EXISTENCE OF RECURSIVE CONSTRAINED OPTIMA IN THE HETEROGENEOUS AGENT NEOCLASSICAL GROWTH MODEL

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This paper establishes the existence of recursive constrained optimal policies, as considered by Dávila et al. (2012), in a neoclassical growth model with idiosyncratic shocks, incomplete insurance markets and production. A constrained planner chooses individual saving and consumption through time, constrained by infinitely many agents' budget constraints, to maximise aggregate welfare. Due to the Inada conditions and an infinite dimensional state and action space, the constrained planner's feasibility correspondences have non-compact image sets. The constrained planner's problem thus does not meet the requirements of standard dynamic optimisation theory used show existence of optimal policies. To address the challenge, first, the paper transforms the recursive problem to a sequential problem and shows existence of sequential constrained optimal policies implies existence of recursive constrained optimal policies. Second, the paper introduces a new existence result for non-compact dynamic optimisation problems and uses the result to verify existence of sequential constrained optimal policies.

KEYWORDS: Neoclassical Growth Models, Incomplete Markets, Heterogeneous Agent, Constrained Planner, Dynamic Optimisation, Existence Result, Recursive Policies, Infinite Dimensional State-Space.

1. INTRODUCTION

The neoclassical growth model with idiosyncratic shocks, incomplete insurance markets and production, also known as the Aiyagari-Huggett (Aiyagari, 1994;

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Please download the latest version of the working paper and the online appendix from `https://github.com/akshayshanker/Existence_of_Social_Optimia_Aiyagari`

Huggett, 1993) model, has developed into a leading model of dynamic macroeconomics.[1] Recently, attention has turned to *optimal* policy in the Aiyagari-Huggett model; and a natural way to formulate an optimal policy problem in this setting is through a constrained planner. Like any realistic government overseeing a market economy, the constrained planner cannot complete insurance markets, but must improve welfare subject to agents' idiosyncratic budget constraints. First introduced in discrete time by Dávila et al. (2012) and continuous time by Nuño and Moll (2017), the constrained planner concept has led to a growing literature on Ramsey style optimal monetary and fiscal policy in incomplete market models (Acikgoz, 2013; Bhandari et al., 2017; Chen et al., 2017; Nuno and Thomas, 2017; Park, 2014).

Despite the importance of the constrained planner for optimal policy analysis, due to mathematical challenges brought on by the infinite dimensional structure of the Aiyagari-Huggett model, existence of constrained optimal policies has not been verified. This paper provides a proof for the existence of discrete time recursive constrained optimal policies as originally considered by Dávila et al. (2012).[2] The paper also presents an easy to verify general result that can be applied to planner problems in other heterogeneous agent models.

From the perspective of applied modellers, the existence result here helps confirm the surprising policy conclusions emerging from computations of constrained optima are sound. In simulations with high income inequality and a wealth distribution resembling actual U.S. data, Dávila et al. (2012) show a decentralised equilibrium in the Aiyagari-Huggett model under-saves compared to the constrained optimum, which justifies saving subsidies. This is in contrast to the long held belief of sub-optimal over-saving in incomplete market models, which justifies capital taxation (see Aiyagari (1995) and discussion by Acikgoz (2013) and Chen et al. (2017)). Moreover, in the high income inequality case, the constrained planner's solution path does not converge to a steady-state, but displays ever increasing wealth inequality. Verifying existence helps confirm such computed solutions are not pathological and creates a foundation for further research on optimal policy dynamics in the Aiyagari-Huggett model.

[1] Macroeconomists use the model to study consumption dynamics (Berger and Vavra, 2015), shapes of wealth distributions (Benhabib et al., 2015), asset pricing (Krusell et al., 2011) and monetary and fiscal policy dynamics (Kaplan et al., 2016; Kaplan and Violante, 2010; Heathcote, 2005; Mckay and Reis, 2016), to name a few topics.

[2] In this basic setting, the government does not consume and there are no net transfers between agents or nominal rigidities.

Mathematical Challenges

Because the constrained planner controls the assets of infinitely many agents through time, both the planner's state, a distribution of agents over assets, and action, a policy function, are infinite dimensional. The literature has made significant progress by establishing infinite dimensional necessary conditions (Dávila et al. (2012) in discrete time and Nuño and Moll (2017) and Nuño (2017) in continuous time). However, continuity and compactness, assumptions used by standard dynamic optimisation theory *to verify existence of solutions*, are more difficult to verify when spaces are infinite dimensional (see Mas-colell and Zame (1991) for an overview of issues in infinite dimensional topology). In the case of the constrained planner, the feasibility correspondence fails to have compact image sets, that is, the image of a compact set under the correspondence will not be compact. The standard assumptions of existing dynamic optimisation theory (Stokey and Lucas (1989), Acemoglu (2009) ch.6 or Stachurski (2009)) are thus not satisfied.

The constrained planner's feasibility correspondences fail to have compact image sets for two reasons. First, as suggested by Dávila et al. (2012), individual agents' asset spaces will not be bounded. We are also unable to justify restrictions such as equicontinuity or monotonicity on the space of policy functions. As such, the image sets of the feasibility correspondences will not be compact in the sup-norm topology or topology of point-wise convergence. At the same time, the recursive problem, the form of the problem considered by Dávila et al. (2012), will not be defined on topological spaces where the feasibility correspondence is compact-valued.

Second, the feasibility correspondences have non-compact image sets because of a discontinuity. The discontinuity arises due to the Inada conditions — as capital converges to zero, interest rates diverge and the variance of feasible asset distributions can diverge to infinity as the mean converges to zero.

To resolve the first challenge, the paper transforms the recursive problem to a sequential problem and uses a novel projection argument to show sequential solutions imply recursive solutions. The sequential planner's problem will be well-defined on the space of square integrable random variables. And with the weak topology, the sequential planner's feasibility correspondences will be compact valued.

While feasibility correspondences for the sequential planner have compact values, due to the discontinuity around zero capital, image sets will still be non-compact. To resolve this second challenge, the paper generalises important new work on the theory of non-compact optimisation (Feinberg et al., 2012, 2013) and introduces an existence result for infinite horizon dynamic optimisation with non-compact feasible correspondences. The main assumption of the result can be verified by checking the variance of feasible sequences of asset distributions leading to a strictly positive per-period pay-off at a time in the future is bounded.

Related Literature

Pathologies similar to the second problem discussed above are encountered in existence proofs of general equilibrium in the Aiyagari model with aggregate shocks (Krusell-Smith models), as discussed in detail by Cao (2016). The solution proposed by Cao (2016) involves solving a sequence of finite horizon problems and showing aggregate capital has a strictly positive lower bound using agents' Euler equations. Instead of verifying a lower-bound for capital,[3] the approach of this paper is to state a general theorem for the infinite horizon dynamic optimization problem on non-compact spaces.

The Aiyagari-Huggett constrained planner problem is not the only dynamic optimisation problem with infinite dimensional state-spaces. A large literature (Boucekkine et al., 2009; Brock et al., 2014; Fabbri et al., 2015) has shown existence and characterised optimal solutions in models of economic geography in continuous time. Lucas and Moll (2014) also solve an infinite dimensional planner's problem to control individual search efforts subject to the law of motion of a density. However, to the best of my knowledge, these models do not encounter the non-compactness of the Aiyagari-Huggett model.

Other models have infinite dimensional states, but with simplifying assumptions, the dynamics of the distribution may only depend on finite dimensional variable (Brunnermeier and Sannikov, 2016; Buera and Moll, 2015; Hopenhayn, 1992; Itskhoki and Moll, 2014; Koren and Tenreyro, 2013; Melitz, 2003).

However, the methodology of this paper is directly relevant for future study of constrained planner problems in extensions and applications of the Aiyagari-Huggett model: this includes Aiyagari-Huggett models incorporating aggregate shocks (Den Haan, 1996; Krusell and Smith, 1998), permanent income shocks Kuhn (2013), endogenous labour supply (Marcet et al., 2007), overlapping generations (Heathcote et al., 2010) or monetary and fiscal policy (Kaplan et al., 2016; Kaplan

[3] As we are concerned with existence of optima as opposed to equilibria, we cannot use the Euler conditions, which in the case of the constrained planner have only been shown to be necessary, to restrict the search for an optimiser since optima may not exist.

and Violante, 2010; Heathcote, 2005; Mckay and Reis, 2016; Nuno and Thomas, 2017).

2. CONSTRAINED PLANNER PROBLEMS

This section presents the recursive and sequential constrained planner's problems in a standard Aiyagari (1994) model. Both Dávila et al. (2012) and Nuño and Moll (2017) formulate their problem as a recursive problem; the exposition here will follow the discrete time version in Dávila et al. (2012), only I place more formal mathematical structure on the model.

In the recursive problem, the constrained planner instructs agents on their next period assets based on their current assets, shock and the aggregate distribution of agents. The recursive problem will be a stationary primitive form infinite horizon dynamic optimisation problem, where the planner selects an action (policy function) to drive a state (wealth distribution). (The distinction between primitive form and reduced form problem is discussed by Sorger (2015), Section 5.1.)

In the sequential problem, the constrained planner instructs agents each period on next period assets based on their history of shocks up to the period. The sequential problem will be a non-stationary reduced form infinite horizon dynamic optimisation problem, where the planner selects a sequence of states (random variables).[4]

The online appendix contains an overview of mathematical concepts used in this paper.

2.1. *The Aiyagari-Hugett Model*

Time is discrete and indexed by $t \in \mathbb{N}$. Let $(I, \mathcal{I}, \zeta)$ be an atom-less probability space indexing the agents of the model. Let A, with $A\colon = [0, \infty)$, be the agents'

[4]In the context of a constrained planner, the terminology 'sequential' and 'recursive' problems is overloaded. The distinction here follows the distinction between 'sequential competitive equilibria' and 'recursive competitive equilibria' made by Miao (2006) and Cao (2016). In contrast to the distinction made here, the term sequential problem is often used to refer to the problem maximising the infinite sum of pay-offs as opposed to the Bellman Operator representation of the same problem. For infinite dimensional and stochastic problems, both sequential and recursive formulations can be written as a deterministic sequence problem (maximising the sum of discounted pay-offs) and using a deterministic Bellman Equation. For example, (12) compared below to (F.23) in the online appendix. This paper uses the term *sequence problem* to refer to a problem such as (12).

asset space[5] and define E as the agents' labour endowment space. Assume E is a closed subset of $\mathbb{R}_+$. Let S, where $S\colon = A \times E$, denote the agents' state space.

