Existence of Solutions to Non-Compact Dynamic Optimization Problems

September 28, 2017

Objective

Present and prove theorem on existence of solutions to a **reduced form** dynamic optimisation problem when feasibility correspondences have **non-compact** image sets and pay-offs are **bounded below**

- Main application and motivation: optimal policies in incomplete market models with heterogeneity

Preliminaries

Semicontinuity

Definition. A function $f: X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is sequentially **upper semi-continuous** if the upper contour sets

$$UC_f(\epsilon) := \{x \in X \mid f(x) \geq \epsilon\}$$

are sequentially closed for all $\epsilon \in \mathbb{R}$.

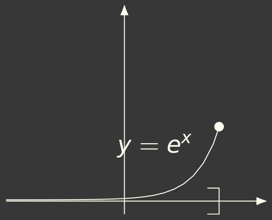
Sup-Compactness

Let D be a subset of $\mathbb{R} \cup \{-\infty, +\infty\}$

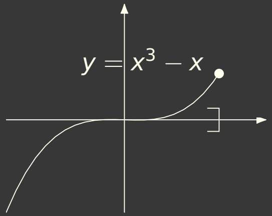
Definition. A function $f : X \to D$ is **sup-compact** if the sets $UC_f(\epsilon)$ are sequentially compact for all $\epsilon \in \mathbb{R}$

If X is not compact and D is bounded below, then f cannot be sup-compact

Definition. A function $f : X \to D$ is **mildly sup-compact** if the sets $UC_f(\epsilon)$ are sequentially compact for all $\epsilon > \inf f$



Mildly Sup-Compact



Sup-Compact

Correspondences

Let (X, τ) and (Y, τ') be topological spaces. A correspondence from a space X to Y is a set valued function denoted by $\Gamma\colon X \twoheadrightarrow Y$.

The image of a subset A of X under the correspondence Γ will be the set

$$\Gamma(A) := \{y \in Y | y \in \Gamma(x) \text{ for some } x \in A\}$$

A correspondence will be called **compact valued** if $\Gamma(x)$ is compact for $x \in X$.

Correspondences

The correspondence Γ is **upper hemi-continuous** if for every x and neighbourhood U of $\Gamma(x)$, there is a neighbourhood V of x such that $z \in V$ implies $\Gamma(z) \subset U$

Upper hemicontinuous correspondences need not be compact valued or have closed graph. Closed graph correspondences also need not be upper hemi-continuous (see Aliprantis and Border (2006), ch. 17). However,

Lemma. If $\Gamma : X \twoheadrightarrow Y$ is upper hemicontinuous and compact valued, then for $C \subset X$ such that C is compact, $\Gamma(C)$ is compact.

See Lemma 17.8 by Aliprantis and Border (2006)) for a proof

Problem Statement

Problem Statement

A non-stationary reduced form economy is a 5-tuple

$$\mathscr{E} := ((\mathbb{X}, \tau), (\mathbb{S}_t)_{t=0}^{\infty}, (\Gamma_t)_{t=0}^{\infty}, (\rho_t)_{t=0}^{\infty}, \beta) \tag{1}$$

consisting of:

- A topological space $(\mathbb{X}, \tau)$
- A collection of state-spaces $(\mathbb{S}_t)_{t=0}^{\infty}$, with $\mathbb{S}_t \subset \mathbb{X}$ for each t
- A collection of non-empty feasibility correspondences $(\Gamma_t)_{t=0}^{\infty}$, with $\Gamma_t \colon \mathbb{S}_t \twoheadrightarrow \mathbb{S}_{t+1}$ for each t
- A collection of per-period pay-offs $(\rho_t)_{t=0}^{\infty}$, with $\rho_t \colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$ and $\inf \rho_t = 0$ for each t
- A discount factor $\beta \in (0, 1)$.

Problem Statement

Define the correspondence of **feasible sequences** $\mathcal{G}_t^T : \mathbb{S}_t \twoheadrightarrow \prod_{i=t}^{T} \mathbb{S}_i$ starting at time t and ending at time T as follows:

$$\mathcal{G}_t^T(x) :\, = \left\{ (x_i)_{i=t}^T \,|\, x_{i+1} \in \Gamma_i(x_i), x_t = x \right\}, \qquad x \in \mathbb{S}_t \quad (2)$$

Let $\mathcal{G}$ denote $\mathcal{G}_0^\infty$ and let $\mathcal{G}^T$ denote $\mathcal{G}_0^T$.

Problem Statement

Define the **value function** $\tilde{V} \colon \mathbb{S}_0 \to \mathbb{R} \cup \{-\infty, +\infty\}$ as follows:

$$\tilde{V}(x) \colon = \sup_{(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)} \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t, x_{t+1}\right) \tag{3}$$

Application

Aiyagari-Huggett optimal policy (roughly)

- let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t=0}^{\infty}, \mathbb{P})$ be a filtered probability space
- $\mathbb{X} = L^2(Z, \mathbb{P})$ with the weak topology
- the state-spaces $\mathbb{S}_t$ are spaces of $\mathscr{F}_t$ measurable random variables (history dependent)
- the correspondences Γ_t does not have compact image sets because of Inada conditions
- feasible sequences $(x_t)_{t=0}^{\infty}$ map histories of shocks to assets
- the pay-off ρ_t integrates pay-offs across all agents given prices that depend on x_t

Assumptions

Fix $x \in \mathbb{S}_0$. Let $\phi_t \colon \mathcal{G}^{t+1}(x) \to \mathbb{R}_+$ denote $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ for each t

The upper contour sets $UC_{\phi_t}(\epsilon)$ of $(x_i)_{i=0}^{t+1} \mapsto \rho_t(x_t, x_{t+1})$ are defined by

$$UC_{\phi_t}(\epsilon) = \{(x_i)_{i=0}^{t+1} \in \mathcal{G}^{t+1}(x) \mid \rho_t(x_t, x_{t+1}) \geq \epsilon\} \tag{4}$$

Assumptions

Standard requirement is for Γ_t to be upper hemicontinuous and compact valued and for $\mathbb{S}_t$ to be a metric space (see by Acemoglu (2009), Assumption 6.2, Kamihigashi (2017), section 6 or Stokey and Lucas (1989), Assumption 4.3, for assumptions used by the standard theory).

Main assumption below relaxes this requirement.

Assumption.3.1 For each $x \in \mathbb{S}_0$ and $t \in \mathbb{N}$, the functions $\phi_t \colon \mathcal{G}^{t+1}(x) \to \mathbb{R}_+$ are mildly sup-compact in the product topology (of τ topology in $\mathbb{X}$)

Assumptions

The next assumption is the standard growth condition (see discussion on Corollary 6.1 by Kamihigashi (2017)).

Assumption.3.2 For each $x \in \mathbb{S}_0$, there exists a sequence of non-negative real numbers $(m_t)_{t=0}^{\infty}$ such that any $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ satisfies

$$\rho_t(x_t, x_{t+1}) \leq m_t, \qquad \forall t \in \mathbb{N} \tag{5}$$

and

$$\sum_{t=0}^{\infty} \beta^t m_t < \infty \tag{6}$$

Assumption.3.3 The functions $(\rho_t)_{t=0}^{\infty}$ are sequentially upper semicontinuous for all $t \in \mathbb{N}$.

