Hand in Assignment 1

Akshay Seethanadi Aravind

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1 Question 1(a)

- The generalized co-ordinates q for the system is defined as

$$q = \begin{bmatrix} P_1 \\ \theta \\ \phi \end{bmatrix} \tag{1}$$

where θ and ϕ are the angles with respect to the x and z axes respectively, P_1 is the position vector for the Helicopter mass m_1.

- The position vector for the hovering mass m_2 is P_2.The position P_2 is defined in terms of P_1, θ, ϕ and L. L is the length of the cable connecting the two masses.

$$P_1 = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix}; P_2 = \begin{bmatrix} p_{11} + L\sin(\phi)\cos(\theta) \\ p_{12} + L\sin(\theta)\sin(\phi) \\ p_{13} - L\cos(\phi) \end{bmatrix} \tag{2}$$

- The first order time derivative of q, P_1 and P_2 are as follows:

$$\dot{P_1} = \begin{bmatrix} \dot{p_{11}} \\ \dot{p_{12}} \\ \dot{p_{13}} \end{bmatrix}; \dot{P_2} = \begin{bmatrix} \dot{p_{11}} + L\dot{\phi}\cos(\phi)\cos(\theta) - L\dot{\theta}\sin(\phi)\sin(\theta) \\ \dot{p_{12}} + L\dot{\phi}\cos(\phi)\sin(\theta) + L\dot{\theta}\cos(\theta)\sin(\phi) \\ \dot{p_{13}} + L\dot{\phi}\sin(\phi) \end{bmatrix}; \dot{q} = \begin{bmatrix} \dot{P_1} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{3}$$

- The Kinetic energy T is calculated as follows:

$$T = \frac{1}{2} * m_1 * \dot{P_1^T} * \dot{P_1} + \frac{1}{2} * m_2 * \dot{P_2^T} * \dot{P_2} \tag{4}$$

- Potential energy V is calculated as follows:

$$V = m_1 * g * \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} * P_1 + m_2 * g * \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} * P_2 \tag{5}$$

- The Lagrange equation therefore can be written as: $LF = T - V$.
- The Euler-Lagrange equation is given as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial LF}{\partial \dot{q}}\right) - \frac{\partial LF}{\partial q} = Q^T \tag{6}$$

  also written as

$$\frac{\partial}{\partial \dot{q}}\left(\frac{\partial LF}{\partial \dot{q}}\right)\ddot{q} + \frac{\partial}{\partial q}\left(\frac{\partial LF}{\partial \dot{q}}\right)\dot{q} - \left(\frac{\partial LF}{\partial q}\right)^T = Q \tag{7}$$

  where $\ddot{q}$ is the second-order time derivative of q and Q is the generalized work generated from external forces acting on the system.

- External forces u acting on the Helicopter along its axes are defined as follows:

$$u = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix} \tag{8}$$

- Amount of work generated Q on the system when moving in the generalized coordinates is given as follows:

$$Q = \left(\frac{\partial P_1}{\partial q}\right)^T u = \nabla_q P_1 u \tag{9}$$

- The equation (7) can be written in the following form:

$$M(q) * \dot{v} = b(q, \dot{q}, u) \tag{10}$$

  where v=$\dot{q}$ and the matrices *M(q)* and b are as follows:

$$M = \frac{\partial}{\partial \dot{q}}\left(\frac{\partial LF}{\partial \dot{q}}\right) \tag{11}$$

$$b(q, \dot{q}, u) = Q - \frac{\partial}{\partial q}\left(\frac{\partial LF}{\partial \dot{q}}\right)\dot{q} + \left(\frac{\partial LF}{\partial q}\right)^T \tag{12}$$

2 Question 1(b)

- The generalized co-ordinates q for the system is defined as

$$q = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{13}$$

where P_1 is the position vector for the Helicopter mass m_1 and P_2 is the position vector for the hovering mass m_2.

