# Existence of Solutions to Non-Compact Dynamic Optimization Problems

September 28, 2017

# Objective

Present and prove theorem on existence of solutions to a **reduced form** dynamic optimisation problem when feasibility correspondences have **non-compact** image sets and pay-offs are **bounded below**

- Main application and motivation: optimal policies in incomplete market models with heterogeneity

# Preliminaries

# Semicontinuity

Definition. A function $f\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is sequentially **upper semi-continuous** if the upper contour sets

$$UC_f(\epsilon)\colon = \{x \in X \mid f(x) \geq \epsilon\}$$

are sequentially closed for all $\epsilon \in \mathbb{R}$.

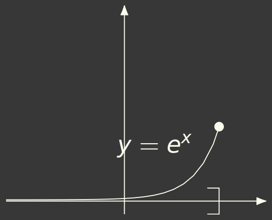
# Sup-Compactness

Let $D$ be a subset of $\mathbb{R} \cup \{-\infty, +\infty\}$

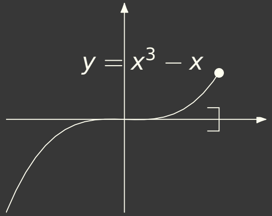
Definition. A function $f : X \to D$ is **sup-compact** if the sets $UC_f(\epsilon)$ are sequentially compact for all $\epsilon \in \mathbb{R}$

If $X$ is not compact and $D$ is bounded below, then $f$ cannot be sup-compact

Definition. A function $f : X \to D$ is **mildly sup-compact** if the sets $UC_f(\epsilon)$ are sequentially compact for all $\epsilon > \inf f$




Mildly Sup-Compact




Sup-Compact

# Correspondences

Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces. A correspondence from a space $X$ to $Y$ is a set valued function denoted by $\Gamma : X \twoheadrightarrow Y$.

The image of a subset $A$ of $X$ under the correspondence $\Gamma$ will be the set

$$\Gamma(A) := \{y \in Y | y \in \Gamma(x) \text{ for some } x \in A\}$$

A correspondence will be called **compact valued** if $\Gamma(x)$ is compact for $x \in X$.

# Correspondences

The correspondence $\Gamma$ is **upper hemi-continuous** if for every $x$ and neighbourhood $U$ of $\Gamma(x)$, there is a neighbourhood $V$ of $x$ such that $z \in V$ implies $\Gamma(z) \subset U$

Upper hemicontinuous correspondences need not be compact valued or have closed graph. Closed graph correspondences also need not be upper hemi-continuous (see Aliprantis and Border (2006), ch. 17). However,

**Lemma.** If $\Gamma\colon X \twoheadrightarrow Y$ is upper hemicontinuous and compact valued, then for $C \subset X$ such that $C$ is compact, $\Gamma(C)$ is compact.

See Lemma 17.8 by Aliprantis and Border (2006)) for a proof

# Problem Statement

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A non-stationary reduced form economy is a 5-tuple

$$\mathscr{E} := ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta) \qquad (1)$$

consisting of:

- A topological space $(\mathbb{X}, \tau)$
- A collection of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$, with $\mathbb{S}_t \subset \mathbb{X}$ for each $t$
- A collection of non-empty feasibility correspondences $(\Gamma_t)_{t=0}^{\infty}$, with $\Gamma_t \colon \mathbb{S}_t \twoheadrightarrow \mathbb{S}_{t+1}$ for each $t$
- A collection of per-period pay-offs $(\rho_t)_{t=0}^{\infty}$, with $\rho_t \colon \operatorname{Gr} \Gamma_t \to \mathbb{R}_+$ and $\inf \rho_t = 0$ for each $t$
- A discount factor $\beta \in (0, 1)$.

# Problem Statement

Define the correspondence of **feasible sequences** $\mathcal{G}_t^T : \mathbb{S}_t \twoheadrightarrow \prod_{i=t}^T \mathbb{S}_i$ starting at time $t$ and ending at time $T$ as follows:

$$\mathcal{G}_t^T(x) := \left\{ (x_i)_{i=t}^T \mid x_{i+1} \in \Gamma_i(x_i), x_t = x \right\}, \qquad x \in \mathbb{S}_t \quad (2)$$

Let $\mathcal{G}$ denote $\mathcal{G}_0^\infty$ and let $\mathcal{G}^T$ denote $\mathcal{G}_0^T$.

