OPTIMAL SAVING AND WORK HOURS IN THE NEOCLASSICAL GROWTH MODEL WITH UNINSURABLE INDIVIDUAL INCOME RISK

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This paper introduces work - leisure choice into a constrained optimal policy problem in a neoclassical growth model with idiosyncratic risk and incomplete markets. The constrained planner cannot complete markets, but must improve welfare subject to agents' budget constraints. As such, rather than addressing the market failure of incomplete markets, the planner addresses the pecuniary externalities of each agents' saving and work decision on interest and wage rates. In an economy calibrated to U.S. wealth and income inequality, the paper finds the constrained planner increases aggregate capital and reduces aggregate hours worked. However, owing to a combination of price and wealth effects, in a constrained efficient allocation, only highly productive individuals increase saving, while less productive individuals reduce saving. Moreover, only the asset poor and unproductive agents reduce work hours, while the wealthy and highly productive increase work hours and reduce leisure time.

1. INTRODUCTION

If a utilitarian social planner wished to improve welfare in an economy with incomplete insurance markets, but could not complete the markets, how should they influence individuals' saving and labour market decisions? In a representative agent economy, the planner outcome corresponds with the market equilibrium and any deviation is distorting. However, Aiyagari (1994) and Huggett (1997) showed a competitive equilibrium without complete markets means people over-save, to self-insure against individual risk, suggesting the need for capital taxation (see Aiyagari (1995) and discussion by Chen et al. (2017)). Precautionary

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over-accumulation in incomplete market economies – the Aiyagari-Huggett effect – has now become conventional wisdom among macroeconomists.

The conventional wisdom was challenged with the work of Dávila et al. (2012) and later Nuno and Thomas (2017) and also Park (2017), who introduce the notion of a constrained planner. The constrained planner is a utilitarian planner who cannot complete markets and is constrained by each individual's budget constraint. The surprising conclusion reached by Dávila et al. (2012) is that a constrained planner may wish to direct wealthy agents to *save more*. Even though the planner cannot complete markets, the planner can still correct for pecuniary externalities of people's saving decisions on wages and interest rates. If the distribution of labor productivity favours the poor, then increasing wages through capital accumulation leads to higher aggregate utility.[1]

This paper extends the analysis of constrained efficiency to also include work hours across the wealth distribution. Understanding the interaction of work hours and saving is important in an incomplete market economy because these decisions both affect wages and prices, which in turn affect aggregate accumulation and output. For instance, Marcet et al. (2007) shows ex-post wealth effects from the rich not working and consuming leisure may be so high that capital productivity and in turn and output is lower than in a complete market economy, contradicting the Aiyagari-Huggett effect. Moreover, government policy typically consists of capital and labour income taxation, and any study of optimality must examine the interaction of these two policy tools.

To study how a constrained planner would alter work hours across the distribution, the paper incorporates labor-leisure choice into the Aiyagari-Huggett model and compute the competitive equilibrium and constrained efficient allocation. Individuals receive idiosyncratic labour productivity shocks each period and are endowed with one unit of time each period. After the realization of uncertainty, individuals decide the proportion of time to spend on work and leisure. Total effective labour supply depends on both on how many hours are worked and by whom; an hour of work by someone who has received a good productivity shock increases effective labour supply more than an hour of work by someone with a poor labour shock.

[1]The constrained planner implements social insurance through correcting for price externalities. In this sense, the planner "sits behind the veil of ignorance" of shock realizations and assumes all agents are ex-ante identical, assigning each agent identical utilitarian weights. As shown by Dávila et al. (2012), ex-post redistribution by the constrained planner may not be a Pareto improvement.

The main finding of this paper is that, in an economy calibrated to wealth and income dispersion of the United States (U.S.), the constrained planner increases capital supply and depresses total hours worked. The constrained planner's decision is based on the observation that the distribution of effective labour favours the poor in the economy calibrated to the U.S., or the poor depend on labour income more than they depend on capital income. Hence increasing wages and depressing interest rates results in an ex-post redistribution of consumption towards those with a high marginal utility of consumption. The constrained efficient allocation can be implemented through a regressive capital subsidy and a progressive labour tax.

The interesting aspect of the result is that while aggregate hours worked falls, total effective labour supply rises. The reason is because not everyone reduces work hours. Because interest rates fall, the marginal utility of consumption for the wealthy rises and induces them work more even though the constrained planner taxes work hours. Since the very wealthy are also the highly productive, effective labour rises.

The result of higher capital and lower hours worked is not general however. The decision of the constrained planner depends on which distribution, capital or effective labour, relatively favours the consumption poor. I also study an unemployment economy (Model I considered by Dávila et al. (2012)). In this economy, there is low income and wealth dispersion and agents are typically employed for two periods. However, since unemployment periods are small, poor agents (those agents with poor labour shocks) tend to be as wealthy as rich agents, the unlucky are poor in labour productivity but are still capital rich — the mass of the distribution of capital slightly favours the poor. I also study the calibration used by Marcet et al. (2007), and in their case, because the poor are both wealth and labour productivity poor, the constrained planner's allocation is identical to the competitive equilibrium.

The quantitative analysis of this paper also re-examines the question of the whether or not the Aiyagari-Huggett effect holds in a model with endogenous labour supply. Both the the Dávila et al. (2012) unemployment economy and the U.S. calibrated economy feature higher capital accumulation compared to the representative agent. This stands in contrast to the effect outlined by Marcet et al. (2007), where those who are ex-post rich work less, leading to an aggregate depression of accumulation (the unlucky do not contribute to effective labour much, so their reduced work hours do not affect the aggregate). In the economy calibrated to the U.S., agents work more than the representative agent, hence increasing the productivity and accumulation of capital. This is because while the ex-post lucky reduce work hours compared to representative agent, the economy also features agents who are neither unemployed nor extremely lucky. These ex-post moderately endowed agents will actually increase work hours compared to a representative agent (see discussion by Pijoan-Mas (2006), section 4.1).

The Dávila et al. (2012) unemployment economy still features lower work hours in the incomplete market equilibrium compared to the representative agent, but the Aiyagari-Huggett effect still does not hold. The key to understanding why is the low persistence of labour productivity shocks. The unlucky are only unlucky for 2 periods, and move back to being lucky before they can run down their capital stocks. By contrast, in the high unemployment persistence environment of Marcet et al. (2007), the unlucky do not hold much capital. Thus, in the Dávila et al. (2012) unemployment economy, more agents accumulate capital and the the fall in capital productivity due to reduced working hours is not enough to counter-act the precautionary saving motive.

Before turning to the related literature, this paper also makes some technical contributions. First, I prove necessity of a functional Euler equation and transversality condition of the infinite dimensional optimization problem for the constrained planner. The proof clarifies some steps regarding the interchange of infinite limits through integrals in the functional differentiation by Dávila et al. (2012), and is the first application of transversality condition to an infinite dimensional optimization problem. The paper also expresses the constrained planner's first order condition in terms of the center of mass of the labour and capital distribution (analogous to the concept of the centre of gravity). And the paper uses the existence result in Shanker (2017) to show the existence of tax- competitive recursive equilibrium, a recursive equilibrium subject to constrained planner taxes.

Related literature

The baseline competitive equilibrium calibration of the model in this paper is identical to Marcet et al. (2007) and Pijoan-Mas (2006). As in this paper, both Marcet et al. (2007) and Pijoan-Mas (2006) specify labour-leisure choice in a separable utility function.

In addition to Dávila et al. (2012) and Nuno and Thomas (2017), who do not consider work hours choice, this paper is closely related to Park (2017). Park (2017) develops a model of work hours dispersion by incorporating human capital investment. By contrast, in this paper, the focus of this paper is on the trade-off across the distribution between unproductive and productive direction of labour.

The issue of work hours across the distribution has been taken up by Boppart et al. (2016), who analyze who will work more and less as productivity grows, albeit in a friction-less complete markets setting. The main results are consistent with the analysis here, in a competitive equilibrium, the ex-ante asset rich (there is no ex-post heterogeneity) exit the work-force first and work less due to strong wealth effects. By contrast, a utilitarian planner (who again "sits behind the veil of ignorance" of the ex-ante wealth heterogeneity), prefers the rich work more. The main difference between this paper and Boppart et al. (2016) is that I consider an incomplete market setting where the planner corrects for price externalities; on the other hand, I do not consider long-run productivity growth implications on the distribution of hours worked.

There is now a growing literature on optimal Ramsey taxation problems in incomplete market models. Examples include Chen et al. (2017), Le Grand and Ragot (2017) and Bhandari et al. (2017). Both Chen et al. (2017) and Le Grand and Ragot (2017) find the optimal tax may be positive, while Bhandari et al. (2017) does not consider capital accumulation in their economy. A key difference between the Ramsey planning literature and this paper is that the constrained planner considers individual taxes that are state depended, that is, they depend on the individual's current asset and shock. By contrast in the Ramsey problem, the policy maker can only implement flat taxes but introduce transfers across agents.

Turning to methodology, this paper is related to ideas in physics called mean-field effects, where each individual's effect on others is captured through aggregate variables, in this case, prices (see Nuño (2017) and Nuno and Thomas (2017)). Finally, the necessity of the transversely condition for arbitrary vector spaces was proved by Cosar and Green (2014), without application. Here I prove a less general result, based on Kamihigashi (2002), but with an application to the constrained planner problem.

2. CONSTRAINED EFFICIENCY IN THE AIYAGARI-HUGGETT MODEL WITH LABOR-LEISURE CHOICE

This section characterizes the competitive equilibrium and constrained efficient allocation in the infinite horizon Aiyagari-Huggett model with Labor-Leisure choice.

