EXISTENCE OF CONSTRAINED OPTIMAL POLICIES IN THE HETEROGENEOUS AGENT GROWTH MODEL

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This paper establishes existence of recursive constrained social optima, considered by Dávila et al. (2012), in a neoclassical growth model with idiosyncratic shocks, incomplete insurance markets and production. A constrained planner chooses individual saving and consumption through time, constrained by infinitely many agents' budget constraints, to maximise aggregate welfare. Due to the structure of the recursive problem and the Inada conditions interacting with an infinite dimensional state and action space, the feasibility correspondences for the constrained planner have non-compact image sets. The constrained planner's problem thus does not meet the requirements of standard dynamic optimisation theory used show existence of optimal policies. To address the challenge, first, the paper transforms the recursive problem to a sequential problem and shows existence of sequential optima implies existence of recursive optima. Second, the paper introduces a new existence result for non-compact dynamic optimisation problems and uses the theorem to verify existence of sequential optima.

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Please download the latest version at `https://github.com/mathuranand/Existence_of_Social_Optimia_Aiyagari`

1. INTRODUCTION

The neoclassical growth model with idiosyncratic shocks, incomplete insurance markets and production, also known as the Aiyagari-Huggett (Aiyagari, 1994; Huggett, 1993) model, has developed into a leading model of dynamic macroeconomics.[2] Recently, macroeconomists have begun studying optimal policy in a general setting of the model. The natural way to formulate optimal policy is via a *constrained planner*; like any realistic government overseeing a market economy, the constrained planner cannot complete insurance markets, but must improve welfare subject to each agents' idiosyncratic budget constraint. The concept of a constrained planner was first introduced in discrete time by Dávila et al. (2012) and continuous time by Nuño and Moll (2017) and has led to a growing literature on optimal monetary and fiscal policy in incomplete market models (Acikgoz, 2013; Chen and Yang, 2017; Bhandari et al., 2017; Park, 2014; Nuno and Thomas, 2017).

Despite the importance of constrained optima for optimal policy analysis, due to the computational and mathematical challenges brought on by the infinite dimensional structure of the Aiyagari-Hugget model, existence of constrained optima has not been verified. This paper provides a proof of existence of constrained optima for the canonical constrained planner as originally considered by Dávila et al. (2012).[1] The paper also presents an easy to verify general result that can be applied to planner problems in other heterogeneous agent models.

From the perspective of applied modellers, the existence result here helps confirm the interesting policy conclusions emerging from computations of constrained optima are sound. In simulations with high income inequality and a wealth distribution resembling actual U.S. data, Dávila et al. (2012) show a decentralised equilibrium in the Aiyagari-Huggett model undersaves compared to the constrained optima, which justifies saving subsidies. This is in contrast to the long held belief of sub-optimal over-saving in incomplete market models, which justifies capital taxation (see Aiyagari (1995) and discussion by Chen and Yang (2017)). Moreover, in the high in-

[2] Macroeconomists today use the model to study consumption dynamics (Berger and Vavra, 2015), shapes of wealth distributions (Benhabib et al., 2015), asset pricing (Krusell et al., 2011) and monetary and fiscal policy dynamics (Kaplan et al., 2016; Kaplan and Violante, 2010; Heathcote, 2005; Mckay and Reis, 2016), to name a few topics.

[1] In this basic setting, the government does not consume and there are no net transfers or nominal rigidities.

come inequality case, the constrained planner's solution path does not converge to a steady-state, but displays ever increasing wealth inequality. Verifying existence helps confirm such computed solutions are not pathological and creates a foundation for further research on optimal policy dynamics in the Aiyagari-Huggett model.

Mathematical Challenges

Because the constrained planner controls the assets of infinitely many agents through time, both the planner's state, a distribution of agents over assets, and action, a policy function, are infinite dimensional. The literature has made significant progress by establishing infinite dimensional necessary conditions (Dávila et al. (2012) in discrete time and Nuño and Moll (2017) and Nuño (2017) in continuous time). However, continuity and compactness, assumptions used by standard dynamic optimisation theory *to verify existence of solutions*, are more difficult to verify when spaces are infinite dimensional (see Mas-colell and Zame (1991) for an overview of issues in infinite dimensional topology). In the case of the constrained planner, the feasibility correspondence fails to have compact image sets, that is, the image of a compact set under the correspondence will not be compact. The standard assumptions of existing dynamic optimisation theory (Stokey and Lucas (1989), Acemoglu (2009) ch.6 or Stachurski (2009)) are thus not satisfied.

The constrained planner's feasibility correspondences fails to have compact image sets for two reasons. First, as suggested by Dávila et al. (2012), individual agents' asset spaces will not be bounded. Moreover, it is difficult to justify restrictions on the space of policy functions, such as equicontinuity or monotonicity, that yield compactness in the sup-norm topology or topology of point-wise convergence. On the other hand, the recursive problem, the form of the problem considered by Dávila et al. (2012), will not be defined on topological spaces where the feasibility correspondence is compact-valued.

To resolve the first challenge, the paper transforms the recursive problem to a sequential problem, where the planner assigns assets to individuals based on their history of shocks. The sequential planner's problem will be well-defined on the space of square integrable random variables. And with the weak topology, the sequential planner's feasibility correspondences will be compact valued. To connect the sequential problem back to the recursive problem, the first innovation of the paper is a novel projection argument to show sequential solutions imply recursive solutions.

Second, while feasibility correspondences for the sequential planner have compact values, the correspondences still have non-compact image sets because of a discontinuity. The discontinuity arises due to the Inada conditions — as capital converges to zero, interest rates diverge and the variance of feasible asset distributions can diverge to infinity as the mean converges to zero. To resolve the second challenge, the paper introduces and applies an existence result for infinite horizon dynamic optimisation that weakens the requirement for feasibility correspondences to have compact image sets. The main assumption of the existence result can be verified by checking the variance of feasible sequences of asset distributions leading to a strictly positive per-period pay-off at a time in the future is bounded.

Related Literature

The existence result in this paper builds on important work on non-compact optimisation (Feinberg et al., 2013) and dynamic programming (Feinberg et al., 2012). To generalise from the requirement of compact image sets, Feinberg et al. (2012) introduce a condition called K-Sup-Compactness,[2] explored further by Feinberg et al. (2013), on the per-period pay-off. However, when utility is bounded below, as in the case of Constant Relative Risk Aversion (CRRA) utility used by Dávila et al. (2012), K-Sup-Compactness is too strong. In particular, when utility is bounded below, K-Sup-Compactness implies compact images. The main assumption of this paper is weaker than K-Sup-Compactness.

The Aiyagari-Huggett model is not the only model with infinite dimensional state-spaces. The growing interest in heterogeneity in economics has led to a variety of such models, studying economic geography, industry dynamics and trade, to name a few topics. Many of these models also study social optimality over infinite dimensional states. For instance, a large literature (Brock et al., 2014; Boucekkine et al., 2009; Fabbri et al., 2015) has shown existence and characterised optimal solutions in models of economic geography in continuous time. Lucas and Moll (2014) also solve an infinite dimensional planner's problem to control individual search efforts subject to the law of motion of a density. However, to the best of my knowl-

[2]Or K-Inf-Compactness for minimisation problems.

edge, these models do not encounter the non-compactness of the Aiyagari-Huggett model. (Note the mathematical challenge for the Aiyagari-Huggett constrained planner is not that the state-space is infinite dimensional *per se*. Extending standard dynamic optimisation arguments from $\mathbb{R}^N$ to an infinite dimensional space is trivial, *if the conditions of compactness and hemicontinuity are satisfied in a suitable topology*.)

On the other hand, the state in a model may be infinite dimensional, but with simplifying assumptions, a planner can still manage the economy through finite dimensional controls. For example, in the industry dynamics model by Hopenhayn (1992), the planner can control total demand (see pp. 1134), in the growth model with financial frictions by Itskhoki and Moll (2014), the planner can control aggregate consumption and in the incomplete markets model with endogenous growth by Brunnermeier and Sannikov (2016), the constrained planner can control a common investment rate across heterogeneous households.[3]

However, extensions of the above models may require the results developed here. Moreover, the methodology of this paper is directly relevant for future study of constrained planner problems in extensions and applications of the Aiyagari-Huggett model: this includes Aiyagari-Huggett models incorporating aggregate shocks (Krusell and Smith, 1998), permanent income shocks Kuhn (2013), endogenous labour supply (Marcet et al., 2007), overlapping generations (Heathcote et al., 2010) or monetary and fiscal policy (Kaplan et al., 2016; Kaplan and Violante, 2010; Heathcote, 2005; Mckay and Reis, 2016; Nuno and Thomas, 2017).

2. CONSTRAINED PLANNER PROBLEMS

This section presents the recursive and sequential constrained planner's problems in a standard Aiyagari (1994) model. Both Dávila et al. (2012) and Nuño and Moll (2017) formulate their problem as a recursive problem; the exposition here will follow the discrete time version in Dávila et al. (2012), only I place more formal mathematical structure on the model.

In the recursive problem, the constrained planner instructs agents on their next period assets based on their current assets, shock and the aggregate distribution of agents. The recursive problem will be a stationary primitive

[3]Other infinite dimensional models where the economy can be collapsed to finite dimensional states include Melitz (2003), Koren and Tenreyro (2013) and Buera and Moll (2015)

form[4] infinite horizon dynamic optimisation problem, where the planner selects an action (policy function) to drive a state (wealth distribution).

In the sequential problem, the constrained planner instructs agents each period on next period assets based on their history of shocks up to the period. The sequential problem will be a non-stationary reduced form infinite horizon dynamic optimisation problem, where the planner selects a sequence of states (random variables).[5]

The online appendix contains an overview of mathematical concepts used in this paper.

2.1. *The Aiyagari Model*

Time is discrete and indexed by $t \in \mathbb{N}$. There are a continuum of identical individuals indexed by $i \in [0,1]$. Let A, with $A: \; = [0, \infty)$, be the agents' asset space[6] and define E as the agents' labour endowment space. Assume $E \subset \mathscr{B}(\mathbb{R}_+)$, where $\mathscr{B}(\mathbb{R}_+)$ are the Borel sub-sets of $\mathbb{R}_+$.

At time zero, each agent i draws an initial asset level x_0^i, with x_0^i taking values in A. In subsequent periods, each agent receives a sequence of labour endowment shocks $(e_t^i)_{t=0}^{\infty}$, with e_t^i taking values in E for each t and i. Assume a common probability space $(\Omega, \Sigma, \mathbb{P})$ for all uncertainty, that is, x_0^i and $(e_t^i)_{t=0}^{\infty}$ for each i are random variables defined on $(\Omega, \Sigma, \mathbb{P})$.

[4]The distinction between primitive form and reduced form problem is discussed by Sorger (2015), Section 5.1.

[5]In the context of a constrained planner, the terminology 'sequential' and 'recursive' problems is overloaded. The distinction here follows the distinction between 'sequential competitive equilibria' and 'recursive competitive equilibria' made by Miao (2006) and Cao (2016). In contrast to the distinction made here, the term sequential problem is often used to refer to the problem maximising the infinite sum of pay-offs as opposed to the Bellman Operator representation of the same problem. For infinite dimensional and stochastic problems, both sequential and recursive formulations can be written as a deterministic sequence problem (maximising the sum of discounted pay-offs) and using a deterministic Bellman Equation. For example, (14) compared to Equation (14) in the online appendix. This paper uses the term *sequence problem* to refer to a problem such as (14).

[6]As in the computations by Dávila et al. (2012), I assume a zero lower bound on assets to simplify the notation. In general, the Aiyagari model allows a strictly negative lower bound, however a zero lower bound is a common assumption, see also Miao (2006) and Cao (2016). The results here can be extended to a model with a negative lower bound, however an additional constraint on the state-space to ensure interest rates are not so high as to violate budget constraints will need to be added.

ASSUMPTION 2.1 The shocks $(e_t^i)_{t=0}^{\infty}$ and x_0^i have finite variance and are independently and identically distributed across i.

The finite variance assumption allows us to work in the L^2 space of square integrable random variables where compact sets are easier to find. The general existence result I present in this paper can also be applied to other topological vector spaces; research on such models is left for further work.

Let μ_0 denote the common joint distribution of x_0^i and e_0^i. That is,

$$\mu_0(B) = \mathbb{P}\{\omega \in \Omega \,|\, x_0^i(\omega), e_0^i(\omega) \in B\}, \qquad B \in \mathscr{B}(S),\, i \in [0,1] \tag{1}$$

Let P denote the joint distribution of x_0^i and $(e_t^i)_{t=0}^{\infty}$.

ASSUMPTION 2.2 For each $i \in [0,1]$, the shocks $(e_t^i)_{t=0}^{\infty}$ are a stationary Markov process with common Markov kernel Q and stationary marginal distribution ψ.