# Problem Statement

Define the **value function** $\tilde{V} : \mathbb{S}_0 \to \mathbb{R} \cup \{-\infty, +\infty\}$ as follows:

$$\tilde{V}(x) := \sup_{(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)} \sum_{t=0}^{\infty} \beta^t \rho_t (x_t, x_{t+1}) \tag{3}$$

# Application

Aiyagari-Huggett optimal policy (roughly)

- let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t=0}^{\infty}, \mathbb{P})$ be a filtered probability space
- $\mathbb{X} = L^2(Z, \mathbb{P})$ with the weak topology
- the state-spaces $\mathbb{S}_t$ are spaces of $\mathscr{F}_t$ measurable random variables (history dependent)
- the correspondences $\Gamma_t$ does not have compact image sets because of Inada conditions
- feasible sequences $(x_t)_{t=0}^{\infty}$ map histories of shocks to assets
- the pay-off $\rho_t$ integrates pay-offs across all agents given prices that depend on $x_t$

# Assumptions

Fix $x \in \mathbb{S}_0$. Let $\phi_t \colon \mathcal{G}^{t+1}(x) \to \mathbb{R}_+$ denote $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ for each $t$

The upper contour sets $UC_{\phi_t}(\epsilon)$ of $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ are defined by

$$UC_{\phi_t}(\epsilon) = \{(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x) \mid \rho_t(x_t, x_{t+1}) \geq \epsilon\} \tag{4}$$

# Assumptions

Standard requirement is for $\Gamma_t$ to be upper hemicontinuous and compact valued and for $\mathbb{S}_t$ to be a metric space (see by Acemoglu (2009), Assumption 6.2, Kamihigashi (2017), section 6 or Stokey and Lucas (1989), Assumption 4.3, for assumptions used by the standard theory).

Main assumption below relaxes this requirement.

**Assumption.**3.1 For each $x \in \mathbb{S}_0$ and $t \in \mathbb{N}$, the functions $\phi_t \colon \mathcal{G}^{t+1}(x) \to \mathbb{R}_+$ are mildly sup-compact in the product topology (of $\tau$ topology in $\mathbb{X}$)

# Assumptions

The next assumption is the standard growth condition (see discussion on Corollary 6.1 by Kamihigashi (2017)).

**Assumption.**3.2 For each $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^{\infty}$ such that any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ satisfies

$$\rho_t(x_t, x_{t+1}) \le m_t, \qquad \forall t \in \mathbb{N} \tag{5}$$

and

$$\sum_{t=0}^{\infty} \beta^t m_t < \infty \tag{6}$$

**Assumption.**3.3 The functions $(\rho_t)_{t=0}^{\infty}$ are sequentially upper semicontinuous for all $t \in \mathbb{N}$.

# Main Theorem

**Theorem.** 3.1 If $\mathscr{E}$ satisfies assumptions 3.1 - 3.3, then for every $x \in S_0$, there will exist $(x_t)_{t=0}^{\infty}$ satisfying $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ such that

$$\tilde{V}(x) = \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t, x_{t+1}\right) < \infty$$

# Proofs

## Proof Premlinaries

Let $(\mathbb{X}, \tau)$ is a topological vector space

Unless otherwise stated, convergence for sequences in $\mathbb{X}$ will be with respect to the $\tau$ topology and convergence for sequences in countable Cartesian products of $\mathbb{X}$ will be in the product topology of the $\tau$ topology on $\mathbb{X}$.

We will use $\mathbf{x}$ to refer to elements of $\mathbb{X}^{\mathbb{N}}$. We can then use $(\mathbf{x}^n)_{n=0}^{\infty}$ to denote a sequence $\{\mathbf{x}^0, \dots, \mathbf{x}^n, \dots\}$, where $(\mathbf{x}^n)_{n=0}^{\infty} \in (\mathbb{X}^{\mathbb{N}})^{\mathbb{N}}$.

Let $U(\mathbf{x}) := \sum_{t=0}^{\infty} \rho_t(x_t, x_{t+1})$.

# Product Topology

Remark. A.1 Let $X = \prod_{i \in F} X_i$ denote a Cartesian product of topological spaces. Let $\pi_i : X \to X_i$ denote the projection map defined as $\pi_i(x) = x_i$ for each $i \in F$.

Recall each projection map will be a continuous function on $X$ when $X$ has the product topology (see section 2.14 by Aliprantis and Border (2006))

Also recall (section 1.8 by Tao (2013)) the image of a (sequentially) compact set under a continuous function is (sequentially) compact.