2.1. *Model environment*

Time is discrete and indexed by t, with $t \in \mathbb{N}$. Let $(I, \mathcal{I}, \zeta)$ be an atom-less probability space indexing agents. Let A, with $A := \mathbb{R}_+$ be the agents' asset space and E, with $E \subset \mathbb{R}_+$ the agents' endowment space. Let S, with $S := A \times E$ denote the agents' state space. Each agent, with $i \in \mathcal{I}$ draws a sequence of shocks $\{a_0^i, e_0^i, e_1^i \dots\}$ where a_0^i is an A valued initial wealth shock and $(e_t^i)_{t=0}^{\infty}$ is an $E^{\mathbb{N}}$ valued sequence of labor endowment shocks. Assume all shocks are defined on a common probability space $(\bar{\Omega}, \Sigma, \bar{\mathbb{P}})$. Also assume:

ASSUMPTION 1 *The shocks satisfy the following conditions:*

1. *for each i, the shocks $(e_t^i)_{t=0}^{\infty}$ are a stationary Markov process with common Markov kernel Q and stationary marginal distribution ψ*
2. *for each t and i, e_t^i and x_0^i has finite variance*
3. *for each i, x_0^i is independent of $(e_t^i)_{t=0}^{\infty}$.*

Agents are endowed with a unit of time each period, which they can divide between labor and leisure.

The *aggregate state* of the economy will be the joint *empirical distribution* of the agents' assets and shocks — this aggregate state affects agents' decisions through prices (wages and interest rates). The *individual state* for agents will be their asset and endowment shock at a particular period.

In this paper, we study a recursive competitive equilibrium (RCE) and a recursive constrained planner's solution; by recursive we mean an equilibrium or solution where at any time t, agents or the planner make decisions based on the period t state of the economy (and state of their own assets and labour shock, in the case of agents). The recursive constrained planner problem or competitive equilibrium will yield a measurable policy function h_t for each t, with $h_t \colon S \to A$ defining a next period asset and a measurable policy function $l_t \colon S \to [0, 1]$ defining leisure at period t. To see how the individual state evolves, a sequence of policy functions $(h_t)_{t=0}^{\infty}$ generates a stochastic sequence of assets for each agent, $(x_t^i)_{t=0}^{\infty}$, by

$$x_{t+1}^i = h_t(x_t^i, e_t^i), \qquad t \in \mathbb{N},\ i \in [0, 1] \tag{1}$$

2.1.1. *Evolution of the Aggregate State Under a Sequence of Policy Functions*

Now we turn to describe how the aggregate state evolves. First, we need to describe how the *theoretical distributions* of assets and shocks evolve in a recursive equilibrium or solution; at time t, let μ_0 denote the common joint distribution of x_0^i and e_0^i.

Since h_t applies to all agents i, the distribution of $\{x_t^i, e_t^i\}$ will be identical across i. Moreover, $\{x_t^i, e_t^i\} \sim \mu_t$ for each i, where $(\mu_t)_{t=0}^{\infty}$ satisfies the recursion

$$\mu_{t+1}(B_A \times B_E) = \int\int \mathbb{1}_{B_A}\{h_t(x,e)\}Q(e, B_E)\mu_t(dx, de), \qquad t \in \mathbb{N} \tag{2}$$

for each t and $B_A \times B_E \in \mathscr{B}(S)$.

The recursion at Equation (2) tells us how the theoretical joint distribution of the shocks and assets evolves. To connect the theoretical distribution to the aggregate state, note the discussion and results by Shanker (2017) in sections x, which state that no aggregate uncertainty (NAU) can apply to this environment. In summary, NAU tells us that the empirical distribution of shock values agrees with the theoretical distribution $\bar{\mathbb{P}}$ almost everywhere. Under NAU, to integrate a function over the empirical distribution of agents, we simply integrate the function over the theoretical distribution — μ_t, which becomes the aggregate state for the economy at time t (see Claim 2.1 by Shanker (2017)).

2.1.2. *Production*

The economy is populated by standard price taking firms with a Cobb-Douglass production function $F(K, L) = AK^{\alpha}L^{1-\alpha}$, where $\alpha \in (0, 1)$. At any time t, because the theoretical and empirical distributions of assets and endowment shocks agree, aggregate capital and effective labour are defined as functions of the aggregate state μ_t and hours worked $1 - l_t$:

$$K(\mu_t) := \int\int x\mu_t(dx, de) \tag{3}$$

and

$$L(\mu_t, l_t) := \int\int (1 - l_t(x, e))e\mu_t(dx, de) \tag{4}$$

The integral in the definition of aggregate effective labour is the weighted "sum" of endowments, where the weights correspond to the how many hours each agent works. Since the firms are price taking, aggregate capital and effective labour give the following interest and wage rate:

$$(5) \qquad r(\mu_t, l_t) := F_1(K(\mu_t), L(\mu_t, l_t)) - \delta, \qquad \delta \in (0,1)$$

and

$$(6) \qquad w(\mu_t, l_t) := F_2(K(\mu_t), L(\mu_t, l_t))$$

where δ is the discount rate.

2.1.3. *Budget Constraints and Utility*

Given the aggregate state μ_t, an agent i with asset x_t^i and endowment shock e_t^i must satisfy their budget constraint

$$(7) \qquad 0 \leqslant h_t(x_t^i, e_t^i) \leqslant (1 + r(\mu_t, l_t))x_t^i + w(\mu_t, l_t)(1 - l_t(x_t^i, e_t^i))e_t^i, \qquad \mathbb{P} - a.e.$$

where $h_t(x_t^i, e_t^i)$ is asset saved at period t for the next period and $l_t(x_t^i, e_t^i)$ are the leisure hours at period t. If x_0^i has finite variance and $r(\mu_t, l_t)$ is real-valued for each t, then if $(x_t^i)_{t=0}^{\infty}$ satisfies (7), x_t^i will have finite variance for each t (see Claim **??** in the online appendix).

Integrating across agents' budget constraints at Equation (7) and using the definition of interest and wages rates, along with homogeneity of the production function (see Theorem 2.1 in Acemoglu (2009)) implies there exits $\bar{K}$ that bounds aggregate capital from above (see discussion at section x in Shanker (2017)).

Agents have an utility function $\nu\colon \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ defined by

$$(8) \qquad \nu(c, l) = \frac{c^{1-\gamma_c}}{1-\gamma_c} + A_l \frac{l^{1-\gamma_l}}{1-\gamma_l}, \qquad \gamma_c, \gamma_l \in \mathbb{R}_+$$

2.2. *Recursive Competitive Equilibrium*

Now that we have set-up the elements of the economy, we turn to describe an RCE. In words, an RCE will be policy functions, that depend only on today's state, such that given the aggregate states generated by the policy functions, the policy functions maximize agents' inter-temporal utility.

I start by defining the inter-temporal problem of agents as a sequential problem, given a sequence of aggregate states $(\mu_t)_{t=0}^{\infty}$ and leisure choices $(l_t)_{t=0}^{\infty}$.[2] The agent's value function at time 0 is:

$$W_0(x_0^i, e_0^i) := \max_{\{x_t^i, q_t^i\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t v(c_t^i, q_{t+1}^i)$$

such that the sequence $\{x_t^i, q_t^i\}_{t=0}^{\infty}$ is measurable with respect to the filtration generated by $\{x_0^i, e_0^i, e_t^i \dots\}$.[3] and subject to (7) to (6) and

$$c_t^i = (1 + r_t(\mu_t, l_t))x_t^i + e_t w(\mu_t, l_t)(1 - q_{t+1}) - x_{t+1}^i \tag{9}$$

Note how the time t leisure choice is given by the random variable q_{t+1}, I have chosen the timing notation so agents' sequential sequential problem has similar notation to the constrained planner's sequential problem, which I discuss to show existence of constrained planner solutions.

Let $\mathbb{Y}_l$ denote the space of real valued measurable functions on S such that if $l \in \mathbb{Y}_l$ then $0 \leqslant l \leqslant 1$. The term W_t, with $W_t \colon S \to \mathbb{R}$ is the time t value function. The recursive representation of the sequential problem, where agents choose functions mapping the current inidividual state to assets and leisure, is characterized by the Bellman Equation:

$$W_t(s) = \max_{x', l' \in \Theta(\mu_t, l_t, s)} v(c, l') + \beta \int W_{t+1}((x', e'))\, Q(e, \mathrm{d}e') \tag{10}$$

where the policy functions are given by

$$h_t(s), l_t(s) = \operatorname*{argmax}_{x', l' \in \Theta(\mu_t, l_t, s)} v(c, l') + \beta \int W_{t+1}((x', e'))\, Q(e, \mathrm{d}e') \tag{11}$$

and where the agents' feasibility correspondence $\Theta \colon \mathbb{M} \times \mathbb{Y} \times S \to \mathbb{R}^2$ is given by

$$\Theta(\mu, l, s) := \left\{x', l' \in A \times [0,1] \mid x' \leqslant (1 + r(\mu, l))x + w(\mu, l))e(1 - l')\right\}$$

and where consumption is

$$c = (1 + r(\mu_t, l_t))x + ew(\mu_t, l_t)(1 - l') - x' \tag{12}$$

[2]The agents' problems takes aggregate interest and wage rates as given, which depend on the aggregate state μ_t and l_t.