We do not need further assumptions, such as boundedness, on E for the proofs in this paper. However, computations by Dávila et al. (2012)) and Nuño and Moll (2017) show a solution with diverging variance, implying a sequence of asset distributions with an increasing upper-bound (see fig.3 and discussion at section 5.4 by Dávila et al. (2012)).[6]As such, A will be unbounded above, even if E is bounded.

At time zero, each agent i, with $i \in I$, draws an initial asset level x_0^i, with x_0^i taking values in A. In subsequent periods, each agent receives a sequence of labour endowment shocks $(e_t^i)_{t=0}^{\infty}$, with e_t^i taking values in E for each t and i. Let P denote the probability law or joint distribution of the sequence of shocks, common across i. Assume all shocks are defined on a common probability space $(\bar{\Omega}, \Sigma, \bar{\mathbb{P}})$, that is, x_0^i and $(e_t^i)_{t=0}^{\infty}$ for each i are random variables defined on $(\bar{\Omega}, \Sigma, \bar{\mathbb{P}})$.

ASSUMPTION 2.1 The shocks satisfy the following conditions:

1. for each i, the shocks $(e_t^i)_{t=0}^{\infty}$ are a stationary Markov process with common Markov kernel Q and stationary marginal distribution ψ
2. for each t and i, e_t^i and x_0^i has finite variance
3. for each i, x_0^i is independent of $(e_t^i)_{t=0}^{\infty}$.

Part 1 of Assumption 2.1 can be relaxed to boundedness of the mean of the endowment shock, however, the stationarity assumption simplifies notation. The finite variance assumption allows us to work in the L^2 space of square integrable random variables where compact sets are easier to find. The general existence result I present in this paper can also be applied to other topological vector spaces; research on such models is left for further work.

Let $Z\colon = A \times E^{\mathbb{N}}$ and let $\mathscr{B}(Z)$ be the Borel sets of Z. For each i, let $z^i\colon \bar{\Omega} \to Z$ denote the map $\omega \mapsto \{x_0^i(\omega), e_0^i(\omega), e_1^i(\omega), \dots\}$, where $\omega \in \Omega$. The following assumption formalises no aggregate uncertainty:

[5]As in the computations by Dávila et al. (2012), I assume a zero lower bound on assets to simplify the notation. In general, the Aiyagari model allows a strictly negative lower bound, however a zero lower bound is a common assumption, see also Miao (2006) and Cao (2016). The results here can be extended to a model with a negative lower bound, however an additional constraint on the state-space to ensure interest rates are not so high as to violate budget constraints will need to be added.

[6]Popoviciu's inequality for variance states the variance of any bounded random variable is bounded.

ASSUMPTION 2.2 For any $B \in \mathscr{B}(Z)$,

$$\zeta\{i \in I \,|\, z^i(\omega) \in B\} = P(B), \qquad \bar{\mathbb{P}} - \text{a.e.}$$

Under no aggregate uncertainty, the empirical distribution of $i \mapsto z^i(\omega)$ for a draw of ω from $\bar{\Omega}$ agrees with the theoretical distribution of z^i with probability one. There is no loss of generality from assuming no aggregate uncertainty; following Sun and Zhang (2009), for any distribution P, there will exist suitable probability spaces and measurable random variables z such that z^i has distribution P for each i and Assumption 2.2 holds.[7]

Under no aggregate uncertainty, at any time t, aggregate variables depend only on the common theoretical distribution of the shocks rather than individual realisations, with probability one. Let μ_0 denote the common joint distribution of x_0^i and e_0^i.

CLAIM 2.1 *Let $g\colon S \to \mathbb{R}$ be a measurable function. If Assumption 2.2 holds, then*

$$\int g(x_0^i, e_0^i)\, \xi(di) = \int \int g(x, e) \mu_0(dx, de) \tag{1}$$

holds $\bar{\mathbb{P}}$-almost everywhere.

The proof is straight-forward and detailed in the online appendix.

Aggregate State

The distribution μ_0 becomes the initial state for the recursive constrained planner problem. The recursive problem we consider is one where the planner selects a measurable policy function h_t for each t, with $h_t\colon S \to A$. Each h_t instructs agents on $t+1$ assets given their time t asset and shock. A sequence of policy functions $(h_t)_{t=0}^{\infty}$ chosen by the constrained planner generates a sequence of assets for each agent, $(x_t^i)_{t=0}^{\infty}$, by

$$x_{t+1}^i = h_t(x_t^i, e_t^i), \qquad t \in \mathbb{N},\ i \in [0,1] \tag{2}$$

[7]In particular, note Z is a complete and separable metric space (theorems 3.37 and 3.38 by Aliprantis and Border (2006)), then apply Corollary 2 and Lemma 1 by Sun and Zhang (2009). See also discussion below Definition 2.1.5 by Sun (2006) on applying the no aggregate uncertainty results to stochastic processes.

Since h_t applies to all agents i, the distribution of $\{x_t^i, e_t^i\}$ will be identical across i. Moreover, $\{x_t^i, e_t^i\} \sim \mu_t$ for each i, where $(\mu_t)_{t=0}^{\infty}$ satisfies the recursion

$$\mu_{t+1}(B_A \times B_E) = \int\int \mathbb{1}_{B_A}\{h_t(x,e)\}Q(e, B_E)\mu_t(dx, de), \qquad t \in \mathbb{N} \tag{3}$$

for each t and $B_A \times B_E \in \mathscr{B}(S)$. See Claim D.1 in the online appendix for the proof.

If each x_t^i has finite variance, using an analogous argument to Claim 2.1, the time t empirical distribution of agents over S will satisfy

$$\int \mathbb{1}_B\{x_t^i, e_t^i\}\zeta(di) = \mu_t(B), \qquad B \in \mathscr{B}(S), \;\; t \in \mathbb{N} \tag{4}$$

with probability one.

Production

Assume a representative firm rents capital (assets) from individuals and hires workers to produce output Y_t:

$$Y_t = F(K(\mu_t), L) - \delta K(\mu_t) \tag{5}$$

where $F \colon \mathbb{R}_+^2 \to \mathbb{R}_+$. When the state is μ_t, using again the LLN argument from Claim 2.1, total capital and labour in the economy is

$$K(\mu_t) \colon = \int\int x\mu_t(dx, de) = \int x_t^i \zeta(di) \tag{6}$$

$$L = \int e \int \mu_t(dx, de) = \int e_t^i \zeta(di) \tag{7}$$

Labour, L, will be constant according to Assumption 2.1.

ASSUMPTION 2.3 The production function F is twice differentiable on $\mathbb{R}_{++}$, homogeneous of degree one, strictly increasing in both arguments, strictly concave and for any $\hat{L} > 0$ and $\hat{K} > 0$ satisfies

1. $\lim_{K\to\infty} F_1(K, \hat{L}) = 0$ and $\lim_{K\to 0} F_1(K, \hat{L}) = \infty$ (Inada conditions)
2. $F(0, \hat{L}) = F(\hat{K}, 0) = 0$
3. $K \mapsto F(K, \hat{L})$ is bijective.

Budget Constraints and Utility

Interest and wage rates in the economy will be

$$r(\mu_t) : = F_1\left(K(\mu_t), L\right) - \delta, \quad w(\mu_t) : = F_2\left(K(\mu_t), L\right)$$

Given the aggregate state μ_t, an agent i with asset x_t^i and endowment shock e_t^i must satisfy their budget constraint

$$0 \leq x_{t+1}^i \leq (1 + r(\mu_t))x_t^i + w(\mu_t)e_t^i \tag{8}$$

where x_{t+1}^i is the next period asset. If x_0^i has finite variance and $r(\mu_t)$ is real-valued for each t, then if $(x_t^i)_{t=0}^{\infty}$ satisfies (8), x_t^i will have finite variance for each t (see Claim D.2 in the online appendix). Consumption for each agent i will be

$$c_t^i = (1 + r(\mu_t))x_t^i + w(\mu_t)e_t^i - x_{t+1}^i$$

Integrating across agents' budget constraints at Equation (8) and using the definition of interest and wages rates, along with homogeneity of the production function (see Theorem 2.1 in Acemoglu (2009)) gives a law of motion for aggregate capital

$$K(\mu_{t+1}) \leq (1 + r(\mu_t))K(\mu_t) + w(\mu_t)L = F(K(\mu_t), L) + (1 - \delta)K(\mu_t) \tag{9}$$

From the law of motion and Assumption 2.3, there exists an upper-bound $\bar{K}$ such that given any initial aggregate level of capital below $\bar{K}$, aggregate capital for wealth distributions satisfying (8) will never exceed $\bar{K}$. That is, if $K(\mu_t) \leq \bar{K}$, then $K(\mu_{t+1}) \leq \bar{K}$ (see Proposition 2.2 and section 6.8 by Acemoglu (2009)).

ASSUMPTION 2.4 The initial wealth distribution μ_0 satisfies $K(\mu_0) \leq \bar{K}$.

Turning to consumer utility, let $v \colon \mathbb{R}_+ \to \mathbb{R}_+$ be each consumer's utility function. Time t utility for agent i will be $v(c_t^i)$.

ASSUMPTION 2.5 The utility function v is strictly increasing, bijective, concave and upper semicontinuous.

For a definition of a competitive equilibrium, see Aiyagari (1994), Dávila et al. (2012), Kuhn (2013), Miao (2002) or Acikgoz (2015).

2.2. *Recursive Constrained Planner*

Let $\mathscr{P}(S)$ denote the space of Borel probability measures on S. The recursive planner's state-space, $\mathbb{M}$, will be a subspace of $\mathscr{P}(S)$ such that each μ, with $\mu \in \mathbb{M}$, satisfies:

1. the marginal distribution across E, $\int \mu(dx, \cdot)$, agrees with ψ
2. the marginal distribution across A, $\int \mu(\cdot, de)$, has finite variance
3. aggregate assets satisfy $\int \int x\mu(dx, de) \in [0, \bar{K}]$.

