EXISTENCE OF RECURSIVE CONSTRAINED OPTIMA IN THE HETEROGENEOUS AGENT NEOCLASSICAL GROWTH MODEL

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Recently, macroeconomists have begun to study optimal policy in heterogeneous agent models with incomplete markets, or Aiyagari-Huggett models. A natural way to formulate an optimal policy problem in the model is by using a constrained planner (Dávila et al., 2012), a planner who cannot complete markets but must improve welfare subject to agents' budget constraints. Despite the relevance of the constrained planner to understanding optimal macroeconomic policy, existence of constrained optima has not been verified. This paper proves existence of recursive constrained optima in the standard Aiyagari-Huggett model. A key challenge the proof overcomes is non-compactness of feasibility correspondences in the constrained planner's dynamic optimisation problem, brought on by the infinite dimensional structure of the Aiyagari-Huggett model.

KEYWORDS: Neoclassical Growth Models, Incomplete Markets, Heterogeneous Agent, Constrained Planner, Dynamic Optimisation, Existence Result, Recursive Policies, Infinite Dimensional State-Space.

1. INTRODUCTION

It is well-known that consumers facing incomplete insurance markets and individual income risk accumulate more capital than in an economy with complete markets (Aiyagari, 1994, 1995; Huggett, 1997). And because the assumptions are

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Please download the latest version of the working paper and the online appendix from `https://github.com/akshayshanker/Existence_of_Social_Optimia_Aiyagari` more realistic, models with incomplete markets and heterogeneous agents, also known as Aiyagari-Huggett models, have been used extensively for positive analysis of consumption dynamics, monetary and fiscal policy (some examples include Heathcote (2005); Berger and Vavra (2015); Kaplan and Violante (2010); Kaplan et al. (2016); McKay and Reis (2016)). Only recently have macroeconomists turned attention to normative questions about the *optimal level* of capital in incomplete market economies. To study optimal policy, Dávila et al. (2012), Nuño and Moll (2017) and Park (2017) use a constrained planner, a utilitarian planner who can only adjust agents' saving and consumption decisions subject to agents' budget constraints, but cannot complete insurance markets or transfer wealth. The surprising conclusion reached by Dávila et al. (2012) is that consumers may under-accumulate capital compared to a constrained planner; when the labour share of income for the poor is high enough, the constrained planner accumulates up to three times more capital than decentralized consumers.

However, despite the relevance of the constrained planner for optimal policy analysis in heterogeneous agent models, existence of constrained optima has not been verified. This paper provides a proof for the existence of discrete time recursive constrained optimal policies in the standard Aiyagari-Huggett model.

The existence result helps confirm the policy conclusions from the study of constrained optima are sound. In the Aiyagari-Huggett and other heterogeneous agent models, the state-space is a distribution of agents. This means the state and control in the constrained planner's dynamic problem is, in general, infinite dimensional — the planner controls a policy function to guide a wealth distribution through time. And we know from our study of topology that verifying the conditions for existence in infinite dimensional spaces is more difficult (see Aliprantis and Border (2006), ch.6 and Mas-colell and Zame (1991)). Indeed, the constrained planner's problem fails to satisfy one of the key ingredients in standard dynamic optimization existence theory, compactness. Moreover, Dávila et al. (2012) compute constrained planner solution paths that may not converge to a steady-state, but display ever increasing wealth inequality. Verifying existence helps confirm such computed solutions are not pathological and creates a foundation for further research on optimal policy in models with wealth distributions and heterogeneity.

1.0.1. *Mathematical Challenges*

The literature has made significant progress by establishing infinite dimensional necessary conditions (Dávila et al. (2012) in discrete time and Nuño and Moll (2017) and Nuño (2017) in continuous time). However, continuity and compactness, assumptions used by standard dynamic optimisation theory to show existence of solutions, are more difficult to verify when spaces are infinite dimensional (see Mas-colell and Zame (1991) for an overview of issues in infinite dimensional topology). In the case of the constrained planner, the feasibility correspondence fails to have compact image sets, that is, the image of a compact set under the correspondence will not be compact. The standard assumptions of existing dynamic optimisation theory (Stokey and Lucas (1989), Acemoglu (2009) ch.6 or Stachurski (2009)) are thus not satisfied.

The constrained planner's feasibility correspondences fail to have compact image sets for two reasons. First, as suggested by Dávila et al. (2012), individual agents' asset spaces will not be bounded. We are also unable to justify restrictions such as equicontinuity or monotonicity on the space of policy functions. As such, the image sets of the feasibility correspondences will not be compact in the sup-norm topology or topology of point-wise convergence. At the same time, the recursive problem, the form of the problem considered by Dávila et al. (2012), will not be defined on topological spaces where the feasibility correspondence is compact-valued.

Second, the feasibility correspondences have non-compact image sets because of a discontinuity. The discontinuity arises due to the Inada conditions — as capital converges to zero, interest rates diverge and the variance of feasible asset distributions can diverge to infinity as the mean converges to zero.

To resolve the first challenge, the paper transforms the recursive problem to a sequential problem and uses a projection argument to show sequential solutions imply recursive solutions. The sequential planner's problem will be well-defined on the space of square integrable random variables. And with the weak topology, the sequential planner's feasibility correspondences will be compact valued.

While feasibility correspondences for the sequential planner have compact values, due to the discontinuity around zero capital, image sets will still be non-compact. To resolve this second challenge, the paper builds on and generalises recent work on the theory of non-compact optimisation (Feinberg et al., 2012, 2013) and introduces an existence result for infinite horizon dynamic optimisation with non-compact feasible correspondences. The main assumption of the result can be verified by checking the variance of feasible asset distributions leading to a strictly positive per-period pay-off at any time in the future is bounded.

1.0.2. *Related Literature*

Pathologies similar to the second problem discussed above are encountered in existence proofs of general equilibrium in the Aiyagari model with aggregate shocks (Krusell-Smith models), as discussed in detail by Cao (2016). The solution proposed by Cao (2016) involves solving a sequence of finite horizon problems and showing aggregate capital has a strictly positive lower bound using agents' Euler equations. Instead of verifying a lower-bound for capital,[1] the approach of this paper is to state a general theorem for the infinite horizon dynamic optimization problem on non-compact spaces.

The Aiyagari-Huggett constrained planner problem is not the only dynamic optimisation problem with infinite dimensional state-spaces. A large literature (Boucekkine et al., 2009; Brock et al., 2014; Fabbri et al., 2015) has shown existence and characterised optimal solutions in models of economic geography in continuous time. Lucas and Moll (2014) also solve an infinite dimensional planner's problem to control individual search efforts subject to the law of motion of a density. However, these models do not encounter the non-compactness of the Aiyagari-Huggett model.

Other macroeconomic models have infinite dimensional states, but with simplifying assumptions, the dynamics of the distribution may only depend on finite dimensional variable. For example, in the industry dynamics model by Hopenhayn (1992), the planner can control total demand (see p. 1134), in the growth model with financial frictions by Itskhoki and Moll (2014), the planner can control aggregate consumption and in the incomplete markets model with endogenous growth by Brunnermeier and Sannikov (2016), the constrained planner can control a common investment rate across heterogeneous households.

However, the methodology of this paper is directly relevant to the growing literature on Ramsey constrained planner problems in heterogeneous agent models (Acikgoz, 2013; Bhandari et al., 2017; Chen et al., 2017; Nuno and Thomas, 2017; Park, 2014) and the future study of constrained planner problems in Aiyagari-Huggett models incorporating aggregate shocks (Den Haan, 1996; Krusell and Smith, 1998), permanent income shocks Kuhn (2013), endogenous labour supply (Marcet et al., 2007), overlapping generations (Heathcote et al., 2010) or monetary and fiscal policy (Kaplan et al., 2016; Heathcote, 2005; McKay and Reis, 2016).

[1] As we are concerned with existence of optima as opposed to equilibria, we cannot use the Euler conditions, which in the case of the constrained planner have only been shown to be necessary, to restrict the search for an optimiser since optima may not exist.

2. CONSTRAINED PLANNER PROBLEMS

This section presents the recursive and sequential constrained planner's problems in a standard Aiyagari (1994) model. Both Dávila et al. (2012) and Nuño and Moll (2017) formulate their problem as a recursive problem; the exposition here will follow the discrete time version in Dávila et al. (2012), only I place more formal mathematical structure on the model.

In the recursive problem, the constrained planner instructs agents on their next period assets based on their current assets, shock and the aggregate distribution of agents. The recursive problem will be a stationary primitive form infinite horizon dynamic optimisation problem, where the planner selects an action (policy function) to drive a state (wealth distribution). (The distinction between primitive form and reduced form problem is discussed by Sorger (2015), section 5.1.)

In the sequential problem, the constrained planner instructs agents each period on next period assets based on their history of shocks up to the period. The sequential problem will be a non-stationary reduced form infinite horizon dynamic optimisation problem, where the planner selects a sequence of states (random variables).[2]

The online appendix contains an overview of mathematical concepts used in this paper.

2.1. *The Aiyagari-Huggett Model*

Time is discrete and indexed by $t \in \mathbb{N}$. Let $(I, \mathcal{I}, \zeta)$ be an atom-less probability space indexing agents. Let A, with $A := \mathbb{R}_+$, be the agents' asset space[3] and define

[2]In the context of a constrained planner, the terminology 'sequential' and 'recursive' problems is overloaded. The distinction here follows the distinction between 'sequential competitive equilibria' and 'recursive competitive equilibria' made by Miao (2006) and Cao (2016). In contrast to the distinction made here, the term sequential problem is often used to refer to the problem maximising the infinite sum of pay-offs as opposed to the Bellman Operator representation of the same problem. For infinite dimensional and stochastic problems, both sequential and recursive formulations can be written as a deterministic sequence problem (maximising the sum of discounted pay-offs) and using a deterministic Bellman Equation. For example, (11) compared below to (F.23) in the online appendix. This paper uses the term *sequence problem* to refer to a problem such as (11).