[3]The functions x_{t+1}^i and q_{t+1}^i are functions defined on the probability space $(\Omega, \mathscr{F}, \bar{\mathbb{P}})$ measurable with respect to the sigma algebra generated by $\{x_0^i, e_0^i, e_1^i, \dots, e_t^i\}$.

Note the policy functions generate a sequence of asset and leisure processes recursively as $x^i_{t+1} = h^i_t(x^i_t, e^i_t)$ and $q_{t+1} = l_t(x^i_t, e^i_t)$. The policy functions are still, however, indexed by t. This is because the value function is non-stationary and depends on future distributions and leisure choices $(\mu_i, l_i)_{i=t}^{\infty}$ at any time t. By contrast, the RCE, which we now turn to define, reduces the dependence on time to dependence on the current state and leisure choice function only.

We'll need some further definitions to formally define the RCE. Let $\mathscr{P}(S)$ denote the space of Borel probability measures on S and let $\mathbb{M}$ be a subspace of $\mathscr{P}(S)$ such that each μ, with $\mu \in \mathbb{M}$, satisfies:

1. the marginal distribution across E, $\int \mu(dx, \cdot)$, agrees with ψ
2. the marginal distribution across A, $\int \mu(\cdot, de)$, has finite variance
3. aggregate assets satisfy $\int\int x\mu(dx, de) \in [0, \bar{K}]$.

Define $\Phi \colon \mathbb{M} \times \mathbb{Y} \to \mathbb{M}$ as

$$\Phi(\mu, h)(B_A \times B_E) := \int\int \mathbb{1}_{B_A}\{h(x, e)\} Q(e, B_E)\mu(dx, de), \qquad B_A \times B_E \in \mathscr{B}(S)$$

where $\mathscr{B}(S)$ are the Borel sets of S. With these definitions, we can define the RCE.

DEFINITION 2.1 (**Recursive Competitive Equilibrium**)

A RCE is a tuple $\{Q, h, l\}$ where $W \colon \mathbb{M} \times S \to \mathbb{R}$, $h \colon \mathbb{M} \times S \to \mathbb{R}$ and $l \colon \mathbb{M} \times S \to \mathbb{R}$ such that:

1. the policy functions satisfy $h(\mu, s), l(\mu, s) \in \Theta(\mu, l(\mu, \cdot)s)$ for each $\mu \in \mathbb{M}$ and $s \in S$ (feasibility)
2. the policy and value functions satisfy (agent maximization)

$$W(\mu, s) = \max_{x', l' \in \Theta(\mu, l(\mu, \cdot), s)} v(c, l') + \beta \int W(\mu', (x', e'))\, Q(e, \mathrm{d}e'), \qquad s \in S, \mu \in \mathbb{M}$$

   and

$$\begin{aligned} & h(\mu, s), l(\mu, s) \\ & = \operatorname*{argmax}_{x', l' \in \Theta(\mu, l(\mu, \cdot), s)} v(c, l') + \beta \int W(\mu', (x', e')) Q(e, \mathrm{d}e'), \qquad s \in S, \mu \in \mathbb{M} \end{aligned}$$

where

$$c = (1 + r(\mu, l(\mu, \cdot))x + ew(\mu, l(\mu, \cdot))(1 - l') - x'$$

and $s = (x, e)$; moreover, the next period state μ' and the set $\Theta(\mu, l(\mu, \cdot), s)$ is taken as given in the maximization problem and $\mu' = \Phi(\mu, h)$ (distribution consistency)

3. the conditions (5) and (6) hold (market clearing).

The existence of an RCE remains an open question. See Cao (2016) for the latest progress in the case without endogenous labour supply.

2.2.1. *Necessary Conditions for Competitive Equilibrium*

The recursive competitive equilibrium satisfies the following necessary conditions for $\mu \in \mathbb{M}$ and $(x, e) \in S$:

$$v_1(c, l(\mu, x, e)) \geqslant \beta(1 + r(\mu', l(\mu', \cdot))) \int v_1\big(c', l(\mu', x', e')\big)\, Q(e, \mathrm{d}e')$$

and

$$v_2(c, l(\mu, x, e)) \geqslant v_1(c, l(\mu, x, e))w(\mu, l(\mu, \cdot))e$$

where

$$c = (1 + r(\mu, l(\mu, \cdot)))x + w(\mu, l(\mu, \cdot))e(1 - l(\mu, x, e)) - x'$$

$$\begin{aligned} c' = {} & (1 + r(\mu', l(\mu', \cdot)))x' + w(\mu', l(\mu', \cdot))e'(1 - x') \\ & - h(\mu', x', e') \end{aligned}$$

$$x' = h(\mu, x, e), \qquad \mu' = \Phi(\mu, h)$$

The necessary conditions are standard. See, for example, Ljungqvist and Sargent (2004), Equations (17.8.11) and (17.8.12) and equation (17.8.5).

2.3. *Recursive Constrained Planner*

The recursive planner's state-space will be the aggregate state space, $\mathbb{M}$, and the constrained planner's action space will be $\mathbb{Y} := \mathbb{Y}_h \times \mathbb{Y}_l$, where $\mathbb{Y}_h$ is the space of positive measurable functions on S.

Define a correspondence Λ, with $\Lambda\colon \mathbb{M} \twoheadrightarrow \mathbb{Y}$, mapping a state to feasible policy functions as follows:

$$\Lambda(\mu) \colon = \begin{cases} h, l \in \mathbb{Y} \mid 0 \leqslant h(x,e) \leqslant (1+r(\mu))x + w(\mu)e(1-l), & \text{if } K(\mu) > 0 \\ h \in \mathbb{Y} \mid h = 0, & \text{if } K(\mu) = 0 \end{cases} \tag{13}$$

The (in) equalities above hold μ - almost everywhere.

The constrained planner's per-period pay-off, $u\colon \operatorname{Gr}\Lambda \to \mathbb{R}_+$, integrates utility across the empirical distribution of agents

$$u(\mu, h, l) \colon = \begin{cases} \int\int \nu\big((1+r(\mu))x + w(\mu)e - h(x,e), l(x,e)\big)\mu(dx, de), & \text{if } K(\mu) > 0 \\ \nu(0,1), & \text{if } K(\mu) = 0 \end{cases}$$

Finally, let $\beta \in (0,1)$ be a discount factor and let V, with $V\colon \mathbb{M} \to \mathbb{R}_+ \cup \{+\infty\}$, denote the constrained planner's value function:

$$V(\mu_0) := \sup_{(\mu_t, h_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t) \tag{14}$$

subject to

$$h_t \in \Lambda(\mu_t), \qquad \mu_{t+1} = \Phi(\mu_t, h_t), \qquad t \in \mathbb{N}, \qquad \mu_0 \text{ given} \tag{15}$$

DEFINITION 2.2 (**Recursive Constrained Planner's Problem**)

Given μ_0, a solution to the recursive constrained planner's problem is a sequence of measurable policy functions $(h_t, l_t)_{t=0}^{\infty}$, with $h_t, l_t\colon S \to A \times [0,1]$ for each t and a sequence of Borel probability measures on S, $(\mu_t)_{t=0}^{\infty}$ satisfying (15) that achieves the value function:

$$V(\mu_0) = \sum_{t=0}^{\infty} \beta^t u(\mu_t, h_t, l_t) \tag{16}$$

THEOREM 2.1 *If assumption* (1) *holds and u is bounded below, then there exists a solution to the recursive constrained planner's problem.*

The existence proof uses the results in Shanker (2017), see the online appendix for details.

2.3.1. *Recursive Constrained Planner's Necessary Conditions*

A solution to the constrained planner's problem satisfies the following necessary conditions. The first is the condition with respect to the capital across all individuals.

$$\begin{aligned} &\nu_1((1+r(\mu_t))a + w(\mu_t)e(1-l_t(a,e)) - h_t(a,e), l_t(a,e)) \\ &\geqslant (1+r(\mu_t))\mathbb{E}\Big\{\nu\big((1+r(\mu_{t+1}))h(a,e) + w(\mu_{t+1})e'(1-l_{t+1}(h(a,e),e')) \\ &\quad - h_{t+1}(h_t(a,e),e'), l_{t+1}(h(a,e),e')\big) \Big| e\Big\} \\ &\quad + \beta F_{K,K}(K_{t+1}, L_{t+1})\Delta(\mu_{t+1}, h_{t+1}, l_{t+1}) \end{aligned} \tag{17}$$

note the only difference between the constrained planner's first order condition and the competitive equilibrium first order condition is the additional term on the RHS (defined below), which captures the external effect of each agent's saving on aggregate welfare via the interest and wage rate. Turning to the first order condition for labour across agents, we have

$$\begin{aligned} &\nu_1((1+r(\mu_t))a + w(\mu_t)el_t(a,e) - h_t(a,e), l_t(a,e))w(\mu_t)e \\ &\quad \leqslant \nu_2((1+r(\mu_t))a + w(\mu_t)el_t(a,e) - h_t(a,e), l_t(a,e)) \\ &\quad - eF_{L,L}(K_{t+1}, L_{t+1})\Delta(\mu_{t+1}, h_{t+1}, l_{t+1}) \end{aligned} \tag{18}$$

The term Δ is a function of the state and planner's policy function, defined by

$$\begin{aligned} \Delta(\mu_t, h_t, l_t) = &\frac{\int \nu'(c_t(a,e), l_t(a,e))a\mu_t(da,de)}{K_t(\mu_t)} \\ &- \frac{\int \nu'(c_t(a,e), l_t(a,e))e(1-l(a,e))\mu_t(da,de)}{L_t(\mu_t)} \end{aligned} \tag{19}$$

Through a change of variables, we can show $\Delta(\mu_t, h_t, l_t) = \mathbf{R}_{K,t} - \mathbf{R}_{L,t}$, where

$$(20) \qquad \mathbf{R}_{K,t} = \frac{\int \tilde{u} G_{K,t}(d\tilde{u})}{K_t}, \mathbf{R}_{L,t} = \frac{\int \tilde{u} G_{L,t}(d\tilde{u})}{L_t}$$

and G_K is a measure on the Borel sets of $\mathbb{R}$ defined by

$$(21) \qquad G_{K,t}(B) := \int \chi\{v_1(c_t(a,e), l_t(a,e)) \in B\} a \, \mu_t(da, de)$$

$$(22) \qquad G_{L,t}(B) := \int \chi\{v_1(c_t(a,e), l_t(a,e)) \in B\}(1 - l_t(a,e)) e \, \mu_t(da, de)$$

for $B \in \mathscr{B}(\mathbb{R})$ and χ is the indicator function.

