EXISTENCE OF RECURSIVE CONSTRAINED OPTIMA IN THE HETEROGENEOUS AGENT GROWTH MODEL

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4 July, 2017

This paper establishes the existence of recursive constrained optimal policies, as considered by Dávila et al. (2012), in a neoclassical growth model with idiosyncratic shocks, incomplete insurance markets and production. A constrained planner chooses individual saving and consumption through time, constrained by infinitely many agents' budget constraints, to maximise aggregate welfare. Due to the structure of the recursive problem and the Inada conditions interacting with an infinite dimensional state and action space, the constrained planner's feasibility correspondences have non-compact image sets. The constrained planner's problem thus does not meet the requirements of standard dynamic optimisation theory used show existence of optimal policies. To address the challenge, first, the paper transforms the recursive problem to a sequential problem and shows existence of sequential constrained optimal policies implies existence of recursive constrained optimal policies. Second, the paper introduces a new existence result for non-compact dynamic optimisation problems and uses the result to verify existence of sequential constrained optimal policies.

[1]Email: akshay.shanker@anu.edu.au. I thank Dan Cao, Wouter den Haan, Chao He, Warwick McKibbin, Ben Moll, John Stachurski, David Stern, Martin Wolf for discussions and suggestions that have been incorporated into this paper. I also thank seminar participants at the London School of Economics Centre for Macroeconomics and ANU Research School of Economics for useful discussions.

Draft only, please download the latest version at `https://github.com/mathuranand/Existence_of_Social_Optimia_Aiyagari`

1. INTRODUCTION

The neoclassical growth model with idiosyncratic shocks, incomplete insurance markets and production, also known as the Aiyagari-Huggett (Aiyagari, 1994; Huggett, 1993) model, has developed into a leading model of dynamic macroeconomics. Macroeconomists use the model to study consumption dynamics (Berger and Vavra, 2015), shapes of wealth distributions (Benhabib et al., 2015), asset pricing (Krusell et al., 2011) and monetary and fiscal policy dynamics (Kaplan et al., 2016; Kaplan and Violante, 2010; Heathcote, 2005; Mckay and Reis, 2016), to name a few topics.

Recently, macroeconomists have begun studying optimal policy in a general setting of the model. A natural way to study optimal policy in the the Aiyagari-Huggett model is to study solutions to a *constrained planner's problem*. Like any realistic government overseeing a market economy, the constrained planner cannot complete insurance markets, but must improve welfare subject to each agents' idiosyncratic budget constraint. First introduced in discrete time by Dávila et al. (2012) and continuous time by Nuño and Moll (2017), the constrained planner concept has led to a growing literature on optimal monetary and fiscal policy in incomplete market models (Acikgoz, 2013; Bhandari et al., 2017; Chen and Yang, 2017; Nuno and Thomas, 2017; Park, 2014).

Despite the importance of the constrained planner for optimal policy analysis, due to the mathematical challenges brought on by the infinite dimensional structure of the Aiyagari-Hugget model, existence of constrained optimal policies has not been verified. This paper provides a proof for the existence of discrete time recursive constrained optimal policies as originally considered by Dávila et al. (2012).[1] The paper also presents an easy to verify general result that can be applied to planner problems in other heterogeneous agent models.

From the perspective of applied modellers, the existence result here helps confirm the surprising policy conclusions emerging from computations of constrained optimal policies are sound. In simulations with high income inequality and a wealth distribution resembling actual U.S. data, Dávila et al. (2012) show a decentralised equilibrium in the Aiyagari-Huggett model under-saves compared to the constrained optimum, which justifies saving

[1] In this basic setting, the government does not consume and there are no net transfers or nominal rigidities.

subsidies. This is in contrast to the long held belief of sub-optimal over-saving in incomplete market models, which justifies capital taxation (see Aiyagari (1995) and discussion by Chen and Yang (2017)). Moreover, in the high income inequality case, the constrained planner's solution path does not converge to a steady-state, but displays ever increasing wealth inequality. Verifying existence helps confirm such computed solutions are not pathological and creates a foundation for further research on optimal policy dynamics in the Aiyagari-Huggett model.

Mathematical Challenges

Because the constrained planner controls the assets of infinitely many agents through time, both the planner's state, a distribution of agents over assets, and action, a policy function, are infinite dimensional. The literature has made significant progress by establishing infinite dimensional necessary conditions (Dávila et al. (2012) in discrete time and Nuño and Moll (2017) and Nuño (2017) in continuous time). However, continuity and compactness, assumptions used by standard dynamic optimisation theory *to verify existence of solutions*, are more difficult to verify when spaces are infinite dimensional (see Mas-colell and Zame (1991) for an overview of issues in infinite dimensional topology). In the case of the constrained planner, the feasibility correspondence fails to have compact image sets, that is, the image of a compact set under the correspondence will not be compact. The standard assumptions of existing dynamic optimisation theory (Stokey and Lucas (1989), Acemoglu (2009) ch.6 or Stachurski (2009)) are thus not satisfied.

The constrained planner's feasibility correspondences fail to have compact image sets for two reasons. First, as suggested by Dávila et al. (2012), individual agents' asset spaces will not be bounded. We are also unable to justify restrictions such as equicontinuity or monotonicity on the space of policy functions. As such, the image sets of the feasibility correspondences will not be compact in the sup-norm topology or topology of point-wise convergence. At the same time, the recursive problem, the form of the problem considered by Dávila et al. (2012), will not be defined on topological spaces where the feasibility correspondence is compact-valued.

Second, the feasibility correspondences have non-compact image sets because of a discontinuity. The discontinuity arises due to the Inada conditions — as capital converges to zero, interest rates diverge and the variance of feasible asset distributions can diverge to infinity as the mean converges to zero.

To resolve the first challenge, the paper transforms the recursive problem to a sequential problem, where the planner assigns assets to individuals based on their history of shocks. The sequential planner's problem will be well-defined on the space of square integrable random variables. And with the weak topology, the sequential planner's feasibility correspondences will be compact valued. To connect the sequential problem back to the recursive problem, the first innovation of the paper is a novel projection argument to show sequential solutions imply recursive solutions.

While feasibility correspondences for the sequential planner have compact values, due to the discontinuity around zero capital, image sets will still be non-compact. To resolve this second challenge, the paper introduces and applies an existence result for infinite horizon dynamic optimisation that weakens the requirement for feasibility correspondences to have compact image sets. The main assumption of the existence result can be verified by checking the variance of feasible sequences of asset distributions leading to a strictly positive per-period pay-off at a time in the future is bounded.

Related Literature

The existence result in this paper builds on important work on non-compact optimisation (Feinberg et al., 2013) and dynamic programming (Feinberg et al., 2012). To generalise from the requirement of compact image sets, Feinberg et al. (2012) introduce a condition called K-Sup-Compactness,[2] explored further by Feinberg et al. (2013), on the per-period pay-off. However, when utility is bounded below, as in the case of Constant Relative Risk Aversion (CRRA) utility used by Dávila et al. (2012), K-Sup-Compactness is too strong. In particular, when utility is bounded below, K-Sup-Compactness implies compact images. The main assumption of this paper is weaker than K-Sup-Compactness.

Pathologies similar to the second problem discussed above are encountered in existence proofs of general equilibrium in the Aiyagari model with aggregate shocks. While the proofs for general equilibrium rely on fixed point arguments as opposed to maximum theorem arguments here, Cao (2016) and

[2]Or K-Inf-Compactness for minimisation problems.

Cheridito and Sagredo (2016) point out divergent interest rates also complicates the proof by Miao (2006) when production satisfies the Inada conditions. The solution proposed by Cao (2016) involves solving a sequence of finite horizon problems and showing aggregate capital has a strictly positive lower bound using agents' Euler equations. By contrast, the solution of this paper is to state a general theorem for the infinite horizon dynamic optimization problem on non-compact spaces.

The Aiyagari-Huggett model is not the only model with infinite dimensional state-spaces. The growing interest in heterogeneity in economics has led to a variety of such models, studying economic geography, industry dynamics and trade, to name a few topics. Many of these models also study social optimality over infinite dimensional states. For instance, a large literature (Boucekkine et al., 2009; Brock et al., 2014; Fabbri et al., 2015) has shown existence and characterised optimal solutions in models of economic geography in continuous time. Lucas and Moll (2014) also solve an infinite dimensional planner's problem to control individual search efforts subject to the law of motion of a density. However, to the best of my knowledge, these models do not encounter the non-compactness of the Aiyagari-Huggett model. (Note the challenge for the Aiyagari-Huggett constrained planner is not that the state-space is infinite dimensional *per se*. Extending standard dynamic optimisation arguments from $\mathbb{R}^N$ to an infinite dimensional space is trivial, *if the conditions of compactness and hemicontinuity are satisfied in a suitable topology*. However, these conditions are harder to satisfy in infinite dimensional spaces.)

On the other hand, the state in a model may be infinite dimensional, but with simplifying assumptions, the dynamics of the distribution may only depend on finite dimensional variables. For example, in the industry dynamics model by Hopenhayn (1992), the planner can control total demand (see p. 1134), in the growth model with financial frictions by Itskhoki and Moll (2014), the planner can control aggregate consumption and in the incomplete markets model with endogenous growth by Brunnermeier and Sannikov (2016), the constrained planner can control a common investment rate across heterogeneous households.[3]

However, extensions of the above models may require the results developed here. Moreover, the methodology of this paper is directly relevant for fu-

[3]Other infinite dimensional models where the economy can be collapsed to finite dimensional states include Melitz (2003), Koren and Tenreyro (2013) and Buera and Moll (2015).

ture study of constrained planner problems in extensions and applications of the Aiyagari-Huggett model: this includes Aiyagari-Huggett models incorporating aggregate shocks (Krusell and Smith, 1998), permanent income shocks Kuhn (2013), endogenous labour supply (Marcet et al., 2007), overlapping generations (Heathcote et al., 2010) or monetary and fiscal policy (Kaplan et al., 2016; Kaplan and Violante, 2010; Heathcote, 2005; Mckay and Reis, 2016; Nuno and Thomas, 2017).

2. CONSTRAINED PLANNER PROBLEMS

This section presents the recursive and sequential constrained planner's problems in a standard Aiyagari (1994) model. Both Dávila et al. (2012) and Nuño and Moll (2017) formulate their problem as a recursive problem; the exposition here will follow the discrete time version in Dávila et al. (2012), only I place more formal mathematical structure on the model.

In the recursive problem, the constrained planner instructs agents on their next period assets based on their current assets, shock and the aggregate distribution of agents. The recursive problem will be a stationary primitive form[4] infinite horizon dynamic optimisation problem, where the planner selects an action (policy function) to drive a state (wealth distribution).

In the sequential problem, the constrained planner instructs agents each period on next period assets based on their history of shocks up to the period. The sequential problem will be a non-stationary reduced form infinite horizon dynamic optimisation problem, where the planner selects a sequence of states (random variables).[5]

The online appendix contains an overview of mathematical concepts used in this paper.

[4]The distinction between primitive form and reduced form problem is discussed by Sorger (2015), Section 5.1.

[5]In the context of a constrained planner, the terminology 'sequential' and 'recursive' problems is overloaded. The distinction here follows the distinction between 'sequential competitive equilibria' and 'recursive competitive equilibria' made by Miao (2006) and Cao (2016). In contrast to the distinction made here, the term sequential problem is often used to refer to the problem maximising the infinite sum of pay-offs as opposed to the Bellman Operator representation of the same problem. For infinite dimensional and stochastic problems, both sequential and recursive formulations can be written as a deterministic sequence problem (maximising the sum of discounted pay-offs) and using a deterministic Bellman Equation. For example, (14) compared below to (14) in the online appendix. This paper uses the term *sequence problem* to refer to a problem such as (14).

2.1. *The Aiyagari Model*

Time is discrete and indexed by $t \in \mathbb{N}$. There are a continuum of identical individuals indexed by $i \in [0,1]$. Let A, with $A := [0,\infty)$, be the agents' asset space[6] and define E as the agents' labour endowment space. Assume $E \subset \mathscr{B}(\mathbb{R}_+)$, where $\mathscr{B}(\mathbb{R}_+)$ are the Borel sub-sets of $\mathbb{R}_+$. Let S, where $S := A \times E$, denote the agents' state space.

At time zero, each agent i draws an initial asset level x_0^i, with x_0^i taking values in A. In subsequent periods, each agent receives a sequence of labour endowment shocks $(e_t^i)_{t=0}^\infty$, with e_t^i taking values in E for each t and i. Assume a common probability space $(\bar{\Omega}, \Sigma, \bar{\mathbb{P}})$ for all uncertainty, that is, x_0^i and $(e_t^i)_{t=0}^\infty$ for each i are random variables defined on $(\bar{\Omega}, \Sigma, \bar{\mathbb{P}})$.

ASSUMPTION 2.1 The shocks satisfy the following conditions:

1. e_t^i and x_0^i has finite variance for each t and i
2. $(e_t^i)_{t=0}^\infty$ and x_0^i are independently and identically distributed across i
3. x_0^i is independent of $(e_t^i)_{t=0}^\infty$ for each i.

The finite variance assumption allows us to work in the L^2 space of square integrable random variables where compact sets are easier to find. The general existence result I present in this paper can also be applied to other topological vector spaces; research on such models is left for further work.

Let μ_0 denote the common joint distribution of x_0^i and e_0^i. That is,

$$\mu_0(B) = \bar{\mathbb{P}}\{\omega \in \bar{\Omega} \,|\, x_0^i(\omega), e_0^i(\omega) \in B\}, \qquad B \in \mathscr{B}(S),\ i \in [0,1] \tag{1}$$

Let P denote the joint distribution of x_0^i and $(e_t^i)_{t=0}^\infty$.

ASSUMPTION 2.2 For each $i \in [0,1]$, the shocks $(e_t^i)_{t=0}^\infty$ are a stationary Markov process with common Markov kernel Q and stationary marginal distribution ψ.

[6]As in the computations by Dávila et al. (2012), I assume a zero lower bound on assets to simplify the notation. In general, the Aiyagari model allows a strictly negative lower bound, however a zero lower bound is a common assumption, see also Miao (2006) and Cao (2016). The results here can be extended to a model with a negative lower bound, however an additional constraint on the state-space to ensure interest rates are not so high as to violate budget constraints will need to be added.

Assumption 2.2 can be relaxed to boundedness of the mean of the endowment shock, however, the stationarity assumption simplifies notation.