- The position vectors P_1 and P_2 are defined along the three Cartesian coordinates as follows:

$$P_1 = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix}; P_2 = \begin{bmatrix} p_{21} \\ p_{22} \\ p_{23} \end{bmatrix} \tag{14}$$

- The first order time derivative of q, P_1 and P_2 are as follows:

$$\dot{P_1} = \begin{bmatrix} \dot{p_{11}} \\ \dot{p_{12}} \\ \dot{p_{13}} \end{bmatrix}; \dot{P_2} = \begin{bmatrix} \dot{p_{21}} \\ \dot{p_{22}} \\ \dot{p_{23}} \end{bmatrix}; \dot{q} = \begin{bmatrix} \dot{P_1} \\ \dot{P_2} \end{bmatrix} \tag{15}$$

- The Kinetic energy T is calculated as follows:

$$T = \frac{1}{2} * m_1 * \dot{P_1^T} * \dot{P_1} + \frac{1}{2} * m_2 * \dot{P_2^T} * \dot{P_2} \tag{16}$$

- Potential energy V is calculated as follows:

$$V = m_1 * g * \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} * P_1 + m_2 * g * \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} * P_2 \tag{17}$$

- The Constraint function C for this model is given as:

$$C = \frac{1}{2}(e^T e - L^2) \tag{18}$$

where

$$e = P_1 - P_2 \tag{19}$$

- The Lagrange equation therefore can be written as: $LF = T - V - Z^T C$, where the Lagrange multiplier $Z^T = Z$ since C is scalar .

- The Euler-Lagrange equation is given as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial LF}{\partial \dot{q}}\right) - \frac{\partial LF}{\partial q} = Q^T \tag{20}$$

$$C(q) = 0 \tag{21}$$

also written as

$$\frac{\partial}{\partial \dot{q}}\left(\frac{\partial LF}{\partial \dot{q}}\right)\ddot{q} + \frac{\partial}{\partial q}\left(\frac{\partial LF}{\partial \dot{q}}\right)\dot{q} - \left(\frac{\partial LF}{\partial q}\right)^T = Q \tag{22}$$

$$C(q) = 0 \tag{23}$$

where $\ddot{q}$ is the second-order time derivative of q and Q is the generalized work generated from external forces acting on the system.

- External forces u acting on the Helicopter along its axes are defined as follows:

$$u = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix} \tag{24}$$

- Amount of work generated Q on the system when moving in the generalized coordinates is given as follows:

$$Q = \left(\frac{\partial P_1}{\partial q}\right)^T u = \nabla_q P_1 u \tag{25}$$

- The equation (22) can be written in the following form:

$$M(q) * \dot{v} = b(q, Z, u) \tag{26}$$

$$C(q) = 0 \tag{27}$$

where v=$\dot{q}$ and the matrices *M(q)* and b are as follows:

$$M = \frac{\partial}{\partial \dot{q}}\left(\frac{\partial LF}{\partial \dot{q}}\right) \tag{28}$$

$$b(q, \dot{q}, u) = Q - \frac{\partial}{\partial q}\left(\frac{\partial LF}{\partial \dot{q}}\right)\dot{q} + \left(\frac{\partial LF}{\partial q}\right)^T \tag{29}$$

the acceleration term $\ddot{q}$ in terms of Z can be calculated as follows:

$$\ddot{q} = \dot{v} = M(q)^{-1} * b(q, Z, u) \tag{30}$$

- **Observation:** The size of the matrices M and b in 1(b) are larger than in 1(a) (6*6 versus 5*5), but it is simpler to calculate them when Cartesian co-ordinates are used to independently define positions of mass 1 and mass 2 as seen in 1(b). Using Polar coordinates and position of mass 1 to define position of mass 2 leads to higher complexity of the equations as in 1(a).

3 Question 2(a)

- Putting the hovering mass model in *1(b)* as follows.

$$\begin{bmatrix} M & a(q) \\ a^T(q) & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ Z \end{bmatrix} = c(q, \dot{q}, u) \tag{31}$$

- Where the variables a and c are

$$a(q) = \left(\frac{\partial C}{\partial q}\right)^T = \nabla_q C \tag{32}$$

$$c(q, \dot{q}, u) = \begin{bmatrix} Q - \frac{\partial}{\partial q}\left(\frac{\partial C}{\partial q}\right) + \nabla_q T - \nabla_q V \\ -\frac{\partial}{\partial q}\left(\frac{\partial C}{\partial q}\dot{q}\right)\dot{q} \end{bmatrix} \tag{33}$$

4 Question 2(b)

- We could observe here that using the explicit form of the Euler-Lagrange equations as in equation (31), we can calculate the acceleration $\ddot{q}$ and the Lagrange multiplier Z terms as a function of q, $\dot{q}$, and C, provided the the Matrix is full rank. Hence, a simulation of the model could be produced for given initial variables *q(0)*, $\dot{q}$*(0)* of masses 1 and 2 and external forces u on the helicopter.

- The complexity in calculating $\ddot{q}$ by using explicit counter part of equation(31) in equation (30) is much less complex if the Lagrange multiplier Z is given or can be calculated somehow.