In words, $G_{K,t}(B)$ is total of capital owned by agents' whose marginal utility falls in B. Thinking of marginal utility as direction, with agents of high marginal utility being poor and agents with low marginal utility being rich, the terms $\mathbf{R}_{K,t}$ and $\mathbf{R}_{L,t}$ are *the centre of mass* of the capital and labour distribution — recall the center of mass is the point at which mass is equally distributed on either side of the point. Thus $\mathbf{R}_{K,t}$ is the marginal utility (or a point on the rich -poor axis) at which there is equal amount of capital on either side. And $\mathbf{R}_{K,t}$ is the marginal utility at which there is equal amount of effective on either.

For the constrained planner, the center of mass interpretation tells us the planner prefers agents save more capital when $\mathbf{R}_{L,t} > \mathbf{R}_{K,t}$ and the distribution of effective labour favours the poor more than the distribution of capital. Conversely, if the distribution of capital favours the poor and $\mathbf{R}_{L,t} < \mathbf{R}_{K,t}$, then the constrained planner prefers agents save less and work more.

2.4. *Implementation of the Constrained Optimum*

To simplify discussion, let us define $\bar{\Delta}_K \colon \mathbb{M} \to \mathbb{R}$ as

$$(23) \qquad \bar{\Delta}_K(\mu_t) = \beta F_{K,K}(K_{t+1}, L_{t+1}) \Delta(\mu_{t+1}, h_{t+1}, l_{t+1})$$

and $\bar{\Delta}_L \colon \mathbb{M} \to \mathbb{R}$ as

$$(24) \qquad \bar{\Delta}_L(\mu_t) = -\beta F_{L,L}(K_{t+1}, L_{t+1}) \Delta(\mu_{t+1}, h_{t+1}, l_{t+1})$$

Theorem 2.1 and section 2.1 in Shanker (2017) shows the constrained planner's policies are recursive, that is the function h_t and l_t depend only on μ_t, and not the entire future sequence of distribution. As such, we are able to write the terms above as functions of the current state, μ_t. A further implication is that knowing only the aggregate state and the planner can implement a capital and labour tax, $\tau_K(\mu_t, a, e)$ and $\tau_L(\mu_t, a, e)$ using

$$\tau_K(\mu_t, a, e) = \frac{\bar{\Delta}_K(\mu_t)}{\mathbb{E}\{v_1[c(h(\mu_t, a, e), l(\mu_t, h(\mu_t, a, e), e'), e'), l(\mu_t, h(\mu_t, a, e), e')] \mid e\}} \tag{25}$$

and

$$\tau_L(\mu_t, a, e) = \frac{e\bar{\Delta}_L(\mu_t)}{v_1[c(a, l(\mu_t, a, e), e), l(\mu_t, a, e)]} \tag{26}$$

Now consider a tax competitive equilibrium, identical to the definition of competitive equilibrium at definition 2.1, with the budget constraint agents replaced by

$$\begin{aligned} &h_t(a_t, e_t) \\ &\quad \leqslant (1 + (1 + r_t(\mu_t))(1 + \tau_K(\mu_t, a, e))a_t + e_t w(\mu_t)(1 + \tau_L(\mu_t, a, e))(1 - l_t(a_t, e_t)) \end{aligned} \tag{27}$$

The following result is the key technical contribution of the paper. The result is particularly striking because the existence of a recursive competitive equilibrium remains an open question.

THEOREM 2.2 *If Assumption 1 holds, then there exists a recursive tax competitive equilibrium.*

The technical details of the proof are in the online appendix. The intuition of the proof is as follows. For each state, we showed the planner can calculate individual state based taxes given by (25) and (26). Given these taxes, we know there exists a recursive constrained planner's solution by Theorem 2.1 in Shanker (2017) and that the planner policy function h and l satisfy the planner's necessary conditions of the planner. These necessary conditions are equivalent to the necessary conditions for individual agents in a tax competitive equilibrium. However, since for the individual agent, it is straightforward to show the necessary conditions are also sufficient, implying the planner's policy functions are also the individual policy function.

Before proceeding to the quantitative analysis, note if the capital tax is negative, that is the planner implements a capital subsidy, then tax is regressive. Since the marginal utility of the rich is lower, the RHS of (25) is higher for the poor. On the other hand, if the labour tax is positive, it is progressive, the rich and more productive are taxed at a higher rate.

3. QUANTITATIVE ANALYSIS

This section computes three steady state economies. While the focus of the analysis here on steady states, in general, a steady state may not exist (see Dávila et al. (2012), model III). However, to illustrate the inter-relationship between hours worked and constrained optima, the steady state analysis here is sufficient. The computational algorithm is a discrete time version of the algorithm used by Nuño and Moll (2017), and is detailed in the online appendix.

3.1. *Economy Calibrated to United States Wealth and Income Inequality*

We begin with an economy with income and wealth inequality roughly matching observed U.S. data. In particular, to model the process $(e_t)_{t=0}^{\infty}$, I used Tauchen's method (see Sargent and Stachurski (2017)) as follows. Consider a sequence of i.i.d. mean zero normal shocks $(\eta_t)_{t=0}^{\infty}$; a three state Markov endowment process was generated from from an AR1 process $z_{t+1} = \rho z_t + \eta_t$ where z_t represents the natural log of the endowment shock. Following Pijoan-Mas (2006), ρ was set to .95 and the standard deviation of each η_t to .3. The generated Markov transition matrix and shock values are presented in Table I. The parameters are chosen to match the calibrations by Pijoan-Mas (2006).[4] The endowment shocks in an simulations are also normalized so the mean of the stationary distribution is one.

Note the endowment process generates a top income shock that is almost 30 times that of the low income shock, the shocks are highly persistent, but there is a non-negligible probability of transitioning from a top shock to a middle and then lower shock. Moreover, periods of being in a low productivity state are relatively long.

This economy generates a high level of precautionary saving, the capital stock in the incomplete market competitive equilibrium is almost double compared to the

[4]The variance parameter is higher than the parameter used by Pijoan-Mas (2006), who uses standard deviation of .22. The high variance ensures a high enough wealth dispersion, yielding negative interest rates and a steady state for the constrained planner. The higher variance also yields a calibration closer to the calibration to the U.S. economy considered by Dávila et al. (2012).

representative agent. However, both hours worked and effective labour supply are similar under a competitive equilibrium and a representative agent.

The constrained planner in this economy requires an even higher level of capital, close to 1.6 times that in the competitive equilibrium, and at this high level of capital, interest rates are negative though wages are close to 20 percent higher. Most interesting is that while the constrained planner reduces overall hours worked, effective labour supply increases by 3 percent.

TABLE I
STEADY STATES FOR U.S. CALIBRATED ECONOMY

	Interest rate	Wage	Capital	Hours	Labour	Wealth gini	Income gini
RA	4.17	—	3.30	33.03	33.03	0.00	0.00
CE.	0.14	2.76	6.09	33.67	33.15	0.64	0.53
CP	-2.07	3.27	10.14	32.96	34.37	0.70	0.56

Simulation results for the representative agent (RA), competitive equilibrium (CE) and constrained planner (CP). Model parameters are $\beta = 0.96$, $\alpha = 0.36$, $\delta = 0.083$, $\gamma_c = 1.458$, $\gamma_l = 2.883$, $A_L = 0.856$, $A = 1.51$ and size of grid is 200. The probability matrix is $[[.96, .04, 0], [.02, .96, .02], [0, .04, .96]]$ and shocks are $[0.1, 0.55, 2.79]$

3.2. *The Low Persistence Unemployment Economy*

In Table II, let us now consider the unemployment economy studied by Dávila et al. (2012). In this economy, there are only two states, a high and low productivity state, with the low productivity state representing unemployment. However, unemployment periods are not very persistent, and unemployed person returns to the high productivity state after an average two periods. The competitive equilibrium now still features precautionary savings, but they are not as large as in the economy calibrated to the U.S. Moreover, and in contrast to the U.S. calibrated economy, incomplete markets result in lower hours worked and labour supply (this is due to the effect studied by Marcet et al. (2007), where ex-post wealth effects induce the rich to work lower hours). In this economy, the constrained planner reduces capital supply by 20 percent, the interest rate is higher under the constrained planner while wages are lower. The constrained planner increases both wages and hours worked.

The qualitative direction of all the variables in this economy (except hours and labour) are similar to the case studied by Dávila et al. (2012). However, notable is the large fall in interests from the representative agent to the competitive equilibrium – due to the lower working hours in the competitive equilibrium economy.

