8. PROBABILITY AND CONDITIONAL EXPECTATION PRELIMINARIES

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. We work with the following definition of conditional expectation (Williams (1991), 9.2)

DEFINITION 8.1 Let $\mathcal{H} \subset \Sigma$ be a sub σ-algebra of Σ and let $x\colon \Omega \to \mathbb{R}^n$ be a random variable. The **conditional expectation** of x given $\mathcal{H}$ is any $\mathcal{H}$-measurable random variable y which satisfies

$$\int_B y \, \mathrm{d}\mathbb{P} = \int_B x \, d\mathbb{P}, \qquad B \in \mathcal{H}$$

If y is a conditional expectation of x given $\mathcal{H}$, we write $y = \mathbb{E}(x|\mathcal{H})$.

Recall the definition of a σ-algebra generated by a family of functions $\{y_i\}_{i \in F}$:

$$\sigma(\{y_i\}_{i \in F}) \colon = \sigma(\cup_{i \in F} \sigma(y_i))$$

Recall also that if $\mathcal{A}$ and $\mathcal{B}$ are independent sub σ-algebras of Σ and if a sub-σ-algebra $\mathcal{C}$ satisfies $\mathcal{C} \subset \mathcal{A}$, then $\mathcal{C}$ and $\mathcal{B}$ will be independent.

The following facts are standard.

FACT 8.1 *Let $\mathcal{G}$ and $\mathcal{H}$ be sub σ-algebras of Σ and let $x\colon \Omega \to \mathbb{R}$ be a random variable. If $\mathcal{H}$ and $\sigma(\mathcal{G}, \sigma(x))$ are independent, then $\mathbb{E}(x|\sigma(\mathcal{G}, \mathcal{H})) = \mathbb{E}(x|\mathcal{G})$.*

PROOF: See section 9.7 in Williams (1991). *Q.E.D.*

FACT 8.2 *If $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are σ-algebras, then $\sigma(\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} \subset \sigma(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$*

PROOF: Recall that for collections of sets $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A} \subset \mathcal{B}$, we have $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$.

Now pick any B such that $B \in \sigma(\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$. We have either $B \in \sigma(\mathcal{A} \cup \mathcal{B})$ or $B \in \mathcal{C}$. If $B \in \sigma(\mathcal{A} \cup \mathcal{B})$, then since $\sigma(\mathcal{A} \cup \mathcal{B}) \subset \sigma(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, $B \in \sigma(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$. Alternatively, if $B \in \mathcal{C}$, then because $\mathcal{C} \subset \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \subset \sigma(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, we can conclude that $B \in \sigma(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$. *Q.E.D.*

FACT 8.3 *(**Doob-Dynkin**) Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. Let $f\colon \Omega \to \mathbb{R}^k$ and $g\colon \Omega \to \mathbb{R}^n$. The generated σ-algebras satisfy $\sigma(f) \subset \sigma(g)$ if and only if there exists a measurable function $h\colon \mathbb{R}^n \to \mathbb{R}^k$ such that $f = h \circ g$.*

PROOF: See Lemma 1.13 by Kallenberg (1997). *Q.E.D.*

FACT 8.4 ***(Jensen's Inequality)***

Let x be a random variable on a probability space $(\Omega, \Sigma, \mathbb{P})$. If $c\colon \mathbb{R} \to \mathbb{R}$ is convex, and $\mathbb{E}|c(x)| < \infty$, then

$$\mathbb{E}(c(x)|\mathcal{H}) \geq c(\mathbb{E}(x|\mathcal{H}))$$

almost everywhere, for any sub σ - algebra $\mathcal{H}$

For a proof, see section 9.7 in Williams (1991).

FACT 8.5 ***(Reverse Fatou's Lemma)*** *If (f_n) is a sequence of real valued functions defined on a measure space (Ω, Σ, μ), and each f_n satisfies $f_n \leq g$ and $f_n \geq 0$, where g is an integrable function on (Ω, Σ, μ), then*

$$\limsup_{n\to\infty} \int f_n d\mu \leq \int \limsup_{n\to\infty} f_n d\mu$$

For a proof, see 5.4 in Williams (1991).

REFERENCES

Kallenberg, O., 1997. Foundations of Modern Probability. Springer-Verlag New York.
Williams, D., 1991. Probability with Martingales. Cambridge University Press.