We do not need further assumptions on E for the proofs in this paper. However, A will, in general, be unbounded above, even if E is bounded. Dávila et al. (2012) assume an upper-bound each period on A, however, simulations by Dávila et al. (2012) (see fig.3 and discussion at section 5.4 in Dávila et al. (2012)) and by Nuño and Moll (2017) show a solution with diverging variance, implying a sequence of asset distributions with an increasing upper-bound.[7]

No Aggregate Uncertainty and the Aggregate State

Under no aggregate uncertainty, aggregate variables only depend on the common theoretical distribution of individual shocks, rather than the realisation of individual shocks. Often, no aggregate uncertainty is an assumption in heterogeneous agent models. Here, I opt for a formal construction following Acemoglu and Jensen (2015).[8] Let λ denote Lebesgue measure:

PROPOSITION 2.1 *Let $g\colon S \to \mathbb{R}$ be a measurable function such that $g(x_0^i, e_0^i)$ has finite variance. If Assumption 2.1 holds, then*

$$\int g(x_0^i, e_0^i)\lambda(di) = \int\int g(x,e)\mu_0(dx,de) \tag{2}$$

holds $\bar{\mathbb{P}}$-almost everywhere.

PROOF: The random variables $g(x_0^i, e_0^i)$ are uncorrelated across i, have a common bounded variance and a common mean $\int\int g(x,e)\mu_0(dx,de)$. We

[7] Popoviciu's inequality for variance states the variance of any bounded random variable is bounded. Dávila et al. (2012) compute a solution path with ever increasing variance that does not converge to an upper-bound.

[8] The IID assumption on individual shocks is intuitively appealing. However, the assumption means the LLN does not hold — discussion of the challenges goes back to Judd (1985). The solution used here and by Acemoglu and Jensen (2015) still faces difficulties in interpretation, in particular, the LHS of (2) is a vector valued Pettis integral, which, unlike the Bochner integral has no link to Lebesgue integral of $i \mapsto g(x_0^i(\omega), e_0^i(\omega))$ evaluated at *a realisation of* $\omega \in \bar{\Omega}$. See section 5.3 in Al-Najjar (2004). Alternatives include Miao (2002) and Miao (2006), who constructs an underlying space following Feldman and Gilles (1985) and drops the IID assumption. Notwithstanding, the results here do not depend on the specific construction of no aggregate uncertainty, so long as the constrained planner's state-space can be represented using a deterministic distribution.

thus satisfy the conditions of Theorem 2 in Uhlig (1996), which gives the result. *Q.E.D.*

As such, the following holds $\bar{\mathbb{P}}$ - almost everywhere:

$$\int \mathbb{1}_B\{x_0^i, e_0^i\}\lambda(di) = \int\int \mathbb{1}_B(x,e)\mu_0(dx,de) = \mu_0(B), \qquad B \in \mathscr{B}(S) \tag{3}$$

The expression says, with probability one, the empirical distribution of agents over S agrees with the theoretical probability distribution of the individual shocks — the constrained planner can use μ_0 to know the mass of agents in any $B \in \mathscr{B}(S)$.

The distribution μ_0 becomes the initial state for the recursive constrained planner problem. The recursive problem we will consider is one where the planner selects measurable policy function for each h_t for each t, with $h_t \colon S \to A$. Each h_t instructs agents on $t+1$ assets given their time t asset and shock. A sequence of policy functions $(h_t)_{t=0}^{\infty}$ chosen by the constrained planner generates a sequence of assets for each agent, $(x_t^i)_{t=0}^{\infty}$, by

$$x_{t+1}^i = h_t(x_t^i, e_t^i), \qquad t \in \mathbb{N},\ i \in [0,1] \tag{4}$$

Since h_t applies to all agents i, the distribution of $\{x_t^i, e_t^i\}$ will be identical across i. Moreover, $\{x_t^i, e_t^i\} \sim \mu_t$ for each i, where $(\mu_t)_{t=0}^{\infty}$ satisfies the recursion

$$\mu_{t+1}(B_A \times B_E) = \int\int \mathbb{1}_{B_A}\{h_t(x,e)\}Q(e, B_E)\mu_t(dx,de), \qquad t \in \mathbb{N} \tag{5}$$

for each t and $B_A \times B_E \in \mathscr{B}(S)$. See Claim 9.1 in the online appendix for the proof.

If each x_t^i has finite variance, once again by Proposition 2.1, the time t empirical distribution of agents over S will satisfy

$$\int \mathbb{1}_B\{x_t^i, e_t^i\}\lambda(di) = \mu_t(B), \qquad B \in \mathscr{B}(S),\ t \in \mathbb{N} \tag{6}$$

with probability one.

Production

Assume a representative firm rents capital (assets) from individuals and hires workers to produce output Y_t:

$$Y_t = F(K(\mu_t), L) - \delta K(\mu_t) \tag{7}$$

where $F\colon \mathbb{R}^2_+ \to \mathbb{R}_+$. When the state is μ_t, using again the LLN argument from Proposition 2.1, total capital and labour in the economy is

$$K(\mu_t) \colon = \int \int x \mu_t(dx, de) = \int x^i_t \lambda(di) \tag{8}$$

$$L = \int e \int \mu_t(dx, de) = \int e^i_t \lambda(di) \tag{9}$$

Labour, L, will be constant according to Assumption 2.2.

ASSUMPTION 2.3 The production function F is differentiable on $\mathbb{R}_{++}$, homogeneous of degree one, strictly concave and for any $\hat{L} > 0$ and $\hat{K} > 0$ satisfies

1. $\lim_{K \to \infty} F_1(K, \hat{L}) = 0$ and $\lim_{K \to 0} F_1(K, \hat{L}) = \infty$ (Inada conditions)
2. $F(0, \hat{L}) = F(\hat{K}, 0) = 0$
3. $K \mapsto F(K, \hat{L})$ is bijective.

Budget Constraints and Utility

Interest and wage rates in the economy will be

$$r(\mu_t) \colon = F_1 \left(K(\mu_t), L\right) - \delta, \quad w(\mu_t) \colon = F_2 \left(K(\mu_t), L\right)$$

Given the aggregate state μ_t, an agent i with asset x^i_t and endowment shock e^i_t must satisfy their budget constraint

$$0 \leq x^i_{t+1} \leq (1 + r(\mu_t))x^i_t + w(\mu_t)e^i_t \tag{10}$$

where x^i_{t+1} is the next period asset. If x^i_0 has finite variance and $r(\mu_t)$ is real-valued for each t, then if $(x^i_t)^{\infty}_{t=0}$ satisfies (10), x^i_t will have finite variance for each t (see Claim 9.3 in the online appendix.) Consumption for each agent i will be

$$c^i_t = (1 + r(\mu_t))x^i_t + w(\mu_t)e^i_t - x^i_{t+1}$$

Integrating across agents' budget constraints at Equation (10) and using the definition of interest and wages rates, along with homogeneity of the production function (see Theorem 2.1 in Acemoglu (2009)) gives a law of motion for aggregate capital

$$K(\mu_{t+1}) \leq (1 + r(\mu_t))K(\mu_t) + w(\mu_t)L = F(K(\mu_t), L) + (1 - \delta)K(\mu_t) \tag{11}$$

From the law of motion above, there exists an upper-bound $\bar{K}$ such that given any initial aggregate level of capital below $\bar{K}$, aggregate capital for wealth distributions satisfying (10) will never exceed $\bar{K}$ (see fig 9.3 in Acemoglu (2009)).

ASSUMPTION 2.4 The initial wealth distribution μ_0 satisfies $K(\mu_0) < \bar{K}$.

Turning to consumer utility, let $\nu\colon \mathbb{R}_+ \to \mathbb{R}_+$ be each consumer's utility function. Time t utility for agent i will be $v(c_t^i)$.

ASSUMPTION 2.5 The utility function ν is strictly increasing, bijective, concave and upper semicontinuous.

I leave out a definition of a competitive equilibrium as it is standard, for example, see Aiyagari (1994), Dávila et al. (2012), Kuhn (2013), Miao (2002) or Acikgoz (2015).

2.2. *Recursive Constrained Planner*

Let $\mathscr{P}(S)$ denote the space of Borel probability measures on S. The recursive planner's state-space, $\mathbb{M}$, will be a subspace of $\mathscr{P}(S)$ such that for each μ, with $\mu \in \mathbb{M}$, satisfies:

1. the marginal distribution across E, $\int \mu(dx, \cdot)$, agrees with ψ
2. the marginal distribution across A, $\int \mu(\cdot, de)$, has finite variance
3. aggregate assets satisfy $\int \int x\mu(dx, de) \in [0, \bar{K}]$.

Let $\mathbb{Y}$ denote the space of measurable functions h where $h\colon S \to A$. The space $\mathbb{Y}$ will be the *action-space* and the constrained planner picks a policy $h_t \in \mathbb{Y}$ for each t and agents' assets transition according to Equation (4).

Define a correspondence Λ, with $\Lambda \colon \mathbb{M} \twoheadrightarrow \mathbb{Y}$, mapping an economy's state to feasible policy functions as follows:

$$\Lambda(\mu) \colon = \begin{cases} h \in \mathbb{Y} \mid 0 \le h(x,e) \le (1 + r(\mu))x + w(\mu)e, & \text{if } K(\mu) > 0 \\ h \in \mathbb{Y} \mid h = 0, & \text{if } K(\mu) = 0 \end{cases} \tag{12}$$

The (in) equalities above hold μ - almost everywhere. We are unable to place restrictions on $\mathbb{Y}$ such that the correspondence Λ has compact image sets in a suitable topology, for details see section 5.

Following Equation (5), given a time t distribution of agents on S, μ, and policy function h, the operator $\Phi \colon \operatorname{Gr}\Lambda \to \mathbb{M}$ is defined by

$$\Phi(\mu, h)(B_A \times B_E) \colon = \int \int \mathbb{1}_{B_A}\{h(x,e)\} Q(e, B_E) \mu(dx, de) \tag{13}$$

where $B_A \times B_E \in \mathscr{B}(S)$, gives the time $t+1$ distribution of agents. We write $\mu_{t+1} = \Phi(\mu_t, h_t)$.

The constrained planner's per-period pay-off, $u \colon \operatorname{Gr}\Lambda \to \mathbb{R}_+$, integrates utility across the empirical distribution of agents

$$u(\mu, h) \colon = \begin{cases} \displaystyle\int \int v\big((1 + r(\mu))x + w(\mu)e - h(x,e)\big) \mu(dx, de) & \text{if } K(\mu) > 0 \\ 0 & \text{if } K(\mu) = 0 \end{cases}$$

It is a straight-forward use of Jensen's inequality (Fact 8.4 in the online appendix) and homogeneity of the production function to show the integral is well-defined and real-valued.

Finally, let $\beta \in (0, 1)$ be a discount factor and let V, with $V \colon \mathbb{M} \to \mathbb{R}_+ \cup \{+\infty\}$, denote the constrained planner's value function:

$$V(\mu_0) \colon = \sup_{(\mu_t, h_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{14}$$

subject to

$$h_t \in \Lambda(\mu_t), \qquad \mu_{t+1} = \Phi(\mu_t, h_t), \qquad t \in \mathbb{N}, \qquad \mu_0 \text{ given} \tag{15}$$

DEFINITION 2.1 (**Recursive Constrained Planner's Problem**)

Given μ_0, a solution to the recursive constrained planner's problem is a sequence of measurable policy functions $(h_t)_{t=0}^{\infty}$, with $h_t\colon S \to A$ for each t and a sequence of Borel probability measures on S, $(\mu_t)_{t=0}^{\infty}$ satisfying (15) that achieves the value function:

$$V(\mu_0) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{16}$$

I now state the main result of this paper:

THEOREM 2.1 *If the recursive constrained planner's problem (Definition 2.1) satisfies Assumptions 2.1 - 2.5, then for any $\mu_0 \in \mathbb{M}$, there exists a solution $(\mu_t, h_t)_{t=0}^{\infty}$.*

Since Λ does not have compact image sets, standard existence results in dynamic optimisation theory fail (see section 5). To prove Theorem 2.1, the paper first defines a sequential planner's problem (section 2.3) and shows existence of a solution for the sequential planner implies existence of a solution for the recursive planner (Theorem 2.2 in section 2.4). The sequential planner's feasibility correspondences will still not have compact image sets around regions where capital is zero. Thus, the paper presents a general existence result for non-compact infinite horizon dynamic optimisation (Theorem 3.1 in section 3), and then checks the sequential planner's problem satisfies the conditions for existence (section 4).

Recursive Policies

If the recursive constrained planner's problem has a solution, $(\mu_t, h_t)_{t=0}^{\infty}$, for each $\mu_0 \in \mathbb{M}$, then following standard arguments, we can show there exists a policy operator $H\colon \mathbb{M} \to \mathbb{Y}$ such that the sequence $(\mu_t, H(\mu_t))_{t=0}^{\infty}$ with $\mu_{t+1} = \Phi(\mu_t, H(\mu_t))$ solves the recursive problem (Corollary 10.1 in the online appendix). Thus, if a solution to the recursive constrained planner's problem exists, then the *policy function* that maps assets and shocks to next period assets depends only on the current distribution.

2.3. *Sequential Constrained Planner*

Recall for any initial state for the recursive problem, $\mu_0 \in \mathbb{M}$, $\{x_0^i, e_0^i\} \sim \mu_0$ for each i and P denotes the joint probability distribution of $\{x_0^i, e_0^i, e_1^i \ldots\}$.

Now construct a separable probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a sequence of random variables $\{x_0, e_0, e_1, \dots\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ such that the distribution of $\{x_0, e_0, e_1, \dots\}$ is P and $(e_t)_{t=0}^{\infty}$ has a recursive representation. That is, $e_{t+1} = G(e_t, \eta_t)$ for each t where G is a measurable function and $(\eta_t)_{t=0}^{\infty}$ is a sequence of I.I.D random variables defined on $(\Omega, \mathscr{F}, \mathbb{P})$.[9] Define $(\mathscr{F}_i)_{i=0}^{\infty}$ as the natural filtration with respect to $\{x_0, e_0, e_1, \dots\}$.[10]

Intuitively, we may view realisations of $\{x_0, e_0, e_1 \dots\}$ as draws from the empirical distribution of individual shock values.

Let $\mathbb{X} \colon = L^2(\Omega, \mathbb{P})$ be the space of square integrable (with respect to $\mathbb{P}$) real-valued functions on Ω. Equip $\mathbb{X}$ with the weak topology. For any $x \in \mathbb{X}$, with $\int x \, \mathrm{d}\mathbb{P} \geq 0$, define

$$\tilde{K}(x) \colon = \int x \, \mathrm{d}\mathbb{P} \tag{17}$$

and if $\int x \mathrm{d}\mathbb{P} > 0$, define

$$\begin{aligned} \tilde{r}(x) &\colon = F_1(\tilde{K}(x), L) - \delta \\ \tilde{w}(x) &\colon = F_2(\tilde{K}(x), L) \end{aligned} \tag{18}$$

For each t, define the time t state-space for the sequential planner:

$$\mathbb{S}_t \colon = \left\{ x \in m\mathscr{F}_t \,\middle|\, 0 \leq x, \int x \, d\mathbb{P} \leq \bar{K} \right\} \tag{19}$$

where $m\mathscr{F}_t \subset \mathbb{X}$ is the space of $\mathscr{F}_t$-measurable random variables.

