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ENERGY INTENSITY, DIRECTED TECHNOLOGICAL CHANGE AND ECONOMIC GROWTH: WILL THE AUTONOMOUS DECLINE OF ENERGY INTENSITY CONTINUE?

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World and U.S. energy intensities have declined over the past century, falling approximately at an average rate of 1.2–1.5 percent a year. The decline has persisted through periods of stagnating or even falling energy prices, suggesting the decline is driven in a large part by autonomous factors, independent of price changes. In this paper, we use directed technical change theory to understand the autonomous decline in energy intensity and ask whether the decline will continue. We show in an economy with no scale effects, where existing knowledge does not make R&D more profitable, energy intensity continues to decline, albeit at a slower rate than output growth, from profit driven energy augmenting innovation. However, in an economy with scale effects, energy intensity eventually stops declining because labor augmenting innovation crowds out energy augmenting innovation. In either case, energy intensity never declines faster than output grows, and so energy use always increases, as long as the wellhead price of energy stays constant.

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The latest version of this paper and the online appendix can be found at `https://github.com/akshayshanker/Energy`

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1. INTRODUCTION

Why does energy intensity fall as an economy grows? Why does energy intensity fall slower than the rate of output growth? Will energy intensity continue to fall in the absence of sustained energy price increases? In this paper, we develop a theoretical understanding of these questions using a model of endogenous growth with directed technical change.

The past century has seen a persistent decline in energy intensity both globally and in many individual countries. Figure 1 shows U.S. energy intensity and the U.S. real energy price from 1900 to 2015. While the energy price has fluctuated, we find no clear increasing trend (see the online appendix for trend tests). And even during periods of falling prices, particularly between 1980 and 2000, energy intensity continued to decline. Figure 2 shows energy intensity and per capita GDP for 100 countries between 1970 and 2010 — per capita GDP growth is associated with a proportional decline in energy intensity. Csereklyei et al. (2016) give a formal analysis and find a negative elasticity between per capita GDP and energy intensity of -.3 as well as conditional and unconditional beta convergence and sigma convergence of energy intensity.

Energy economists have suspected autonomous factors *tied to growth*, rather than increasing prices, have played a role in the world-wide decline of energy intensity.[1] Indeed, climate policy models,[2] and projections of future energy intensity, for example by the International Energy Association (IEA, 2016) and the U.S. Energy Information Administration (EIA, 2017), both assume the autonomous decline of energy intensity will continue.

However, despite energy intensity being "central to the achievement of a range of policy goals, including energy security, economic growth and environmental sustainability" and despite the vast literature incorporating endogenous technical change in models estimating the costs of climate policy (Goulder and Schneider, 1999; Nordhaus, 2002; Jakeman et al., 2004; Popp, 2004), there has been no formal analytical study of what, in terms of economic incentives, could drive declining energy intensity and whether the decline will continue.

[1]The climate policy modeling literature refers to the autonomous decline of energy intensity as Autonomous Energy Efficiency Improvement (AEEI). Though extensively debated, estimates of AEEI for the U.S. range from .5% to 2% per year (See Williams et al. (1990), Löschel (2002), Stern (2004), Sue Wing and Eckaus (2007) and Webster et al. (2008)). Newell et al. (1999), who use micro-level data on efficiency of air-conditioners, find price increases induce technical change but "autonomous drives of energy efficiency explain up to 62% of total changes in energy efficiency". On the other hand, Kaufmann (2004) using a co-integration analysis finds no left-over deterministic trend.

[2]See Popp et al. (2010) for an overview of the use of AEEI in models with exogenous technical change. Some models with induced technical change also incorporate and exogenous efficiency

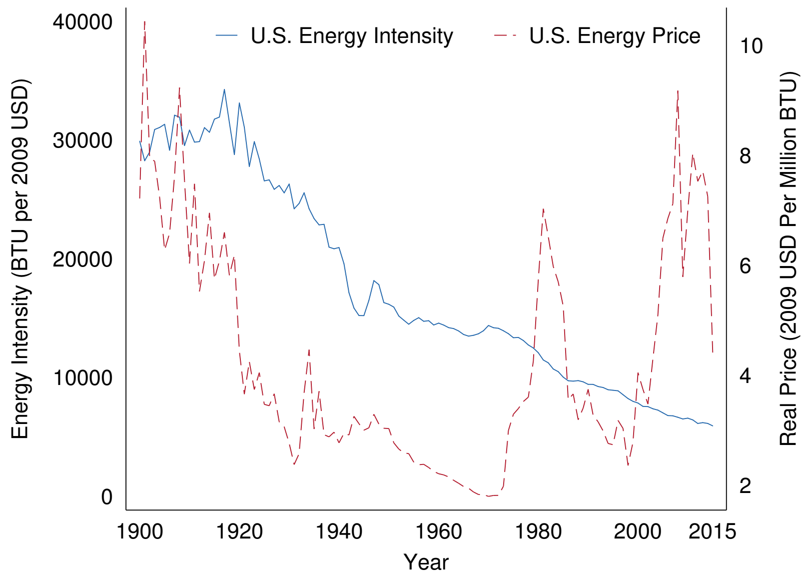

FIGURE 1.— U.S. energy intensity (including traditional fuels) and wellhead real price index of energy between 1900 and 2015. Energy intensity fell at an average rate of 1.2 percent per year between 1900 and 1950 and 1.5 percent per year between 1950 and 2015. See the online appendix for a detailed description of the data sources.

Our analysis incorporates directed technical change (Acemoglu, 1998, 2002), where innovation can augment labor or energy, in a Schumpetarian endogenous growth model with quality improvements. The quality improvement model can be traced to the work of Aghion and Howitt (1992), however, our treatment with a large number of firms most closely follows the framework by Acemoglu (2009), ch. 14.[3] Following established estimates in the literature (see Stern and Kander (2012) and also Van der Werf (2008)), we assume an elasticity of substitution between energy and labor services of less than one.

1.1. *Main Findings*

Our first finding is that falling energy intensity without increasing wellhead energy prices is *inconsistent* with balanced growth. Rather, an autonomous decline of energy intensity implies a falling cost share of energy and a non-constant, but pos-

trend, for example, Popp (2004).

[3]The structure of our modeling is also similar to Acemoglu and Cao (2015), except Acemoglu and Cao (2015) also incorporate radical innovations while we incorporate directed technical change.

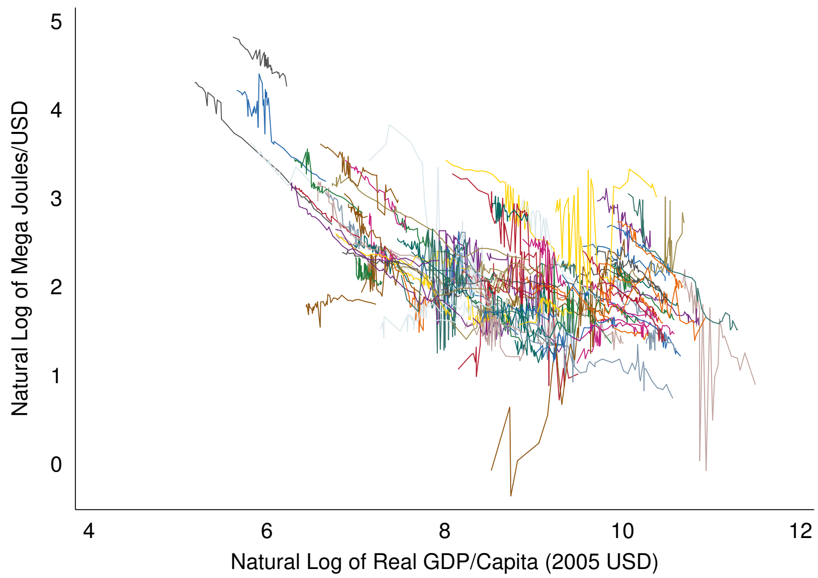

FIGURE 2.— Energy intensity and real GDP/capita for countries between 1970 and 2010.

sibly converging growth rate of output. The finding holds in a general setting with neoclassical production. Since we can no longer use balanced growth paths (BGP), as in the standard text-book treatment of directed technical change, we proceed to formalise asymptotic behaviour.

Motivated by evidence (Bloom et al., 2017) for new innovations being more difficult to find as technologies advance, we begin in an economy without scale effects (also known as path dependence or standing on shoulders effects). Under no scale effects, the existing level of knowledge does not improve the profitability of research. We show asymptotic convergence to a growth path where energy intensity falls at a constant rate due to investment in energy augmenting technologies. Consistent with the data, energy intensity declines slower than output grows, and we express the asymptotic elasticity between energy intensity and output as a function of only the elasticity of substitution between energy and labor.

To understand incentives to undertake energy augmenting research, recall profits of monopolistically competitive entrepreneurs are the product of the price a new product can command and the number of units of the new product that can be sold, the market size (see Acemoglu (2009), ch. 15 for a detailed explanation of these market size and price effects). As energy use increases along a growing economy (and wellhead energy prices do not trend upwards) the price of energy services falls, reducing profitability and incentives for energy augmenting research.

However, the market size of energy increases, holding up incentives to conduct energy augmenting research.

Nonetheless, the rate of decline of energy intensity is slower than the rate of output growth because energy augmenting technologies advance at a slower rate than labor augmenting technologies. While the market size effect of increasing energy use is sufficient to maintain energy augmenting innovation, the price effect of falling energy prices relative to the labor wage (figure 3) is stronger than the market size effect, of greater per capita energy use, because the elasticity of substitution is less than one. The stronger price effect means greater incentives to undertake labor augmenting research and in turn a faster rate of labor saving innovation.

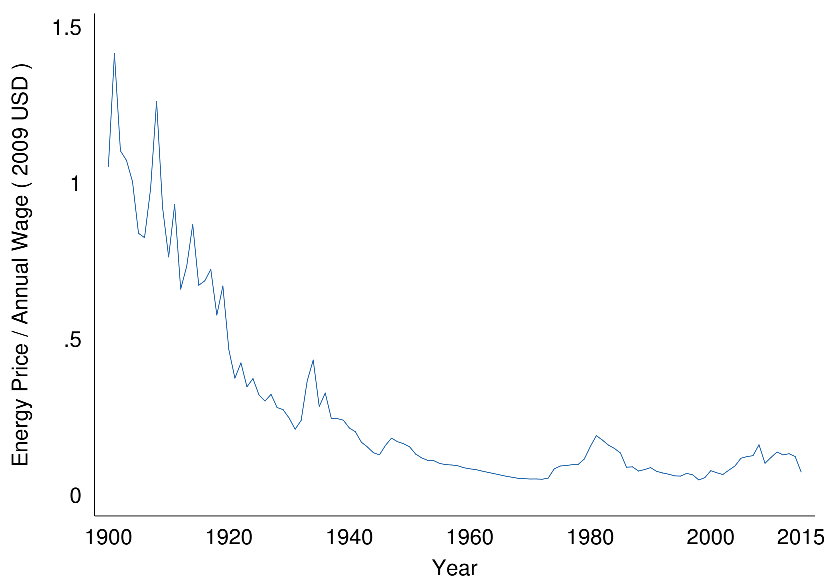

FIGURE 3.— The ratio of U.S. real energy price to real annual wage per employee shows a declining trend, creating stronger incentives for labor saving innovation.

The model with no scale effects can also explain convergence of energy intensities conditional on output. Two countries starting with similar GDP per-capita but where one country has a higher energy intensity will both converge to the same energy intensity - GDP per-capita relationship.

We also examine an economy under scale effects. We find an economy under scale effects possesses a BGP. While autonomous energy intensity can occur outside a BGP, energy intensity, in the absence of price changes eventually converges to a strictly positive constant. The reasons is that scale effects compound the relatively stronger incentive to undertake labor augmenting research arising from the price effect; the incentives to undertake labor-augmenting research are strong enough to crowd out all energy-augmenting research, leading to an eventual end to autonomous declines of energy intensity.

While the model with no scale effects accounts for the observed trends in energy intensity, we cannot rule out a model with scale effects. If we assume sufficiently many countries, including the U.S., begin with energy intensities away from the long-run BGP, then a scale effects model can account for the observed falls in energy intensity associated with growth. One piece of evidence even favoring a scale effects economy is the relatively high elasticity of energy intensity with respect to output in the cross-section for high-income countries — as income increases across high-income countries energy intensity falls at a relatively slow rate. It turns out that a high elasticity can only be supported in a model with a persistent autonomous decline of energy intensity if the elasticity of substitution is higher than previous estimates in the literature (see Stern and Kander (2012) or Hassler et al. (2016), we discuss these issues in detail in Section 3).

1.2. *Related Literature*

Our paper complements Hassler et al. (2016) and André and Smulders (2014) who also analyze models with endogenous energy- and labor-augmenting technical change. Hassler et al. (2016) provide a calibrated and tractable framework to forecast the role of energy intensity improvements in generating possibilities of growth where energy is scarce. André and Smulders (2014) uses a directed technical change model to account for stylized facts in U.S. energy use. However, both authors do not explicitly analyze the role and potential for autonomous drivers of falling energy intensity to persist along a growth path. In particular, both papers also only study innovation under scale effects and use a Hotelling rule featuring an exponentially increasing energy price, which becomes the driver of long-run energy efficiency improvements.

Our key finding that a model with scale effects cannot feature a constant growth rate of energy-augmenting technical change along an asymptotic growth path is analogous to the main finding of Acemoglu (2003). Acemoglu (2003) shows how under scale effects, long-run technical change can only be labor-augmenting, rather than capital-augmenting for interest rates to remain constant. Our model under scale effects is similar to Acemoglu's, except with capital replaced by energy. A specification with no scale effects is ruled out by Acemoglu precisely because it would lead to a fall in capital intensity given constant interest rates — just as energy intensity falls with constant energy prices in the model presented here with no scale effects.