Main Theorem

Theorem. 3.1 If $\mathscr{E}$ satisfies assumptions 3.1 - 3.3, then for every $x \in S_0$, there will exist $(x_t)_{t=0}^{\infty}$ satisfying $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ such that

$$\tilde{V}(x) = \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t, x_{t+1}\right) < \infty$$

Proofs

Proof Premlinaries

Let $(\mathbb{X}, \tau)$ be a topological vector space

Unless otherwise stated, convergence for sequences in $\mathbb{X}$ will be with respect to the τ topology and convergence for sequences in countable Cartesian products of $\mathbb{X}$ will be in the product topology of the τ topology on $\mathbb{X}$.

We will use $\mathbf{x}$ to refer to elements of $\mathbb{X}^{\mathbb{N}}$. We can then use $(\mathbf{x}^n)_{n=0}^{\infty}$ to denote a sequence $\{\mathbf{x}^0, \ldots, \mathbf{x}^n, \ldots\}$, where $(\mathbf{x}^n)_{n=0}^{\infty} \in (\mathbb{X}^{\mathbb{N}})^{\mathbb{N}}$.

Let $U(\mathbf{x}) := \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$.

Product Topology

Remark. A.1 Let $X = \prod_{i \in F} X_i$ denote a Cartesian product of topological spaces. Let $\pi_i \colon X \to X_i$ denote the projection map defined as $\pi_i(x) = x_i$ for each $i \in F$.

Recall each projection map will be a continuous function on X when X has the product topology (see section 2.14 by Aliprantis and Border (2006))

Also recall (section 1.8 by Tao (2013)) the image of a (sequentially) compact set under a continuous function is (sequentially) compact.

If a set C with $C \subset X$ is (sequentially) compact in the product topology, then $\pi_i(C)$ will be (sequentially) compact.

Lemma A.1

Lemma. A.1 Let Assumption 3.2 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\mathbf{x}^n)_{n=0}^{\infty}$ is a sequence with $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ for $B > 0$, then there exists a sub-sequence $(\mathbf{x}^{n_k})_{k=0}^{\infty}$ such that for all $t \in \mathbb{N}$

$$\lim_{k \to \infty} \rho_t(x_t^{n_k}, x_{t+1}^{n_k}) \to c_t$$

where $c_t \in \mathbb{R}_+$ for each t and $c_t > 0$ for at-least one t.

Proof of Lemma A.1

Proof. By Assumption 3.2, for each t and n,

$$m_t \geq \rho_t(x_t^n, x_{t+1}^n) \geq 0 \tag{7}$$

Accordingly, for each n, $(\rho_t(x_t^n, x_{t+1}^n))_{t=0}^{\infty}$ will belong to the set $\prod_{t=0}^{\infty}[0, m_t]$, which by Tychonoff's Theorem (see Proposition 1.8.12 by Tao (2010)) will be compact in the product topology.

There then exists a sub-sequence of $(\mathbf{x}^n)_{n=0}^{\infty}$, $(\mathbf{x}^{n_k})_{k=0}^{\infty}$, such that $(\rho(x_t^{n_k}, x_{t+1}^{n_k}))_{k=0}^{\infty}$ converges for each t.

Proof of Lemma A.1

Let $c_t := \lim_{k\to\infty} \rho(x_t^{n_k}, x_{t+1}^{n_k})$ and note

$$B = \lim_{k\to\infty} \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t^{n_k}, x_{t+1}^{n_k}\right)$$

$$= \sum_{t=0}^{\infty} \lim_{k\to\infty} \beta^t \rho_t \left(x_t^{n_k}, x_{t+1}^{n_k}\right) = \sum_{t=0}^{\infty} \beta^t c_t \quad (8)$$

Since (7) holds, and $\sum_{t=0}^{\infty} \beta^t m_t < \infty$ by Assumption 3.2, we can pass limits through in the second equality using dominated convergence theorem (see Corollary 7.3.15 by Stachurski (2009))

If B is strictly positive, the above means there is at least one $c_t > 0$.

Lemma A.2

Lemma. A.2

Let x satisfy $x \in \mathbb{S}_0$. If $(\mathbf{x}^n)_{n=0}^{\infty}$ is a sequence with $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and for some t

$$\rho_t(x_t^n, x_{t+1}^n) \to c_t$$

with $c_t > 0$, then there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$.

Proof. There exists ι such that $\epsilon := c_t - \iota$ is strictly positive

For N large enough and any $n > N$, $\rho_t(x_t^n, x_{t+1}^n) \in [\epsilon, c_t + \iota]$, implying $\rho_t(x_t^n, x_{t+1}^n) \geq \epsilon$ and $(x_i^n)_{i=0}^{t+1} \in UC_{\phi_t}(\epsilon)$.

Lemma A.3

Lemma. A.3
Let assumptions 3.1- 3.3 hold and let x satisfy $x \in \mathbb{S}_0$. If $(\mathbf{x}^n)_{n=0}^{\infty}$ is a sequence such that $\mathbf{x}^n \in \mathcal{G}(x)$ for each $n \in \mathbb{N}$ and $U(\mathbf{x}^n) \to B$ where $B > 0$, then:

1. $(\mathbf{x}^n)_{n=0}^{\infty}$ has a convergent sub-sequence with a limit $\mathbf{x} \in \mathcal{G}(x)$, and
2. $B \leq U(\mathbf{x}) < \infty$.

Proof of Lemma A.3

Proof. Let x satisfy $x \in \mathbb{S}_0$ and let $(\mathbf{x}^n)_{n=0}^{\infty}$ be a sequence such that $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ where $B > 0$.

By Lemma A.1 there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that for each $t \in \mathbb{N}$, $c_t := \lim_{j \to \infty} \rho_t(x_t^{n_j}, x_{t+1}^{n_j}) > 0$ for at-least one t

Re-label $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ to $(\mathbf{x}^n)_{n=0}^{\infty}$, and let P denote the subset of $\mathbb{N}$ such that $t \in P$ if and only if $c_t > 0$

- The set P will be non-empty, but could be finite or infinite.

Proof of Lemma A.3

We consider first the case when P is infinite and then the case when P is finite.

Suppose P is infinite and consider any $t \in \mathbb{N}$. There will exist $k > t$ such that $c_k > 0$

By Lemma A.2, there exists N and $\epsilon > 0$ such that for all $n > N$, $(x_i^n)_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$

Proof of Lemma A.3

By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology

The space $\pi_t(UC_{\phi_k}(\epsilon))$ will also be sequentially compact by the argument in Remark A.1

Let $\Xi_t := \{x_1^0, \ldots, x_t^N\} \cup \pi_t(UC_{\phi_k}(\epsilon))$

Since $\{x_1^0, \ldots, x_t^N\}$ is sequentially compact, Ξ_t will be sequentially compact

- Note $x_t^n \in \Xi_t$ for each $n \in \mathbb{N}$

Proof of Lemma A.3

Since t was arbitrary, can construct a Ξ_t as above for every $t \in \mathbb{N}$

Let $\Xi := \prod_{t \in \mathbb{N}} \Xi_t$

Using the Sequential Tychonoff Theorem (Proposition 1.8.12 by Tao (2010)), Ξ will be sequentially compact

Since for each t, $x_t^n \in \Xi_t$ for each n, $\mathbf{x}^n \in \Xi$ for each n, there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ converging to $\mathbf{x}$, with $\mathbf{x} \in \Xi$

Proof of Lemma A.3

We now confirm $\mathbf{x} \in \mathcal{G}(x)$ by showing $x_{t+1} \in \Gamma_t(x_t)$ for all $t \in \mathbb{N}$

Pick any $t \in \mathbb{N}$, there will be a k satisfying $k > t$ such that $c_k > 0$

By Lemma A.2, there exists $\epsilon > 0$ and J such that for all $j > J$ we have $(x_i^{n_j})_{i=0}^{k+1} \in UC_{\phi_k}(\epsilon)$

Proof of Lemma A.3

By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact, moreover, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$ by the definition of $UC_{\phi_k}(\epsilon)$ at (4), frame 15.