[3]As in the computations by Dávila et al. (2012), I assume a zero lower bound on assets to simplify the notation. In general, the Aiyagari model allows a strictly negative lower bound, however a zero lower bound is a common assumption, see also Miao (2006) and Cao (2016). The results here can be extended to a model with a negative lower bound, however an additional constraint on the state-space to ensure interest rates are not so high as to violate budget constraints will need to be added.

E as the agents' labour endowment space. Assume E is a closed subset of $\mathbb{R}_+$. Let S, where $S := A \times E$, denote the agents' state space.

We do not need further assumptions, such as boundedness, on E for the proofs in this paper. However, computations by Dávila et al. (2012) and Nuño and Moll (2017) show a solution with diverging variance, implying a sequence of asset distributions with an increasing upper-bound (see fig.3 and discussion at section 5.4 by Dávila et al. (2012)).[4]As such, A will be unbounded above, even if E is bounded.

At time zero, each agent i, with $i \in I$, draws an initial asset level x_0^i, with x_0^i taking values in A. Each agent also receives a sequence of labour endowment shocks $(e_t^i)_{t=0}^{\infty}$, with e_t^i taking values in E for each t and i. Let P denote the probability law or joint distribution of the sequence of shocks, common across i. Assume all shocks are defined on a common probability space $(\bar{\Omega}, \Sigma, \bar{\mathbb{P}})$, that is, x_0^i and $(e_t^i)_{t=0}^{\infty}$ for each i are random variables defined on $(\bar{\Omega}, \Sigma, \bar{\mathbb{P}})$.

ASSUMPTION 2.1 The shocks satisfy the following conditions:

1. for each i, the shocks $(e_t^i)_{t=0}^{\infty}$ are a stationary Markov process with common Markov kernel Q and stationary marginal distribution ψ
2. for each t and i, e_t^i and x_0^i has finite variance
3. for each i, x_0^i is independent of $(e_t^i)_{t=0}^{\infty}$.

Part 1 of Assumption 2.1 can be relaxed to boundedness of the mean of the endowment shock, however, the stationarity assumption simplifies notation. The finite variance assumption allows us to work in the L^2 space of square integrable random variables where compact sets are easier to find.

Let $Z := A \times E^{\mathbb{N}}$ and let $\mathscr{B}(Z)$ be the Borel sets of Z. For each i, let $z^i \colon \bar{\Omega} \to Z$ denote the map $\omega \mapsto \{x_0^i(\omega), e_0^i(\omega), e_1^i(\omega), \dots\}$, where $\omega \in \bar{\Omega}$. The following assumption formalises no aggregate uncertainty:

ASSUMPTION 2.2 For any $B \in \mathscr{B}(Z)$,

$$\zeta\{i \in I \,|\, z^i(\omega) \in B\} = P(B), \qquad \bar{\mathbb{P}} - \text{a.e.}$$

The assumption says the empirical distribution of $i \mapsto z^i(\omega)$ for a draw of ω from $\bar{\Omega}$ agrees with the common theoretical distribution of z^i with probability one. There

[4]Popoviciu's inequality for variance states the variance of any bounded random variable is bounded.

is no loss of generality in assuming no aggregate uncertainty; following Sun and Zhang (2009), for any distribution P, there will exist suitable probability spaces and a measurable random variable z such that z^i has distribution P for each i and Assumption 2.2 holds.[5]

2.1.1. *Aggregate State*

Let μ_0 denote the common joint distribution of x_0^i and e_0^i. The distribution μ_0 becomes the initial state for the recursive constrained planner problem. The recursive problem we consider is one where the planner selects a measurable policy function h_t for each t, with $h_t\colon S \to A$. Each h_t instructs agents on $t+1$ assets given their time t asset and shock. A sequence of policy functions $(h_t)_{t=0}^{\infty}$ chosen by the constrained planner generates a sequence of assets for each agent, $(x_t^i)_{t=0}^{\infty}$, by

$$x_{t+1}^i = h_t(x_t^i, e_t^i), \qquad t \in \mathbb{N},\ i \in [0,1] \tag{1}$$

Since h_t applies to all agents i, the distribution of $\{x_t^i, e_t^i\}$ will be identical across i. Moreover, $\{x_t^i, e_t^i\} \sim \mu_t$ for each i, where $(\mu_t)_{t=0}^{\infty}$ satisfies the recursion

$$\mu_{t+1}(B_A \times B_E) = \int\int \mathbb{1}_{B_A}\{h_t(x,e)\}Q(e, B_E)\mu_t(dx, de), \qquad t \in \mathbb{N} \tag{2}$$

for each t and $B_A \times B_E \in \mathscr{B}(S)$. See Claim D.1 in the online appendix for the proof.

Under no aggregate uncertainty, at any time t, aggregate variables depend only on the common theoretical distribution of the shocks rather than individual realisations, with probability one. The next claim is straightforward to prove (see the online appendix) and confirms the intuition.

CLAIM 2.1 *Let $(\mu_t)_{t=0}^{\infty}$ and $(x_t^i)_{t=0}^{\infty}$ satisfy* (2) *for a sequence of policy functions $(h_t)_{t=0}^{\infty}$. If Assumption 2.2 holds, then for each t and measurable policy function g,*

$$\int g(x_t^i, e_t^i)\, \xi(di) = \int\int g(x,e)\mu_t(dx, de) \tag{3}$$

holds $\bar{\mathbb{P}}$-almost everywhere.

[5]In particular, note Z is a complete and separable metric space (theorems 3.37 and 3.38 by Aliprantis and Border (2006)), then apply Corollary 2 and Lemma 1 by Sun and Zhang (2009). See also discussion below Definition 2.1.5 by Sun (2006) on applying the no aggregate uncertainty results to stochastic processes.

2.1.2. *Production*

Assume a representative firm rents capital (assets) from individuals and hires workers to produce output Y_t:

$$Y_t = F(K(\mu_t), L) - \delta K(\mu_t) \tag{4}$$

where $F\colon \mathbb{R}^2_+ \to \mathbb{R}_+$. When the state is μ_t, using again the LLN argument from Claim 2.1, total capital and labour in the economy is

$$K(\mu_t) := \int\int x\mu_t(dx, de) = \int x^i_t \zeta(di) \tag{5}$$

$$L = \int e \int \mu_t(dx, de) = \int e^i_t \zeta(di) \tag{6}$$

Labour, L, will be constant according to Assumption 2.1.

ASSUMPTION 2.3 The production function F is twice differentiable on $\mathbb{R}_{++}$, homogeneous of degree one, strictly increasing in both arguments, strictly concave and for any $\hat{L} > 0$ and $\hat{K} > 0$ satisfies

1. $\lim_{K\to\infty} F_1(K, \hat{L}) = 0$ and $\lim_{K\to 0} F_1(K, \hat{L}) = \infty$ (Inada conditions)
2. $F(0, \hat{L}) = F(\hat{K}, 0) = 0$
3. $K \mapsto F(K, \hat{L})$ is bijective.

2.1.3. *Budget Constraints and Utility*

Interest and wage rates in the economy will be

$$r(\mu_t) := F_1\left(K(\mu_t), L\right) - \delta, \quad w(\mu_t)\colon = F_2\left(K(\mu_t), L\right)$$

Given the aggregate state μ_t, an agent i with asset x^i_t and endowment shock e^i_t must satisfy their budget constraint

$$0 \leq x^i_{t+1} \leq (1 + r(\mu_t))x^i_t + w(\mu_t)e^i_t \tag{7}$$

where x^i_{t+1} is the next period asset. If x^i_0 has finite variance and $r(\mu_t)$ is real-valued for each t, then if $(x^i_t)^\infty_{t=0}$ satisfies (7), x^i_t will have finite variance for each t (see Claim D.2 in the online appendix). Consumption for each agent i will be

$$c^i_t = (1 + r(\mu_t))x^i_t + w(\mu_t)e^i_t - x^i_{t+1}$$

Integrating across agents' budget constraints at Equation (7) and using the definition of interest and wages rates, along with homogeneity of the production function (see Theorem 2.1 in Acemoglu (2009)) implies

$$K(\mu_{t+1}) \leq (1 + r(\mu_t))K(\mu_t) + w(\mu_t)L = F(K(\mu_t), L) + (1 - \delta)K(\mu_t) \tag{8}$$

From (8) and Assumption 2.3, there exists an upper-bound $\bar{K}$ such that given any initial aggregate level of capital below $\bar{K}$, aggregate capital for wealth distributions satisfying (7) will never exceed $\bar{K}$. That is, if $K(\mu_t) \leq \bar{K}$, then $K(\mu_{t+1}) \leq \bar{K}$ (see Proposition 2.2 and section 6.8 by Acemoglu (2009)).

ASSUMPTION 2.4 The initial wealth distribution μ_0 satisfies $K(\mu_0) \leq \bar{K}$.

Turning to consumer utility, let $\nu\colon \mathbb{R}_+ \to \mathbb{R}_+$ be each consumer's utility function. Time t utility for agent i will be $\nu(c_t^i)$.

ASSUMPTION 2.5 The utility function ν is strictly increasing, bijective, concave and upper semicontinuous.

For a definition of a competitive equilibrium, see Aiyagari (1994), Dávila et al. (2012), Kuhn (2013), Miao (2002) or Acikgoz (2015).

2.2. *Recursive Constrained Planner*

Let $\mathscr{P}(S)$ denote the space of Borel probability measures on S. The recursive planner's state-space, $\mathbb{M}$, will be a subspace of $\mathscr{P}(S)$ such that each μ, with $\mu \in \mathbb{M}$, satisfies:

1. the marginal distribution across E, $\int \mu(dx, \cdot)$, agrees with ψ
2. the marginal distribution across A, $\int \mu(\cdot, de)$, has finite variance
3. aggregate assets satisfy $\int\int x\mu(dx, de) \in [0, \bar{K}]$.

