Lecture 4: Faraday's law and induction

Learning outcomes

Assessable

- Use Faraday's law and Lenz's law to calculate induced current and emf.
- Calculate energy transfer due to induction.

Understanding

- Visualize and conceptually explain induced currents and electric fields.

1. Summary

This lecture will cover Faraday's law, Lenz's law, and induction and energy transfer.

Consider a loop of conducting wire connected to an ammeter. If we move a bar magnet towards the loop, we will observe:

1. A current appears only when the magnetic is moving relative to the loop.
2. Faster movement causes a greater current.
3. The direction of current differs depending on whether the north or south pole of the magnet is towards the loop.

The current produced in the loop is called **induced current**. The work done per unit charge to produce the current is called the **induced emf**.

If we use a second loop of conducting wire connected to a current source (battery) to generate a magnetic dipole near the loop connected to the ammeter, we observe the same actions as above with the addition that:

4. A current is only observed in the first loop when during brief periods when the current in the second loop is switch on an off. No induced current is observed in the first loop when the current in the second loop is in a steady state.

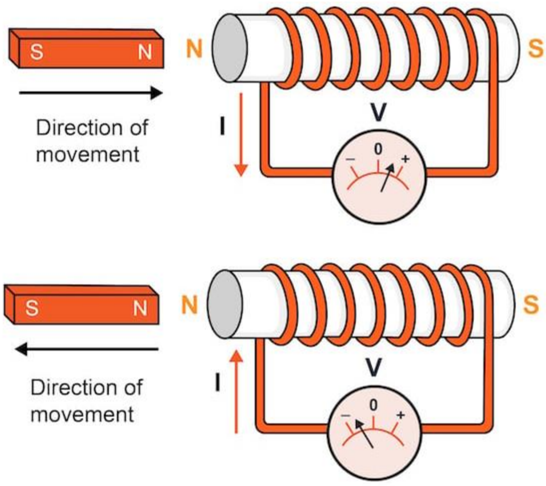

Figure 1: A bar magnetic being used to induced a current in a coil. [Image: All About Circuits]

2. Faraday's law of induction

Induced emf and current is caused by changing the **amount of magnetic field** passing through the conducting loop. This is **Faraday's law of induction**.

We refer to the *amount of magnetic field* as **magnetic flux** which is defined as:

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \tag{4.1}$$

where $\vec{B}$ is the magnetic field enclosed by a conducting loop of area A.

> In other units, you may see electric flux defined similarly as:
>
> $$\Phi_E = \int \vec{E} \cdot d\vec{A}$$

In the case where the magnetic field is perpendicular to the plane of the loop, we can write the dot product as $B\ dA\ \cos 0 = B\ dA$. If the magnetic field is uniform, then $\vec{B}$ can be brought out the front of the integral and we get:

$$\Phi_B = BA \tag{4.2}$$

From eqtn 4.2 we can see that the unit of magnetic flux must be tesla-square metres, which is also called **weber**.

$$1\ weber = 1\ Wb = 1\ T \cdot m^2 \tag{4.3}$$

> **Faraday's law of induction:**
> The magnitude of emf $\mathcal{E}$ induced in a conducting loop is equal to the rate at which the magnetic flux Φ_B through the loop changes with time.

Formally, we write Faraday's law as:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{4.4}$$

The minus sign indicated that the induced emf **acts to oppose the change of flux** in the loop.

In an ideal (closely packed) coil of N turns, the equation becomes:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \tag{4.5}$$

There are many ways to change the flux through a coil, including:

1. Change the magnitude of B,
2. Change the area of the coil, or the region of the coil exposed to the magnetic field, and
3. Change the angle of the plane of the coil relative to $\vec{B}$.

2.1. Example – induced emf due to a solenoid

A long solenoid S has diameter D of 3.2 cm, 220 turns/cm and carries a current i = 1.5 A. Within the solenoid, we place a smaller coil of wire of diameter d = 2.1 cm and 130 turns. The current in the solenoid is switched off and decreases to zero over a period of 25 ms. What is the magnitude of the emf in the smaller coil?

From solenoids (lecture 3) we know that the initial magnetic field is

$$B = \mu_0 in$$

So,

$$\Phi_B = (\mu_0 in)A = (4\pi \times 10^{-7})(1.5)(22000)(3.464 \times 10^{-4}) = 1.44 \times 10^{-5}\, Wb$$

Since the current in the solenoid decreases at a steady rate, the magnetic flux will decrease at a steady rate and so we can write.

$$\frac{d\Phi_B}{dt} = \frac{\Delta\Phi_B}{\Delta t} = \frac{(0 - 1.44 \times 10^{-5})}{0.025} = -5.76 \times 10^{-4}\, Wb/s = -5.76 \times 10^{-4}\, V$$

We're only interested in the magnitude for this question, so we'll ignore the minus sign.

$$\mathcal{E} = N\frac{d\Phi_B}{dt} = (130)(5.76 \times 10^{-4}) = 7.5 \times 10^{-2} V = 75\, mV$$

3. Lenz's Law

Lenz's law:
An induced current has a direction such that the magnetic field due to the current opposes the change in the magnetic flux that induces the current.