For each t, define the feasibility correspondence $\Gamma_t \colon \mathbb{S}_t \to \mathbb{S}_{t+1}$:

$$\Gamma_t(x) \colon = \begin{cases} y \in \mathbb{S}_{t+1} \,\middle|\, 0 \leq y \leq (1 + \tilde{r}(x))\, x + \tilde{w}(x)\, e_t & \text{if } \tilde{K}(x) > 0 \\ y \in \mathbb{S}_{t+1} \,\middle|\, y = 0 & \text{if } \tilde{K}(x) = 0 \end{cases} \tag{20}$$

For each t, define the time t pay-offs $\rho_t \colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$:

$$\rho_t(x, y) \colon = \begin{cases} \displaystyle\int v\left((1 + \tilde{r}(x))\, x + \tilde{w}(x)\, e_t - y\right) \mathrm{d}\mathbb{P} & \text{if } \tilde{K}(x) > 0 \\ 0 & \text{if } \tilde{K}(x) = 0 \end{cases} \tag{21}$$

[9]One construction is to let $\Omega = A \times [0,1]^{\mathbb{N}}$ where η_t is defined as the function $(\omega_i)_{i=0}^{\infty} \mapsto \omega_t$ on Ω. See Borovkov (2003), Section 17.1.

[10]That is, $\mathscr{F}_0$ is the σ-algebra generated by x_0 and for each $i \geq 1$, $\mathscr{F}_i$ is the σ-algebra generated by $\{x_0, e_0, \dots, e_{i-1}\}$.

Finally, let $\tilde{V}$, with $\tilde{V}\colon \mathbb{S}_0 \to \mathbb{R}_+ \cup \{+\infty\}$ denote the time 0 sequential planner's value function:

$$\tilde{V}(x_0) \colon = \sup_{(x_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$$

subject to

$$x_{t+1} \in \Gamma_t(x_t), \qquad \forall t \in \mathbb{N}, \qquad x_0 \in \mathbb{S}_0 \text{ given} \tag{22}$$

DEFINITION 2.2 (**Sequential Constrained Planner's Problem**)

Given $x_0 \in \mathbb{S}_0$, a solution to the sequential constrained planner's problem is a sequence of random variables $(x_t)_{t=0}^{\infty}$ satisfying (22) that achieve the sequential planner's value function:

$$\tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \tag{23}$$

2.4. *Sequential Solution Implies Recursive Solution*

The Bellman equation for the sequential planner's problem will be a *non-stationary Bellman equation* with a policy operator at each t, $\Theta_t \colon \mathbb{S}_t \to \mathbb{S}_{t+1}$. The optimal sequence will follow $x_{t+1} = \Theta_t x_t$. However, we cannot use the Bellman equation characterisation of the sequential problem to show existence of recursive policies because x_{t+1} still depends on the history $\{x_0, e_0, \ldots, e_t\}$. By contrast, a recursive solution requires x_{t+1} to be $\{x_t, e_t\}$ measurable; that is, we require a function $h_t \colon S \to A$ such that $x_{t+1}(\omega) = h_t(x_t(\omega), e_t(\omega))$ for $\omega \in \Omega$. To convert a sequential solution to a recursive solution, the following procedure projects a sequential solution back onto its previous period, furnishing the required measurability from properties of conditional expectation (see 9.2 by Williams, 1991 or ch. IV by Çinlar (2011)).

Given x_0 satisfying $x_0 \in \mathbb{S}_0$, let $(y_t)_{t=0}^{\infty}$ be a solution to the sequential planner's problem. Construct a candidate sequence, $(x_t)_{t=0}^{\infty}$, as follows:

$$\begin{aligned} & x_0 = y_0, \qquad x_1 = \mathbb{E}(y_1 | \sigma(x_0, e_0)) \\ & \text{and} \\ & x_{t+1} = \mathbb{E}(y_t | \sigma(x_t, e_t)), \qquad \forall t \in \mathbb{N} \end{aligned} \tag{24}$$

The term $\sigma(x_t, e_t)$ denotes the σ-algebra generated by x_t and e_t. And $\mathbb{E}(y_t | \sigma(x_t, e_t))$ denotes the conditional expectation of y_t with respect to x_t and e_t.

PROPOSITION 2.2 *Let Assumptions 2.1 - 2.5 hold. If $(y_t)_{t=0}^{\infty}$ is a solution to the sequential problem (Definition 2.2), then $(x_t)_{t=0}^{\infty}$ defined by* (24) *is a solution to the sequential problem.*

See the appendix for a proof.

THEOREM 2.2 *Let Assumptions 2.1 - 2.5 hold. If there exists a solution to the sequential problem (Definition 2.2), then there exists a solution to the recursive problem (Definition 2.1) and $V(\mu_0) = \tilde{V}(x_0)$.*

The complete proof is in the appendix and proceeds as follows. Let $(y_t)_{t=0}^{\infty}$ solve the sequential problem and let $(x_t)_{t=0}^{\infty}$ be defined by (24). Since x_{t+1} is $\sigma(x_t, e_t)$ measurable, $x_{t+1} = h_t(x_t, e_t)$ for a measurable function h_t for each t. For each t, define

$$\mu_t(B) \colon = \mathbb{P}\{x_t, e_t \in B\}, \qquad B \in \mathscr{B}(S) \tag{25}$$

The remainder of the proof verifies $(\mu_t, h_t)_{t=0}^{\infty}$ solves the recursive problem.

3. EXISTENCE THEOREMS

I now introduce a general existence result for an infinite horizon dynamic optimisation problem with non-compact feasibility correspondences on arbitrary topological spaces. After stating the general result, this section shows how to verify the conditions for the result on L^2 spaces.

We require the following definition for the main assumption. Let (X, τ) be a topological space,

DEFINITION 3.1 A function $f \colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is **mildly sup-compact** if the upper contour sets

$$UC_f(\epsilon) \colon = \{x \in X \,|\, f(x) \geq \epsilon\} \tag{26}$$

are sequentially compact for all $\epsilon > \inf f$.

A discussion on the relationship between mild sup-compactness, sup-compactness and upper semicontinuity is given in the online appendix.

3.1. *General Existence Theorem*

A non-stationary reduced form economy is a 5-tuple

$$\mathscr{E}: := ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta)$$

consisting of:

1. A topological space $(\mathbb{X}, \tau)$
2. A collection of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$, with $\mathbb{S}_t \subset \mathbb{X}$ for each t
3. A collection of non-empty feasibility correspondences $(\Gamma_t)_{t=0}^{\infty}$, with $\Gamma_t \colon \mathbb{S}_t \twoheadrightarrow \mathbb{S}_{t+1}$ for each t
4. A collection of per-period pay-offs $(\rho_t)_{t=0}^{\infty}$, with $\rho_t \colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$ for each t
5. A discount factor $\beta \in (0, 1)$.

Define the correspondence of **feasible sequences** $\mathcal{G}_t^T \colon \mathbb{S}_t \twoheadrightarrow \prod_{i=t}^{T} \mathbb{S}_i$ starting at time t and ending at time T as follows:

$$\mathcal{G}_t^T(x) := \left\{ (x_i)_{i=t}^{T} \mid x_{i+1} \in \Gamma_i(x_i), x_t = x \right\}, \qquad x \in \mathbb{S}_t \tag{27}$$

Let $\mathcal{G}$ denote $\mathcal{G}_0^{\infty}$ and let $\mathcal{G}^T$ denote $\mathcal{G}_0^T$.

Define the **value function** $\tilde{V} \colon \mathbb{S}_0 \to \mathbb{R} \cup \{-\infty, \infty\}$ as follows:

$$\tilde{V}(x) := \sup_{(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \tag{28}$$

Recall a compact-valued upper hemicontinuous correspondence has compact image sets (see Mathematical Preliminaries in the online appendix). The first assumption below is the main assumption of the paper, it relaxes the standard requirement for Γ_t to be upper hemicontinuous and compact valued and for $\mathbb{S}_t$ to be a metric space (see by Acemoglu (2009), Assumption 6.2, Kamihigashi (2017), section 6 or Stokey and Lucas (1989), Assumption 4.3. for assumptions used by the standard theory).

ASSUMPTION 3.1 For each $x \in \mathbb{S}_0$ and $t \in \mathbb{N}$, the functions $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}^{t+1}(x)$ are mildly sup-compact in the product topology (of τ topology in $\mathbb{X}$).

The next assumption is the standard growth condition (see discussion on Corollary 6.1 by Kamihigashi (2017)).

ASSUMPTION 3.2 For each $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^{\infty}$ such that any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ satisfies

$$\rho_t(x_t, x_{t+1}) \leq m_t, \qquad \forall t \in \mathbb{N} \tag{29}$$

and

$$\sum_{t=0}^{\infty} \beta^t m_t < \infty \tag{30}$$

ASSUMPTION 3.3 The functions ρ_t are sequentially upper semicontinuous for all $t \in \mathbb{N}$.

THEOREM 3.1 *If $\mathscr{E}$ satisfies Assumption 3.1 - 3.3, then the value function will satisfy $\tilde{V} < \infty$ and for every $x \in S_0$, there will exist $(x_t)_{t=0}^{\infty}$ satisfying $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ such that $\tilde{V}(x) = \sum_{t=0}^{\infty} \beta^t \rho_t (x_t, x_{t+1})$.*

See the appendix for a proof. The proof for Theorem 3.1 follows a product space approach rather than iteration of the Bellman equation. (Despite the mild sup-compactness condition, a proof using iteration of a non-stationary Bellman operator is still not possible as the Bellman operator will not maintain semicontinuity of the value function.) In essence, the proof works by showing feasible paths of states that converge to the supremum of the problem belong a compact space in the product topology (of the topology τ in $\mathbb{X}$). By contrast, the standard assumption of semicontinuity and compact-valued correspondences requires that *all* feasible sequences belong to a compact space in the product topology. A further discussion of how standard theory uses these assumptions to verify existence is in section of the online appendix.

3.2. *Checking Mild Sup-Compactness in L^2 Spaces*

Let $(\Omega, \Sigma, \varphi)$ be a finite separable measure space and let $\mathbb{X} = L^2(\Omega, \varphi)$ be the space of real-valued measurable function on Ω with

$$\|x\| := \left(\int x^2 \, d\varphi \right)^{\frac{1}{2}} < \infty$$

Equip $\mathbb{X}$ with the weak topology. Recall a sequence $(x_n)_{n=0}^{\infty}$ with $x_n \in \mathbb{X}$ for each n converges in the weak topology if $\int x_n h \, \mathrm{d}\mathbb{P} \to \int xh \, \mathrm{d}\mathbb{P}$ for each $h \in \mathbb{X}$.[11]

Unless otherwise stated, convergence and topological notions will be with respect to the weak topology.

ASSUMPTION 3.4 The state-spaces $\mathbb{S}_t$ are sequentially closed in $\mathbb{X}$ for all $t \in \mathbb{N}$.

ASSUMPTION 3.5 The correspondences Γ_t have a sequentially closed graph for all $t \in \mathbb{N}$.

ASSUMPTION 3.6 For each $t \in \mathbb{N}$, $\epsilon > 0$ and $x \in \mathbb{S}_0$, there exists a constant $\bar{M}$ such that if $(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $u_t(x_t, x_{t+1}) \geq \epsilon$, then $\|x_i\| \leq \bar{M}$ for each $i \in \{0, \dots, t+1\}$.

PROPOSITION 3.1 *Consider $\mathscr{E}$ where $\mathbb{X} = L^2(\Omega, \varphi)$ and τ is the weak topology. If $\mathscr{E}$ satisfies Assumptions 3.3, 3.5 and 3.6, then $\mathscr{E}$ satisfies Assumption 3.1.*

To summarise, the following serves as a checklist to show existence of the value function on an L^2 space:

1. Check the growth condition, Assumption 3.2
2. Check sequential semicontinuity of ρ_t for each t, Assumption 3.3
3. Check $\mathbb{S}_t$ is sequentially closed for each t, Assumption 3.4
4. Check the sequentially closed graph property of Γ_t for each t, Assumption 3.5
5. Check feasible sequences give a positive per-period pay-offs at some time in the future have a finite norm, Assumption 3.6.

4. EXISTENCE OF CONSTRAINED OPTIMA

Consider the case of the sequential constrained planner in the Aiyagari-Huggett model presented in section 2.3. Let Assumptions 2.1 - 2.4 hold and let

$$\mathscr{E} = ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta)$$

where:

[11] See ch. 5.14 and 13.8 by Aliprantis and Border (2005) or ch 5.10 by Luenberger (1968).

1. $\mathbb{X} = L^2(\Omega, \mathbb{P})$
2. The topology τ is the weak topology
3. The sequence of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$ are defined by (19)
4. The sequence of correspondences $(\Gamma_t)_{t=0}^{\infty}$ are defined by (20)
5. The sequence of pay-offs $(\rho_t)_{t=0}^{\infty}$ are defined by (21).

PROPOSITION 4.1 ***(Checking Assumption 3.2)*** *For any $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^{\infty}$ such that $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ and any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ satisfies $\rho_t(x_t, x_{t+1}) \leq m_t$ for each t.*

PROPOSITION 4.2 ***(Checking Assumption 3.3)*** *The functions $(\rho_t)_{t=0}^{\infty}$ are sequentially upper semicontinuous for each t.*

PROPOSITION 4.3 ***(Checking Assumption 3.4)*** *The state spaces $\mathbb{S}_t$ are sequentially closed for each t.*

PROPOSITION 4.4 ***(Checking Assumption 3.5)*** *The correspondences $(\Gamma_t)_{t=0}^{\infty}$ have closed graph for each t.*

PROPOSITION 4.5 ***(Checking Assumption 3.6)*** *For any $t \in \mathbb{N}$, $\epsilon > 0$ and $x \in \mathbb{S}_0$, there exists a constant $\bar{M}$ such that if $(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$, then*

$$\|x_i\| \leq \bar{M}$$

for all $i \in \{0, 1, \ldots, t+1\}$.

We are now ready to verify existence of recursive constrained optima.

PROOF OF THEOREM 2.1: Recall the setting of section 2.1 where μ_0 is the initial state of the economy and recall the random variables $\{x_0, e_0, e_1, \ldots\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ are as defined in section 2.3. By assumptions 2.1 - 2.4 and propositions 4.1 - 4.5, the economy $\mathscr{E}$ satisfies assumptions 3.2 - 3.6.

Since assumptions 3.4 - 3.6 satisfy the conditions for Proposition 3.1, $\mathscr{E}$ satisfies Assumption 3.1. As such, $\mathscr{E}$ satisfies assumptions 3.1 - 3.3 and the conditions for Theorem 3.1.

By Theorem 3.1, there exists a sequence $(y_t)_{t=0}^{\infty}$ solving the Sequential Planner's problem (Definition 2.2) such that $\tilde{V}(x_0) < \infty$. By Proposition 2.2, the sequence $(x_t)_{t=0}^{\infty}$ defined by (24) also solves the sequential planners problem. Moreover, there exists a sequence of measurable policy functions $(h_t)_{t=0}^{\infty}$ with $h_t \colon S \to A$ and $x_{t+1} = h_t(x_t, e_t)$ for each t.