Let $\mathbb{Y}$ denote the space of measurable functions h where $h\colon S \to A$. The space $\mathbb{Y}$ will be the *action-space* and the constrained planner picks a policy $h_t \in \mathbb{Y}$ for each t and agents' assets transition according to Equation (1).

Define a correspondence Λ, with $\Lambda\colon \mathbb{M} \twoheadrightarrow \mathbb{Y}$, mapping a state to feasible policy functions as follows:

$$\Lambda(\mu)\colon = \begin{cases} h \in \mathbb{Y} \,|0 \leq h(x,e) \leq (1 + r(\mu))x + w(\mu)e, & \text{if } K(\mu) > 0 \\ h \in \mathbb{Y} \,\big|h = 0, & \text{if } K(\mu) = 0 \end{cases} \tag{9}$$

The (in) equalities above hold μ - almost everywhere.

Following Equation (2), given a time t empirical distribution of agents on S, μ, and policy function h, the operator $\Phi\colon \operatorname{Gr}\Lambda \to \mathbb{M}$ defined by

$$\Phi(\mu,h)(B_A \times B_E) := \int\int \mathbb{1}_{B_A}\{h(x,e)\}Q(e,B_E)\mu(dx,de) \tag{10}$$

where $B_A \times B_E \in \mathscr{B}(S)$, gives the time $t+1$ empirical distribution of agents. We write $\mu_{t+1} = \Phi(\mu_t, h_t)$.

The constrained planner's per-period pay-off, $u\colon \operatorname{Gr}\Lambda \to \mathbb{R}_+$, integrates utility across the empirical distribution of agents

$$u(\mu,h)\colon = \begin{cases} \int\int v\big((1+r(\mu))x + w(\mu)e - h(x,e)\big)\mu(dx,de), & \text{if } K(\mu) > 0 \\ 0, & \text{if } K(\mu) = 0 \end{cases}$$

It is a straight-forward use of Jensen's inequality (fact C.4 in the online appendix) and homogeneity of the production function to show the integral is well-defined and real-valued.

Finally, let $\beta \in (0,1)$ be a discount factor and let V, with $V\colon \mathbb{M} \to \mathbb{R}_+ \cup \{+\infty\}$, denote the constrained planner's value function:

$$V(\mu_0) := \sup_{(\mu_t,h_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{11}$$

subject to

$$h_t \in \Lambda(\mu_t), \qquad \mu_{t+1} = \Phi(\mu_t, h_t), \qquad t \in \mathbb{N}, \qquad \mu_0 \text{ given} \tag{12}$$

DEFINITION 2.1 (**Recursive Constrained Planner's Problem**)

Given μ_0, a solution to the recursive constrained planner's problem is a sequence of measurable policy functions $(h_t)_{t=0}^{\infty}$, with $h_t\colon S \to A$ for each t and a sequence of Borel probability measures on S, $(\mu_t)_{t=0}^{\infty}$ satisfying (12) that achieves the value function:

$$V(\mu_0) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{13}$$

I now state the main result of this paper:

THEOREM 2.1 *If the recursive constrained planner's problem (Definition 2.1) satisfies assumptions 2.1 - 2.5, then for any $\mu_0 \in \mathbb{M}$, there exists a solution $(\mu_t, h_t)_{t=0}^{\infty}$.*

The proof is in section 4. Since Λ does not have compact image sets, standard existence results in dynamic optimisation theory fail (section 5). To prove Theorem 2.1, the paper first defines a sequential planner's problem (section 2.3) and shows existence of a solution for the sequential planner implies existence of a solution for the recursive planner (Theorem 2.2 in section 2.4). The sequential planner's feasibility correspondences will still not have compact image sets around regions where capital is zero. Thus, the paper presents a general existence result for non-compact infinite horizon dynamic optimisation (Theorem 3.1 in section 3), and then checks the sequential planner's problem satisfies the conditions for existence (section 4).

2.2.1. *Recursive Policies*

If the recursive constrained planner's problem has a solution, $(\mu_t, h_t)_{t=0}^{\infty}$, for each $\mu_0 \in \mathbb{M}$, then following standard arguments (see Corollary E.1 in the online appendix), there exists a policy operator $H\colon \mathbb{M} \to \mathbb{Y}$ such that the sequence $(\mu_t, H(\mu_t))_{t=0}^{\infty}$ with $\mu_{t+1} = \Phi(\mu_t, H(\mu_t))$ solves the recursive problem. Thus, if a solution to the recursive constrained planner's problem exists, then the *policy function* that maps assets and shocks to next period assets depends only on the current distribution.

2.3. *Sequential Constrained Planner*

Construct a countably generated[6] probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a sequence of random variables $\{x_0, e_0, e_1, \dots\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ such that $(e_t)_{t=0}^{\infty}$ is a Markov process with Kernel Q that can be written as a stochastic recursive sequence (see section C.4 of the online appendix) and $\{x_0, e_0\} \sim \mu_0$. Intuitively, we may view realisations of $\{x_0, e_0, e_1 \dots\}$ as draws from the empirical distribution of individual shock values. Define $(\mathscr{F}_i)_{i=0}^{\infty}$ as the natural filtration with respect to $\{x_0, e_0, e_1, \dots\}$.[7]

Let $L^2(\Omega, \mathbb{P})$ be the space of square integrable (with respect to $\mathbb{P}$) real-valued functions on Ω. Equip $L^2(\Omega, \mathbb{P})$ with the weak topology. For any $x \in L^2(\Omega, \mathbb{P})$, if

[6]The σ-algebra $\mathscr{F}$ is generated by a countable collection of subsets of Ω.

[7]That is, $\mathscr{F}_0$ is the σ-algebra generated by x_0 and for each $i \geq 1$, $\mathscr{F}_i$ is the σ-algebra generated by $\{x_0, e_0, \dots, e_{i-1}\}$.

$\int x \mathrm{d}\mathbb{P} > 0$, define

$$\begin{aligned} \tilde{r}(x) &: = F_1(\mathbb{E}x, L) - \delta \\ \tilde{w}(x) &:= F_2(\mathbb{E}x, L) \end{aligned} \tag{14}$$

For each t, define the time t state-space for the sequential planner:

$$\mathbb{S}_t := \left\{ x \in m\mathscr{F}_t \,\middle|\, 0 \leq x, \int x \,\mathrm{d}\mathbb{P} \leq \bar{K} \right\} \tag{15}$$

where $m\mathscr{F}_t \subset L^2(\Omega, \mathbb{P})$ is the space of $\mathscr{F}_t$-measurable random variables.

For each t, define the feasibility correspondence $\Gamma_t \colon \mathbb{S}_t \to \mathbb{S}_{t+1}$:

$$\Gamma_t(x) := \begin{cases} y \in \mathbb{S}_{t+1} \,|\, 0 \leq y \leq (1 + \tilde{r}(x))\, x + \tilde{w}(x)\, e_t, & \text{if } \mathbb{E}x > 0 \\ y \in \mathbb{S}_{t+1} \,|\, y = 0, & \text{if } \mathbb{E}x = 0 \end{cases} \tag{16}$$

For each t, define the time t pay-offs $\rho_t \colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$:

$$\rho_t(x, y) := \begin{cases} \displaystyle\int v\left((1 + \tilde{r}(x))\, x + \tilde{w}(x)\, e_t - y\right) \mathrm{d}\mathbb{P}, & \text{if } \mathbb{E}x > 0 \\ 0, & \text{if } \mathbb{E}x = 0 \end{cases} \tag{17}$$

Finally, let J, with $J \colon \mathbb{S}_0 \to \mathbb{R}_+ \cup \{+\infty\}$ denote the time 0 sequential planner's value function:

$$J(x_0) := \sup_{(x_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$$

subject to

$$x_{t+1} \in \Gamma_t(x_t), \qquad \forall t \in \mathbb{N}, \qquad x_0 \in \mathbb{S}_0 \text{ given} \tag{18}$$

DEFINITION 2.2 (**Sequential Constrained Planner's Problem**)

Given $x_0 \in \mathbb{S}_0$, a solution to the sequential constrained planner's problem is a sequence of random variables $(x_t)_{t=0}^{\infty}$ satisfying (18) that achieve the sequential planner's value function:

$$J(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \tag{19}$$

2.4. *Sequential Solution Implies Recursive Solution*

Let $(x_t)_{t=0}^{\infty}$ be a solution for the sequential problem. Note for each t, the value of $x_{t+1}(\omega)$ depends on the value of the value of the history $\{x_0(\omega), e_0(\omega), \ldots, e_t(\omega)\}$ for $\omega \in \Omega$. By contrast, a recursive solution requires x_{t+1} to be $\{x_t, e_t\}$ measurable; that is, we require a function $h_t \colon S \to A$ such that $x_{t+1}(\omega) = h_t(x_t(\omega), e_t(\omega))$. The following procedure projects each period's sequential solution back onto the previous period, furnishing the required measurability from properties of conditional expectation (see 9.2 by Williams (1991) or ch. IV, 1 by Çinlar (2011)).

Given x_0 satisfying $x_0 \in \mathbb{S}_0$, let $(y_t)_{t=0}^{\infty}$ be a solution to the sequential planner's problem. Construct a candidate sequence, $(x_t)_{t=0}^{\infty}$, as follows:

$$\begin{aligned} & x_0 = y_0, \qquad x_1 = \mathbb{E}(y_1 |\, x_0, e_0) \\ & \text{and} \\ & x_{t+1} = \mathbb{E}(y_t |\, x_t, e_t), \qquad\qquad \forall t \in \mathbb{N} \end{aligned} \tag{20}$$

PROPOSITION 2.1 *Let assumptions 2.1 - 2.5 hold. If $(y_t)_{t=0}^{\infty}$ is a solution to the sequential problem (Definition 2.2), then $(x_t)_{t=0}^{\infty}$ defined by* (20) *is a solution to the sequential problem.*

See the online appendix for a proof.

