Z(f) \neq \varnothing$. Write $Y \cap Z(f) = Y_1 \cup \cdots \cup Y_m$, a union of its irreducible components. Let $i \in \{1, \ldots, m\}$ be arbitrary. $Y_i \subseteq Y$, so let $\mathfrak{p}$ be the ideal in $A(Y)$ corresponding to Y_i. Then $\dim Y_i = \dim A(Y)/\mathfrak{p}$, since closed irreducible subsets of Y_i correspond to prime ideals of A containing $I(Y_i)$, and $I(Y_i) \supseteq I(Y)$, so those prime ideals correspond to prime ideals in $A(Y)$ containing $\mathfrak{p}$, which correspond to prime ideals in $A(Y)/\mathfrak{p}$. Since Y is a variety, $A(Y)$ is an integral domain and a finitely generated k-algebra. Therefore it suffices to show that height $\mathfrak{p} = 1$.

$Y_i \subseteq Z(f)$, so $[f] \in \mathfrak{p}$. Prime ideals of $A(Y)$ containing $[f]$ are in bijective correspondence with prime ideals of A containing $(I(Y), f)$. Therefore if there is a prime ideal $\mathfrak{q} \subsetneq \mathfrak{p}$, then there is a corresponding prime ideal $\widetilde{\mathfrak{q}}$ such that $(I(Y), f) \subseteq \widetilde{\mathfrak{q}} \subsetneq I(Y_i)$, and therefore $Y_i \subsetneq Z(\widetilde{\mathfrak{q}}) \subseteq Z((I(Y), f))$. However, since $Z((I(Y), f)) = Y \cap Z(f)$, this contradicts the unique decomposition of $Y \cap Z(f)$ into irreducible components. Therefore $\mathfrak{p}$ is a minimal prime ideal containing $[f]$. $A(Y)$ is a Noetherian ring. $Y \not\subseteq Z(f)$, so $[f] \neq 0$, and $A(Y)$ is an integral domain, so $[f]$ is not a zero-divisor. Also, $[f]$ is not a unit: if it were then for all $y \in Y$, we would have $f(y) \neq 0$, meaning $Y \cap Z(f) = \varnothing$. Therefore, by Krull's Hauptidealsatz, height $\mathfrak{p} = 1$. $\blacksquare$