An Introduction to Computational Macroeconomics

Dynamic Programming: Chapter 2

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Introduction

Summary of this lecture:

- TBA
- TBA

Markov Dynamics

Our next task is to review Markov dynamics

- An essential workhorse for countless models in
  - economics
  - finance,
  - operations research, etc.
- Very general – can handle most processes of interest
- Elegant theory
- Natural fit for dynamic programming (Markov decisions)

Reminders

Recall that $(\lambda, v) \in \mathbb{C} \times \mathbb{C}^n$ is an **eigenpair** of $n \times n$ matrix A if

$$v \neq 0 \quad \text{and} \quad Av = \lambda v$$

Fix eigenvalue λ and let

$$E_\lambda := \{w \in \mathbb{C}^n : w = 0 \text{ or } (\lambda, w) \text{ is an eigenpair of } A\}$$

Ex. Show that E_λ is a linear subspace of $\mathbb{C}^n$

Proof: If $v, w \in E_\lambda$ and $\alpha, \beta \in \mathbb{C}$, then

$$A(\alpha w + \beta v) = \alpha A w + \beta A v = \alpha \lambda w + \beta \lambda v = \lambda(\alpha w + \beta v)$$

Hence $\alpha w + \beta v \in E_\lambda$

Implication: $\exists$ continuum of eigenvectors paired with λ

So what can we say about uniqueness?

Let (λ, v) be an eigenpair for A

We say that v has (geometric) **multiplicity one** if $\dim E_\lambda = 1$

In other words,

$$w \in E_\lambda \implies w = \alpha v \text{ for some } \alpha \in \mathbb{C}$$

In a sense, there is "just one" eigenvector corresponding to λ, since any other is a scalar multiple

Nonnegative Matrices

We call a matrix A **nonnegative** and write $A \geqslant 0$ if all the elements of A are nonnegative

A is called **positive**, and we write $A \gg 0$, if every element of A is strictly positive

A nonnegative square matrix A is called **irreducible** if

$$\sum_{k \in \mathbb{N}} A^k \gg 0$$

(Stronger than nonnegativity but weaker than positivity)

A nonnegative square matrix P is called **stochastic** if $P\mathbb{1} = \mathbb{1}$

Let A be $n \times n$

It is <u>not</u> always true that $r(A)$ is an eigenvalue of A

Example. Let

$$A := \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Eigenvalues of A are $\{-1, 0\}$

Hence $r(A) = |-1| = 1$, which is not an eigenvalue of A

However, when $A \geqslant 0$, we have the following result

Theorem. (Perron–Frobenius) If $A \geqslant 0$, then $r(A)$ is an eigenvalue of A with nonnegative, real-valued right and left eigenvectors

In particular, there exists

- a nonnegative, nonzero column vector e s.t. $Ae = r(A)e$
- a nonnegative, nonzero row vector ε s.t. $\varepsilon A = r(A)\varepsilon$

If A **is irreducible**, then these eigenvalues are everywhere positive and have multiplicity of one

If A **is positive**, then with e and ε such that $\langle \varepsilon, e \rangle = 1$, we have

$$r(A)^{-t} A^t \to e\,\varepsilon^\top \qquad (t \to \infty)$$

Bounds on the Spectral Radius

Fix $n \times n$ matrix $A = (a_{ij})$ and set

- $\mathrm{rs}_i(A) :=$ the i-th row sum of A and
- $\mathrm{cs}_j(A) :=$ the j-th column sum of A

Lemma. If $A \geqslant 0$, then

1. $\min_i \mathrm{rs}_i(A) \leqslant r(A) \leqslant \max_i \mathrm{rs}_i(A)$ and
2. $\min_j \mathrm{cs}_j(A) \leqslant r(A) \leqslant \max_j \mathrm{cs}_j(A)$

Ex. Prove this. (Solved exercise in text)

Ex. Show that P is stochastic $\implies r(P) = 1$

Existence of Stationary Distributions

Let P be a stochastic matrix

Ex. Prove: $\exists$ a row vector $\psi \in \mathbb{R}^n_+$ such that $\psi \mathbb{1} = 1$ and $\psi P = \psi$

Proof:

By the Perron–Frobenius theorem, there exists a nonzero, nonnegative row vector φ satisfying $\varphi P = \varphi$

Since φ is nonzero, $\varphi \mathbb{1} > 0$

Setting $\varphi := \varphi / (\varphi \mathbb{1})$ gives the desired vector

The vector φ is called then **stationary distribution** of P

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Uniqueness of Stationary Distributions

Ex. Prove: If P is also **irreducible**, then the stationary vector ψ is everwhere positive and unique

Proof of Positivity: See Perron–Frobenius theorem

Proof of Uniqueness: Let $\varphi \geqslant 0$ satisfy $\varphi \mathbb{1} = 1$ and $\varphi P = \varphi$