By Theorem 2.2, $(h_t)_{t=0}^{\infty}$ and $(\mu_t)_{t=0}^{\infty}$ defined by Equation (25) solve the recursive problem and

$$V(\mu_0) = \tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) < \infty \tag{31}$$

Q.E.D.

5. DISCUSSION

5.1. *Non-Compactness of the Feasibility Correspondence*

Recall a correspondence $\Lambda \colon \mathbb{X} \twoheadrightarrow \mathbb{Y}$ has compact image sets if $\Lambda(C)$ is compact for compact C.

There are two reasons why the constrained planner's feasibility correspondences will not have compact image sets. The first concerns the structure of the recursive problem and the second concerns the behaviour of interest rates around regions where capital is zero.

5.1.1. *Structure of Recursive Problem*

In the recursive problem, to show Λ is has compact image sets, we need to place some further restrictions on the space $\mathbb{Y}$. I consider three topologies: the sup-norm topology, the topology of point-wise convergence and L^p weak topology.

For compactness in the sup-norm topology, the Arzela-Arscoli Theorem states that a uniformly bounded, equicontinuous family of functions on a compact interval will be compact. However, A will not be bounded and policy functions may not be bounded. A possible approaching could be restricting $\mathbb{M}$ to measures on a compact support. For each $\mu \in \mathbb{M}$, we can then restrict policy functions in $\Lambda(\mu)$ to be defined on the bounded support of μ. If the mean of μ is positive, then policy functions in $\Gamma(\mu)$ will also be bounded. Notwithstanding the pathologies (see below) as interest rates diverge, to now use the Arzela-Arscoli Theorem, we also need to restrict feasible policy functions in each period to an equicontinuous family of functions. This line of argument has so far not yielded success.

On the other hand, we could once again restrict $\mathbb{M}$ to measures on a compact support and try to verify compactness in the topology of point-wise convergence using Helly's selection theorem. To do this, we need to restrict $\mathbb{Y}$ to the space of monotone policy functions. In the context of a proof for general equilibrium in a Aiyagari-Huggett model with aggregate shocks, the approach by Cao (2016) is to justify monotonicity using the necessary Euler equations of individual agents. However, in the case of an optimisation problem, we cannot restrict the search for an optimiser based on necessary conditions for the constrained planner *because a solution for the constrained planner may not exist*. In particular, there may be sequences of functions outside the space of monotone functions converging to the supremum.

The online appendix gives further detail of pathologies in the weak topology if we let $\mathbb{Y}$ be the space of square integrable functions on S, where Φ fails to be well-defined.

Note the pathologies in the recursive problem also prevent the use of the non-compactness existence result of section 3. This is because we cannot place restrictions such as monotonicity or equicontinuity on policy functions in the upper contour sets of Assumption 3.1.

5.1.2. *Non-Compactness Near Zero Capital*

Let us consider the second pathology, concerning the behaviour of interest rates near zero capital, in the context of the sequential problem. Consider the setting and notation of section 4. Once we move to L^2 space with the weak topology, the feasibility correspondence will be compact valued and have a closed graph. The sequential problem will also be well-defined. However, since the state-space is not compact, the correspondences will not be upper hemicontinuous. In particular, there will exist a compact set $C \subset \mathbb{S}_t$ such that $\Gamma_t(C)$ is not compact. (Recall from section 8.1 in the online appendix that the image of a compact set under a compact-valued and upper hemicontinuous is compact.)

For the following claim, assume $F(K, L) = K^{\alpha - 1}$ and $\alpha = .3$. Furthermore, assume $x_0 \in \mathbb{S}_0$, the initial assets for the economy are a uniform random variable on the interval $[0, 1]$. Assume the random variable e_0 is large enough to satisfy $\tilde{w}(x_0)e_0 > 1$.

CLAIM 5.1 *There exists a compact set C, satisfying $C \subset \mathbb{S}_1$, such that the image set $\Gamma_1(C)$ is not compact.*

The details of the claim are in the appendix. Roughly, we can construct a sequence of assets in period $t = 1$ whose mean converge to zero but do not converge uniformly, leaving a smaller and smaller measure of agents with positive assets taken over to period $t = 2$. In period $t = 2$, interest rates will diverge to infinity, the smaller and smaller measure of agents with positive assets can now accumulate assets which diverge to infinity, while, in the mean, aggregate assets still converge to zero. The scenario occurs despite assets of each agent in period $t = 0$ being bounded above.

5.2. *Relationship to K-Sup-Compactness*

To relax compactness *and* continuity requirements on the feasibility correspondence, Feinberg et al. (2012) introduce a condition (assumption W* in Feinberg et al. (2012)), later defined as K-Sup-Compactness by Feinberg et al. (2013) (Definition 1.1), on per-period pay-offs.[12] Assumption 3.1 of this paper is a generalisation of a condition called K-Sup-Compactness.

Recall the definition of sup-compact from the online appendix, section 8.2. While Feinberg et al. (2012) consider stationary problems, for the sequential constrained planner's setting, K-Sup-Compactness of each per-period pay-off ρ_t becomes:

ASSUMPTION 5.1 (**K-Sup-Compact**) Let $t \in \mathbb{N}$. If C is a sequentially compact subset of $\mathbb{S}_t$, then the function $\{x_t, x_{t+1}\} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}_t^{t+1}(C)$ is sup-compact.

The assumption allows the Bellman Equation (in our case, a non-stationary Bellman Equation) to preserve semicontinuity (see Theorem 2 in Feinberg et al. (2012) and Lemma 2.5 in Feinberg et al. (2013)).

With utility bounded below, ρ_t will not satisfy K-Sup-Compactness. To see why, note $\mathcal{G}_t^{t+1}(C) = \{x, y \mid y \in \Gamma_t(x), x \in C\}$. Moreover, the upper-contour set of the function $\{x_t, x_{t+1}\} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}_t^{t+1}(C)$ when $\epsilon = 0$ will be:

$$\{x, y \mid y \in \Gamma_t(x), x \in C, \rho_t(x, y) \geq 0\}$$
$$= \{x, y \mid y \in \Gamma_t(x), x \in C\} = \mathcal{G}_t^{t+1}(C)$$

[12] Feinberg et al. (2013) use term K-Inf-Compactness, as they work with minimisaiton problems.

For the constrained planner, K-Sup-Compactness of ρ_t will then imply compact $\mathcal{G}_t^{t+1}(C)$ for compact C. However, Claim 5.1 constructs an example where $x_n \in C$ and $y_n \in \Gamma(x_n)$ such that the norm of y_n diverges, implying non-compact $\mathcal{G}_t^{t+1}(C)$.

6. CONCLUSION

This paper proved existence of recursive constrained optima in a standard Aiyagari (1994) model, as considered by Dávila et al. (2012). The results here only apply to problems where the planner's pay-offs are bounded below. The assumption was maintained in the paper so the results here can be directly applied to Dávila et al. (2012), who, in their computations, assume CRRA utility that is bounded below. Moreover, a key technical contribution of the paper was a general existence result that overcomes the non-trivial difficulties when a non-compact dynamic optimisation problems has pay-offs bounded below. An immediate path for further work is to explore the application of the already developed non-compact dynamic optimisation theory for unbounded pay-offs (Feinberg et al., 2012) to heterogeneous agent models.

In addition to applying the existence proof to other heterogeneous agent models as mentioned in the introduction, there are two more important technical areas of research. The first concerns computational methods which are known to converge to the true constrained optima. The second concerns asymptotic properties (stochastic stability) of the solution path.

APPENDIX

Proofs for Section 2.4

The following claim will be used in the proofs for section 2.4.

CLAIM 6.1 *Let $(y_t)_{t=0}^{\infty}$ be a solution to the sequential problem. If $(x_t)_{t=0}^{\infty}$ is a sequence of random variables defined by* (24)*, then*

$$x_t :\, = \mathbb{E}(y_t | \sigma(x_{t-1}, e_{t-1})) = \mathbb{E}(y_t | \sigma(x_t, e_t)), \qquad \forall t \in \mathbb{N}$$

See the online appendix for a proof.

PROOF OF PROPOSITION 2.2: We show the sequence $(x_t)_{t=0}^{\infty}$ is feasible and then show $(x_t)_{t=0}^{\infty}$ achieves the sequential planner's value function.

Before proceeding, note the following holds due to the Tower Property of conditional expectation,[13]

$$\int y_t \, \mathrm{d}\mathbb{P} = \int \mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) \, \mathrm{d}\mathbb{P} = \int x_t \, \mathrm{d}\mathbb{P}, \qquad t \in \mathbb{N} \tag{32}$$

To show feasibility of $(x_t)_{t=0}^{\infty}$, we verify $x_{t+1} \in \Gamma_t(x_t)$ for each t, where Γ_t is defined by (20). First, $x_{t+1} = h_t(x_t, e_t)$ for a measurable function h_t for each t with x_0 given. Thus each x_t can be written as a measurable function of $x_0, \dots, e_{t-1}$, implying $x_t \in m\mathscr{F}_t$ for each t. Moreover, by (32), $\int x_t \, \mathrm{d}\mathbb{P} = \int y_t \, \mathrm{d}\mathbb{P}$ and since $\int y_t \, \mathrm{d}\mathbb{P} \in [0, \bar{K}]$ for each t, we have $\int x_t \, \mathrm{d}\mathbb{P} \in [0, \bar{K}]$ for each t. And by positivity of conditional expectation,[14] since $y_t \geq 0$, we must have $x_t = \mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) \geq 0$.

Next, we check x_{t+1} satisfies the budget constraints in the definition of $\Gamma_t(x_t)$, given by (20), for each t. Set any $t \in \mathbb{N}$. There are two cases to consider: first $\int x_t \, \mathrm{d}\mathbb{P} > 0$ and second $\int x_t \, \mathrm{d}\mathbb{P} = 0$. Suppose $\int x_t \, \mathrm{d}\mathbb{P} > 0$. By (32), we have $\int y_t \, \mathrm{d}\mathbb{P} = \int x_t \, \mathrm{d}\mathbb{P} > 0$. Since $(y_t)_{t=0}^{\infty}$ is a solution to the sequential planner's problem, we have $y_{t+1} \in \Gamma_t(y_t)$ and thus, $y_{t+1} \leq (1 + \tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t$. To show $x_{t+1} \leq (1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t$, consider,

$$\begin{aligned}
x_{t+1} &= \mathbb{E}(y_{t+1}|\sigma(x_t, e_t)) \\
&\leq \mathbb{E}((1 + \tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t|\sigma(x_t, e_t)) \\
&= (1 + \tilde{r}(x_t))\mathbb{E}(y_t|\sigma(x_t, e_t)) + \tilde{w}(x_t)e_t \\
&= (1 + \tilde{r}(x_t))\mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) + \tilde{w}(x_t)e_t \\
&= (1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t
\end{aligned}$$

where, noting (32), the third line follows from

$$\tilde{r}(y_t) = F_1\left(\int y_t \, \mathrm{d}\mathbb{P}, L\right) = F_1\left(\int x_t \, \mathrm{d}\mathbb{P}, L\right) = \tilde{r}(x_t)$$

A similar argument shows $\tilde{w}(y_t) = \tilde{w}(x_t)$. The fourth line follows from Claim 6.1 and the final line follows from the definition of x_t by (**??**).

[13]See footnote 18.
[14]See 9.7 d) by Williams (1991).

On the other hand, consider the case when $\int x_t \,\mathrm{d}\mathbb{P} = 0$. We have $\int y_t = \int x_t \,\mathrm{d}\mathbb{P} = 0$ by (32). As such, since $y_{t+1} \in \Gamma_t(y_t)$ and noting the definition of Γ_t at (20), $\int x_{t+1} \,\mathrm{d}\mathbb{P} = \int y_{t+1} \,\mathrm{d}\mathbb{P} = 0$. Since $x_{t+1} \geq 0$, x_{t+1} must then satisfy $x_{t+1} = 0$.[15]

To re-cap, for each $t \in \mathbb{N}$, x_t is $\mathscr{F}_t$ measurable and satisfies $\int x_t \leq K$ and $x_t \geq 0$. Hence $x_t \in \mathbb{S}_t$. Moreover, x_{t+1} satisfies the budget constraints in the definition of Γ_t for each t. Thus $x_{t+1} \in \Gamma_t(x_t)$ for each t.

Next, we check $\rho_t(x_t, x_{t+1}) \geq \rho_t(y_t, y_{t+1})$ for each t. Select any t and consider the case $\int x_t \,\mathrm{d}\mathbb{P} > 0$. We have

$$\begin{aligned}
\rho_t(x_t, x_{t+1}) &= \int \nu\left((1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1}\right) \,\mathrm{d}\mathbb{P} \\
&= \int \nu\Big((1+\tilde{r}(y_t))\mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) + \tilde{w}(y_t)e_t \\
&\quad - \mathbb{E}(y_{t+1}|\sigma(x_t, e_t))\Big) \,\mathrm{d}\mathbb{P} \\
&= \int \nu\left(\mathbb{E}\left[(1+\tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t - y_{t+1}|\sigma(x_t, e_t)\right]\right) \,\mathrm{d}\mathbb{P} \\
&\geq \int \mathbb{E}\left(\nu((1+\tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t - y_{t+1})|\sigma(x_t, e_t)\right) \,\mathrm{d}\mathbb{P} \\
&= \rho_t(y_t, y_{t+1})
\end{aligned}$$

where the second line is due to the definition x_t and x_{t+1}. The third line follows from Claim 6.1, the fourth line follows from Jensen's inequality (Fact 8.4 in the online appendix) and the final line is due to the Tower Property.[16]

If $\int x_t \,\mathrm{d}\mathbb{P} = 0$, then $\rho_t(x_t, x_{t+1}) = 0$ by definition of ρ_t. Since $\int y_t \,\mathrm{d}\mathbb{P} = 0$ by (32), $\rho_t(y_t, y_{t+1}) = 0$. Conclude

$$\tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(y_t, y_{t+1}) \leq \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \tag{33}$$

Since $\tilde{V}(x_0)$ achieved the supremum of all pay-offs from feasible sequences, and $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} \in \Gamma_t(x_t)$ for each t, we must have $\tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$, allowing us to conclude $(x_t)_{t=0}^{\infty}$ is a solution to the sequential problem.

Q.E.D.

[15] See Theorem 1.1.20 by Tao (2010)

[16] See footnote 18.

PROOF OF THEOREM 2.2: Consider the setting of sections 2.1 - 2.3 and note μ_0 is the initial state of the economy for the recursive planner. Let $(y_t)_{t=0}^{\infty}$ be a solution to the sequential problem (Definition 2.2). Since $y_0 = x_0$ and $\{x_0, e_0, e_1, \dots\}$ have a joint distribution P, $\{y_0, e_0\} \sim \mu_0$.

