APMA 4301 Project: Discretizing Navier-Stokes Equations

AKSHAY SRIPADA, as4928

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1 Introduction

The Navier-Stokes equations are a set of partial differential equations that represent fluid motion. The equations were developed by Claude-Louis Navier and George Gabriel Stokes. They can be viewed as the Newton's second law for fluid motion. The set of equations have a wide variety of applications in the fields of weather prediction, ocean currents, fluid dynamics and aerodynamics. For the scope of this project, only incompressible flow is considered. Incompressible flow are the class of fluid that do not change density over time and space [1,2]. The general form of the equations in two dimensions (x, y) are given bellow:

$$\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} - \frac{\partial (u^2)}{\partial x} - \frac{\partial (uv)}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial P}{\partial y} - \frac{\partial (uv)}{\partial x} - \frac{\partial (v^2)}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

Where:

- u - Velocity in the x-direction
- v - Velocity in the y-direction
- t - Time
- P - Pressure
- Re - Reynolds Number

The first two equations are the momentum equations in the x and y directions. The third equation represents the conservation of mass and is known as the continuity equation. It is evident that these sets of equations are non-linear and thus very difficult to solve. The existence and uniqueness of a solution for the equations are still an open problem, it is listed as one of the Millennium Prize Problems. Even though we do not completely understand these equations they are widely used. There are a lot of simpler cases of the equations which can be derived using particular boundary conditions and other restrictions. This project will explore the different types of flows between two parallel plates, also known as Couette Flow. The main aspects are how the plate velocity and pressure gradient affect the flow. Figure 1 shows the basic setup for flow between two plates. The cross section is assumed to be $x \in [0,1]$ and $y \in [0,1]$. So both L and D are 1 for this whole study.

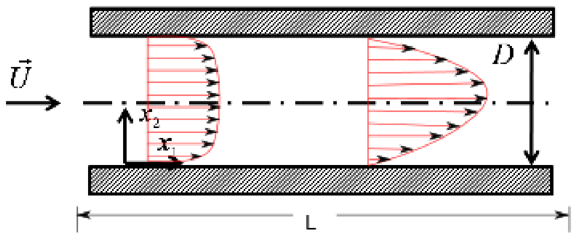


Figure 1: Basic setup of the fluid flow problem between two parallel plates

2 Steady State Coutte Flow ($u_t = 0$)

Couette flow is the flow of fluid between two parallel plates, one of which is moving relative to the other. The flow occurs due to the drag force between the stationary fluid and the moving plate. The presence of a pressure gradient also introduces another flow that introduces flow. The simplest case to consider is where the fluid has no velocity in the y-direction ($v = 0$) and thus the continuity equation simplifies to:

$$\frac{\partial u}{\partial x} = 0 \tag{4}$$

This indicates that the velocity u is only changing in the y-direction. Thus the momentum equation reduces to:

$$0 = -\frac{dP}{dx} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2}\right) \tag{5a}$$

$$u_{yy} = Re\frac{dP}{dx} \tag{5b}$$

The equation is very simple in this case. Since the velocity only changes in y-direction and the pressure gradient is kept constant, we can easily derive the analytical solution by integrating the equation twice. The following equation shows the general solution:

$$u(y) = \frac{Re}{2}\frac{dP}{dx}y^2 + c_1 y + c_2 \tag{6}$$

Where c_1 and c_2 depend on the boundary conditions.
We can implement a simple finite difference formulation to obtain a formulation for the velocity which can be solved iteratively using Gauss-Sidel. The numerical approximation is shown below:

$$u_j = \frac{u_{j-1} + u_{j+1}}{2} - \frac{h^2}{2Re}\frac{dP}{dx} \tag{7}$$

Where the j's represent the grid locations in the y-direction and h is the step size used in the grid. Using the numerical approximation we can estimate the velocity profile for the three cases of Coutte flow.

2.1 Moving Top Plate

The equations stated above can be used to model the velocity profile of a fluid flow between two parallel plates with the top plate moving with a initial velocity. The boundary conditions for this case simplify to $u(y=0)=0$ and $u(y=1)=u_0$. Reynolds number is assumed to be 100 in default cases, but there are studies for different values in later sections. Thus the analytical solution is:

$$u(y) = u_0 y - \frac{Re}{2}\frac{dP}{dx}y(1-y) \tag{8}$$

The Figure 2 compares the approximation to the analytical solution. The simplest case would be where the pressure gradient is zero. This results in a straight line as the only force acting on the fluid is the friction from the moving top plate.