THEOREM 2.2 *Let assumptions 2.1 - 2.5 hold. If there exists a solution to the sequential problem (Definition 2.2), then there exists a solution to the recursive problem (Definition 2.1) and $V(\mu_0) = J(x_0)$.*

The complete proof is in the online appendix. The proof proceeds as follows. Let $(y_t)_{t=0}^{\infty}$ solve the sequential problem and let $(x_t)_{t=0}^{\infty}$ be defined by (20). Since x_{t+1} is $\{x_t, e_t\}$ measurable, $x_{t+1} = h_t(x_t, e_t)$ for a measurable function h_t for each t. For each t, define μ_t as

$$\mu_t(B) = \mathbb{P}\left\{x_t, e_t \in B\right\}, \qquad\qquad B \in \mathscr{B}(S) \tag{21}$$

The remainder of the proof verifies $(\mu_t, h_t)_{t=0}^{\infty}$ solves the recursive problem.

3. EXISTENCE THEOREM

I now introduce a general existence result for an infinite horizon dynamic optimisation problem with non-compact feasibility correspondences on arbitrary topological spaces. After stating the general result, this section shows how to verify the conditions for the result on L^2 spaces.

Let (X, τ) be a topological space,

DEFINITION 3.1 A function $f\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is **mildly sup-compact** if the upper contour sets

$$UC_f(\epsilon) : = \{x \in X \mid f(x) \geq \epsilon\} \tag{22}$$

are sequentially compact for all $\epsilon > \inf f$.

Study of sup-compact (or inf-compact) functions can be traced to Rockafellar and Moreau (see discussion of early literature by Roger and Wets (1973)), however, to the best of my knowledge, the term mildly sup-compact is new. A discussion on the relationship between mild sup-compactness, sup-compactness and upper semicontinuity is given in the online appendix.

3.1. *General Existence Theorem*

A non-stationary reduced form economy is a 5-tuple

$$\mathscr{E} := ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta)$$

consisting of:

1. A topological space $(\mathbb{X}, \tau)$
2. A collection of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$, with $\mathbb{S}_t \subset \mathbb{X}$ for each t
3. A collection of non-empty feasibility correspondences $(\Gamma_t)_{t=0}^{\infty}$, with $\Gamma_t\colon \mathbb{S}_t \twoheadrightarrow \mathbb{S}_{t+1}$ for each t
4. A collection of per-period pay-offs $(\rho_t)_{t=0}^{\infty}$, with $\rho_t\colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$ for each t
5. A discount factor $\beta \in (0, 1)$.

Define the correspondence of **feasible sequences** $\mathcal{G}_t^T\colon \mathbb{S}_t \twoheadrightarrow \prod_{i=t}^{T} \mathbb{S}_i$ starting at time t and ending at time T as follows:

$$\mathcal{G}_t^T(x) : = \left\{(x_i)_{i=t}^{T} \mid x_{i+1} \in \Gamma_i(x_i), x_t = x\right\}, \qquad x \in \mathbb{S}_t \tag{23}$$

Let $\mathcal{G}$ denote $\mathcal{G}_0^\infty$ and let $\mathcal{G}^T$ denote $\mathcal{G}_0^T$.

Define the **value function** $J\colon \mathbb{S}_0 \to \mathbb{R} \cup \{-\infty, +\infty\}$ as follows:

$$J(x) := \sup_{(x_t)_{t=0}^\infty \in \mathcal{G}(x)} \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t, x_{t+1}\right) \tag{24}$$

Recall a compact-valued upper hemicontinuous correspondence has compact image sets (see Mathematical Preliminaries in the online appendix). The first assumption below is the main assumption of the existence theorem, it relaxes the standard requirement for Γ_t to be upper hemicontinuous and compact valued and for $\mathbb{S}_t$ to be a metric space (see Acemoglu (2009), Assumption 6.2, Kamihigashi (2017), section 6 or Stokey and Lucas (1989), Assumption 4.3, for assumptions used by the standard theory).

ASSUMPTION 3.1 For each $x \in \mathbb{S}_0$ and $t \in \mathbb{N}$, the functions $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}^{t+1}(x)$ are mildly sup-compact in the product topology (of τ topology in $\mathbb{X}$).

The next assumption is the standard growth condition (see discussion on Corollary 6.1 by Kamihigashi (2017)).

ASSUMPTION 3.2 For each $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^\infty$ such that any $(x_t)_{t=0}^\infty \in \mathcal{G}(x)$ satisfies

$$\rho_t(x_t, x_{t+1}) \leq m_t, \qquad \forall t \in \mathbb{N} \tag{25}$$

and

$$\sum_{t=0}^{\infty} \beta^t m_t < \infty \tag{26}$$

ASSUMPTION 3.3 The functions $(\rho_t)_{t=0}^\infty$ are sequentially upper semicontinuous for all $t \in \mathbb{N}$.

THEOREM 3.1 *If $\mathscr{E}$ satisfies assumptions 3.1 - 3.3, then for every $x \in \mathbb{S}_0$, there will exist $(x_t)_{t=0}^\infty$ satisfying $(x_t)_{t=0}^\infty \in \mathcal{G}(x)$ such that*

$$J(x) = \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t, x_{t+1}\right) < \infty$$

.

See the appendix for a proof. The proof works by showing feasible paths of states converging to the supremum of the problem belong to a compact space in the product topology (of the topology τ in $\mathbb{X}$). By contrast, the standard assumptions of hemicontinuity and compact-valued correspondences requires that *all* feasible sequences belong to a compact space in the product topology. A further discussion of how standard theory uses these assumptions to verify existence is in section F.1 of the online appendix.

3.2. *Checking Mild Sup-Compactness in L^2 Spaces*

Let $(\Omega, \Sigma, \varphi)$ be a finite countably generated measure space and let $\mathbb{X} = L^2(\Omega, \varphi)$ be the space of real-valued measurable function on Ω with

$$\|x\| := \left(\int x^2 \, d\varphi \right)^{\frac{1}{2}} < \infty$$

Equip $\mathbb{X}$ with the weak topology. Recall a sequence $(x_n)_{n=0}^{\infty}$ with $x_n \in \mathbb{X}$ for each n converges in the weak topology if $\int x_n h \, \mathrm{d}\mathbb{P} \to \int xh \, \mathrm{d}\mathbb{P}$ for each $h \in \mathbb{X}$.[8]

Unless otherwise stated, convergence and topological notions will be with respect to the weak topology.

ASSUMPTION 3.4 The state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$ are sequentially closed in $\mathbb{X}$ for all $t \in \mathbb{N}$.

ASSUMPTION 3.5 The correspondences $(\Gamma_t)_{t=0}^{\infty}$ have a sequentially closed graph for all $t \in \mathbb{N}$.

ASSUMPTION 3.6 For each $t \in \mathbb{N}$, $\epsilon > 0$ and $x \in \mathbb{S}_0$, there exists a constant $\bar{M}$ such that if $(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $u_t(x_t, x_{t+1}) \geq \epsilon$, then $\|x_i\| \leq \bar{M}$ for each $i \in \{0, \ldots, t+1\}$.

PROPOSITION 3.1 *Consider $\mathscr{E}$ where $\mathbb{X} = L^2(\Omega, \varphi)$ and τ is the weak topology. If $\mathscr{E}$ satisfies assumptions 3.3 - 3.6, then $\mathscr{E}$ satisfies Assumption 3.1.*

[8]See 5.14 and 13.8 by Aliprantis and Border (2006) or 5.10 by Luenberger (1968).

4. EXISTENCE OF CONSTRAINED OPTIMA

Consider the case of the sequential constrained planner of section 2.3. Let assumptions 2.1 - 2.5 hold and let

$$\mathscr{E} = ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta)$$

where:

1. $\mathbb{X} = L^2(\Omega, \mathbb{P})$
2. The topology τ is the weak topology
3. The sequence of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$ are defined by (15)
4. The sequence of correspondences $(\Gamma_t)_{t=0}^{\infty}$ are defined by (16)
5. The sequence of pay-offs $(\rho_t)_{t=0}^{\infty}$ are defined by (17).

PROPOSITION 4.1 ***(Checking Assumption 3.2)*** *For any $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^{\infty}$ such that $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ and any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ satisfies $\rho_t(x_t, x_{t+1}) \leq m_t$ for each t.*

PROPOSITION 4.2 ***(Checking Assumption 3.3)*** *The functions $(\rho_t)_{t=0}^{\infty}$ are sequentially upper semicontinuous for each t.*

PROPOSITION 4.3 ***(Checking Assumption 3.4)*** *The state spaces $(\mathbb{S}_t)_{t=0}^{\infty}$ are sequentially closed for each t.*

PROPOSITION 4.4 ***(Checking Assumption 3.5)*** *The correspondences $(\Gamma_t)_{t=0}^{\infty}$ have closed graph for each t.*

PROPOSITION 4.5 ***(Checking Assumption 3.6)*** *For any $t \in \mathbb{N}$, $\epsilon > 0$ and $x \in \mathbb{S}_0$, there exists a constant $\bar{M}$ such that if $(x_i)_{t=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$, then*

$$\|x_i\| \leq \bar{M}$$

for all $i \in \{0, 1, \dots, t+1\}$.

The proofs for Propositions 4.1 and 4.3 are in the online appendix. The remaining proofs are in the appendix below.

We are now ready to verify existence of recursive constrained optima.

PROOF OF THEOREM 2.1: Recall the setting of section 2.1 where μ_0 is the initial state of the economy and let the random variables $\{x_0, e_0, e_1, \dots\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ be as defined in section 2.3. By assumptions 2.1 - 2.5 and propositions 4.1 - 4.5, the economy $\mathscr{E}$ satisfies assumptions 3.2 - 3.6.