Construct $(x_t)_{t=0}^{\infty}$ according to (24); recall $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} = h_t(x_t, e_t)$ for a sequence of measurable functions $(h_t)_{t=0}^{\infty}$ with $h_t \colon S \to A$ for each t. Define a sequence of Borel probability measures $(\mu_t)_{t=0}^{\infty}$ using (25) and note by definition that μ_t will the distribution of $\{x_t, e_t\}$ for each t.

We show $(\mu_t, h_t)_{t=0}^{\infty}$ solves the recursive problem in two steps. First we verify $(\mu_t, h_t)_{t=0}^{\infty}$ is feasible for the recursive problem (part 1) and then verify the sum of discounted pay-offs from $(\mu_t, h_t)_{t=0}^{\infty}$ dominates the sum of discounted pay-offs from any other feasible sequence of distributions and policy functions (part 2).

Part 1: Show $(\mu_t, h_t)_{t=0}^{\infty}$ satisfies feasibility for the recursive problem

This part shows $(\mu_t, h_t)_{t=0}^{\infty}$ satisfies (15), that is, $h_t \in \Lambda(\mu_t)$ and $\mu_{t+1} = \Phi(\mu_t, h_t)$ for each t.

Fix any $t \in \mathbb{N}$. To confirm $h_t \in \Lambda(\mu_t)$, we consider two cases: when $\int\int x\mu_t(dx, de) > 0$ and when $\int\int x\mu_t(dx, de) = 0$. First suppose $\int\int x\mu_t(dx, de) > 0$, we show

$$\mu_t\{a, e \in S \mid h_t(a, e) \notin [0, (1 + r(\mu_t))a + w(\mu_t)e]\} = 0$$

The condition says the policy function h_t satisfies agents' budget constraints μ_t - almost everywhere. Using the definition of μ_t by Equation (25),

$$\begin{aligned}
&\mu_t\{a, e \in S \mid h_t(a, e) \notin [0, (1 + r(\mu_t))a + w(\mu_t)e]\} \\
&= \mathbb{P}\{\omega \in \Omega \mid h_t(x_t(\omega), e_t(\omega)) \\
&\quad \notin [0, (1 + \tilde{r}(x_t))x_t(\omega) + \tilde{w}(x_t)e_t(\omega)]\} \\
&= \mathbb{P}\{\omega \in \Omega \mid x_{t+1}(\omega) \\
&\quad \notin [0, (1 + \tilde{r}(x_t))x_t(\omega) + \tilde{w}(x_t)e_t(\omega)]\} \\
&= 0
\end{aligned} \tag{34}$$

The first equality uses the following observation, which holds because μ_t is the joint distribution of $\{x_t, e_t\}$:

$$\int\int x\mu_t(dx, de) = \int x_t \, \mathrm{d}\mathbb{P} > 0 \tag{35}$$

whence,

$$r(\mu_t) = F_1\left(\int\int x\mu_t(dx,de), L\right) = F_1\left(\int x_t \, \mathrm{d}\mathbb{P}, L\right) = \tilde{r}(x_t) \tag{36}$$

An identical arguments shows $\tilde{w}(x_t) = w(\mu_t)$. The second equality in (34) follows from $x_{t+1} = h_t(x_t, e_t)$. The final equality is true because $\int x_t \, \mathrm{d}\mathbb{P} > 0$ and because the sequence $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} \in \Gamma_t(x_t)$ for each t. Thus, $0 \leq h_t(a,e) \leq (1 + r(\mu_t))a + w(\mu_t)e$ for μ_t almost everywhere, implying $h_t \in \Lambda(\mu_t)$ if $\int\int x\mu_t(dx,de) > 0$.

Now suppose $\int\int x\mu_t(dx,de) = 0$. Observe

$$\int x \int \mu_t(dx,de) = \int x_t \, \mathrm{d}\mathbb{P} = 0 \tag{37}$$

Since $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} \in \Gamma_t(x_t)$ for each t, $x_{t+1} = 0$. Whence,

$$\mu_t\{h_t(a,e) \neq 0\} = \mathbb{P}\{h_t(x_t,e_t) \neq 0\} = \mathbb{P}\{x_{t+1} \neq 0\} = 0$$

Thus $h_t = 0$ for μ_t almost everywhere and $h_t \in \Lambda(\mu_t)$ if $\int\int x\mu_t(dx,de) = 0$.

Now we show $\mu_{t+1} = \Phi(\mu_t, h_t)$ for each t. Let $B \in \mathscr{B}(S)$, where $B = B_A \times B_E$ for $B_A \in \mathscr{B}(A)$ and $B_E \in \mathscr{B}(E)$. Use the definition of μ_{t+1} to write

$$\begin{aligned}
\mu_{t+1}(B_A \times B_E) :&= \mathbb{P}\{x_{t+1} \in B_A, e_{t+1} \in B_E\} \\
&= \mathbb{E}\left\{\mathbb{E}\left\{\mathbb{1}_{B_A}(x_{t+1}) \times \mathbb{1}_{B_E}(e_{t+1}) \mid \mathscr{F}_{t+1}\right\}\right\} \\
&= \mathbb{E}\left\{\mathbb{1}_{B_A}(x_{t+1}) \times \mathbb{E}\left\{\mathbb{1}_{B_E}(e_{t+1}) \mid \mathscr{F}_{t+1}\right\}\right\} \\
&= \mathbb{E}\left\{\mathbb{1}_{B_A}(x_{t+1}) \times \mathbb{E}\left\{\mathbb{1}_{B_E}(e_{t+1}) | \sigma(x_0, e_0, \dots, e_t)\right\}\right\} \\
&= \mathbb{E}\left\{\mathbb{1}_{B_A}(x_{t+1}) \times \mathbb{E}\left\{\mathbb{1}_{B_E}(e_{t+1}) | \sigma(e_0, \dots, e_t)\right\}\right\} \\
&= \mathbb{E}\left\{\mathbb{1}_{B_A}(x_{t+1}) \times \mathbb{E}\left\{\mathbb{1}_{B_E}(e_{t+1}) | \sigma(e_t)\right\}\right\} \\
&= \mathbb{E}\left\{\mathbb{1}_{B_A}(x_{t+1}) \times \mathbb{P}\left\{e_{t+1} \in B_E \,\middle|\, e_t\right\}\right\} \\
&= \int\int \mathbb{1}_{B_A}\{h_t(x,e)\} \, Q(e, B_E)\mu_t(dx,de)
\end{aligned}$$

The first equality is given by the standard definition of conditional probability[17] and the Tower Property.[18] In the second equality, we 'pull out' the term

[17] Williams (1991), 9.9.

[18] Williams (1991), 9.7, i) and note the trivial σ-algebra is a subset of $\mathscr{F}_{t+1}$.

$\mathbb{1}_{B_A}(x_{t+1})$ from the conditional expectation since x_{t+1} is $\mathscr{F}_{t+1}$ measurable.[19] The fourth inequality is due to independence of x_0 and $\sigma(e_0, \dots, e_t)$ by 3) of Assumption 2.1.[20] The fifth inequality is due to the Markov property. The sixth equality is due to the definition of conditional probability.[21] The final line follows from the definition of a Markov kernel[22] and because μ_t is the distribution of $\{x_t, e_t\}$.

Part 2: Show $(\mu_t, h_t)_{t=0}^{\infty}$ achieves the value function for the recursive problem

We have shown $(\mu_t, h_t)_{t=0}^{\infty}$ satisfies feasibility. Our next task is to show

$$\sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \geq \sum_{t=0}^{\infty} \beta^t u(\tilde{\mu}_t, \tilde{h}_t)$$

holds for any other sequence $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$ feasible for the recursive problem.

Let $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$ be any other sequence of Borel probability measures on S and measurable policy functions $\tilde{h}_t \colon S \to A$ satisfying $\tilde{\mu}_0 = \mu_0$ and (15). Construct a sequence of A valued random variables $(\tilde{x}_t)_{t=0}^{\infty}$ by letting $\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t)$ for each $t > 0$ and with $\tilde{x}_0 = x_0$ given. The sequence of random variables $(\tilde{x}_t)_{t=0}^{\infty}$ will be defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. By Claim 9.2 in the online appendix, $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$ for each t. Moreover, using an analogous argument to Claim 9.3 in the online appendix, each $\tilde{x}_t$ will have finite variance and hence $\tilde{x}_t \in L^2(\Omega, \mathbb{P})$ for each t.

Our strategy is to show $(\tilde{x}_t)_{t=0}^{\infty}$ is feasible for the sequential problem and show $u(\tilde{\mu}_t, \tilde{h}_t) = \rho_t(\tilde{x}_t, \tilde{x}_{t+1})$ and $u(\mu_t, h_t) = \rho_t(x_t, x_{t+1})$ for each t. The proof will then be complete since, noting $(x_t)_{t=0}^{\infty}$ is a solution for the sequential problem (Proposition 2.2), $u(\mu_t, h_t) = \rho_t(x_t, x_{t+1}) \geq \rho_t(\tilde{x}_t, \tilde{x}_{t+1}) = u(\tilde{\mu}_t, \tilde{h}_t)$ for each t.

To check $(\tilde{x}_t)_{t=0}^{\infty}$ satisfies $\tilde{x}_{t+1} \in \Gamma_t(\tilde{x}_t)$ for each t, we check the conditions stated at (20) for each t. First, we confirm $(\tilde{x}_t)_{t=0}^{\infty}$ is adapted to the filtration $(\mathscr{F}_t)_{t=0}^{\infty}$. Proceed by induction, let $t = 1$ and consider:

$$\tilde{x}_1 = \tilde{h}_1(x_0, e_0)$$

[19] Williams (1991), 9.7, j.
[20] Williams (1991), 9.7, k).
[21] Williams (1991), 9.9.
[22] Equation (5.5) by Çinlar (2011).

Since $\tilde{h}_1$ is measurable, by the Doob-Dynkin Lemma (see Fact 8.3 in the online appendix), $\tilde{x}_1$ will be $\sigma(x_0, e_0)$ measurable. Now suppose $\tilde{x}_t$ is $\sigma(x_0, e_0, \ldots, e_{t-1})$ measurable. Consider

$$\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t) = \tilde{h}_t(g(x_0, e_0, \ldots, e_{t-1}), e_t)$$

for some measurable function $g\colon A \times E^t \to A$. Once again, since $\tilde{h}_t$ is Borel measurable, using the Doob-Dynkin Lemma, $\tilde{x}_{t+1}$ is $\sigma(x_0, e_0, \ldots, e_t)$ measurable. By the Principle of Induction, $(\tilde{x}_t)_{t=0}^{\infty}$ is adapted to the filtration $(\mathscr{F}_t)_{t=0}^{\infty}$.

To confirm $\int \tilde{x}_t \,\mathrm{d}\mathbb{P} \in [0, \bar{K}]$ for each t, since $\tilde{\mu}_t \in \mathbb{M}$ and $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$, we have

$$\int \tilde{x}_t \,\mathrm{d}\mathbb{P} = \int\int x\tilde{\mu}_t(dx, de) \in [0, \bar{K}] \tag{38}$$

Now we turn to show the sequence $(\tilde{x}_t)_{t=0}^{\infty}$ satisfies the agent budget constraints for each t. Fix any $t \in \mathbb{N}$ and suppose $\int \tilde{x}_t \,\mathrm{d}\mathbb{P} > 0$. We have

$$\begin{aligned}
\mathbb{P}\{\tilde{x}_{t+1} \notin [0, (1+\tilde{r}(\tilde{x}_t))\tilde{x}_t + \tilde{w}(\tilde{x}_t)e_t]\} &= \mathbb{P}\{\tilde{h}_t(\tilde{x}_t, e_t) \\
&\quad \notin [0, (1+\tilde{r}(\tilde{x}_t))\tilde{x}_t + \tilde{w}(\tilde{x}_t)e_t]\} \\
&= \tilde{\mu}_t\{\tilde{h}_t(x, e) \\
&\quad \notin [0, (1+r(\tilde{\mu}_t))x + w(\tilde{\mu}_t)e]\} \\
&= 0
\end{aligned}$$

The final equality holds because $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$ and because $\tilde{h}_t$ satisfies the feasibility condition shown at (12) for $\tilde{\mu}_t$ - almost everywhere.

On the other hand, suppose $\int \tilde{x}_t \,\mathrm{d}\mathbb{P} = 0$. We have $\int \tilde{x}_t \,\mathrm{d}\mathbb{P} = \int\int x\tilde{\mu}_t(dx, de) = 0$. Since $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$ satisfies $\tilde{h}_t \in \Lambda(\tilde{\mu}_t)$ for each t, $\tilde{h}_t(x, e) = 0$ for $\tilde{\mu}_t$ almost everywhere. Whence,

$$\mathbb{P}\left\{\tilde{x}_{t+1} \neq 0\right\} = \mathbb{P}\left\{\tilde{h}_t(x_t, e_t) \neq 0\right\} = \tilde{\mu}_t\left\{\tilde{h}_t(x, e) \neq 0\right\} = 0$$

The first equality holds because we defined $\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t)$. The second inequality holds because $\tilde{\mu}_t$ is the joint distribution of $\{\tilde{x}_t, e_t\}$. As such, for each t, $\tilde{x}_{t+1}$ satisfies all the conditions stated in the definition of the feasibility correspondence, (20), for $\tilde{x}_{t+1} \in \Gamma_t(\tilde{x}_t)$.

To complete the proof, for each t,

$$
\begin{aligned}
u(\mu_t, h_t) &= \int v((1 + r(\mu_t))x + w(\mu_t)e - h_t(x, e))\mu_t(dx, de) \\
&= \int v((1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - h_t(x_t, e_t))\, \mathrm{d}\mathbb{P} \\
&= \int v((1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1})\, \mathrm{d}\mathbb{P} \\
&= \rho_t(x_t, x_{t+1})
\end{aligned} \tag{39}
$$

And similarly, $u(\tilde{\mu}_t, \tilde{h}_t) = \rho_t(\tilde{x}_t, \tilde{x}_{t+1})$ for each t. As such, conclude

$$
\sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \geq \sum_{t=0}^{\infty} \beta^t \rho_t(\tilde{x}_t, \tilde{x}_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(\tilde{\mu}_t, \tilde{h}_t) \tag{40}
$$

where the inequality follows since $(x_t)_{t=0}^{\infty}$ is a solution to the sequential problem and its discounted sum of pay-offs dominate the discounted sum of pay-offs from $(\tilde{x}_t)_{t=0}^{\infty}$.

Finally, since any arbitrary feasible sequence $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$, with $\tilde{\mu}_0 = \mu_0$, satisfies (40), we have $V(\mu_0) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t)$. Moreover, since $(x_t)_{t=0}^{\infty}$ solves the sequential planner's problem, the first equality of (40) implies $V(\mu_0) = \tilde{V}(x_0)$.

Q.E.D.

Proofs for section 3

Recall the setting and notation of section 3.1, where $(\mathbb{X}, \tau)$ is a topological vector space. Throughout this section, unless otherwise stated, convergence for sequences in $\mathbb{X}$ will be with respect to the τ topology and convergence for sequences in countable Cartesian products of $\mathbb{X}$ will be in the product topology of the τ topology on $\mathbb{X}$.