Let $\mathbb{Y}$ denote the space of measurable functions h where $h \colon S \to A$. The space $\mathbb{Y}$ will be the *action-space* and the constrained planner picks a policy $h_t \in \mathbb{Y}$ for each t and agents' assets transition according to Equation (2).

Define a correspondence Λ, with $\Lambda \colon \mathbb{M} \twoheadrightarrow \mathbb{Y}$, mapping a state to feasible policy functions as follows:

$$\Lambda(\mu) \colon = \begin{cases} h \in \mathbb{Y} \; | 0 \leq h(x,e) \leq (1 + r(\mu))x + w(\mu)e, & \text{if } K(\mu) > 0 \\ h \in \mathbb{Y} \; \big| h = 0, & \text{if } K(\mu) = 0 \end{cases} \tag{10}$$

The (in) equalities above hold μ - almost everywhere. We are unable to place restrictions on $\mathbb{Y}$ such that the correspondence Λ has compact image sets in a suitable topology, for details see section 5.

Following Equation (3), given a time t empirical distribution of agents on S, μ, and policy function h, the operator $\Phi \colon \operatorname{Gr} \Lambda \to \mathbb{M}$ defined by

$$\Phi(\mu, h)(B_A \times B_E) \colon = \int \int \mathbb{1}_{B_A}\{h(x,e)\} Q(e, B_E) \mu(dx, de) \tag{11}$$

where $B_A \times B_E \in \mathscr{B}(S)$, gives the time $t+1$ empirical distribution of agents. We write $\mu_{t+1} = \Phi(\mu_t, h_t)$.

The constrained planner's per-period pay-off, $u \colon \operatorname{Gr} \Lambda \to \mathbb{R}_+$, integrates utility across the empirical distribution of agents

$$u(\mu, h) \colon = \begin{cases} \displaystyle\int \int v\big((1 + r(\mu))x + w(\mu)e - h(x,e)\big)\mu(dx, de), & \text{if } K(\mu) > 0 \\ 0, & \text{if } K(\mu) = 0 \end{cases}$$

It is a straight-forward use of Jensen's inequality (fact C.4 in the online appendix) and homogeneity of the production function to show the integral is well-defined and real-valued.

Finally, let $\beta \in (0,1)$ be a discount factor and let V, with $V\colon \mathbb{M} \to \mathbb{R}_+ \cup \{+\infty\}$, denote the constrained planner's value function:

$$V(\mu_0)\colon = \sup_{(\mu_t,h_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{12}$$

subject to

$$h_t \in \Lambda(\mu_t), \qquad \mu_{t+1} = \Phi(\mu_t, h_t), \qquad\qquad t \in \mathbb{N}, \qquad \mu_0 \,\text{given} \tag{13}$$

DEFINITION 2.1 (**Recursive Constrained Planner's Problem**)

Given μ_0, a solution to the recursive constrained planner's problem is a sequence of measurable policy functions $(h_t)_{t=0}^{\infty}$, with $h_t\colon S \to A$ for each t and a sequence of Borel probability measures on S, $(\mu_t)_{t=0}^{\infty}$ satisfying (13) that achieves the value function:

$$V(\mu_0) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{14}$$

I now state the main result of this paper:

THEOREM 2.1 *If the recursive constrained planner's problem (Definition 2.1) satisfies assumptions 2.1 - 2.5, then for any $\mu_0 \in \mathbb{M}$, there exists a solution $(\mu_t, h_t)_{t=0}^{\infty}$.*

The proof is in section 4. Since Λ does not have compact image sets, standard existence results in dynamic optimisation theory fail (section 5). To prove Theorem 2.1, the paper first defines a sequential planner's problem (section 2.3) and shows existence of a solution for the sequential planner implies existence of a solution for the recursive planner (Theorem 2.2 in section 2.4). The sequential planner's feasibility correspondences will still not have compact image sets around regions where capital is zero. Thus, the paper presents a general existence result for non-compact infinite horizon dynamic optimisation (Theorem 3.1 in section 3), and then checks the sequential planner's problem satisfies the conditions for existence (section 4).

Recursive Policies

If the recursive constrained planner's problem has a solution, $(\mu_t, h_t)_{t=0}^{\infty}$, for each $\mu_0 \in \mathbb{M}$, then following standard arguments, there exists a policy operator $H\colon \mathbb{M} \to \mathbb{Y}$ such that the sequence $(\mu_t, H(\mu_t))_{t=0}^{\infty}$ with $\mu_{t+1} = \Phi(\mu_t, H(\mu_t))$ solves the recursive problem (Corollary E.1 in the online appendix). Thus, if a solution to the recursive constrained planner's problem exists, then the *policy function* that maps assets and shocks to next period assets depends only on the current distribution.

2.3. *Sequential Constrained Planner*

Construct a countably generated[8] probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a sequence of random variables $\{x_0, e_0, e_1, \dots\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ such that $(e_t)_{t=0}^{\infty}$ is a Markov process with Kernel Q that can be written as a stochastic recursive sequence (see section C.4 of the online appendix) and $\{x_0, e_0\} \sim \mu_0$. Intuitively, we may view realisations of $\{x_0, e_0, e_1 \dots\}$ as draws from the empirical distribution of individual shock values. Define $(\mathscr{F}_i)_{i=0}^{\infty}$ as the natural filtration with respect to $\{x_0, e_0, e_1, \dots\}$.[9]

Let $\mathbb{X} := L^2(\Omega, \mathbb{P})$ be the space of square integrable (with respect to $\mathbb{P}$) real-valued functions on Ω. Equip $\mathbb{X}$ with the weak topology. For any $x \in \mathbb{X}$, with $\int x \, \mathrm{d}\mathbb{P} \geq 0$, define

$$\tilde{K}(x) := \int x \, \mathrm{d}\mathbb{P} \tag{15}$$

and if $\int x \mathrm{d}\mathbb{P} > 0$, define

$$\begin{aligned} \tilde{r}(x) &:= F_1(\tilde{K}(x), L) - \delta \\ \tilde{w}(x) &:= F_2(\tilde{K}(x), L) \end{aligned} \tag{16}$$

For each t, define the time t state-space for the sequential planner:

$$\mathbb{S}_t := \left\{ x \in m\mathscr{F}_t \,\middle|\, 0 \leq x, \int x \, d\mathbb{P} \leq \bar{K} \right\} \tag{17}$$

where $m\mathscr{F}_t \subset \mathbb{X}$ is the space of $\mathscr{F}_t$-measurable random variables.

For each t, define the feasibility correspondence $\Gamma_t \colon \mathbb{S}_t \to \mathbb{S}_{t+1}$:

$$\Gamma_t(x) := \begin{cases} y \in \mathbb{S}_{t+1} \,|\, 0 \leq y \leq (1 + \tilde{r}(x))\, x + \tilde{w}(x)\, e_t, & \text{if } \tilde{K}(x) > 0 \\ y \in \mathbb{S}_{t+1} \,|\, y = 0, & \text{if } \tilde{K}(x) = 0 \end{cases} \tag{18}$$

[8]The σ-algebra $\mathscr{F}$ is generated by a countable collection of subsets of Ω.

[9]That is, $\mathscr{F}_0$ is the σ-algebra generated by x_0 and for each $i \geq 1$, $\mathscr{F}_i$ is the σ-algebra generated by $\{x_0, e_0, \dots, e_{i-1}\}$.

For each t, define the time t pay-offs $\rho_t\colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$:

$$\rho_t(x,y) := \begin{cases} \int \nu\left((1+\tilde{r}(x))\,x + \tilde{w}(x)\,e_t - y\right)\,\mathrm{d}\mathbb{P}, & \text{if } \tilde{K}(x) > 0 \\ 0, & \text{if } \tilde{K}(x) = 0 \end{cases} \tag{19}$$

Finally, let $\tilde{V}$, with $\tilde{V}\colon \mathbb{S}_0 \to \mathbb{R}_+ \cup \{+\infty\}$ denote the time 0 sequential planner's value function:

$$\tilde{V}(x_0) := \sup_{(x_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$$

subject to

$$x_{t+1} \in \Gamma_t(x_t), \qquad \forall t \in \mathbb{N}, \qquad x_0 \in \mathbb{S}_0 \text{ given} \tag{20}$$

DEFINITION 2.2 (**Sequential Constrained Planner's Problem**)

Given $x_0 \in \mathbb{S}_0$, a solution to the sequential constrained planner's problem is a sequence of random variables $(x_t)_{t=0}^{\infty}$ satisfying (20) that achieve the sequential planner's value function:

$$\tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \tag{21}$$

2.4. *Sequential Solution Implies Recursive Solution*

Let $(x_t)_{t=0}^{\infty}$ be a solution for the sequential problem. Note for each t, x_{t+1} depends on the history $\{x_0, e_0, \dots, e_t\}$. By contrast, a recursive solution requires x_{t+1} to be $\{x_t, e_t\}$ measurable; that is, we require a function $h_t\colon S \to A$ such that $x_{t+1}(\omega) = h_t(x_t(\omega), e_t(\omega))$ for $\omega \in \Omega$. The following procedure projects each period's sequential solution back onto the previous period, furnishing the required measurability from properties of conditional expectation (see 9.2 by Williams (1991) or ch. IV, 1 by Çinlar (2011)).

Given x_0 satisfying $x_0 \in \mathbb{S}_0$, let $(y_t)_{t=0}^{\infty}$ be a solution to the sequential planner's problem. Construct a candidate sequence, $(x_t)_{t=0}^{\infty}$, as follows:

$$\begin{aligned} &x_0 = y_0, \qquad x_1 = \mathbb{E}(y_1 | \sigma(x_0, e_0)) \\ &\text{and} \\ &x_{t+1} = \mathbb{E}(y_t | \sigma(x_t, e_t)), \qquad \forall t \in \mathbb{N} \end{aligned} \tag{22}$$

The term $\sigma(x_t, e_t)$ denotes the σ-algebra generated by x_t and e_t. And $\mathbb{E}(y_t|\sigma(x_t, e_t))$ denotes the conditional expectation of y_t with respect to x_t and e_t.