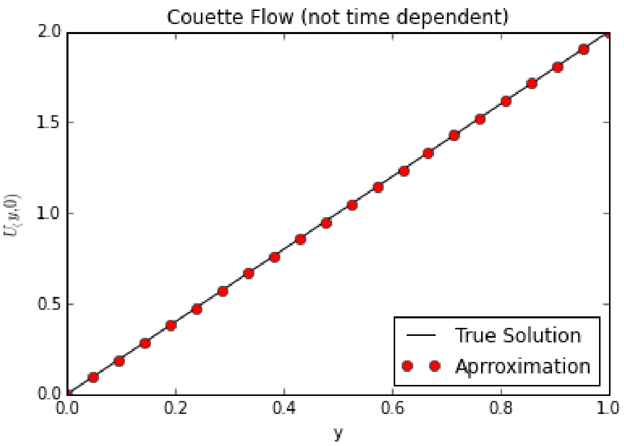


Figure 2: Solution for moving top plate with no pressure gradient

2.2 Moving Bottom Plate

Another scenario is the velocity profile of a fluid flow between two parallel plates with the bottom plate moving with a initial velocity. The boundary conditions for this case simplify to

$u(y=0)=u_0$ and $u(y=1)=0$. Thus the analytical solution is:

$$u(y) = u_0(1-y) - \frac{Re}{2}\frac{dP}{dx}y(y-1) \tag{9}$$

Figure 3 compares the approximation to the analytical solution. The simplest case would be where the pressure gradient is zero. This results in a straight line as the only force acting on the fluid is the friction from the moving top plate. But the slope of the line is the exact opposite when compared to the previous case.

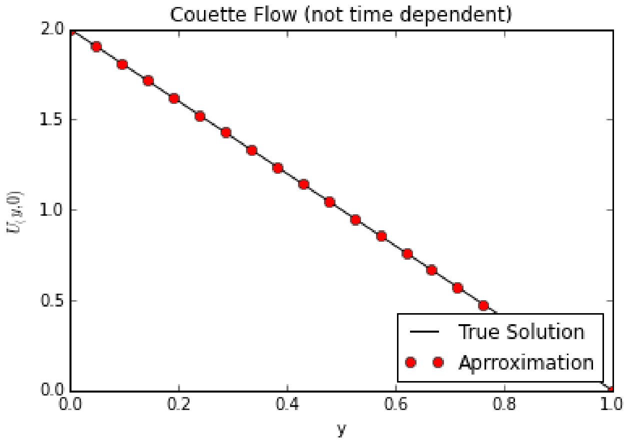


Figure 3: Solution for moving bottom plate with no pressure gradient

2.3 Stationary Plates

This scenario is not technically a type of Couette flow, but is an interesting case because the fluid flow is intiated due to the pressure gradient. The boundary conditions for this case simplify to $u(y=0)=0$ and $u(y=1)=0$. The pressure gradient is set to be $-5Pa$. This means that the left side is at higher pressure and the right side is at lower pressure. The fluid flows from the higher potential to the lower potential side. Thus the analytical solution is:

$$u(y) = \frac{Re}{2}\frac{dP}{dx}y(y-1) \tag{10}$$

The Figure 4 compares the approximation to the analytical solution. The resulting velocity profile looks like a parabola with the maximum velocity at $y=0.5$.

2.4 Pressure Gradients

The approximations shown above are accurate but the cases are very simple and usually resolve to simple straight lines or parabolas. The more interesting cases are where we have a pressure gradient and this affects the velocity profile. We have already seen that in the stationary plates case but it is interesting to see how an added pressure gradient affects the velocity profile when the plates are also moving. Figure 5 shows how the change in pressure gradient affects the velocity profile for the three different cases mentioned above. The velocity profiles