TABLE II
DAVILLA ET AL. (2012) LOW UNEMPLOYMENT PERSISTENCE ECONOMY

	Interest rate	Wage	Capital	Hours	Labour	Wealth gini	Income gini
RA	4.17	—	5.11	51.14	51.14	0.00	0.00
CE	0.18	2.75	5.89	30.89	32.29	0.05	0.08
CP	1.77	2.50	4.77	32.83	34.20	0.11	0.08

Model parameters are $\beta = 0.96$, $\alpha = 0.36$, $\delta = 0.083$, $\gamma_c = 2$, $\gamma_l = 0.5$, $A_L = 2$, $A = 1.51$ and size of grid is 20. The probmatrix is $[[0.62, 0.38], [0.02, 0.98]]$ and shocks are $[0.104, 1.048]$

3.3. *The High Persistence Unemployment Economy*

Marcet et al. (2007) used an economy with high unemployment persistence to show that wealth effects which induce the productive to worked less lead to a lower return on capital which overcomes the precautionary saving motive. Table III presents the results for the results of the constrained planner in an economy with calibrations similar to the ones used by Marcet et al. (2007). The main difference of note between the Marcet et al. (2007) economy and the unemployment economy presented by Dávila et al. (2012) is the lower probability of moving from the unemployed state to the employed state. As expected, total working hours, capital and labour supply are lower in the competitive equilibrium than in the first best. However, in this economy, the constrained planner’s allocation is close to the competitive equilibrium. While the constrained planner increases capital slightly, hours worked and the wage rate remain unchanged.

TABLE III

MARCET ET. AL. (2007) HIGH UNEMPLOYMENT PERSISTENCE ECONOMY

	Interest rate	Wage	Capital	Hours	Labour	Wealth gini	Income gini
RA	5.82	—	4.20	51.09	51.09	0.00	0.00
CE	5.75	2.07	3.04	34.47	36.69	0.11	0.10
CP	5.72	2.07	3.05	34.47	36.68	0.11	0.10

Simulation results for the representative agent (RA), competitive equilibrium (CE) and constrained planner (CP). Model parameters are $\eta = 0.945$, $\alpha = 0.36$, $\delta = 0.083$, $\gamma_c = 2$, $\gamma_l = 0.5$, $A_L = 2$, $A = 1.51$ and size of grid is 750. The probmatrix is $[[0.09, 0.91][0.06, 0.94]]$ and shocks are $[0.0266045, 1.06417992]$.

4. DISCUSSION

Compared to the incomplete market competitive equilibrium, three factors determine how work and saving behaviour across the wealth and endowment distribution in the constrained planner's allocation differs from the competitive equiibrium. First is the effect of the added $\bar{\Delta}_K$ and $\bar{\Delta}_L$ terms in the first order conditions (17) and (19). Second are the changes in prices, the interest rate and wage induced by altered saving and work behaviour of all agents. Third are wealth effects: changes in marginal utility of consumption under the constrained planner's allocation affects the ex-post decision of how much to work.

4.1. *Center of mass and direction of the constrained planner's tax*

Let us first consider the Δ terms in the constrained planner's first order conditions. When $\bar{\Delta}_K$ is positive (negative), then the planner implements a saving subsidy (tax), giving a greater incentive for agents to save (less). And when $\bar{\Delta}_L$ is positive (negative), the planner implements an employment subsidy (tax), giving agents incentive to work or (less). The sign of $\bar{\Delta}_K$ depends on the center of mass of the capital and labour distribution. If the centre of mass of labour productivity (capital) is more towards the the poor, then the constrained planner prefers agents save more (less) on average.

The term $\bar{\Delta}_L$ has the opposite sign to $\bar{\Delta}_K$, by equation (24). This is because when the planner prefers agents save more on average, the planner will prefer agents work less. The reason is intuitive, the constrained planner wishes interest rates to fall

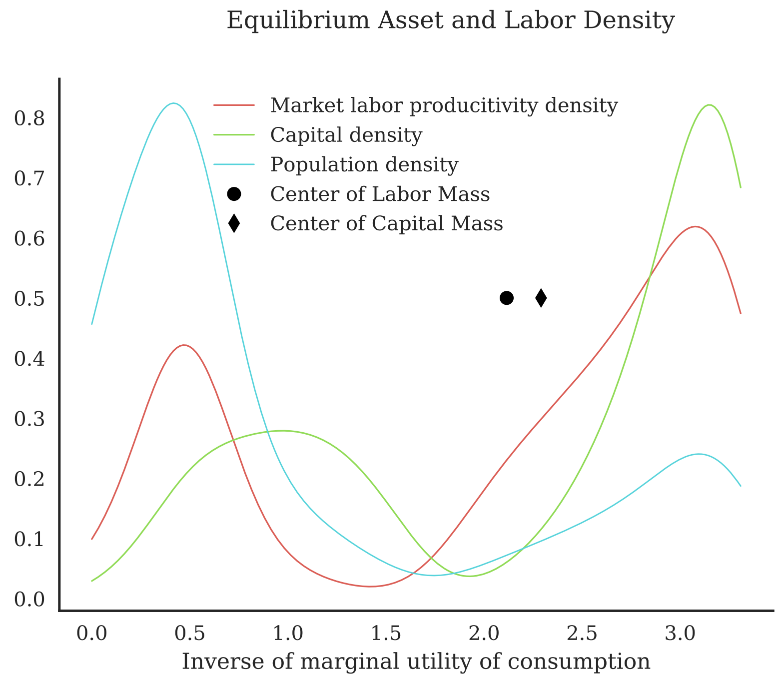

Figure 1: U.S. Calibrated Economy: competitive equilibrium density of capital, labour and population on utility. All mass is normalized to one.

and wages to rise, achieving this through an increase in capital and fall in hours worked.

To understand how the labour and capital distribution effects the planner's Δ term, consider Figure 1. The figure shows the density of capital, population and labour on utility. Poor agents are on the left of the axis contains poor agents, and the right of the axis contains rich agents. Integrating under any of the densities over an interval gives us the total mass of capital, effective labor or population in that interval of utility (mathematically, the densities are the densities of the G_K and G_L distribution defined at (21) and (22)).

Most of the population density is concentrated towards the poor. These agents have some labour endowment, resulting in a large mass of effective labor on the left of the axis in of Figure 1. However, note, while the 'working poor' provide labour in the market, they do not accumulate much capital. The density of capital is concentrated towards the consumption rich. The mass of agents towards the rich is small, however, these agents are highly productive — their mass of labor productivity is higher than the effective labour provided by the 'working poor'. Notable, however, is that these agents also save more and accumulate more capital. The fact that agents at the top are highly productive and accumulate more capital, while agents at the bottom have almost no capital but are still moderately endowed with effective labour means the center of mass of labour tilts towards the poor. The poor are relatively more dependent on labor, and hence, the constrained planner wishes to improve wages and depress capital returns.

Let us now contrast the U.S. calibrated economy with the unemployment economy with low shock persistence. The capital, labor and population densities of this economy are shown in Figure 3 in the appendix. Note the mass of agents who have received a low employment shock populate the region of the axis just left of 0.8, where there is no labour productivity. This time the center of mass of capital favours the poor. This is because the unlucky in this economy not only have a bad productivity shock, but do not provide any labor to the market. The unlucky, however, still hold capital. Since unemployment shocks are not persistent, the unlucky are only temporarily unlucky and move back to the highly productive state before they can run down their capital stocks.

Finally, Figure 3 in the appendix shows the distribution of capital, labor and population for the unemployment economy with high unemployment persistence. Compared to Figure 3, the main difference is the lack of capital mass for agents with low labour productivity. The capital mass and labour productivity is distributed among the same agents who are not unemployed. Since the spells of unemployment are long, unlucky agents will run down their capital stocks in order to consume before they encounter a high productivity shock. This not only leads to a center of mass of labour and capital that is almost equal, but leads to a lower aggregate capital stock.

4.2. *Price and wealth effects*

Now we turn to the second factor shaping how the constrained planner's allocation differs from the competitive equilibrium — wage rates and interest rates. While in the U.S. calibrated economy, the constrained planner implements a capital tax, as shown in Figure 5 in the Appendix, only the most productive agents respond to the subsidy by increasing saving. This is because the saving subsidy is lower for consumption poor agents, and the unluckier agents respond more to the fall in interest rates by reducing saving.

The price effects also show in the labor market decisions agents make. Figure 2 shows the hours worked by agents for different asset levels and shock values. We can see that at lower levels of assets (below 5), agents with lower labour productivity decrease their hours worked in response to the labour income tax. However, for agents with the highest labour productivity, hours of work actually increase. This is due to the rise in wages which come from a large capital stock and induce the most productive agents to work more even though they pay the highest labor income taxes.