If a set $C$ with $C \subset X$ is (sequentially) compact in the product topology, then $\pi_i(C)$ will be (sequentially) compact.

# Lemma A.1

**Lemma.** A.1 Let Assumption 3.2 hold and let $x$ satisfy $x \in \mathbb{S}_0$. If $(\mathbf{x}^n)_{n=0}^{\infty}$ is a sequence with $\mathbf{x}^n \in \mathcal{G}(x)$ for each $n$ and $U(\mathbf{x}^n) \to B$ for $B > 0$, then there exists a sub-sequence $(\mathbf{x}^{n_k})_{k=0}^{\infty}$ such that for all $t \in \mathbb{N}$

$$\lim_{k \to \infty} \rho_t(x_t^{n_k}, x_{t+1}^{n_k}) \to c_t$$

where $c_t \in \mathbb{R}_+$ for each $t$ and $c_t > 0$ for at-least one $t$.

# Proof of Lemma A.1

**Proof.** By Assumption 3.2, for each $t$ and $n$,

$$m_t \geq \rho_t(x_t^n, x_{t+1}^n) \geq 0 \tag{7}$$

Accordingly, for each $n$, $(\rho_t(x_t^n, x_{t+1}^n))_{t=0}^{\infty}$ will belong to the set $\prod_{t=0}^{\infty}[0, m_t]$, which by Tychonoff's Theorem (see Proposition 1.8.12 by Tao (2010)) will be compact in the product topology.

There then exists a sub-sequence of $(\mathbf{x}^n)_{n=0}^{\infty}$, $(\mathbf{x}^{n_k})_{k=0}^{\infty}$, such that $(\rho(x_t^{n_k}, x_{t+1}^{n_k}))_{k=0}^{\infty}$ converges for each $t$.

# Proof of Lemma A.1

Let $c_t := \lim_{k\to\infty} \rho(x_t^{n_k}, x_{t+1}^{n_k})$ and note

$$B = \lim_{k\to\infty} \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t^{n_k}, x_{t+1}^{n_k}\right)$$
$$= \sum_{t=0}^{\infty} \lim_{k\to\infty} \beta^t \rho_t \left(x_t^{n_k}, x_{t+1}^{n_k}\right) = \sum_{t=0}^{\infty} \beta^t c_t \quad (8)$$

Since (7) holds, and $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ by Assumption 3.2, we can pass limits through in the second equality using dominated convergence theorem (see Corollary 7.3.15 by Stachurski (2009))

If B is strictly positive, the above means there is at least one $c_t > 0$.

# Lemma A.2

**Lemma.** A.2
Let $x$ satisfy $x \in \mathbb{S}_0$. If $(\mathbf{x}^n)_{n=0}^{\infty}$ is a sequence with $\mathbf{x}^n \in \mathcal{G}(x)$ for each $n$ and for some $t$

$$\rho_t(x_t^n, x_{t+1}^n) \to c_t$$

with $c_t > 0$, then there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$.

**Proof.** There exists $\iota$ such that $\epsilon := c_t - \iota$ is strictly positive

For $N$ large enough and any $n > N$, $\rho_t(x_t^n, x_{t+1}^n) \in [\epsilon, c_t + \iota]$, implying $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$ and $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$.

# Lemma A.3

**Lemma.** A.3
Let assumptions 3.1- 3.3 hold and let $x$ satisfy $x \in \mathbb{S}_0$. If $(\mathbf{x}^n)_{n=0}^{\infty}$ is a sequence such that $\mathbf{x}^n \in \mathcal{G}(x)$ for each $n \in \mathbb{N}$ and $U(\mathbf{x}^n) \to B$ where $B > 0$, then:

1. $(\mathbf{x}^n)_{n=0}^{\infty}$ has a convergent sub-sequence with a limit $\mathbf{x} \in \mathcal{G}(x)$, and
2. $B \leq U(\mathbf{x}) < \infty$.

# Proof of Lemma A.3

**Proof.** Let $x$ satisfy $x \in \mathbb{S}_0$ and let $(\mathbf{x}^n)_{n=0}^{\infty}$ be a sequence such that $\mathbf{x}^n \in \mathcal{G}(x)$ for each $n$ and $U(\mathbf{x}^n) \to B$ where $B > 0$.