PROPOSITION 2.1 *Let assumptions 2.1 - 2.5 hold. If $(y_t)_{t=0}^{\infty}$ is a solution to the sequential problem (Definition 2.2), then $(x_t)_{t=0}^{\infty}$ defined by* (22) *is a solution to the sequential problem.*

See the online appendix for a proof.

THEOREM 2.2 *Let assumptions 2.1 - 2.5 hold. If there exists a solution to the sequential problem (Definition 2.2), then there exists a solution to the recursive problem (Definition 2.1) and $V(\mu_0) = \tilde{V}(x_0)$.*

The complete proof is in the online appendix. The proof proceeds as follows. Let $(y_t)_{t=0}^{\infty}$ solve the sequential problem and let $(x_t)_{t=0}^{\infty}$ be defined by (22). Since x_{t+1} is $\sigma(x_t, e_t)$ measurable, $x_{t+1} = h_t(x_t, e_t)$ for a measurable function h_t for each t. For each t, define μ_t as

$$\mu_t(B) = \mathbb{P}\{x_t, e_t \in B\}, \qquad B \in \mathscr{B}(S) \tag{23}$$

The remainder of the proof verifies $(\mu_t, h_t)_{t=0}^{\infty}$ solves the recursive problem.

3. EXISTENCE THEOREM

I now introduce a general existence result for an infinite horizon dynamic optimisation problem with non-compact feasibility correspondences on arbitrary topological spaces. After stating the general result, this section shows how to verify the conditions for the result on L^2 spaces.

Let (X, τ) be a topological space,

DEFINITION 3.1 A function $f\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is **mildly sup-compact** if the upper contour sets

$$UC_f(\epsilon) :=\{x \in X \mid f(x) \geq \epsilon\} \tag{24}$$

are sequentially compact for all $\epsilon > \inf f$.

Study of sup-compact (or inf-compact) functions can be traced to Rockafellar and Moreau (see discussion of early literature by Roger and Wets (1973)), however, to the best of my knowledge, the term mildly sup-compact is new. A discussion on the relationship between mild sup-compactness, sup-compactness and upper semicontinuity is given in the online appendix.

3.1. *General Existence Theorem*

A non-stationary reduced form economy is a 5-tuple

$$\mathscr{E}: = ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta)$$

consisting of:

1. A topological space $(\mathbb{X}, \tau)$
2. A collection of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$, with $\mathbb{S}_t \subset \mathbb{X}$ for each t
3. A collection of non-empty feasibility correspondences $(\Gamma_t)_{t=0}^{\infty}$, with $\Gamma_t \colon \mathbb{S}_t \twoheadrightarrow \mathbb{S}_{t+1}$ for each t
4. A collection of per-period pay-offs $(\rho_t)_{t=0}^{\infty}$, with $\rho_t \colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$ for each t
5. A discount factor $\beta \in (0, 1)$.

Define the correspondence of **feasible sequences** $\mathcal{G}_t^T \colon \mathbb{S}_t \twoheadrightarrow \prod_{i=t}^{T} \mathbb{S}_i$ starting at time t and ending at time T as follows:

$$\mathcal{G}_t^T(x): = \left\{ (x_i)_{i=t}^{T} \mid x_{i+1} \in \Gamma_i(x_i), x_t = x \right\}, \qquad x \in \mathbb{S}_t \tag{25}$$

Let $\mathcal{G}$ denote $\mathcal{G}_0^{\infty}$ and let $\mathcal{G}^T$ denote $\mathcal{G}_0^T$.

Define the **value function** $\tilde{V} \colon \mathbb{S}_0 \to \mathbb{R} \cup \{-\infty, +\infty\}$ as follows:

$$\tilde{V}(x): = \sup_{(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \tag{26}$$

Recall a compact-valued upper hemicontinuous correspondence has compact image sets (see Mathematical Preliminaries in the online appendix). The first assumption below is the main assumption of the paper, it relaxes the standard requirement for Γ_t to be upper hemicontinuous and compact valued and for $\mathbb{S}_t$ to be a metric space (see by Acemoglu (2009), Assumption 6.2, Kamihigashi (2017), section 6 or Stokey and Lucas (1989), Assumption 4.3, for assumptions used by the standard theory).

ASSUMPTION 3.1 For each $x \in \mathbb{S}_0$ and $t \in \mathbb{N}$, the functions $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}^{t+1}(x)$ are mildly sup-compact in the product topology (of τ topology in $\mathbb{X}$).

The next assumption is the standard growth condition (see discussion on Corollary 6.1 by Kamihigashi (2017)).

ASSUMPTION 3.2 For each $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^{\infty}$ such that any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ satisfies

$$\rho_t(x_t, x_{t+1}) \leq m_t, \qquad \forall t \in \mathbb{N} \tag{27}$$

and

$$\sum_{t=0}^{\infty} \beta^t m_t < \infty \tag{28}$$

ASSUMPTION 3.3 The functions $(\rho_t)_{t=0}^{\infty}$ are sequentially upper semicontinuous for all $t \in \mathbb{N}$.

THEOREM 3.1 *If $\mathscr{E}$ satisfies assumptions 3.1 - 3.3, then for every $x \in S_0$, there will exist $(x_t)_{t=0}^{\infty}$ satisfying $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ such that*

$$\tilde{V}(x) = \sum_{t=0}^{\infty} \beta^t \rho_t\left(x_t, x_{t+1}\right) < \infty$$

.

The proof works by showing feasible paths of states that converge to the supremum of the problem belong to a compact space in the product topology (of the topology τ in $\mathbb{X}$). By contrast, the standard assumptions of hemicontinuity and compact-valued correspondences requires that *all* feasible sequences belong to a compact space in the product topology. A further discussion of how standard theory uses these assumptions to verify existence is in section F.1 of the online appendix.

3.2. *Checking Mild Sup-Compactness in L^2 Spaces*

Let $(\Omega, \Sigma, \varphi)$ be a finite countably measure space and let $\mathbb{X} = L^2(\Omega, \varphi)$ be the space of real-valued measurable function on Ω with

$$\|x\| := \left(\int x^2 \, d\varphi\right)^{\frac{1}{2}} < \infty$$

Equip $\mathbb{X}$ with the weak topology. Recall a sequence $(x_n)_{n=0}^{\infty}$ with $x_n \in \mathbb{X}$ for each n converges in the weak topology if $\int x_n h \, \mathrm{d}\mathbb{P} \to \int xh \, \mathrm{d}\mathbb{P}$ for each $h \in \mathbb{X}$.[10]

Unless otherwise stated, convergence and topological notions will be with respect to the weak topology.

[10] See 5.14 and 13.8 by Aliprantis and Border (2006) or 5.10 by Luenberger (1968).

ASSUMPTION 3.4 The state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$ are sequentially closed in $\mathbb{X}$ for all $t \in \mathbb{N}$.

ASSUMPTION 3.5 The correspondences $(\Gamma_t)_{t=0}^{\infty}$ have a sequentially closed graph for all $t \in \mathbb{N}$.

ASSUMPTION 3.6 For each $t \in \mathbb{N}$, $\epsilon > 0$ and $x \in \mathbb{S}_0$, there exists a constant $\bar{M}$ such that if $(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $u_t(x_t, x_{t+1}) \geq \epsilon$, then $\|x_i\| \leq \bar{M}$ for each $i \in \{0, \ldots, t+1\}$.

PROPOSITION 3.1 *Consider $\mathscr{E}$ where $\mathbb{X} = L^2(\Omega, \varphi)$ and τ is the weak topology. If $\mathscr{E}$ satisfies assumptions 3.3 - 3.6, then $\mathscr{E}$ satisfies Assumption 3.1.*

4. EXISTENCE OF CONSTRAINED OPTIMA

Consider the case of the sequential constrained planner of section 2.3. Let assumptions 2.1 - 2.5 hold and let

$$\mathscr{E} = ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta)$$

where:

1. $\mathbb{X} = L^2(\Omega, \mathbb{P})$
2. The topology τ is the weak topology
3. The sequence of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$ are defined by (17)
4. The sequence of correspondences $(\Gamma_t)_{t=0}^{\infty}$ are defined by (18)
5. The sequence of pay-offs $(\rho_t)_{t=0}^{\infty}$ are defined by (19).

PROPOSITION 4.1 ***(Checking Assumption 3.2)*** *For any $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^{\infty}$ such that $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ and any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ satisfies $\rho_t(x_t, x_{t+1}) \leq m_t$ for each t.*

PROPOSITION 4.2 ***(Checking Assumption 3.3)*** *The functions $(\rho_t)_{t=0}^{\infty}$ are sequentially upper semicontinuous for each t.*

PROPOSITION 4.3 ***(Checking Assumption 3.4)*** *The state spaces $(\mathbb{S}_t)_{t=0}^{\infty}$ are sequentially closed for each t.*

PROPOSITION 4.4 ***(Checking Assumption 3.5)*** *The correspondences $(\Gamma_t)_{t=0}^{\infty}$ have closed graph for each t.*

PROPOSITION 4.5 ***(Checking Assumption 3.6)*** *For any* $t \in \mathbb{N}$, $\epsilon > 0$ *and* $x \in \mathbb{S}_0$, *there exists a constant* $\bar{M}$ *such that if* $(x_i)_{t=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ *and* $\rho_t(x_t, x_{t+1}) \geq \epsilon$, *then*

$$\|x_i\| \leq \bar{M}$$

for all $i \in \{0, 1, \dots, t+1\}$.

We are now ready to verify existence of recursive constrained optima.

PROOF OF THEOREM 2.1: Recall the setting of section 2.1 where μ_0 is the initial state of the economy and let the random variables $\{x_0, e_0, e_1, \dots\}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ be as defined in section 2.3. By assumptions 2.1 - 2.5 and propositions 4.1 - 4.5, the economy $\mathscr{E}$ satisfies assumptions 3.2 - 3.6.

Since assumptions 3.3 - 3.6 satisfy the conditions for Proposition 3.1, $\mathscr{E}$ satisfies Assumption 3.1. As such, $\mathscr{E}$ satisfies assumptions 3.1 - 3.3 and the conditions for Theorem 3.1.