Assumption 2.2 can be relaxed to boundedness of the mean of the endowment shock, however, the stationarity assumption simplifies notation.

We do not need further assumptions on E for the proofs in this paper. However, A will, in general, be unbounded above, even if E is bounded. Dávila et al. (2012) assume an upper-bound each period on A, however, simulations by Dávila et al. (2012) (see fig.3 and discussion at section 5.4 in Dávila et al. (2012)) and by Nuño and Moll (2017) show a solution with diverging variance, implying a sequence of asset distributions with an increasing upper-bound.[7]

No Aggregate Uncertainty and the Aggregate State

Under no aggregate uncertainty, aggregate variables only depend on the common theoretical distribution of individual shocks, rather than the realisation of individual shocks. Often, no aggregate uncertainty is an assumption in heterogeneous agent models. Here, I opt for a formal construction following Acemoglu and Jensen (2015).[8]Let λ denote Lebesgue measure:

[7]Popoviciu's inequality for variance states the variance of any bounded random variable is bounded. Dávila et al. (2012) compute a solution path with ever increasing variance that does not converge to an upper-bound.

[8]The IID assumption on individual shocks is intuitively appealing. However, the assumption means the LLN does not hold — discussion of the challenges goes back to Judd

PROPOSITION 2.1 *Let $g\colon S \to \mathbb{R}$ be a measurable function such that $g(x_0^i, e_0^i)$ has finite variance. If Assumption 2.1 holds, then*

$$\int g(x_0^i, e_0^i)\lambda(di) = \int\int g(x,e)\mu_0(dx,de) \tag{2}$$

holds $\mathbb{P}$-almost everywhere.

PROOF: The random variables $g(x_0^i, e_0^i)$ are uncorrelated across i, have a common bounded variance and a common mean $\int\int g(x,e)\mu_0(dx,de)$. We thus satisfy the conditions of Theorem 2 in Uhlig (1996), which gives the result. *Q.E.D.*

As such, the following holds $\mathbb{P}$ - almost everywhere:

$$\int \mathbb{1}_B\{x_0^i, e_0^i\}\lambda(di) = \int\int \mathbb{1}_B(x,e)\mu_0(dx,de) = \mu_0(B), \qquad B \in \mathscr{B}(S) \tag{3}$$

The expression says, with probability one, the empirical distribution of agents over S agrees with the theoretical probability distribution of the individual shocks — the constrained planner can use μ_0 to know the mass of agents in any $B \in \mathscr{B}(S)$.

The distribution μ_0 becomes the initial state for the recursive constrained planner problem. The recursive problem we will consider is one where the planner selects measurable policy function for each h_t for each t, with $h_t\colon S \to A$. Each h_t instructs agents on $t+1$ assets given their time t asset and shock. A sequence of policy functions $(h_t)_{t=0}^{\infty}$ chosen by the constrained planner generates a sequence of assets for each agent, $(x_t^i)_{t=0}^{\infty}$, by

$$x_{t+1}^i = h_t(x_t^i, e_t^i), \qquad t \in \mathbb{N},\ i \in [0,1] \tag{4}$$

(1985). The solution used here and by Acemoglu and Jensen (2015) still faces difficulties in interpretation, in particular, the LHS of (2) is a vector valued Pettis integral, which, unlike the Bochner integral has no link to Lebesgue integral of $i \mapsto g(x_0^i(\omega), e_0^i(\omega))$ evaluated at *a realisation of* $\omega \in \Omega$. See section 5.3 in Al-Najjar (2004). Alternatives include Miao (2002) and Miao (2006), who constructs an underlying space following Feldman and Gilles (1985) and drops the IID assumption. Notwithstanding, the results here do not depend on the specific construction of no aggregate uncertainty, so long as the constrained planner's state-space can be represented using a deterministic distribution.

Since h_t applies to all agents i, the distribution of $\{x_t^i, e_t^i\}$ will be identical across i. Moreover, $\{x_t^i, e_t^i\} \sim \mu_t$ for each i, where $(\mu_t)_{t=0}^{\infty}$ satisfies the recursion

$$\mu_{t+1}(B_A \times B_E) = \int \int \mathbb{1}_{B_A}\{h_t(x,e)\} Q(e, B_E) \mu_t(dx, de), \qquad t \in \mathbb{N} \tag{5}$$

for each t and $B_A \times B_E \in \mathscr{B}(S)$. See Claim 9.1 in the online appendix for the proof.

If each x_t^i has finite variance, once again by Proposition 2.1, the time t empirical distribution of agents over S will satisfy

$$\int \mathbb{1}_B \{x_t^i, e_t^i\} \lambda(di) = \mu_t(B), \qquad B \in \mathscr{B}(S), \; t \in \mathbb{N} \tag{6}$$

with probability one.

Production

Assume a representative firm rents capital (assets) from individuals and hires workers to produce output Y_t:

$$Y_t = F(K(\mu_t), L) - \delta K(\mu_t) \tag{7}$$

where $F \colon \mathbb{R}^2_+ \to \mathbb{R}_+$. When the state is μ_t, using again the LLN argument from Proposition 2.1, total capital and labour in the economy is

$$K(\mu_t) \colon = \int \int x \mu_t(dx, de) = \int x_t^i \lambda(di) \tag{8}$$

$$L = \int e \int \mu_t(dx, de) = \int e_t^i \lambda(di) \tag{9}$$

Labour, L, will be constant according to Assumption 2.2.

ASSUMPTION 2.3 The production function F is differentiable on $\mathbb{R}_{++}$, homogeneous of degree one, strictly concave and for any $\hat{L} > 0$ and $\hat{K} > 0$ satisfies

1. $\lim_{K \to \infty} F_1(K, \hat{L}) = 0$ and $\lim_{K \to 0} F_1(K, \hat{L}) = \infty$ (Inada conditions)
2. $F(0, \hat{L}) = F(\hat{K}, 0) = 0$
3. $K \mapsto F(K, \hat{L})$ is bijective.

Budget Constraints and Utility

Interest and wage rates in the economy will be

$$r(\mu_t) := F_1(K(\mu_t), L) - \delta, \quad w(\mu_t) := F_2(K(\mu_t), L)$$

Given the aggregate state μ_t, an agent i with asset x_t^i and endowment shock e_t^i must satisfy their budget constraint

$$0 \leq x_{t+1}^i \leq (1 + r(\mu_t))x_t^i + w(\mu_t)e_t^i \tag{10}$$

where x_{t+1}^i is the next period asset. If x_0^i has finite variance and $r(\mu_t)$ is real-valued for each t, then if $(x_t^i)_{t=0}^{\infty}$ satisfies (10), x_t^i will have finite variance for each t (see Claim 9.2 in the online appendix.) Consumption for each agent i will be

$$c_t^i = (1 + r(\mu_t))x_t^i + w(\mu_t)e_t^i - x_{t+1}^i$$

Integrating across agents' budget constraints at Equation (10) and using the definition of interest and wages rates, along with homogeneity of the production function (see Theorem 2.1 in Acemoglu (2009)) gives a law of motion for aggregate capital

$$K(\mu_{t+1}) \leq (1 + r(\mu_t))K(\mu_t) + w(\mu_t)L = F(K(\mu_t), L) + (1 - \delta)K(\mu_t) \tag{11}$$

From the law of motion above, there exists an upper-bound $\bar{K}$ such that given any initial aggregate level of capital below $\bar{K}$, aggregate capital for wealth distributions satisfying (10) will never exceed $\bar{K}$ (see fig 9.3 in Acemoglu (2009)).

ASSUMPTION 2.4 The initial wealth distribution μ_0 satisfies $K(\mu_0) < \bar{K}$.

Turning to consumer utility, let $v \colon \mathbb{R}_+ \to \mathbb{R}_+$ be each consumer's utility function. Time t utility for agent i will be $v(c_t^i)$.

ASSUMPTION 2.5 The utility function v is strictly increasing, bijective, concave and upper semicontinuous.

I leave out a definition of a competitive equilibrium as it is standard, for example, see Aiyagari (1994), Dávila et al. (2012), Kuhn (2013), Miao (2002) or Acikgoz (2015).

2.2. *Recursive Constrained Planner*

Let $\mathscr{P}(S)$ denote the space of Borel probability measures on S. The recursive planner's state-space, $\mathbb{M}$, will be a subspace of $\mathscr{P}(S)$ such that for each μ, with $\mu \in \mathbb{M}$, satisfies:

a) the marginal distribution across E, $\int \mu(dx, \cdot)$, agrees with ψ
b) the marginal distribution across A, $\int \mu(\cdot, de)$, has finite variance
c) aggregate assets satisfy $\int \int x\mu(dx, de) \in [0, \bar{K}]$.

Let $\mathbb{Y}$ denote the space of measurable functions h where $h\colon S \to A$. The space $\mathbb{Y}$ will be the *action-space* and the constrained planner picks a policy $h_t \in \mathbb{Y}$ for each t and agents' assets transition according to Equation (4).

Define a correspondence Λ, with $\Lambda\colon \mathbb{M} \twoheadrightarrow \mathbb{Y}$, mapping an economy's state to feasible policy functions as follows:

$$\Lambda(\mu) \colon = \begin{cases} h \in \mathbb{Y} \; | 0 \leq h(x,e) \leq (1 + r(\mu))x + w(\mu)e, & \text{if } K(\mu) > 0 \\ h \in \mathbb{Y} \; | h = 0, & \text{if } K(\mu) = 0 \end{cases} \tag{12}$$

The (in) equalities above hold μ - almost everywhere. We are unable to place restrictions on $\mathbb{Y}$ such that the correspondence Λ has compact image sets in a suitable topology, for details see section 5.

Following Equation (5), given a time t distribution of agents on S, μ, and policy function h, the operator $\Phi\colon \operatorname{Gr}\Lambda \to \mathbb{M}$ is defined by

$$\Phi(\mu, h)(B_A \times B_E) \colon = \int \int \mathbb{1}_{B_A}\{h(x,e)\} Q(e, B_E)\mu(dx, de) \tag{13}$$

where $B_A \times B_E \in \mathscr{B}(S)$, gives the time $t+1$ distribution of agents. We write $\mu_{t+1} = \Phi(\mu_t, h_t)$.

The constrained planner's per-period pay-off, $u\colon \operatorname{Gr}\Lambda \to \mathbb{R}_+$, integrates utility across the empirical distribution of agents

$$u(\mu, h) \colon = \begin{cases} \displaystyle\int \int v\big((1 + r(\mu))x + w(\mu)e - h(x,e)\big)\mu(dx, de) & \text{if } K(\mu) > 0 \\ 0 & \text{if } K(\mu) = 0 \end{cases}$$

It is a straight-forward use of Jensen's inequality (Fact 8.4 in the online appendix) and homogeneity of the production function to show the integral is well-defined and real-valued.

Finally, let $\beta \in (0,1)$ be a discount factor and let V, with $V\colon \mathbb{M} \to \mathbb{R}_+ \cup \{+\infty\}$, denote the constrained planner's value function:

$$V(\mu_0) := \sup_{(\mu_t, h_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{14}$$

subject to

$$h_t \in \Lambda(\mu_t), \qquad \mu_{t+1} = \Phi(\mu_t, h_t), \qquad t \in \mathbb{N}, \qquad \mu_0 \text{ given} \tag{15}$$

DEFINITION 2.1 (**Recursive Constrained Planner's Problem**)

Given μ_0, a solution to the recursive constrained planner's problem is a sequence of measurable policy functions $(h_t)_{t=0}^{\infty}$, with $h_t\colon S \to A$ for each t and a sequence of Borel probability measures on S, $(\mu_t)_{t=0}^{\infty}$ satisfying (15) that achieves the value function:

$$V(\mu_0) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{16}$$

I now state the main result of this paper:

THEOREM 2.1 *If the recursive constrained planner's problem (Definition 2.1) satisfies Assumptions 2.1 - 2.5, then for any $\mu_0 \in \mathbb{M}$, there exists a solution $(\mu_t, h_t)_{t=0}^{\infty}$.*

Since Λ does not have compact image sets, standard existence results in dynamic optimisation theory fail (see section 5). To prove Theorem 2.1, the paper first defines a sequential planner's problem (section 2.3) and shows existence of a solution for the sequential planner implies existence of a solution for the recursive planner (Theorem 2.2 in section 2.4). The sequential planner's feasibility correspondences will still not have compact image sets around regions where capital is zero. Thus, the paper presents a general existence result for non-compact infinite horizon dynamic optimisation (Theorem 3.1 in section 3), and then checks the sequential planner's problem satisfies the conditions for existence (section 4).