Since assumptions 3.3 - 3.6 satisfy the conditions for Proposition 3.1, $\mathscr{E}$ satisfies Assumption 3.1. As such, $\mathscr{E}$ satisfies assumptions 3.1 - 3.3 and the conditions for Theorem 3.1.

By Theorem 3.1, there exists $(y_t)_{t=0}^{\infty}$ solving the sequential planner's problem (Definition 2.2) such that $J(x_0) < \infty$. By Proposition 2.1, $(x_t)_{t=0}^{\infty}$ defined by (20) also solves the sequential planner's problem. Moreover, there exists a sequence of measurable policy functions $(h_t)_{t=0}^{\infty}$ with $h_t \colon S \to A$ and $x_{t+1} = h_t(x_t, e_t)$ for each t. By Theorem 2.2, $(h_t)_{t=0}^{\infty}$ and $(\mu_t)_{t=0}^{\infty}$ defined by (21) solve the recursive problem and

$$V(\mu_0) = J(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) < \infty \tag{27}$$

Q.E.D.

5. DISCUSSION

5.1. *Non-Compactness of the Feasibility Correspondence*

There are two reasons why the constrained planner's feasibility correspondences will not have compact image sets. The first concerns the structure of the recursive problem and the second concerns the behaviour of interest rates around regions where capital is zero.

5.1.1. *Structure of Recursive Problem*

In the recursive problem, to show Λ has compact image sets, we need to place some further restrictions on the space $\mathbb{Y}$ and equip $\mathbb{Y}$ with a suitable topology. In the sup-norm topology, the Arzela-Arscoli Theorem (see example 6.2, Mas-colell and Zame (1991)) states uniformly bounded, equicontinuous family of functions on a compact interval will be compact. However, A will not be bounded and policy functions may not be bounded. As such, $\Lambda(\mu)$ cannot be compact valued as it is defined here.

A possible approach could be restricting $\mathbb{M}$ to measures on a compact support. For each $\mu \in \mathbb{M}$, we can then restrict policy functions in $\Lambda(\mu)$ to be defined on the bounded support of μ. If the mean of μ is positive, then policy functions in $\Lambda(\mu)$ will also be bounded. Notwithstanding the pathologies (see below) as interest rates diverge, to now use the Arzela-Arscoli Theorem, we also need to restrict feasible policy functions in each period to an equicontinuous family of functions. Similarly, to use Helly's Selection Theorem (see Cao (2016), Lemma 11) to verify compactness in the product topology on $\mathbb{Y}$, we need to restrict $\mathbb{Y}$ to the space of monotone policy functions.

However, we cannot use the Euler equations for the constrained planner to restrict the search for an optimiser to the space of an equicontinuous or monotone family of functions; the Euler equations have only been shown to be necessary *and a solution for the constrained planner may not exist*. In particular, there may be pathological sequences of policy functions outside the equicontinuous or monotone family converging to the supremum.

The online appendix gives further detail of pathologies in the weak topology if we let $\mathbb{Y}$ be the space of square integrable functions on S, where Φ fails to be defined.

Note the pathologies in the recursive problem also prevent the use of the non-compact existence result of section 3. This is because we cannot place restrictions such as monotonicity or equicontinuity on policy functions in the upper contour sets of Assumption 3.1.

5.1.2. *Non-Compactness Near Zero Capital*

Consider the setting and notation of section 4. Once we move to L^2 space with the weak topology, the feasibility correspondence will be compact valued and have a closed graph. The sequential problem will also be well-defined. However, there will exist a compact set $C \subset \mathbb{S}_t$ such that $\Gamma_t(C)$ is not compact. (Recall from section C.1 in the online appendix that the image of a compact set under a compact-valued and upper hemicontinuous is compact.)

For the following claim, assume $F(K, L) = \alpha^{-1}K^{\alpha}$ and $\alpha = .3$. Furthermore, assume $x_0 \in \mathbb{S}_0$, the initial assets for the economy are a uniform random variable on the interval $[0, 1]$. Assume the random variable e_0 is large enough to satisfy $\tilde{w}(x_0)e_0 > 1$.

CLAIM 5.1 *There exists a compact set C, satisfying $C \subset \mathbb{S}_1$, such that the image set $\Gamma_1(C)$ is not compact.*

The proof is in the online appendix. Roughly, we can construct a sequence of asset distributions in $\Gamma_1(C)$ whose means converge but variances diverge. This is because a smaller and smaller measure of agents can accumulate assets that go to infinity due to higher and higher interest rates as aggregate assets converge to zero.

5.2. *Relationship to K-Sup-Compactness*

To relax compactness and continuity requirements on the feasibility correspondence, Feinberg et al. (2012) introduce a condition (assumption W* in Feinberg et al. (2012)), later defined as K-Sup-Compactness by Feinberg et al. (2013) (Definition 1.1), on per-period pay-offs.[9] Recall the definition of sup-compact from the online appendix, section C.2. While Feinberg et al. (2012) consider stationary problems, for the sequential constrained planner's setting, K-Sup-Compactness of each per-period pay-off ρ_t becomes:

ASSUMPTION 5.1 (**K-Sup-Compact**) Let $t \in \mathbb{N}$. If C is a sequentially compact subset of $\mathbb{S}_t$, then the function $\{x_t, x_{t+1}\} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}_t^{t+1}(C)$ is sup-compact.

The assumption allows the Bellman Equation (in our case, a non-stationary Bellman Equation) to preserve semicontinuity (see Theorem 2 in Feinberg et al. (2012) and Lemma 2.5 in Feinberg et al. (2013)).

With utility bounded below, ρ_t will not satisfy K-Sup-Compactness. To see why, note $\mathcal{G}_t^{t+1}(C) = \{x, y \,|\, y \in \Gamma_t(x), x \in C\}$. Moreover, the upper-contour set of the function $\{x_t, x_{t+1}\} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}_t^{t+1}(C)$ when $\epsilon = 0$ will be:

$$\begin{aligned}&\{x, y \,|\, y \in \Gamma_t(x), x \in C, \rho_t(x, y) \geq 0\} \\ &\qquad\qquad = \{x, y \,|\, y \in \Gamma_t(x), x \in C\} = \mathcal{G}_t^{t+1}(C)\end{aligned}$$

For the constrained planner, K-Sup-Compactness of ρ_t will then imply compact $\mathcal{G}_t^{t+1}(C)$ for compact C. However, Claim 5.1 constructs an example where $x_n \in C$ and $y_n \in \Gamma(x_n)$ such that the norm of y_n diverges, implying non-compact $\mathcal{G}_t^{t+1}(C)$.[10] As such, when utility is bounded below, the main assumption of this paper is weaker than K-Sup-Compactness.

[9] Feinberg et al. (2013) use term K-Inf-Compactness, as they work with minimisation problems.

[10] The constrained planner's sequential problem will satisfy the stronger condition where Assumption 3.1 holds for the stated functions on $\mathcal{G}^{t+1}(C)$. The stronger condition gives semicontinuity of the value function and compactness of policy correspondences. The details are a work in progress and available on request.

6. CONCLUSION

This paper proved existence of recursive constrained optima in a standard Aiyagari (1994) model, as considered by Dávila et al. (2012). The results here only apply to problems where the planner's pay-offs are bounded below — a key technical contribution of the paper was a general existence result that overcomes the difficulties when a non-compact dynamic optimisation problems has pay-offs bounded below. Some paths for further research include studying the application of non-compact dynamic optimisation theory for unbounded pay-offs (Feinberg et al., 2012) to heterogeneous agent models, studying asymptotic properties (stochastic stability) of the constrained planner's solution path and studying optimal policy in the variety of incomplete market models. Finally, existence of recursive competitive equilibria in the Aiyagari-Huggett model away from the steady state remains an open question (Cao (2016) proves existence of a generalised recursive equilibrium); if a competitive equilibrium can be stated as a solution to a modified constrained planner, then the approach of this paper can be used to verify existence of recursive competitive equilibria.

APPENDIX A: PROOFS

A.1. *Proofs for Section 3*

Recall the setting and notation of section 3.1, where $(\mathbb{X}, \tau)$ is a topological vector space. Throughout this section, unless otherwise stated, convergence for sequences in $\mathbb{X}$ will be with respect to the τ topology and convergence for sequences in countable Cartesian products of $\mathbb{X}$ will be in the product topology of the τ topology on $\mathbb{X}$.

We will use $\mathbf{x}$ to refer to elements of $\mathbb{X}^{\mathbb{N}}$. We can then use $(\mathbf{x}^n)_{n=0}^{\infty}$ to denote a sequence $\{\mathbf{x}^0, \dots, \mathbf{x}^n, \dots\}$, where $(\mathbf{x}^n)_{n=0}^{\infty} \in (\mathbb{X}^{\mathbb{N}})^{\mathbb{N}}$.

REMARK A.1 Let $X = \prod_{i \in F} X_i$ denote a Cartesian product of topological spaces. Let $\pi_i \colon X \to X_i$ denote the projection map defined as $\pi_i(x) = x_i$ for each $i \in F$. Recall each projection map will be a continuous function on X when X has the product topology (see section 2.14 by Aliprantis and Border (2006)). Also recall (section 1.8 by Tao (2013)) the image of a (sequentially) compact set under a continuous function is (sequentially) compact. Accordingly, if a set C with $C \subset X$ is (sequentially) compact in the product topology, then $\pi_i(C)$ will be (sequentially) compact.

Finally, let the function $\phi_t \colon \mathcal{G}^{t+1}(x) \to \mathbb{R}_+$ denote $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ for each t and let $U(\mathbf{x}) := \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$.

LEMMA A.1 *Let Assumption 3.2 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\mathbf{x}^n)_{n=0}^{\infty}$ is a sequence with $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ for $B > 0$, then there exists a sub-sequence $(\mathbf{x}^{n_k})_{k=0}^{\infty}$ such that for all $t \in \mathbb{N}$*

$$\lim_{k \to \infty} \rho_t(x_t^{n_k}, x_{t+1}^{n_k}) \to c_t$$

where $c_t \in \mathbb{R}_+$ for each t and $c_t > 0$ for at-least one t.