As such, the sub-sequence $(x_i^{n_j})_{i=0}^{k+1}$ converges to $(x_i)_{i=0}^{k+1}$, with $(x_i)_{i=0}^{k+1} \in \mathcal{G}^{k+1}(x)$, allowing us to conclude $x_{t+1} \in \Gamma_t(x_t)$

Since the t was arbitrary, $x_{t+1} \in \Gamma_t(x_t)$ for each $t \in \mathbb{N}$ and $\mathbf{x} \in \mathcal{G}(x)$.

Proof of Lemma A.3

Now assume P is finite. P will have a maximum element, which we now call k

By Lemma A.2, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that $(x_t^n)_{t=0}^{k+1} \in UC_{\phi_k}(\epsilon)$ for each $n > N$

By Assumption 3.1, $UC_{\phi_k}(\epsilon)$ will be sequentially compact in the product topology

As such, there exists a sub-sequence $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ such that $(x_t^{n_j})_{j=0}^{\infty}$ for each $t \leq k+1$

Define $(x_t)_{t=0}^{\infty}$ by setting $x_t = \lim_{j \to \infty} x_t^{n_j}$ for $t \leq k+1$ and picking any $x_{t+1} \in \Gamma_t(x_t)$ for $t \geq k+1$.

Proof of Lemma A.3

To confirm $(x_t)_{t=0}^{\infty}$ is feasible, we check $x_{t+1} \in \Gamma_t(x_t)$ for each t

Once again, note by definition, $UC_{\phi_k}(\epsilon) \subset \mathcal{G}^{k+1}(x)$

Since $UC_{\phi_k}(\epsilon)$ is sequentially compact, $(x_t)_{t=0}^{k+1} \in \mathcal{G}(x)$ and $x_{t+1} \in \Gamma_t(x_t)$ for all t satisfying $t \leq k$

On the other hand, if $t > k$, by construction, $x_{t+1} \in \Gamma_t(x_t)$, confirming $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$

Proof of Lemma A.3

Re-label $(\mathbf{x}^{n_j})_{j=0}^{\infty}$ to $(\mathbf{x}^{n})_{n=0}^{\infty}$

To prove part 2 of the lemma, by Assumption 3.2,

$$\rho_t(x_t^n, x_{t+1}^n) \leq m_t$$

for each t and n, where $\sum_{t=0}^{\infty} \beta^t m_t < \infty$.

Fatou's Lemma[1] gives

$$B = \limsup_{n\to\infty} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{\infty} \limsup_{n\to\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) < \infty \quad (9)$$

[1] See 5.4 b) by Williams (1991) and let $\Omega = \mathbb{Z}_+$ and μ be the counting measure. Also see Equation (1.1) and discussion by Kamihigashi (2017).

Proof of Lemma A.3

Upper-semicontinuity of ρ_t (Assumption 3.3) and the growth condition (Assumption 3.2) imply

$$\limsup_{n\to\infty} \rho_t(x_t^n, x_{t+1}^n) \leq \rho_t(x_t, x_{t+1}) \leq m_t, \qquad t \in \mathbb{N} \tag{10}$$

Note the growth condition implies

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \rho_t(x_t, x_{t+1}) \leq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t m_t < \infty \tag{11}$$

Thus, combine (10) with (9) and conclude

$$B \leq \sum_{t=0}^{\infty} \limsup_{n\to\infty} \beta^t \rho_t(x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) = U(\mathbf{x}) < \infty$$

□

Proof of Theorem 3.1

Theorem. 3.1 If $\mathscr{E}$ satisfies assumptions 3.1 - 3.3, then for every $x \in \mathbb{S}_0$, there will exist $(x_t)_{t=0}^{\infty}$ satisfying $(x_t)_{t=0}^{\infty} \in \mathcal{G}(x)$ such that

$$\tilde{V}(x) = \sum_{t=0}^{\infty} \beta^t \rho_t \left(x_t, x_{t+1}\right) < \infty$$

Proof. Fix $x \in \mathbb{S}_0$. If $U(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{G}(x)$, then our solution will be any $\mathbf{x} \in \mathcal{G}(x)$.

Proof of Theorem 3.1

Next, suppose at-least one $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ satisfies $U(\mathbf{x}) > 0$

By Assumption 3.2, there exists a sequence of real numbers $(m_t)_{t=0}^{\infty}$ such that $\rho_t\left(x_t, x_{t+1}\right) \leq m_t$ for any $\mathbf{x}$ in $\mathcal{G}(x)$ and

$$\bar{B} := \sum_{t=0}^{\infty} \beta^t m_t < \infty$$

Proof of Theorem 3.1

Any $\mathbf{x}$ with $\mathbf{x} \in \mathcal{G}(x)$ will satisfy

$$U(\mathbf{x}) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \leq \bar{B}$$

Now, consider the set $I := \{U(\mathbf{x}) \mid \mathbf{x} \in \mathcal{G}(x)\}$

- I will be a subset of $\mathbb{R} \cup \{-\infty, \infty\}$ and so must have a supremum

Let $B := \sup I$ and note $0 \leq B \leq \bar{B} < \infty$

Proof of Theorem 3.1

Construct a sequence $(\mathbf{x}^n)_{n=0}^{\infty}$ with $\mathbf{x}^n \in \mathcal{G}(x)$ for each n and $U(\mathbf{x}^n) \to B$ as follows:

- for every $n \in \mathbb{N}$, take $\mathbf{x}^n$ such that $B - U(\mathbf{x}^n) < \frac{1}{n+1}$

Such a sequence exists, otherwise for some n, $U(\mathbf{x}) \leq B - \frac{1}{n+1}$ for all $\mathbf{x} \in \mathcal{G}(x)$ and B will not be the supremum of I.

Proof of Theorem 3.1

Since $U(\mathbf{x}^n) \to B$, by Lemma A.3, there exists $\mathbf{x} \in \mathcal{G}(x)$ such that $U(\mathbf{x}) \geq B$. Since B was the supremum for I, conclude

$$U(\mathbf{x}) = B = \tilde{V}(x) < \infty$$

$\square$

Application: Existence of Recursive Constrained Optima in the Aiyagari-Huggett Model

Model setup

Aiyagari (1994) model, constrained planner as considered by Dávila et al. (2012)

- time is discrete and indexed by t, with $t \in \mathbb{N}$
- A, with $A := [0, \infty)$, denotes asset space
- E, with $E \subset \mathbb{R}_+$ denotes labour endowment space

Let $S := A \times E$

Uncertainty

Consider square integrable random variables $\{x_0, e_0, e_1, \ldots\}$, with x_0 taking values in A and with e_t taking values in E for each t

The random variables $\{x_0, e_0, e_1, \ldots\}$ defined on common probability space $(\Omega, \Sigma, \mathbb{P})$

Let P denote the probability law or joint distribution of $\{x_0, e_0, e_1, \ldots\}$

Let μ_0 denote the joint distribution of x_0 and e_0

Assumption. 2.1

The shocks satisfy the following conditions:

1. the shocks $(e_t)_{t=0}^{\infty}$ are iid a with common distribution ψ
2. x_0 is independent of $(e_t)_{t=0}^{\infty}$.