Finally, our paper relates to empirical work by Sue Wing (2008) showing that most of the decline in U.S. energy intensity before the 1970s oil price shock can be attributed to structural change towards sectors with lower energy intensity, while falls in intensity during the period of increasing prices can be attributed to within sector declines in intensity. McKibbin et al. (2004) take a similar view, arguing that the relationship between output and energy use depends on changes in the shares of different sectors in an economy, rather than just on an energy efficiency trend within sectors. These arguments may at first seem inconsistent with our argument, where structural change implies an increase in the quantity ratio of energy services to output. However, energy-augmenting technical change in models with high aggregation such as ours should be seen as any change in the economy that allows more output per unit of energy, including technical change leading to structural change,[4] than strictly as 'energy efficiency' within specific industries. The results of Sue Wing (2008) also point to an important difference between price induced technological change and autonomous technological change, whereas in our aggregated model, both are captured by the same variable.

2. SCHUMPETERIAN GROWTH MODEL OF ENERGY AND DIRECTED TECHNICAL CHANGE

2.1. *The Model Economy*

Consider a continuous time economy where t, with $t \in \mathbb{R}_+$, indexes time. Assume a risk neutral representative consumer, so preferences over a time path for consumption are[5]

$$\int_{t=0}^{\infty} e^{-\rho t} C(t)\, \mathrm{d}t \tag{1}$$

where ρ is the discount factor and $C(t)$ is consumption at time t. The consumer faces a standard dynamic optimization problem and chooses a path of consumption and assets, $\{a(t)\}_{t=0}^{\infty}$, given a path of interest rates, $\{r(t)\}_{t=0}^{\infty}$ and wage rates $\{w_L(t)\}_{t=0}^{\infty}$ to maximize (1). For any t, the resource constraint for the economy is

$$Z(t) + C(t) + X(t) + \kappa(t)E(t) \leqslant Y(t) \tag{2}$$

[4]The definition of energy services in an aggregated directed technical change model does not map to any particular sector in the economy, but may resemble a notion broader than "useful work" within an economy, which has shown an increasing trend, see Warr et al. (2010).

[5]Linear utility simplifies analysis of the dynamics in the model. Aghion and Howitt (1992) makes the same assumption as does Acemoglu (2003) when analyzing dynamics. Under a constant relative risk aversion (CRRA) utility function, the main results of our paper holds conditional on the assumption that any equilibrium path is an asymptotic balanced growth path (ABGP). However, we were unable to verify that every equilibrium path is an ABGP in the case of CRRA utility.

where $Z(t)$ is the total level of R&D, $X(t)$ is total expenditure on machine varieties, $E(t)$ energy use and $\kappa(t)$ is the extraction cost of energy, which is exogenous. We use an exogenous extraction cost as our focus is on autonomous endogenous technical change — we are interested in the response of technological change *given a path of prices*.[6]

At each t, final goods $Y(t)$ are produced using two intermediate goods: an energy good $Y_E(t)$ and a labor good $Y_L(t)$:

$$Y(t) = \left(\gamma_E Y_E(t)^{\frac{\epsilon-1}{\epsilon}} + \gamma_L Y_L(t)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} \tag{3}$$

where ϵ is the elasticity of substitution and we assume $\epsilon < 1$. Competitive firms produce intermediate goods using a continuum of machines indexed by i, where $i \in [0, 1]$. The intermediate production functions are

$$Y_E(t) = (1-\alpha)^{-1} E(t)^{\alpha} \int_0^1 q_E(i,t)^{\alpha} x_E(i,t \mid q_E(i,t))^{1-\alpha} \, \mathrm{d}i \tag{4}$$

and

$$Y_L(t) = (1-\alpha)^{-1} L(t)^{\alpha} \int_0^1 q_L(i,t)^{\alpha} x_L\left(i,t \mid q_L(i,t)\right)^{1-\alpha} \, \mathrm{d}i \tag{5}$$

Let j index the sectors E and L. In the equations above, $q_j(i,t)$ denotes the highest quality of machine i at time t in sector j and $x_j(i,t \mid q_j(i,t))$ is the quantity of machine type i in sector j with quality $q_j(i,t)$.

We assume labor supply is fixed with $L(t) = L$ and $L > 0$ for all t. The energy sector uses a fuel $E(t)$, which can be extracted at a fixed cost $\kappa(t)$.[7]

The price of final output is normalized to one and all prices will be stated in terms of the final good price. We assume machines of all varieties have a constant production cost $(1-\alpha)$, and monopolists who own patents for the varieties make and sell machines to the intermediate producers.

Let $p_j^x(i,t|q)$ denote the price of machine i in sector j at time t with quality q. And let $p_j(t)$ denote the price of the intermediate good $Y_j(t)$.

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[6]Some other researchers e.g. André and Smulders (2014) assume the price of energy follows a Hotelling rule. However, observed energy prices are inconsistent with a simple Hotelling rule which would imply an exponentially growing price (Hamilton, 2009).

[7]Under population growth, a model with a constant asymptotic growth rate of output must have no scale effects (Jones, 1995). The asymptotic behaviour of a no scale effects model with population growth is similar to the main results of this paper, however, in the case with population growth, we were unable to show a rational expectations equilibrium is an asymptotic growth path.

Turning to machine qualities, we assume a quality ladder for highest quality machines as follows

$$q_j(i,t) = \lambda^{n_j(i,t)} q_j(i,0), \qquad j \in \{E, L\}, i \in [0,1], t \in \mathbb{R}_+ \tag{6}$$

where $q_j(0,i)$ is the quality of machine i at time 0 and $n_j(i,t)$ equals the random number of incremental innovations on the machine variety up to time t. The arrival of a new innovation improves the machine quality by a factor λ.

New entrants engage in research to improve machine varieties. New entrants who have a successful innovation own a perpetual patent on the machine variety, however, once a new variety has been invented, the improved quality captures the whole market for the variety — Schumpeterian creative destruction. The following assumption[8] ensures the firm with the highest quality machine can charge the unconstrained monopoly price:

ASSUMPTION 1 $\lambda \geqslant \left(\frac{1}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}}$

Final goods are used to undertake R&D which can be directed towards energy or labor machine improvements. A prospective entrant expending $z_j(i,t)$ units of R&D to improve the equality of machine i in sector j at time t generates a flow rate of machine improvement equal to $\eta_j b_j(i,t)$, where η_j is an exogenous parameter and[9]

$$b_j(i,t) := \frac{z_j(i,t)}{q_j(i,t)}$$

The denominator implies higher quality machines are more difficult to improve upon, cancelling out scale effects embedded in the quality ladder.

Total R&D expenditure will be

$$Z(t) = \int_0^1 z_E(i,t)\,\mathrm{d}i + \int_0^1 z_L(i,t)\,\mathrm{d}i \tag{7}$$

[8]See Acemoglu (2009) sections 12.3.3 and 14.1.2 for discussion.

[9]Recall the flow rate tells us the probability of a machine improvement occurring during a small period of time Δt given R&D expenditure $z_j(i,t)$ is $\frac{\eta_j z_j(i,t)}{q_j(i,t)}\Delta t$. Formally,

$$\mathbb{P}\{n_j(t+\Delta t, i) - n_j(t,i) = 1\} = \frac{\eta_j z_j(i,t)}{q_j(i,t)}\Delta t + o(\Delta t), \qquad j \in \{E, L\}$$

2.2. *Equilibrium Characterization*

Objects in the model economy can be arranged in a tuple $\mathscr{E}$, where

$$\mathscr{E} := ((\Omega, \mathscr{F}, \mathbb{P}), q_E, q_L, v_E, v_L, p_E^x, p_L^x, x_L, x_E, z_E, z_L, C, X, E, Z, r, w, a)$$

and

1. the tuple $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space
2. the path of leading machine qualities, q_E and q_L, are $(\mathbb{R}_+^{[0,1]})^{\mathbb{R}_+}$ valued random variables defined on $(\Omega, \mathscr{F}, \mathbb{P})$
3. the functions $p_E^x \colon [0,1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ and $p_L^x \colon [0,1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ are paths of monopolist prices
4. the functions $p_E \colon \mathbb{R}_+ \to \mathbb{R}_+$ and $p_L \colon \mathbb{R}_+ \to \mathbb{R}_+$ are intermediate goods prices
5. the functions $x_E \colon [0,1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ and $x_L \colon [0,1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ are machine quantities
6. paths for consumption, total expenditure on machines, energy, total R&D and energy that satisfy $C, X, Z, E \in \mathscr{C}^1(\mathbb{R}_+)$ where $\mathscr{C}^1(\mathbb{R}_+)$ is the space of all real valued continuously differentiable functions on $\mathbb{R}_+$
7. the functions $z_E \colon \mathbb{R}_+ \to \mathbb{R}_+$ and, $z_L \colon \mathbb{R}_+ \to \mathbb{R}_+$ are paths for R&D
8. the functions $v_E \colon [0,1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ and $v_L \colon [0,1] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ are paths for the value functions
9. the functions $r \colon \mathbb{R}_+ \to \mathbb{R}_+$, $w \colon \mathbb{R}_+ \to \mathbb{R}_+$ and $a \colon \mathbb{R}_+ \to \mathbb{R}_+$ are paths for the interest rate, wage rate and assets.

A formal definition of equilibrium is given in the Appendix; we now turn to documenting the familiar static equilibrium conditions, given technology.

Profit maximization by final goods producers gives

$$\gamma_E Y(t)^{\frac{1}{\epsilon}} Y_E(t)^{-\frac{1}{\epsilon}} = p_E(t), \qquad \gamma_L Y(t)^{\frac{1}{\epsilon}} Y_L(t)^{-\frac{1}{\epsilon}} = p_L(t) \tag{8}$$

At each t, profit maximization by the intermediate goods producers gives machine demands for the highest quality machine of type i

$$x_E(i, t \mid q_E(i,t)) = p_E(t)^{\frac{1}{\alpha}} p_E^x(i, t \mid q_E(i,t))^{\frac{1}{\alpha}} q_E(i,t) E(t) \tag{9}$$

$$x_L(i, t \mid q_L(i,t)) = p_L(t)^{\frac{1}{\alpha}} p_L^x(i, t \mid q_L(i,t))^{\frac{1}{\alpha}} q_L(i,t) L \tag{10}$$

and first order conditions for energy and labor

$$\alpha(1-\alpha)^{-1} p_E(t) E(t)^{\alpha - 1} \int_0^1 q_E(i,t)^{\alpha} x_E(i, t \mid q_L(i,t))^{1-\alpha} \, \mathrm{d}i = \kappa(t) \tag{11}$$

$$\alpha(1-\alpha)^{-1} p_L(t) L^{\alpha-1} \int_0^1 q_L(i,t)^{\alpha} x_L(i,t \,|\, q_L(i,t))^{1-\alpha}\, \mathrm{d}i = w(t) \tag{12}$$

Since the cost of machine varieties is $1-\alpha$ and by Assumption 1, monopolists who own the highest quality machine production technologies set $p_j^x(i,t \,|\, q_j(i,t)) = 1$, machine demands for the highest quality machine of type i are then

$$x_E(i,t \,|\, q_E(i,t)) = p_E(t)^{\frac{1}{\alpha}} q_E(i,t) E(t), \qquad x_L(i,t \,|\, q_L(i,t)) = p_L(t)^{\frac{1}{\alpha}} q_L(i,t) L \tag{13}$$

Profits for monopolists who own the leading edge machines are

$$\pi_E(i,t \,|\, q_E(i,t)) = \alpha p_E(t)^{\frac{1}{\alpha}} q_E(i,t) E(t), \qquad \pi_L(i,t \,|\, q_L(i,t)) = \alpha p_L(t)^{\frac{1}{\alpha}} q_L(i,t) L \tag{14}$$

Owners of lower equality machines receive zero profits.