We will use $\mathbf{x}$ to refer to elements of $\mathbb{X}^{\mathbb{N}}$. We can then use $(\mathbf{x}^n)_{n=0}^{\infty}$ to denote a sequence $\{\mathbf{x}^0, \dots, \mathbf{x}^n, \dots\}$, where $(\mathbf{x}^n)_{n=0}^{\infty} \in (\mathbb{X}^{\mathbb{N}})^{\mathbb{N}}$.

REMARK 6.1 Let $X = \prod_{i \in F} X_i$ denote a Cartesian product of topological spaces. Let $\pi_i \colon X \to X_i$ denote the projection map defined as $\pi_i(x) = x_i$ for each $i \in F$. Recall each projection map will be a continuous function on X when X has the product topology (see section 2.14 by Aliprantis and Border (2005)). Also recall (see section 1.8 by Tao (2013)) that the image of a (sequentially) compact set under a continuous function is (sequentially) compact. Accordingly, if a set C with $C \subset X$ is (sequentially) compact in the product topology, then $\pi_i(C)$ will be (sequentially) compact.

Finally, let the function $\phi_t \colon \mathcal{G}^{t+1}(x) \to \mathbb{R}_+$ denote $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ for each t and let $U(\mathbf{x}) \colon = \sum_{t=0}^{\infty} \rho_t(x_t, x_{t+1})$.

LEMMA 6.1 *Let Assumption 3.2 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty}$ is a sequence with $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each n and $U(\boldsymbol{x}^n) \to B$ for $B > 0$, then there exists a sub-sequence $(\boldsymbol{x}^{n_k})_{k=0}^{\infty}$ such that for all $t \in \mathbb{N}$*

$$\lim_{k\to\infty} \rho_t(x_t^{n_k}, x_{t+1}^{n_k}) \to c_t$$

where $c_t \in \mathbb{R}_+$ for each t and $c_t > 0$ for at-least one t.

PROOF: By Assumption 3.2, for each t and n,

$$m_t \geq \rho_t(x_t^n, x_{t+1}^n) \geq 0 \tag{41}$$

Accordingly, for each n, $(\rho_t(x_t^n, x_{t+1}^n))_{t=0}^{\infty}$ will belong to the set $\prod_{t=0}^{\infty} [0, m_t]$, which by Tychonoff's Theorem (see Proposition 1.8.12 by Tao (2010)) will be compact in the product topology. There then exists a sub-sequence of $(\mathbf{x}^n)_{n=0}^{\infty}$, $(\mathbf{x}^{n_k})_{k=0}^{\infty}$, such that $(\rho(x_t^{n_k}, x_{t+1}^{n_k}))_{k=0}^{\infty}$ converges for each t. Let $c_t = \lim_{k\to\infty} \rho(x_t^{n_k}, x_{t+1}^{n_k})$ and note

$$B = \lim_{k\to\infty} \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t^{n_k}, x_{t+1}^{n_k}\right) = \sum_{t=0}^{\infty} \lim_{k\to\infty} \beta^t \rho_t \left(x_t^{n_k}, x_{t+1}^{n_k}\right) = \sum_{t=0}^{\infty} \beta^t c_t \tag{42}$$

Since (41) holds, and $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ by Assumption 3.2, we can pass limits through in the second equality using dominated convergence theorem (see Corollary 7.3.15 by Stachurski (2009)). If B is strictly positive, the above means there is at least one $c_t > 0$.

Q.E.D.

LEMMA 6.2 *Let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty}$ is a sequence with $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each n and for some t*

$$\rho_t(x_t^n, x_{t+1}^n) \to c_t$$

with $c_t > 0$, then there exists an $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$.

PROOF: There exists ι such that $\epsilon: = c_t - \iota$ is strictly positive. For N large enough and any $n > N$, $\rho_t(x_t^n, x_{t+1}^n) \in [\epsilon, c_t + \iota]$, $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$ implying $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$. *Q.E.D.*

LEMMA 6.3 *Let Assumption 3.1 and Assumption 3.2 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty}$ is a sequence such that $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each $n \in \mathbb{N}$ and $U(\boldsymbol{x}^n) \to B$ where $B > 0$, then:*

1. *$(\boldsymbol{x}^n)_{n=0}^{\infty}$ has a convergent sub-sequence with a limit $\boldsymbol{x} \in \mathcal{G}(x)$, and*
2. *$B \leq U(\boldsymbol{x}) < \infty$.*

PROOF: Let x satisfy $x \in \mathbb{S}_0$ and let $(\mathbf{x}^n)_{n=0}^{\infty}$ be a sequence such that $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ where $B > 0$. By Lemma 6.1 there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that for each $t \in \mathbb{N}$, $\lim_{j \to \infty} \rho_t(x_t^{n_j}, x_{t+1}^{n_j}) = c_t$ and $c_t > 0$ for at-least one t. Re-label $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ to $(\mathbf{x}^n)_{n=0}^{\infty}$, and let P denote the subset of $\mathbb{N}$ such that $t \in P$ if and only if $c_t > 0$. The set P will be non-empty, but could be finite or infinite.

To part 1 of the lemma, consider first the case when P is infinite and then the case when P is finite.

Suppose P is infinite and consider any $t \in \mathbb{N}$. There will exist $k > t$ such that $c_k > 0$. By Lemma 6.2, there exists N and $\epsilon > 0$ such that for all $n > N$, $(x_i^n)_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$.

By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology. The space $\pi_t(UC_{\phi_k}(\epsilon))$ will also be sequentially compact by the argument in Remark 6.1. Let $\Xi_t: = \{x_1^0, \dots, x_t^N\} \cup \pi_t(UC_{\phi_k}(\epsilon))$. Since $\{x_1^0, \dots, x_t^N\}$ is sequentially compact, Ξ_t will be sequentially compact. Moreover, note $x_t^n \in \Xi_t$ for each $n \in \mathbb{N}$.

We can construct a Ξ_t as above for every $t \in \mathbb{N}$. Now let $\Xi: = \prod_{t \in \mathbb{N}} \Xi_t$. Using the Sequential Tychonoff Theorem (see Proposition 1.8.12 by Tao (2010)),

Ξ will be sequentially compact. Since for each t, $x_t^n \in \Xi_t$ for each n, $\mathbf{x}^n \in \Xi$ for each n. There then exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ converging to $\mathbf{x}$, with $\mathbf{x} \in \Xi$.

We now confirm $\mathbf{x} \in \mathcal{G}(x)$ by showing $x_{t+1} \in \Gamma_t(x_t)$ for all $t \in \mathbb{N}$. Pick any $t \in \mathbb{N}$, there will be a k satisfying $k > t$ such that $c_k > 0$. By Lemma 6.2, there exists $\epsilon > 0$ and J such that for all $j > J$ we have $(x_i^{n_j})_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$. By 3.1, the upper-contour set $UC_{\phi_k}(\epsilon)$ will be sequentially compact, moreover, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$ by the definition of $UC_{\phi_k}(\epsilon)$ at (26). As such, the sub-sequence $(x_i^{n_j})_{i=0}^{k+1}$ converges to $(x_i)_{i=0}^{k+1}$, with $(x_i)_{i=0}^{k+1} \in \mathcal{G}^{k+1}(x)$, allowing us to conclude $x_{t+1} \in \Gamma(x_t)$. Since the t was arbitrary, $x_{t+1} \in \Gamma_t(x_t)$ for each $t \in \mathbb{N}$ and $\mathbf{x} \in \mathcal{G}(x)$.

Now assume P is finite; P will have a maximum element, which we now call k. By Lemma 6.2, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that $(x_t^n)_{t=0}^{k+1} \in UC_{\phi_k}(\epsilon)$ for each $n > N$. By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology. As such, there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that $(x_t^{n_j})_{j=0}^{\infty}$ for each $t \leq k+1$. Define $(x_t)_{t=0}^{\infty}$ by setting $x_t = \lim_{j\to\infty} x_t^{n_j}$ for $t \leq k+1$ and picking any $x_{t+1} \in \Gamma_t(x_t)$ for $t \geq k+1$.

To confirm $(x_t)_{t=0}^{\infty}$ is feasible, let us check $x_{t+1} \in \Gamma_t(x_t)$ for each t. Once again, note by definition, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$. Since $UC_{\phi_k}(\epsilon)$ is sequentially compact, $(x_t)_{t=0}^{k+1} \in \mathcal{G}(x)$ and $x_{t+1} \in \Gamma_t(x_t)$ for all t satisfying $t \leq k$. On the other hand, if $t > k$, by construction, $x_{t+1} \in \Gamma_t(x_t)$, confirming $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$.

To prove part 2 of the lemma, by Assumption 3.2,

$$\rho_t(x_t^n, x_{t+1}^n) \leq m_t$$

for each t and n, where $\sum_{t=0}^{\infty} \beta^t m_t < \infty$. Whence Fatou's Lemma[23]

$$B = \limsup_{n\to\infty} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{\infty} \limsup_{n\to\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) < \infty \tag{43}$$

Upper-semicontinuity of ρ_t (Assumption 3.3) and the growth condition (Assumption 3.2) imply

$$\limsup_{n\to\infty} \rho_t(x_t^n, x_{t+1}^n) \leq \rho_t(x_t, x_{t+1}) \leq m_t, \qquad t \in \mathbb{N} \tag{44}$$

[23]See fact **??** in the online appendix and let $\Omega = \mathbb{Z}_+$ and μ be the counting measure. Also see Equation (1.1) and discussed by Kamihigashi (2017).

To complete the proof, combine (44) with (43) and conclude

$$B \leq \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = U(\mathbf{x}) < \infty$$

Q.E.D.

PROOF OF THEOREM 3.1: Fix $x \in \mathbb{S}_0$. If $U(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{G}(x)$, then our solution will be any $\mathbf{x} \in \mathcal{G}(x)$.

Next, suppose at-least one $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ satisfies $U(\mathbf{x}) > 0$. By Assumption 3.2, there exists a sequence of real numbers $(m_t)_{t=0}^{\infty}$ such that $\rho_t(x_t, x_{t+1}) \leq m_t$ for any $\mathbf{x}$ in $\mathcal{G}(x)$ and

$$\bar{B} := \sum_{t=0}^{\infty} \beta^t m_t < \infty$$

Any $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ will satisfy

$$U(\mathbf{x}) = \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t \rho_t(x_t, x_{t+1}) \leq \bar{B}$$

Consider the set $I := \{U(\mathbf{x}) \mid \mathbf{x} \in \mathcal{G}(x)\}$. The set I will be a subset of $\mathbb{R} \cup \{-\infty, \infty\}$ and so must have a supremum. Let $B := \sup I$ and note $0 \leq B \leq \bar{B} < \infty$.

Construct a sequence $(\mathbf{x}^n)_{n=0}^{\infty}$ with $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ as follows: for every $n \in \mathbb{N}$, take $\mathbf{x}^n$ such that $B - U(\mathbf{x}^n) < \frac{1}{n+1}$. Such a sequence exists, otherwise for some n, $U(\mathbf{x}) \leq B - \frac{1}{n+1}$ for all $\mathbf{x} \in \mathcal{G}(x)$ and B will not be the supremum of I.

Since $U(\mathbf{x}^n) \to B$, by Lemma 6.3, there exists $\mathbf{x} \in \mathcal{G}(x)$ such that $U(\mathbf{x}) \geq B$. Since B was the supremum for I, conclude

$$U(\mathbf{x}) = B = \tilde{V}(x) < \infty$$

Q.E.D.

PROOF OF PROPOSITION 3.1: Let x satisfy $x \in S_0$. Fix any $t \in \mathbb{N}$ and any ϵ satisfying $\epsilon > 0$. We show the upper contour sets $UC_{\phi_t}(\epsilon)$ defined by

$$UC_{\phi_t}(\epsilon) = \{(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x) \mid \rho_t(x_t, x_{t+1}) \geq \epsilon\} \tag{45}$$

are sequentially compact. In particular, we first show $UC_{\phi_t}(\epsilon)$ is a sequentially closed sub-set of $\mathbb{X}^{t+1}$ and then show $UC_{\phi_t}(\epsilon)$ is contained within a compact and metrizable set.

To show $UC_{\phi_t}(\epsilon)$ is sequentially closed in $\mathbb{X}^{t+1}$, take any sequence $(\mathbf{x}^n)_{n=0}^{\infty}$ with $\mathbf{x}^n \in UC_{\phi_t}(\epsilon)$ for each n that converges to $\mathbf{x} = (x_i)_{i=0}^{t+1}$ point-wise. Note $x_i^n \in \mathbb{S}_i$ for each $i \leq t+1$ and n. Since each $\mathbb{S}_i$ is sequentially closed (Assumption 3.4), $x_i \in \mathbb{S}_i$.

By Assumption 3.5, each Γ_i has a sequentially closed graph, and thus $x_{i+1} \in \Gamma_i(x_i)$ for each $i \leq t+1$. Noting the definition of $\mathcal{G}(x)^{t+1}$ by (27), conclude $\mathbf{x} \in \mathcal{G}(x)^{t+1}$.

We now confirm $\rho_t(x_t, x_{t+1}) \geq \epsilon$. By upper semi-continuity of ρ_t (Assumption 3.3), $UC_{\rho_t}(\epsilon) = \{(x,y) \in \operatorname{Gr}\Gamma_t \mid \rho_t(x,y) \geq \epsilon\}$ is sequentially closed. The sequence $(x_i^n)_{i=0}^{t+1}$ will satisfy $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$ and thus $\{x_t^n, x_{t+1}^n\} \in UC_{\rho_t}(\epsilon)$ for each n. Moreover, $x_{t+1} \in \Gamma_t(x_t)$. Accordingly, $\{x_t, x_{t+1}\} \in UC_{\rho_t}(\epsilon)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$. We conclude $\mathbf{x} \in UC_{\phi_t}(\epsilon)$ and $UC_{\phi_t}(\epsilon)$ is sequentially closed.

Since $\mathscr{E}$ satisfies Assumption 3.6, there will exist $\bar{M}$ such that if $(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$, then $\|x_i\| \leq \bar{M}$ for $i \in \{0, \dots, t+1\}$. Whence $UC_{\phi_t}(\epsilon)$ will be a sub-set of the space $B_{\bar{M}} := \Pi_{i=0}^{t+1}\{x_i \in \mathbb{S}_i \mid \|x_i\| \leq \bar{M}\}$.

For each $i \leq t+1$, the space $\{x_i \in \mathbb{S}_i \mid \|x_i\| \leq \bar{M}\}$ will be compact by Alaoglu's Theorem.[24] Next, $L^2(\Omega, \mathbb{P})$ is a separable space since $\mathscr{F}$ is separable.[25] As such, since $L^2(\Omega, \mathbb{P})$ is reflexive, the spaces $\{x_i \in \mathbb{S}_i \mid \|x_i\| \leq \bar{M}\}$ are metrizable and sequentially compact.[26] Moreover, by the Sequential Tychonoff's Theorem,[27] the space $B_{\bar{M}}$ will be sequentially compact in the product topology (of weak topology on $\mathbb{X}$). By the argument in the preceding paragraph, $UC_{\phi_t}(\epsilon)$ is a sequentially closed sub-set of $B_{\bar{M}}$, allowing us to conclude $UC_{\phi_t}(\epsilon)$ is sequentially compact.