State-Space

Let $\mathscr{P}(S)$ denote the space of Borel probability measures on S

The recursive planner's *state-space*:

$$\mathbb{M} := \left\{ \mu \in \mathscr{P}(S) \,\middle|\, \psi = \int \mu(dx, \cdot), \int \int x^2 \mu(dx, de) < \infty \right.$$
$$\left. \int \int x \mu(dx, de) \in \left[0, \bar{K}\right] \right\}$$

where $\bar{K} > 0$

Aggregate Action Space

Let $\mathbb{Y}$ denote the space of measurable functions h where $h\colon S \to A$

The space $\mathbb{Y}$ will be the *action-space*

Production

Given $\mu \in \mathbb{M}$, output produced according to production function $F\colon \mathbb{R}_+^2 \to \mathbb{R}_+$:

$$Y = F(K(\mu), L) - \delta K(\mu) \tag{12}$$

where $K := \mathbb{M} \to \mathbb{R}_+$

$$K(\mu) := \int \int x\mu(dx, de) \tag{13}$$

and

$$L := \int e \int \mu(dx, de) \tag{14}$$

Production

Assumption. 2.3 The production function F is twice differentiable on $\mathbb{R}_{++}$, homogeneous of degree one, strictly increasing in both arguments, strictly concave and for any $\hat{L} > 0$ and $\hat{K} > 0$ satisfies

1. $\lim_{K\to\infty} F_1(K,\hat{L}) = 0$ and $\lim_{K\to 0} F_1(K,\hat{L}) = \infty$ (Inada conditions)
2. $F(0,\hat{L}) = F(\hat{K},0) = 0$
3. $K \mapsto F(K,\hat{L})$ is bijective.

Interest and wage rates

$$r(\mu) := F_1\left(K(\mu), L\right) - \delta, \quad w(\mu) := F_2\left(K(\mu), L\right)$$

Feasibility Correspondence

Define feasibility correspondence Λ, with $\Lambda \colon \mathbb{M} \twoheadrightarrow \mathbb{Y}$, mapping a state to feasible policy functions:

$$\Lambda(\mu) \colon = \begin{cases} h \in \mathbb{Y} \; \middle| 0 \leq h(x, e) \leq (1 + r(\mu))x + w(\mu)e, & \text{if } K(\mu) > 0 \\ h \in \mathbb{Y} \; \middle| h = 0, & \text{if } K(\mu) = 0 \end{cases} \tag{15}$$

the (in)equalities above hold μ-a.e.

Transition Function

Define operator $\Phi \colon \operatorname{Gr}\Lambda \to \mathbb{M}$

$$\Phi(\mu, h)(B_A \times B_E) \colon = \psi(B_E) \int \int \chi_{B_A}\{h(x,e)\}\mu(dx, de) \quad (16)$$

where $B_A \times B_E \in \mathscr{B}(S)$

We write $\mu_{t+1} = \Phi(\mu_t, h_t)$

Remark. 1 $\Phi \colon \operatorname{Gr}\Lambda \to \mathbb{M}$ is well-defined

Proof of Remark 1.

Proof.

We check $\Phi(\mu, h) \in \mathbb{M}$ for any $\mu \in \mathbb{M}$ and $h \in \mathbb{Y}$

First we check $\Phi(\mu, h)(A, \cdot) = \psi$. Note

$$\begin{aligned} \Phi(\mu, h)(A \times B_E) : &= \psi(B_E) \int \int \chi_A\{h(x, e)\} \mu(dx, de) \\ &= \psi(B_E) \end{aligned}$$

for $B_E \in \mathscr{B}(E)$

Proof of Remark 1.

Next, let $\mu' = \Phi(\mu, h)(A \times B_E)$, we confirm $\int \int x \mu'(dx, de) \in [0, \bar{K}]$. Note

$$\int \int x \mu'(dx, de) = \int x \, \mathrm{d}\mu \circ h^{-1}$$
$$= \int \int h(a, e) \mu(da, de)$$

where since $h \in \Lambda(\mu)$, we must have $\int \int h(a, e) \mu(da, de) \geq 0$. Similar argument confirms variance of $\mu'(\cdot, E)$ is finite.

Moreover, from the law of motion and Assumption 2.3, there exists an upper-bound $\bar{K}$ such that given any initial aggregate level of capital below $\bar{K}$, aggregate capital for wealth distributions satisfying feasibility will never exceed $\bar{K}$

That is, if $\int \int x \mu(dx, de) \leq \bar{K}$, then $\int \int x \mu'(dx, de) \leq \bar{K}$ (see Proposition 2.2 and section 6.8 by Acemoglu (2009)).

Constrained Planner's Pay-off

Define the constrained planner's per-period pay-off, $u\colon \operatorname{Gr}\Lambda \to \mathbb{R}_+$:

$$u(\mu, h) := \begin{cases} \int \int \nu\big(R(\mu)x + w(\mu)e - h(x,e)\big)\mu(dx, de), & \text{if } K(\mu) > 0 \\ 0, & \text{if } K(\mu) = 0 \end{cases}$$

where $R(\mu) := 1 + r(\mu)$ and $\nu := \mathbb{R}_+ \to \mathbb{R}_+$

Assumption.2.5 The function ν is strictly increasing, bijective, concave and upper semicontinuous

Constrained Planner's Dynamic Problem

Let $\beta \in (0,1)$ be a discount factor and let V, with $V\colon \mathbb{M} \to \mathbb{R}_+ \cup \{+\infty\}$, denote the recursive constrained planner's (RCP) value function:

$$V(\mu_0) \colon = \sup_{(\mu_t, h_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{17}$$

subject to

$$h_t \in \Lambda(\mu_t), \qquad \mu_{t+1} = \Phi(\mu_t, h_t)$$
$$t \in \mathbb{N}, \qquad \mu_0 \text{ given} \tag{18}$$

Definition. 2.1 (**RCP Solution**) Given μ_0, a solution to the RCP is a sequence of measurable policy functions $(h_t)_{t=0}^{\infty}$, with $h_t \colon S \to A$ for each t and a sequence of Borel probability measures on S, $(\mu_t)_{t=0}^{\infty}$ satisfying (18) that achieves the value function:

$$V(\mu_0) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{19}$$

Theorem. 2.1 If the RCP (Definition 2.1) satisfies assumptions 2.1 - 2.5, then for any $\mu_0 \in \mathbb{M}$, there exists a solution $(\mu_t, h_t)_{t=0}^{\infty}$.