Define $Q_E : = \int_0^1 q_E(i,t)\, \mathrm{d}i$ and $Q_L : = \int_0^1 q_L(i,t)\, \mathrm{d}i$ as the average qualities of leading machines. Intermediate output, using (4) and (5) with (13) becomes

$$Y_E(t) = (1-\alpha)^{-1} E(t) p_E(t)^{\frac{1-\alpha}{\alpha}} Q_E(t), \qquad Y_L(t) = (1-\alpha)^{-1} L p_L(t)^{\frac{1-\alpha}{\alpha}} Q_L(t) \tag{15}$$

Next, using the intermediate demand conditions, (8), we can write the ratio of prices as

$$\frac{p_E(t)}{p_L(t)} = \gamma \left(\frac{Y_E(t)}{Y_L(t)}\right)^{-\frac{1}{\epsilon}} = \gamma \left(\frac{E(t)}{L}\right)^{-\frac{1}{\epsilon}} \left(\frac{p_E(t)}{p_L(t)}\right)^{-\frac{1-\alpha}{\alpha\epsilon}} \left(\frac{Q_E(t)}{Q_L(t)}\right)^{-\frac{1}{\epsilon}}$$

and solving for the ratio of prices gives

$$\frac{p_E(t)}{p_L(t)} = \gamma^{\frac{\alpha\epsilon}{\sigma}} \left(\frac{E(t) Q_E(t)}{L Q_L(t)}\right)^{-\frac{\alpha}{\sigma}} \tag{16}$$

where $\sigma = 1 + \alpha(\epsilon - 1)$ and $\gamma = \frac{\gamma_E}{\gamma_L}$ is the elasticity of substitution between the factors of production. Using (15), we can also derive a ratio of intermediate goods

$$\frac{Y_E(t)}{Y_L(t)} = \left(\frac{E(t) Q_E(t)}{L Q_L(t)}\right)^{\frac{\epsilon\alpha}{\sigma}} \gamma^{\frac{\epsilon(1-\alpha)}{\sigma}} \tag{17}$$

Next, insert (8) into (15), using the expression for final output [10] and the ratio of intermediate goods above (17) to derive

$$Y(t) = (1-\alpha)^{-1} \left[\gamma_E^{\frac{\epsilon}{\sigma}} \left(E(t) Q_E(t)\right)^{\frac{\sigma-1}{\sigma}} + \gamma_L^{\frac{\epsilon}{\sigma}} \left(L Q_L(t)\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{18}$$

[10] Note $Y(t) = \left(1 + \gamma^{-1} \left(\frac{Y_L(t)}{Y_E(t)}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} Y_E(t) \gamma_E$

We now derive a condition characterizing energy intensity. Use (11) and the definition of $Y_E(t)$ to arrive at

$$\alpha\theta_E(t)\frac{Y(t)}{E(t)} = \kappa(t) \tag{19}$$

where

$$\theta_E(t) := \frac{p_E(t)Y_E(t)}{Y(t)} = \left(1 + \gamma^{-\frac{\epsilon}{\sigma}}\left(\frac{E(t)Q_E(t)}{Q_L(t)L}\right)^{\frac{1-\sigma}{\sigma}}\right)^{-1} \tag{20}$$

is the cost share of energy services in final production. The second equality results from using (8) and (17). Use (20) along with (18) to write (19) as

$$\gamma_E^{\frac{\epsilon}{\sigma(\sigma-1)}}\alpha(1-\alpha)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y(t)}{E(t)}\right)^{\frac{1}{\sigma}} Q_E(t)^{\frac{\sigma-1}{\sigma}} = \kappa(t) \tag{21}$$

Equation (21) tells us that energy-augmenting technical change is energy saving. The appendix gives further comparative static results on how energy augmenting technologies effect energy use.

Now we turn to the dynamic aspects of equilibrium concerning technology choice. Since consumers are risk neutral, the interest rate in the economy will be ρ for all t. For the no scale effects model, let $v_j(i, t \mid \lambda q)$ denote the value of a successful innovation of machine i in sector j with quality q at time t. We have

$$v_j(i, t \mid \lambda q) = \mathbb{E}_t \int_{s=t}^{T} e^{-\rho s}\pi(i, s \mid \lambda q)\, \mathrm{d}s \tag{22}$$

where T is the random stopping time after which a new innovation replaces the incumbent. Free entry and exit implies[11]

$$\eta_j \frac{v_j(i, t \mid \lambda q)}{q} \leqslant 1, \qquad j \in \{E, L\} \tag{23}$$

holds for all t and $i \in [0, 1]$ and the inequality is an equality of $z_j(i, t) > 0$. Assuming the value function is differentiable implies $\dot{v}_j(i, t \mid q) = 0$. The Hamilton Jacobi Bellman (HJB) equation[12] for $v_j(t, i \mid q)$ is

$$\dot{v}_j(i, t \mid q) = \big(\rho + \eta_j b_j(i, t)\big)\, v_j(i, t \mid q) - \pi_j(i, t \mid q) \tag{24}$$

[11]Recall that expenditure of $z_j(i, t)$ units of final good on R&D generates a flow rate of $\eta_j \frac{z_j(i,t)}{q_j(i,t)}$. Since the price of the final good is normalised to one the value of spending one unit of the final good on research is $\frac{\eta_j v_j(i,t \mid \lambda q)}{q(i,t)} - 1$, which should not be strictly positive.

[12]See Acemoglu (2009), Equation (14.13).

Immediately giving, using (23) and (14),

$$\eta_E b_E(t) + \rho \geqslant \eta_E \lambda \alpha p_E(t)^{\frac{1}{\alpha}} E(t), \qquad \eta_L b_L(t) + \rho \geqslant \eta_L \lambda \alpha p_L(t)^{\frac{1}{\alpha}} L \tag{25}$$

where, once again, the inequality is an equality of $z_j(i,t) > 0$. Note the above equation implies $b_j(t)$ is no longer conditioned on the machine variety — there is a common rate of innovation for all machines in a sector.

Finally, for each i, $q_j(i,t)$, is a random process. However, at teach t, the average machine quality $Q_j(t)$ for each sector j will be deterministic,[13] determined by the innovation rate b_j as follows[14]

$$\dot{Q}_j(t) = b_j(t) Q_j(t), \qquad j \in \{E, L\} \tag{26}$$

2.3. *Main Theoretical Results*

An equilibrium growth path is a balanced growth path if output and consumption both grow at a constant rate g, energy grows at a constant rate g_E, and Q_E and Q_L grow at constant rates g_{Q_E} and g_{Q_L} respectively. For any path, $\varphi(t)$, we will use $\hat{\varphi}(t)$ to refer to the growth rate of $\varphi(t)$ at time t, that is $\hat{\varphi}(t) = \frac{\dot{\varphi}(t)}{\varphi(t)}$, and we use $\varphi(t) \to \bar{\varphi}(t)$ to mean $\lim_{t\to\infty} \varphi(t) = \bar{\varphi}(t)$ for a path $\bar{\varphi}(t)$.[15]

PROPOSITION 2.1 *Let an economy $\mathscr{E}$ be an equilibrium. If $\hat{Q}_E(t) > c$ for all t, where $c > 0$, and $\dot{\kappa}(t) = 0$, then $\hat{Y}(t)$, $\hat{E}(t)$, $\hat{Q}_L(t)$ and $\hat{Q}_E(t)$ cannot be constant.*

PROOF: By (18), $Y(t) = \tilde{F}(Q_E(t)E(t), Q_L(t)L)$ where $\tilde{F}$ is a homogeneous of degree one production function. Using Euler's Theorem, $g_{Q_E} + g_E = g_{Q_L}$ must hold for g, g_E, g_{Q_E} and g_{Q_L} to be constant (See Claim A.1 in the Appendix). However, $g_{Q_E} + g_E = g_{Q_L}$ yields a contradiction, since Equation (20) implies $\theta_E(t)$ is constant, which in turn implies $\hat{Q}_E(t) = 0$ and $E(t)/Y(t)$ is constant by (21) and (38).

Q.E.D.

Thus a BGP cannot feature autonomous energy intensity improvements. We study

[13]More precisely, the average quality will equal $\mathbb{E}q(i,t)$ with probability one (Sun and Zhang, 2009), this is refereed to as no aggregate uncertainty (NAU) and in this paper we view NAU as an assumption without providing a formal proof.

[14]Acemoglu and Cao (2015) (Footnote 23) provide a concise derivation of Equation (26).

[15]We say $\varphi(t) \to \bar{\varphi}(t)$ if and only if for every $\epsilon > 0$, there exists T such that for all $t > T$, $|\varphi(t) - \bar{\varphi}(t)| < \epsilon$.

the possibility of autonomous energy intensity improvements along *asymptotic* balanced growth paths (ABGP).[16]

DEFINITION 2.1 An equilibrium economy $\mathscr{E}$ is an asymptotic balanced growth path (ABGP) if $\hat{Y}(t) \to g$, $\hat{C}(t) \to g$, $\hat{E}(t) \to g_E$, $\hat{Q}_E(t) \to g_{Q_E}$, $\hat{Q}_L(t) \to g_{Q_L}$, $b_E(t) \to b_E^\star$ and $b_L(t) \to b_L^\star$, where all limits are real valued.

ASSUMPTION 2 *The economy $\mathscr{E}$ satisfies*

$$\eta_L \alpha \gamma_L^{\frac{\epsilon}{\sigma-1}} (1-\theta_E(0))^{\frac{1}{1-\sigma}} L > \rho \tag{27}$$

Note by (20), we have $\alpha p_L(t)^{\frac{1}{\alpha}} L = \alpha \gamma_L^{\frac{\epsilon}{\sigma-1}} (1-\theta_E(t))^{\frac{1}{1-\sigma}} L$ for each t. Thus, by the free entry and exit condition and (25), the assumption above ensures the growth rate of labor augmenting technologies and output is positive.

We now show an ABGP for the model with no scale effects features autonomous energy intensity improvements.

THEOREM 2.1 *If an equilibrium $\mathscr{E}$ is an ABGP, satisfies Assumption 2 and $\dot{\kappa}(t) = 0$, then*

1. $\hat{Y}(t) \to \hat{Q}_L(t) \to g$ *with* $g > 0$
2. $\frac{\hat{Q}_E(t)}{\hat{Q}_L(t)} \to \frac{1-\theta_E(t)}{2-\theta_E(t)-\sigma}$
3. $\theta_E(t) \to 0$
4. $\hat{E}(t) - \hat{Y}(t) \to \hat{Y}(t)\frac{(1-\theta_E(t))(\sigma-1)}{2-\theta_E(t)-\sigma}$

The proof of the theorem is in the appendix. Part 1. of the theorem says the growth of output converges to the growth rate of labor-augmenting technologies, this happens because the cost share of labor converges to 100% of output (part 3.). Part 2. tells us the growth rate of energy-augmenting technologies converges to a constant rate that is lower than the growth rate of labor-augmenting technologies, since we have assumed $\sigma < 1$.

Part 4. tells us energy intensity declines, but at a slower rate than output growth. Energy intensity declines because of energy-augmenting technical change (recall equation (21)). And the rate of decline is slower than the rate of output growth since energy-augmenting technologies grow slower than labor augmenting technologies (because of the price effect discussed at equation (35)). On the other hand,

[16] The definition of BGP and ABGP varies. Hassler et al. (2016) define a BGP in the same way as us. However, Acemoglu (2003) uses the term BGP to refer to what we define at ABGP. The distinction matters in our model since BGPs cannot feature autonomous intensity improvements while ABGPs can.

labor services become an increasingly important contributor to output as the share of energy services falls to zero; thus output growth converges to the rate of growth of labor augmenting technologies (see decomposition of output growth at equation (42) in the Appendix).

The following result is an implication of Theorem 2.1, part 2, and says if initially there are no energy augmenting technical advances (and energy intensity does not fall), there there exists some time T after which energy augmenting technologies will begin to advance.

COROLLARY 2.1 *Let $\mathscr{E}$ be an equilibrium allocation. If*

$$\eta_E \alpha \gamma_E^{\frac{\epsilon}{\sigma-1}} \theta_E(0)^{\frac{1}{1-\sigma}} E(0) - \rho < 0 \tag{28}$$

then there exists $T >\geqslant 0$ such that for all $t > T$,

$$\eta_E b_E(t) = \eta_E \alpha \gamma_E^{\frac{\epsilon}{\sigma-1}} \theta_E(t)^{\frac{1}{1-\sigma}} E(t) - \rho > 0$$

Figure 4 shows ABGPs (in bold), assuming $\kappa(t)$ remains constant, for the regime with no scale effects. To understand the direction of ABGPs intuitively, note output is growing and $b_E(t) > 0$, hence energy intensity must be falling. And we verified in Theorem 2.1 that the elasticity of E/Y to Y must converge to $\frac{\sigma-1}{2-\sigma}$.

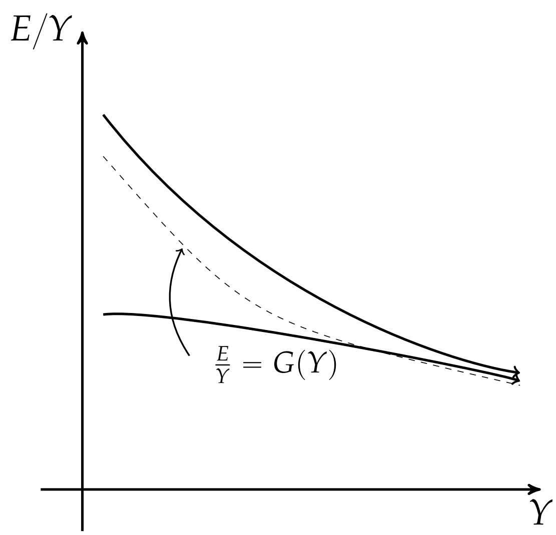

FIGURE 4.— ABGPs for a regime with no scale effects (in bold). All ABGPs feature declining energy intensity, but must converge to the dotted curve.

Below, we show that for a given level of output energy intensity for converges for all ABGPs (conditional convergence).

PROPOSITION 2.2 *Any equilibrium with no scale effects is an ABGP with*

$$\frac{E(t)}{Y(t)} \to G(Y(t))$$

where $G\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ *is a monotone decreasing function satisfying* $\lim_{x\to\infty} G(x) \to 0$*.*

The online appendix studies the dynamics transitional dynamics of ABGPs in the detail, in particular why ABGPs starting below $G(Y)$ in figure 4 must cross $G(Y)$ as they converge. The online appendix also shows any equilibrium path in an ABGP.

For conditional convergence of energy to take place, ABGPs starting with higher energy intensity must experience a faster decline in energy intensity relative to output. For the next result, let $\{E, Y, Q_E, Q_L, \theta_E\}$ and $\{\tilde{E}, \tilde{Y}, \tilde{Q}_E, \tilde{Q}_L, \tilde{\theta}_E\}$ be two equilibrium paths with no scale effects.

