Hand in Assignment 2

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October 6, 2022

Estimators and least squares

1 Question 1: Unbiased estimate

- The unbiased estimator of θ, $\hat{\theta}$ is given as follows:

$$\hat{\theta} = E\left[\frac{1}{N-1}\sum_{i=1}^{N-1}\overline{x}_i\right] \tag{1}$$

where $\overline{x}_i$ is the average of the sample x_i. The sample mean $\overline{x}_i$ is given as:

$$\overline{x}_i = \frac{\theta}{2} \tag{2}$$

- Therefore the equation (1) can be written as:

$$\hat{\theta} = \frac{1}{N-1}E\left[\sum_{i=1}^{N-1}\left(\frac{\theta}{2}\right)\right] \tag{3}$$

$$= \frac{1}{N-1}\left[N-1 * \frac{\theta}{2}\right] \tag{4}$$

$$\hat{\theta} = \frac{\theta}{2} \tag{5}$$

which is the same as the population mean. Therefore $\hat{\theta}$ is the unbiased estimator for θ.

2 Question2: Linear least squares

2.1 2a: Least-squares estimate

- The linear system is given as:

$$y[k] = \theta u[k] + e[k] \tag{6}$$

the prediction is given as

$$\hat{y}[k] = \theta u[k] \tag{7}$$

with $e[k]$ belonging to $N(1, \sigma^2)$. The least-square estimate for θ is given as:

$$\hat{\theta}_{N-1} = argmin V_{N-1}(\theta) \tag{8}$$

where $V_{N-1}(\theta)$ is calculated as follows:

$$\begin{aligned}
V_{N-1}(\theta) &= \frac{1}{N}\sum_{k=0}^{N-1}(y(k) - \hat{y}(k(\theta)))^2 && (9)\\
&= \frac{1}{N}\sum_{k=0}^{N-1}(y(k) - \theta u[k])^2 && (10)\\
&= \frac{1}{N}\left[\sum_{k=0}^{N-1} y^2[k]\right] + \theta\frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]\theta^T - 2\theta\frac{1}{N}\left[\sum_{k=0}^{N-1} y[k]u[k]\right] && (11)\\
&= \frac{1}{N}\left[\sum_{k=0}^{N-1} y^2[k]\right] + \theta R_n \theta^T - 2\theta f_n && (12)
\end{aligned}$$

where $R_n = \frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]$ and $f_n = \frac{1}{N}\left[\sum_{k=0}^{N-1} y[k]u[k]\right]$

- Then equation(8) is minimum when

$$\hat{\theta}_{N-1} = R_n^{-1} f_n \tag{13}$$

$$\hat{\theta}_{N-1} = \frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]^{-1}\frac{1}{N}\left[\sum_{k=0}^{N-1} y[k]u[k]\right] \tag{14}$$

2.2 2b: Bias

- From equation (14) we have:

$$\begin{aligned}
\hat{\theta}_{N-1} &= \frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]^{-1}\frac{1}{N}\left[\sum_{k=0}^{N-1} y[k]u[k]\right] && (15)\\
&= \frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]^{-1}\frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]\left(u[k] + e[k]\right)\right] && (16)
\end{aligned}$$

the estimate for the above equation is calculated as follows

$$\mathbb{E}\left[\hat{\theta}_{N-1}\right] = \mathbb{E}\left[\frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]^{-1} \frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]\left(\theta u[k] + e[k]\right)\right]\right] \tag{17}$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]^{-1} \frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1} u[k]\left(\theta u[k] + e[k]\right)\right] \tag{18}$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]^{-1} \frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1} \theta u[k]u[k]^T + u[k]e[k]\right] \tag{19}$$

simplifying the above equation we get the estimate in the form:

$$\mathbb{E}\left[\hat{\theta}_{N-1}\right] = \theta + \frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]^{-1} \frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1} u[k]e[k]\right] \tag{20}$$

- since the noise $e[k]$ has mean of 1 the term, $\mathbb{E}\left[\sum_{k=0}^{N-1} u[k]e[k]\right] \neq 0$. Therefore, $\mathbb{E}\left[\hat{\theta}_{N-1}\right]$ is not equal to θ and the estimate is said to be biased. The bias equation is given as:

$$\frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]u[k]^T\right]^{-1} \frac{1}{N}\left[\sum_{k=0}^{N-1} u[k]\right] \tag{21}$$

2.3 2c: If $u[k] = 0$

- If $u[k] = 0$, then the estimate $\hat{\theta}$ doesn't exist and the output signal $y[k]$ does not contain any input variable other than the noise $e[k]$.