Sequential Constrained Planner's Problem

Consider Sequential Constrained Planner (SCP)

Let $\mathbb{X} := L^2(\Omega, \mathbb{P})$ be the space of square integrable (with respect to $\mathbb{P}$) real-valued functions on Ω

- Equip $\mathbb{X}$ with the weak topology

Define $(\mathscr{F}_i)_{i=0}^{\infty}$ as the natural filtration with respect to $\{x_0, e_0, e_1, \dots\}$

SCP State-Space

Time t state-space:

$$\mathbb{S}_t := \left\{ x \in m\mathscr{F}_t \,\middle|\, 0 \leq x, \int x \,\mathrm{d}\mathbb{P} \leq \bar{K} \right\} \tag{20}$$

where $m\mathscr{F}_t \subset \mathbb{X}$ is the space of $\mathscr{F}_t$-measurable random variables

Inequalities hold $\mathbb{P}$ almost everywhere

Sequential Constrained Planner's Problem

For any $x \in \mathbb{X}$, with $\int x \, \mathrm{d}\mathbb{P} \geq 0$, define $\tilde{K} := \mathbb{S}_t \to \mathbb{R}_+$:

$$\tilde{K}(x) = \int x \, \mathrm{d}\mathbb{P} \tag{21}$$

For $x \in \mathbb{X}$ with $\int x \mathrm{d}\mathbb{P} > 0$, define

$$\begin{aligned} \tilde{r}(x) &:= F_1(\tilde{K}(x), L) - \delta \\ \tilde{w}(x) &:= F_2(\tilde{K}(x), L) \end{aligned} \tag{22}$$

SCP Feasibility Correspondence

Time t feasibility correspondence $\Gamma_t \colon \mathbb{S}_t \to \mathbb{S}_{t+1}$:

$$\Gamma_t(x) := \begin{cases} y \in \mathbb{S}_{t+1} \,\middle|\, 0 \leq y \leq (1 + \tilde{r}(x))\, x + \tilde{w}(x)\, e_t, & \text{if } \tilde{K}(x) > 0 \\ y \in \mathbb{S}_{t+1} \,\middle|\, y = 0, & \text{if } \tilde{K}(x) = 0 \end{cases} \tag{23}$$

SCP Pay-Offs

Time t pay-offs $\rho_t\colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$:

$$\rho_t(x,y) := \begin{cases} \displaystyle\int \nu\left(\left(1+\tilde{r}(x)\right)x + \tilde{w}(x)\, e_t - y\right)\, \mathrm{d}\mathbb{P}, & \text{if } \tilde{K}(x) > 0 \\ 0, & \text{if } \tilde{K}(x) = 0 \end{cases}$$

SCP Value Function

Let $\tilde{V}$, with $\tilde{V} : \mathbb{S}_0 \to \mathbb{R}_+ \cup \{+\infty\}$ denote the time 0 sequential planner's value function:

$$\tilde{V}(x_0) := \sup_{(x_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$$

subject to

$$x_{t+1} \in \Gamma_t(x_t), \qquad \forall t \in \mathbb{N}, \qquad x_0 \in \mathbb{S}_0 \text{ given} \tag{24}$$

SCP Problem

Definition. 2.2 Given $x_0 \in \mathbb{S}_0$, a solution to the sequential constrained planner's problem is a sequence of random variables $(x_t)_{t=0}^{\infty}$ satisfying (24) that achieves the sequential planner's value function:

$$\tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \tag{25}$$

SCP Solution Implies RCP Solution

Theorem. 2.2 Let assumptions 2.1 - 2.5 hold. If there exists a solution to the sequential problem (Definition 2.2), then there exists a solution to the recursive problem (Definition 2.1) and $V(\mu_0) = \tilde{V}(x_0)$.

SCP Solution Implies RCP Solution

Given x_0 satisfying $x_0 \in \mathbb{S}_0$, let $(y_t)_{t=0}^{\infty}$ be a solution to the sequential planner's problem. Construct a candidate sequence, $(x_t)_{t=0}^{\infty}$:

$$\begin{aligned} & x_0 = y_0, \qquad x_1 = \mathbb{E}(y_1 | \sigma(x_0, e_0)) \\ & \text{and} \\ & x_{t+1} = \mathbb{E}(y_t | \sigma(x_t, e_t)), \qquad \forall t \in \mathbb{N} \end{aligned} \tag{26}$$

The term $\sigma(x_t, e_t)$ denotes the σ-algebra generated by x_t and e_t.

SCP Solution Implies RCP Solution

Since x_{t+1} is $\sigma(x_t, e_t)$ measurable, $x_{t+1} = h_t(x_t, e_t)$ for a measurable function h_t for each t.[2] For each t, define μ_t as

$$\mu_t(B) = \mathbb{P}\left\{x_t, e_t \in B\right\}, \qquad B \in \mathscr{B}(S) \tag{27}$$

We will show $(\mu_t, h_t)_{t=0}^{\infty}$ solves the recursive problem

[2] Note $\sigma(x_{t+1}) \subset \sigma(x_t, e_t)$, then use Doob-Dynkin (Lemma 1.13 by Kallenberg).

Proofs

Claim D.3

Claim. D.3 Let $(y_t)_{t=0}^{\infty}$ be a solution to the SCP problem. If $(x_t)_{t=0}^{\infty}$ is a sequence of random variables defined by (26), then

$$x_t :\,= \mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) = \mathbb{E}(y_t|\sigma(x_t, e_t)), \qquad \forall t \in \mathbb{N}$$

Proof of Claim D.3

Proof. Use the definition of conditional expectation from 9.2 by Williams (1991).

We show $\mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1}))$ is $\sigma(x_t, e_t)$ measurable and satisfies

$$\int_B \mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1}))\,\mathrm{d}\mathbb{P} = \int_B y_t\,\mathrm{d}\mathbb{P}$$

for $B \in \sigma(x_t, e_t)$

Proof of Claim D.3

To show $\mathbb{E}(y_t | \sigma(x_{t-1}, e_{t-1}))$ is $\sigma(x_t, e_t)$ measurable, observe

$$x_t = \mathbb{E}(y_t | \sigma(x_{t-1}, e_{t-1})) \tag{28}$$

and note $\sigma(x_t) \subset \sigma(x_t, e_t)$, thus $x_t^{-1}(B) \in \sigma(x_t, e_t)$ for any $B \in \mathscr{B}(\mathbb{R})$

Next, since e_t is independent, we have (use 9.7 k) by Williams (1991))

$$\mathbb{E}(y_t | \sigma(x_{t-1}, e_{t-1})) = \mathbb{E}(y_t | \sigma(x_{t-1}, e_{t-1}, e_t))$$

Proof of Claim D.3

Moreover, $\sigma(x_t, e_t) \subset \sigma(x_{t-1}, e_{t-1}, e_t)$ by the Doob-Dynkin Lemma since x_t is $\sigma(x_{t-1}, e_{t-1})$ measurable by definition of x_t

Now take any B satisfying $B \in \sigma(x_t, e_t)$. Since $B \in \sigma(x_{t-1}, e_{t-1}, e_t)$,

$$\int_B \mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1}))\,\mathrm{d}\mathbb{P}$$
$$= \int_B \mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1}, e_t))\,\mathrm{d}\mathbb{P} = \int_B y_t\,\mathrm{d}\mathbb{P}$$

as was to be shown to prove the claim. □

Propositon 2.1

Proposition. 2.1 Let assumptions 2.1 - 2.5 hold. If $(y_t)_{t=0}^{\infty}$ is a solution to the SCP problem (Definition 2.2), then $(x_t)_{t=0}^{\infty}$ defined by (26) is a solution to the SCP problem.

Proof. We show the sequence $(x_t)_{t=0}^{\infty}$ is feasible and achieves the sequential planner's value function.