PROOF: By Assumption 3.2, for each t and n,

$$m_t \geq \rho_t(x_t^n, x_{t+1}^n) \geq 0 \tag{A.28}$$

Accordingly, for each n, $(\rho_t(x_t^n, x_{t+1}^n))_{t=0}^{\infty}$ will belong to the set $\prod_{t=0}^{\infty}[0, m_t]$, which by Tychonoff's Theorem (see Proposition 1.8.12 by Tao (2010)) will be compact in the product topology. There then exists a sub-sequence of $(\mathbf{x}^n)_{n=0}^{\infty}$, $(\mathbf{x}^{n_k})_{k=0}^{\infty}$, such that $(\rho(x_t^{n_k}, x_{t+1}^{n_k}))_{k=0}^{\infty}$ converges for each t. Let $c_t := \lim_{k\to\infty} \rho(x_t^{n_k}, x_{t+1}^{n_k})$ and note

$$B = \lim_{k\to\infty} \sum_{t=0}^{\infty} \beta^t \rho_t\left(x_t^{n_k}, x_{t+1}^{n_k}\right) = \sum_{t=0}^{\infty} \lim_{k\to\infty} \beta^t \rho_t\left(x_t^{n_k}, x_{t+1}^{n_k}\right) = \sum_{t=0}^{\infty} \beta^t c_t \tag{A.29}$$

Since (A.28) holds, and $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ by Assumption 3.2, we can pass limits through in the second equality using dominated convergence theorem (see Corollary 7.3.15 by Stachurski (2009)). If B is strictly positive, the above means there is at least one $c_t > 0$.

Q.E.D.

LEMMA A.2 *Let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty}$ is a sequence with $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each n and for some t*

$$\rho_t(x_t^n, x_{t+1}^n) \to c_t$$

with $c_t > 0$, then there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$.

PROOF: There exists ι such that $\epsilon := c_t - \iota$ is strictly positive. For N large enough and any $n > N$, $\rho_t(x_t^n, x_{t+1}^n) \in [\epsilon, c_t + \iota]$, implying $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$ and $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$.

Q.E.D.

LEMMA A.3 *Let assumptions 3.1- 3.3 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty}$ is a sequence such that $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each $n \in \mathbb{N}$ and $U(\boldsymbol{x}^n) \to B$ where $B > 0$, then:*

1. *$(\boldsymbol{x}^n)_{n=0}^{\infty}$ has a convergent sub-sequence with a limit $\boldsymbol{x} \in \mathcal{G}(x)$, and*
2. *$B \leq U(\boldsymbol{x}) < \infty$.*

PROOF: Let x satisfy $x \in \mathbb{S}_0$ and let $(\mathbf{x}^n)_{n=0}^{\infty}$ be a sequence such that $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ where $B > 0$. By Lemma A.1 there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that for each $t \in \mathbb{N}$, $c_t := \lim_{j\to\infty} \rho_t(x_t^{n_j}, x_{t+1}^{n_j}) > 0$ for at-least one t. Re-label $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ to $(\mathbf{x}^n)_{n=0}^{\infty}$, and let P denote the subset of $\mathbb{N}$ such that $t \in P$ if and only if $c_t > 0$. The set P will be non-empty, but could be finite or infinite.

To prove part 1 of the lemma, consider first the case when P is infinite and then the case when P is finite.

Suppose P is infinite and consider any $t \in \mathbb{N}$. There will exist $k > t$ such that $c_k > 0$. By Lemma A.2, there exists N and $\epsilon > 0$ such that for all $n > N$, $(x_i^n)_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$.

By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology. The space $\pi_t(UC_{\phi_k}(\epsilon))$ will also be sequentially compact by the argument in Remark A.1. Let $\Xi_t := \{x_1^0, \dots, x_t^N\} \cup \pi_t(UC_{\phi_k}(\epsilon))$. Since $\{x_1^0, \dots, x_t^N\}$ is sequentially compact, Ξ_t will be sequentially compact. Moreover, note $x_t^n \in \Xi_t$ for each $n \in \mathbb{N}$.

Since t was arbitrary, we can construct a Ξ_t as above for every $t \in \mathbb{N}$. Now let $\Xi := \prod_{t \in \mathbb{N}} \Xi_t$. Using the Sequential Tychonoff Theorem (Proposition 1.8.12 by Tao (2010)), Ξ will be sequentially compact. Since for each t, $x_t^n \in \Xi_t$ for each n, $\mathbf{x}^n \in \Xi$ for each n. There then exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ converging to $\mathbf{x}$, with $\mathbf{x} \in \Xi$.

We now confirm $\mathbf{x} \in \mathcal{G}(x)$ by showing $x_{t+1} \in \Gamma_t(x_t)$ for all $t \in \mathbb{N}$. Pick any $t \in \mathbb{N}$, there will be a k satisfying $k > t$ such that $c_k > 0$. By Lemma A.2, there exists $\epsilon > 0$ and J such that for all $j > J$ we have $(x_i^{n_j})_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$. By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact, moreover, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$ by the definition of $UC_{\phi_k}(\epsilon)$ at (22). As such, the sub-sequence $(x_i^{n_j})_{i=0}^{k+1}$ converges to $(x_i)_{i=0}^{k+1}$, with $(x_i)_{i=0}^{k+1} \in \mathcal{G}^{k+1}(x)$, allowing us to conclude $x_{t+1} \in \Gamma_t(x_t)$. Since the t was arbitrary, $x_{t+1} \in \Gamma_t(x_t)$ for each $t \in \mathbb{N}$ and $\mathbf{x} \in \mathcal{G}(x)$.

Now assume P is finite; P will have a maximum element, which we now call k. By Lemma A.2, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that $(x_t^n)_{t=0}^{k+1} \in UC_{\phi_k}(\epsilon)$ for each $n > N$. By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology. As such, there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that $(x_t^{n_j})_{j=0}^{\infty}$ for each $t \leq k+1$. Define $(x_t)_{t=0}^{\infty}$ by setting $x_t = \lim_{j \to \infty} x_t^{n_j}$ for $t \leq k+1$ and picking any $x_{t+1} \in \Gamma_t(x_t)$ for $t \geq k+1$.

To confirm $(x_t)_{t=0}^{\infty}$ is feasible, let us check $x_{t+1} \in \Gamma_t(x_t)$ for each t. Once again, note by definition, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$. Since $UC_{\phi_k}(\epsilon)$ is sequentially compact, $(x_t)_{t=0}^{k+1} \in \mathcal{G}(x)$ and $x_{t+1} \in \Gamma_t(x_t)$ for all t satisfying $t \leq k$. On the other hand, if $t > k$, by construction, $x_{t+1} \in \Gamma_t(x_t)$, confirming $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$.

Re-label $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ to $(\mathbf{x}^n)_{n=0}^{\infty}$. To prove part 2 of the lemma, by Assumption 3.2,

$$\rho_t(x_t^n, x_{t+1}^n) \leq m_t$$

for each t and n, where $\sum_{t=0}^{\infty} \beta^t m_t < \infty$. Whence Fatou's Lemma[11] gives

$$B = \limsup_{n \to \infty} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{\infty} \limsup_{n \to \infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) < \infty \tag{A.30}$$

Upper-semicontinuity of ρ_t (Assumption 3.3) and the growth condition (Assumption 3.2) imply

$$\limsup_{n \to \infty} \rho_t(x_t^n, x_{t+1}^n) \leq \rho_t(x_t, x_{t+1}) \leq m_t, \qquad t \in \mathbb{N} \tag{A.31}$$

[11]See fact C.5 in the online appendix and let $\Omega = \mathbb{Z}_+$ and μ be the counting measure. Also see Equation (1.1) and discussion by Kamihigashi (2017).

To complete the proof, combine (A.31) with (A.30) and conclude

$$B \leq \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = U(\mathbf{x}) < \infty$$

Q.E.D.

PROOF OF THEOREM 3.1: Fix $x \in \mathbb{S}_0$. If $U(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{G}(x)$, then our solution will be any $\mathbf{x} \in \mathcal{G}(x)$.

Next, suppose at-least one $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ satisfies $U(\mathbf{x}) > 0$. By Assumption 3.2, there exists a sequence of real numbers $(m_t)_{t=0}^{\infty}$ such that $\rho_t(x_t, x_{t+1}) \leq m_t$ for any $\mathbf{x}$ in $\mathcal{G}(x)$ and

$$\bar{B} := \sum_{t=0}^{\infty} \beta^t m_t < \infty$$

Any $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ will satisfy

$$U(\mathbf{x}) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \leq \bar{B}$$

Consider the set $I := \{U(\mathbf{x}) \mid \mathbf{x} \in \mathcal{G}(x)\}$. The set I will be a subset of $\mathbb{R} \cup \{-\infty, \infty\}$ and so must have a supremum. Let $B := \sup I$ and note $0 \leq B \leq \bar{B} < \infty$.

Construct a sequence $(\mathbf{x}^n)_{n=0}^{\infty}$ with $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ as follows: for every $n \in \mathbb{N}$, take $\mathbf{x}^n$ such that $B - U(\mathbf{x}^n) < \frac{1}{n+1}$. Such a sequence exists, otherwise for some n, $U(\mathbf{x}) \leq B - \frac{1}{n+1}$ for all $\mathbf{x} \in \mathcal{G}(x)$ and B will not be the supremum of I.

Since $U(\mathbf{x}^n) \to B$, by Lemma A.3, there exists $\mathbf{x} \in \mathcal{G}(x)$ such that $U(\mathbf{x}) \geq B$. Since B was the supremum for I, conclude

$$U(\mathbf{x}) = B = J(x) < \infty$$

Q.E.D.