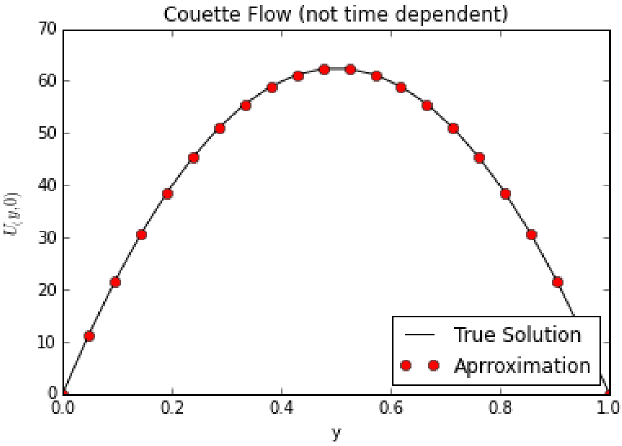
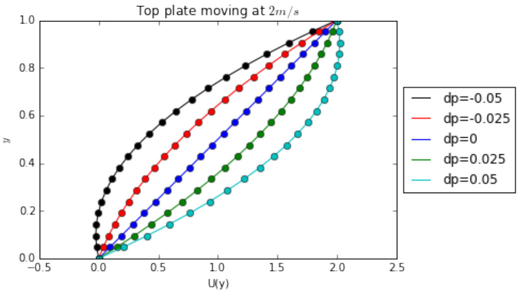
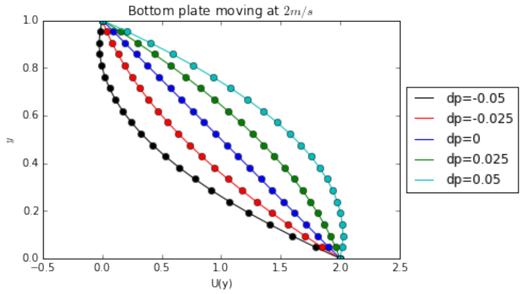


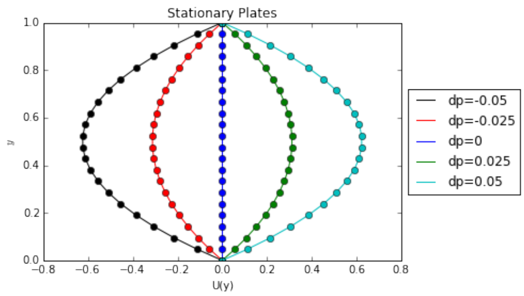
Figure 4: Solution for stationary plates with pressure gradient of $5Pa$