Before proceeding, note the following holds due to the Tower Property (Williams (1991) 9.7i)) of conditional expectations.

$$\int y_t \, \mathrm{d}\mathbb{P} = \int \mathbb{E}(y_t | \sigma(x_{t-1}, e_{t-1})) \, \mathrm{d}\mathbb{P} = \int x_t \, \mathrm{d}\mathbb{P}, \qquad t \in \mathbb{N} \tag{29}$$

Proof of Propositon 2.1

To show feasibility of $(x_t)_{t=0}^{\infty}$, we verify $x_{t+1} \in \Gamma_t(x_t)$ for each t

First, $x_{t+1} = h_t(x_t, e_t)$ for a measurable function h_t for each t with x_0 given

- x_t can be written as a measurable function of $x_0, \dots, e_{t-1}$, implying $x_t \in m\mathscr{F}_t$ for each t

Moreover, by (29), $\int x_t \, \mathrm{d}\mathbb{P} = \int y_t \, \mathrm{d}\mathbb{P}$ and since $\int y_t \, \mathrm{d}\mathbb{P} \in [0, \bar{K}]$ for each t, $\int x_t \, \mathrm{d}\mathbb{P} \in [0, \bar{K}]$

By positivity of conditional expectation,[3] since $y_t \geq 0$, $x_t = \mathbb{E}(y_t | \sigma(x_{t-1}, e_{t-1})) \geq 0$

Thus $x_t \in \mathbb{S}_t$ for each t

---

[3] See 9.7 d) by Williams (1991).

Proof of Propositon 2.1

Now we check x_{t+1} satisfies the budget constraints in the definition of $\Gamma_t(x_t)$

Set any $t \in \mathbb{N}$. There are two cases to consider:

- first $\int x_t \, \mathrm{d}\mathbb{P} > 0$
- second $\int x_t \, \mathrm{d}\mathbb{P} = 0$

Suppose $\int x_t \, \mathrm{d}\mathbb{P} > 0$

By (29), we have $\int y_t \, \mathrm{d}\mathbb{P} = \int x_t \, \mathrm{d}\mathbb{P} > 0$

Since $(y_t)_{t=0}^{\infty}$ is a solution to the sequential planner's problem, we have $y_{t+1} \in \Gamma_t(y_t)$ and thus, $y_{t+1} \leq (1 + \tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t$

Proof of Propositon 2.1

To show $x_{t+1} \leq (1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t$, consider,

$$\begin{aligned}
x_{t+1} &= \mathbb{E}(y_{t+1}|\sigma(x_t, e_t)) \\
&\leq \mathbb{E}((1+\tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t|\sigma(x_t, e_t)) \\
&= (1+\tilde{r}(x_t))\mathbb{E}(y_t|\sigma(x_t, e_t)) + \tilde{w}(x_t)e_t \\
&= (1+\tilde{r}(x_t))\mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) + \tilde{w}(x_t)e_t \\
&= (1+\tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t
\end{aligned}$$

where, noting (29), the third line follows from

$$\tilde{r}(y_t) = F_1\left(\int y_t \,\mathrm{d}\mathbb{P}, L\right) = F_1\left(\int x_t \,\mathrm{d}\mathbb{P}, L\right) = \tilde{r}(x_t)$$

A similar argument shows $\tilde{w}(y_t) = \tilde{w}(x_t)$. The fourth line follows from Claim D.1 and the final line follows from the definition of x_t

Proof of Propositon 2.1

On the other hand, suppose $\int x_t \, \mathrm{d}\mathbb{P} = 0$

We have $\int y_t = \int x_t \, \mathrm{d}\mathbb{P} = 0$ by (29)

As such, since $y_{t+1} \in \Gamma_t(y_t)$ and noting the definition of Γ_t at (17) in the main text, $\int x_{t+1} \, \mathrm{d}\mathbb{P} = \int y_{t+1} \, \mathrm{d}\mathbb{P} = 0$

Since $x_{t+1} \geq 0$, x_{t+1} satisfies $x_{t+1} = 0$[4]

[4] See Theorem 1.1.20 by Tao (2010).

Proof of Propositon 2.1

To re-cap, for each t, x_t is $\mathscr{F}_t$ measurable and satisfies $\int x_t \leq \bar{K}$ and $x_t \geq 0$

Hence $x_t \in \mathbb{S}_t$

Moreover, x_{t+1} satisfies the budget constraints in the definition of Γ_t for each t. Thus $x_{t+1} \in \Gamma_t(x_t)$ for each t

Proof of Propositon 2.1

Next, we check $\rho_t(x_t, x_{t+1}) \geq \rho_t(y_t, y_{t+1})$ for each t. Select any t and consider the case $\int x_t \, \mathrm{d}\mathbb{P} > 0$. We have

$$
\begin{aligned}
\rho_t(x_t, x_{t+1}) &= \int \nu\left((1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1}\right) \mathrm{d}\mathbb{P} \\
&= \int \nu\Big((1 + \tilde{r}(y_t))\mathbb{E}(y_t|\sigma(x_{t-1}, e_{t-1})) + \tilde{w}(y_t)e_t \\
&\quad - \mathbb{E}(y_{t+1}|\sigma(x_t, e_t))\Big) \mathrm{d}\mathbb{P} \\
&= \int \nu\left(\mathbb{E}\left[(1 + \tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t - y_{t+1}|\sigma(x_t, e_t)\right]\right) \mathrm{d}\mathbb{P} \\
&\geq \int \mathbb{E}\left(\nu((1 + \tilde{r}(y_t))y_t + \tilde{w}(y_t)e_t - y_{t+1})|\sigma(x_t, e_t)\right) \mathrm{d}\mathbb{P} \\
&= \rho_t(y_t, y_{t+1})
\end{aligned}
$$

Proof of Propositon 2.1

The second line is due to the definition x_t and x_{t+1}

The third line follows from Claim D.1, the fourth line follows from Jensen's inequality (See 9.7 by Williams (1991) in the Mathematical Preliminaries)

The final line is due to the Tower Property

Proof of Propositon 2.1

If $\int x_t \,\mathrm{d}\mathbb{P} = 0$, then $\rho_t(x_t, x_{t+1}) = 0$ by definition of ρ_t. Since $\int y_t \,\mathrm{d}\mathbb{P} = 0$ by (29), $\rho_t(y_t, y_{t+1}) = 0$.

Conclude

$$\tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(y_t, y_{t+1}) \leq \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$$

Since $\tilde{V}(x_0)$ achieved the supremum of all pay-offs from feasible sequences and $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} \in \Gamma_t(x_t)$ for each t, $\tilde{V}(x_0) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1})$

- conclude $(x_t)_{t=0}^{\infty}$ is a solution to the SCP problem

Proof of Theorem 2.2

Theorem. 2.2 Let assumptions 2.1 - 2.5 hold. If there exists a solution to the sequential problem (Definition 2.2), then there exists a solution to the recursive problem (Definition 2.1) and $V(\mu_0) = \tilde{V}(x_0)$.

Proof.(Proof of Theorem 2.2)

Let $(y_t)_{t=0}^{\infty}$ solve the sequential problem and let $(x_t)_{t=0}^{\infty}$ be defined by (26)

Since x_{t+1} is $\sigma(x_t, e_t)$ measurable, $x_{t+1} = h_t(x_t, e_t)$ for a measurable function h_t for each t. For each t, define μ_t as

$$\mu_t(B) = \mathbb{P}\left\{x_t, e_t \in B\right\}, \qquad B \in \mathscr{B}(S) \tag{30}$$

Proof of Theorem 2.2

The remainder of the proof verifies $(\mu_t, h_t)_{t=0}^{\infty}$ solves the recursive problem

- First we verify $(\mu_t, h_t)_{t=0}^{\infty}$ is feasible for the recursive problem (part 1)
- Second, we verify the sum of discounted pay-offs from $(\mu_t, h_t)_{t=0}^{\infty}$ dominates the sum of discounted pay-offs from any other feasible sequence of distributions and policy functions (part 2).