Recursive Policies

If the recursive constrained planner's problem has a solution, $(\mu_t, h_t)_{t=0}^{\infty}$, for each $\mu_0 \in \mathbb{M}$, then following standard arguments, we can show there exists a policy operator $H\colon \mathbb{M} \to \mathbb{Y}$ such that the sequence $(\mu_t, H(\mu_t))_{t=0}^{\infty}$ with $\mu_{t+1} = \Phi(\mu_t, H(\mu_t))$ solves the recursive problem (Corollary 10.1 in the online appendix). Thus, if a solution to the recursive constrained planner's problem exists, then the *policy function* that maps assets and shocks to next period assets depends only on the current distribution.

2.3. *Sequential Constrained Planner*

Recall for any initial state for the recursive problem, $\mu_0 \in \mathbb{M}$, $\{x_0^i, e_0^i\} \sim \mu_0$ for each i and P denotes the joint probability distribution of $\{x_0^i, e_0^i, e_1^i \dots\}$. Define $Z\colon = A \times E^{\mathbb{N}}$ and consider the probability space $(Z, \mathscr{B}(Z), P)$. Let $\{x_0, e_0, e_1 \dots\}$ be a sequence of random variables representing draws from $(Z, \mathscr{B}(Z), P)$. By Proposition 2.1, realisations of the random variable $\{x_0, e_0, e_1 \dots\}$ are draws from the empirical distribution of individual shock values in Z.[9]

Let $\mathbb{X}\colon = L^2(Z, P)$ be the space of square integrable (with respect to P) real-valued functions on Z. Equip $\mathbb{X}$ with the weak topology and define $(\mathscr{F}_i)_{i=0}^{\infty}$ as the natural filtration with respect to $\{x_0, e_0, e_1, \dots\}$.[10]

For any $x \in \mathbb{X}$, with $\int x \geq 0$, define

$$\tilde{K}(x)\colon = \int x \, \mathrm{d}P \tag{17}$$

and if $\int x \mathrm{d}P > 0$, define

$$\begin{aligned} \tilde{r}(x)\colon &= F_1(\tilde{K}(x), L) - \delta \\ \tilde{w}(x)\colon &= F_2(\tilde{K}(x), L) \end{aligned} \tag{18}$$

For each t, define the time t state-space for the sequential planner:

$$\mathbb{S}_t\colon = \left\{ x \in m\mathscr{F}_t \,\middle|\, 0 \leq x, \int x \, dP \leq \bar{K} \right\} \tag{19}$$

where $m\mathscr{F}_t \subset \mathbb{X}$ is the space of $\mathscr{F}_t$-measurable functions.

[9] Formally the random variable is the *canonical random variable*, that is, for a realisation $z \in Z$, $x_0(z), e_0(z), e_1(z), \dots = \mathrm{id}_Z(z)$, where id_Z is the identity function on Z.

[10] That is, $\mathscr{F}_0$ is the σ-algebra generated by x_0 and for each $i \geq 1$, $\mathscr{F}_i$ is the σ-algebra generated by $\{x_0, e_0, \dots, e_{i-1}\}$.

For each t, define the feasibility correspondence $\Gamma_t \colon \mathbb{S}_t \to \mathbb{S}_{t+1}$:

$$(20) \qquad \Gamma_t(x) := \begin{cases} y \in \mathbb{S}_{t+1} \,\middle|\, 0 \leq y \leq (1 + \tilde{r}(x))\, x + \tilde{w}(x)\, e_t & \text{if } \tilde{K}(x) > 0 \\ y \in \mathbb{S}_{t+1} \,\middle|\, y = 0 & \text{if } \tilde{K}(x) = 0 \end{cases}$$

For each t, define the time t pay-offs $\rho_t \colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$:

$$(21) \qquad \rho_t(x, y) := \begin{cases} \int \nu\left((1 + \tilde{r}(x))\, x + \tilde{w}(x)\, e_t - y\right) \mathrm{d}P & \text{if } \tilde{K}(x) > 0 \\ 0 & \text{if } \tilde{K}(x) = 0 \end{cases}$$

Finally, let $\tilde{V}$, with $\tilde{V} \colon \mathbb{S}_0 \to \mathbb{R}_+ \cup \{+\infty\}$ denote the time 0 sequential planner's value function:

$$\tilde{V}(x_0) := \sup_{(x_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$$

subject to

$$(22) \qquad x_{t+1} \in \Gamma_t(x_t), \qquad \forall t \in \mathbb{N}, \qquad x_0 \in \mathbb{S}_0 \text{ given}$$

DEFINITION 2.2 (**Sequential Constrained Planner's Problem**)

Given $x_0 \in \mathbb{S}_0$, a solution to the sequential constrained planner's problem is a sequence of random variables $(x_t)_{t=0}^{\infty}$ satisfying (22) that achieve the sequential planner's value function:

$$(23) \qquad \tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$$

2.4. *Sequential Solution Implies Recursive Solution*

Our objective in this section is to show existence of solutions to the recursive planner's problem by showing existence to the sequential problem. The Bellman equation for the sequential planner's problem will be a *non-stationary Bellman equation* with a policy operator at each t, $\Theta_t \colon \mathbb{S}_t \to \mathbb{S}_{t+1}$. The optimal sequence will follow $x_{t+1} = \Theta_t x_t$. However, x_{t+1} still depends on the history $\{x_0, e_0, \ldots, e_t\}$. That is, by contrast, a recursive solution requires x_{t+1} to be $\{x_t, e_t\}$ measurable; that is, we require a function

$h_t \colon S \to A$ such that $x_{t+1}(z) = h_t(x_t(z), e_t(z))$ for $z \in Z$. To convert a sequential solution to a recursive solution, the following procedure projects a sequential solution back onto its previous period, furnishing the required measurability from properties of conditional expectation.

Given x_0 satisfying $x_0 \in \mathbb{S}_0$, let $(y_t)_{t=0}^{\infty}$ be a solution to the sequential planner's problem. Construct a candidate sequence, $(x_t)_{t=0}^{\infty}$, as follows:

$$
\begin{aligned}
&x_0 = y_0, \qquad x_1 = \mathbb{E}(y_1 | \sigma(x_0, e_0)) \\
&\text{and} \\
&x_{t+1} = \mathbb{E}(y_t | \sigma(x_t, e_t)), \qquad\qquad \forall t \in \mathbb{N}
\end{aligned}
\tag{24}
$$

The term $\sigma(x_t, e_t)$ denotes the σ-algebra generated by x_t and e_t. And $\mathbb{E}(y_t | \sigma(x_t, e_t))$ denotes the conditional expectation of y_t with respect to x_t and e_t.

PROPOSITION 2.2 *Let Assumptions 2.1 - 2.5 hold. If $(y_t)_{t=0}^{\infty}$ is a solution to the sequential problem (Definition 2.2), then $(x_t)_{t=0}^{\infty}$ defined by* (24) *is a solution to the sequential problem.*

See the appendix for a proof.

THEOREM 2.2 *Let Assumptions 2.1 - 2.5 hold. If there exists a solution to the sequential problem (Definition 2.2), then there exists a solution to the recursive problem (Definition 2.1) and $V(\mu_0) = \tilde{V}(x_0)$.*

The complete proof is in the appendix and proceeds as follows. Let $(y_t)_{t=0}^{\infty}$ solve the sequential problem and let $(x_t)_{t=0}^{\infty}$ be defined by (24). Since x_{t+1} is $\sigma(x_t, e_t)$ measurable, $x_{t+1} = h_t(x_t, e_t)$ for a measurable function h_t for each t. For each t, define

$$
\mu_t(B) \colon = P\{x_t, e_t \in B\}, \qquad\qquad B \in \mathscr{B}(S)
\tag{25}
$$

The remainder of the proof verifies $(\mu_t, h_t)_{t=0}^{\infty}$ solves the recursive problem.

3. EXISTENCE THEOREM FOR NON-STATIONARY REDUCED FORM INFINITE HORIZON DYNAMIC OPTIMISATION

We require the following definition for the main assumption. Let (X, τ) be a topological space, DEFINITION 3.1 A function $f\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is **mildly sup-compact** if the upper contour sets

$$UC_f(\epsilon) :\, = \{x \in X \mid f(x) \geq \epsilon\} \tag{26}$$

are sequentially compact for all $\epsilon > \inf f$.

A discussion on the relationship between mild sup-compactness, sup-compactness and upper semicontinuity is given in the online appendix.

3.1. *General Existence Theorem*

A non-stationary reduced form economy is a 5-tuple

$$\mathscr{E} :\, :\, = ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta)$$

consisting of:

1. A topological space $(\mathbb{X}, \tau)$
2. A collection of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$, with $\mathbb{S}_t \subset \mathbb{X}$ for each t
3. A collection of non-empty feasibility correspondences $(\Gamma_t)_{t=0}^{\infty}$, with $\Gamma_t\colon \mathbb{S}_t \twoheadrightarrow \mathbb{S}_{t+1}$ for each t
4. A collection of per-period pay-offs $(\rho_t)_{t=0}^{\infty}$, with $\rho_t\colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$ for each t
5. A discount factor $\beta \in (0, 1)$.

Define the correspondence of **feasible sequences** $\mathcal{G}_t^T : \mathbb{S}_t \twoheadrightarrow \prod_{i=t}^{T} \mathbb{S}_i$ starting at time t and ending at time T as follows:

$$\mathcal{G}_t^T(x) :\, = \left\{ (x_i)_{i=t}^{T} \mid x_{i+1} \in \Gamma_i(x_i), x_t = x \right\}, \qquad x \in \mathbb{S}_t \tag{27}$$

Let $\mathcal{G}$ denote $\mathcal{G}_0^{\infty}$ and let $\mathcal{G}^T$ denote $\mathcal{G}_0^T$.

Define the **value function** $\tilde{V}\colon \mathbb{S}_0 \to \mathbb{R} \cup \{-\infty, \infty\}$ as follows:

$$\tilde{V}(x) :\, = \sup_{(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \tag{28}$$

Recall a compact-valued upper hemicontinuous correspondence has compact image sets (see Mathematical Preliminaries in the online appendix).

The first assumption below relaxes the standard requirement for Γ_t to be upper hemicontinuous and compact valued and for $\mathbb{S}_t$ to be a metric space. The assumptions are stated, for example, by Acemoglu (2009), Assumption 6.2, Kamihigashi (2017), section 6 or Stokey and Lucas (1989), Assumption 4.3. A discussion of how standard theory uses these assumptions to verify existence is in section of the online appendix.

ASSUMPTION 3.1 For each $x \in \mathbb{S}_0$ and $t \in \mathbb{N}$, the functions $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}^{t+1}(x)$ are mildly sup-compact in the product topology (of τ topology in $\mathbb{X}$).

The next assumption is the standard growth condition (see discussion on Corollary 6.1 by Kamihigashi (2017)).

ASSUMPTION 3.2 For each $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^{\infty}$ such that any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ satisfies

$$\rho_t(x_t, x_{t+1}) \leq m_t, \qquad \forall t \in \mathbb{N} \tag{29}$$

and

$$\sum_{t=0}^{\infty} \beta^t m_t < \infty \tag{30}$$

ASSUMPTION 3.3 The functions ρ_t are sequentially upper semicontinuous for all $t \in \mathbb{N}$.

THEOREM 3.1 *If $\mathscr{E}$ satisfies Assumption 3.1 - 3.3, then the value function will satisfy $\tilde{V} < \infty$ and for every $x \in S_0$, there will exist $(x_t)_{t=0}^{\infty}$ satisfying $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ such that $\tilde{V}(x) = \sum_{t=0}^{\infty} \beta^t \rho_t\left(x_t, x_{t+1}\right)$.*

See the appendix for a proof. The proof for Theorem 3.1 follows a product space approach rather than iteration of the Bellman equation. (Despite the mild sup-compactness condition, a proof using iteration of a non-stationary Bellman operator is still not possible as the Bellman operator will not maintain semicontinuity of the value function.) In essence, the proof works by showing feasible paths of states that converge to the supremum of the problem belong a compact space in the product topology (of the topology τ in $\mathbb{X}$). By contrast, the standard assumption of semicontinuity and compact-valued correspondences requires that *all* feasible sequences belong to a compact space in the product topology.

The proof is in the online appendix.

3.2. *Checking Mild Sup-Compactness in L^2 Spaces*

Let (Z, Σ, φ) be a finite measure space and let $\mathbb{X} = L^2(Z, \varphi)$ with norm

$$\|x\|: \ = \left(\int x^2 \, d\varphi \right)^{\frac{1}{2}}$$

Equip $\mathbb{X}$ with the weak topology (see the online appendix). Unless otherwise stated, convergence and topological notions will be with respect to the weak topology.

ASSUMPTION 3.4 The state spaces $\mathbb{S}_t$ are sequentially closed for each t.

ASSUMPTION 3.5 The correspondences Γ_t have a sequentially closed graph for each t.