By Theorem 3.1, there exists $(y_t)_{t=0}^{\infty}$ solving the sequential planner's problem (Definition 2.2) such that $\tilde{V}(x_0) < \infty$. By Proposition 2.1, $(x_t)_{t=0}^{\infty}$ defined by (22) also solves the sequential planner's problem. Moreover, there exists a sequence of measurable policy functions $(h_t)_{t=0}^{\infty}$ with $h_t \colon S \to A$ and $x_{t+1} = h_t(x_t, e_t)$ for each t. By Theorem 2.2, $(h_t)_{t=0}^{\infty}$ and $(\mu_t)_{t=0}^{\infty}$ defined by (23) solve the recursive problem and

$$V(\mu_0) = \tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) < \infty \tag{29}$$

Q.E.D.

5. DISCUSSION

5.1. *Non-Compactness of the Feasibility Correspondence*

There are two reasons why the constrained planner's feasibility correspondences will not have compact image sets. The first concerns the structure of the recursive problem and the second concerns the behaviour of interest rates around regions where capital is zero.

5.1.1. *Structure of Recursive Problem*

In the recursive problem, to show Λ has compact image sets, we need to place some further restrictions on the space $\mathbb{Y}$ and equip $\mathbb{Y}$ with a suitable topology. For compactness in the sup-norm topology, the Arzela-Arscoli Theorem (see example 6.2, Mas-colell and Zame (1991)) states uniformly bounded, equicontinuous family of functions on a compact interval will be compact. However, A will not be bounded and policy functions may not be bounded. A possible approach could be restricting $\mathbb{M}$ to measures on a compact support. For each $\mu \in \mathbb{M}$, we can then restrict policy functions in $\Lambda(\mu)$ to be defined on the bounded support of μ. If the mean of μ is positive, then policy functions in $\Gamma(\mu)$ will also be bounded. Notwithstanding the pathologies (see below) as interest rates diverge, to now use the Arzela-Arscoli Theorem, we also need to restrict feasible policy functions in each period to an equicontinuous family of functions. This line of argument has so far not yielded success.

On the other hand, we could once again restrict $\mathbb{M}$ to measures on a compact support and try to verify compactness in the topology of point-wise convergence using Helly's selection theorem (see Cao (2016), Lemma 11). To do this, we need to restrict $\mathbb{Y}$ to the space of monotone policy functions. In the context of a proof for general equilibrium in a Aiyagari-Huggett model with aggregate shocks, the approach by Cao (2016) is to justify monotonicity using the necessary Euler equations of individual agents. However, in the case of an optimisation problem, we cannot restrict the search for an optimiser based on necessary conditions *because a solution for the constrained planner may not exist*. In particular, there may be sequences of functions outside the space of monotone functions converging to the supremum.

The online appendix gives further detail of pathologies in the weak topology if we let $\mathbb{Y}$ be the space of square integrable functions on S, where Φ fails to be defined.

Note the pathologies in the recursive problem also prevent the use of the non-compact existence result of section 3. This is because we cannot place restrictions such as monotonicity or equicontinuity on policy functions in the upper contour sets of Assumption 3.1.

5.1.2. *Non-Compactness Near Zero Capital*

Consider the setting and notation of section 4. Once we move to L^2 space with the weak topology, the feasibility correspondence will be compact valued and have a closed graph. The sequential problem will also be well-defined. However, there will exist a compact set $C \subset \mathbb{S}_t$ such that $\Gamma_t(C)$ is not compact. (Recall from section C.1 in the online appendix that the image of a compact set under a compact-valued and upper hemicontinuous is compact.)

For the following claim, assume $F(K, L) = \alpha^{-1}K^{\alpha}$ and $\alpha = .3$. Furthermore, assume $x_0 \in \mathbb{S}_0$, the initial assets for the economy are a uniform random variable on the interval $[0, 1]$. Assume the random variable e_0 is large enough to satisfy $\tilde{w}(x_0)e_0 > 1$.

CLAIM 5.1 *There exists a compact set C, satisfying $C \subset \mathbb{S}_1$, such that the image set $\Gamma_1(C)$ is not compact.*

The proof is in the online appendix. Roughly, we can construct a sequence of asset distributions in $\Gamma_1(C)$ whose means converge but variances diverge. This is because a smaller and smaller measure of agents can accumulate assets that go to infinity due to higher and higher interest rates as aggregate assets converge to zero.

5.2. *Relationship to K-Sup-Compactness*

To relax compactness and continuity requirements on the feasibility correspondence, Feinberg et al. (2012) introduce a condition (assumption W* in Feinberg et al. (2012)), later defined as K-Sup-Compactness by Feinberg et al. (2013) (Definition 1.1), on per-period pay-offs.[11] Recall the definition of sup-compact from the online appendix, section C.2. While Feinberg et al. (2012) consider stationary problems, for the sequential constrained planner's setting, K-Sup-Compactness of each per-period pay-off ρ_t becomes:

ASSUMPTION 5.1 (**K-Sup-Compact**) Let $t \in \mathbb{N}$. If C is a sequentially compact subset of $\mathbb{S}_t$, then the function $\{x_t, x_{t+1}\} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}_t^{t+1}(C)$ is sup-compact.

The assumption allows the Bellman Equation (in our case, a non-stationary Bellman Equation) to preserve semicontinuity (see Theorem 2 in Feinberg et al. (2012) and Lemma 2.5 in Feinberg et al. (2013)).

With utility bounded below, ρ_t will not satisfy K-Sup-Compactness. To see why, note $\mathcal{G}_t^{t+1}(C) = \{x, y \,|\, y \in \Gamma_t(x), x \in C\}$. Moreover, the upper-contour set of the

[11] Feinberg et al. (2013) use term K-Inf-Compactness, as they work with minimisation problems.

function $\{x_t, x_{t+1}\} \mapsto \rho_t(x_t, x_{t+1})$ on $\mathcal{G}_t^{t+1}(C)$ when $\epsilon = 0$ will be:

$$\{x, y \mid y \in \Gamma_t(x), x \in C, \rho_t(x, y) \geq 0\}$$
$$= \{x, y \mid y \in \Gamma_t(x), x \in C\} = \mathcal{G}_t^{t+1}(C)$$

For the constrained planner, K-Sup-Compactness of ρ_t will then imply compact $\mathcal{G}_t^{t+1}(C)$ for compact C. However, Claim 5.1 constructs an example where $x_n \in C$ and $y_n \in \Gamma(x_n)$ such that the norm of y_n diverges, implying non-compact $\mathcal{G}_t^{t+1}(C)$.[12] As such, when utility is bounded below, the main assumption of this paper is weaker than K-Sup-Compactness.

6. CONCLUSION

This paper proved existence of recursive constrained optima in a standard Aiyagari (1994) model, as considered by Dávila et al. (2012). The results here only apply to problems where the planner's pay-offs are bounded below. The assumption was maintained in the paper so the results here can be directly applied to Dávila et al. (2012), who, in their computations, assume CRRA utility that is bounded below. Moreover, a key technical contribution of the paper was a general existence result that overcomes the difficulties when a non-compact dynamic optimisation problems has pay-offs bounded below.

Some paths for further work are:

1. Explore the application of the already developed non-compact dynamic optimisation theory for unbounded pay-offs (Feinberg et al., 2012) to heterogeneous agent models.
2. Explore existence and policy implications of constrained optima in the variety of Aiyagari-Huggett style and other heterogeneous agent models.
3. Design computational methods known to converge to the true constrained optima.
4. Establish asymptotic properties (stochastic stability) of the solution path.
5. Establish existence to recursive competitive equilibria in the Aiyagari-Huggett model outside the steady state (current results, such as Acikgoz (2015), Miao (2002) or Kuhn (2013) focus on stationary steady states). The

[12] The constrained planner's sequential problem will satisfy the stronger condition where Assumption 3.1 holds for the stated functions on $\mathcal{G}^{t+1}(C)$. The stronger condition gives semi-continuity of the value function and compactness of policy correspondences. The details are a work in progress and available on request.

equilibria will be a special case of the one considered by Miao (2006) and Cao (2016) where aggregate uncertainty is removed.

6. Explore the relationship between constrained optima and competitive equilibria, i.e. when does one imply the other?

APPENDIX A: PROOFS

A.1. *Proofs for Section 3*

Recall the setting and notation of section 3.1, where $(\mathbb{X}, \tau)$ is a topological vector space. Throughout this section, unless otherwise stated, convergence for sequences in $\mathbb{X}$ will be with respect to the τ topology and convergence for sequences in countable Cartesian products of $\mathbb{X}$ will be in the product topology of the τ topology on $\mathbb{X}$.

We will use $\mathbf{x}$ to refer to elements of $\mathbb{X}^{\mathbb{N}}$. We can then use $(\mathbf{x}^n)_{n=0}^{\infty}$ to denote a sequence $\{\mathbf{x}^0, \dots, \mathbf{x}^n, \dots\}$, where $(\mathbf{x}^n)_{n=0}^{\infty} \in (\mathbb{X}^{\mathbb{N}})^{\mathbb{N}}$.

REMARK A.1 Let $X = \prod_{i \in F} X_i$ denote a Cartesian product of topological spaces. Let $\pi_i \colon X \to X_i$ denote the projection map defined as $\pi_i(x) = x_i$ for each $i \in F$. Recall each projection map will be a continuous function on X when X has the product topology (see section 2.14 by Aliprantis and Border (2006)). Also recall (section 1.8 by Tao (2013)) the image of a (sequentially) compact set under a continuous function is (sequentially) compact. Accordingly, if a set C with $C \subset X$ is (sequentially) compact in the product topology, then $\pi_i(C)$ will be (sequentially) compact.

Finally, let the function $\phi_t \colon \mathcal{G}^{t+1}(x) \to \mathbb{R}_+$ denote $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ for each t and let $U(\mathbf{x})\colon = \sum_{t=0}^{\infty} \rho_t(x_t, x_{t+1})$.