Proof of Theorem 2.2 — Part 1

This part shows $(\mu_t, h_t)_{t=0}^{\infty}$ satisfies (18), that is, $h_t \in \Lambda(\mu_t)$ and $\mu_{t+1} = \Phi(\mu_t, h_t)$ for each t.

Fix any $t \in \mathbb{N}$. To confirm $h_t \in \Lambda(\mu_t)$, we consider two cases: when $\int \int x \mu_t(dx, de) > 0$ and when $\int \int x \mu_t(dx, de) = 0$

Proof of Theorem 2.2 — Part 1

First suppose $\int \int x \mu_t(dx, de) > 0$, we show

$$\mu_t \{ a, e \in S \mid h_t(a, e) \notin [0, (1 + r(\mu_t))a + w(\mu_t)e] \} = 0$$

The condition says the policy function h_t satisfies agents' budget constraints μ_t - almost everywhere

Proof of Theorem 2.2 — Part 1

Using the definition of μ_t,

$$\begin{aligned}
&\mu_t\big\{a, e \in S \,|\, h_t(a, e) \notin [0, (1 + r(\mu_t))a + w(\mu_t)e]\big\} \\
&= \mathbb{P}\big\{\omega \in \Omega \,|\, h_t(x_t(\omega), e_t(\omega)) \\
&\quad \notin [0, (1 + \tilde{r}(x_t))x_t(\omega) + \tilde{w}(x_t)e_t(\omega)]\big\} \\
&= \mathbb{P}\big\{\omega \in \Omega \,|\, x_{t+1}(\omega) \\
&\quad \notin [0, (1 + \tilde{r}(x_t))x_t(\omega) + \tilde{w}(x_t)e_t(\omega)]\big\} \\
&= 0
\end{aligned} \tag{31}$$

Proof of Theorem 2.2 — Part 1

The first equality uses the following observation, which holds because μ_t is the joint distribution of $\{x_t, e_t\}$:

$$\int \int x \mu_t(dx, de) = \int x_t \, \mathrm{d}\mathbb{P} > 0 \tag{32}$$

whence,

$$r(\mu_t) = F_1 \left(\int \int x \mu_t(dx, de), L \right) = F_1 \left(\int x_t \, \mathrm{d}\mathbb{P}, L \right) = \tilde{r}(x_t) \tag{33}$$

Proof of Theorem 2.2 — Part 1

Now suppose $\int \int x \mu_t(dx, de) = 0$. Observe

$$\int x \int \mu_t(dx, de) = \int x_t \, \mathrm{d}\mathbb{P} = 0 \tag{34}$$

Since $(x_t)_{t=0}^{\infty}$ satisfies $x_{t+1} \in \Gamma_t(x_t)$ for each t, $x_{t+1} = 0$.[5]
Whence,

$$\mu_t \{h_t(a, e) \neq 0\} = \mathbb{P}\{h_t(x_t, e_t) \neq 0\} = \mathbb{P}\{x_{t+1} \neq 0\} = 0$$

Thus $h_t = 0$ for μ_t almost everywhere and $h_t \in \Lambda(\mu_t)$ if $\int \int x \mu_t(dx, de) = 0$

[5] See Theorem 1.1.20 by Tao (2010).

Proof of Theorem 2.2 — Part 1

Now we show $\mu_{t+1} = \Phi(\mu_t, h_t)$ for each t. Let $B \in \mathscr{B}(S)$, where $B = B_A \times B_E$ for $B_A \in \mathscr{B}(A)$ and $B_E \in \mathscr{B}(E)$. Use the definition of μ_{t+1} to write

$$\begin{aligned}
\mu_{t+1}(B_A \times B_E) :&= \mathbb{P}\left\{x_{t+1} \in B_A, e_{t+1} \in B_E\right\} \\
&= \psi(B_E) \int \chi_{B_A}\left\{x_{t+1}\right\} \mathrm{d}\mathbb{P} \\
&= \psi(B_E) \int \int \chi_{B_A}\left\{h_t(x, e)\right\} \mu_t(dx, de)
\end{aligned}$$

Proof of Theorem 2.2 — Part 2

We have shown $(\mu_t, h_t)_{t=0}^{\infty}$ satisfies feasibility. Our next task is to show

$$\sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \geq \sum_{t=0}^{\infty} \beta^t u(\tilde{\mu}_t, \tilde{h}_t)$$

holds for any other sequence $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$ feasible for the recursive problem

Proof of Theorem 2.2 — Part 2

Let $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$ be any other sequence of Borel probability measures on S and measurable policy functions $\tilde{h}_t \colon S \to A$ satisfying $\tilde{\mu}_0 = \mu_0$ and (18)

Construct a sequence of A valued random variables $(\tilde{x}_t)_{t=0}^{\infty}$ by letting $\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t)$ for each $t > 0$ and with $\tilde{x}_0 = x_0$ given

The sequence of random variables $(\tilde{x}_t)_{t=0}^{\infty}$ will be defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$

Note $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$ for each t (see Remark D.4 below)

Moreover, each $\tilde{x}_t$ will have finite variance and hence $\tilde{x}_t \in L^2(\Omega, \mathbb{P})$ for each t

Proof of Theorem 2.2 — Part 2

Our strategy is to show $(\tilde{x}_t)_{t=0}^{\infty}$ is feasible for the sequential problem and show $u(\tilde{\mu}_t, \tilde{h}_t) = \rho_t(\tilde{x}_t, \tilde{x}_{t+1})$ and $u(\mu_t, h_t) = \rho_t(x_t, x_{t+1})$ for each t

The proof will then be complete since, noting $(x_t)_{t=0}^{\infty}$ is a solution for the sequential problem (Proposition 2.1), $u(\mu_t, h_t) = \rho_t(x_t, x_{t+1}) \geq \rho_t(\tilde{x}_t, \tilde{x}_{t+1}) = u(\tilde{\mu}_t, \tilde{h}_t)$ for each t

Proof of Theorem 2.2 — Part 2

To check $(\tilde{x}_t)_{t=0}^{\infty}$ satisfies $\tilde{x}_{t+1} \in \Gamma_t(\tilde{x}_t)$ for each t, we check the conditions stated at (23) for each t

First, we confirm $(\tilde{x}_t)_{t=0}^{\infty}$ is adapted to the filtration $(\mathscr{F}_t)_{t=0}^{\infty}$

Proceed by induction, let $t = 1$ and consider:

$$\tilde{x}_1 = \tilde{h}_1(x_0, e_0)$$

Proof of Theorem 2.2 — Part 2

Since $\tilde{h}_1$ is measurable, by the Doob-Dynkin Lemma (Lemma 1.13 by Kallenberg), $\tilde{x}_1$ will be $\sigma(x_0, e_0)$ measurable

Now suppose $\tilde{x}_t$ is $\sigma(x_0, e_0, \ldots, e_{t-1})$ measurable

Consider

$$\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t) = \tilde{h}_t(g(x_0, e_0, \ldots, e_{t-1}), e_t)$$

for some measurable function $g \colon A \times E^t \to A$

Once again, since $\tilde{h}_t$ is Borel measurable, using the Doob-Dynkin Lemma, $\tilde{x}_{t+1}$ is $\sigma(x_0, e_0, \ldots, e_t)$ measurable