3 Question3: Curve fitting

3.1 3a

- The parameters that best fit the data are $\hat{\theta} = \begin{bmatrix} \hat{a} = -3.1160 \\ \hat{b} = 7.1840 \end{bmatrix}$.

  Refer figure1 for the best fit plot.

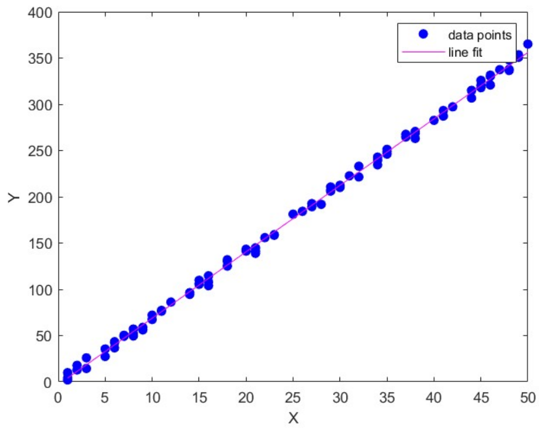

Figure 1

3.2 3b

- The parameters of the polynomial that best fit the data are $\hat{\theta} = \begin{bmatrix} \hat{a} = -2.5289 \\ \hat{b} = 6.4412 \\ \hat{c} = 2.8151 \end{bmatrix}$
- The parameters of the line that fit the data are $\hat{\theta} = \begin{bmatrix} \hat{a} = -1.1359e + 03 \\ \hat{b} = 146.7885 \end{bmatrix}$
- The prediction error of polynomial fit was: **8.3508** and the prediction error of line fit was: **282758.6138**. The polynomial Refer figure2 for the best fit plots.

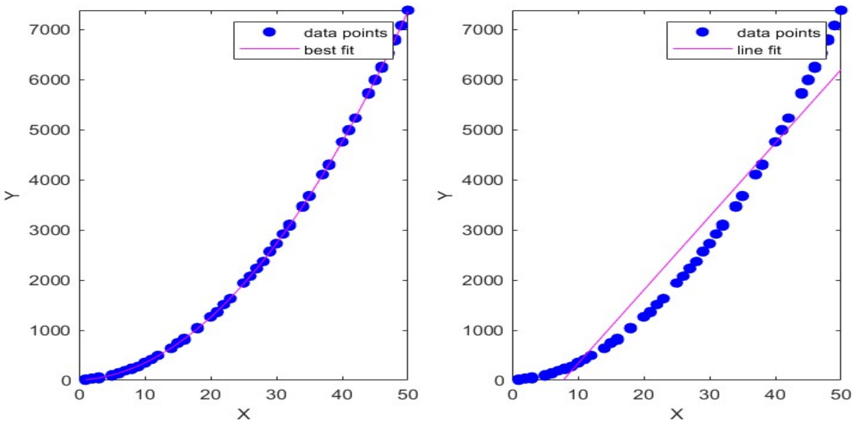

Figure 2

Identification of linear models for dynamical systems

4 Question1:One-step ahead predictor

4.1 1(a)

- The model structure is **ARMAX**.

4.2 1(b): Plant model and Noise model

For the model structure $y(t) + a_1 y(t-1) + a_2 y(t-2) = b_0 u(t) + e(t) + c_1 e(t-1)$

- The expressions of the plant model G and noise model H are as follows:

$$G(t|t{-}1,\theta) = \frac{b_0}{1 + a_1\left(\frac{y(t-1)}{y(t)}\right) + a_2\left(\frac{y(t-2)}{y(t)}\right)};\; H(t|t{-}1,\theta) = \frac{1 + c_1\left(\frac{e(t-1)}{e(t)}\right)}{1 + a_1\left(\frac{y(t-1)}{y(t)}\right) + a_2\left(\frac{y(t-2)}{y(t)}\right)} \tag{22}$$