PROOF OF PROPOSITION 3.1: Let x satisfy $x \in S_0$. Fix any $t \in \mathbb{N}$ and any ϵ satisfying $\epsilon > 0$. We show the upper contour sets $UC_{\phi_t}(\epsilon)$ defined by

$$UC_{\phi_t}(\epsilon) = \{(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x) \mid \rho_t(x_t, x_{t+1}) \geq \epsilon\} \tag{A.32}$$

are sequentially compact. In particular, we first show $UC_{\phi_t}(\epsilon)$ is a sequentially closed sub-set of $\mathbb{X}^{t+1}$ and then show $UC_{\phi_t}(\epsilon)$ is contained within a compact and metrizable set.

To show $UC_{\phi_t}(\epsilon)$ is sequentially closed in $\mathbb{X}^{t+1}$, take any sequence $(\mathbf{x}^n)_{n=0}^{\infty}$ with $\mathbf{x}^n \in UC_{\phi_t}(\epsilon)$ for each n that converges to $\mathbf{x} = (x_i)_{i=0}^{t+1}$ point-wise. Note $x_i^n \in \mathbb{S}_i$ for each $i \leq t+1$ and n. Since each $\mathbb{S}_i$ is sequentially closed (Assumption 3.4), $x_i \in \mathbb{S}_i$.

By Assumption 3.5, each Γ_i has a sequentially closed graph, and thus $x_{i+1} \in \Gamma_i(x_i)$ for each $i \leq t+1$. Noting the definition of $\mathcal{G}(x)^{t+1}$ by (23), conclude $\mathbf{x} \in \mathcal{G}(x)^{t+1}$.

We now confirm $\rho_t(x_t, x_{t+1}) \geq \epsilon$. By upper semi-continuity of ρ_t (Assumption 3.3), $UC_{\rho_t}(\epsilon) = \{(x,y) \in \operatorname{Gr}\Gamma_t \mid \rho_t(x,y) \geq \epsilon\}$ is sequentially closed. The sequence $(x_i^n)_{i=0}^{t+1}$ will satisfy $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$ and thus $\{x_t^n, x_{t+1}^n\} \in UC_{\rho_t}(\epsilon)$ for each n. Moreover, $x_{t+1} \in \Gamma_t(x_t)$. Accordingly, $\{x_t, x_{t+1}\} \in UC_{\rho_t}(\epsilon)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$. We conclude $\mathbf{x} \in UC_{\phi_t}(\epsilon)$ and $UC_{\phi_t}(\epsilon)$ is sequentially closed.

Since $\mathscr{E}$ satisfies Assumption 3.6, there will exist $\bar{M}$ such that if $(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$, then $\|x_i\| \leq \bar{M}$ for $i \in \{0, \dots, t+1\}$. Whence $UC_{\phi_t}(\epsilon)$ will be a sub-set of the space $B_{\bar{M}} := \Pi_{i=0}^{t+1}\{x_i \in \mathbb{S}_i \mid \|x_i\| \leq \bar{M}\}$.

For each $i \leq t+1$, the space $\{x_i \in \mathbb{S}_i \mid \|x_i\| \leq \bar{M}\}$ will be compact by Alaoglu's Theorem.[12] Next, $L^2(\Omega, \mathbb{P})$ is a separable space since $\mathscr{F}$ is separable.[13] As such, since $L^2(\Omega, \mathbb{P})$ is reflexive, the spaces $\{x_i \in \mathbb{S}_i \mid \|x_i\| \leq \bar{M}\}$ are metrizable and sequentially compact.[14] Moreover, by the Sequential Tychonoff's Theorem,[15] the space $B_{\bar{M}}$ will be sequentially compact in the product topology (of weak topology on $\mathbb{X}$). By the argument in the preceding paragraph, $UC_{\phi_t}(\epsilon)$ is a sequentially closed subset of $B_{\bar{M}}$, allowing us to conclude $UC_{\phi_t}(\epsilon)$ is sequentially compact.

Q.E.D.

A.2. *Proofs for Section 4*

Recall point-wise inequalities in $\mathbb{X}$ hold $\mathbb{P}$ - almost everywhere and convergence of (x^n) with $x^n \in \mathbb{X}$ for each n will be with respect to the weak topology.

Recall the definition of sequential upper semicontinuity from section C.2 in the online appendix. The proof for the following claim is standard and placed in the online appendix.

CLAIM A.1 *Let (Ω, Σ, μ) be a finite measure space. Consider a function $g \colon \mathbb{R} \to \mathbb{R}$. Define $G \colon L^2(\Omega, \mu) \to \mathbb{R}$ as*

$$G(s) := \int g(s)\, d\mu, \qquad s \in L^2(\Omega, \mu)$$

If g is concave and upper semicontinuous, then G will be weak sequentially upper semicontinuous.

PROOF OF PROPOSITION 4.2: Set any $t \in \mathbb{N}$ and consider sequences $(x^n)_{n=0}^{\infty}$ and $(y^n)_{n=0}^{\infty}$ with $\{x^n, y^n\} \in \operatorname{Gr}\Gamma_t$ for each n. Let $x^n \to x$ and $y^n \to y$ with $y \in \Gamma_t(x)$. To verify sequential upper semicontinuity, we show

$$\begin{aligned}\limsup_{n\to\infty} \rho_t(x^n, y^n) &= \limsup_{n\to\infty} \int \nu((1+\tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n)\, \mathrm{d}\mathbb{P} \\ &\leq \rho_t(x,y)\end{aligned} \tag{A.33}$$

[12] Mas-colell and Zame (1991), section 6.1 or Theorem 6.21 by Aliprantis and Border (2006).

[13] Ex. 1.3.9 by Tao (2010).

[14] See discussion proceeding Corollary 1.9.16 by Tao (2010) or Theorem 6.30 by Aliprantis and Border (2006).

[15] Proposition 1.8.12 by Tao (2010).

By Assumption 2.5, ν is concave and continuous. To use the statement made by Claim A.1, consider a continuous concave extension of ν to $\bar{\nu}$, where $\bar{\nu}\colon \mathbb{R} \to \mathbb{R}$ and $\bar{\nu}|_{\mathbb{R}_+} = \nu$.[16] The mapping $s \mapsto \int \bar{\nu}(s)\,\mathrm{d}\mathbb{P}$ for $s \in L^2(\Omega, \mathbb{P})$ will be sequentially upper semicontinuous since $\bar{\nu}$ is concave and upper semicontinuous. As such, for any sequence in $L^2(\Omega, \mathbb{P})$ satisfying $f^n \to f$ weakly,

$$\limsup_{n\to\infty} \int \bar{\nu}(f^n)\,\mathrm{d}\mathbb{P} \le \int \bar{\nu}(f)\,\mathrm{d}\mathbb{P} \tag{A.34}$$

Let $f^n := (1+\tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n$ and note $f^n \in L^2(\Omega, \mathbb{P})$ for each n. First, we show (A.33) for the case $\int x\,\mathrm{d}\mathbb{P} > 0$. If $\int x\,\mathrm{d}\mathbb{P} > 0$, then

$$\begin{aligned}\int f^n h\,\mathrm{d}\mathbb{P} &= (1+\tilde{r}(x^n))\int x^n h\,\mathrm{d}\mathbb{P} + \tilde{w}(x^n)\int e_t h\,\mathrm{d}\mathbb{P} - \int y^n h\,\mathrm{d}\mathbb{P} \\ &\to (1+\tilde{r}(x))\int xh\,\mathrm{d}\mathbb{P} + \tilde{w}(x)\int e_t h\,\mathrm{d}\mathbb{P} - \int yh\,\mathrm{d}\mathbb{P}\end{aligned}$$

for any $h \in L^2(\Omega, \mathbb{P})$. Thus f^n converges weakly to $f := (1+\tilde{r}(x))x + \tilde{w}(x)e_t - y$, implying by (A.34),

$$\limsup_{n\to\infty} \int \nu(f^n)\,\mathrm{d}\mathbb{P} \le \int \nu(f)\,\mathrm{d}\mathbb{P} = \rho_t(x,y)$$

If $\int x\,\mathrm{d}\mathbb{P} = 0$, then

$$\begin{aligned}\limsup_{n\to\infty} \int \nu(f^n)\,\mathrm{d}\mathbb{P} &\le \limsup_{n\to\infty} \nu\left(\int (1+\tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n\,\mathrm{d}\mathbb{P}\right) \\ &\le \lim_{n\to\infty} \nu(F(\mathbb{E}x^n, L) + (1-\delta)\mathbb{E}x^n) \\ &= 0 = \rho_t(x,y)\end{aligned}$$

where the first inequality follows from Jensen's inequality (fact C.4 in the online appendix). The second inequality follows from Assumption 2.3 on homogeneity of the production function (recall Equation (8)).

Q.E.D.

PROOF OF PROPOSITION 4.4: Recall the definition of closed graph correspondences from the Mathematical Preliminaries section of the online appendix. Set t and suppose $(x^n, y^n)_{n=0}^{\infty}$ satisfies $y^n \in \Gamma_t(x^n)$ for each n. Suppose $(x^n)_{n=0}^{\infty}$ converges to $x \in \mathbb{S}_t$ and $(y^n)_{n=0}^{\infty}$ converges to $y \in \mathbb{S}_{t+1}$.

We show $y \in \Gamma_t(x)$ by checking both the cases stated in the definition of Γ_t at Equation (16): either $\int x\,\mathrm{d}\mathbb{P} = 0$ or $\int x\,\mathrm{d}\mathbb{P} > 0$. First let $\int x\,\mathrm{d}\mathbb{P} > 0$, we show $y \le (1+\tilde{r}(x))x + \tilde{w}(x)e_t$ for $\mathbb{P}$-almost everywhere. Suppose by contradiction

$$\mathbb{P}\left\{y > (1+\tilde{r}(x))x + \tilde{w}(x)e_t\right\} > 0$$

[16]See Corollary 8.3.10 by Borwein and Vanderwerff (2010).