LEMMA A.1 *Let Assumption 3.2 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty}$ is a sequence with $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each n and $U(\boldsymbol{x}^n) \to B$ for $B > 0$, then there exists a sub-sequence $(\boldsymbol{x}^{n_k})_{k=0}^{\infty}$ such that for all $t \in \mathbb{N}$*

$$\lim_{k \to \infty} \rho_t(x_t^{n_k}, x_{t+1}^{n_k}) \to c_t$$

where $c_t \in \mathbb{R}_+$ for each t and $c_t > 0$ for at-least one t.

PROOF: By Assumption 3.2, for each t and n,

$$m_t \geq \rho_t(x_t^n, x_{t+1}^n) \geq 0 \tag{A.30}$$

Accordingly, for each n, $(\rho_t(x_t^n, x_{t+1}^n))_{t=0}^{\infty}$ will belong to the set $\prod_{t=0}^{\infty}[0, m_t]$, which by Tychonoff's Theorem (see Proposition 1.8.12 by Tao (2010)) will be compact in the product topology. There then exists a sub-sequence of $(\mathbf{x}^n)_{n=0}^{\infty}$, $(\mathbf{x}^{n_k})_{k=0}^{\infty}$, such that $(\rho(x_t^{n_k}, x_{t+1}^{n_k}))_{k=0}^{\infty}$ converges for each t. Let $c_t \colon = \lim\limits_{k\to\infty} \rho(x_t^{n_k}, x_{t+1}^{n_k})$ and note

$$B = \lim_{k\to\infty} \sum_{t=0}^{\infty} \beta^t \rho_t\left(x_t^{n_k}, x_{t+1}^{n_k}\right) = \sum_{t=0}^{\infty} \lim_{k\to\infty} \beta^t \rho_t\left(x_t^{n_k}, x_{t+1}^{n_k}\right) = \sum_{t=0}^{\infty} \beta^t c_t \tag{A.31}$$

Since (A.30) holds, and $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ by Assumption 3.2, we can pass limits through in the second equality using dominated convergence theorem (see Corollary 7.3.15 by Stachurski (2009)). If B is strictly positive, the above means there is at least one $c_t > 0$.

Q.E.D.

LEMMA A.2 *Let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty}$ is a sequence with $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each n and for some t*

$$\rho_t(x_t^n, x_{t+1}^n) \to c_t$$

with $c_t > 0$, then there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$.

PROOF: There exists ι such that $\epsilon \colon = c_t - \iota$ is strictly positive. For N large enough and any $n > N$, $\rho_t(x_t^n, x_{t+1}^n) \in [\epsilon, c_t + \iota]$, implying $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$ and $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$.

Q.E.D.

LEMMA A.3 *Let assumptions 3.1- 3.3 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\boldsymbol{x}^n)_{n=0}^{\infty}$ is a sequence such that $\boldsymbol{x}^n \in \mathcal{G}(x)$ for each $n \in \mathbb{N}$ and $U(\boldsymbol{x}^n) \to B$ where $B > 0$, then:*

1. *$(\boldsymbol{x}^n)_{n=0}^{\infty}$ has a convergent sub-sequence with a limit $\boldsymbol{x} \in \mathcal{G}(x)$, and*
2. *$B \leq U(\boldsymbol{x}) < \infty$.*

PROOF: Let x satisfy $x \in S_0$ and let $(\mathbf{x}^n)_{n=0}^{\infty}$ be a sequence such that $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ where $B > 0$. By Lemma A.1 there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that for each $t \in \mathbb{N}$, $c_t := \lim_{j\to\infty} \rho_t(x_t^{n_j}, x_{t+1}^{n_j}) > 0$ for at-least one t. Re-label $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ to $(\mathbf{x}^n)_{n=0}^{\infty}$, and let P denote the subset of $\mathbb{N}$ such that $t \in P$ if and only if $c_t > 0$. The set P will be non-empty, but could be finite or infinite.

To prove part 1 of the lemma, consider first the case when P is infinite and then the case when P is finite.

Suppose P is infinite and consider any $t \in \mathbb{N}$. There will exist $k > t$ such that $c_k > 0$. By Lemma A.2, there exists N and $\epsilon > 0$ such that for all $n > N$, $(x_i^n)_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$.

By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology. The space $\pi_t(UC_{\phi_k}(\epsilon))$ will also be sequentially compact by the argument in Remark A.1. Let $\Xi_t := \{x_1^0, \dots, x_t^N\} \cup \pi_t(UC_{\phi_k}(\epsilon))$. Since $\{x_1^0, \dots, x_t^N\}$ is sequentially compact, Ξ_t will be sequentially compact. Moreover, note $x_t^n \in \Xi_t$ for each $n \in \mathbb{N}$.

Since t was arbitrary, we can construct a Ξ_t as above for every $t \in \mathbb{N}$. Now let $\Xi := \prod_{t\in\mathbb{N}} \Xi_t$. Using the Sequential Tychonoff Theorem (Proposition 1.8.12 by Tao (2010)), Ξ will be sequentially compact. Since for each t, $x_t^n \in \Xi_t$ for each n, $\mathbf{x}^n \in \Xi$ for each n. There then exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ converging to $\mathbf{x}$, with $\mathbf{x} \in \Xi$.

We now confirm $\mathbf{x} \in \mathcal{G}(x)$ by showing $x_{t+1} \in \Gamma_t(x_t)$ for all $t \in \mathbb{N}$. Pick any $t \in \mathbb{N}$, there will be a k satisfying $k > t$ such that $c_k > 0$. By Lemma A.2, there exists $\epsilon > 0$ and J such that for all $j > J$ we have $(x_i^{n_j})_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$. By 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact, moreover, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$ by the definition of $UC_{\phi_k}(\epsilon)$ at (24). As such, the sub-sequence $(x_i^{n_j})_{i=0}^{k+1}$ converges to $(x_i)_{i=0}^{k+1}$, with $(x_i)_{i=0}^{k+1} \in \mathcal{G}^{k+1}(x)$, allowing us to conclude $x_{t+1} \in \Gamma(x_t)$. Since the t was arbitrary, $x_{t+1} \in \Gamma_t(x_t)$ for each $t \in \mathbb{N}$ and $\mathbf{x} \in \mathcal{G}(x)$.

Now assume P is finite; P will have a maximum element, which we now call k. By Lemma A.2, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that $(x_t^n)_{t=0}^{k+1} \in UC_{\phi_k}(\epsilon)$ for each $n > N$. By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology. As such, there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that $(x_t^{n_j})_{j=0}^{\infty}$ for each $t \leq k+1$. Define $(x_t)_{t=0}^{\infty}$ by setting $x_t = \lim_{j\to\infty} x_t^{n_j}$ for $t \leq k+1$ and picking any $x_{t+1} \in \Gamma_t(x_t)$ for $t \geq k+1$.

To confirm $(x_t)_{t=0}^{\infty}$ is feasible, let us check $x_{t+1} \in \Gamma_t(x_t)$ for each t. Once again, note by definition, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$. Since $UC_{\phi_k}(\epsilon)$ is sequentially compact, $(x_t)_{t=0}^{k+1} \in \mathcal{G}(x)$ and $x_{t+1} \in \Gamma_t(x_t)$ for all t satisfying $t \leq k$. On the other hand, if $t > k$, by construction, $x_{t+1} \in \Gamma_t(x_t)$, confirming $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$.

To prove part 2 of the lemma, by Assumption 3.2,

$$\rho_t(x_t^n, x_{t+1}^n) \leq m_t$$

for each t and n, where $\sum_{t=0}^{\infty} \beta^t m_t < \infty$. Whence Fatou's Lemma[13] gives

$$B = \limsup_{n\to\infty} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{\infty} \limsup_{n\to\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) < \infty \tag{A.32}$$

Upper-semicontinuity of ρ_t (Assumption 3.3) and the growth condition (Assumption 3.2) imply

$$\limsup_{n\to\infty} \rho_t(x_t^n, x_{t+1}^n) \leq \rho_t(x_t, x_{t+1}) \leq m_t, \qquad t \in \mathbb{N} \tag{A.33}$$

To complete the proof, combine (A.33) with (A.32) and conclude

$$B \leq \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = U(\mathbf{x}) < \infty$$

Q.E.D.

PROOF OF THEOREM 3.1: Fix $x \in \mathbb{S}_0$. If $U(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{G}(x)$, then our solution will be any $\mathbf{x} \in \mathcal{G}(x)$.

Next, suppose at-least one $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ satisfies $U(\mathbf{x}) > 0$. By Assumption 3.2, there exists a sequence of real numbers $(m_t)_{t=0}^{\infty}$ such that $\rho_t\,(x_t, x_{t+1}) \leq m_t$ for any $\mathbf{x}$ in $\mathcal{G}(x)$ and

$$\bar{B} :\, = \sum_{t=0}^{\infty} \beta^t m_t < \infty$$

Any $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ will satisfy

$$U\,(\mathbf{x}) = \sum_{t=0}^{\infty} \beta^t \rho_t\,(x_t, x_{t+1}) \leq \bar{B}$$

[13]See fact C.5 in the online appendix and let $\Omega = \mathbb{Z}_+$ and μ be the counting measure. Also see Equation (1.1) and discussion by Kamihigashi (2017).

Consider the set $I := \{U(\mathbf{x}) \mid \mathbf{x} \in \mathcal{G}(x)\}$. The set I will be a subset of $\mathbb{R} \cup \{-\infty, \infty\}$ and so must have a supremum. Let $B := \sup I$ and note $0 \leq B \leq \bar{B} < \infty$.

Construct a sequence $(\mathbf{x}^n)_{n=0}^{\infty}$ with $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ as follows: for every $n \in \mathbb{N}$, take $\mathbf{x}^n$ such that $B - U(\mathbf{x}^n) < \frac{1}{n+1}$. Such a sequence exists, otherwise for some n, $U(\mathbf{x}) \leq B - \frac{1}{n+1}$ for all $\mathbf{x} \in \mathcal{G}(x)$ and B will not be the supremum of I.

Since $U(\mathbf{x}^n) \to B$, by Lemma A.3, there exists $\mathbf{x} \in \mathcal{G}(x)$ such that $U(\mathbf{x}) \geq B$. Since B was the supremum for I, conclude

$$U(\mathbf{x}) = B = \tilde{V}(x) < \infty$$

Q.E.D.