CLAIM 2.1 If $\tilde{Y}(t) = Y(t)$ and $\tilde{E}(t) > E(t)$, then

1. $\frac{\hat{\tilde{Q}}_E(t)}{\hat{\tilde{Q}}_L(t)} > \frac{\hat{Q}_E(t)}{\hat{Q}_L(t)}$
2. $\hat{\tilde{Q}}_E(t) > \hat{Q}_E(t)$
3. $\frac{\hat{\tilde{E}}(t)-\hat{\tilde{Y}}(t)}{\hat{\tilde{Y}}(t)} < \frac{\hat{E}(t)-\hat{Y}(t)}{\hat{Y}(t)}$

2.4. *Scale Effects Economy*

The main difference between the scale scale effects and no scale effects economy detailed above are innovation incentives. Assume a constant unit measure of scientists who undertake research directed towards energy or labor machine improvements. If a scientist works to improve the equality of machine i in sector j, the flow rate of machine improvement is η_j. We assume prospective entrants can only choose to work in a sector j instead of choosing the specific machine variety i to improve. Once an entrant chooses j, they are randomly allocated to a machine variety i with no congestion, such that each variety i has at most one scientist allocated to it at t. Let s_j denote the measure of scientists working to improve machines in sector j. The flow rate of innovations in sector j will be $s_j\eta_j$.[17]

To characterize incentives to innovate under scale effects, let

$$V_j(t) \colon = \int_0^1 v_j(i, t \,|\, q_j(i,t))\, \mathrm{d}i\,, \qquad j \in \{E, L\} \tag{29}$$

[17]The random allocation of scientists across machine varieties ensures all varieties experience innovation in equilibrium. Acemoglu et al. (2012) make a similar assumption in discrete time. The assumption here would not be required if we assumed a product improvement environment, however, the Schumpeterian environment simplifies exposition of dynamics.

and let

$$\Pi_j(t) = \int_0^1 \pi_j(i,t \,|\, q_j(i,t))\, \mathrm{d}i\,, \qquad j \in \{E, L\} \tag{30}$$

The free entry and exit conditions under scale effects are

$$\begin{aligned} V_E(t) &\geqslant V_L(t), && \text{if} \quad s_E > 0 \\ V_L(t) &\geqslant V_E(t), && \text{if} \quad s_L > 0 \end{aligned} \tag{31}$$

The definition of an economy and equilibrium are identical to the no scale effects case, with b_j replaced by s_j and the free entry and exit conditions specified by (31) instead of (23). We use $\mathscr{E}_S$ to denote a scale effects economy.

The first scale effects result tells us sustained autonomous energy intensity improvements are not possible.

THEOREM 2.2 *If $\mathscr{E}_S$ is an ABGP and $\dot{\kappa}(t) = 0$, then $\hat{Y}(t) \to \hat{Q}_L(t) \to g$ where $g = \eta_L$, $\theta(t) \to \theta_E^\star$ where $\theta_E^\star > 0$ and $\hat{E}(t) - \hat{Y}(t) \to 0$.*

Moreover, if the energy extraction costs are low enough or energy-augmenting technologies are advanced enough, then there exists a BGP with no energy-augmenting research. For the next result, let $\kappa(t) = \kappa$ for all t.

PROPOSITION 2.3 *Let $\mathscr{E}_S$ be an equilibrium. If $\dot{\kappa}(t) = 0$ for all t, then there exists a balanced growth path with $\eta_L s_L(t) = \eta_L$ and $s_E(t) = 0$ for all t if and only if*

$$\left(\frac{\kappa}{Q_E(0)}\right)^{1-\sigma} \leqslant \frac{(\alpha(1-\alpha))^{1-\sigma}\, \gamma^{\frac{\epsilon}{1-\sigma}}}{\gamma^{\frac{2\epsilon}{\sigma-1}} \frac{\rho+\eta_L}{\rho-\eta_L} + 1}, \qquad t \geqslant 0 \tag{32}$$

Combining (38) and (19) with (32) implies

$$\frac{E(0)}{Y(0)} \leqslant \frac{\alpha \kappa^{-1}}{\gamma^{\frac{2\epsilon}{\sigma-1}} \frac{\rho+\eta_L}{\rho-\eta_L} + 1} := \phi \tag{33}$$

must hold at time 0 for for a BGP to exist. Noting (37), if Q_E is constant along the BGP, then θ_E remains constant, and thus, by (19), $\frac{E}{Y}$ remains constant and satisfies (33) along the BGP.

Turning now to convergence under scale effects, consider Figure 5, which shows ABGPs for a scale effects regime with a constant real energy price. The dotted line is the BGP where $\frac{E}{Y} = \phi$, where ϕ is implied by Equation (33). By Proposition 2.3, any economy with energy intensity below ϕ has a unique BGP with no energy-augmenting technical change. However, again by proposition 2.3, if energy intensity is strictly greater than ϕ, energy-augmenting technologies must advance. And by Theorem 2.2, Q_E and hence energy intensity must converge to a constant.

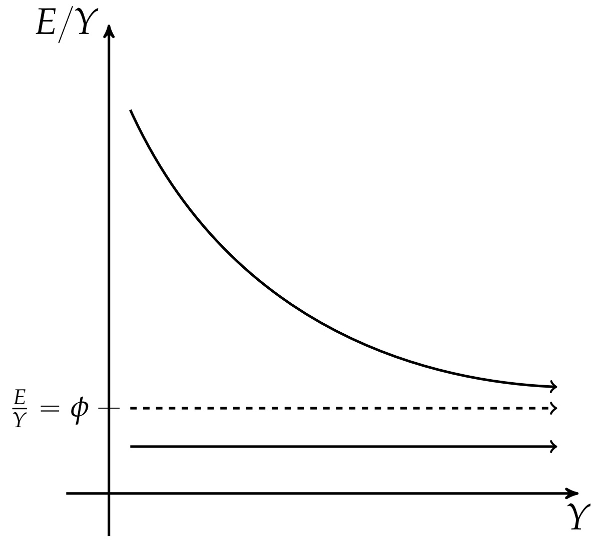

FIGURE 5.— ABGPs for a regime with scale effects. All ABGPs must converge to a BGP where energy intensity is constant.

3. DISCUSSION OF MAIN QUESTIONS

3.1. *Why Energy Intensity Falls Slower than Output Growth and Stops Under Scale Effects*

First, let us see why as an economy grows and energy augmenting technologies advance, there are continued incentives to undertake energy augmenting research in the no scale effects economy; incentives to innovate (equation (25)) can be split into the product of the market size and price effect:

$$\eta_E b_E(t) = \eta_E \lambda \overbrace{\alpha p_E(t)^{\frac{1}{\alpha}}}^{\text{Price effect}} \underbrace{E(t)}_{\text{Market size effect}} - \rho \tag{34}$$

and, by equation by (20) and noting (50) in the appendix, $\alpha p_E(t)^{\frac{1}{\alpha}} = \alpha \gamma_E^{\frac{\epsilon}{\sigma-1}} \theta_E(t)^{\frac{1}{1-\sigma}}$. Thus along an ABGP, since energy fuel prices are constant and energy use expands, the price of energy services falls. (By contrast, suppose E was constant, then along a BGP, Q_E grows at the same rate as Y and the intermediate price stays constant.) At the same time, the market size of E increases, sustaining incentives to innovate.

However, the relative incentive to innovate favours labor augmenting research.

With innovation in both sectors, by (25), relative innovation is given by

$$\frac{\rho + \eta_E b_E(t)}{\rho + \eta_L b_L(t)} = \frac{\eta_E}{\eta_L} \overbrace{\left(\frac{p_E(t)}{p_L(t)}\right)^{\frac{1}{\alpha}}}^{\text{Price effect}} \underbrace{\frac{E(t)}{L}}_{\text{Market size effect}} = \frac{\eta_E}{\eta_L} \left(\frac{Q_E(t)}{Q_L(t)}\right)^{-\frac{1}{\sigma}} \left(\frac{E(t)}{L(t)}\right)^{\frac{\sigma-1}{\sigma}} \tag{35}$$

As energy use grows, the response of the relative rates of innovation can be split into first order effects, how relative rates of innovation respond keeping $\frac{Q_E(t)}{Q_L(t)}$ fixed and 'second order effects', how relative rates of innovation respond after $\frac{Q_E(t)}{Q_L(t)}$ has adjusted in response to the first order effects. Now, keep the ratio $\frac{Q_E(t)}{Q_L(t)}$ fixed and consider the 'first order effects' of expanding energy use on relative research profitability. The market size effect increases profitability for energy-augmenting research, while an increase in energy use reduces the relative price of the energy intermediate, leading to a fall in energy research profitability. If $\sigma < 1$, then the price effect is stronger and the first order effects lead to a fall of profitability to energy research.

Turning to the second order effects, to maintain the free entry/exit conditions, b_E must fall, and then, from the second equality in (35), $\frac{Q_E(t)}{Q_L(t)}$ falls, leading to a second order price effect running in the opposite direction to the first order price effect (recall again Equation (14)). Under no scale effects, b_E falls till it converges to a strictly positive constant and the second order effects from the falling $\frac{Q_E(t)}{Q_L(t)}$ ratio are sufficient to keep relative incentives to do research in both sectors in balance. Innovation continues in both sectors.

The relative rates of innovation under a rational expectations equilibrium in the case with scale effects are more difficult to characterize using current profit ratios. However, we will consider the current profit ratios to gain intuition of the theoretical results we proved in the previous section 2.4. Suppose that for innovation to occur in both sectors, we must have

$$\begin{aligned} \frac{\rho + \eta_E s_E(t)}{\rho + \eta_L s_L(t)} &= \frac{\eta_E \Pi_E(t)}{\eta_L \Pi_L(t)} \\ &= \frac{\eta_E}{\eta_L} \overbrace{\frac{p_E(t)}{p_L(t)}^{\frac{1}{\alpha}}}^{\text{Price effect}} \underbrace{\frac{E(t)}{L}}_{\text{Market size effect}} \overbrace{\frac{Q_E(t)}{Q_L(t)}}^{\text{Scale effect}} = \frac{\eta_E}{\eta_L} \left(\frac{Q_E(t)}{Q_L(t)}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{E(t)}{L(t)}\right)^{\frac{\sigma-1}{\sigma}} \end{aligned} \tag{36}$$

As energy use grows, once again, because of the price effect, the first order effects lead to a fall in relative profitability to do energy research. However, there are now two second order effects; on the one hand, the subsequent fall in $\frac{Q_E(t)}{Q_L(t)}$ from the increased profitability of labor-augmenting research pushes up energy-augmenting research profitability through the second order price effect. On the other hand, as $\frac{Q_E(t)}{Q_L(t)}$ falls, the scale effect increases the profitability of labor-augmenting research. The second order scale effect runs in *the same direction as the first order price effect*, pushing down relative profitability for energy research even further. Whether the relative profits for energy- and labor-augmenting research remain in a constant ratio or whether relative profits for energy keep falling depends on how fast energy use grows relative to technology. From (20), consider

$$(37) \qquad \left(\frac{Q_E(t)E(t)}{Q_L(t)L}\right)^{\frac{\sigma-1}{\sigma}} = \frac{\theta_E(t)}{1-\theta_E(t)}\gamma^{\frac{-\epsilon}{\sigma}}$$

Now use (19) and (21), to arrive at

$$(38) \qquad \theta_E(t) = (\alpha(1-\alpha))^{\sigma-1}\kappa(t)^{1-\sigma}\gamma_E^{\frac{\epsilon}{\sigma-1}}Q_E(t)^{\sigma-1}$$

Thus, if $Q(t) \to \infty$, then we must have $\frac{\rho+\eta_E s_E(t)}{\rho+\eta_L s_L(t)} \to 0$, which cannot hold since since $s_L \leqslant s$. Continued energy augmenting technological advancement cannot now occur along any ABGP since incentives to invest in labour augmenting technologies increase at such high a rate, from the path dependence generated by scale effects, that investment in energy augmenting technologies must keep decreasing for the no arbitrage condition to hold.

3.2. *Scale Effects or Not?*

We solve Equation (21) for Q_E. In discrete time, adding a stationary — but probably serially correlated — error term, we have:

$$(39) \qquad Q_{E,t} + u_{E,t} = A_E\kappa_t^{\frac{\sigma}{\sigma-1}}\left(\frac{E_t}{Y_t}\right)^{\frac{1}{\sigma-1}}$$

where $A_E = \gamma_E^{-\frac{\epsilon}{(\sigma-1)^2}}\alpha^{-\frac{\sigma}{\sigma-1}}(1-\alpha)^{-1}$.[18] We compute the RHS of (39) using annual U.S. data from 1900 to 2015 as described in the online appendix and assumming that $\sigma = .5$ and apply the Hodrick-Prescott filter to obtain the estimate of $Q_{E,t}$ plotted in Figure 6.[19]

[18]As A_E only shifts estimates of $Q_{E,t}$ without affecting its growth rate – and A_L similarly affects $Q_{L,t}$, we set $A_E = A_L = 1$ in the following.

