Applied Signal Processing
Hand-In Problem 2
FIR Differentiator Design using Local Models

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This problem concerns estimating the speed of an object from noisy samples of the position. The data file (`hip2.mat`) for the problem contains two variables: `true_position` representing samples of the true position and `noisy_position` representing the samples of the measured position which are noisy. The position is the distance, in meters, traveled by a truck from the start point, and the sampling interval Δt is 1 second. By definition, the velocity is the derivative of the position, so a very natural estimate of the velocity from sampled position data is:

$$\hat{v}(n) = \frac{p(n) - p(n-1)}{\Delta t}, \tag{1}$$

where Δt is the sampling interval and $p(n)$ is the sampled position. This is called the Euler approximation of the derivative. From (1) it is clear that the derivative approximation $\hat{v}(n)$ can be seen as the output after FIR filtering with the impulse response

$$h_{\text{Euler}}(n) = \begin{cases} 1/\Delta t, & n = 0 \\ -1/\Delta t, & n = 1 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Apply this Euler approximation to the true and measured signals respectively (use `conv` in MATLAB), and plot the result. When plotting the output from the filter disregard the last sample. (The last sample is extremely large compared to the others. We will come back to this later on.) Apparently, differentiation using sampled data is an extremely noise-sensitive task. To improve the estimator we can make use of the fact that the position signal is slowly varying while the noise is fairly quickly changing and on average is zero.

Your task is to design a FIR filter that produces an output which performs approximate differentiation and at the same time reduce the effect of the noise.

To accomplish this you need to select an appropriate local model structure which involves selecting the family of basis functions $f_i(n)$ and how many to use, i.e. the local model order p. Following the notation in the lecture notes a local model is given by

$$\hat{y}(n+m, \boldsymbol{\alpha}) = \sum_{i=0}^{p-1} \alpha f_i(m) \tag{3}$$

If $\hat{y}(n+m_0, \boldsymbol{\alpha})$ is the position of the vehicle at time index $n+m_0$ given by the local model, then the velocity of the vehicle at time index $n+m_0$ is given by

$$\hat{v}(n+m_0) = \frac{\partial}{\partial m}\hat{y}(n+m, \boldsymbol{\alpha})\bigg|_{m=m_0} = \sum_{i=0}^{p-1} \alpha_i f_i'(m) \tag{4}$$

where $f_i'(m) = \frac{\partial}{\partial m} f_i(m)$.

Following the notation in the lecture notes the impulse response for a derivative FIR filter of length $2M+1$ is given by

$$\begin{bmatrix} h(M) & \dots & h(0) & \cdots & h(-M) \end{bmatrix} = (\boldsymbol{f}'(m_0))^T (\boldsymbol{R}^T \boldsymbol{W} \boldsymbol{R})^{-1} \boldsymbol{R}^T \boldsymbol{W} \tag{5}$$

where

$$\boldsymbol{R} = \begin{bmatrix} f_0(-M) & f_1(-M) & \cdots & f_{p-1}(-M) \\ f_0(-M+1) & f_1(-M+1) & \cdots & f_{p-1}(-M+1) \\ \vdots & \vdots & \dots & \vdots \\ f_0(0) & f_1(0) & \cdots & f_{p-1}(0) \\ \vdots & \vdots & \dots & \vdots \\ f_0(M) & f_1(M) & \cdots & f_{p-1}(M) \end{bmatrix} \tag{6}$$

and

$$(\boldsymbol{f}'(m))^T = \begin{bmatrix} f_0'(m) & f_1'(m) & \cdots & f_{p-1}'(m) \end{bmatrix} \tag{7}$$

and $\boldsymbol{W}$ is diagonal matrix with positive scalars on the diagonal.

Design task and investigations Your task is to design a low-pass FIR differentiator filter using the local model approach when the basis functions are selected as monomials and

$$\begin{aligned} f_0(m) &= 1, \quad i = 0 \\ f_i(m) &= m^i, \quad i = 1, \dots, p-1 \end{aligned} \tag{8}$$

If m_0 is selected to M then the filter output

$$\hat{v}(n) = \sum_{k=0}^{2M-1} h(k) p(n-k) \tag{9}$$

will be a causal estimate of the velocity at time sample n when $p(n)$ is the measured position at sample n.

Investigate the MATLAB script `hip2_lmfir.m`. The file is available at the assignment page on Canvas. Some code in the script need to be complemented to get it fully functional. It is your task to do this. Look for `%%TBC` remarks in the file. Upload your final `hip2_lmfir.m` file together with your report. After you have completed the code. Investigate the following:

- Consider the frequency function of the designed filter with $M = 30, m_0 = 0, p = 3$. Zoom in on the magnitude function for $0 < f/f_s < 0.01$. For low frequencies the magnitude is almost linear. What is the slope near frequency zero for the designed filter and what slope would an ideal differentiator filter have?

- Investigate the effect of different model orders $p = 1, 2, 3, 4, 5$ for $M = 30$ and $m_0 = M$.

- Investigate the effect of different window sizes $M = 10, 20, 30, 40$ for $p = 3$ and $m_0 = 0$.

- Investigate the effect of different choices for $m_0 = 0, 10, \dots, M$ for $p = 3$ and $M = 30$.

- Use the `w1` in the design which corresponds to a Chebyshev window weight vector and compare with the `w0` rectangular window weight vector. Investigate the effect for $m_0 = 0$, $p = 2, 3, 4$ and $M = 30$.

You can use the signal `vhat_true_0` as a good estimate to the true velocity and compare the output from the designed filters with this signal.

In your report answer the following set of questions and include a motivation for the answer:

Questions

1. Give an argument to why the signal `vhat_true_0` is a good estimate of the true velocity.

2. What is the minimum model order p required in order to get any meaningful solution, i.e. not only the zero solution?

3. Summarize the investigation regarding the effect of the M and p parameter choices.

4. Explicitly compare the result for these two designs

   - $M = 30, m_0 = M, p = 3$
   - $M = 30, m_0 = M, p = 2$

   and explain the reason to why the designs give different results.

5. At the end of the filter output very large oscillations occur. Why? *Hint: test filtering with the two signals*

   ```
   y1 = [noisy_position; flip(noisy_position)];
   y2 = [noisy_position; zeros(length(noisy_position),1)];
   ```

   Why do/don't we get similar behavior?

6. What is the maximum speed of the vehicle found using the observed signal and "true" signal?

Good Luck!