Q.E.D.

[24] Mas-colell and Zame (1991), section 6.1 or Theorem 6.21 by Aliprantis and Border (2005).

[25] Ex. 1.3.9 by Tao (2010).

[26] See discussion proceeding Corollary 1.9.16 by Tao (2010) or Theorem 6.40 by Aliprantis and Border (2005).

[27] Proposition 1.8.12 by Tao (2010).

Proofs for Section 4

Recall point-wise inequalities in $\mathbb{X}$ hold $\mathbb{P}$ - almost everywhere and convergence of (x^n) with $x^n \in \mathbb{X}$ for each n will be with respect to the weak topology.

PROOF OF PROPOSITION 4.1: Let x satisfy $x \in \mathbb{S}_0$. By Assumption 2.4, aggregate capital is bounded above by $\bar{K}$. As such, for any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ and for any t, we can use Jensen's inequality (fact 8.4 in the online appendix) to arrive at

$$\rho_t(x_t, x_{t+1}) \\ \leq \nu\left((1+\tilde{r}(x_t))\tilde{K}(x_t) + \tilde{w}(x_t)L\right) \leq \nu(F(\bar{K}, L) + (1-\delta)\bar{K})$$

where the second inequality follows from homogeneity of degree one of the production function.

Let $m_t \colon = \nu(F(\bar{K}, L) + (1-\delta)\bar{K})$ for all t. As such, for any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$,

$$\rho_t(x_t, x_{t+1}) \leq m_t$$

for all t. Since m_t is a constant, the sequence $(m_t)_{t=0}^{\infty}$ will satisfy $\sum_{t=0}^{\infty} \beta^t m_t < \infty$.

Q.E.D.

Recall the definition of sequential upper semicontinuity from section 8.2 in the online appendix.

CLAIM 6.2 *Let (Ω, Σ, μ) be a finite measure space. Consider a function $g \colon \mathbb{R} \to \mathbb{R}$. Define $G \colon L^2(\Omega, \mu) \to \mathbb{R}$ as*

$$G(s) = \int g(s)\, d\mu, \qquad s \in L^2(\Omega, \mu)$$

If g is concave and upper semicontinuous, then G will be weak sequentially upper semicontinuous.

PROOF: The functional G will be upper semicontinuous with respect to norm convergence in $L^1(\Omega, \mu)$ and concave (Proposition 6.3.1 by Borwein and Vanderwerff (2010)). Since, by Holder's inequality, norm convergence in $L^2(\Omega, \mu)$ implies norm convergence in $L^1(\Omega, \mu)$, G will be upper semicontinuous with respect to norm convergence in $L^2(\Omega, \mu)$. As such, the upper-contour sets of G will be convex (Fact 2.1.7, Borwein and Vanderwerff (2010)) and norm-closed. Since norm closed convex sets in $L^2(\Omega, \mu)$ are also weak sequentially closed (see Theorem 2.5.16 and discussion by Megginson (1998) or Theorem 8.13 by Alt and Nürnberg (2016)), the upper contour sets of G will be weak sequentually closed. By the definition of upper semi-continuity in the online appendix, we conclude G will be weak sequentially upper semicontinuous.

Q.E.D.

PROOF OF PROPOSITION 4.2: Set any $t \in \mathbb{N}$ and consider sequences $(x^n)_{n=0}^{\infty}$ and $(y^n)_{n=0}^{\infty}$ with $\{x^n, y^n\} \in \operatorname{Gr}\Gamma_t$ for each n. Let $x^n \to x$ and $y^n \to y$ with $y \in \Gamma_t(x)$. To verify sequential upper semicontinuity, we show

$$\begin{aligned} \limsup_{n\to\infty} \rho_t(x^n, y^n) &= \limsup_{n\to\infty} \int \nu((1 + \tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n)\, \mathrm{d}\mathbb{P} \\ &\leq \rho_t(x, y) \end{aligned} \tag{46}$$

By Assumption 2.5, ν is concave and continuous. To use the statement made by Claim 6.2, consider a continuous concave extension of ν to $\bar{\nu}$, where $\bar{\nu}\colon \mathbb{R} \to \mathbb{R}$ and $\bar{\nu}|_{\mathbb{R}_+} = \nu$.[28] The mapping $s \mapsto \int \bar{\nu}(s)\, \mathrm{d}\mathbb{P}$ for $s \in L^2(\Omega, \mathbb{P})$ will be sequentially upper semicontinuous since $\bar{\nu}$ is concave and upper semicontinuous. As such, for any sequence in $L^2(\Omega, \mathbb{P})$ satisfying $f^n \to f$ weakly,

$$\limsup_{n\to\infty} \int \bar{\nu}(f^n)\, \mathrm{d}\mathbb{P} \leq \int \bar{\nu}(f)\, \mathrm{d}\mathbb{P} \tag{47}$$

Let $f^n\colon = (1 + \tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n$ and note $f^n \in L^2(\Omega, \mathbb{P})$ for each n. First, we show (46) for the case $\int x\, \mathrm{d}\mathbb{P} > 0$. If $\int x\, \mathrm{d}\mathbb{P} > 0$, then

$$\begin{aligned} \int f^n h\, \mathrm{d}\mathbb{P} &= (1 + \tilde{r}(x^n)) \int x^n h\, \mathrm{d}\mathbb{P} + \tilde{w}(x^n) \int e_t h\, \mathrm{d}\mathbb{P} - \int y^n h\, \mathrm{d}\mathbb{P} \\ &\to (1 + \tilde{r}(x)) \int x h\, \mathrm{d}\mathbb{P} + \tilde{w}(x) \int e_t h\, \mathrm{d}\mathbb{P} - \int y h\, \mathrm{d}\mathbb{P} \end{aligned}$$

[28]See Corollary 8.3.10 by Borwein and Vanderwerff (2010).

for any $h \in L^2(\Omega, \mathbb{P})$. Thus f^n converges weakly to $f := (1 + \tilde{r}(x))x + \tilde{w}(x)e_t - y$, implying by (47),

$$\limsup_{n\to\infty} \int \nu(f^n)\,\mathrm{d}\mathbb{P} \leq \int \nu(f)\,\mathrm{d}\mathbb{P} = \rho_t(x,y)$$

If $\int x\,\mathrm{d}\mathbb{P} = 0$, then

$$\begin{aligned}\limsup_{n\to\infty} \int \nu(f^n)\,\mathrm{d}\mathbb{P} &\leq \limsup_{n\to\infty} \nu\left(\int (1+\tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n\,\mathrm{d}\mathbb{P}\right)\\ &\leq \lim_{n\to\infty} \nu(F(\tilde{K}(x^n), L) + (1-\delta)\tilde{K}(x^n))\\ &= 0 = \rho_t(x,y)\end{aligned}$$

where the first inequality follows from Jensen's inequality. The second inequality follows from Assumption 2.3 on homogeneity of the production function (recall Equation (11)).

Q.E.D.

PROOF OF PROPOSITION 4.3: Recall the definition of $\mathbb{S}_t$:

$$\mathbb{S}_t := \left\{x \in m\mathscr{F}_t \,\middle|\, 0 \leq x\,, \int x\,\mathrm{d}\mathbb{P} \leq \bar{K}\right\}$$

Consider a sequence $(x^n)_{n=0}^{\infty}$ with $x^n \in \mathbb{S}_t$ for each n and $x^n \to x^\star$ with $x^\star \in \mathbb{X}$. We show $x^\star \in \mathbb{S}_t$.

Let $L^2(\Omega, \mathscr{F}_t, \mathbb{P})$ denote the space of $\mathscr{F}_t$ measurable functions on Ω square integrable with respect to $\mathbb{P}$. Observe $x^n \in m\mathscr{F}_t$, thus $x^n \in L^2(\Omega, \mathscr{F}_t, \mathbb{P})$ for each n. Since $L^2(\Omega, \mathscr{F}_t, \mathbb{P}) \subset \mathbb{X}$ and $L^2(\Omega, \mathscr{F}_t, \mathbb{P})$ is a Banach space (Proposition 1.3.7 by Tao (2010)), $L^2(\Omega, \mathscr{F}_t, \mathbb{P})$ will be a norm closed subspace of $\mathbb{X}$. Moreover, $L^2(\Omega, \mathscr{F}_t, \mathbb{P})$ is convex, thus by Mazur's Lemma, $L^2(\Omega, \mathscr{F}_t, \mathbb{P})$ will be a weak sequentially closed sub-space of $\mathbb{X}$ (see Theorem 2.5.16 and discussion by Megginson (1998) or Theorem 8.13 by Alt and Nürnberg (2016)). As such, $x^\star \in m\mathscr{F}_t$.

Next, the space $\{x \in \mathbb{X} \,|\, x \geq 0\}$ is norm closed and convex, thus $\{x \in \mathbb{X} \,|\, x \geq 0\}$ will be weak sequentially closed, implying $x^\star \geq 0$. To conclude, suppose by contradiction $x^\star \notin \mathbb{S}_t$, then $\int x^\star\,\mathrm{d}\mathbb{P} > \bar{K}$. However, since x^n converges weakly, $\int x^n\,\mathrm{d}\mathbb{P} \to \int x^\star\,\mathrm{d}\mathbb{P} > \bar{K}$, yielding a contradiction since $x^n \notin \mathbb{S}_t$ for some n. We conclude $x^\star \in \mathbb{S}_t$ and $\mathbb{S}_t$ is weak sequentially closed.

Q.E.D.

PROOF OF PROPOSITION 4.4: Set t and suppose $(x^n, y^n)_{n=0}^{\infty}$ satisfies $y^n \in \Gamma_t(x^n)$ for each n. Suppose $(x^n)_{n=0}^{\infty}$ converges to $x \in \mathbb{S}_t$ and $(y^n)_{n=0}^{\infty}$ converges to $y \in \mathbb{S}_{t+1}$.

We show $y \in \Gamma_t(x)$ by checking both the cases stated in the definition of Γ_t at Equation (20): either $\int x \, \mathrm{d}\mathbb{P} = 0$ or $\int x \, \mathrm{d}\mathbb{P} > 0$. First let $\int x \, \mathrm{d}\mathbb{P} > 0$, we show $y \leq (1 + \tilde{r}(x))x + \tilde{w}(x)e_t$ for $\mathbb{P}$-almost everywhere. Suppose by contradiction

$$\mathbb{P}\left\{y > (1 + \tilde{r}(x))x + \tilde{w}(x)e_t\right\} > 0$$

Let $B\colon = \{\omega \in \Omega \,|\, y(\omega) > (1 + \tilde{r}(x))x(\omega) + \tilde{w}(x)e_t(\omega)\}$, we have $\mathbb{P}(B) > 0$ and

$$\int \mathbb{1}_B y \, \mathrm{d}\mathbb{P} > \int \mathbb{1}_B \times [(1 + \tilde{r}(x))x + \tilde{w}(x)e_t] \, \mathrm{d}\mathbb{P} \tag{48}$$

Since $\tilde{K}(x^n) \to \tilde{K}(x)$ and $\tilde{K}(x) > 0$, there exists N such that for all $n > N$, $\tilde{K}(x^n) > 0$. And for the tail sequence $(x^n)_{n=N+1}^{\infty}$, $\tilde{r}(x^n) = F_1(\tilde{K}(x^n), L)$ will converge, implying

$$\begin{aligned}(1 + \tilde{r}(x^n)) \int x^n h \, \mathrm{d}\mathbb{P} + \tilde{w}(x^n) \int h e_t \, \mathrm{d}\mathbb{P} \\ \to (1 + \tilde{r}(x)) \int x h \, \mathrm{d}\mathbb{P} + \tilde{w}(x) \int h e_t \, \mathrm{d}\mathbb{P}\end{aligned} \tag{49}$$

for any function h satisfying $h \in L^2(\Omega, \mathbb{P})$. In particular, let $h = \mathbb{1}_B$, and note $y^n \in \Gamma_t(x^n)$; by the feasibility condition at (20), we write

$$\int \mathbb{1}_B y^n \, \mathrm{d}\mathbb{P} \leq (1 + \tilde{r}(x^n)) \int \mathbb{1}_B x^n \, \mathrm{d}\mathbb{P} + \tilde{w}(x^n) \int \mathbb{1}_B e_t \, \mathrm{d}\mathbb{P}$$

for each $n > N$. Since the weak inequality above will be preserved under the limits of real-valued sequences, we arrive at

$$\int \mathbb{1}_B y \, \mathrm{d}\mathbb{P} \leq (1 + \tilde{r}(x)) \int \mathbb{1}_B x \, \mathrm{d}\mathbb{P} + \tilde{w}(x) \int \mathbb{1}_B e_t \, \mathrm{d}\mathbb{P} \tag{50}$$

However, (50) is a contradiction to (48) and we conclude

$$y \leq (1 + \tilde{r}(x))x + \tilde{w}(x)e_t$$

Now suppose $\int x^n \,\mathrm{d}\mathbb{P} \to 0$. Note

$$\int y^n \,\mathrm{d}\mathbb{P} \leq \int (1+\tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t \,\mathrm{d}\mathbb{P} = F(\tilde{K}(x^n), L) + (1-\delta)\tilde{K}(x^n)$$

The above equality follows from homogeneity of degree one of the production function (Assumption 2.3 and recall discussion preceding Equation (11)). Since $\tilde{K}(x^n) = \int x^n \,\mathrm{d}\mathbb{P} \to 0$, we have $F(\tilde{K}(x^n), L) \to 0$ by Assumption 2.3, and

$$0 = \lim_{n\to\infty} \int y^n = \int y \,\mathrm{d}\mathbb{P} \tag{51}$$

Since $y \in \mathbb{S}_{t+1}$, $y \geq 0$ and (51) implies $y = 0$ for $\mathbb{P}$-almost everywhere.[29]

Thus we have checked $x_t \in \Gamma_t(x)$ under both the cases stated in the definition of Γ_t at (20), completing the proof.

Q.E.D.

For the following lemma, consider the setting and notation of the sequential planner's problem in section 4.

LEMMA 6.4 *Fix x with $x \in \mathbb{S}_0$, $\epsilon > 0$ and $t \in \mathbb{N}$. If assumptions 2.1 - 2.3 hold, then there exists $\bar{r} \in \mathbb{R}_+$ such that for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have $\tilde{r}(x_i) \leq \bar{r}$ for each $i \leq t$.*

PROOF: Fix x with $x \in \mathbb{S}_0$, $\epsilon > 0$ and $t \in \mathbb{N}$. Select $(x_i)_{i=0}^{\infty}$ satisfying $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$.