ASSUMPTION 3.6 For each $t \in \mathbb{N}$, $\epsilon > 0$ and $x \in \mathbb{S}_0$, there exists a constant $\bar{M}$ such that if $(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $u_t(x_t, x_{t+1}) \geq \epsilon$, then $\|x_i\| \leq \bar{M}$ for each $i \in \{0, \ldots, t+1\}$.

PROPOSITION 3.1 *Consider $\mathscr{E}$ where $\mathbb{X} = L^2(Z, \varphi)$ and τ is the weak topology. If $\mathscr{E}$ satisfies Assumptions 3.3, 3.5 and 3.6, then $\mathscr{E}$ satisfies Assumption 3.1.*

To summarise, the following serves as a checklist to show existence of the value function on an L^2 space:

1. Check the growth condition, Assumption 3.2
2. Check sequential semicontinuity of ρ_t for each t, Assumption 3.3
3. Check $\mathbb{S}_t$ is sequentially closed for each t, Assumption 3.4
4. Check the sequentially closed graph property of Γ_t for each t, Assumption 3.5
5. Check feasible sequences give a positive per-period pay-offs at some time in the future have a finite norm, Assumption 3.6.

4. EXISTENCE OF CONSTRAINED OPTIMA

Consider the case of the sequential constrained planner in the Aiyagari-Huggett model presented in section 2.3. Let Assumptions 2.1 - 2.4 hold and let

$$\mathscr{E} = ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta)$$

where:

1. $\mathbb{X} = L^2(Z, P)$, where $Z = A \times E^{\mathbb{N}}$ and P is the common joint distribution of $\{x_0^i, e_0^i, e_1^i, \dots\}$
2. The topology τ is the weak topology
3. The sequence of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$ are defined by (19)
4. The sequence of correspondences $(\Gamma_t)_{t=0}^{\infty}$ are defined by (20)
5. The sequence of pay-offs $(\rho_t)_{t=0}^{\infty}$ are defined by (21).

PROPOSITION 4.1 ***(Checking Assumption 3.2)*** *For any $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^{\infty}$ such that $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ and any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ satisfies $\rho_t(x_t, x_{t+1}) \leq m_t$ for each t.*

PROPOSITION 4.2 ***(Checking Assumption 3.3)*** *The functions $(\rho_t)_{t=0}^{\infty}$ are sequentially upper semicontinuous for each t*

PROPOSITION 4.3 ***(Checking Assumption 3.4)*** *The state spaces $\mathbb{S}_t$ are sequentially closed for each t*

PROPOSITION 4.4 ***(Checking Assumption 3.5)*** *The correspondences $(\Gamma_t)_{t=0}^{\infty}$ have closed graph for each t.*

PROPOSITION 4.5 ***(Checking Assumption 3.6)*** *For any $t \in \mathbb{N}$, $\epsilon > 0$ and $x \in \mathbb{S}_0$, there exists a constant $\bar{M}$ such that if $(x_i)_{t=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$, then*

$$\|x_i\| \leq \bar{M}$$

for all $i \in \{0, 1, \dots, t+1\}$.

We are now ready to verify existence of recursive constrained optima.

PROOF OF THEOREM 2.1: Given μ_0, let P and $\{x_0, e_0, e_1, \dots\}$ be as defined in section 2.3. By assumptions 2.1 - 2.4 and propositions 4.1 - 4.5, the economy $\mathscr{E}$ satisfies Assumptions 3.2 - 3.6.

Since Assumptions 3.4 - 3.6 satisfy the conditions for Proposition 3.1, $\mathscr{E}$ satisfies Assumption 3.1. As such, $\mathscr{E}$ satisfies Assumptions 3.1 - 3.3 and the conditions for Theorem 3.1.

By Theorem 3.1, there exists a sequence $(y_t)_{t=0}^{\infty}$ solving the Sequential Planner's problem (Definition 2.2) such that $\tilde{V}(x_0) < \infty$. By Proposition 2.2, the sequence $(x_t)_{t=0}^{\infty}$ defined by (24) also solves the sequential planners problem. Moreover, there exists a sequence of measurable policy functions $(h_t)_{t=0}^{\infty}$ with $h_t\colon S \to A$ and $x_{t+1} = h_t(x_t, e_t)$ for each t.

By Theorem 2.2, $(h_t)_{t=0}^{\infty}$ and $(\mu_t)_{t=0}^{\infty}$ defined by Equation (25) solve the recursive problem and

$$V(\mu_0) = \tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) < \infty \tag{31}$$

Q.E.D.

5. DISCUSSION

5.1. *Non-Compactness of the Feasibility Correspondence*

Recall a correspondence $\Lambda\colon \mathbb{X} \twoheadrightarrow \mathbb{Y}$ has compact image sets if $\Lambda(C)$ is compact for compact C.

There are two reasons why the constrained planner's feasibility correspondences will not have compact image sets. The first concerns the structure of the recursive problem and the second concerns the behaviour of interest rates around regions where capital is zero.

5.1.1. *Structure of Recursive Problem*

In the recursive problem, to show Λ is has compact image sets, we need to place some further restrictions on the space $\mathbb{Y}$. I consider three topologies: the sup-norm topology, the topology of point-wise convergence and L^p weak topology.

For compactness in the sup-norm topology, the Arzela-Arscoli Theorem states that a uniformly bounded, equicontinuous family of functions on a compact interval will be bounded. However, A will not be bounded and policy functions may not be compact. A possible approaching could be restricting $\mathbb{M}$ to measures on a compact support. For each $\mu \in \mathbb{M}$, we can then restrict policy functions in $\Gamma(\mu)$ to be defined on the bounded support of μ. If the mean of μ is positive, then policy functions in $\Gamma(\mu)$ will also be bounded. Notwithstanding the pathologies (see below) as interest rates diverge, to now use the Arzela-Arscoli Theorem, we also need to restrict feasible policy functions in each period to an equicontinuous family of functions. This line of argument has so far not yielded success.

On the other hand, we could once again restrict $\mathbb{M}$ to measures on a compact support and try to verify compactness in the topology of point-wise convergence using Helly's selection theorem. To do this, we would need to restricting $\mathbb{Y}$ to the space of monotone policy functions. In the context of a proof for general equilibrium in a Aiyagari-Huggett model with aggregate shocks, the approach by Cao (2016) is to justify monotonicity using the necessary Euler equations of individual agents. However, in the case of an optimisation problem, we cannot restrict the search for an optimiser based on necessary conditions for the constrained planner *because a solution for the constrained planner may not exist*. In particular, there may be sequences of functions outside the space of monotone functions converging to the supremum.

The online appendix gives further detail of pathologies in the weak topology if we let $\mathbb{Y}$ be the space of square integrable functions on S, where Φ fails to be well-defined.

Note the pathologies in the recursive problem also prevent the use of the non-compactness existence result of section 3. This is because we cannot place restrictions such as monotonicity or equicontinuity on policy functions in the upper contour sets of Assumption 3.1.

5.1.2. *Non-compactness Near Zero Capital*

Let us consider the second pathology, concerning the behaviour of interest rates near zero capital, in the context of the sequential problem. Consider the setting and notation of section 4. Once we move to L^2 space with the weak topology, the feasibility correspondence will be compact valued and have a closed graph. The sequential problem will also be well-defined. However, since the state-space is not compact, the correspondences will not be upper hemicontinuous. In particular, there will exist a compact set $C \subset \mathbb{S}_t$ such that $\Gamma_t(C)$ is not compact. (Recall from section 8.1 in the online appendix that the image of a compact set under a compact-valued and upper hemicontinuous is compact.)

For the following claim, assume $F(K, L) = K^{\alpha-1}$ and $\alpha = 3$. Furthermore, assume $x_0 \in \mathbb{S}_0$, the initial assets for the economy are a uniform random variable on the interval $[0, 1]$. Assume the random variable e_0 is large enough to satisfy $\tilde{w}(x_0)e_0 > 1$.

CLAIM 5.1 *There exists a compact set C, satisfying $C \subset \mathbb{S}_1$, such that the image set $\Gamma_1(C)$ is not compact.*

The details of the claim are in the appendix. Roughly, we can construct a sequence of assets in period $t = 1$ that converge to zero in mean but do not converge uniformly, leaving a smaller and smaller measure of agents with positive assets taken over to period $t = 2$. In period $t = 2$, interest rates will diverge to infinity, the smaller and smaller measure of agents with positive assets can now accumulate assets which diverge to infinity, while, in the mean, aggregate assets still converge to zero. The scenario occurs despite assets of each agent in period $t = 0$ being bounded above.

5.2. *Relationship to K-Sup-Compactness*

To relax compactness *and* continuity requirements on the feasibility correspondence, Feinberg et al. (2012) introduce a condition (assumption W* in Feinberg et al. (2012)), later defined as K-Sup-Compactness by Feinberg et al. (2013) (Definition 1.1), on per-period pay-offs.[11] Assumption 3.1 of this paper is a generalisation of a condition called K-Sup-Compactness.

Recall the definition of sup-compact from the online appendix, section 8.2. While Feinberg et al. (2012) consider stationary problems, for the sequential constrained planner's setting, K-Sup-Compactness of each per-period pay-off ρ_t becomes:

ASSUMPTION 5.1 (**K-Sup-Compact**) Let $t \in \mathbb{N}$. If C is a sequentially compact subset of $\mathbb{S}_t$, then the function $\{x_t, x_{t+1}\} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}_t^{t+1}(C)$ is sup-compact.

The assumption allows the Bellman Equation (in our case, a non-stationary Bellman Equation) to preserve semicontinuity (see Theorem 2 in Feinberg et al. (2012) and Lemma 2.5 in Feinberg et al. (2013)).

With utility bounded below, ρ_t will not satisfy K-Sup-Compactness. To see why, note $\mathcal{G}_t^{t+1}(C) = \{x, y \mid y \in \Gamma_t(x), x \in C\}$. Moreover, the upper-contour

[11] Feinberg et al. (2013) use term K-Inf-Compactness, as they work with minimisaiton problems.

set of the function $\{x_t, x_{t+1}\} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}_t^{t+1}(C)$ when $\epsilon = 0$ will be:

$$\{x, y \,|\, y \in \Gamma_t(x), x \in C, \rho_t(x, y) \geq 0\} \\ = \{x, y \,|\, y \in \Gamma_t(x), x \in C\} = \mathcal{G}_t^{t+1}(C)$$

For the constrained planner, K-Sup-Compactness of ρ_t will then imply compact $\mathcal{G}_t^{t+1}(C)$ for compact C. However, Claim 5.1 constructs an example where $x_n \in C$ and $y_n \in \Gamma(x_n)$ such that the norm of y_n diverges, implying non-compact $\mathcal{G}_t^{t+1}(C)$.

The constrained planner's problem will satisfy a milder K-Sup-Compact condition, where the functions in Assumption 5.1 are mildly sup-compact. However, mild K-Sup-Compactness is too weak, and the Bellman Operator will not preserve semicontinuity.[12]

6. CONCLUSION

The paper presented a proof of existence for constrained optima in a standard Aiyagari (1994) model. While the constrained planner's problem is a natural way to understand optimal policy in heterogeneous agent models, mathematical difficulties with the standard recursive problem, arising from the infinite dimensional state and action space, means existing dynamic programming theory fails. The paper addressed the mathematical challenges by first shifting to a sequential characterisation of the constrained planner's problem and showing sequential solutions are also recursive solutions. The paper then presented a new existence result for infinite horizon dynamic optimisation on non-compact state-spaces to show existence of a solution to the sequential problem.

An immediate path for further work is to develop computational methods which are known to converge to the true constrained optima and to study stability properties of the solution path.

[12] An inventory control problem with bounded costs will satisfy mild K-Sup-Compactness, but will nonetheless fail to have a solution since the Bellman Operator will not preserve semi-continuity. Details are available on request

APPENDIX

Proofs for Section 2.4

CLAIM 6.1 *Let $(y_t)_{t=0}^{\infty}$ be a solution to the sequential problem. If $(x_t)_{t=0}^{\infty}$ is a sequence of random variables defined by* (24), *then*

$$x_t := \mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) = \mathbb{E}(y_t|\sigma(x_t, e_t)) \qquad \forall t \in \mathbb{N}$$

See the online appendix for a detailed proof.

PROOF OF PROPOSITION 2.2: We first show the sequence $(x_t)_{t=0}^{\infty}$ is feasible and then show $(x_t)_{t=0}^{\infty}$ achieves the sequential planner's value function.