(a) Top plate moving



(b) Bottom plate moving

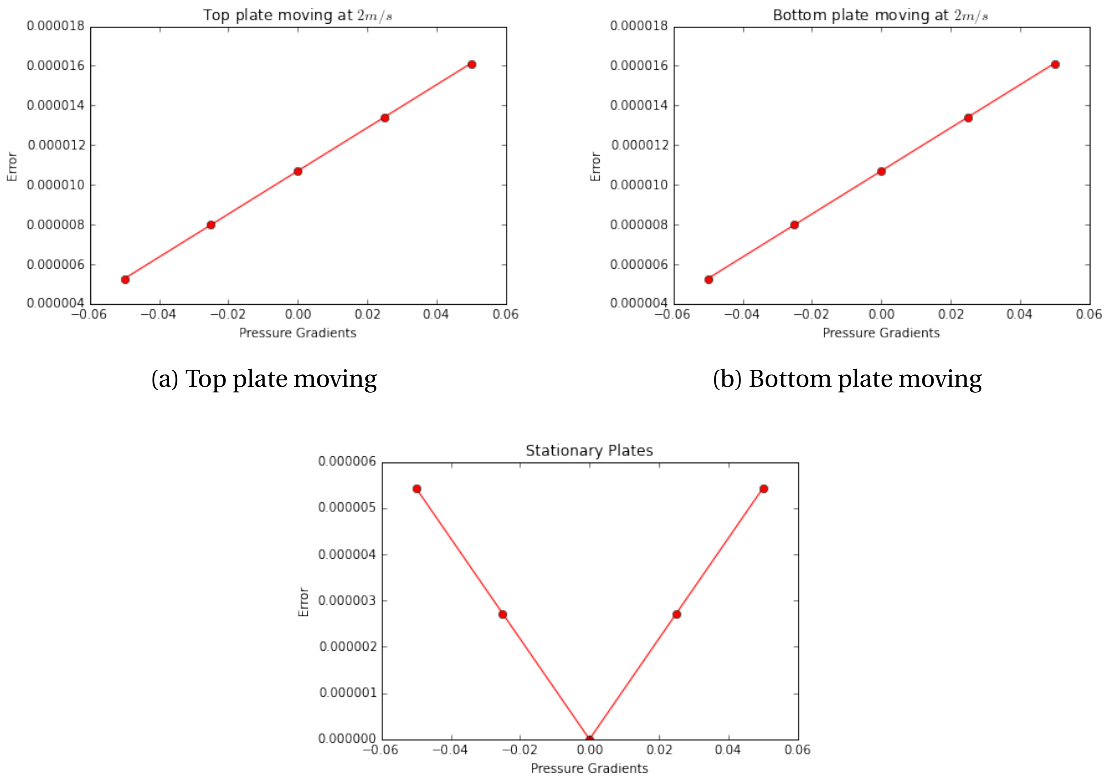


(c) Stationary plates

Figure 5: Velocity profile for different pressure gradients

are no longer simple straight lines as the pressure gradient introduces the non-linear features.

Figure 6 shows the change in error when the pressure gradients are changed. Since the pressure gradient is a constant it does affect the accuracy of the solution, but as we can see from the figure it is not a huge factor.



(a) Top plate moving

(b) Bottom plate moving

(c) Stationary plates

Figure 6: Error for different pressure gradients

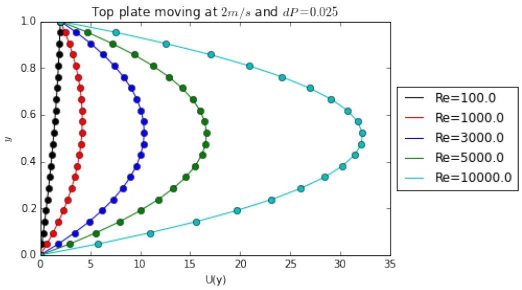
2.5 Reynolds Number

The other major factor in the velocity profile is the Reynolds number. Physically the Reynolds number represents how viscous a fluid is, the more viscous the fluid, higher the resistance to moementum. But we often see in fluid dynamics that high Reynolds number indicate turbulent flow and this usually means that there are some features in the flow that are very hard to accurately track. Figure 7 shows how different Reynolds number affects the velocity profile of the fluid.

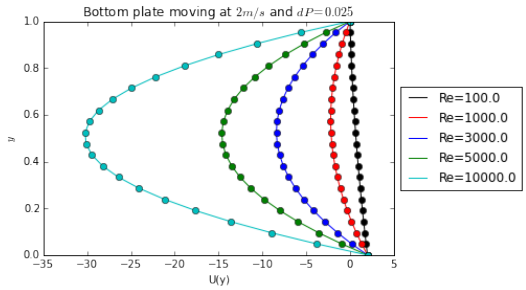
The model is very simple and thus the error is affected more by the Reynolds number. Figure 8 show how the accuracy of the model is affected with the change in the Reynolds number. We saw in Figure 6 that the pressure gradients do not have a big impact on the error, but the Reynolds Number does play a major role. There is an increase order of magnitude of the error when the flow transitions from a Laminar flow ($Re < 2300$) to a Turbulent flow ($Re > 4000$).

3 Time dependent Coutte Flow

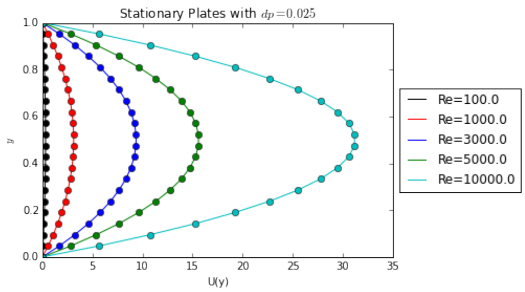
The next class of problems in this study are the same problems mentioned above but now with a time dependence. We can study the features as they evolve in time. We still study the problems only in one dimension, thus we have the same continuity equation, and the



(a) Top plate moving



(b) Bottom plate moving



(c) Stationary plates

Figure 7: Velocity profile for different Reynolds Number

velocity is restricted to the y-direction. For time dependent problems, we no longer assume that $u_t = 0$ but we assume there is no pressure gradient and the velocity in y-direction is 0 ($v = 0$) and thus we get the following diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} \tag{11}$$

The analytical solution [3] in the general form is shown below:

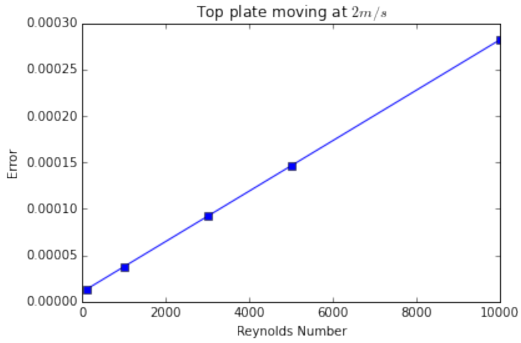
$$u = u_0\left(c_1 + c_2 erf\left(\frac{\sqrt{Re}y}{2\sqrt{t}}\right)\right) \tag{12}$$

Where c_1 and c_2 depend on the boundary conditions of the given problem and the erf function is the Gaussian error function defined as the following:

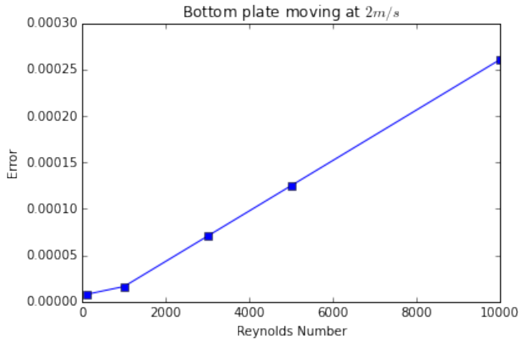
$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx \tag{13}$$

We implement the Crank-Nicholson method to numerically solve the diffusion or elliptical equation. Crank-Nicholson was chosen as this method is implicit and it allows for much larger time steps than other explicit methods [4]. The general setup is explained in further details in the equations below:

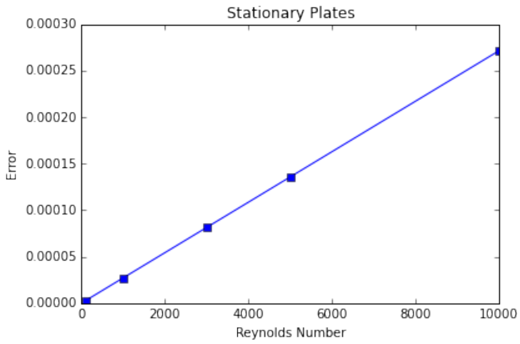
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{2Re}(f(U_j^n) + f(U_j^{n+1})) \tag{14a}$$



(a) Top plate moving



(b) Bottom plate moving



(c) Stationary plates

Figure 8: Error for different Reynolds Numbers

$$f(U_j^n) + f(U_j^{n+1}) = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n + U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{\Delta y^2} \tag{14b}$$

Using the finite differences we can write an update formula and represent it in a matrix form of $AU^{n+1} = f(t_n, U^n)$

$$-cU_{j-1}^{n+1} + (1+2c)U_j^{n+1} - cU_{j+1}^{n+1} = cU_{j-1}^n + (1-2c)U_j^n + cU_{j+1}^n \tag{15a}$$

$$c = \frac{\Delta t}{2Re\Delta y^2} \tag{15b}$$

$$A = \begin{bmatrix} 1+2c & -c & & & & \\ -c & 1+2c & -c & & & \\ & -c & 1+2c & -c & & \\ & & \ddots & \ddots & \ddots & \\ & & & -c & 1+2c & -c \\ & & & & -c & 1+2c \end{bmatrix} \tag{15c}$$

$$f(t_n, U^n) = \begin{bmatrix} c(g_0(t_n) + g_0(t_{n+1})) + (1-2c)U_1^n + cU_2^n \\ cU_1^n + (1-2c)U_2^n + cU_3^n \\ cU_2^n + (1-2c)U_3^n + cU_4^n \\ \vdots \\ cU_{m-2}^n + (1-2c)U_{m-1}^n + cU_m^n \\ cU_{m-1}^n + (1-2c)U_m^n + c(g_1(t_n) + g_1(t_{n+1})) \end{bmatrix} \tag{15d}$$

The most interesting part of these time dependent flow models are the Boundary Layers. These effects aren't evident in the steady state problems but we can see the Boundary layers evolve over time. We can have two different types of boundary layers depending on the Reynolds Number. The Laminar boundary layer is smooth and occurs at lower Reynolds number. But at high reynolds number or even when the boundary layer thickens we can transition to a Turbulent boundary layer which is more unstable and usually contains swirls. We examine these features and effect of Reynolds number further in the following sections.