[19]Hassler et al. (2016) assume there is no error in their equivalent of (39). As a result, they find that only very low values of σ are compatible with a reasonable time path for $Q_{E,t}$. But given our

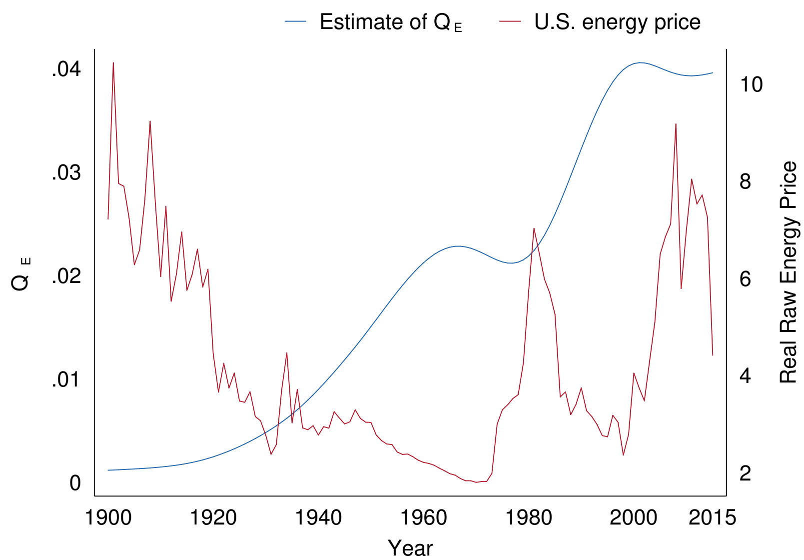

FIGURE 6.— Smoothed estimate of $Q_E(t)$, estimated assuming $\sigma = .5$, and the U.S. energy price

Before 1950, the growth rate of Q_E was positive, while prices declined, suggesting an autonomous decline of energy intensity. After 1960, and before 1980, the growth rate of Q_E declines to under zero by the early 1970s, before rising and falling again in a lagged response to prices.

Both the no scale effects and scale effects regimes could be consistent with the path of Q_E. If we suppose the U.S. economy was sufficiently energy inefficient in the early 1900s, there could have been autonomous improvements in Q_E under a scale effects regime (recall Proposition 2.3). The growth of $Q_E(t)$ does decline below zero just before the price shocks of the 1970s and the early 2000s, however, this does not imply a model with no autonomous energy efficiency improvements along an ABGP. Both scale effects and no scale effects regimes predict such a decline, where the decline in the growth rate of $Q_E(t)$ is associated with a fall in energy prices. To see why, for a given level of $Q_E(t)$ and $Q_L(t)$, when prices fall, energy consumption increases, and recalling (35) and (36), energy-augmenting technical change becomes relatively less profitable due to the stronger price effect.

model is very simple, energy intensity likely adjusts slowly to its equilibrium value; and variables are measured with error – the addition of an error term seems more reasonable to us than not. However, values above .8, even with smoothing, give unreasonable jumps and declines in line with the observation made by Hassler et al. (2016). As such, we maintain a reasonable value of σ to be between 0 and .5.

Both the no scale effects and scale effects regime can also be consistent with paths of energy intensity across countries. Both scale and no scale effects give convergence of energy intensities across countries given a level of output — consistent with conditional convergence shown by Csereklyei et al. (2016) and the increasingly 'tighter' relationship between output per capita and energy intensity seen in Figure 2 and Figure B.8 in the appendix.

Second, both regimes can explain why many countries experience no declines in energy intensity. Figure 2 and Figure B.8 in the appendix shows that many countries, such as Brazil, Italy and Spain, with low energy intensities with close to constant paths of energy intensity. Under no scale effects, initially, energy intensity may be so low given output such that (28) holds and no energy augmenting innovation occurs. On the other hand, under scale effects, countries showing constant energy intensity through time may be on their long-run BGP, to which all countries will converge to.

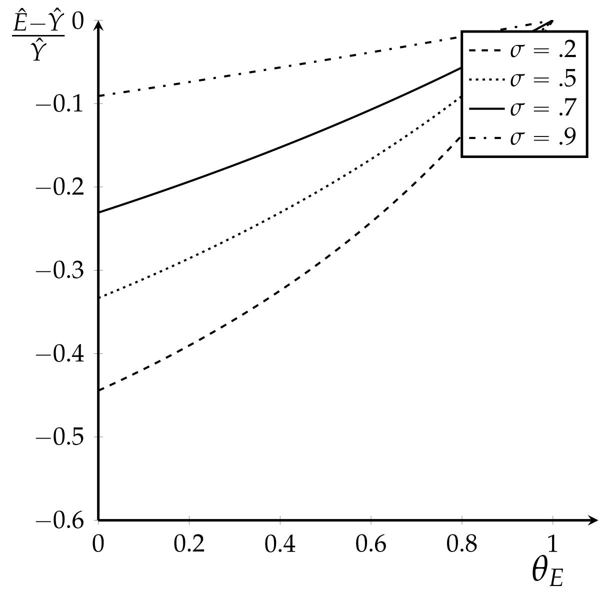

FIGURE 7.— Elasticity of Energy Intensity to Output along the 'stable path', all ABGPs in the model with no scale effects converge to this 'stable path'.

Perhaps one piece of evidence in favor of the scale effects scenario is the low absolute elasticity of energy intensity to output through time and in the cross-section for countries with low energy intensity. In the cross-section, the elasticity between per-capita output and energy rises above -10% and is even positive in 1971 and 1981 (see Table 2 and Figure B.9 in the appendix) for countries with low energy intensity. Consider the elasticity of energy intensity to output towards which ABGPs converge to, given by Theorem 2.1,

$$\hat{E} - \hat{Y} = \hat{Y}\frac{(\sigma - 1)\,(1 - \theta_E(t))}{2 - \theta_E(t) - \sigma}$$

With constant prices, the low in absolute value elasticities seen in the cross-section of energy intensity to output can only be predicted along the ABGP in the no scale effects model if σ is approximately .9 (see Figure 7). However, such elasticities of substitution are higher than the estimates in the literature, which range from close to zero (Hassler et al., 2016) to .68 (Stern and Kander, 2012). An elasticity of substitution of .9 is also not consistent with a reasonable evolution of $Q_E(t)$. As figure B.10 in the appendix suggests, even with smoothing applied, a high σ implies Q_E rises and falls almost by 1000% between 1970 and 2015; it is difficult to attribute Q_E in this case to technical change.

4. CONCLUSION

We develop an endogenous model of energy and economic growth with directed technical change to understand the economic drivers and future potential for autonomous energy efficiency improvements. Our analysis focused on understanding the trade-offs firms face between undertaking energy-augmenting and non-energy-augmenting research. Our key finding was that unless energy prices rise, there is a relatively stronger incentive to undertake labor-augmenting research, since the price of labor rises relative to the price of energy. Whether or not labor-augmenting research completely crowds out energy-augmenting research depends on whether or not innovation possibilities depend on scale effects; with scale effects, the relatively stronger incentive for labor-augmenting research is compounded and autonomous energy efficiency improvements are eventually crowded out. On the other hand, in a model without scale effects, autonomous energy efficiency improvements continue at a constant rate relative to output, albeit at a slower rate than labor-augmenting technical change.

Despite the sharp theoretical predictions of the directed technical change model, we were unable to identify whether a model with scale effects or no scale effects more accurately accounts for past trends. This is because under both models, one, if energy intensity is high enough, autonomous energy efficiency improvements take place, and two, fluctuating prices have led to notable fluctuations in the estimate of energy-augmenting technology, which prevents us from determining whether energy intensity is converging to a constant or not. Some mild evidence in favor of the scale effects model is the fact that the fall in energy intensity relative to output in the cross-section of high income countries is small, suggesting energy intensities in the cross-section converge to a constant.

APPENDIX A: TECHNICAL APPENDIX

A.1. *Definition of Equilibrium*

DEFINITION A.1 An equilibrium is an allocation $\mathscr{E}$ satisfying:

1. no aggregate uncertainty holds and for $j \in \{E, L\}$ and $i \in [0,1]$, the processes $(q_j(i,t))_{t=0}^{\infty}$ satisfy (6), where the arrival rate of innovations is identical across i in a sector $j \in \{E, L\}$: given by b_j for the no scale effects regime or s_j for the scale effects regime
2. at each $t \in \mathbb{R}_+$ and $i \in [0,1]$ given $p_E(t)$ and $p_L(t)$, monopolists who own blueprints of machine type i with quality q pick prices, $p_E^x(i,t,q)$ and $p_L^x(i,t,q)$, to maximize profits
3. at each t and for each draw from the random variable q_E and q_L, given $p_E(t)$, $p_E^x(t)$, $p_L^x(t)$ and $p_L(t)$, intermediate goods producers choose $E(t)$, $L(t)$ and $x_E(t,i,q_E(i,t))$ and $x_L(t,i,q_L(i,t))$ to maximize profits
4. final goods markets producers maximize profits given $p_E(t)$ and $p_L(t)$ and the market for intermediate goods clears
5. given r and w, consumers choose a to maximize their inter-temporal utility
6. the corporate asset market clears, that is $a(t) = V_L(t) + V_E(t)$, where V_L and V_E are defined by (29)
7. given q_E and q_L, the value functions satisfy
$$v_j(i,t \mid q(i,t)) = \mathbb{E}_t \int_{s=t}^{T} e^{-\rho s} \pi(i,s \mid q(i,t)) \, \mathrm{d}s, \qquad j \in \{E, L\}, \mathbb{P} - a.e.$$
where $\pi(i,s \mid q)$ is given in terms of the paths of prices, energy use and a quality q by (14)[20]
8. without scale effects, $s_E(t) = s_L(t) = 0$ and the free entry and exit condition
$$\eta_j \frac{v_j(i,t \mid q(i,t))}{q(i,t)} = 1, \qquad j \in \{E, L\}, \mathbb{P} - a.e.$$
holds
9. with scale effects, $Z_E(t) = Z_L(t) = 0$ and the free entry and exit, (31), holds
10. the resource constraint (2) holds and $L(t) = L$ for all t.

A.2. *Proofs of Main Results*

For the following claim, let $\tilde{F} \colon \mathbb{R}_+^2 \to \mathbb{R}$ be homogeneous of degree one, increasing in both arguments and differentiable, let $Y(t) = \tilde{F}(X(t), Z(t))$ and let $\hat{Y}(t)$, $\hat{X}(t)$ and $\hat{Z}(t)$ denote the growth rates of $Y(t)$, $X(t)$ and $Z(t)$ respectively.

[20]Recall expectations here are taken over T, where T is the random stopping time after which variety i in sector j experiences an innovation and the incumbent experiences zero profits thereafter. The distribution of T is pinned down entirely by the paths b_j and s_j for $j \in \{E, L\}$

The following claim is used in the proof of Proposition 2.1.

CLAIM A.1 If $\hat{Y}$, $\hat{X}$ and $\hat{Z}$ are constant, then $\hat{X} = \hat{Z}$.

PROOF: Let g_Y, g_X and g_Z be the constant growth rates of $Y(t)$, $X(t)$ and $Z(t)$. Taking time derivatives of $Y(t)$ gives

$$\dot{Y}(t) = \tilde{F}_1(X(t), Z(t))\dot{X}(t) + \tilde{F}_2(X(t), Z(t))\dot{Z}(t)$$

and thus

$$\hat{Y}(t) = \frac{\tilde{F}_1(X(t), Z(t))X(t)}{\tilde{F}(X(t), Z(t))}\hat{X}(t) + \frac{\tilde{F}_2(X(t), Z(t))Z(t)}{\tilde{F}(X(t), Z(t))}\hat{Z}(t): = (1 - \tilde{\theta}(t))\hat{X}(t) + \tilde{\theta}(t)\hat{Z}(t)$$

where, by Euler's Theorem (theorem 2.1 in Acemoglu (2009)),

$$(1 - \tilde{\theta}(t)) + \tilde{\theta}(t) = \frac{\tilde{F}_1\left(X(t), Z(t)\right)X(t) + \tilde{F}_2\left(X(t), Z(t)\right)Z(t)}{\tilde{F}\left(X(t), Z(t)\right)} = 1$$

Now suppose by contradiction that $g_X \neq g_Z$. First consider the case $g_X < g_Z$. We must have

$$g_Y = (1 - \tilde{\theta}(t))g_X + \tilde{\theta}(t)g_Z < (1 - \tilde{\theta}(t))g_Z + \tilde{\theta}(t)g_Z = g_Z \tag{40}$$

Since g_Y, g_X and g_Z are the growth rates of $Y(t)$, $X(t)$ and $Z(t)$,

$$Y(0)e^{tg_Y} = \tilde{F}\left(X(0)e^{tg_X}, Z(0)e^{tg_Z}\right)$$

dividing the LHS and RHS by e^{tg_Y} and by homogeneity of the function $\tilde{F}$, we have

$$\begin{aligned} Y(0) &= e^{t(g_Z - g_Y)}\tilde{F}\left(X(0)e^{t(g_X - g_Z)}, Z(0)\right) \\ &< \tilde{F}\left(X(0), Z(0)\right) = Y(0) \end{aligned}$$

where the inequality follows from our assumption $g_X < g_Z$, Equation (40) which says $g_Y < g_Z$, the fact that $\tilde{F}$ is increasing and observing $e^x < 1$ for any $x < 0$. However, $Y(0) < Y(0)$ is a contradiction, implying $g_X \geqslant g_Z$. The case to rule out $g_X > g_Z$ is symmetric by replacing X with Z in the above steps, establishing $g_X \neq g_Z$.

Q.E.D.

