An Introduction to Computational Macroeconomics

Dynamic Programming: Chapter 3

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Introduction

Summary of this lecture:

- Introduction to partial orders
- Pointwise orders
- Order-preserving maps
- Fixed points and order
- Monotone Markov chains

Order

Order theory needed for optimality and fixed point results

Let P be a nonempty set

A **partial order** on a P is a binary relation $\preceq$ on $P \times P$ satisfying, for any p, q, r in P,

$p \preceq p$,

$p \preceq q$ and $q \preceq p$ implies $p = q$ and

$p \preceq q$ and $q \preceq r$ implies $p \preceq r$

(Reflexivity, antisymmetry, transitivity)

We call $(P, \preceq)$ (or just P) a **partially ordered set**

Ex.

1. Show that the usual order $\leqslant$ on $\mathbb{R}$ is a partial order on $\mathbb{R}$
2. Given set M, show that $\subset$ is a partial order on $\wp(M)$

A partial order $\preceq$ on P is called a **total order** if

$$\text{either } p \preceq q \text{ or } q \preceq p \text{ for all } p, q \in P$$

Example. $\leqslant$ is a total order on $\mathbb{R}$

Ex. Prove: $\subset$ is not a total order on $\wp(M)$ when $|M| > 1$

Pointwise Partial Orders

Given set M and f, g in $\mathbb{R}^M$, we set

$$f \leqslant g \iff f(x) \leqslant g(x) \text{ for all } x \in M$$

Ex. Show $\leqslant$ is a partial order on $\mathbb{R}^M$

Similarly, for $n \times k$ matrices $A = (a_{ij})$ and $B = (b_{ij})$,

$$A \leqslant B \iff a_{ij} \leqslant b_{ij} \text{ for all } i, j$$

Ex. Show that $\leqslant$ is a partial order on $\mathbb{M}^{n \times k} :=$ all $n \times k$ matrices

Both are called **pointwise partial orders**

Special case: **pointwise order** for vectors

Recall $[n] := \{1, \ldots, n\}$

For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we write

$$x \leqslant y \text{ if } x_i \leqslant y_i \text{ for all } i \in [n]$$

Ex. Prove: for $a, b \in \mathbb{R}^n$ and sequence (x_k) in $\mathbb{R}^n$, we have

$$a \leqslant x_k \leqslant b \text{ for all } k \in \mathbb{N} \text{ and } x_k \to x \text{ implies } a \leqslant x \leqslant b$$

- Hint: $x_k \to x$ implies $x_{i,k} \to x_i$ for $i \in [n]$

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Strict inequalities

We write

- $f \ll g$ if $f(x) < g(x)$ for all $x \in M$
- $x \ll y$ if $x_i < y_i$ for all $i \in [n]$
- $A \ll B$ if $a_{ij} < b_{ij}$ for all i, j

These are <u>not</u> partial orders

Ex. Why is $f \ll g$ not a partial order on $\mathbb{R}^M$?

Pointwise partial order $\leqslant$ on $\mathbb{R}^2$:

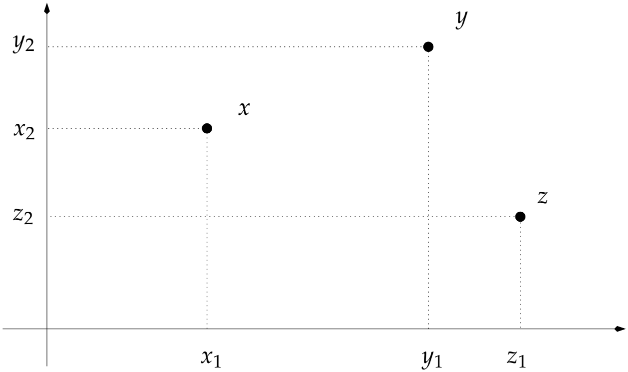

Figure: Pointwise we have $x \leqslant y$ and $x \ll y$ but not $z \leqslant y$

Ex. Prove: If B is $m \times k$ and $B \geqslant 0$, then

$$|Bx| \leqslant B|x| \text{ for all } k \times 1 \text{ column vectors } x$$

Proof: Fix $B \in \mathbb{M}^{m \times k}$ with $b_{ij} \geqslant 0$ for all i, j