3.1 Moving Top Plate

Similar to how we analyzed the steady state flow we can study the time dependent velocity profile for the flow with a moving top plate. The boundary conditions are the same but we need to introduce an initial condition $u(0, y) = u_0$. The mentioned conditions also represent a case where both the plates were moving at the initial velocity but the bottom plate is decelerated to zero velocity. Using the boundary conditions we can derive the analytical solution to be:

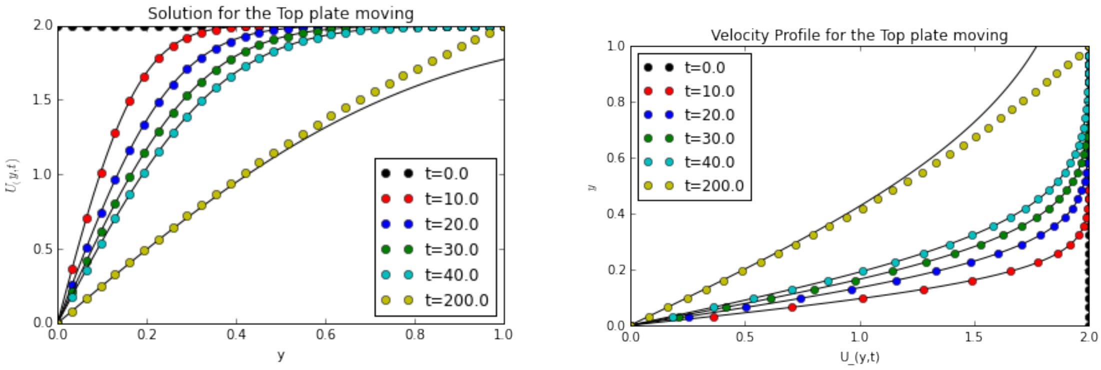
$$u = u_0 erf\left(\frac{\sqrt{Re}y}{2\sqrt{t}}\right) \tag{16}$$

Figure 9 compares the approximated solution and the velocity profile using the Crank-Nicholson method and the true solution. The approximation is accurate and we can see that the fluid is dragged along by the top plate. The momentum is transferred slowly to the rest of the fluid and the boundary layer thickens over time.The velocity profile is evolving towards the steady state solution shown in the previous section.

3.2 Moving Bottom Plate

The other type of Couette flow is when the bottom plate is moving. The boundary conditions are similar to the steady state problem, but we do need a initial condition for the time dependent problem. The initial condition for this case is $u(0, y) = 0$. This is different from the initial conditions for the top plate moving. Using the boundary conditions we obtain a analytical solutions as shown below:

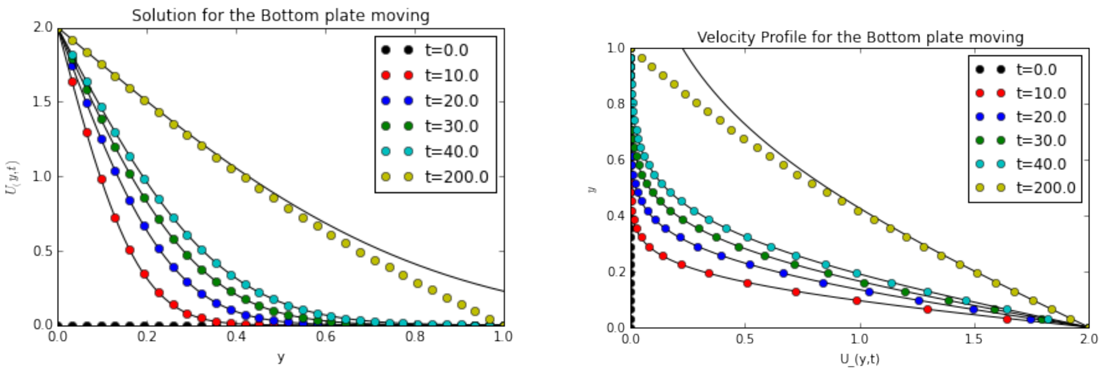
$$u = u_0\left(1 - erf\left(\frac{\sqrt{Re}y}{2\sqrt{t}}\right)\right) \tag{17}$$



(a) Solution (b) Velocity Profile

Figure 9: Time dependent solution for moving top plate

Figure 10 compares the approximated solution and the velocity profile using the Crank-Nicholson method and the true solution. The approximation is accurate and we can see that the momentum of the plate is slowly transferred to the fluid and as time passes the boundary layer thickens. Eventually the velocity profile looks at lot like the profile we saw with the steady state solutions.