By the Perron–Frobenius theorem, $\varphi = \alpha \psi$ for some $\alpha > 0$

But then $\varphi \mathbb{1} = \alpha \psi \mathbb{1} = 1$ and $\psi \mathbb{1} = 1$

Hence $\alpha = 1$

Hence $\varphi = \psi$

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```
julia> P = [0.2 0.8;
            0.1 0.9]
2×2 Matrix{Float64}:
    0.2  0.8
    0.1  0.9

julia> using QuantEcon

julia> mc = MarkovChain(P)
    Discrete Markov Chain
    stochastic matrix of type Matrix{Float64}:
    [0.2 0.8; 0.1 0.9]

julia> is_irreducible(mc)
true

julia> stationary_distributions(mc)
    1-element Vector{Vector{Float64}}:
    [0.1111111111111111, 0.8888888888888888]
```

A Local Spectral Radius Theorem

Lemma. Let $\| \cdot \|$ be any norm on $\mathbb{R}^n$

If A is $n \times n$, $A \geqslant 0$ and $h \gg 0$, then

$$\|A^k h\|^{1/k} \to r(A) \qquad (k \to \infty)$$

Interpretation: eventually, for any positive h, the norm of the vector $A^k h$ grows at rate $r(A)$

Remarks

- Similar to Gelfand's formula
- but uses a vector norm rather than a matrix norm

Lake Model of Employment

An illustration of the Perron–Frobenius theorem

We analyze a model of employment and unemployment flows in a large population

The model is sometimes called a "lake model"

Two pools of workers:

- those who are currently employed and
- those who are currently unemployed but still seeking work

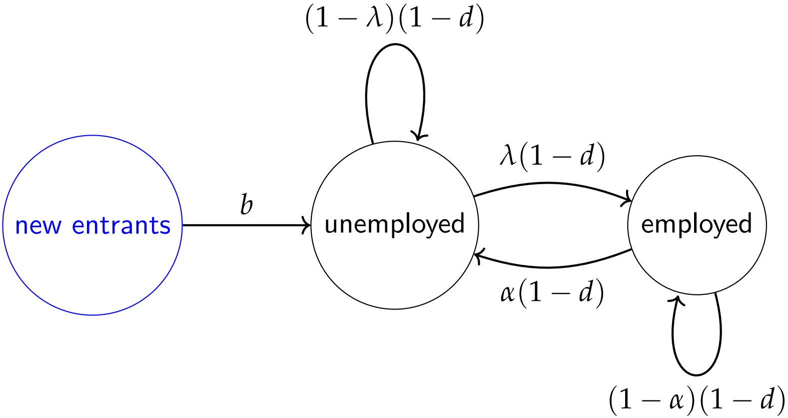
FP theorem helps us analyze dynamics

The flows between states are as follows:

- workers **exit** the labor market at rate d
- new workers **enter** the labor market at rate b
- employed workers **separate** from their jobs and become unemployed at rate α
- unemployed workers **find jobs** at rate λ

We assume that all of these parameters lie in $(0, 1)$

New workers are initially unemployed


$(1-\lambda)(1-d)$
new entrants
b
unemployed
$\lambda(1-d)$
$\alpha(1-d)$
employed
$(1-\alpha)(1-d)$

Let

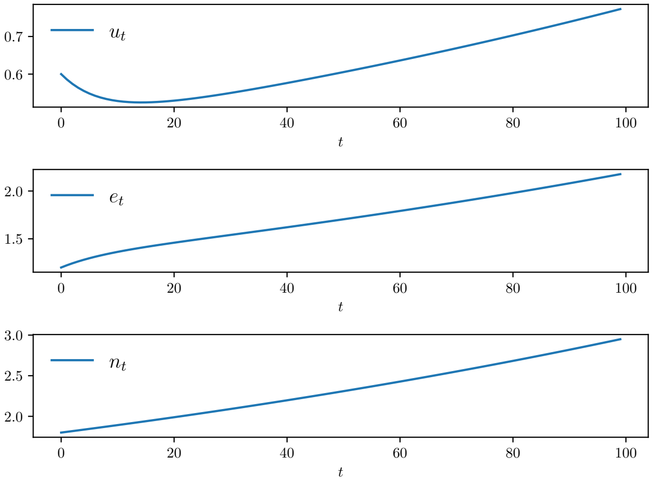
- $u_t :=$ number of unemployed workers at time t
- $e_t :=$ number of employed workers
- $n_t := e_t + u_t :=$ total population of workers

Dynamics are

$$u_{t+1} = (1-d)\alpha e_t + (1-d)(1-\lambda)u_t + bn_t$$

and

$$e_{t+1} = (1-d)(1-\alpha)e_t + (1-d)\lambda u_t$$



u_t
0.7
0.6
e_t
2.0
1.5
n_t
3.0
2.5
2.0
0
20
40
60
80
100
t

Can we say more about the dynamics of this system?

For example,

- what long run unemployment rate should we expect?
- do outcomes depend on the initial conditions u_0 and e_0?
- Or are there general statements we can make?

Let's organize the linear system for (e_t) and (u_t) by setting

$$x_t := \begin{pmatrix} u_t \\ e_t \end{pmatrix}$$

and

$$A := \begin{pmatrix} (1-d)(1-\lambda)+b & (1-d)\alpha+b \\ (1-d)\lambda & (1-d)(1-\alpha) \end{pmatrix}$$