Before proceeding, note the following holds due to the Tower Property of conditional expectation,[13]

$$\int y_t \,\mathrm{d}P = \int \mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1}))\,\mathrm{d}P = \int x_t \,\mathrm{d}P, \qquad t \in \mathbb{N} \tag{32}$$

To show feasibility, first we need to verify $x_{t+1} \in \Gamma_t(x_t)$ for each t, where Γ_t is defined by (20). To check $x_t \in \mathbb{S}_t$, note $x_{t+1} = h_t(x_t, e_t)$ for a measurable function h_t for each t with x_0 given. Thus each x_t can be written as a measurable function of $x_0, \ldots, e_{t-1}$, implying $x_t \in m\mathscr{F}_t$ for each t.

Furthermore, by (32), $\int x_t \,\mathrm{d}P = \int y_t \,\mathrm{d}P$ and since $\int y_t \,\mathrm{d}P \in [0, \bar{K}]$ for each t, we have $x_t \in [0, \bar{K}]$ for each t, allowing us to conclude $x_t \in \mathbb{S}_t$.

Set any $t \in \mathbb{N}$. To check individual assets satisfy feasibility, there are two cases to consider: first $\int x_t \,\mathrm{d}P > 0$ and second $\int x_t \,\mathrm{d}P = 0$. In either case, since $y_t \in \Gamma_t(y_t)$ for each t, we have $y_t \geq 0$. By properties of conditional expectation, [14] we will also have $x_{t+1} = \mathbb{E}(y_t|\sigma(x_t, e_t)) \geq 0$ for each t.

Suppose $\int x_t \,\mathrm{d}P > 0$. By (32), we have $\int y_t \,\mathrm{d}P = \int x_t \,\mathrm{d}P > 0$. Since $(y_t)_{t=0}^{\infty}$ is a solution to the sequential planner's problem, we have $y_{t+1} \in \Gamma_t(y_t)$ and thus, $y_{t+1} \leq (1 + \tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t$. To show $x_{t+1} \leq (1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t$,

[13]See 9.7 i) by Williams (1991)

[14]See 9.7 d) by Williams (1991)

consider,

$$
\begin{aligned}
x_{t+1} &= \mathbb{E}(y_{t+1}|\sigma(x_t, e_t)) \\
&\leq \mathbb{E}((1+\tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t|\sigma(x_t, e_t)) \\
&= (1+\tilde{r}(x_t))\mathbb{E}(y_t|\sigma(x_t, e_t)) + \tilde{w}(x_t)e_t \\
&= (1+\tilde{r}(x_t))\mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) + \tilde{w}(x_t)e_t \\
&= (1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t
\end{aligned}
$$

where, noting (32), the third line follows from

$$
\tilde{r}(y_t) = F_1\left(\int y_t \,\mathrm{d}P, L\right) = F_1\left(\int x_t \,\mathrm{d}P, L\right) = \tilde{r}(x_t)
$$

A similar argument shows $\tilde{w}(y_t) = \tilde{w}(x_t)$. The fourth line follows from Claim 6.1 and the final line follows from the definition of x_t.

On the other hand, consider the case when $\int x_t \,\mathrm{d}P = 0$. We have $\int y_t = \int x_t \,\mathrm{d}P = 0$ by (32). As such, by (20), $\int x_{t+1} \,\mathrm{d}P = \int y_{t+1} \,\mathrm{d}P = 0$. Since $x_{t+1} \geq 0$, x_{t+1} must then satisfy $x_{t+1} = 0$. Thus we have satisfied all the conditions, stated at (20), for x_{t+1} to belong to $\Gamma_t(x_t)$.

Next, we check $\rho_t(x_t, x_{t+1}) \geq \rho_t(y_t, y_{t+1})$ for each t. Select any t and consider the case $\int x_t \,\mathrm{d}P > 0$. We have

$$
\begin{aligned}
\rho_t(x_t, x_{t+1}) &= \int \nu\left((1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1}\right) \mathrm{d}P \\
&= \int \nu\big((1+\tilde{r}(y_t))\mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) + \tilde{w}(y_t)e_t \\
&\quad - \mathbb{E}(y_{t+1}|\sigma(x_t, e_t))\big) \,\mathrm{d}P \\
&= \int \nu\left(\mathbb{E}\left[(1+\tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t - y_{t+1}|\sigma(x_t, e_t)\right]\right) \mathrm{d}P \\
&\geq \int \mathbb{E}\left(\nu((1+\tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t - y_{t+1})|\sigma(x_t, e_t)\right) \mathrm{d}P \\
&= \rho_t(y_t, y_{t+1})
\end{aligned}
$$

where the second line is due to the definition x_t and x_{t+1}. The third line follows from Claim 6.1, the fourth line follows from Jensen's inequality (Fact 8.4 in the online appendix) and the final line is due to the Tower Property.

If $\int x_t \,\mathrm{d}P = 0$, then $\rho_t(x_t, x_{t+1}) = 0$ by definition of ρ_t. Since $\int y_t \,\mathrm{d}P = 0$ by (32), $\rho_t(y_t, y_{t+1}) = 0$. We can then conclude

$$\tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(y_t, y_{t+1}) \leq \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \tag{33}$$

Since $\tilde{V}(x_0)$ achieved the supremum of all pay-offs from feasible sequences, and $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} \in \Gamma_t(x_t)$ for each t, we must have $\tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$, allowing us to conclude $(x_t)_{t=0}^{\infty}$ is a solution to the sequential problem.

Q.E.D.

PROOF OF THEOREM 2.2: Let P denote the common joint probability distribution of the individual shocks as in section 2.1 and let μ_0 be the initial state of the economy for the recursive planner. Let $\{x_0, e_0, e_1, \dots\}$ be the canonical random variables on $(Z, \mathscr{B}(Z), P)$ as described in section 2.2 and recall the joint distribution of $\{x_0, e_0, e_1, \dots\}$ will be P and the joint distribution of $\{x_0, e_0\}$ will be μ_0.

Let $(y_t)_{t=0}^{\infty}$ be a solution to the sequential problem (Definition 2.2). Construct a sequence $(x_t)_{t=0}^{\infty}$ according to (24); recall $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} = h_t(x_t, e_t)$ for a sequence of measurable functions $(h_t)_{t=0}^{\infty}$ with $h_t \colon S \to A$ for each t. Define a sequence of Borel probability measures $(\mu_t)_{t=0}^{\infty}$ using (25) and note by definition that μ_t will the distribution of $\{x_t, e_t\}$ for each t.

The proof proceeds in two steps. To show $(\mu_t, h_t)_{t=0}^{\infty}$ solve the recursive problem, we will first show $(\mu_t, h_t)_{t=0}^{\infty}$ are feasible (part 1) and then show the sum of discounted pay-offs from $(\mu_t, h_t)_{t=0}^{\infty}$ dominate the sum of discounted pay-offs from any other feasible sequence of distributions and policy functions (part 2).

Part 1: Show $(\mu_t, h_t)_{t=0}^{\infty}$ satisfies feasibility for the recursive problem

Our task for this part is to show $(\mu_t, h_t)_{t=0}^{\infty}$ satisfies (15) for each $t \in \mathbb{N}$.

Fix any $t \in \mathbb{N}$, to show $h_t \in \Lambda(\mu_t)$, we have to consider two cases: when $\int \int x \mu_t(dx, de) > 0$ and when $\int \int x \mu_t(dx, de) = 0$. First suppose $\int \int x \mu_t(dx, de) > 0$, we will show

$$\mu_t\{a, e \in S \mid h_t(a, e) > (1 + r(\mu_t))a + w(\mu_t)e\} = 0$$

The condition says the policy function h_t satisfies agents' budget constraints μ_t - almost everywhere. Using the definition of μ_t by Equation (25), we have

$$\begin{aligned}
&\mu_t\{a, e \in S \,|\, h_t(a,e) > (1 + r(\mu_t))a + w(\mu_t)e\} \\
&= P\{\omega \in Z \,|\, h_t(x_t(\omega), e_t(\omega)) \\
&\quad > (1 + \tilde{r}(x_t))x_t(\omega) + \tilde{w}(x_t)e_t(\omega)\} \\
&= P\{\omega \in Z \,|\, x_{t+1}(\omega) \\
&\quad > (1 + \tilde{r}(x_t))x_t(\omega) + \tilde{w}(x_t)e_t(\omega)\} \\
&= 0
\end{aligned} \tag{34}$$

The first equality also uses the following observation, which holds because μ_t is the joint distribution of $\{x_t, e_t\}$

$$\int \int x\mu_t(dx, de) = \int x_t \, \mathrm{d}P > 0 \tag{35}$$

whence,

$$r(\mu_t) = F_1\left(\int \int x\mu_t(dx, de), L\right) = F_1\left(\int x_t \, \mathrm{d}P, L\right) = \tilde{r}(x_t) \tag{36}$$

An identical arguments shows $\tilde{w}(x_t) = w(\mu_t)$. The second equality in (34) follows from the assumption of the proposition that $x_{t+1} = h_t(x_t, e_t)$. The final equality is true because $\int x_t \, \mathrm{d}P > 0$ and because the sequence $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} \in \Gamma_t(x_t)$ for each t. Thus, $h_t(a,e) \leq (1 + r(\mu_t))a + w(\mu_t)e$ for μ_t almost everywhere, implying $h_t \in \Lambda(\mu_t)$ if $\int \int x\mu_t(dx, de) > 0$.

Now suppose $\int \int x\mu_t(dx, de) = 0$, we have,

$$\int x \int \mu_t(dx, de) = \int x_t \, \mathrm{d}P = 0 \tag{37}$$

Since $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} \in \Gamma_t(x_t)$ for each t, $x_{t+1} = 0$. Furthermore,

$$\mu_t\left\{h_t(a,e) \neq 0\right\} = P\left\{h_t(x_t, e_t) \neq 0\right\} = 0$$

Thus $h_t = 0$ for μ_t almost everywhere and $h_t \in \Lambda(\mu_t)$ if $\int \int x\mu_t(dx, de) = 0$.

Now we turn to showing $\mu_{t+1} = \Phi(\mu_t, h_t)$ for each t. Let $B \in \mathscr{B}(A \times E)$, where $B = B_A \times B_E$ for some $B \in \mathscr{B}(A)$ and $B_E \in \mathscr{B}(E)$. Use the definition of μ_{t+1} to write

$$\begin{aligned}
\mu_{t+1}(B) &= P\{x_{t+1} \in B_A, e_{t+1} \in B_E\} \\
&= P\{h_t(x_t, e_t) \in B_A, G(e_t, \eta_{t+1}) \in B_E\} \\
&= \mathbb{E}\left[\mathbb{1}_B\{h_t(x_t, e_t), G(e_t, \eta_{t+1})\}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{B_A}\{h_t(x_t, e_t)\} \times \mathbb{1}_{B_E}\{G(e_t, \eta_{t+1})\}\right]
\end{aligned}$$

We have used the recursive formulation of the Markov process in the second line, Equation (2) of the mathematical preliminaries in the online appendix.

Note the joint distribution of e_t and x_t is μ_t and the distribution of the shock η_{t+1} is ψ. Furthermore, since η_{t+1} is independent of e_t and x_t,

$$\begin{aligned}
\mu_{t+1}(B) &= \mathbb{E}\left[\mathbb{1}_{B_A}\{h_t(x_t, e_t)\} \times \mathbb{1}_{B_E}\{G(e_t, \eta_{t+1})\}\right] \\
&= \int \int \int \mathbb{1}_{B_A}\{h_t(a, e)\} \\
&\quad \times \mathbb{1}_{B_E}\{G(e, \eta)\}\mu_t(da, de)\psi(d\eta) \\
&= \int \int \mathbb{1}_{B_A}\{h_t(a, e)\} \\
&\quad \times \left[\int \mathbb{1}_{B_E}\{G(e, \eta)\}\psi(d\eta)\right]\mu_t(da, de) \\
&= \int \int \mathbb{1}_{B_A}\{h_t(a, e)\}Q(e, B_E)\mu_t(da, de)
\end{aligned}$$

the second equality uses the recursive formulation of the Markov process (Equation (2)). The final equality is the RHS of Equation (13), allowing us to conclude $\mu_{t+1} = \Phi(\mu_t, h_t)$.

Part 2: Show $(\mu_t, h_t)_{t=0}^{\infty}$ achieves the value function for the recursive problem

We have so far shown $(\mu_t, h_t)_{t=0}^{\infty}$ satisfies feasibility. Our next task is to show

$$\sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \geq \sum_{t=0}^{\infty} \beta^t u(\tilde{\mu}_t, \tilde{h}_t) \tag{38}$$

holds for any other sequence of feasible Borel probability measures and policy functions $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$.