By the Principle of Induction, $(\tilde{x}_t)_{t=0}^{\infty}$ is adapted to the filtration $(\mathscr{F}_t)_{t=0}^{\infty}$

Proof of Theorem 2.2 — Part 2

To confirm $\int \tilde{x}_t \, \mathrm{d}\mathbb{P} \in [0, \bar{K}]$ for each t, since $\tilde{\mu}_t \in \mathbb{M}$ and $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$, we have

$$\int \tilde{x}_t \, \mathrm{d}\mathbb{P} = \int \int x \tilde{\mu}_t(dx, de) \in [0, \bar{K}] \tag{35}$$

Proof of Theorem 2.2 — Part 2

To confirm $\int \tilde{x}_t \, \mathrm{d}\mathbb{P} \in [0, \bar{K}]$ for each t, since $\tilde{\mu}_t \in \mathbb{M}$ and $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$, we have

$$\int \tilde{x}_t \, \mathrm{d}\mathbb{P} = \int \int x \tilde{\mu}_t(dx, de) \in [0, \bar{K}] \tag{36}$$

Proof of Theorem 2.2 — Part 2

Now show the sequence $(\tilde{x}_t)_{t=0}^{\infty}$ satisfies the agent budget constraints for each t. Fix any $t \in \mathbb{N}$ and suppose $\int \tilde{x}_t \, \mathrm{d}\mathbb{P} > 0$

We have

$$\begin{aligned}
\mathbb{P}\{\tilde{x}_{t+1} \notin [0, (1 + \tilde{r}(\tilde{x}_t))\tilde{x}_t + \tilde{w}(\tilde{x}_t)e_t]\} &= \mathbb{P}\big\{\tilde{h}_t(\tilde{x}_t, e_t) \\
&\quad \notin [0, (1 + \tilde{r}(\tilde{x}_t))\tilde{x}_t + \tilde{w}(\tilde{x}_t)e_t]\big\} \\
&= \tilde{\mu}_t\big\{\tilde{h}_t(x, e) \\
&\quad \notin [0, (1 + r(\tilde{\mu}_t))x + w(\tilde{\mu}_t)e]\big\} \\
&= 0
\end{aligned}$$

Final equality holds because $\tilde{\mu}_t$ is the distribution of $\{\tilde{x}_t, e_t\}$ and because $\tilde{h}_t$ satisfies the feasibility condition shown at (15) $\tilde{\mu}_t$ - almost everywhere

Proof of Theorem 2.2 — Part 2

On the other hand, suppose $\int \tilde{x}_t \, \mathrm{d}\mathbb{P} = 0$

We have $\int \tilde{x}_t \, \mathrm{d}\mathbb{P} = \int \int x \tilde{\mu}_t(dx, de) = 0$

Since $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$ satisfies $\tilde{h}_t \in \Lambda(\tilde{\mu}_t)$ for each t, $\tilde{h}_t(x, e) = 0$ for $\tilde{\mu}_t$ almost everywhere

Whence,

$$\mathbb{P}\{\tilde{x}_{t+1} \neq 0\} = \mathbb{P}\left\{\tilde{h}_t(x_t, e_t) \neq 0\right\} = \tilde{\mu}_t\left\{\tilde{h}_t(x, e) \neq 0\right\} = 0$$

The first equality holds because we defined $\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t)$

The second inequality holds because $\tilde{\mu}_t$ is the joint distribution of $\{\tilde{x}_t, e_t\}$

Proof of Theorem 2.2 — Part 2

As such, for each t, $\tilde{x}_{t+1}$ satisfies all the conditions stated in the definition of the feasibility correspondence, (23), for $\tilde{x}_{t+1} \in \Gamma_t(\tilde{x}_t)$

To complete the proof, for each t,

$$\begin{aligned} u(\mu_t, h_t) &= \int \nu((1 + r(\mu_t))x + w(\mu_t)e - h_t(x, e))\mu_t(dx, de) \\ &= \int \nu((1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - h_t(x_t, e_t))\, \mathrm{d}\mathbb{P} \\ &= \int \nu((1 + \tilde{r}(x_t))x_t + \tilde{w}(x_t)e_t - x_{t+1})\, \mathrm{d}\mathbb{P} \\ &= \rho_t(x_t, x_{t+1}) \end{aligned} \tag{37}$$

Proof of Theorem 2.2 — Part 2

And similarly, $u(\tilde{\mu}_t, \tilde{h}_t) = \rho_t(\tilde{x}_t, \tilde{x}_{t+1})$ for each t. As such, conclude

$$\sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, x_{t+1}) \geq \sum_{t=0}^{\infty} \beta^t \rho_t(\tilde{x}_t, \tilde{x}_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(\tilde{\mu}_t, \tilde{h}_t) \tag{38}$$

where the inequality follows since $(x_t)_{t=0}^{\infty}$ is a solution to the sequential problem and its discounted sum of pay-offs dominate the discounted sum of pay-offs from $(\tilde{x}_t)_{t=0}^{\infty}$

Proof of Theorem 2.2 — Part 2

Finally, since any arbitrary feasible sequence $(\tilde{\mu}_t, \tilde{h}_t)_{t=0}^{\infty}$, with $\tilde{\mu}_0 = \mu_0$, satisfies (38), we have $V(\mu_0) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t)$

Moreover, since $(x_t)_{t=0}^{\infty}$ solves the sequential planner's problem, the first equality of (38) implies $V(\mu_0) = \tilde{V}(x_0)$ □

For the following claim, consider the setting and notation in the proof for Theorem (2.2), part 2.

Remark. D.4 If $\tilde{x}_{t+1} = \tilde{h}_t(\tilde{x}_t, e_t)$ for each $t > 0$ and $\tilde{x}_0 = x_0$, then $\{\tilde{x}_t, e_t\} \sim \mu_t$ for each $t \in \mathbb{N}$.

Proof. We use a proof by induction

Let $t = 0$, since $\tilde{x}_0 = x_0$ and $\tilde{\mu}_0 = \mu_0$, by the construction of $\{x_0, e_0, e_1, \dots\}$ in section 2.3, we have $\{\tilde{x}_0, e_0\} \sim \tilde{\mu}_0$

Now make the inductive assumption $\{\tilde{x}_t, e_t\} \sim \tilde{\mu}_t$

We show $\{\tilde{x}_{t+1}, e_{t+1}\} \sim \tilde{\mu}_{t+1}$

Let $B_A \times B_E \in \mathscr{B}(S)$ and observe

$$\begin{aligned} \mathbb{P}\{\tilde{x}_{t+1} \in B_A, e_{t+1} \in B_E\} &= \mathbb{E}\{\chi_{B_A}(\tilde{x}_{t+1}) \times \chi_{B_E}(e_{t+1})\} \\ &= \psi(B_E) \int \int \chi_{B_A}(\tilde{h}_t(x, e))\tilde{\mu}_t(dx, de) \\ &= \Phi(\tilde{\mu}_t, \tilde{h}_t)(B_A, B_E) = \tilde{\mu}_{t+1}(B_A \times B_E) \end{aligned}$$

To conclude, since $\{\tilde{x}_0, e_0\} \sim \mu_0$ and $\{\tilde{x}_{t+1}, e_{t+1}\} \sim \mu_{t+1}$ if $\{\tilde{x}_t, e_t\} \sim \mu_t$, by the Principle of Induction, $\{\tilde{x}_t, e_t\} \sim \mu_t$ for each $t \in \mathbb{N}$. $\square$

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