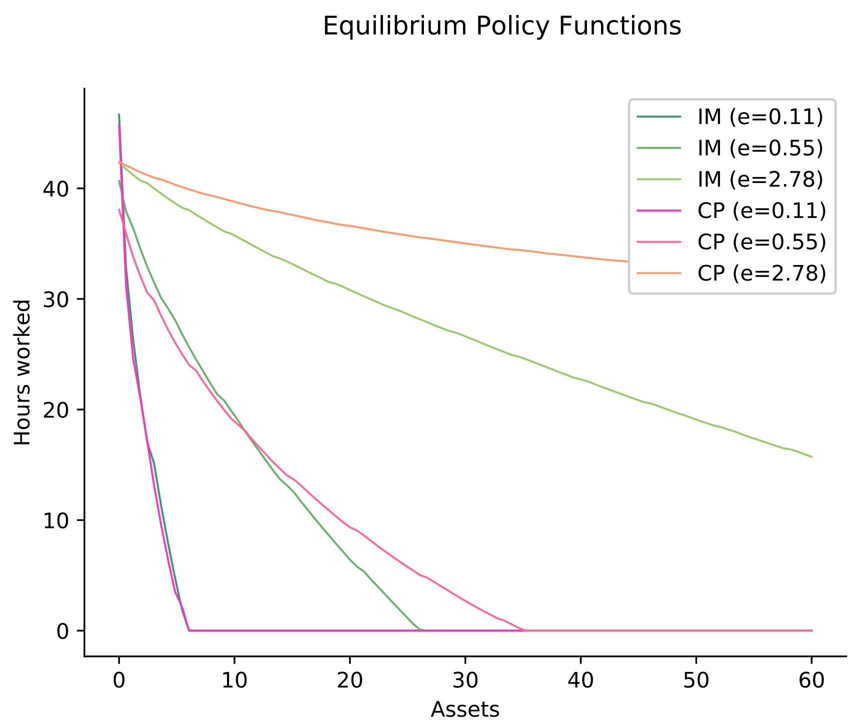

Figure 2: U.S. Calibrated Economy: comparison of policy functions of hours worked for competitive equilibrium (IM) and constrained planner (CP) allocation.

Most intriguing, however, is the increase in hours of work under the constrained planner allocation work for asset rich agents. For instance, an agent with medium productivity ($e = .55$) but with assets over 15 actually increases work hours under the constrained planner's allocation. This is because asset wealthy agents are all ex-post worse off from a decline in interest rates, their marginal utility of consumption rises, inducing a reduction in leisure time. Because these wealth and price effects hit only the more productive, effective labor supply rises, while total hours worked fall from the reduced labor supply of the unproductive.

5. CONCLUSION

This paper considered optimal saving and work hours in a heterogeneous agent growth model with uninsured idiosyncratic risk. The paper considered optimality from the point of view of a constrained planner, a planner who cannot complete markets or initiate net transfers between agents, but can guide each agents' saving decisions and work hours. The paper showed the constrained planner can implement the constrained optima using individual state based taxation; the existence of recursive equilibrium under this tax regime was also proved.

The main applied findings of the paper are consistent with Dávila et al. (2012), introducing work hours does not qualitatively change their conclusions. In particular, for an economy calibrated to observed U.S. wealth and income inequality, the constrained planner increases aggregate capital. When hours of work are introduced, the paper finds the constrained planner wishes to lower aggregate hours of work. Both the rise in capital and fall in hours of work are targetting at increasing wages and lowering interest rates, resulting in a redistribution of consumption from the consumption rich to consumption poor.

The aggregate differences between the constrained planner's allocation and incomplete market equilibrium hide, however, the diversity how people of different levels of productivity and wealth respond to the taxes and subsidies implemented by the constrained planner. Most notably, because of the rise in wages, the most productive workers increase work hours despite a progressive tax on labor and the capital wealthy also increase work because of ex-post wealth effects from the fall in interest rates.

The analysis of is only a first attempt to understand work hours across wealth distributions and how they should be influenced by policy. A more ambitious analysis would involve linking incomplete market heterogeneous agent models to work by Boppart and Krusell (2017), who use a broader class of consumption - leisure preferences to explain the long term trend of declining work hours in along a balanced growth path (traditional preferences, such as the ones use here, cannot account for these falling work hours.). Moreover, this paper only examined choices at the intensive margin, that is, choices about how much to work. There is also scope for further research on how individuals make choices at the intensive margin, that is, whether or not to withdraw from the work force altogether.

On the technical front, existence of recursive competitive equilibrium away from the steady state for the standard heterogeneous agent growth model remains unresolved. Some progress has been on the existence of steady states in the heterogeneous agent model with endogenous labour supply (Zhu, 2017), however, there is scope for refining results regarding convergence of Euler equation iterations for the income fluctuation problem with endogenous labour supply (results analogous to Li and Stachurski (2014)).

APPENDIX A: SEQUENTIAL CONSTRAINED PLANNER'S PROBLEM

A.1. *Definition of the Sequential Constrained Planner's Problem*

Now I set up the sequential constrained planner's problem. Remaining consistent with the notation in Shanker (2017), let $\mathbb{X} = L^2(\Omega, \mathbb{P})^2$. Let $\mathbb{S}_t$ be defined by:

$$\mathbb{S}_t := \left\{ x, q \in (m\mathscr{F}_t^+)^2 \,\middle|\, \int x \,\mathrm{d}\mathbb{P} \leqslant \bar{K}, 0 \leqslant q \leqslant 1 \right\} \tag{A.28}$$

The state at time $t+1$ will be the $\mathscr{F}_t$ measurable functions chosen by the planner; this includes the asset x_{t+1} that is taken to time $t+1$ and the leisure hours worked at time t. For each t, define the feasibility correspondence $\Gamma_t \colon \mathbb{S}_t \to \mathbb{S}_{t+1}$:

(A.29)

$$\Gamma_t\,(x, m) = \begin{cases} y, q \in \mathbb{S}_{t+1} \,\big|\, 0 \leqslant y \leqslant (1 + \tilde{r}(x,q))x + \tilde{w}(x,q)e_t(1-q), & \text{if } \mathbb{E}x > 0 \\ y, q \in \mathbb{S}_{t+1} \,\big|\, x = 0, & \text{if } \mathbb{E}x = 0 \end{cases}$$

Note the hours worked does not effect the space of feasible choices.

For each t, define the time t pay-offs $\rho_t \colon \operatorname{Gr}\Gamma_t \to \mathbb{R}_+$:

(A.30)

$$\rho_t\,(x, m, y, q) = \begin{cases} \displaystyle\int \nu((1 + \tilde{r}(x,q))x + \tilde{w}(x,q)e_t(1-q) - y, y_2)\,\mathrm{d}\mathbb{P}, & \text{if } \mathbb{E}x_1 > 0 \\ \nu(0,1), & \text{if } \mathbb{E}x_1 = 0 \end{cases}$$

Note how while hours worked at time $t-1$ form the utility for time t, they affect the pay-offs at time $t-1$ but do not affect the pay-offs at time t.

Finally, let J, with $J \colon \mathbb{S}_0 \to \mathbb{R}_+ \cup \{+\infty\}$ denote the time 0 sequential planner's value function:

$$J(x(0)) := \sup_{(x_t, q_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, q_t, x_{t+1}, q_{t+1})$$

subject to

$$x_{t+1}, q_{t+1} \in \Gamma_t(x_t), \qquad \forall t \in \mathbb{N}, \qquad x_0 \in \mathbb{S}_0 \text{ given} \tag{A.31}$$

DEFINITION A.1 (**Sequential Constrained Planner's Problem**)

Given $x_0 \in \mathbb{S}_0$, a solution to the sequential constrained planner's problem is a sequence of random variables $(x_t, q_t)_{t=0}^{\infty}$ satisfying (A.31) that achieve the sequential planner's value function:

$$J(x(0)) = \sum_{t=0}^{\infty} \beta^t \rho_t(x_t, q_t, x_{t+1}, q_{t+1}) \tag{A.32}$$

A.2. *Necessary Conditions*

Now we proceed to analyze the necessary conditions. I would like to make some comments on notation for the following section before proceeding. While we defined ρ_t with all its arguments formally, that is, under an optimal sequence, the time t pay-off is $\rho_t(x_t, q_t, x_{t+1}, q_{t+1})$, since the pay-off does not depend on q_t, I will abuse notation to save space and write the time t pay-off below as $\rho_t(x_t, x_{t+1}, q_{t+1})$. Let us also define the following for each t along an optimal sequence $(x_t, q_t)_{t=0}^{\infty}$:

$$\begin{aligned} K_t &:= \int x_t \,\mathrm{d}\mathbb{P} \\ L_t &:= \int (1 - q_{t+1}) e_t \,\mathrm{d}\mathbb{P} \\ c_t &:= (1 + \tilde{r}(x_t, q_{t+1}))x_t + \tilde{w}(x_t, q_{t+1})(1 - q_{t+1}) - x_{t+1} \end{aligned}$$

The following proposition is a necessary condition for dynamic optimization.

PROPOSITION A.1 *If $(x_t, q_t)_{t=0}^{\infty}$ is a solution to the sequential constrained planner's problem, then*

$$x_{t+1} = \operatorname*{argmax}_{y,q \in \Gamma_t(x_t, q_t)} \rho_t(x_t, y, q) + \beta \rho_{t+1}(y, x_{t+2}, q_{t+2}), \qquad t\ in \mathbb{N}$$

The proof for this proposition is standard and thus omitted. (See for example, section 5.2 by Sorger (2015). The technical challenge now is to show the necessary condition of proposition A.1 implies a functional Euler equation. To prepare for the differentiability assumption, let

$$\varphi_t(y, q) := \rho_t(x_t, y, q) + \beta \rho_{t+1}(y, x_{t+2}, q_{t+2}), \qquad t \in \mathbb{N} \tag{A.33}$$

We have the following definition. Let X be a vector space, Y a normed space, and T a transformation defined on a domain $D \subset X$ and having range $R \subset Y$

DEFINITION A.2 Let $x \in D$ and let h be a arbitrary in X. If the limit

$$\delta T(x;h) = \lim_{\lambda \to 0} \frac{1}{\lambda}[T(x + \lambda h) - T(x)] \tag{A.34}$$

exists, it is called the *Gateaux differential of T at x with increment h.*

Now let $\delta\varphi_{t,1}(x_{t+1}, q_{t+1}; \mathbb{1}_B c_t)$ denote the Gateaux differential of φ with respect to its first argument with increment $\mathbb{1}_B c_t$ and let $\delta\varphi_{t,2}(x_{t+1}, q_{t+1}; \mathbb{1}_B(1 - q_{t+1}))$ denote the Gateaux differential of φ with respect to its second argument with increment $\mathbb{1}_B(1 - q_{t+1})$. Where $B \in \mathscr{F}$.