As such, let $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$ be any other sequence of Borel probability measures on S and measurable policy functions $h_t\colon S \to A$ satisfying $\tilde{\mu}_0 = \mu_0$ and (15). Construct a sequence of A valued random variables $(\tilde{x}_t)_{t=0}^{\infty}$ by letting $\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t)$ for each $t > 0$ and with $\tilde{x}_0 = x_0$ given. The sequence of random variables $(\tilde{x}_t)_{t=0}^{\infty}$ will be defined on the probability space $(Z, \mathscr{B}(Z), P)$. By Claim 9.1 in the online appendix, $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$ for each t. Moreover, using an analogous argument to Claim 9.2 in the online appendix, each $\tilde{x}_t$ will have finite variance and hence $\tilde{x}_t \in L^2(Z, P)$ for each t.

Our strategy is to show $(\tilde{x}_t)_{t=0}^{\infty}$ is feasible for the sequential problem and show $u(\tilde{\mu}_t, \tilde{h}_t) = \rho_t(\tilde{x}_t, \tilde{x}_{t+1})$ for each t. The proof will then be complete since, noting that $(x_t)_{t=0}^{\infty}$ is a solution for the sequential problem, $u(\mu_t, h_t) = \rho_t(x_t, x_{t+1}) \geq \rho_t(\tilde{x}_t, \tilde{x}_{t+1}) = u(\tilde{\mu}_t, \tilde{h}_t)$ for each t.

To check $(\tilde{x}_t)_{t=0}^{\infty}$ is feasible for the sequential problem, first, let us check whether $(\tilde{x}_t)_{t=0}^{\infty}$ is adapted to the filtration $(\mathscr{F}_t)_{t=0}^{\infty}$. We proceed by induction; let $t = 1$ and consider:

$$\tilde{x}_1 = \tilde{h}_1(x_0, e_0)$$

Since $\tilde{h}_1$ is measurable, by the Doob-Dynkin Lemma (see Fact 8.3 in the online appendix), $\tilde{x}_1$ will be $\sigma(x_0, e_0)$ measurable. Now suppose x_t is $\sigma(x_0, e_0, \ldots, e_{t-1})$ measurable. Consider

$$\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t) = \tilde{h}_t(g(x_0, e_0, \ldots, e_{t-1}), e_t)$$

for some measurable function $g\colon A \times E^t \to A$. Once again, since $\tilde{h}_t$ is Borel measurable, using the Doob-Dynkin lemma, $\tilde{x}_{t+1}$ is $\sigma(x_0, e_0, \ldots, e_t)$ measurable. By the principle of induction, each $\tilde{x}_{t+1}$ is then $\sigma(\tilde{x}_0, e_0, e_1, \ldots, e_t)$ measurable and $(\tilde{x}_t)_{t=0}^{\infty}$ is adapted to the filtration $(\mathscr{F}_t)_{t=0}^{\infty}$.

To confirm $\tilde{x}_t \in \mathbb{S}_t$ for each t, we also need to confirm $\int \tilde{x}_t \, \mathrm{d}P \in [0, \bar{K}]$ for each t; note since $\tilde{\mu}_t \in \mathbb{M}$ and $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$,

$$\int \tilde{x}_t \, \mathrm{d}P = \int \int x \tilde{\mu}_t(dx, de) \in [0, \bar{K}] \tag{39}$$

Thus we conclude that $\tilde{x}_t \in \mathbb{S}_t$ for each t.

Now we turn to show the sequence $(\tilde{x}_t)_{t=0}^{\infty}$ satisfies $\tilde{x}_{t+1} \in \Gamma_t(\tilde{x}_t)$ for each t. Fix any $t \in \mathbb{N}$ and suppose $\int \tilde{x}_t \, \mathrm{d}P > 0$. We have

$$
\begin{aligned}
P\{\tilde{x}_{t+1} > (1+\tilde{r}(\tilde{x}_t))\tilde{x}_t + \tilde{w}(\tilde{x}_t)e_t\} &= P\{\tilde{h}_t(\tilde{x}_t, e_t) \\
&\quad > (1+\tilde{r}(\tilde{x}_t))\tilde{x}_t + \tilde{w}(\tilde{x}_t)e_t\} \\
&= \tilde{\mu}_t\{\tilde{h}_t(x,e) \\
&\quad > (1+r(\tilde{\mu}_t))x + w(\tilde{\mu}_t)e\} \\
&= 0
\end{aligned}
$$

The final equality holds because $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$ and because $\tilde{h}_t$ satisfies the feasibility condition (12) for $\tilde{\mu}_t$ - almost everywhere.

On the other hand, suppose $\int \tilde{x}_t \, \mathrm{d}P = 0$. We have $\int \tilde{x}_t \, \mathrm{d}P = \int \int x \tilde{\mu}_t(dx, de) = 0$. Since $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$ satisfies $\tilde{h}_t \in \Lambda(\tilde{\mu}_t)$ for each t, $\tilde{h}_t(x,e) = 0$ for $\tilde{\mu}_t$ almost everywhere. Whence,

$$
P\left\{\tilde{x}_{t+1} \neq 0\right\} = P\left\{\tilde{h}_t(x_t, e_t) \neq 0\right\} = \tilde{\mu}_t\left\{\tilde{h}_t(a,e) \neq 0\right\} = 0
$$

The first equality holds because we defined $\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t)$. The second inequality holds because $\tilde{\mu}_t$ is the joint distribution of $\{\tilde{x}_t, e_t\}$. As such, we can now conclude $(\tilde{x}_t)_{t=0}^{\infty}$ satisfies $\tilde{x}_{t+1} \in \Gamma_t(\tilde{x}_t)$ for each t.

To complete the proof, note for each t

$$
\begin{aligned}
u(\mu_t, h_t) &= \int v((1+r(\mu_t))a + w(\mu_t)e - h_t(a,e))\mu_t(da, de) \\
&= \int v((1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - h_t(x_t, e_t)) \, \mathrm{d}P \\
&= \int v((1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1}) \, \mathrm{d}P \\
&= \rho_t(x_t, x_{t+1})
\end{aligned}
\tag{40}
$$

And similarly, we have $u(\tilde{\mu}_t, \tilde{h}_t) = \rho_t(\tilde{x}_t, \tilde{x}_{t+1})$ for each t. As such, we conclude

$$
\sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \geq \sum_{t=0}^{\infty} \beta^t \rho_t(\tilde{x}_t, \tilde{x}_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(\tilde{\mu}_t, \tilde{h}_t) \tag{41}
$$

Where the inequality follows since $(x_t)_{t=0}^{\infty}$ is a solution to the sequential problem and its discounted sum of pay-offs dominate the discounted sum of pay-offs from $(\tilde{x}_t)_{t=0}^{\infty}$.

Finally, since any arbitrary feasible sequence $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$, with $\tilde{\mu}_0 = \mu_0$, satisfies (41), we have $V(\mu_0) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t)$. Moreover, since $(x_t)_{t=0}^{\infty}$ solves the sequential planner's problem, the first equality of (41) implies $V(\mu_0) = \tilde{V}(x_0)$.

Q.E.D.

Proofs for section 3

We will use $\mathbf{x}$ to refer to elements of $X^{\mathbb{N}}$. We can then use $(\mathbf{x}^n)_{n=0}^{\infty}$ to denote a sequence $\{\mathbf{x}^0, \ldots, \mathbf{x}^n, \ldots\}$, where $(\mathbf{x}^n)_{n=0}^{\infty} \in (X^{\mathbb{N}})^{\mathbb{N}}$.

REMARK 6.1 Let $X = \prod_{i \in F} X_i$ denote a Cartesian product of topological spaces. Let $\pi_i \colon X \to X_i$ denote projection maps defined as $\pi_i(x) = x_i$ for each $i \in F$. Recall each projection map will be a continuous function on X when X has the product topology. Accordingly, if X is sequentially compact in the product topology, and F is countable, then $\pi_i(X) = X_i$ will be compact.

Finally, let the function $\eta_t \colon \mathcal{G}^{t+1}(x) \to \mathbb{R}_+$ denote $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ for each t and let $U(\mathbf{x}) \colon = \sum_{t=0}^{\infty} \rho_t(x_t, x_{t+1})$.

The following lemmas will be used in the proof.

LEMMA 6.1 *Let assumption 3.2 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty}$ is a sequence with $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each n and $U(\boldsymbol{x}^n) \to B$ for $B > 0$, then there exists a sub-sequence $(\boldsymbol{x}^{n_k})_{k=0}^{\infty}$ such that for all $t \in \mathbb{N}$*

$$\lim_{k \to \infty} \rho_t(x_t^{n_k}, x_{t+1}^{n_k}) \to c_t$$

where $c_t \in \mathbb{R}_+$ for each t and $c_t > 0$ for at-least one t.

PROOF: By assumption 3.2, for each t and n,

$$m_t \geq \rho_t(x_t^n, x_{t+1}^n) \geq 0 \tag{42}$$

Accordingly, for each n, $(\rho_t(x_t^n, x_{t+1}^n))_{t=0}^{\infty}$ will belong to the set $\prod_{t=0}^{\infty} [0, m_t]$, which by Tychonoff's theorem will be compact in the product topology. There then exists a sub-sequence of $(\mathbf{x}^n)_{n=0}^{\infty}$, $(\mathbf{x}^{n_k})_{k=0}^{\infty}$, such that

$(\rho(x_t^{n_k}, x_{t+1}^{n_k}))_{k=0}^{\infty}$ converges for each t. Let $c_t = \lim_{k\to\infty} \rho(x_t^{n_k}, x_{t+1}^{n_k})$ and note

$$B = \lim_{k\to\infty} \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \rho_t \left(x_t^{n_k}, x_{t+1}^{n_k}\right) \tag{43}$$

$$= \lim_{T\to\infty} \sum_{t=0}^{T} \lim_{n\to\infty} \beta^t \rho_t \left(x_t^{n_k}, x_{t+1}^{n_k}\right) = \sum_{t=0}^{\infty} \beta^t c_t$$

Since (42) holds, and $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ by assumption 3.2, we can pass limits through in the second equality using dominated convergence theorem. If B is strictly positive, the above means there is at least one $c_t > 0$. *Q.E.D.*

LEMMA 6.2 *Let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty} \subset \mathcal{G}(x)$ and for some t*

$$\rho_t(x_t^n, x_{t+1}^n) \to c_t \tag{44}$$

with $c_t > 0$, then there exists an $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, $(x_i^n)_{i=0}^{t+1} \in UC_{\eta_t}(\epsilon)$

PROOF: There will exist ι small enough such that the term $\epsilon: = c_t - \iota$ will be strictly positive. For N large enough and any $n > N$, $\rho_t(x_t^n, x_{t+1}^n) \in [\epsilon, c_t + \iota]$, or $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$ and hence $(x_i^n)_{i=0}^{t+1} \in UC_{\eta_t}(\epsilon)$. *Q.E.D.*

LEMMA 6.3 *Let Assumption 3.2 and Assumption 3.1 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)$ is a sequence such that $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each $n \in \mathbb{N}$ and $U(\boldsymbol{x}^n) \to B$ where $B > 0$, then:*

1. *$(\boldsymbol{x}^n)_{n=0}^{\infty}$ has a convergent sub-sequence with a limit $\boldsymbol{x} \in \mathcal{G}(x)$, and*
2. *$U(\boldsymbol{x}) \geq B$.*

PROOF: Let x satisfy $x \in \mathbb{S}_0$ and let $(\mathbf{x}^n)_{n=0}^{\infty}$ be a sequence such that $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ where $B > 0$. By Lemma 6.1 we can find a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that for each $t \in \mathbb{N}$, $\rho_t(x_t^{n_j}, x_{t+1}^{n_j})$ converges to c_t and $c_t > 0$ for at-least one t. Re-label $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ to $(\mathbf{x}^n)_{n=0}^{\infty}$, and let P denote the subset of $\mathbb{N}$ such that $t \in P$ if and only if $c_t > 0$. The set P will be non-empty, but could be finite or infinite.

We will now prove part 1 of the Lemma, considering first the case when P is infinite and then the case when P is finite.

Suppose P is infinite and consider any $t \in \mathbb{N}$. There will exist $k > t$ such that $c_k > 0$. By Lemma 6.2, there will exist a natural number N and $\epsilon > 0$ such that for all $n > N$, we have $(x_i^n)_{i=0}^{k+1} \in UC_{\eta_k}(\epsilon)$.

By Assumption 3.1, $UC_{\eta_k}(\epsilon)$ will be sequentially compact in the product topology. The space $\pi_t(UC_{\eta_k}(\epsilon))$ will also be sequentially compact, by the argument in Remark 6.1. Define $\Xi_t \colon = \{x_1^0, \dots, x_t^N\} \cup \pi_t(UC_{\eta_k}(\epsilon))$. Since $\{x_1^0, \dots, x_t^N\}$ is sequentially compact, Ξ_t will be sequentially compact. Moreover, note $x_t^n \in \Xi_t$ for each n.