4.3 1(c): One step ahead predictor

- The one step ahead predictor for given ARMAX model is as follows:

$$\hat{y}(t|t-1,\theta) = H^{-1}(q)\ G(q)\ u(t) + \left(1 - H^{-1}(q)\right)\ y(t) \tag{23}$$

$$\hat{y}(t|t-1,\theta) = \left(\frac{1+c_1q^{-1}}{a_1q^{-1}+a_2q^{-2}}\right)^{-1}\left(\frac{b_0}{a_1q^{-1}+a_2q^{-2}}\right)u(t) + \left[1 - \left(\frac{1+c_1q^{-1}}{a_1q^{-1}+a_2q^{-2}}\right)^{-1}\right]y(t) \tag{24}$$

simplifying the equation we get

$$\hat{y}(t|t-1,\theta) = \left[\frac{b_0}{1+c_1q^{-1}}\right]u(t) + \left[\frac{1+c_1q^{-1}-a_1q^{-1}-a_2q^{-2}}{1+c_1q^{-1}}\right]y(t) \tag{25}$$

4.4 1(d) Linearity of the predictor

- The equation (25) can be reduced to:

$$\hat{y}(t|t-1,\theta) = b_0u(t) + (y(t-1) - \hat{y}(t|t-2,\theta))\,c_1 + y(t)\left(1 - a_1q^{-1} - a_2q^{-1}\right) \tag{26}$$

it is in the form of

$$\hat{y}(t|t-1,\theta) = \theta^T\varphi(t,\theta) \tag{27}$$

From (26) and (27) the prediction $\hat{y}(t|t-1,\theta)$ is nonlinear in θ, since the vector $\varphi(t,\theta)$ now depends on θ.

5 Question2: Prediction or simulation?

5.1 2(a): Model structure

- The model structure is **OE**.

5.2 2(b): One step ahead predictor

- The 1-step ahead predictor is as follows:

$$y(t) = \frac{b_0}{1+a_1\left(\frac{y(t-1)}{y(t)}\right)}u(t) + e(t) \tag{28}$$

$$\hat{y}(t|t-1,\theta) = \frac{b_0}{1+a_1\left(\frac{y(t-1)}{y(t)}\right)}u(t) \tag{29}$$

rearranging (29) we get;

$$\hat{y}(t|t-1,\theta) = \left(1 - \frac{y(t-1)}{y(t)}\right)\hat{y}(t|t-1,\theta) + b_0 u(t) \tag{30}$$

from (30) since for an OE model, the predictor $\hat{y}$ does not contain the previous values of the prediction error $(y(t) - \hat{y}(t|t-1,\theta))$, but rather contains the input signal $u(t)$ and the previous values $\hat{y}(t|t-1,\theta)$, the prediction may not yield good results if the output $y(t)$ has high transient behavior.

6 Question3: Identification of an ARX model

6.1 3(a): One step ahead predictor

- The parameters of each candidate ARX model, using the estimation data are as follows.

$$for\ 12(a): \quad \hat{\theta} = \begin{bmatrix} \hat{b}_0 = 0.0688 \\ \hat{a}_1 = -0.9590 \\ \hat{a}_2 = 0.3567 \end{bmatrix} \tag{31}$$

$$for\ 12(b): \quad \hat{\theta} = \begin{bmatrix} \hat{b}_0 = 0.0113 \\ \hat{b}_1 = 0.9946 \\ \hat{a}_1 = -0.8923 \\ \hat{a}_2 = 0.3116 \end{bmatrix} \tag{32}$$

$$for\ 12(c): \quad \hat{\theta} = \begin{bmatrix} \hat{b}_1 = 1 \\ \hat{a}_1 = -1 \\ \hat{a}_2 = 0.6 \\ \hat{a}_3 = -0.3 \end{bmatrix} \tag{33}$$

$$\tag{34}$$

6.2 3(b): Root Mean Square Error

- The **prediction errors** for the ARX models are as follows:

$$for\ 12(a): 1.0582 \tag{35}$$

$$for\ 12(b): 0.31664 \tag{36}$$

$$for\ 12(c): 1.2215e^{-15} \tag{37}$$

- The **simulation errors** for the ARX models are as follows:

$$for\ 12(a) : 1.3941 \tag{38}$$

$$for\ 12(b) : 0.41933 \tag{39}$$

$$for\ 12(c) : 1.9358e^{-15} \tag{40}$$

- For tboth prediction and simulation the equation ***12c*** is the best model for the given data, since the error $(y(t) - \hat{y}(t|t-1, \theta))$ and it's RMS value are the least for ***12c***.