ASSUMPTION 2 *If $(x_t, q_t)_{t=0}^{\infty}$ is a solution to the sequential constrained planner's problem, then the Gateaux differentials $\delta\varphi_1(x_{t+1}, q_{t+1}; \mathbb{1}_B c_t)$ and $\delta\varphi_2(x_{t+1}, q_{t+1}; \mathbb{1}_B(1 - q_{t+1}))$ exist for each t and $B \in \mathscr{F}$.*

PROPOSITION A.2 *If Assumption 2 holds, then*

$$\begin{aligned} \delta\varphi_{t,1}(x_{t+1}, q_{t+1}; \mathbb{1}_B c_t) = {} & \mathbb{E}\big[(\nu_1(c_{t+1}, q_{t+1})(1 + \tilde{r}(K_{t+1}, L_{t+1})) \\ & - \nu_1(c_t, x_{t+1}))\mathbb{1}_B c_t\big] \\ & + \mathbb{E}\big[F_{KK}(K_{t+1}, L_{t+1})x_{t+1} \\ & + F_{KL}(K_{t+1}, L_{t+1})(1 - q_{t+2})e_{t+1}\big]\mathbb{E}\mathbb{1}_B c_t \end{aligned} \tag{A.35}$$

$$\begin{aligned} \delta\varphi_{t,2}(x_{t+1}, q_{t+1}; \mathbb{1}_B(1 - q_{t+1})) = {} & \mathbb{E}\big[(v_1(c_t, x_{t+1})\tilde{w}(K_t, L_t)e_t \\ & + v_2(c_t, q_{t+1}))\mathbb{1}_B(1 - q_{t+1})\big] \\ & + \mathbb{E}\big[(F_{KL}(K_t, L_t)x_{t+1} \\ & + F_{LL}(K_t, L_t)(1 - q_{t+1})e_t\big]\mathbb{E}\mathbb{1}_B(1 - q_{t+1})e_t \end{aligned} \tag{A.36}$$

The following theorem will prove the sequential planner's necessary conditions then become:

$$\nu_1(c_t, q_{t+1}) \geqslant (1 + r(x_t, q_{t+1}))\mathbb{E}_t\nu_1(c_{t+1}, q_{t+2}) + \beta F_{KK}(K_t, L_t)\Delta(x_{t+1}, q_{t+2}) \tag{A.37}$$

$$\nu_1(c_t, q_{t+1})w(x_t, q_{t+1})e_t \leqslant \nu_2(c_t, q_{t+1}) - e_t F_{LL}(K_{t+1}, L_{t+1})\Delta(x_t, q_{t+1}) \tag{A.38}$$

where

$$\bar{\Delta}(x_t, q_{t+1}) :\, = \mathbb{E}\frac{\nu_1(c_t, q_{t+1})x_t}{\mathbb{E}x_t} - \mathbb{E}\frac{\nu_1(c_t, q_{t+1})e_t(1 - q_{t+1})}{\mathbb{E}(1 - q_{t+1})e_t}, \qquad t \in \mathbb{N} \tag{A.39}$$

THEOREM A.1 *If Assumption 2 holds and $c_t > 0$ and $q_{t+1} < 1$, then* (A.37) *and* (A.38) *hold.*

PROOF: Set $t \in \mathbb{N}$. Note Proposition A.1, then, by Theorem 2 of section 7 by Luenberger (1968) we must have

$$\text{(A.40)} \quad \delta\varphi_{t,1}(x_{t+1}, q_{t+1}; h - x_{t+1}) \geqslant 0, \qquad \forall h \in \Gamma_t(x_t \mid q = q_{t+1})$$

and

$$\text{(A.41)} \quad \delta\varphi_{t,2}(x_{t+1}, q_{t+1}; h - q_{t+1}) \geqslant 0, \qquad \forall h \in \Gamma_t(x_t \mid y = x_{t+1})$$

if the differentials exist, where $\Gamma_t(x_t \mid q = q_{t+1}) = \{y \mid y, q_{t+1} \in \Gamma_t(x_t)\}$. And $\Gamma_t(x_t \mid y = x_{t+1}) = \{q \mid x_{t+1}, q \in \Gamma_t(x_t)\}$. In particular, pick any $B \in \mathscr{F}$ and let

$$h_1 = \mathbb{1}_B(1 + r(x_t, q_{t+1}))x_t + w(x_t, q_{t+1})e_t(1 - q_{t+1}) + \mathbb{1}_{\Omega\setminus B}x_{t+1}$$

$$h_2 = \mathbb{1}_B + \mathbb{1}_{\Omega\setminus B}q_{t+1}$$

Note $h_1, h_2 \in \Gamma_t(x_t)$ and $x_{t+1} - h_1 = \mathbb{1}_B c_t$ and $q_{t+1} - h_2 = \mathbb{1}_B(1 - q_{t+1})$. As such, noting Assumption 2, the Gateaux differential $\varphi_{t,1}(x_{t+1}, q_{t+1}; h_1)$ exists and by (A.40) we now derive

$$\begin{aligned}
\delta\varphi_1(x_{t+1}, q_{t+1}; \mathbb{1}_B c_t) &= \mathbb{E}\big[\,(\nu_1(c_{t+1}, q_{t+2})(1 + \tilde{r}(K_{t+1}, L_{t+1})) - \nu_1(c_t, q_{t+1}))\,\mathbb{1}_B c_t\big] \\
&\quad + \mathbb{E}\big[F_{KK}(K_{t+1}, L_{t+1})x_{t+1} \\
&\quad + F_{KL}(K_{t+1}, L_{t+1})(1 - q_{t+2})e_{t+1}\big]\mathbb{E}\mathbb{1}_B c_t \\
&= \mathbb{E}\mathbb{E}_t\big[\,(\nu_1(c_{t+1}, q_{t+2})(1 + \tilde{r}(K_{t+1}, L_{t+1})) - \nu_1(c_t, q_{t+1}))\,\mathbb{1}_B c_t\big] \\
&\quad + \mathbb{E}\big[F_{KK}(K_{t+1}, L_{t+1})q_{t+1} \\
&\quad + F_{KL}(K_{t+1}, L_{t+1})(1 - q_{t+2})e_{t+1}\big]\mathbb{E}\mathbb{1}_B c_t \\
&= \mathbb{E}\big[\,(\mathbb{E}_t\{\nu_1(c_{t+1}, q_{t+2})(1 + \tilde{r}(K_{t+1}, L_{t+1}))\} - \nu_1(c_t, q_{t+1}))\,\mathbb{1}_B c_t\big] \\
&\quad + \mathbb{E}\Big[K_{t+1}F_{KK}(K_{t+1}, L_{t+1})\frac{x_{t+1}}{K_{t+1}} \\
&\quad + L_{t+1}F_{KL}(K_{t+1}, L_{t+1})\frac{(1 - q_{t+2})e_{t+1}}{L_{t+1}}\Big]\mathbb{E}\mathbb{1}_B c_t \\
&= \mathbb{E}\big[\,(\mathbb{E}_t\{\nu_1(c_{t+1}, q_{t+2})(1 + \tilde{r}(K_{t+1}, L_{t+1}))\} - \nu_1(c_t, q_{t+1}))\,\mathbb{1}_B c_t\big] \\
&\quad + \bar{\Delta}(x_{t+1}, q_{t+2})F_{KK}(K_{t+1}, L_{t+1})K_{t+1}\mathbb{E}\mathbb{1}_B c_t \\
&= \mathbb{E}\big[\mathbb{1}_B c_t\big(\mathbb{E}_t\{\nu_1(c_{t+1}, q_{t+2})(1 + \tilde{r}(K_{t+1}, L_{t+1}))\} - \nu_1(c_t, q_{t+1}) \\
&\quad + \bar{\Delta}(x_{t+1}, q_{t+2})F_{KK}(K_{t+1}, L_{t+1})K_{t+1}\big)\big] \leqslant 0
\end{aligned}$$