Since $x_i \in \Gamma_{i-1}(x_{i-1})$, by the feasibility correspondence (Equation (20)) and homogeneity of degree one (Assumption 2.3 and recall discussion preceding Equation (11)) of the production function F, we have

$$\begin{aligned}\tilde{K}(x_i) = \int x_i \,\mathrm{d}\mathbb{P} &\leq \int (1+\tilde{r}(x_{i-1}))x_{i-1} + \tilde{w}(x_{i-1})e_{i-1} \,\mathrm{d}\mathbb{P} \\ &= (1 + F_1(\tilde{K}(x_{i-1}), L) - \delta)\tilde{K}(x_{i-1}) \\ &\quad + F_2(\tilde{K}(x_{i-1}), L)L \\ &= F(\tilde{K}(x_{i-1}), L) + (1-\delta)\tilde{K}(x_{i-1})\end{aligned} \tag{52}$$

for each $i \in \mathbb{N}$.

[29] If $y \geq 0$, then $y = 0$ if and only if $\int f \, d\mathbb{P} = 0$. See Theorem 1.1.20 by Tao (2010).

Define $\hat{F}(K) := F(K, L) + (1-\delta)K$ and note $\hat{F}$ will be strictly increasing. By (52),

$$\tilde{K}(x_i) \leq \hat{F}(\tilde{K}(x_{i-1})), \qquad i \in \mathbb{N} \tag{53}$$

As such, for any $k > 1$, by a simple inductive argument (Claim 9.4 in the online appendix), we can show

$$\tilde{K}(x_k) \leq \hat{F}^{k-i}(\tilde{K}(x_i)), \qquad \forall\, i \leq k \tag{54}$$

Next, since ν is concave, from Jensen's inequality (fact 8.4 in the online appendix),

$$\begin{aligned}
\epsilon \leq \rho_t(x_t, x_{t+1}) &= \int \nu\left((1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1}\right) \mathrm{d}\mathbb{P} \\
&\leq \nu\left(\int (1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t \,\mathrm{d}\mathbb{P}\right) \\
&= \nu(\hat{F}(\tilde{K}(x_t)))
\end{aligned} \tag{55}$$

Note the inverse of v, v^{-1}, is also increasing since v is increasing. (The inverse of ν exists by Assumption 2.5.) From (55), $v^{-1}(\epsilon) \leq \hat{F}(\tilde{K}(x_t))$. And, by (54),

$$\nu^{-1}(\epsilon) \leq \hat{F}(\tilde{K}(x_t)) \leq \hat{F}^{t-i+1}(\tilde{K}(x_i)), \qquad \forall\, i \leq t \tag{56}$$

Next, Let G^j denote the inverse of $\hat{F}^j$. Since $\hat{F}$ is strictly increasing, by (56), we have $\tilde{K}(x_i) \geq G^{t-i+1}(\nu^{-1}(\epsilon))$ for each $i \leq t$. Define

$$\underline{K} := \min_{i \in \{0,\ldots,t\}} \{G^{t-i+1}(\nu^{-1}(\epsilon))\}$$

and note $\tilde{K}(x_i) \geq \underline{K}$ for each $i \leq t$.

Finally, let $\bar{r} := F_1(\underline{K}, L) - \delta$. Note $F_1(K, L)$ is decreasing in the first argument since F is concave and conclude

$$\tilde{r}(x_i) = F_1(\tilde{K}(x_i), L) - \delta \leq F_1(\underline{K}, L) - \delta := \bar{r}, \qquad \forall\, i \leq t$$

Since $\bar{r}$ depends only on t and ϵ, the above will hold for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$.

Q.E.D.

PROOF OF PROPOSITION 4.5: Fix any x satisfying $x \in \mathbb{S}_0$, $\epsilon > 0$ and t. By Lemma 6.4, there exists $\bar{r} \in \mathbb{R}_+$ such that for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have

$$r(x_i) \leq \bar{r}, \qquad \forall\, i \leq t$$

Moreover, since aggregate capital will be bounded from above, the maximum possible wage rate will be bounded above by a constant, which we now denote as $\bar{w}$.

Let $(x_i)_{i=0}^{\infty}$ be any sequence satisfying $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$. For any $i \in \{1, \dots, t+1\}$,

$$\begin{aligned}
x_i &\leq (1+\bar{r})x_{i-1} + \bar{w}e_{i-1} \\
&\leq (1+\bar{r})^2 x_{i-2} + \bar{w}e_{i-1} + (1+\bar{r})\bar{w}e_{i-2} \\
&\qquad\qquad\vdots \\
&\leq (1+\bar{r})^i x + \bar{w}\sum_{j=0}^{i-1}(1+\bar{r})^j e_{i-j-1}
\end{aligned}$$

Let $W_i :\,= \bar{w}\sum_{j=0}^{i-1}(1+\bar{r})^j e_{i-j-1}$ and note $\|W_i\|$ will be finite. Next, since $x_i \geq 0$,

$$x_i \leq (1+\bar{r})^i x + W_i \implies (x_i)^2 \leq \Big((1+\bar{r})^i x + W_i\Big)^2$$

As such, for all $i \in \{1, \dots, t+1\}$,

$$\begin{aligned}
\|x_i\| &\leq \|(1+\bar{r})^i x + W_i\| \\
&\leq (1+\bar{r})^i \|x\| + \|W_i\| \\
&:\,= \hat{M}_i \in \mathbb{R}
\end{aligned}$$

To conclude, let $\hat{M} :\,= \max\{\|x\|, \hat{M}_1, \dots, M_{t+1}\}$. The scalar $\hat{M}$ depends only on x, $\bar{r}$, $\bar{w}$, t and ϵ. As such, for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ that satisfies $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have $\|x_i\| \leq \hat{M}$ for each $i \leq t+1$, as was to be shown.

Q.E.D.

REFERENCES

Acemoglu, D., 2009. Introduction to Modern Economic Growth. Princeton University Press, Princeton, New Jersey.

Acemoglu, D., Jensen, M. K., 2015. Robust Comparative Statics in Large Dynamic Economies. Journal of Political Economy 123 (3).

Acikgoz, O., 2013. Transitional Dynamics and Long-run Optimal Taxation Under Incomplete Markets. Working Paper.
URL http://mpra.ub.uni-muenchen.de/50160/

Acikgoz, O., 2015. On the Existence and Uniqueness of Stationary Equilibrium in Bewley Economies with Production. Working Paper.
URL https://mpra.ub.uni-muenchen.de/71066/

Aiyagari, S. R., 1994. Uninsured Idiosyncratic Risk and Aggregate Saving. The Quarterly Journal of Economics 109 (3), 659–684.

Aiyagari, S. R., 1995. Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints and Constant Discounting. Journal of Political Economy 103 (6), 1158–1175.

Al-Najjar, N. I., 2004. Aggregation and the law of large numbers in large economies. Games and Economic Behavior 47 (1), 1–35.

Aliprantis, C. D., Border, K. C., 2005. Infinite Dimensional Analysis: A Hitchhiker's Guide. Springer-Verlag, Berlin.

Alt, H. W., Nürnberg, R., 2016. Linear Functional Analysis: An Application-Oriented Introduction. Universitext. Springer London.
URL https://books.google.com.au/books?id=qyCcDAEACAAJ

Benhabib, J., Bisin, A., Zhu, S., 2015. The Wealth Distribution in Bewley Economies with Capital Income Risk. Journal of Economic Theory 159, 489–515.

Berger, D., Vavra, J., 2015. Consumption Dynamics During Recessions. Econometrica 83 (1), 101–154.

Bhandari, A., Evans, D., Sargent, T. J., 2017. Optimal Fiscal-Monetary Policy with Redistribution. Working Paper.
URL http://www.tomsargent.com/research/begs2.pdf

Borovkov, A. A., 2003. Probability Theory, 4th Edition. Springer-Verlag, London.

Borwein, J. M., Vanderwerff, J. D., 2010. Convex Functions: Constructions , Characterizations and Counterexamples. Cambridge University Press, New York.

Boucekkine, R., Camacho, C., Zou, B., jan 2009. Bridging the Gap Between Growth Theory and the New Economic Geography: the Spatial Ramsey Model. Macroeconomic Dynamics 13 (01), 20.

Brock, W. A., Xepapadeas, A., Yannacopoulos, A. N., 2014. Optimal Agglomerations in Dynamic Economics. Journal of Mathematical Economics 53, 1–15.

Brunnermeier, M. K., Sannikov, Y., 2016. On The Optimal Inflation Rate. American Economic Review 106 (5), 484–489.

Buera, B. F. J., Moll, B., 2015. Aggregate Implications of a Credit Crunch: The Importance of Heterogeneity. American Economic Journal. Macroeconomics 7 (3), 1–42.

Cao, D., 2016. Existence of Generalized Recursive Equilibrium in Krusell and Smith (1998). Working Paper.
URL http://faculty.georgetown.edu/dc448/KSExistenceV17.pdf

Chen, Y., Yang, C., 2017. Aiyagari Meets Ramsey: Optimal Capital Taxation with Incom-

plete Markets. Working Paper.
URL https://dx.doi.org/10.20955/wp.2017.003

Cheridito, P., Sagredo, J., 2016. Existence of Sequential Competitive Equilibrium in Krusell–Smith type Economies. Working Paper.
URL https://papers.ssrn.com/sol3/papers.cfm?abstract{_}id=2794629

Çinlar, E., 2011. Probability and Stochastics. Springer-Verlag New York.

Dávila, J., Hong, J. H., Krusell, P., Ríos-Rull, J.-V., 2012. Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks. Econometrica 80 (6), 2431–2467.

Fabbri, G., Faggian, S., Freni, G., 2015. On the Mitra-wan Forest Management Problem in Continuous Time. Journal of Economic Theory 157, 1001–1040.

Feinberg, E. A., Kasyanov, P. O., Zadoianchuk, N. V., 2012. Average Cost Markov Decision Processes with Weakly Continuous Transition Probabilities. Mathematics of Operations Research 37 (4), 591–607.

Feinberg, E. a., Kasyanov, P. O., Zadoianchuk, N. V., 2013. Berge's Theorem for Non-Compact Image Sets. Journal of Mathematical Analysis and Applications 397 (1), 255–259.

Feldman, M., Gilles, C., 1985. An Expository Note on Individual Risk Without Aggregate Uncertainty. Journal of Economic Theory 35 (1), 26–32.

Heathcote, J., 2005. Fiscal Policy with Heterogeneous Agents and Incomplete Markets. Review of Economic Studies 72 (1), 161–188.

Heathcote, J., Storesletten, K., Violante, G. L., 2010. The Macroeconomic Implications of Rising Wage Inequality in the United States. Journal of Political Economy 118 (4), 681–722.

Hopenhayn, H. A., 1992. Entry, Exit, and firm Dynamics in Long Run Equilibrium. Econometrica 60 (5), 1127–1150.

Huggett, M., 1993. The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies. Journal of Economic Dynamics and Control 17, 953–969.

Itskhoki, O., Moll, B., 2014. Optimal Development Policies with Financial Frictions. Working Paper.
URL http://www.nber.org/papers/w19994

Judd, K. L., may 1985. On the Performance of Patents. Econometrica 53 (3), 567–585.

Kamihigashi, T., 2017. A Generalisation of Fatou's Lemma for Extended Real-Valued Functions on sigma-Finite Measure spaces: with an Application to Infinite-Horizon Optimization in Discrete Time. Journal of Inequalities and Applications 2017 (1), 24.

Kaplan, G., Moll, B., Violante, G. L., 2016. Monetary Policy According to HANK. Working Paper, 1–61.

Kaplan, G., Violante, G. L., 2010. How Much Consumption Insurance Beyond Self-Insurance? American Economic Journal: Macroeconomics 2 (2), 53–87.

Koren, M., Tenreyro, S., feb 2013. Technological Diversification. American Economic Review 103 (1), 378–414.

Krusell, P., Mukoyama, T., Smith, A. A., 2011. Asset prices in a Huggett economy. Journal of Economic Theory 146 (3), 812–844.

Krusell, P., Smith, A. A., 1998. Income and Wealth Heterogeneity in the Macroeconomy. Journal of Political Economy 106 (5), 867–896.

Kuhn, M., 2013. Recursive Equilibriua in an Aiyagari-Style Economy with Permanent Income Shocks. International Economic Review 54 (3), 807–835.

Lucas, R. E., Moll, B., 2014. Knowledge Growth and the Allocation of Time. Journal of Political Economy 122 (1).

Luenberger, D. G., 1968. Optimization by vector space methods. Wiley New York.

Marcet, A., Obiols-Homs, F., Weil, P., 2007. Incomplete Markets, Labor Supply and Capital Accumulation. Journal of Monetary Economics 54 (8), 2621–2635.

Mas-colell, A., Zame, W. R., 1991. Equilibrium Theory in Infinite Dimensional Spaces. In: Hildenbrand, W., Sonnenschein, H. (Eds.), Handbook of Mathematical Economics. pp. 1835–1890.

Mckay, A., Reis, R., 2016. The Role of Automatic Stabilizers in the U.S. Business Cycle. Econometrica 84 (1), 141–194.

Megginson, R. E., 1998. An Introduction to Banach Spaces. Springer-Verlag New York.

Melitz, M. J., 2003. The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity. Econometrica 71 (6), 1695–1725.

Miao, J., 2002. Stationary Equilibria of Economies with a Continuum of Heterogeneous Consumers. Working Paper.
URL `http://people.bu.edu/miaoj/shockid48.pdf`

Miao, J., 2006. Competitive Equilibria of Economies with a Continuum of Consumers and Aggregate Shocks. Journal of Economic Theory 128 (1), 274–298.

Nuño, G., 2017. Optimal Social Policies in Mean Field Games. Applied Mathematics & Optimization Forthcomin.

Nuño, G., Moll, B., 2017. Controlling a Distribution of Heterogeneous Agents, 1–37.

Nuno, G., Thomas, C., 2017. Monetary Policy with Heterogeneous Agents. Working Paper.
URL `http://ideas.repec.org/p/red/sed013/356.html`

Park, Y., 2014. Optimal Taxation in a Limited Commitment Economy. Review of Economic Studies 81 (2), 884–918.

Sorger, G., 2015. Dynamic Economic Analysis: Deterministic Models in Discrete Time. Cambridge University Press.
URL `https://books.google.com.au/books?id=yfaboAEACAAJ`

Stachurski, J., 2009. Economic Dynamics: Theory and Computation. MIT Press Books, Cambridge, MA.

Stokey, N., Lucas, R., 1989. Recursive Methods in Economic Dynamics. Harvard University Press, Cambridge, MA.

Tao, T., 2010. Epsilon of Room, One: Volume 117 of Graduate Studies in Mathematics. An epsilon of room. American Mathematical Soc.
URL `https://books.google.com.au/books?id=DhWarYBllZAC`

Tao, T., 2013. Compactness and contradiction. Americal Mathematical Society.

Uhlig, H., 1996. A Law of Large Numbers for Large Economies. Economic Theory 8 (1), 41–50.

Williams, D., 1991. Probability with Martingales. Cambridge University Press, Cambridge.