Let $B := \{\omega \in \Omega \,|\, y(\omega) > (1+\tilde{r}(x))x(\omega) + \tilde{w}(x)e_t(\omega)\}$, we have $\mathbb{P}(B) > 0$ and

$$\int \mathbb{1}_B y \,\mathrm{d}\mathbb{P} > \int \mathbb{1}_B \times [(1+\tilde{r}(x))x + \tilde{w}(x)e_t] \,\mathrm{d}\mathbb{P} \tag{A.35}$$

Since $\mathbb{E}x^n \to \mathbb{E}x$ and $\mathbb{E}x > 0$, there exists N such that for all $n > N$, $\mathbb{E}x^n > 0$. And for the tail sequence $(x^n)_{n=N+1}^{\infty}$, $\tilde{r}(x^n) = F_1(\mathbb{E}x^n, L)$ converges, implying

$$(1+\tilde{r}(x^n)) \int x^n h \,\mathrm{d}\mathbb{P} + \tilde{w}(x^n) \int h e_t \,\mathrm{d}\mathbb{P} \to (1+\tilde{r}(x)) \int x h \,\mathrm{d}\mathbb{P} + \tilde{w}(x) \int h e_t \,\mathrm{d}\mathbb{P} \tag{A.36}$$

for any function h satisfying $h \in L^2(\Omega, \mathbb{P})$. In particular, let $h = \mathbb{1}_B$, and note $y^n \in \Gamma_t(x^n)$; by the feasibility condition at (16), we write

$$\int \mathbb{1}_B y^n \,\mathrm{d}\mathbb{P} \leq (1+\tilde{r}(x^n)) \int \mathbb{1}_B x^n \,\mathrm{d}\mathbb{P} + \tilde{w}(x^n) \int \mathbb{1}_B e_t \,\mathrm{d}\mathbb{P}$$

for each $n > N$. Since the weak inequality above will be preserved under the limits of real-valued sequences, we arrive at

$$\int \mathbb{1}_B y \,\mathrm{d}\mathbb{P} \leq (1+\tilde{r}(x)) \int \mathbb{1}_B x \,\mathrm{d}\mathbb{P} + \tilde{w}(x) \int \mathbb{1}_B e_t \,\mathrm{d}\mathbb{P} \tag{A.37}$$

However, (A.37) is a contradiction to (A.35) and we conclude

$$y \leq (1+\tilde{r}(x))x + \tilde{w}(x)e_t$$

Now suppose $\int x^n \,\mathrm{d}\mathbb{P} \to 0$. Note

$$\int y^n \,\mathrm{d}\mathbb{P} \leq \int (1+\tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t \,\mathrm{d}\mathbb{P} = F(\mathbb{E}x^n, L) + (1-\delta)\mathbb{E}x^n$$

The above equality follows from homogeneity of degree one of the production function (Assumption 2.3 and recall discussion preceding Equation (8)). Since $\mathbb{E}x^n = \int x^n \,\mathrm{d}\mathbb{P} \to 0$, we have $F(\mathbb{E}x^n, L) \to 0$ by Assumption 2.3, and

$$0 = \lim_{n\to\infty} \int y^n = \int y \,\mathrm{d}\mathbb{P} \tag{A.38}$$

Since $y \in \mathbb{S}_{t+1}$, $y \geq 0$ and (A.38) implies $y = 0$ for $\mathbb{P}$-almost everywhere.[17]

Thus we have checked $x_t \in \Gamma_t(x)$ under both the cases stated in the definition of Γ_t at (16), completing the proof.

Q.E.D.

[17]If $y \geq 0$, then $y = 0$ if and only if $\int y \,\mathrm{d}\mathbb{P} = 0$. See Theorem 1.1.20 by Tao (2010).

For the following lemma, consider the setting and notation of the sequential planner's problem in section 4.

LEMMA A.4 *Fix x with $x \in \mathbb{S}_0$, $\epsilon > 0$ and $t \in \mathbb{N}$. If assumptions 2.1 - 2.5 hold, then there exists $\bar{r} \in \mathbb{R}_+$ such that for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have $\tilde{r}(x_i) \leq \bar{r}$ for each $i \leq t$.*

PROOF: Fix x with $x \in \mathbb{S}_0$, $\epsilon > 0$ and $t \in \mathbb{N}$. Select $(x_i)_{i=0}^{\infty}$ satisfying $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$.

Since $x_i \in \Gamma_{i-1}(x_{i-1})$, by the feasibility correspondence (Equation (16)) and homogeneity of degree one (Assumption 2.3 and recall discussion preceding Equation (8)) of the production function F, we have

$$
\begin{aligned}
\mathbb{E}x_i = \int x_i \,\mathrm{d}\mathbb{P} &\leq \int (1 + \tilde{r}(x_{i-1}))x_{i-1} + \tilde{w}(x_{i-1})e_{i-1}\,\mathrm{d}\mathbb{P} \\
&= (1 + F_1(\mathbb{E}x_{i-1}, L) - \delta)\mathbb{E}x_{i-1} \\
&\quad + F_2(\mathbb{E}x_{i-1}, L)L \\
&= F(\mathbb{E}x_{i-1}, L) + (1-\delta)\mathbb{E}x_{i-1}
\end{aligned} \tag{A.39}
$$

for each $i \in \mathbb{N}$.

Define $\hat{F}(K) := F(K, L) + (1-\delta)K$ and note $\hat{F}$ will be strictly increasing. By (A.39),

$$
\mathbb{E}x_i \leq \hat{F}(\mathbb{E}x_{i-1}), \qquad i \in \mathbb{N} \tag{A.40}
$$

As such, for any $k > 1$, by a simple inductive argument (Claim D.5 in the online appendix), we can show

$$
\mathbb{E}x_k \leq \hat{F}^{k-i}(\mathbb{E}x_i), \qquad \forall\, i \leq k \tag{A.41}
$$

Next, since ν is concave, from Jensen's inequality (fact C.4 in the online appendix),

$$
\begin{aligned}
\epsilon \leq \rho_t(x_t, x_{t+1}) &= \int \nu\left((1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1}\right)\,\mathrm{d}\mathbb{P} \\
&\leq \nu\left(\int (1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t\,\mathrm{d}\mathbb{P}\right) \\
&= \nu(\hat{F}(\mathbb{E}x_t))
\end{aligned} \tag{A.42}
$$

Note the inverse of v, v^{-1}, is also increasing since v is increasing. (The inverse of ν exists by Assumption 2.5.) From (A.42), $v^{-1}(\epsilon) \leq \hat{F}(\mathbb{E}x_t)$. And, by (A.41),

$$
\nu^{-1}(\epsilon) \leq \hat{F}(\mathbb{E}x_t) \leq \hat{F}^{t-i+1}(\mathbb{E}x_i), \qquad \forall\, i \leq t \tag{A.43}
$$

Next, Let G^j denote the inverse of $\hat{F}^j$. Since $\hat{F}$ is strictly increasing, by (A.43), we have $\mathbb{E}x_i \geq G^{t-i+1}(\nu^{-1}(\epsilon))$ for each $i \leq t$. Define

$$
\underline{K} := \min_{i \in \{0, \ldots, t\}} \{G^{t-i+1}(\nu^{-1}(\epsilon))\}
$$

and note $\mathbb{E}x_i \geq \underline{K}$ for each $i \leq t$.

Finally, let $\bar{r} := F_1(\underline{K}, L) - \delta$. Note $F_1(K, L)$ is decreasing in the first argument since F is concave and conclude

$$\tilde{r}(x_i) = F_1(\mathbb{E}x_i, L) - \delta \leq F_1(\underline{K}, L) - \delta := \bar{r}, \qquad \forall\, i \leq t$$

Since $\bar{r}$ depends only on t and ϵ, the above will hold for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$.

Q.E.D.

Proof of proposition 4.5: Fix any x satisfying $x \in \mathbb{S}_0$, $\epsilon > 0$ and t. By Lemma A.4, there exists $\bar{r} \in \mathbb{R}_+$ such that for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have

$$r(x_i) \leq \bar{r}, \qquad \forall\, i \leq t$$

Moreover, since aggregate capital will be bounded from above, the maximum possible wage rate will be bounded above by a constant, which we now denote as $\bar{\varpi}$.

Let $(x_i)_{i=0}^{\infty}$ be any sequence satisfying $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$. For any $i \in \{1, \ldots, t+1\}$,

$$\begin{aligned} x_i &\leq (1+\bar{r})x_{i-1} + \bar{\varpi} e_{i-1} \\ &\leq (1+\bar{r})^2 x_{i-2} + \bar{\varpi} e_{i-1} + (1+\bar{r})\bar{\varpi} e_{i-2} \\ &\quad \vdots \\ &\leq (1+\bar{r})^i x + \bar{\varpi} \sum_{j=0}^{i-1} (1+\bar{r})^j e_{i-j-1} \end{aligned}$$

Let $W_i := \bar{\varpi} \sum_{j=0}^{i-1} (1+\bar{r})^j e_{i-j-1}$ and note $\|W_i\|$ will be finite. Next, since $x_i \geq 0$,

$$x_i \leq (1+\bar{r})^i x + W_i \implies (x_i)^2 \leq \Big((1+\bar{r})^i x + W_i\Big)^2$$

As such, for all $i \in \{1, \ldots, t+1\}$,

$$\begin{aligned} \|x_i\| &\leq \|(1+\bar{r})^i x + W_i\| \\ &\leq (1+\bar{r})^i \|x\| + \|W_i\| \\ &:= \hat{M}_i \in \mathbb{R} \end{aligned}$$

To conclude, let $\hat{M} := \max\{\|x\|, \hat{M}_1, \ldots, \hat{M}_{t+1}\}$. The scalar $\hat{M}$ depends only on x, $\bar{r}$, $\bar{\varpi}$, t and ϵ. As such, for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ that satisfies $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have $\|x_i\| \leq \hat{M}$ for each $i \leq t+1$.

Q.E.D.

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