With these definitions, we can write the dynamics as

$$x_{t+1} = Ax_t$$

As a result, $x_t = A^t x_0$, where $x_0 = (u_0 \; e_0)^\top$

Ex. With $g := b - d$, show that

$$n_{t+1} = (1 + g)n_t \text{ for all } t$$

Ex. Prove that $r(A) = 1 + g$

$$\text{PF theorem} \implies 1 + g \text{ is an eigenvalue of } A \text{ !}$$

Ex. Show that $\mathbb{1}^\top := (1\ 1)$ is a left eigenvector of $1 + g$

Ex. Prove that the unique vector $\bar{x}$ satisfying

$$A\bar{x} = r(A)\bar{x} \text{ and } \mathbb{1}^\top \bar{x}$$

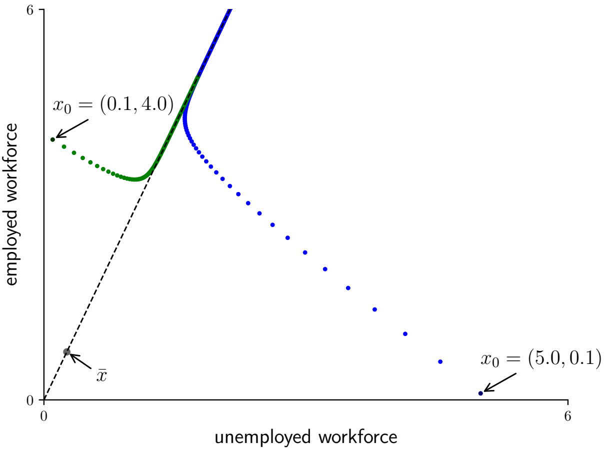
is given by

$$\bar{x} := \begin{pmatrix} \bar{u} \\ \bar{e} \end{pmatrix}$$

with

$$\bar{u} := \frac{1 + g - (1-d)(1-\alpha)}{1 + g - (1-d)(1-\alpha) + (1-d)\lambda}$$

and $\bar{e} := 1 - \bar{u}$



6
0
0
6
$x_0 = (0.1, 4.0)$
$x_0 = (5.0, 0.1)$
$\bar{x}$
employed workforce
unemployed workforce

Let

$$D := \{x \in \mathbb{R}^2 : x = \alpha \bar{x} \text{ for some } \alpha > 0\}$$

Shown as a dashed black line in the last figure

The two time paths are of the form $(x_t)_{t \geqslant 0} = (A^t x_0)_{t \geqslant 0}$

In both cases, the paths converge to D over time

Suggests all paths are "eventually almost" multiples of $\bar{x}$

How can we explain this strong regularity?

From the Perron–Frobenius theorem, since $A \gg 0$, we have

$$A^t \approx r(A)^t \cdot \bar{x}\mathbb{1}^\top = (1+g)^t \begin{pmatrix} \bar{u} & \bar{u} \\ \bar{e} & \bar{e} \end{pmatrix} \quad \text{for large } t$$

As a result, for any initial condition $x_0 = (u_0 \; e_0)^\top$, we have

$$A^t x_0 \approx (1+g)^t \begin{pmatrix} \bar{u} & \bar{u} \\ \bar{e} & \bar{e} \end{pmatrix} \begin{pmatrix} u_0 \\ e_0 \end{pmatrix}$$

$$= (1+g)^t (u_0 + e_0) \begin{pmatrix} \bar{u} \\ \bar{e} \end{pmatrix} = n_t \bar{x},$$

where $n_t = (1+g)^t n_0$ and $n_0 = u_0 + e_0$

Regardless of x_0, state scales along $\bar{x}$ at rate of population growth

We can give an additional interpretation to $\bar{u}$ and $\bar{e}$

Since $n_t =$ size of the workforce,

$$\text{unemployment rate} = u_t / n_t$$

As just shown, for large t, we have $u_t \approx n_t \bar{u}$

Hence

$$\text{unemployment rate } \approx (n_t \bar{u}) / n_t = \bar{u}$$

Hence $\bar{u}$ is the long term rate of unemployment

Similarly, $\bar{e}$ is the long run employment rate

Extensions

Further analysis: think about how α, λ, b and d are determined

For the hiring rate λ, we could use the job search model

In particular, with w^* as the reservation wage, we could set

$$\lambda = \mathbb{P}\{w_t \geqslant w^*\} = \sum_{w \in \mathsf{W}} \varphi(\mathbb{1}\{w \geqslant w^*\})$$

Doing so would allow us to study the crucial rate λ in terms of fundamental primitives, such as

- unemployment compensation
- impatience of individual agents, etc.