PROOF OF PROPOSITION 3.1: Let x satisfy $x \in S_0$. Fix any $t \in \mathbb{N}$ and any ϵ satisfying $\epsilon > 0$. We show the upper contour sets $UC_{\phi_t}(\epsilon)$ defined by

$$UC_{\phi_t}(\epsilon) = \{(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x) \mid \rho_t(x_t, x_{t+1}) \geq \epsilon\} \tag{A.34}$$

are sequentially compact. In particular, we first show $UC_{\phi_t}(\epsilon)$ is a sequentially closed sub-set of $\mathbb{X}^{t+1}$ and then show $UC_{\phi_t}(\epsilon)$ is contained within a compact and metrizable set.

To show $UC_{\phi_t}(\epsilon)$ is sequentially closed in $\mathbb{X}^{t+1}$, take any sequence $(\mathbf{x}^n)_{n=0}^{\infty}$ with $\mathbf{x}^n \in UC_{\phi_t}(\epsilon)$ for each n that converges to $\mathbf{x} = (x_i)_{i=0}^{t+1}$ point-wise. Note $x_i^n \in \mathbb{S}_i$ for each $i \leq t+1$ and n. Since each $\mathbb{S}_i$ is sequentially closed (Assumption 3.4), $x_i \in \mathbb{S}_i$.

By Assumption 3.5, each Γ_i has a sequentially closed graph, and thus $x_{i+1} \in \Gamma_i(x_i)$ for each $i \leq t+1$. Noting the definition of $\mathcal{G}(x)^{t+1}$ by (25), conclude $\mathbf{x} \in \mathcal{G}(x)^{t+1}$.

We now confirm $\rho_t(x_t, x_{t+1}) \geq \epsilon$. By upper semi-continuity of ρ_t (Assumption 3.3), $UC_{\rho_t}(\epsilon) = \{(x, y) \in \operatorname{Gr} \Gamma_t \mid \rho_t(x, y) \geq \epsilon\}$ is sequentially closed. The sequence $(x_i^n)_{i=0}^{t+1}$ will satisfy $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$ and thus $\{x_t^n, x_{t+1}^n\} \in UC_{\rho_t}(\epsilon)$ for each n. Moreover, $x_{t+1} \in \Gamma_t(x_t)$. Accordingly, $\{x_t, x_{t+1}\} \in UC_{\rho_t}(\epsilon)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$. We conclude $\mathbf{x} \in UC_{\phi_t}(\epsilon)$ and $UC_{\phi_t}(\epsilon)$ is sequentially closed.

Since $\mathscr{E}$ satisfies Assumption 3.6, there will exist $\bar{M}$ such that if $(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$, then $\|x_i\| \leq \bar{M}$ for $i \in \{0, \ldots, t+1\}$. Whence $UC_{\phi_t}(\epsilon)$ will be a sub-set of the space $B_{\bar{M}} := \Pi_{i=0}^{t+1}\{x_i \in \mathbb{S}_i \mid \|x_i\| \leq \bar{M}\}$.

For each $i \leq t+1$, the space $\{x_i \in \mathbb{S}_i \mid \|x_i\| \leq \bar{M}\}$ will be compact by Alaoglu's Theorem.[14] Next, $L^2(\Omega, \mathbb{P})$ is a separable space since $\mathscr{F}$ is separable.[15] As such, since $L^2(\Omega, \mathbb{P})$ is reflexive, the spaces $\{x_i \in \mathbb{S}_i \mid \|x_i\| \leq \bar{M}\}$ are metrizable and sequentially compact.[16] Moreover, by the Sequential Tychonoff's Theorem,[17] the space $B_{\bar{M}}$ will be sequentially compact in the product topology (of weak topology on $\mathbb{X}$). By the argument in the preceding paragraph, $UC_{\phi_t}(\epsilon)$ is a sequentially closed sub-set of $B_{\bar{M}}$, allowing us to conclude $UC_{\phi_t}(\epsilon)$ is sequentially compact.

Q.E.D.

A.2. *Proofs for Section 4*

Recall point-wise inequalities in $\mathbb{X}$ hold $\mathbb{P}$ - almost everywhere and convergence of (x^n) with $x^n \in \mathbb{X}$ for each n will be with respect to the weak topology.

Recall the definition of sequential upper semicontinuity from section C.2 in the online appendix. The proof for the following claim is standard and placed in the online appendix.

CLAIM A.1 *Let (Ω, Σ, μ) be a finite measure space. Consider a function $g\colon \mathbb{R} \to \mathbb{R}$. Define $G\colon L^2(\Omega, \mu) \to \mathbb{R}$ as*

$$G(s)\colon = \int g(s)\, d\mu, \qquad s \in L^2(\Omega, \mu)$$

If g is concave and upper semicontinuous, then G will be weak sequentially upper semicontinuous.

PROOF OF PROPOSITION 4.2: Set any $t \in \mathbb{N}$ and consider sequences $(x^n)_{n=0}^{\infty}$ and $(y^n)_{n=0}^{\infty}$ with $\{x^n, y^n\} \in \operatorname{Gr}\Gamma_t$ for each n. Let $x^n \to x$ and $y^n \to y$ with $y \in \Gamma_t(x)$. To verify sequential upper semicontinuity, we show

$$\begin{aligned} \limsup_{n\to\infty} \rho_t(x^n, y^n) &= \limsup_{n\to\infty} \int \nu((1+\tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n)\, \mathrm{d}\mathbb{P} \\ &\leq \rho_t(x,y) \end{aligned} \tag{A.35}$$

[14] Mas-colell and Zame (1991), section 6.1 or Theorem 6.21 by Aliprantis and Border (2006).

[15] Ex. 1.3.9 by Tao (2010).

[16] See discussion proceeding Corollary 1.9.16 by Tao (2010) or Theorem 6.30 by Aliprantis and Border (2006).

[17] Proposition 1.8.12 by Tao (2010).

By Assumption 2.5, ν is concave and continuous. To use the statement made by Claim A.1, consider a continuous concave extension of ν to $\bar{v}$, where $\bar{v}\colon \mathbb{R} \to \mathbb{R}$ and $\bar{v}|_{\mathbb{R}_+} = \nu$.[18] The mapping $s \mapsto \int \bar{v}(s)\,\mathrm{d}\mathbb{P}$ for $s \in L^2(\Omega, \mathbb{P})$ will be sequentially upper semicontinuous since $\bar{v}$ is concave and upper semicontinuous. As such, for any sequence in $L^2(\Omega, \mathbb{P})$ satisfying $f^n \to f$ weakly,

$$\limsup_{n\to\infty} \int \bar{v}(f^n)\,\mathrm{d}\mathbb{P} \leq \int \bar{v}(f)\,\mathrm{d}\mathbb{P} \tag{A.36}$$

Let $f^n \colon = (1 + \tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n$ and note $f^n \in L^2(\Omega, \mathbb{P})$ for each n. First, we show (A.35) for the case $\int x\,\mathrm{d}\mathbb{P} > 0$. If $\int x\,\mathrm{d}\mathbb{P} > 0$, then

$$\begin{aligned}\int f^n h\,\mathrm{d}\mathbb{P} &= (1 + \tilde{r}(x^n)) \int x^n h\,\mathrm{d}\mathbb{P} + \tilde{w}(x^n) \int e_t h\,\mathrm{d}\mathbb{P} - \int y^n h\,\mathrm{d}\mathbb{P} \\ &\to (1 + \tilde{r}(x)) \int xh\,\mathrm{d}\mathbb{P} + \tilde{w}(x) \int e_t h\,\mathrm{d}\mathbb{P} - \int yh\,\mathrm{d}\mathbb{P}\end{aligned}$$

for any $h \in L^2(\Omega, \mathbb{P})$. Thus f^n converges weakly to $f \colon = (1 + \tilde{r}(x))x + \tilde{w}(x)e_t - y$, implying by (A.36),

$$\limsup_{n\to\infty} \int \nu(f^n)\,\mathrm{d}\mathbb{P} \leq \int \nu(f)\,\mathrm{d}\mathbb{P} = \rho_t(x, y)$$

If $\int x\,\mathrm{d}\mathbb{P} = 0$, then

$$\begin{aligned}\limsup_{n\to\infty} \int \nu(f^n)\,\mathrm{d}\mathbb{P} &\leq \limsup_{n\to\infty} \nu\left(\int (1 + \tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t - y^n\,\mathrm{d}\mathbb{P}\right) \\ &\leq \lim_{n\to\infty} \nu(F(\tilde{K}(x^n), L) + (1-\delta)\tilde{K}(x^n)) \\ &= 0 = \rho_t(x, y)\end{aligned}$$

where the first inequality follows from Jensen's inequality (fact C.4 in the online appendix). The second inequality follows from Assumption 2.3 on homogeneity of the production function (recall Equation (9)).

Q.E.D.

[18] See Corollary 8.3.10 by Borwein and Vanderwerff (2010).

PROOF OF PROPOSITION 4.4: Recall the definition of closed graph correspondences from the Mathematical Preliminaries section of the online appendix. Set t and suppose $(x^n, y^n)_{n=0}^{\infty}$ satisfies $y^n \in \Gamma_t(x^n)$ for each n. Suppose $(x^n)_{n=0}^{\infty}$ converges to $x \in \mathbb{S}_t$ and $(y^n)_{n=0}^{\infty}$ converges to $y \in \mathbb{S}_{t+1}$.