PROOF OF THEOREM 2.1: We first derive growth equations that hold in equilibrium. Take time derivatives of $\theta_E(t)$, defined by (20), to write

$$\hat{\theta}_E(t) = \frac{\sigma - 1}{\sigma}(1 - \theta(t)) \left(\hat{Q}_E(t) + \hat{E}(t) - \hat{Q}_L(t)\right) \tag{41}$$

Similarly, take time derivatives of $Y(t)$, defined by (18), to write

$$\hat{Y}(t) = (1 - \theta_E(t))\, \hat{Q}_L(t) + \theta_E(t) \left(\hat{Q}_E(t) + \hat{E}(t)\right) \tag{42}$$

Now, note from (19), we have $\hat{\theta} = \hat{E} - \hat{Y}$. Use this expression in Equation (41) to write

$$\hat{E}(t) = \frac{\sigma - 1}{\sigma}(1 - \theta_E(t))(\hat{B}(t) + \hat{E}(t)) + \hat{Y}(t) \tag{43}$$

Next, combine (43) with (42) to write

$$\hat{E}(t) = \frac{\sigma}{1 - \theta_E(t)} \hat{Q}_E(t) - \hat{B}(t) \tag{44}$$

Let $\Gamma(t)$ define the ratio of relative profitability between energy and labor-augmenting research as follows:

$$\Gamma(t) :\, = \frac{\rho + \eta_E b_E(t)}{\rho + \eta_L b_L(t)} \tag{45}$$

Recall along an equilibrium path, innovation in both sectors, no arbitrage requires

$$\Gamma(t) = \left(\frac{Q_E(t)}{Q_L(t)}\right)^{-\frac{1}{\sigma}} \left(\frac{E(t)}{L(t)}\right)^{\frac{\sigma-1}{\sigma}} \tag{46}$$

And thus,

$$\hat{\Gamma}(t) = \frac{1 - \sigma}{\sigma} \hat{E}(t) - \hat{B}(t) \left(\frac{1}{\sigma}\right) \tag{47}$$

Combine (44) with (47),

$$\begin{aligned}
\hat{\Gamma}(t) &= \frac{\sigma - 1}{\sigma} \hat{E} - \frac{\hat{B}(t)}{\sigma} \\
&= \frac{\sigma - 1}{\sigma} \left(\frac{\sigma}{1 - \theta(t)} \hat{Q}_E(t) - \hat{B}(t)\right) - \frac{\hat{B}(t)}{\sigma} \\
&= \frac{\sigma - 1}{1 - \theta_E(t)} \hat{Q}_E(t) - \hat{B}(t)
\end{aligned} \tag{48}$$

Along an ABGP, $\hat{\Gamma}(t) \to 0$, and using the above, recalling $\hat{B}(t) = \hat{Q}_E(t) - \hat{Q}_L(t)$, deduce

$$\frac{\hat{Q}_E(t)}{\hat{Q}_L(t)} \to \frac{1-\theta(t)}{2-\theta(t)-\sigma} \tag{49}$$

Since $\hat{Q}_L(t)$ is positive and bounded below by a positive constant, we must have $\hat{Q}_E(t)$ is positive and bounded below by a positive constant. Accordingly, $\theta_E(t) \to 0$ by (38). By (42), $\hat{Y}(t) \to \hat{Q}_L(t)$. Now, noting (21), taking time derivatives, we arrive at

$$\hat{E}(t) - \hat{Y}(t) = (\sigma - 1)\,\hat{Q}_E(t) \to \hat{Y}(t)\frac{(1-\theta(t))\,(\sigma-1)}{2-\theta(t)-\sigma} \to \hat{Y}(t)\frac{\sigma-1}{2-\sigma}$$

Q.E.D.

We now prepare some preliminary notation before turning to the proof of theorem 2.2. Noting $p_L(t) = \gamma_L\left(\frac{Y(t)}{Y_L(t)}\right)^{\frac{1}{\epsilon}}$, we have

$$\begin{aligned} p_L(t) = \gamma_L\left(\frac{Y(t)}{Y_L(t)}\right)^{\frac{1}{\epsilon}} &= \gamma_L\left(\gamma_L + \gamma_E\left(\frac{Y_E}{Y_L}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{1}{\epsilon-1}} \\ &= \gamma_L^{\frac{\epsilon}{\epsilon-1}}\left(1 + \gamma^{\frac{\epsilon}{\sigma}}\left(\frac{Q_E(t)E(t)}{Q_L(t)L}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\epsilon-1}} \end{aligned} \tag{50}$$

where the third equality uses (17). Using (20), we can then write

$$\alpha p_L(t)^{\frac{1}{\alpha}} L = \alpha\gamma_L^{\frac{\epsilon}{\sigma-1}}\left(1-\theta_E(t)\right)^{\frac{1}{1-\sigma}} L \tag{51}$$

similarly

$$\alpha p_E(t)^{\frac{1}{\alpha}} E(t) = \alpha\gamma_E^{\frac{\epsilon}{\sigma-1}}\theta_E(t)^{\frac{1}{1-\sigma}} E(t) \tag{52}$$

Now define

$$J_L(t) := \mathbb{E}_t \int_t^{T_L(t)} e^{-\rho s}\alpha p_L(s)^{\frac{1}{\alpha}} L\,\mathrm{d}s, \quad J_E(t) := \mathbb{E}_t \int_t^{T_E(t)} e^{-\rho s}\alpha p_E(s)^{\frac{1}{\alpha}} E(s)\,\mathrm{d}s$$

where $T_j(t)$ is the random stopping time after which an incumbent is replaced by a new entrant. Note the distribution of $T_j(t)$ does not depend on the individual machine i, since each machine experiences a common innovation, hence replacement rate $s_j\eta_j$. Recalling the definition of $V_L(t)$ from (29),

$$\begin{aligned} V_L(t) = \int_0^1 v_L\left(i, t \,|\, q\right)\, \mathrm{d}i &= \int_0^1 \mathbb{E}_t \int_{s=t}^{T_L(t)} e^{-\rho s}\pi_L(s)\,\mathrm{d}s\,\mathrm{d}i \\ &= \int_0^1 q(i,t)\mathbb{E}_t \int_{s=t}^{T_L(t)} e^{-\rho s}\alpha p_L(s)^{\frac{1}{\alpha}} L\,\mathrm{d}s\,\mathrm{d}i \\ &= Q_L(t)\mathbb{E}_t \int_{s=t}^{T_L(t)} e^{-\rho s}\alpha p_L(s)^{\frac{1}{\alpha}} L\,\mathrm{d}s \\ &= Q_L(t)J_L(t) \end{aligned}$$

The HJB equation for $J_L(t)$ will be

$$\dot{J}_L(t) = \left(\rho + \eta_L s_L(t)\right) J_L(t) - \alpha p_L(t)^{\frac{1}{\alpha}} L \tag{53}$$

and similarly, the HJB equation for $J_E(t)$ becomes

$$\dot{J}_E(t) = \left(\rho + \eta_E s_E(t)\right) J_E(t) - \alpha p_E(t)^{\frac{1}{\alpha}} E(t) \tag{54}$$

LEMMA A.1 *If an equilibrium with scale effects is an asymptotic balanced growth path with* $s_E^\star > 0$ *and* $s_L^\star > 0$, *then* $\eta_L s_L^\star = \rho + 2\eta_E s_E^\star$.

PROOF: By Equation (38), since $Q_E(t) \to \infty$, $\theta_E(t) \to 0$. Now, if $\theta(t)_E \to 0$, using (51), we have

$$\alpha p_L(t)^{\frac{1}{\alpha}} L \to \alpha \gamma_L^{\frac{\epsilon}{\sigma-1}} L \tag{55}$$

which implies $J_L(t)$ converges to a constant. Note both the terms on the RHS of the HJB equation for $J_L(t)$, Equation (53), converge, implying $\dot{J}_L(t)$ converges. But since $J_L(t)$ converges to a constant, we must have $\dot{J}_L(t) \to 0$. Once again by the HJB condition, Equation (53),

$$J_L(t) \to \frac{\alpha \gamma_L^{\frac{\epsilon}{\sigma-1}} L}{\rho + s_E^\star} \tag{56}$$

Also note

$$\hat{J}_L(t) = \frac{\dot{J}_L(t)}{J_L(t)} \tag{57}$$

The denominator converges a positive constant, while numerator converges to zero, implying $\hat{J}_L(t) \to 0$. Next, from the free entry and exit condition, since $s_E^\star > 0$ and $s_L^\star > 0$, there exists $\bar{T}$ such that

$$J_E(t) = \frac{Q_L(t)J_L(t)}{Q_E(t)}, \qquad t > \bar{T} \tag{58}$$

Taking time derivatives, we have

$$\hat{J}_L(t) - \hat{J}_E(t) = \hat{Q}_E(t) - \hat{Q}_L(t), \qquad t > \bar{T} \tag{59}$$

Rearranging gives

$$\hat{J}_E(t) = \hat{Q}_L(t) - \hat{Q}_E(t) + \hat{J}_L(t) \to \eta_L s_L^\star - \eta_E s_E^\star \tag{60}$$

where $\hat{J}_L(t) \to 0$ by the argument proceeding Equation (57). Now use the above with the HJB condition for $J_E(t)$, Equation (54),

$$\frac{\alpha p_E(t)^{\frac{1}{\alpha}} E(t)}{J_E(t)} = \rho + \eta_E s_E(t) - \hat{J}_E(t) \to \rho + 2\eta_E s_E^\star - \eta_L s_L^\star \tag{61}$$

Note the limit is a (possibly zero) constant. Moreover, by the HJB condition for $J_L(t)$, Equation (53),

$$\frac{\alpha p_L(t)^{\frac{1}{\alpha}} L}{J_L} = \rho + \eta_L s_L^\star - \hat{J}_L(t) \to \rho + \eta_L s_L^\star \tag{62}$$

since $\hat{J}_E(t) \to 0$. As such,

$$\begin{aligned} \frac{\theta_E(t)}{1-\theta_E(t)}\gamma^{-\epsilon\sigma} &= \left(\frac{Q_E(t)E(t)}{Q_L(t)L}\right)^{\frac{\sigma-1}{\sigma}} \\ &= \frac{p_E^{\frac{1}{\alpha}} Q_E(t)E(t)}{p_L^{\frac{1}{\alpha}} Q_L(t)L} = \frac{p_E^{\frac{1}{\alpha}} J_L(t)E(t)}{p_L^{\frac{1}{\alpha}} J_E(t)L} \to \frac{\rho + 2\eta_E s_E^\star - \eta_L s_L^\star}{\rho + \eta_L s_L^\star} \end{aligned}$$

where the first equality comes from (37), the second equality uses (16), the third uses the free entry and exit conditions (31) and convergence at the final step follows from (61) and (62) and noting the denominator converges to a strictly positive constant. Thus, if $\theta_E(t) \to 0$, then $\eta_L s_L^\star = \rho + 2\eta_E s_E^\star$.

Q.E.D.

CLAIM A.2 If an equilibrium with scale effects is an asymptotic balanced growth path with $s_E^\star > 0$ and $s_L^\star > 0$, then there exists $\bar{T}$ and M, with $M < \infty$ such that

$$\mathbb{E}_t \int_{s=t}^{T_E(t)} e^{\int_t^s \hat{E}(\bar{s}) - \eta_E s_E(\bar{s}) - \rho \, \mathrm{d}\bar{s}} \leqslant M < \infty, \qquad t > \bar{T}$$

PROOF: The distribution of $T_E(t)$ is

$$\mathbb{P}\left(T_E(t) \leqslant t + s\right) = 1 - e^{-\int_0^s \eta_E s_E(s_1 + t)\, \mathrm{d}s_1}$$

and the density of $T_E(t)-t$ is $\eta_E s_E(s)e^{-\int_0^s \eta_E s_E(s_1+t)\mathrm{d}s_1}$. Allowing us to write

$$\begin{aligned}\mathbb{E}_t\int_{s=t}^{T_E(t)} e^{\int_t^s \hat{E}(\bar{s})-\eta_E s_E(\bar{s})-\rho\,\mathrm{d}\bar{s}}\,\mathrm{d}s &= \mathbb{E}_t\int_0^{T_E(t)-t} e^{\int_0^s \hat{E}(\bar{s}+t)-\eta_E s_E(\bar{s}+t)-\rho\,\mathrm{d}\bar{s}}\,\mathrm{d}s\\ &= \eta_E s_E(s)\int_0^\infty\int_0^T e^{\int_0^s \hat{E}(\bar{s}+t)-\eta_E s_E(\bar{s}+t)-\rho\,\mathrm{d}\bar{s}}\,\mathrm{d}s\, e^{-\int_0^T \eta_E s_E(s_1+t)\mathrm{d}s_1}\,\mathrm{d}T\end{aligned}$$

Define

$$c := \lim_{t\to\infty}\{\hat{E}(t)-2\eta_E s_E(t)-\rho\} \tag{63}$$

We now show $c<0$. Taking growth rates across equation (21) gives $\hat{E}(t)-\hat{Y}(t)=(\sigma-1)\hat{Q}_E(t)$, and thus $\hat{E}(t)-\hat{Y}(t)\to\eta_E s_E^\star(\sigma-1)$, and $\hat{E}(t)\to\eta_E s_E^\star(\sigma-1)+\eta_L s_L^\star$ since $\hat{Y}(t)\to\eta_L s_L^\star$ by (42). If $s_L^\star=0$, then

$$c=\eta_E s_E^\star(\sigma-1)+\eta_L s_L^\star-2\eta_E s_E^\star-\rho=\eta_E s_E^\star(\sigma-3)-\rho<0 \tag{64}$$

On the other hand, if $s_L^\star>0$, then by Lemma A.1, $\eta_L s_L^\star=\rho+2\eta_E s_E^\star$

$$c=\eta_E s_E^\star(\sigma-1)+\eta_L s_L^\star-\eta_E 2s_E^\star-\rho=s_E^\star(\sigma-1)<0 \tag{65}$$

Next, note there exists $\epsilon>0$ such that $c+\epsilon<0$. Moreover, since $\hat{E}(t)-\eta_E s_E(t)-\rho\to\eta_E s_E^\star(\sigma-2)+s_L^\star-\rho$, there exists $\bar{T}$ such that for all $t>\bar{T}$, we have

$$\hat{E}(t)-\eta_E s_E(t)-\rho<\eta_E s_E^\star(\sigma-2)+\eta_L s_L^\star-\rho+\frac{\epsilon}{2}:=c_1 \tag{66}$$

and

$$-\eta_E s_E(t)<-\eta_E s_E^\star+\frac{\epsilon}{2}:=c_2 \tag{67}$$

The above two inequalities give

$$\begin{aligned}&\eta_E s_E(s)\int_0^\infty\int_0^T e^{\int_0^s \hat{E}(\bar{s})-\eta_E s_E(\bar{s})-\rho\,\mathrm{d}\bar{s}}\,\mathrm{d}s\, e^{-\int_0^T \eta_E s_E(s_1)\mathrm{d}s_1}\,\mathrm{d}T\\ &\leqslant \eta_E s_E(s)\int_0^\infty\int_0^T e^{\int_0^s c_1\,\mathrm{d}\bar{s}}\,\mathrm{d}s\, e^{\int_0^T c_2\mathrm{d}s_1}\,\mathrm{d}T=\int_0^\infty\frac{e^{T(c_1+c_2)}-e^{Tc_2}}{c_1}\,\mathrm{d}T:=M<\infty\end{aligned} \tag{68}$$

where the first inequality follows from monotonicity of the exponential function and noting (66) and (67). The second inequality comes from solving the inside integrals and the final inequality comes from noting $c_1+c_2=c+\epsilon<0$.