We can construct a Ξ_t as above for every $t \in \mathbb{N}$. Now let $\Xi \colon = \prod_{t \in \mathbb{N}} \Xi_t$. Using the Sequential Tychonoff Theorem (see Proposition 1.8.12 by Tao (2010)), Ξ will be sequentially compact. Since for each t, $x_t^n \in \Xi_t$ for each n, $\mathbf{x}^n \in \Xi$ for each n. There then exists a sub-sequence $(\mathbf{x}^{n_j})$ converging to $\mathbf{x}$, with $\mathbf{x} \in \Xi$.

We will now confirm $\mathbf{x}$ belongs to $\mathcal{G}(x)$ by showing $x_{t+1} \in \Gamma_t(x_t)$ for any $t \in \mathbb{N}$. Pick any $t \in \mathbb{N}$, there will be a k satisfying $k > t$ such that $c_k > 0$. By Lemma 6.2, there exists $\epsilon > 0$ and J such that for all $j > J$ we have $(x_i^{n_j})_{i=0}^{\leq k+1} \in UC_{\eta_k}(\epsilon)$. The upper-contour set $UC_{\eta_k}(\epsilon)$ will be sequentially compact, moreover, $UC_{\eta_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$ by the definition of $UC_{\eta_k}(\epsilon)$ at (26). As such, the sub-sequence $(x_i^{n_j})_{i=0}^{k+1}$ converges to $(x_i)_{i=0}^{k+1}$, with $(x_i)_{i \leq k+1} \in \mathcal{G}^{k+1}(x)$, allowing us to conclude $x_{t+1} \in \Gamma(x_t)$. Since the t we selected was an arbitrary $\mathbb{N}$, $x_{t+1} \in \Gamma_t$ for each t and $\mathbf{x}$ belongs to $\mathcal{G}(x)$.

Now assume P is finite; P will have a maximum element, which we now call k. By Lemma 6.2, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that $(x_t^n)_{t=0}^{k+1} \in UC_{\eta_k}(\epsilon)$ for each $n > N$. By Assumption 3.1 and Lemma 6.2, $UC_{\eta_k}(\epsilon)$ will be sequentially compact in the product topology. As such, there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that $(x_t^{n_j})_{j=0}^{\infty}$ converges point-wise for each $t \leq k+1$. Define $(x_t)_{t=0}^{\infty}$ by setting $x_t = \lim_{j \to \infty} x_t^{n_j}$ for $t \leq k+1$ and picking any $x_{t+1} \in \Gamma_t(x_t)$ for $t \geq k+1$.

Now we confirm $(x_t)_{t=0}^{\infty}$ is feasible by confirming $x_{t+1} \in \Gamma_t(x_t)$ for each t. Once again, note by definition, $UC_{\eta_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$. Since $UC_{\eta_k}(\epsilon)$ is sequentially compact, $(x_t)_{t=0}^{k+1} \in \mathcal{G}(x)$ and $x_{t+1} \in \Gamma_t(x_t)$ for all t satisfying $t \leq k$. On the other hand, if $t > k$, by construction, $x_{t+1} \in \Gamma_t(x_t)$, confirming $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$.

To prove part 2 of the Lemma, by Assumption 3.2,

$$\rho_t(x_t^n, x_{t+1}^n) \leq m_t$$

for each t and n, where $\sum_{t=0}^{\infty} \beta^t m_t < \infty$. Whence the Fatou's Lemma[15] gives

$$B = \limsup_{n\to\infty} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{\infty} \limsup_{n\to\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) < \infty \tag{45}$$

Furthermore, upper-semicontinuity of ρ_t and Assumption 3.2 implies

$$\limsup_{n\to\infty} \rho_t(x_t^n, x_{t+1}^n) \leq \rho_t(x_t, x_{t+1}) \leq m_t \tag{46}$$

To complete the proof, combine (46) with (45) to conclude

$$B \leq \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) < \infty$$

Q.E.D.

PROOF OF THEOREM 3.1: Fix $x \in \mathbb{S}_0$. If $U(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{G}(x)$, then our solution will be any $\mathbf{x} \in \mathcal{G}(x)$.

Next, suppose we know of at-least one $\mathbf{x} \in \mathcal{G}(x)$ such that $U(\mathbf{x}) > 0$. Assumption 3.2 tells us there exists a sequence of real numbers $(m_t)_{t=0}^{\infty}$, with the sequence $(\beta^t m_t)_{t=0}^{\infty}$ summable, such that $\rho_t\left(x_t, x_{t+1}\right) \leq m_t$ for any $\mathbf{x}$ in $\mathcal{G}(x)$. As such, let $B = \sum_{t=0}^{\infty} \beta^t m_t$, and note $B < \infty$. Any $\mathbf{x}$ in $\mathcal{G}(x)$ will satisfy

$$U\left(\mathbf{x}\right) = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \rho_t\left(x_t, x_{t+1}\right) \leq B$$

Consider the set $I := \{U\left(\mathbf{x}\right), |\mathbf{x} \in \mathcal{G}\left(x\right)\}$; I will be a subset of $\mathbb{R} \cup \{-\infty, \infty\}$, and so must have a supremum. Moreover, the supremum of I must be less than or equal to the real number B and strictly greater than 0. Now use B to denote the supremum of I.

We can construct a sequence $\{\mathbf{x}^0 \dots \mathbf{x}^n \dots\} \subset \mathcal{G}\left(x\right)$ with $U\left(\mathbf{x}^n\right) \to B$: for every $n \in \mathbb{N}$, take $(\mathbf{x}^n)_{n=0}^{\infty}$ such that $B - U\left(\mathbf{x}^n\right) < \frac{1}{n+1}$. Such a sequence exists, otherwise for some n, $U\left(\mathbf{x}\right) \leq B - \frac{1}{n+1}$ for all $\mathbf{x} \in \mathcal{G}\left(x\right)$ and B will

[15] See (1.1) in Kamihigashi (2017) and let $\Omega = \mathbb{Z}_+$ and μ be the counting measure.

not be the least upper-bound of I. [16] We will now denote $\{\mathbf{x}^0 \dots \mathbf{x}^n \dots\}$ as $(\mathbf{x}^n)_{n=0}^{\infty}$.

By Lemma 6.3, there will exist $\mathbf{x} \in \mathcal{G}(x)$ such that $U(\mathbf{x}) \geq B$; since B was the supremum for I, we can deduce

$$U(\mathbf{x}) = B = \sup_{(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = \tilde{V}(x) \tag{47}$$

Q.E.D.

PROOF OF PROPOSITION 3.1: Before beginning, let $\eta_t \colon \mathcal{G}^{t+1}(x) \to \mathbb{R}_+$ denote $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ for each t.

Let x satisfy $x \in S_0$. Pick any $t \in \mathbb{N}$ and any ϵ satisfying $\epsilon > 0$. We will show the upper contour sets UC_{η_t} defined by

$$UC_{\eta_t} = \{(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x) \mid \rho_t(x_t, x_{t+1}) \geq \epsilon\} \tag{48}$$

are sequentially compact. In particular, we will show UC_{η_t} is a sequentially closed sub-set of $\mathbb{X}^{t+1}$ and then show UC_{η_t} is contained within a compact and metrizable set.

To show UC_{η_t} is closed in $\mathbb{X}^{t+1}$, take any sequence $(\mathbf{x}^n)$ with $\mathbf{x}^n \in UC_{\eta_t}$ for each n that converges to $\mathbf{x} = (x_i)_{i=0}^{t+1}$ point-wise for each i. Note $x_i^n \in \mathbb{S}_i$ for each $i \leq t+1$ and n. Since each $\mathbb{S}_t$ is closed, we must have $x_i \in \mathbb{S}_i$.

By assumption 3.5, each Γ_i will have a closed graph, and thus x_{i+1} will be in $\Gamma_i(x_i)$ for each $i \leq t+1$. Noting the definition of $\mathcal{G}(x)^{t+1}$ in Equation (27), we can conclude $\mathbf{x} \in \mathcal{G}(x)^{t+1}$.

We now need to confirm $\rho_t(x_t, x_{t+1}) \geq \epsilon$. Note $UC_{\rho_t} = \{(x, y) \in \operatorname{Gr}\Gamma_t \mid \rho_t(x, y) \geq \epsilon\}$ is sequentially closed by upper semi-continuity of ρ_t. The time t pay-off from the sequence $(x_i^n)_{i=0}^{t+1}$ will satisfy $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$, and thus $\{x_t^n, x_{t+1}^n\} \in UC_{\rho_t}$ for each n. Moreover, $x_{t+1} \in \Gamma_t(x_t)$. Accordingly, by upper semi-continuity of ρ_t, $\{x_t, x_{t+1}\} \in UC_{\rho_t}$, and we conclude $\rho_t(x_t, x_{t+1}) \geq \epsilon$.

Since $\mathscr{E}$ satisfies Assumption 3.6, there will exist $\bar{M}$ such that if $(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$, then $\|x_i\| \leq \bar{M}$ for $i \in \{0, \dots, t+1\}$. Whence

[16] If there exists x such that $U(x) = B$, then our sequence x^n is made, however we are not sure yet that such a maximiser exists.

UC_{η_t} will be a sub-set of the space $B_{\bar{M}}: = \{x \mid \|x\| \leq \bar{M}\}^{t+1}$, which will be compact by Alaoglu's Theorem. Since the weak topology is metrizable on bounded subsets of L^2, $B_{\bar{M}}$ will be sequentially compact (see discussion on topology in the online appendix). Finally, by the argument above, UC_{η_t} will be a closed sub-set of $B_{\bar{M}}$, allowing us to conclude UC_{η_t} is sequentially compact.

Q.E.D.

Proofs for Section 4

Recall point-wise inequalities in $\mathbb{X}$ hold P - almost everywhere and convergence of (x^n) with $x^n \in \mathbb{X}$ for each n will be with respect to the weak topology.

PROOF OF PROPOSITION 4.1: Let x satisfy $x \in \mathbb{S}_0$. By Assumption 2.4, aggregate capital is bounded above by $\bar{K}$. As such, for any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ and for any t, we can use Jensen's inequality (Fact 8.4 in the online appendix) to arrive at

$$\rho_t(x_t, x_{t+1}) \\ \leq \nu\left((1+\tilde{r}(x_t))\tilde{K}(x_t) + \tilde{w}(x_t)L\right) \leq \nu(F(\bar{K}, L) + (1-\delta)\bar{K})$$

where the second inequality follows from homogeneity of degree one of the production function.

Let $m_t: = \nu(F(\bar{K}, L) + (1-\delta)\bar{K})$ for all t. As such, for any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$,

$$\rho_t(x_t, x_{t+1}) \leq m_t$$

for all t. Since m_t is a constant, the sequence $(m_t)_{t=0}^{\infty}$ will satisfy $\sum_{t=0}^{\infty} \beta^t m_t < \infty$.

Q.E.D.

CLAIM 6.2 *Let (Z, Σ, μ) be a finite measure space. Let D be a convex subset of $\mathbb{R}$ and consider a function $g\colon D \to \mathbb{R}$. Define $G\colon L^2(Z, \mu) \to \mathbb{R}$ as*

$$G(s) = \int g(s)\, d\mu, \qquad s \in L^2(Z, \mu)$$

If g is concave and upper semicontinuous, then G will be weak sequentially upper semicontinuous.

PROOF: The functional G will be upper semicontinuous with respect to norm convergence in $L^1(Z,\mu)$ (Proposition 6.3.1 by Borwein and Vanderwerff (2010)). Moreover, G will be concave. Since, by Holder's inequality, norm convergence in $L^2(Z,\mu)$ implies norm convergence in $L^1(Z,\mu)$, G will be upper semicontinuous with respect to norm convergence in $L^2(Z,\mu)$. As such, the upper-contour sets of G will be convex (Fact 2.1.7, Borwein and Vanderwerff (2010)) and norm-closed. Since norm closed convex sets in $L^2(Z,\mu)$ are also weakly closed (an implication of Mazur's Lemma, see Theorem 1.3 in Amar, Afif Ben, O'Regan (2016)),[17] the upper contour sets of G will be weakly closed. Moreover, since the upper contour of G sets are convex, they will be weak sequentially closed (an implication of the Krein-Smulian Thoerem, Corollary 2.7.13 by Megginson (1998)). We can now conclude G will be weak sequentially upper semicontinuous.

Q.E.D.

PROOF OF PROPOSITION 4.2: Recall the definition of sequential upper semicontinuity from section 8.2 in the online appendix.