We show $y \in \Gamma_t(x)$ by checking both the cases stated in the definition of Γ_t at Equation (18): either $\int x \, \mathrm{d}\mathbb{P} = 0$ or $\int x \, \mathrm{d}\mathbb{P} > 0$. First let $\int x \, \mathrm{d}\mathbb{P} > 0$, we show $y \leq (1 + \tilde{r}(x))x + \tilde{w}(x)e_t$ for $\mathbb{P}$-almost everywhere. Suppose by contradiction

$$\mathbb{P}\left\{y > (1 + \tilde{r}(x))x + \tilde{w}(x)e_t\right\} > 0$$

Let $B \colon = \{\omega \in \Omega \,|\, y(\omega) > (1 + \tilde{r}(x))x(\omega) + \tilde{w}(x)e_t(\omega)\}$, we have $\mathbb{P}(B) > 0$ and

$$\int \mathbb{1}_B y \, \mathrm{d}\mathbb{P} > \int \mathbb{1}_B \times [(1 + \tilde{r}(x))x + \tilde{w}(x)e_t] \, \mathrm{d}\mathbb{P} \tag{A.37}$$

Since $\tilde{K}(x^n) \to \tilde{K}(x)$ and $\tilde{K}(x) > 0$, there exists N such that for all $n > N$, $\tilde{K}(x^n) > 0$. And for the tail sequence $(x^n)_{n=N+1}^{\infty}$, $\tilde{r}(x^n) = F_1(\tilde{K}(x^n), L)$ converges, implying

$$\begin{aligned}(1 + \tilde{r}(x^n)) \int x^n h \, \mathrm{d}\mathbb{P} + \tilde{w}(x^n) \int h e_t \, \mathrm{d}\mathbb{P} \\ \to (1 + \tilde{r}(x)) \int x h \, \mathrm{d}\mathbb{P} + \tilde{w}(x) \int h e_t \, \mathrm{d}\mathbb{P}\end{aligned} \tag{A.38}$$

for any function h satisfying $h \in L^2(\Omega, \mathbb{P})$. In particular, let $h = \mathbb{1}_B$, and note $y^n \in \Gamma_t(x^n)$; by the feasibility condition at (18), we write

$$\int \mathbb{1}_B y^n \, \mathrm{d}\mathbb{P} \leq (1 + \tilde{r}(x^n)) \int \mathbb{1}_B x^n \, \mathrm{d}\mathbb{P} + \tilde{w}(x^n) \int \mathbb{1}_B e_t \, \mathrm{d}\mathbb{P}$$

for each $n > N$. Since the weak inequality above will be preserved under the limits of real-valued sequences, we arrive at

$$\int \mathbb{1}_B y \, \mathrm{d}\mathbb{P} \leq (1 + \tilde{r}(x)) \int \mathbb{1}_B x \, \mathrm{d}\mathbb{P} + \tilde{w}(x) \int \mathbb{1}_B e_t \, \mathrm{d}\mathbb{P} \tag{A.39}$$

However, (A.39) is a contradiction to (A.37) and we conclude

$$y \leq (1 + \tilde{r}(x))x + \tilde{w}(x)e_t$$

Now suppose $\int x^n \, \mathrm{d}\mathbb{P} \to 0$. Note

$$\int y^n \, \mathrm{d}\mathbb{P} \leq \int (1 + \tilde{r}(x^n))x^n + \tilde{w}(x^n)e_t \, \mathrm{d}\mathbb{P} = F(\tilde{K}(x^n), L) + (1 - \delta)\tilde{K}(x^n)$$

The above equality follows from homogeneity of degree one of the production function (Assumption 2.3 and recall discussion preceding Equation (9)). Since $\tilde{K}(x^n) = \int x^n \, \mathrm{d}\mathbb{P} \to 0$, we have $F(\tilde{K}(x^n), L) \to 0$ by Assumption 2.3, and

$$0 = \lim_{n\to\infty} \int y^n = \int y \, \mathrm{d}\mathbb{P} \tag{A.40}$$

Since $y \in \mathbb{S}_{t+1}$, $y \geq 0$ and (A.40) implies $y = 0$ for $\mathbb{P}$-almost everywhere.[19]

Thus we have checked $x_t \in \Gamma_t(x)$ under both the cases stated in the definition of Γ_t at (18), completing the proof.

Q.E.D.

For the following lemma, consider the setting and notation of the sequential planner's problem in section 4.

LEMMA A.4 *Fix x with $x \in \mathbb{S}_0$, $\epsilon > 0$ and $t \in \mathbb{N}$. If assumptions 2.1 - 2.5 hold, then there exists $\bar{r} \in \mathbb{R}_+$ such that for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have $\tilde{r}(x_i) \leq \bar{r}$ for each $i \leq t$.*

PROOF: Fix x with $x \in \mathbb{S}_0$, $\epsilon > 0$ and $t \in \mathbb{N}$. Select $(x_i)_{i=0}^{\infty}$ satisfying $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$.

Since $x_i \in \Gamma_{i-1}(x_{i-1})$, by the feasibility correspondence (Equation (18)) and homogeneity of degree one (Assumption 2.3 and recall discussion preceding Equation (9)) of the production function F, we have

$$\begin{aligned} \tilde{K}(x_i) = \int x_i \, \mathrm{d}\mathbb{P} &\leq \int (1 + \tilde{r}(x_{i-1}))x_{i-1} + \tilde{w}(x_{i-1})e_{i-1} \, \mathrm{d}\mathbb{P} \\ &= (1 + F_1(\tilde{K}(x_{i-1}), L) - \delta)\tilde{K}(x_{i-1}) \\ &\quad + F_2(\tilde{K}(x_{i-1}), L)L \\ &= F(\tilde{K}(x_{i-1}), L) + (1-\delta)\tilde{K}(x_{i-1}) \end{aligned} \tag{A.41}$$

for each $i \in \mathbb{N}$.

Define $\hat{F}(K) := F(K, L) + (1-\delta)K$ and note $\hat{F}$ will be strictly increasing. By (A.41),

$$\tilde{K}(x_i) \leq \hat{F}(\tilde{K}(x_{i-1})), \qquad i \in \mathbb{N} \tag{A.42}$$

[19] If $y \geq 0$, then $y = 0$ if and only if $\int y \, \mathrm{d}\mathbb{P} = 0$. See Theorem 1.1.20 by Tao (2010).

As such, for any $k > 1$, by a simple inductive argument (Claim D.5 in the online appendix), we can show

$$\tilde{K}(x_k) \leq \hat{F}^{k-i}(\tilde{K}(x_i)), \qquad \forall\, i \leq k \tag{A.43}$$

Next, since ν is concave, from Jensen's inequality (fact C.4 in the online appendix),

$$\begin{aligned} \epsilon \leq \rho_t(x_t, x_{t+1}) &= \int \nu\left((1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1}\right) \mathrm{d}\mathbb{P} \\ &\leq \nu\left(\int (1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t \,\mathrm{d}\mathbb{P}\right) \\ &= \nu(\hat{F}(\tilde{K}(x_t))) \end{aligned} \tag{A.44}$$

Note the inverse of v, v^{-1}, is also increasing since v is increasing. (The inverse of ν exists by Assumption 2.5.) From (A.44), $v^{-1}(\epsilon) \leq \hat{F}(\tilde{K}(x_t))$. And, by (A.43),

$$\nu^{-1}(\epsilon) \leq \hat{F}(\tilde{K}(x_t)) \leq \hat{F}^{t-i+1}(\tilde{K}(x_i)), \qquad \forall\, i \leq t \tag{A.45}$$

Next, Let G^j denote the inverse of $\hat{F}^j$. Since $\hat{F}$ is strictly increasing, by (A.45), we have $\tilde{K}(x_i) \geq G^{t-i+1}(\nu^{-1}(\epsilon))$ for each $i \leq t$. Define

$$\underline{K} := \min_{i \in \{0,\dots,t\}} \{G^{t-i+1}(\nu^{-1}(\epsilon))\}$$

and note $\tilde{K}(x_i) \geq \underline{K}$ for each $i \leq t$.

Finally, let $\bar{r} := F_1(\underline{K}, L) - \delta$. Note $F_1(K, L)$ is decreasing in the first argument since F is concave and conclude

$$\tilde{r}(x_i) = F_1(\tilde{K}(x_i), L) - \delta \leq F_1(\underline{K}, L) - \delta := \bar{r}, \qquad \forall\, i \leq t$$

Since $\bar{r}$ depends only on t and ϵ, the above will hold for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$.

Q.E.D.

PROOF OF PROPOSITION 4.5: Fix any x satisfying $x \in \mathbb{S}_0$, $\epsilon > 0$ and t. By Lemma A.4, there exists $\bar{r} \in \mathbb{R}_+$ such that for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ satisfying $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have

$$r(x_i) \leq \bar{r}, \qquad \forall\, i \leq t$$

Moreover, since aggregate capital will be bounded from above, the maximum possible wage rate will be bounded above by a constant, which we now denote as $\bar{w}$.

Let $(x_i)_{i=0}^{\infty}$ be any sequence satisfying $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ and $\rho_t(x_t, x_{t+1}) \geq \epsilon$. For any $i \in \{1, \ldots, t+1\}$,

$$\begin{aligned} x_i &\leq (1+\bar{r})x_{i-1} + \bar{w}e_{i-1} \\ &\leq (1+\bar{r})^2 x_{i-2} + \bar{w}e_{i-1} + (1+\bar{r})\bar{w}e_{i-2} \\ &\qquad\qquad\qquad \vdots \\ &\leq (1+\bar{r})^i x + \bar{w}\sum_{j=0}^{i-1}(1+\bar{r})^j e_{i-j-1} \end{aligned}$$

Let $W_i : = \bar{w}\sum_{j=0}^{i-1}(1+\bar{r})^j e_{i-j-1}$ and note $\|W_i\|$ will be finite. Next, since $x_i \geq 0$,

$$x_i \leq (1+\bar{r})^i x + W_i \implies (x_i)^2 \leq \Big((1+\bar{r})^i x + W_i\Big)^2$$

As such, for all $i \in \{1, \ldots, t+1\}$,

$$\begin{aligned} \|x_i\| &\leq \|(1+\bar{r})^i x + W_i\| \\ &\leq (1+\bar{r})^i\|x\| + \|W_i\| \\ &: = \hat{M}_i \in \mathbb{R} \end{aligned}$$

To conclude, let $\hat{M} : = \max\{\|x\|, \hat{M}_1, \ldots, \hat{M}_{t+1}\}$. The scalar $\hat{M}$ depends only on x, $\bar{r}$, $\bar{w}$, t and ϵ. As such, for any $(x_i)_{i=0}^{\infty} \in \mathcal{G}(x)$ that satisfies $\rho_t(x_t, x_{t+1}) \geq \epsilon$, we have $\|x_i\| \leq \hat{M}$ for each $i \leq t+1$.

Q.E.D.

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