Q.E.D.

PROOF OF THEOREM 2.2: If $s_E^\star > 0$, then there exists $\bar{T}$ such that for $t > \bar{T}$, the no arbitrage condition holds

$$\frac{J_E(t)Q_E(t)}{J_L(t)Q_L(t)} \geqslant 1, \qquad t > \bar{T} \tag{69}$$

to prove the theorem, we will show this condition cannot hold if $Q_E(t) \to \infty$ and $E(t)/Y(t) \to 0$. From the definition of $J_E(t)$ and $J_L(t)$, we have

$$\begin{aligned}
\frac{J_E(t)Q_E(t)}{J_L(t)Q_L(t)} &= \frac{Q_E(t)\mathbb{E}_t \int_{s=t}^{T_E(t)} \pi_E(s)^{\frac{1}{\alpha}} E(s) e^{-\rho s}\, \mathrm{d}s}{Q_L(t) J_L(t)} \\
&= \kappa(t)(1-\theta_E(t))^{1-\sigma} L \left(\frac{\theta_E(t)}{1-\theta_E(t)}\right)^{\frac{1}{1-\sigma}} \\
&\quad \times \frac{E(t)Q_E(t)}{LQ_L(t)} \frac{\mathbb{E}_t \int_{s=t}^{T(t)} e^{\int_{\bar{s}=t}^{s} \hat{E}(\bar{s}) - \eta_E s_E(\bar{s}) - \rho \, \mathrm{d}\bar{s}}\, \mathrm{d}s}{J_L(t)} \\
&= \kappa(t)\, (1-\theta_E(t))^{1-\sigma}\, L \left(\frac{\theta_E(t)}{1-\theta_E(t)}\right) \\
&\quad \times \frac{\mathbb{E}_t \int_{s=t}^{T(t)} e^{\int_{\bar{s}=t}^{s} \hat{E}(\bar{s}) - \eta_E s_E(\bar{s}) - \rho \, \mathrm{d}\bar{s}}\, \mathrm{d}s}{J_L(t)}
\end{aligned} \tag{70}$$

where we have omitted constants in front of the RHS above for simplicity. The second equality above uses (52) and (38) to derive

$$\begin{aligned}
\pi_E(s)^{\frac{1}{\alpha}} E(s) = \kappa(t)\theta_E(s)^{\frac{1}{1-\sigma}} E(s) &= Q_E(s)^{-1} E(s) \\
&= \kappa(t) Q_E(t)^{-1} E(t) e^{\int_{\bar{s}=t}^{s} \hat{E}(\bar{s}) - \eta_E s_E(\bar{s})\, \mathrm{d}\bar{s}} \\
&= \kappa(t)\theta_E(t)^{\frac{1}{1-\sigma}} E(t) e^{\int_{\bar{s}=t}^{s} \hat{E}(\bar{s}) - \eta_E s_E(\bar{s})\, \mathrm{d}\bar{s}}
\end{aligned}$$

for $s > t$. Note once again, we have omitted constants in front of the RHS for simplicity. The third equality at (70) uses (37).

To complete the proof, by Claim A.2, $\mathbb{E}_t \int_{s=t}^{T} e^{\int_{\bar{s}=t}^{s} \hat{E}(\bar{s}) - \eta_E s_E(\bar{s}) - \rho \, \mathrm{d}\bar{s}}\, \mathrm{d}s < M$, where $M < \infty$. As such,

$$\frac{J_E(t)Q_E(t)}{J_L(t)Q_L(t)} \leqslant \kappa(t)\, (1-\theta_E(t))^{\frac{1}{1-\sigma}}\, L \left(\frac{\theta_E(t)}{1-\theta_E(t)}\right) \frac{M}{J_L(t)}$$

Recall $\theta_E(t) \to 0$ by (38) if $Q_E(t) \to \infty$ and $J_L(t)$ converges to a constant, $\frac{J_E(t)Q_E(t)}{J_L(t)Q_L(t)} \to 0$. However, this contradicts (69).

Q.E.D.

PROOF OF PROPOSITION 2.3: We first prove the *if* statement of the proposition.

Part 1: If

Suppose (32) holds, we will show an allocation satisfying 1.- 4. of Definition A.1 with $\eta_L s_L(t) = \eta_L$ for all t is an equilibrium (satisfies 6. of Definition A.1) and is a BGP. To show the allocation satisfies 6. of Definition A.1, it is sufficient to confirm the free entry and exit condition holds along the allocation path, that is,

$$\frac{J_L(t)Q_L(t)}{J_E(t)Q_E(t)} \geqslant 1, \qquad \forall\, t \geqslant 0 \tag{71}$$

Since $s_E = 0$, $Q_E(t) = Q_E(0)$ for all t and we must have, using Equation (38),

$$\theta_E(t) = \theta_E^\star := (\alpha(1-\alpha))^{\sigma-1}\,\kappa(t)^{1-\sigma}\gamma_E^{\frac{\epsilon}{\sigma-1}}Q_E(0)^{\sigma-1}, \qquad \forall\, t \geqslant 0 \tag{72}$$

By (51), $p_L(t)$ remains constant. As such,

$$J_L(t) = \frac{\alpha p_L(t)^{\frac{1}{\alpha}}L}{\rho+\eta_L} = \frac{\alpha\gamma_L^{\frac{\epsilon}{\sigma-1}}\left(1-\theta_E^\star\right)^{\frac{1}{1-\sigma}}L}{\rho+\eta_L}, \qquad \forall\, t \geqslant 0 \tag{73}$$

Since the replacement rate is zero for energy-augmenting innovations,

$$\begin{aligned}
J_E(t) &= \int_t^\infty \alpha p_E(t)^{\frac{1}{\alpha}}E(t)e^{-\rho s}\,\mathrm{d}s \\
&= \alpha\gamma_E^{\frac{\epsilon}{\sigma-1}}\theta_E^{\star\,\frac{1}{1-\sigma}}\int_0^\infty E(t)e^{-\rho s}\,\mathrm{d}s \\
&= \alpha\gamma_E^{\frac{\epsilon}{\sigma-1}}\theta_E^{\star\,\frac{1}{1-\sigma}}E(t)\int_t^\infty e^{(\eta_L-\rho)s}\,\mathrm{d}s \\
&= \frac{\alpha\gamma_E^{\frac{\epsilon}{\sigma-1}}\theta_E^{\star\,\frac{1}{1-\sigma}}E(t)}{\rho-\eta_L}
\end{aligned}$$

The second equality uses (52). For the third equality, note when $\hat{Q}_E(t) = 0$, $\hat{E}(t) = \eta_L = \hat{Y}(t)$ for all t by (21) and (42). The third equality follows from solving the integral.

We can now write

$$\begin{aligned}
\frac{J_L(t)Q_L(t)}{J_E(t)Q_E(t)} &= \gamma^{\frac{-\epsilon}{\sigma-1}}\left(\frac{1-\theta_E^\star}{\theta_E^\star}\right)^{\frac{1}{1-\sigma}}\left(\frac{LQ_L(t)}{E(t)Q_E(t)}\right)\frac{\rho-\eta_L}{\rho+\eta_L} \\
&= \gamma^{\frac{-2\epsilon}{\sigma-1}}\left(\frac{1-\theta_E^\star}{\theta_E^\star}\right)\frac{\rho-\eta_L}{\rho+\eta_L} \\
&= \frac{J_L(0)Q_L(0)}{J_E(0)Q_E(0)}
\end{aligned} \tag{74}$$

for all $t \geqslant 0$. The second equality follows from (37). The third equality follows from our observation that $\theta_E(t)$ is constant for all $t \geqslant 0$, given by (72).

Using algebra, and (72), we can verify

$$\gamma^{\frac{-2\epsilon}{\sigma-1}}\left(\frac{1-\theta_E^\star}{\theta_E^\star}\right)\frac{\rho-\eta_L}{\rho+\eta_L} \geqslant 1 \Longleftrightarrow \left(\frac{\kappa}{Q_E(0)}\right)^{1-\sigma} \leqslant \frac{\left(\alpha(1-\alpha)\right)^{1-\sigma}\gamma^{\frac{\epsilon}{1-\sigma}}}{\gamma^{\frac{2\epsilon}{\sigma-1}}\frac{\rho+\eta_L}{\rho-\eta_L}+1} \tag{75}$$

And thus, if (32) holds, then $\frac{J_L(0)Q_L(0)}{J_E(0)Q_E(0)} \geqslant 1$, which by (74), implies (71). To confirm the equilibrium allocation is a BGP, note by (21) and (42), $\hat{Y}(t) = \eta_L$ and $\hat{E}(t) = \eta_L$ for all t. Moreover, $s_L(t) = 1$, $s_E(t) = 0$, $\hat{Q}_L(t) = \eta_L$ and $\hat{Q}_E(t) = 0$ for all t.

Now we turn to the *only if* statement of the proposition.

Part 2: Only If

Let an equilibrium allocation be a BGP with $\eta_L s_L(t) = \eta_L$. We show (32) must hold. Since $s_E(t) = 0$ for all t, (74) must hold along the growth path by the argument between proceeding Equation (71) above. Because the path is an equilibrium path, the free entry and exit conditions (71) must hold, and since (74) holds,

$$\gamma^{\frac{-2\epsilon}{\sigma-1}}\left(\frac{1-\theta_E^\star}{\theta_E^\star}\right)\frac{\rho-\eta_L}{\rho+\eta_L} \geqslant 1 \tag{76}$$

which, by (75), implies (32) holds. *Q.E.D.*

A.3. *Convergence of Growth Paths*

PROOF OF PROPOSITION 2.2: Along an asymptotic balanced growth path $z_E(t) \to z_E^\star$ and $z_L(t) \to z_E^\star$, where by Theorem 2.1, $z_E^\star > 0$ and $z_L^\star > 0$.

Recall

$$\begin{aligned}
\eta_E \alpha p_E(t)^{\frac{1}{\alpha}} E(t) &= \eta_E \alpha \gamma_E^{\frac{\epsilon}{\sigma-1}} \theta_E(t)^{\frac{1}{1-\sigma}} E(t) \\
&= \eta_E \alpha \gamma_E^{\frac{\epsilon}{\sigma-1}} \theta_E(t)^{\frac{2-\sigma}{1-\sigma}} \frac{E(t)}{\theta_E(t)} \\
&= \eta_E \alpha^2 \kappa(t)^{-1} \gamma_E^{\frac{\epsilon}{\sigma-1}} \theta_E(t)^{\frac{2-\sigma}{1-\sigma}} Y(t)
\end{aligned} \tag{77}$$

where the first equality is from Equation (52), the second equality follows from dividing through and multypying by $\theta_E(t)$ and the final equality follows Equation (19). Now, by Equation (49) in the proof of Theorem 2.1,

$$\frac{\hat{Q}_E(t)}{\hat{Q}_L(t)} \to \frac{1-\theta_E(t)-\sigma}{2-\sigma-\theta_E(t)} \tag{78}$$

But since $\frac{\hat{Q}_E(t)}{\hat{Q}_L(t)} = \frac{\eta_E z_E(t)}{\eta_L z_L(t)}$, we must have

$$\begin{aligned} \frac{\eta_E z_E(t)}{\eta_L z_L(t)} &\to \frac{1-\theta_E(t)-\sigma}{2-\sigma-\theta_E(t)} \\ \implies \frac{\eta_E \alpha \kappa(t)^{-1} \gamma_E^{\frac{\epsilon}{\sigma-1}} \theta_E(t)^{\frac{2-\sigma}{1-\sigma}} Y(t) - \rho}{\eta_L \gamma_L^{\frac{\epsilon}{\sigma-1}} (1-\theta_E(t))^{\frac{1}{1-\sigma}} L - \rho} &\to \frac{1-\theta_E(t)-\sigma}{2-\sigma-\theta_E(t)} \end{aligned}$$

The implication follows from the no arbitrage conditions, Equation (77) and (51). The above implies

$$Y(t) \to \frac{\frac{1-\theta_E(t)-\sigma}{2-\sigma-\theta_E(t)} \left(\alpha \eta_L \gamma_L^{\frac{\epsilon}{\sigma-1}} (1-\theta_E(t)^{\frac{1}{1-\sigma}} L - \rho \right) + \rho}{\alpha^2 \gamma_E^{\frac{\epsilon}{1-\sigma}} \theta_E(t)^{\frac{2-\sigma}{1-\sigma}}} := H(\theta_E(t))$$

where $H\colon (0, \theta_E(0)) \to \mathbb{R}_+$. Note H is differentiable. By assumption, $\theta_E(0) + \sigma < 1$, which can be shown to imply H has a negative derivative and hence H is decreasing. Since H is decreasing, H is injective. Moreover, since $(0, \theta_E(0))$ is open, by the open mapping theorem, H has a continuous inverse, which we now denote as $G : \mathbb{R}_+ \to (0, \theta_E(0))$.