Set any $t \in \mathbb{N}$ and consider sequences $(x^n)_{n=0}^{\infty}$ and $(y^n)_{n=0}^{\infty}$ with $\{x^n, y^n\} \in \operatorname{Gr}\Gamma_t$ for each n. Let $x^n \to x$ and $y^n \to y$ weakly with $y \in \Gamma_t(x)$. To verify sequential upper semicontinuity, we will show

$$\begin{aligned}\limsup_{n\to\infty} \rho_t(x^n, y^n) &= \limsup_{n\to\infty} \int \nu((1+\tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n)\, \mathrm{d}P \\ &\leq \rho_t(x,y)\end{aligned} \tag{49}$$

By Claim 6.2, the mapping $s \mapsto \int \nu(s)\, \mathrm{d}P$ for $s \in L^2(Z,P)$ will be sequentially upper semicontinuous since ν is concave and upper semicontinuous (Assumption 2.5). As such, for any sequence in $L^2(Z,\mu)$ satisfying $f^n \to f$ weakly, we have

$$\limsup_{n\to\infty} \int v(f^n)\, \mathrm{d}P \leq \int v(f)\, \mathrm{d}P \tag{50}$$

Let $f^n \colon = (1+\tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n$ and note $f^n \in L^2(Z,P)$ for each n.

[17] Also see Corrolary 5.99 in Aliprantis and Border (2005) and note the norm and weak topologies are consistent since $L^2(Z,\mu)$ is reflexive.

First, we show (49) for the case $\int x \, \mathrm{d}P > 0$. If $\int x \, \mathrm{d}P > 0$, then f^n converges weakly to $f := (1 + \tilde{r}(x))x + \tilde{w}(x)e_t - y$, implying by (50),

$$\limsup_{n \to \infty} \int \nu(f^n) \, \mathrm{d}P \leq \int \nu(f^n) \, \mathrm{d}P = \rho_t(x, y)$$

On the other hand, if $\int x \, \mathrm{d}P = 0$, then

$$\begin{aligned}
\limsup_{n \to \infty} \int \nu(f^n) \, \mathrm{d}P &\leq \limsup_{n \to \infty} \nu \left(\int (1 + \tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n \, \mathrm{d}P \right) \\
&\leq \lim_{n \to \infty} \nu(F(\tilde{K}(x^n), L) + (1 - \delta)\tilde{K}(x^n)) \\
&= 0 = \rho_t(x, y)
\end{aligned}$$

where the first inequality follows from Jensen's inequality. The second inequality follows from Assumption 2.3 on homogeneity of the production function (recall Equation (11)).

Q.E.D.

PROOF OF PROPOSITION 4.3: The space $\{x \in \mathbb{X} \,|\, x \geq 0\}$ is norm closed and convex, thus by Mazur's Lemma, $\{x \in \mathbb{X} \,|\, x \geq 0\}$ will also be weakly closed (Theorem 1.3 in Amar, Afif Ben, O'Regan (2016)). Moreover, since $L^2(Z, P)$ is a reflexive space, weakly closed convex sets will be weak sequentially closed (Corollary 2.7.13 in Megginson (1998)). Now take any $x^n \in \mathbb{S}_t$ with x^n converging weakly in $\mathbb{X}$, we have $\mathbb{S}_t \subset \{x \in \mathbb{X} \,|\, x \geq 0\}$, thus $x \in \{x \in \mathbb{X} \,|\, x \geq 0\}$.

Suppose by contradiction $x \notin \mathbb{S}_t$, then $\int x > \bar{K}$. However, since x^n converges weakly; we have $\int x^n \, \mathrm{d}P \to \int x \, \mathrm{d}P > \bar{K}$, yielding a contradiction since we have $x^n \notin \mathbb{S}_t$ for some n. We then conclude $x \in \mathbb{S}_t$ and $\mathbb{S}_t$ is weak sequentially closed.

Q.E.D.

PROOF OF PROPOSITION 4.4: Set t and suppose $(x^n, y^n)_{n=0}^{\infty}$ satisfies $y^n \in \Gamma_t(x^n)$ for each n. Suppose $(x^n)_{n=0}^{\infty}$ converges to $x \in \mathbb{S}_0$ and $(y^n)_{n=0}^{\infty}$ converges to $y \in \mathbb{S}_1$. We will show $y \in \Gamma_t(x)$.

We have either $\int x \, \mathrm{d}P = 0$ or $\int x \, \mathrm{d}P > 0$. First let $\int x \, \mathrm{d}P > 0$, we wish to show $y \leq (1 + r(x))x + w(x)e_t$ for P-almost everywhere. Suppose by contradiction

$$P\{y > (1 + r(x))x + w(x)e_t\} > 0$$

Let $B\colon = \{z \in Z | y(z) > (1 + r(x))x(z) + w(x)e_t(z)\}$, we will have $P(B) > 0$. Moreover,

$$\int \mathbb{1}_B y \, \mathrm{d}P > \int \mathbb{1}_B \times [(1 + r(x))x + w(x)e_t] \, \mathrm{d}P \tag{51}$$

Next, note $\tilde{r}(x^n) = F_1(\tilde{K}(x^n), L)$ will converge, implying

$$\begin{aligned}(1 + \tilde{r}(x^n)) \int x^n h \, \mathrm{d}P + \tilde{w}(x^n) \int h e_t \, \mathrm{d}P \\ \to (1 + \tilde{r}(x)) \int x h \, \mathrm{d}P + \tilde{w}(x) \int h e_t \, \mathrm{d}P\end{aligned} \tag{52}$$

for any function h satisfying $h \in L^2(Z, P)$. In particular, letting $h = \mathbb{1}_B$, and noting $y^n \in \Gamma_t(x^n)$, by (20), we will have

$$\int \mathbb{1}_B y^n \, \mathrm{d}P \leq (1 + \tilde{r}(x^n)) \int \mathbb{1}_B x^n \, \mathrm{d}P + \tilde{w}(x^n) \int \mathbb{1}_B e_t \, \mathrm{d}P$$

for each n. Since the weak inequality above will be preserved under the limits of real-valued sequences, we arrive at

$$\int \mathbb{1}_B y \, \mathrm{d}P \leq (1 + \tilde{r}(x)) \int \mathbb{1}_B x \, \mathrm{d}P + \tilde{w}(x) \int \mathbb{1}_B e_t \, \mathrm{d}P \tag{53}$$

However, (53) is a contradiction to (51), allowing us to conclude

$$y \leq (1 + \tilde{r}(x))x + \tilde{w}(x)e_t$$

Now suppose $\int x^n \, \mathrm{d}P \to 0$. Note

$$\int y^n \, \mathrm{d}P \leq \int (1 + \tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t \, \mathrm{d}P = F(\tilde{K}(x^n), L) + (1 - \delta)\tilde{K}(x^n)$$

The above equality follows from homogeneity of degree one of the production function (see Equation (54) below). Since $\tilde{K}(x^n) = \int x^n \, \mathrm{d}P \to 0$, we have $F(\tilde{K}(x^n), L) \to 0$ by Assumption 2.3, and

$$0 = \lim_{n \to \infty} \int y^n = \int y \, \mathrm{d}P$$

Since $y \geq 0$, we must have $y = 0$ for P-almost everywhere, allowing us to conclude $y \in \Gamma_t(x)$.

Q.E.D.

LEMMA 6.4 *Consider the setting and notation of the sequential planner's problem in section 4. Fix x with $x \in \mathbb{S}_0$, an $\epsilon > 0$ and $t \in \mathbb{N}$. If assumptions 2.1 - 2.3 hold, then there exists $\bar{r} \in \mathbb{R}_+$ such that for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have $\tilde{r}(x_i) \leq \bar{r}$ for each $i \leq t$.*

PROOF: Fix any $x \in \mathbb{S}_0$, $\epsilon > 0$ and $t \in \mathbb{N}$ and select $(x_i)_{i=0}^{\infty}$ satisfying $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$.

Since $x_i \in \Gamma_{i-1}(x_{i-1})$, by the feasibility correspondence (Equation (20)) and homogeneity of degree one (Assumption 2.3) of the production function F, we have

$$
\begin{aligned}
\tilde{K}(x_i) = \int x_i \, \mathrm{d}P &\leq \int (1 + \tilde{r}(x_{i-1}))x_{i-1} + \tilde{w}(x_{i-1})e_{i-1} \, \mathrm{d}P \\
&= (1 + F_1(\tilde{K}(x_{i-1}), L) - \delta)\tilde{K}(x_{i-1}) \\
&\quad + F_2(\tilde{K}(x_{i-1}), L)L \\
&= F(\tilde{K}(x_{i-1}), L) + (1 - \delta)\tilde{K}(x_{i-1})
\end{aligned} \tag{54}
$$

for each $i \in \mathbb{N}$.

Define $\hat{F}(K) := F(K, L) + (1 - \delta)K$. By (54), we can write

$$
\tilde{K}(x_i) \leq \hat{F}(\tilde{K}(x_{i-1})), \qquad i \in \mathbb{N} \tag{55}
$$

Since $(\rho_t)_{t=0}^{\infty}$ is concave, from Jensen's inequality,

$$
\begin{aligned}
\epsilon \leq \rho_t(x_t, x_{t+1}) &= \int \nu \left((1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1} \right) \, \mathrm{d}P \\
&\leq \nu \left(\int (1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t \, dP \right) \\
&= \nu(\hat{F}(\tilde{K}(x_t)))
\end{aligned} \tag{56}
$$

Note the inverse of v, v^{-1}, is also increasing since v is increasing. (The inverse of ν exists by Assumption 2.5.) From Equation (56), $v^{-1}(\epsilon) \leq \hat{F}(\tilde{K}(x_t))$. And, by (55), since $\hat{F}$ is increasing, we arrive at

$$
v^{-1}(\epsilon) \leq \hat{F}(\tilde{K}(x_t)) \leq \hat{F}^2(\tilde{K}(x_{t-1}))
$$

Moreover, for any i satisfying $i \leq t$,

$$
\nu^{-1}(\epsilon) \leq \hat{F}(\tilde{K}(x_t)) \leq F^{t-i+1}(\tilde{K}(x_i)) \tag{57}
$$

Next, Let G^j denote the inverse of $\hat{F}^j$. Since $\hat{F}$ is strictly increasing, by (57), we have $\tilde{K}(x_i) \geq G^{t-i+1}(\nu^{-1}(\epsilon))$ for each $i \leq t$. Define

$$\underline{K} \colon = \min_{i \leq t} \{G^{t-i+1}(\nu^{-1}(\epsilon))\}$$

and note $\tilde{K}(x_i) \geq \underline{K}$ for each $i \leq t$.

Finally, define $\bar{r} \colon = F_1(\underline{K}, L) - \delta$. Note $F_1(K, L)$ is decreasing in the first argument since F is concave. We can conclude

$$\tilde{r}(x_i) = F_1(\tilde{K}(x_i), L) - \delta \leq F_1(\underline{K}, L) - \delta \colon = \bar{r}, \qquad \forall\, i \leq t$$

Since $\bar{r}$ depends only on t and ϵ, the above will hold for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$. *Q.E.D.*

PROOF OF PROPOSITION 4.5: Fix any x satisfying $x \in \mathbb{S}_0$, $\epsilon > 0$ and t. Suppose $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfies $\rho(x_t, x_{t+1}) \geq \epsilon$. By Lemma 6.4, for any $i \leq t$, we know $r(x_i) \leq \bar{r}$. Since aggregate capital will be bounded from above, the maximum possible wage rate will be bounded above by a constant, which we now denote as $\bar{w}$. For all $i \leq t+1$, we have

$$\begin{aligned}
\underline{a} \leq x_i &\leq (1+\bar{r})x_{i-1} + \bar{w}e_i \\
&\leq (1+\bar{r})^2 x_{i-2} + \bar{w}e_i + (1+\bar{r})\bar{w}e_{i-1} \\
&\leq (1+\bar{r})^i x_0 + \bar{w}\sum_{j=1}^{i}(1+\bar{r})^{i-j} e_{i-j}
\end{aligned}$$

Let $W \colon = \bar{w}\sum_{j=1}^{i}(1+\bar{r})^{i-j}e_{i-j}$ to simplify notation and note $\|W\|$ will be finite. Next since $x_i \geq 0$,

$$x_i \leq (1+\bar{r})^i x_0 + W \implies (x_i)^2 \leq \Big((1+\bar{r})^i x_0 + W\Big)^2 \tag{58}$$

As such, letting $M = \|x_0\|$, we can write

$$\begin{aligned}
\|x_i\| &\leq \|(1+\bar{r})^i x_0 + W\| \\
&\leq (1+\bar{r})^i \|x_0\| + \|W\| \\
&\leq (1+\bar{r})^i M + \|W\| \\
&\colon = \hat{M}_i \in \mathbb{R}
\end{aligned}$$

There will be such a $\hat{M}_i$ for each $i \leq t+1$. Moreover, the constants M_i will depend only on x, t and ϵ and so the above bound will hold for any feasible sequence $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho(x_t, x_{t+1}) \geq \epsilon$.

Q.E.D.

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