By Lemma A.1 there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that for each $t \in \mathbb{N}$, $c_t := \lim_{j \to \infty} \rho_t(x_t^{n_j}, x_{t+1}^{n_j}) > 0$ for at-least one $t$

Re-label $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ to $(\mathbf{x}^n)_{n=0}^{\infty}$, and let $P$ denote the subset of $\mathbb{N}$ such that $t \in P$ if and only if $c_t > 0$

- The set $P$ will be non-empty, but could be finite or infinite.

# Proof of Lemma A.3

We consider first the case when $P$ is infinite and then the case when $P$ is finite.

Suppose $P$ is infinite and consider any $t \in \mathbb{N}$. There will exist $k > t$ such that $c_k > 0$

By Lemma A.2, there exists $N$ and $\epsilon > 0$ such that for all $n > N$, $(x_i^n)_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$

# Proof of Lemma A.3

By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology

The space $\pi_t(UC_{\phi_k}(\epsilon))$ will also be sequentially compact by the argument in Remark A.1

Let $\Xi_t := \{x_1^0, \ldots, x_t^N\} \cup \pi_t(UC_{\phi_k}(\epsilon))$

Since $\{x_1^0, \ldots, x_t^N\}$ is sequentially compact, $\Xi_t$ will be sequentially compact

- Note $x_t^n \in \Xi_t$ for each $n \in \mathbb{N}$

# Proof of Lemma A.3

Since $t$ was arbitrary, can construct a $\Xi_t$ as above for every $t \in \mathbb{N}$

Let $\Xi := \prod_{t \in \mathbb{N}} \Xi_t$

Using the Sequential Tychonoff Theorem (Proposition 1.8.12 by Tao (2010)), $\Xi$ will be sequentially compact

Since for each $t$, $x_t^n \in \Xi_t$ for each $n$, $\mathbf{x}^n \in \Xi$ for each $n$, there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ converging to $\mathbf{x}$, with $\mathbf{x} \in \Xi$

# Proof of Lemma A.3

We now confirm $\mathbf{x} \in \mathcal{G}(x)$ by showing $x_{t+1} \in \Gamma_t(x_t)$ for all $t \in \mathbb{N}$

Pick any $t \in \mathbb{N}$, there will be a $k$ satisfying $k > t$ such that $c_k > 0$

By Lemma A.2, there exists $\epsilon > 0$ and $J$ such that for all $j > J$ we have $(x_i^{n_j})_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$

# Proof of Lemma A.3

By 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact, moreover, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$ by the definition of $UC_{\phi_k}(\epsilon)$ at (4), frame 15.

As such, the sub-sequence $(x_i^{n_j})_{i=0}^{k+1}$ converges to $(x_i)_{i=0}^{k+1}$, with $(x_i)_{i=0}^{k+1} \in \mathcal{G}^{k+1}(x)$, allowing us to conclude $x_{t+1} \in \Gamma(x_t)$

Since the $t$ was arbitrary, $x_{t+1} \in \Gamma_t(x_t)$ for each $t \in \mathbb{N}$ and $\mathbf{x} \in \mathcal{G}(x)$.

# Proof of Lemma A.3

Now assume $P$ is finite. $P$ will have a maximum element, which we now call $k$

By Lemma A.2, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that $(x_t^n)_{t=0}^{k+1} \in UC_{\phi_k}(\epsilon)$ for each $n > N$

By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology

As such, there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that $(x_t^{n_j})_{j=0}^{\infty}$ for each $t \leq k+1$

Define $(x_t)_{t=0}^{\infty}$ by setting $x_t = \lim_{j \to \infty} x_t^{n_j}$ for $t \leq k+1$ and picking any $x_{t+1} \in \Gamma_t(x_t)$ for $t \geq k+1$.

# Proof of Lemma A.3

To confirm $(x_t)_{t=0}^{\infty}$, we check $x_{t+1} \in \Gamma_t(x_t)$ for each $t$

Once again, note by definition, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$

Since $UC_{\phi_k}(\epsilon)$ is sequentially compact, $(x_t)_{t=0}^{k+1} \in \mathcal{G}(x)$ and $x_{t+1} \in \Gamma_t(x_t)$ for all $t$ satisfying $t \leq k$

On the other hand, if $t > k$, by construction, $x_{t+1} \in \Gamma_t(x_t)$, confirming $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$

# Proof of Lemma A.3

To prove part 2 of the lemma, by Assumption 3.2,

$$\rho_t(x_t^n, x_{t+1}^n) \leq m_t$$

for each $t$ and $n$, where $\sum_{t=0}^{\infty} \beta^t m_t < \infty$.