Finally, we show for any $\epsilon > 0$, there exists T such that for all $t > T$,$|G(Y(t)) - \theta_E(t)| < \epsilon$. There exists $\delta > 0$ such that $|G(x) - G(y)| < \epsilon$ for $|x - y| < \delta$. Moreover, there exists T such that for all $t > T$, we have $|Y(t) - H(\theta_E(t))| < \delta$, giving $|G(Y(t)) - \theta_E(t)| < \epsilon$. This establishes that $G(Y(t)) \to \theta_E(t)$ and in view of Equation (19), implies $\frac{E(t)}{Y(t)} \to G(Y(t))$.

Q.E.D.

A.4. *Comparative Statics of Equilibrium Energy Use and Technology*

In this section, we study the effect of energy augmenting technology on energy use. For a given level of output and real energy price, $\kappa(t)$, higher $Q_E(t)$ leads to lower energy use. Given a real energy price, we can decompose the effect of energy-augmenting technical change on energy use into the change in energy use given the level of output and the change in energy use as output increases given a level of energy intensity:

$$\frac{\partial E}{\partial Q_E} = \Phi \overbrace{\frac{\partial Y}{\partial Q_E} Q_E^{\sigma-1}}^{\text{Output effect}} + \Phi \overbrace{(\sigma-1) Y Q_E^{\sigma-2}}^{\text{Efficiency effect}} \tag{79}$$

where $\Phi = \gamma_E^{\frac{\epsilon}{\sigma-1}} \alpha^\sigma (1-\alpha)^{\sigma-1}$ and we have dropped the time index for simplicity. The efficiency effect is always negative, and the output is always positive. The following proposition tells us when increased energy efficiency results in a fall in energy use.

PROPOSITION A.1 *If* $\theta_E < (1-\sigma)(1-\alpha)^{\frac{1-\sigma}{\sigma}}$, *then* $\frac{\partial E}{\partial Q_E} < 0$.

PROOF: Evaluate the derivative on the right hand side of (79) and simplify the equation

$$\frac{\partial E}{\partial Q_E} = \Phi Q_E^{\sigma-2} Y \left(\gamma_E^{\frac{\epsilon}{\sigma}} Y^{\frac{1-\sigma}{\sigma}} (Q_E E)^{\frac{\sigma-1}{\sigma}} + \sigma - 1 \right) \tag{80}$$

Now note the expression for output (18) and θ_E (20) to arrive at the result.

Q.E.D.

When real energy prices are fixed, the result implies the rebound effect is less than 1 if the cost-share of energy is less than $(1-\sigma)(1-\alpha)^{\frac{1-\sigma}{\sigma}}$.[21]

[21] In a similar exogenous technical change setting, the rebound is less than 1 if $\theta_E < (1-\sigma)$ (Saunders, 2015).

B.0.1. *Energy Intensity for Selected Countries 1900-2010*

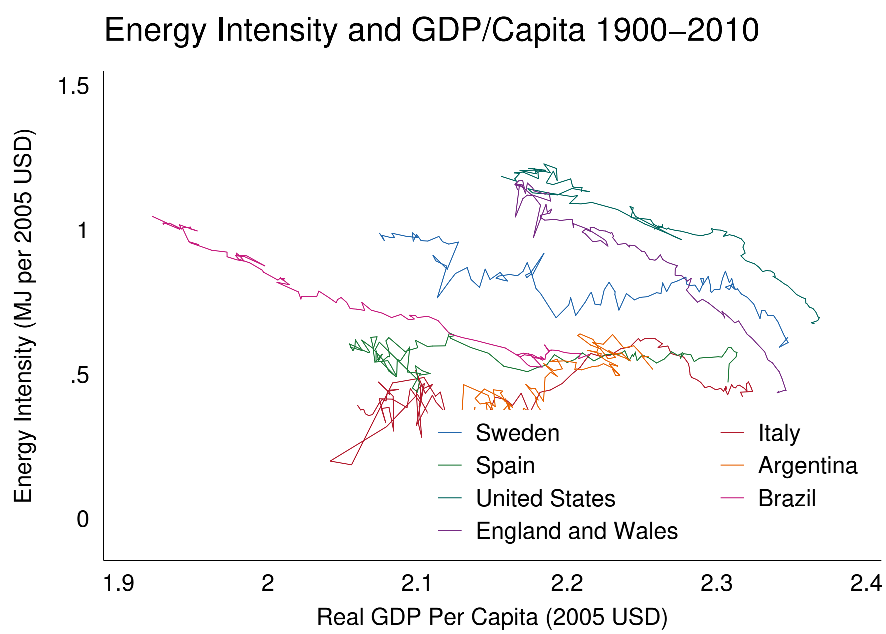

FIGURE B.8.— Energy intensity and real GDP per capita for selected countries between 1900 and 2010.

B.1. *Cross-sectional Energy Intensity and Output*

Details of the cross-sectional data used in this section are in Csereklyei et al. (2016). Figure B.9 gives a scatter plot of the natural log of energy intensity against the natural log of GDP per capita in PPP dollars at ten year intervals between 1971 and 2010. The fitted regression line is a cubic polynomial; we compared the Akaike Information Criteria (AIC) and the Bayesian Information Criteria (BIC) for a linear model and cubic model, and found the cubic has the lowest score on both criteria.

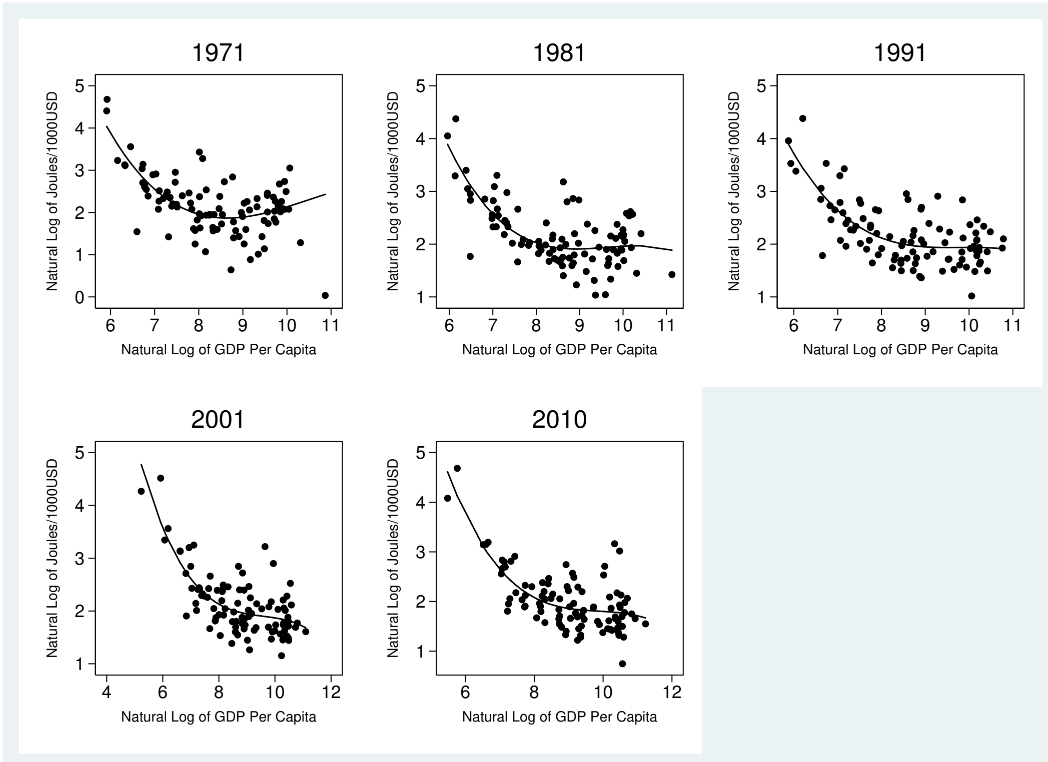

FIGURE B.9.— Natural log of energy intensity and natural log of output for a cross section of countries.

Using the polynomial coefficients from table B.1, we can estimate the elasticity of energy intensity to output in the cross-section. The elasticities are given in table B.1. There is a negative relationship between energy intensity and output, however, for lower levels of output, the fall in energy intensity associated with an increase in output is higher. For example, in 2010, for countries at a GDP per capita of 2,000 USD, a 1 percent increase in output is associated with a .5 percent fall in energy intensity, while for countries with a GDP per capita of 30,000, a 1 percent increase in output is associated with .06 percent fall in energy intensity. At least in the cross section of countries, the elasticity of energy intensity to output is not constant and becomes close to zero amongst richer countries.

	1971	1981	1991	2001	2010
Ln(Y)	-11.28	-12.25**	-9.413*	-8.740**	-9.215**
	(6.947)	(4.368)	(4.274)	(2.744)	(3.073)
Ln(Y) X Ln(Y)	1.111	1.280*	0.960	0.906**	0.931*
	(0.865)	(0.525)	(0.517)	(0.335)	(0.368)
Ln(Y) X Ln(Y) X Ln(Y)	-0.0354	-0.0443*	-0.0326	-0.0316*	-0.0315*
	(0.0355)	(0.0208)	(0.0206)	(0.0134)	(0.0145)
Constant	39.19*	40.84***	32.67**	30.22***	32.34***
	(18.40)	(11.97)	(11.62)	(7.385)	(8.432)
Observations	98	99	99	99	99

Standard errors in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

TABLE B.1
CUBIC POLYNOMIAL REGRESSION TABLE

USD/ capita	1971	1981	1971	2001	2010
2,000	-0.527	-0.470	-0.470	-0.444	-0.522
10,000	0.176	0.055	-0.026	-0.093	-0.082
20,000	0.310	0.068	0.010	-0.093	-0.043
30,000	0.320	0.060	-0.013	-0.135	-0.063

TABLE B.2
ELASTICITY OF ENERGY INTENSITY TO OUTPUT IN THE CROSS-SECTION

B.2. *Estimates of Q_E for the U.S.*

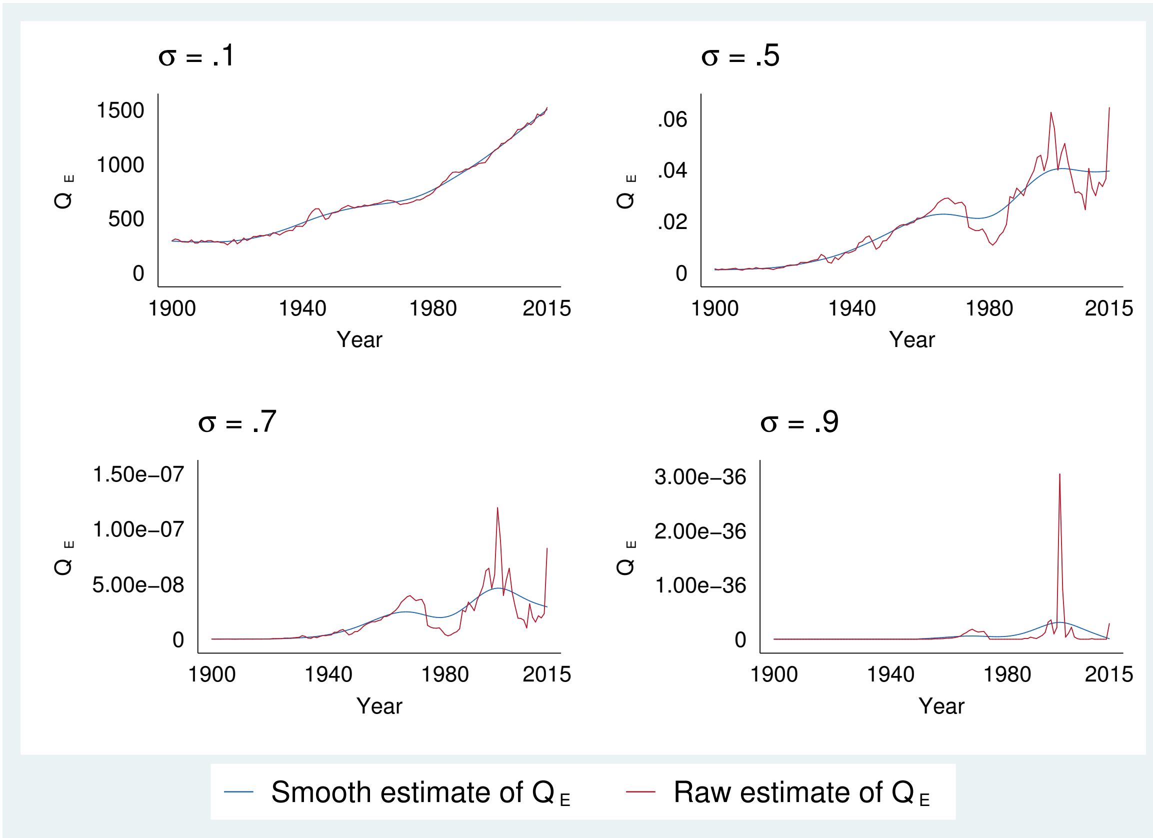

FIGURE B.10.— Smoothed and raw estimate of Q_E for different values of σ

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