Where the first equality follows from the definition of $\delta\varphi_1(x_{t+1}, q_{t+1}; \mathbb{1}_B c_t)$. The second equality uses the Tower Property of conditional expectations (see Williams (1991), 9.7 i)). The third equality follows from 'taking out what is known' (see Williams (1991), 9.7 j) since $\mathbb{1}_B c_t$ is $\mathscr{F}_t$ measurable and also using $F_{KK}(K, L)K + F_{LK}(K, L)L = 0$. The fourth equality follows from the definition of $\Delta_t(x_{t+1}, q_{t+2})$ at (A.39). The final equality follows from linearity of expectations. Since the inequality must hold for any $B \in \mathscr{F}_t$ and $c_t > 0$ by assumption made by the theorem, we have

$$\mathbb{E}_t\{\nu_1(c_{t+1}, q_{t+2})(1 + \tilde{r}(K_t, L_t))\} - \nu_1(c_t, q_{t+1}) + \Delta F_{KK}(K_{t+1}, L_{t+1})K_{t+1} \leqslant 0$$

for $\mathbb{P}$ a.e., which immediately gives (A.37). Similarly,

$$\begin{aligned}
\delta\varphi_2(x_{t+1}, q_{t+1}; \mathbb{1}_B(1 - q_{t+1})) &= \mathbb{E}\big[(v_1(c_t, x_{t+1})\tilde{w}(K_t, L_t)e_t \\
&\quad + v_2(c_t, q_{t+1}))\mathbb{1}_B(1 - q_{t+1})\big] \\
&\quad + \mathbb{E}\big[(F_{KL}(K_t, L_t)x_{t+1} \\
&\quad + F_{LL}(K_t, L_t)(1 - q_{t+1})e_t\big]\mathbb{E}\mathbb{1}_B(1 - q_{t+1})e_t \\
&= \mathbb{E}\big[(v_1(c_t, x_{t+1})\tilde{w}(K_t, L_t)e_t \\
&\quad + v_2(c_t, q_{t+1}))\mathbb{1}_B(1 - q_{t+1})\big] \\
&\quad + \mathbb{E}\big[(K_t F_{KL}(K_t, L_t)\frac{x_t}{K_t} \\
&\quad + L_t F_{LL}(K_t, L_t)\frac{(1 - q_{t+1})e_t}{L_t}\big]\mathbb{E}\mathbb{1}_B(1 - q_{t+1})e_t \\
&= \mathbb{E}\big[\mathbb{1}_B e_t(1 - q_{t+1})\big(v_1(c_t, x_{t+1})\tilde{w}(K_t, L_t)e_t \\
&\quad + v_2(c_t, q_{t+1}) - L_t F_{LL}(K_t, L_t)\bar{\Delta}(x_t, q_{t+1})\big)\big] \leqslant 0
\end{aligned}$$

once again, since the inequality above must hold for any $B \in \mathscr{F}_t$, we have

$$v_1(c_t, x_{t+1})\tilde{w}(K_t, L_t)e_t + v_2(c_t, q_{t+1}) - L_t F_{LL}(K_t, L_t)\bar{\Delta}(x_t, q_{t+1})e_t \leqslant 0$$

which immediately gives (A.38) as required.

Q.E.D.

APPENDIX B: ADDITIONAL FIGURES

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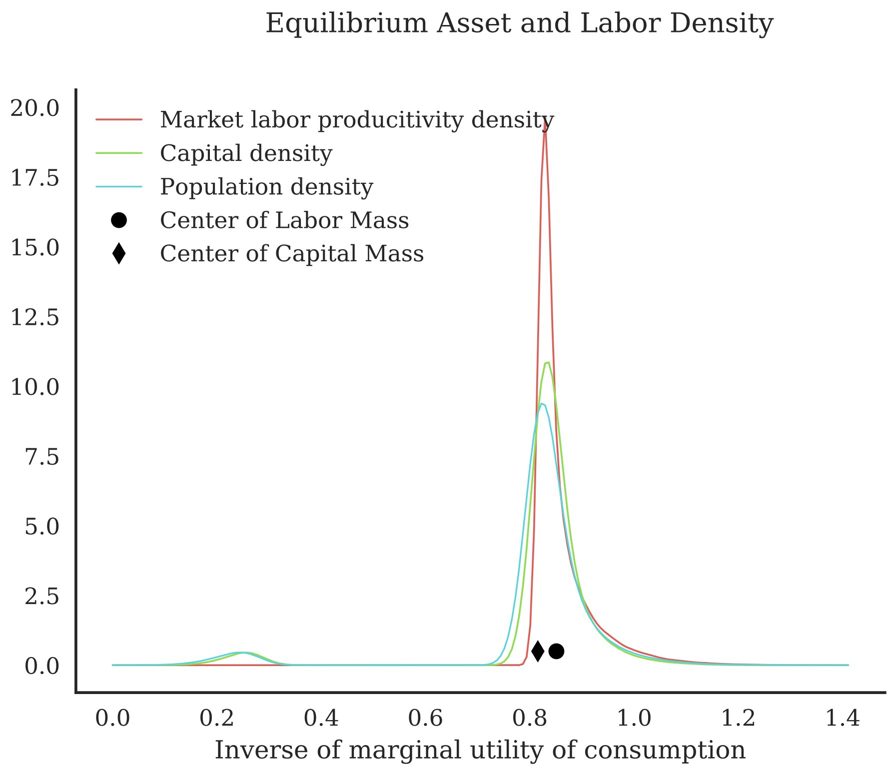

Figure 3: Low Unemployment Persistence Economy: competitive equilibrium density of capital, labour and population on utility. All mass is normalized to one.

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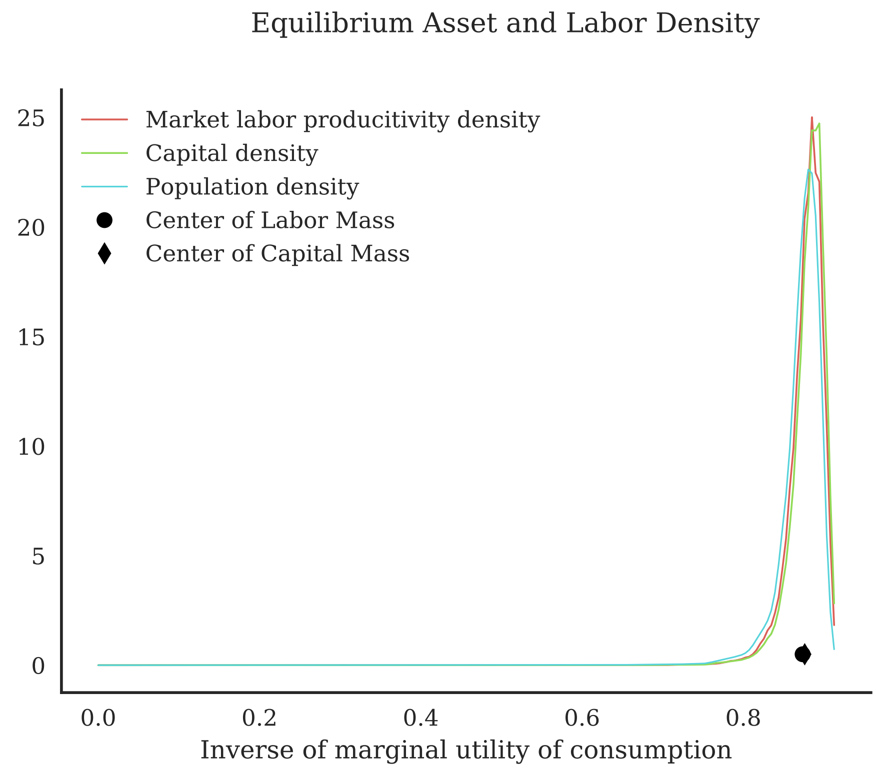

Figure 4: High Unemployment Persistence Economy: competitive equilibrium density of capital, labour and population on utility. All mass is normalized to one.

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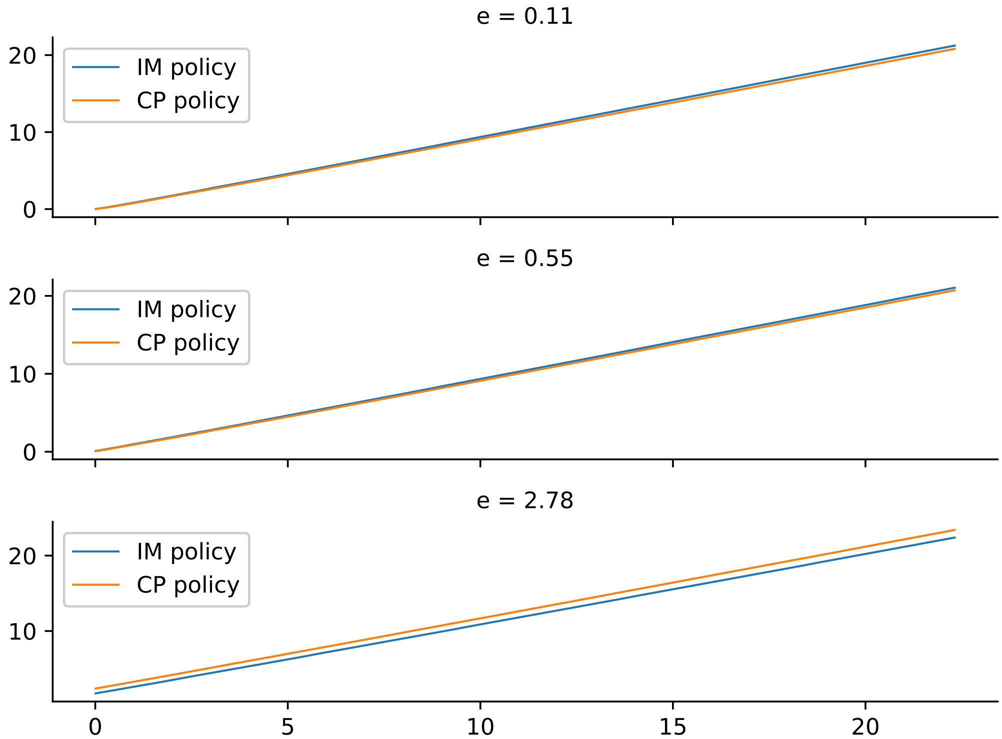

Figure 5: U.S. Calibrated Economy: competitive equilibrium policy functions for next period assets

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