(a) Solution (b) Velocity Profile

Figure 10: Time dependent solution for moving bottom plate

3.3 Error and Stability

The Crank-Nicholson method is second order accurate in both time and space. Figure 11 shows a convergence study for the solutions for the case where the bottom plate has a velocity. The convergence is very similar for the case with the top plate moving.

The Crank-Nicholson method is stable for diffusion equations, but in this model the stability does depend on the Reynolds Number. The following equations show the Von-Nueman analysis and the amplification factor for these particular class of problems.

$$g(\xi) = \frac{1+z}{1-z} \tag{18a}$$

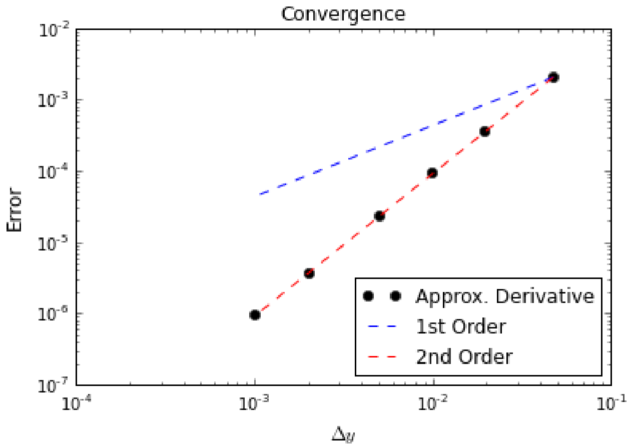


Figure 11: Convergence Study

$$z = \frac{\Delta t}{Re\Delta y^2}\left(cos(\xi\Delta y) - 1\right) \tag{18b}$$

The z value ranges from $\left(\frac{-2\Delta t}{Re\Delta y^2}, 0\right)$. Since z values are always negative we conclude that the method is unconditionally stable. Even with this finding Reynolds number is still a very interesting parameter to study since the transition from Laminar to Turbulent flow always introduces very interesting but complicated features. Figure 12 shows how the error is affected by the change in Reynolds number for the case with the moving bottom plate. We do observe that the error does worsen when the flow gets in to the turbulent region, but since we are only studying 1-D affects we don't see too much discrepancy.

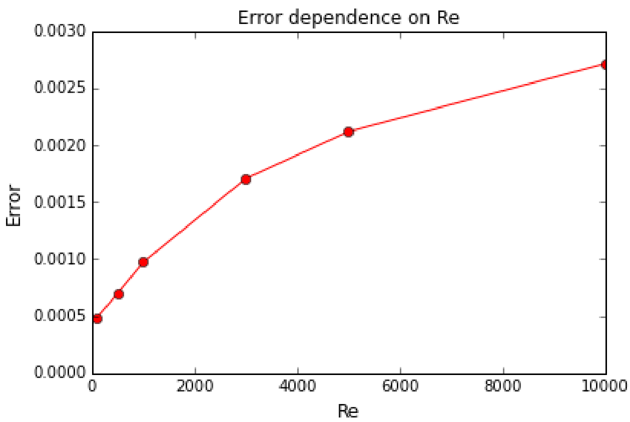


Figure 12: Change in error with change in Reynolds Number

4 Conclusion

The study successfully shows good numerical methods to model Couette flow in one dimension for both steady state and time dependent problems. The tools developed [5] help understand how the basic concepts of how fluid flow develops between two parallel plates. Various scenarios can be easily modeled. Important factors such as pressure gradients and Reynolds numbers are also introduced and used to analyze the stability of the numerical model. There are additional methods such as Back Euler implemented to solve the diffusion equation.

The study needs to be expanded to 2-D problems to study more interesting features such as velocity curls which blow up when transitioning from Laminar to Turbulent flow. The code developed has a section dedicated to this based on some code found by a earlier project [6], but unfortunately the code couldn't be implemented in time. There are numerous challenges for numerical approximations since 2-D Navier-Stokes equation involve non-linear terms. There are various applications of staggered grids that have shown promising results for handling non-linear velocity approximations.

References

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