Fix $i \in [m]$ and $x \in \mathbb{R}^k$

By the triangle inequality, we have $|\sum_j b_{ij} x_j| \leqslant \sum_j b_{ij} |x_j|$

Stacking these inequalities yields

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Lemma. Given a finite set D and f, g in $\mathbb{R}^D$, we have

$$|\max_{z \in D} f(z) - \max_{z \in D} g(z)| \leqslant \max_{z \in D} |f(z) - g(z)|$$

Proof: Fixing $f, g \in \mathbb{R}^D$, we have

$$f = f - g + g \leqslant |f - g| + g \qquad \text{(pointwise)}$$

$$\therefore \quad \max f \leqslant \max(|f - g| + g) \leqslant \max |f - g| + \max g$$

$$\therefore \quad \max f - \max g \leqslant \max |f - g|$$

Reversing the roles of f and g proves the claim

Let $(P, \preceq)$ and $(Q, \trianglelefteq)$ be partially ordered sets

$T \colon P \to Q$ is called **order-preserving** if

$$p, p' \in P \text{ and } p \preceq p' \quad \implies \quad Tp \trianglelefteq Tp'$$

Special case $(Q, \trianglelefteq) = (\mathbb{R}, \leqslant)$: we call $h \in \mathbb{R}^P$

- **increasing** if $p \preceq p'$ implies $h(p) \leqslant h(p')$ and
- **decreasing** if $p \preceq p'$ implies $h(p) \geqslant h(p')$

Symbol $i\mathbb{R}^P$ denotes all increasing functions in $\mathbb{R}^P$

Example. Let $\leqslant$ denote the pointwise partial order on $\mathbb{R}^n$

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Tx = Ax + b$

If $A \geqslant 0$, then T is order preserving on $\mathbb{R}^n$

<u>Proof:</u> Fix $x \leqslant y$

Then $y - x \geqslant 0$

$$\therefore \quad A(y - x) \geqslant 0$$

$$\therefore \quad Ax \leqslant Ay$$

$$\therefore \quad Tx \leqslant Ty$$

Standard definitions:

Suppose $h \colon P \to Q$ with $P, Q \subset \mathbb{R}$

We call h

- **strictly increasing** if

$$x < y \text{ implies } h(x) < h(y)$$

and

- **strictly decreasing** if

$$x < y \text{ implies } h(x) > h(y)$$

Parametric Monotonicity

Let $(P, \preceq)$ be a partially ordered set

Given two self-maps S and T on a set P, we set

$$S \preceq T \iff Sp \preceq Tp \text{ for every } p \in P$$

We say that T **dominates** S on P

Ex. Show that $\preceq$ is a partial order on

$$\mathcal{S}_P := P^P := \text{ set of all self-maps on } P$$

Proof of antisymmetry of $\preceq$ on $\mathcal{S}_P$:

Let $(P, \preceq)$ and $S, T \in \mathcal{S}_P$ be as defined above

Suppose $S \preceq T$ and $T \preceq S$

Fix any $p \in P$

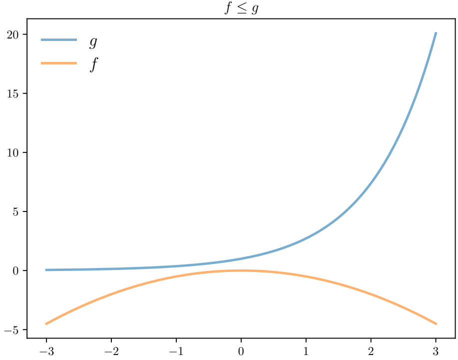
We have $Sp \preceq Tp$ and $Tp \preceq Sp$

Since $\preceq$ is antisymmetric on P, we have $Sp = Tp$

Since p was arbitrary, $S = T$

Hence $\preceq$ is antisymmetric on $\mathcal{S}_P$

Example. If $(\preceq, P) = (\leqslant, \mathbb{R})$, then $\leqslant$ is the pointwise partial order over functions



Example. Let $P = \mathbb{R}^n_+$ with the pointwise order on vectors

Let

- $Sx = Ax + b$ and
- $Tx = Bx + b$

Ex. Show that $A \leqslant B$ implies that T dominates S on P

Proof: Fix $x \in P$

Since $A \leqslant B$ and $x \geqslant 0$, we have $Ax \leqslant Bx$

Hence $Sx \leqslant Tx$

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Conjecture: If $S \preceq T$, then the fixed points of T will be larger

This is <u>not</u> true in general...