Fatou's Lemma[1] gives

$$B = \limsup_{n \to \infty} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{\infty} \limsup_{n \to \infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) < \infty \quad (9)$$

[1] See 5.4 b) by Williams (1991) and let $\Omega = \mathbb{Z}_+$ and $\mu$ be the counting measure. Also see Equation (1.1) and discussion by Kamihigashi (2017).

# Proof of Lemma A.3

Upper-semicontinuity of $\rho_t$ (Assumption 3.3) and the growth condition (Assumption 3.2) imply

$$\limsup_{n\to\infty} \rho_t(x_t^n, x_{t+1}^n) \leq \rho_t(x_t, x_{t+1}) \leq m_t, \qquad t \in \mathbb{N} \qquad (10)$$

To complete the proof, combine (10) with (9) and conclude

$$B \leq \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = U(\mathbf{x}) < \infty$$

□

# Proof of Theorem 3.1

**Theorem.** 3.1 If $\mathscr{E}$ satisfies assumptions 3.1 - 3.3, then for every $x \in S_0$, there will exist $(x_t)_{t=0}^{\infty}$ satisfying $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ such that

$$\tilde{V}(x) = \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t, x_{t+1}\right) < \infty$$

**Proof.** Fix $x \in \mathbb{S}_0$. If $U(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{G}(x)$, then our solution will be any $\mathbf{x} \in \mathcal{G}(x)$.

# Proof of Theorem 3.1

Next, suppose at-least one $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ satisfies $U(\mathbf{x}) > 0$

By Assumption 3.2, there exists a sequence of real numbers $(m_t)_{t=0}^{\infty}$ such that $\rho_t(x_t, x_{t+1}) \leq m_t$ for any $\mathbf{x}$ in $\mathcal{G}(x)$ and

$$\bar{B} := \sum_{t=0}^{\infty} \beta^t m_t < \infty$$

# Proof of Theorem 3.1

Any $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ will satisfy

$$U(\mathbf{x}) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \leq \bar{B}$$

Now, consider the set $I := \{U(\mathbf{x}) \mid \mathbf{x} \in \mathcal{G}(x)\}$

- $I$ will be a subset of $\mathbb{R} \cup \{-\infty, \infty\}$ and so must have a supremum

Let $B := \sup I$ and note $0 \leq B \leq \bar{B} < \infty$

# Proof of Theorem 3.1

Construct a sequence $(\mathbf{x}^n)_{n=0}^{\infty}$ with $\mathbf{x}^n \in \mathcal{G}(x)$ for each $n$ and $U(\mathbf{x}^n) \to B$ as follows:

- for every $n \in \mathbb{N}$, take $\mathbf{x}^n$ such that $B - U(\mathbf{x}^n) < \frac{1}{n+1}$

Such a sequence exists, otherwise for some $n$, $U(\mathbf{x}) \leq B - \frac{1}{n+1}$ for all $\mathbf{x} \in \mathcal{G}(x)$ and $B$ will not be the supremum of $I$.

# Proof of Theorem 3.1

Since $U(\mathbf{x}^n) \to B$, by Lemma A.3, there exists $\mathbf{x} \in \mathcal{G}(x)$ such that $U(\mathbf{x}) \geq B$. Since $B$ was the supremum for $I$, conclude

$$U(\mathbf{x}) = B = \tilde{V}(x) < \infty$$

$\square$

Acemoglu, D. (2009). *Introduction to Modern Economic Growth*. Princeton University Press, Princeton, New Jersey.

Aliprantis, C. D. and Border, K. C. (2006). *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer-Verlag, Berlin.

Kamihigashi, T. (2017). A Generalisation of Fatou's Lemma for Extended Real-Valued Functions on sigma-Finite Measure spaces: with an Application to Infinite-Horizon Optimization in Discrete Time. *Journal of Inequalities and Applications*, 2017(1):24.

Stachurski, J. (2009). *Economic Dynamics: Theory and Computation*. MIT Press Books, Cambridge, MA.

Stokey, N. and Lucas, R. (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press, Cambridge, MA.

Tao, T. (2010). *Epsilon of Room, One: Volume 117 of Graduate Studies in Mathematics*. American Mathematical Soc.

Tao, T. (2013). *Compactness and contradiction*. Americal Mathematical Society.

Williams, D. (1991). *Probability with Martingales*. Cambridge University Press.