Algebraic Geometry Exercises

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Contents

1 Hartshorne – varieties

1.1 Affine varieties

Exercise 1. (a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

Solution. (a) $y - x^2$ is irreducible, so $I(Y) = (y - x^2)$. Therefore $A(Y) = k[x, y]/(y - x^2) = k[x]$.

(b) Similarly, $A(Z) = k[x, y]/(xy - 1) = k[x, x^{-1}]$. Suppose there existed an isomorphism $\phi : k[x, x^{-1}] \to k[t]$. x, x^{-1} and all non-zero elements of k are units in $k[x, x^{-1}]$. Therefore, their images under ϕ must be units. However, $k[t]^\times = k \setminus \{0\}$, so there is no element of $k[x, x^{-1}]$ which maps to t, since the elements of $k[x, x^{-1}]$ are polynomials in x and x^{-1} and ϕ is a ring homomorphism. This contradicts the injectivity of ϕ.

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Exercise 2. *The twisted cubic curve.* Let $Y \subseteq \mathbb{A}^3$ be the set $\{(t, t^2, t^3) : t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t$, $y = t^2$, $z = t^3$.

Solution. It is easy to verify that $Y = Z(y-x^2, z-x^3)$. Then $A/(y-x^2, z-x^3) = k[x]$, which is a principal ideal domain, so in particular an integral domain, meaning $(y - x^2, z - x^3)$ is prime, so Y is an affine variety. Also, this tells us that $I(Y) = (y-x^2, z-x^2)$ since all prime ideals are radical, so $y-x^2$ and $z-x^2$

are generators for $I(Y)$. We already saw that $A(Y) = A/(y - x^2, z - x^3) = k[x]$. To see that Y has dimension 1, observe that $\dim A(Y) = \operatorname{trdeg}_k k(x) = 1$. ■

Exercise 3. Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

Solution.

$$
\begin{aligned}
Y &= Z(x^2 - yz, xz - x) \\
&= Z(x^2 - yz) \cap (Z(x) \cup Z(z-1)) \\
&= (Z(x^2 - yz) \cap Z(x)) \cup (Z(x^2 - yz) \cap Z(z-1)) \\
&= ((Z(y) \cup Z(z)) \cap Z(x)) \cup (Z(x^2 - y) \cap Z(z-1)) \\
&= (Z(y) \cap Z(x)) \cup (Z(z) \cap Z(x)) \cup (Z(x^2 - y) \cap Z(z-1)) \\
&= Z(x, y) \cup Z(x, z) \cup Z(x^2 - y, z - 1).
\end{aligned}
$$

These are irreducible because (x, y), (x, z) and $(x^2 - y, z - 1)$ are prime ideals of $k[x, y, z]$. ■

Exercise 4. If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Solution. The complement of $Z(y - x)$ is open in $\mathbb{A}^2$ by definition, but can be seen not to be open in $\mathbb{A}^1 \times \mathbb{A}^1$. The topology on $\mathbb{A}^1 \times \mathbb{A}^1$ is generated by sets of the form $U \times V$, where U and V are open subsets of $\mathbb{A}^1$. An open subset of $\mathbb{A}^1$ is either empty or the complement of a finite set of points. However, $Z(y - x)$ is infinite, so its complement cannot be written as a union of sets of the form $U \times V$. ■

Exercise 5. Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

Solution. We already know that for all algebraic sets $Y \subseteq \mathbb{A}^n$, the coordinate ring $A(Y)$ is finitely generated. We will show that $A(Y)$ has no nilpotent elements. Suppose $f \in A(Y)$ is nilpotent. We can think of f as a polynomial such that $f^r \in I(Y)$ for some positive integer r. Since Y is algebraic, $I(Y)$ is radical, so $f \in I(Y)$, meaning f is the zero element of $A(Y)$.

Now we show that if B is a finitely generated k-algebra with no nilpotent elements then it is isomorphic to a coordinate ring. B is finitely generated, say by n elements, so there is a surjective homomorphism $\phi : A \to B$. It is surjective, so by the isomorphism theorem $A/\ker\phi \cong B$. Let $Y = Z(\ker\phi)$. ϕ is a homomorphism, so if $f^r \in \ker\phi$, then that means $\phi(f^r) = \phi(f)^r = 0$, implying $f \in \ker \phi$, since B has no nilpotent elements. In other words $\ker \phi$ is a radical ideal. Therefore $A(Y) = A/I(Y) = A/\ker \phi \cong B$. ■

Exercise 6. Any non-empty open subset of an irreducible topological space is dense and irreducible. If Y is subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Solution. Let Z be an irreducible topological space and $U \subseteq Z$ be non-empty and open. Let $V = Z \setminus U$ be the complement of U. V is closed. We can write $Z = \overline{U} \cup V$, a union of closed sets. Z is irreducible, so either $\overline{U} = Z$ or $V = Z$. U was non-empty, so we cannot have $V = Z$. Therefore $\overline{U} = Z$. Suppose, for contradiction, that U is not irreducible. Then we could write $U = U_1 \cup U_2$, where U_1 and U_2 are closed proper subsets of U. Then U_1 and U_2 are also closed in Z, and therefore their union U is closed in Z. Then we could write $Z = U \cup V$. Since Z is irreducible, either $U = Z$ or $V = Z$, but U is non-empty so we know $V \neq Z$. Therefore $U = Z$, but then U is irreducible.

Suppose, for contradiction, that $\overline{Y}$ is not irreducible. Then we can write $\overline{Y} = Y_1 \cup Y_2$, where Y_1 and Y_2 are closed proper subsets of $\overline{Y}$. For $i = 1, 2$, $X \setminus \overline{Y}$ and $\overline{Y} \setminus Y_i$ are open, so $X \setminus Y_i = (X \setminus Y) \cup (\overline{Y} \setminus Y_i)$ is open. Therefore Y_i is closed, so $Y \cap Y_i$ is closed in Y. Then, we can write $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$. Irreducibility of Y implies, without loss of generality, that $Y \subseteq Y_1$. Overall we have $Y \subseteq Y_1 \subsetneq \overline{Y}$, a contradiction, since $\overline{Y}$ is the minimal closed set containing Y. ■