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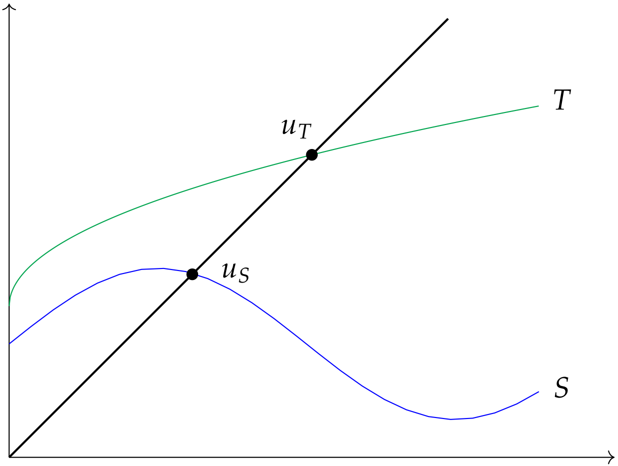
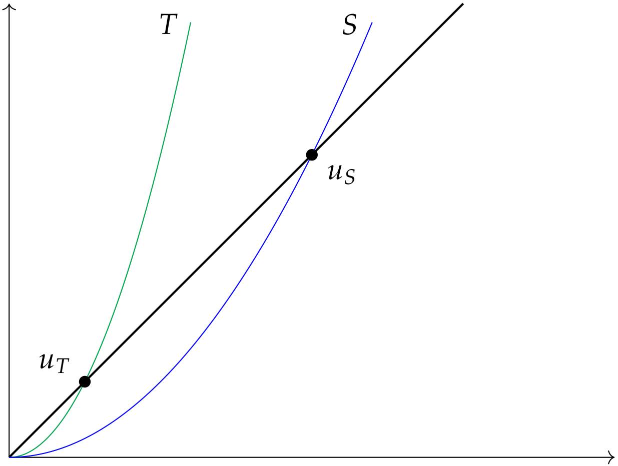
Sometimes true:



And sometimes false:



One difference: in the first case, T is globally stable

This leads us to our next result

Proposition. Let S and T be self-maps on $M \subset \mathbb{R}^n$

Let $\leqslant$ be the pointwise partial order over vectors

If

1. T dominates S on M and
2. T is order-preserving and globally stable on M,

then the unique fixed point of T dominates any fixed point of S

Proof: Assume the conditions

Let

- u_T be the unique fixed point of T and
- u_S be any fixed point of S

Since $S \leqslant T$, we have $u_S = Su_S \leqslant Tu_S$

Applying T to both sides of $u_S \leqslant Tu_S$ gives $Tu_S \leqslant T^2 u_S$

But then $u_S \leqslant T^2 u_S$

Continuing in this fashion yields $u_S \leqslant T^k u_S$ for all $k \in \mathbb{N}$

Since $\leqslant$ is preserved under limits and T is globally stable,

$$u_S \leqslant \lim_k T^k u_S = u_T$$

Example. Recall that, in the job search model,

$$h^* = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, h^* \right\} \varphi(w')$$

We found h^* as the fixed point of $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$g(h) = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(w')$$

In the exercise, you showed that g is a contraction map on $\mathbb{R}_+$

Ex. Prove that the optimal continuation value h^* is increasing in β

Proof: Fix $\beta_1 \leqslant \beta_2$ and let

- $h_i^* :=$ fixed point corresponding to β_i
- $g_i :=$ fixed point map corresponding to β_i

Since $\beta_1 \leqslant \beta_2$, we have $g_1(h) \leqslant g_2(h)$ for all $h \in \mathbb{R}_+$

In addition,

1. g_2 is a contraction (so globally stable) and
2. g_2 is increasing

Hence $h_1^* \leqslant h_2^*$

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