The magnetic field produced by an induced current will act to oppose the motion of the magnetic or change of magnetic flux. The direction of the induced current can the be found using the right-hand-rule.

Lenz's law is the reason for the negative sign in eqtn 4.4 and 4.5.

3.1. Example – induced current due to a changing uniform magnetic field

Figure shows a conducting loop comprising a half-circle of radius r = 0.2 m and three straight sections. The half circle lies in a uniform magnetic field that is directed into the page. The magnitude of the magnetic field is $B = 4t^2 + 2t + 3$. An ideal battery with emf $\mathcal{E}_{batt}$ = 2 V is connected to the loop. The resistance of the loop is 2 ohms.

(a) What are the magnitude and direction of the emf at t = 10s?

(b) What is the current in the loop at t = 10 s?

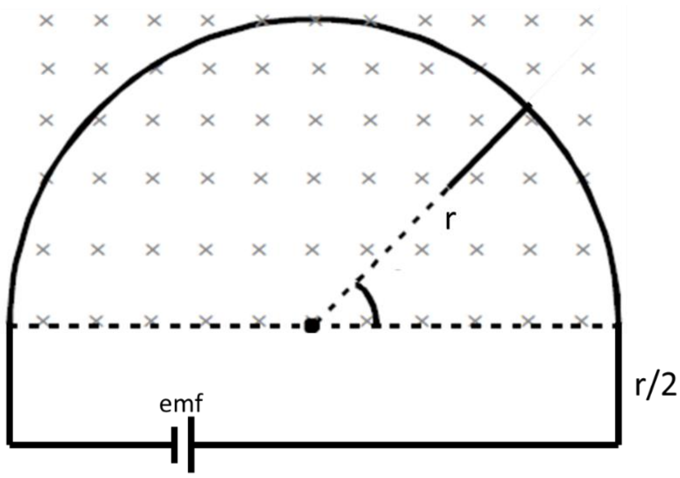

Figure 2: Loop for example question.

Using Faraday's law:

$$\mathcal{E} = \frac{d\Phi_B}{dt} = \frac{d(BA)}{dt} = A\frac{dB}{dt}$$

Since the magnetic field is only in the half-circle portion of the loop, the area is $\frac{1}{2}\pi r^2$.

$$\mathcal{E} = A\frac{dB}{dt} = \frac{\pi r^2}{2}\frac{d}{dt}(4t^2 + 2t + 3) = \frac{\pi r^2}{2}(8t + 2)$$

So, at t = 10 s:

$$\mathcal{E} = \frac{\pi r^2}{2}(8(10) + 2) = 5.152\ V$$

Since the flux is into the page, and the induced emf acts to oppose the change, the direction of the induced field must be out of the page. Thus, the induced current is anti-clockwise around the loop (and so is the induced emf).

There are two emfs, the induced emf and the emf due to the battery. The battery emf also drives a current anti-clockwise around the loop, so:

$$i = \frac{\mathcal{E}_{net}}{R} = \frac{\mathcal{E}_{induced} + \mathcal{E}_{batt}}{R} = \frac{5.152 + 2}{2} = 1.58\ A$$

3.2. Example – induced emf due to a changing non-uniform current

A rectangular loop of wire of height (y-axis) H = 2 m and width (x-axis) W = 3 m is in a nonuniform varying magnetic field. The field is perpendicular to the plane of the loop and is directed into the page. The magnitude of the field is given by $B = 4t^2x^2$. What is the magnitude and direction of the induced emf at t = 0.1 s?

Since the field is perpendicular to the area of the loop, the dot product is simply $B\ dA$. Since the magnetic field varies with x but not y, the differential area becomes $dA = H\ dx$.

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \int B\ dA = \int BH\ dx = \int 4t^2x^2\ H\ dx$$

$$\Phi_B = 4t^2H\int_0^3 x^2\,dx = 4t^2H\left[\frac{x^3}{3}\right]_0^3 = 72t^2$$

Using Faraday's law:

$$\mathcal{E} = \frac{d\Phi_B}{dt} = \frac{d(72t^2)}{dt} = 144t$$

So, at t = 0.1 s:

$$\mathcal{E} = 144(0.1) = 14.4\,V$$

The flux is into the page and is increasing with time. The emf will act to oppose the increase and so is directed out of the page. Using the right-hand-rule we find that the induced current is anti-clockwise around the loop, and so is the induced emf.

4. Inductors and energy transfer

Consider moving a bar magnet near a conducting loop. Because (due to Lenz's law) the induced magnetic field resists the motion of the magnet, a force needs to be applied to the magnet, and so work is done. The energy/work you expend in moving the magnet near the loop ends up as energy in the loop. (Which would be dissipated as heat due to the resistance of the loop if you had no other devices connected.)

Consider a rectangular loop of wire being moved out of a uniform magnetic field as shown in Figure 3. To pull the loop out at constant velocity $\vec{v}$ you must apply a constant force $\vec{F}$ because a magnetic force of equal magnitude and direction acts to oppose the motion. The rate at which you do work (the power) is given by:

$$P = Fv$$

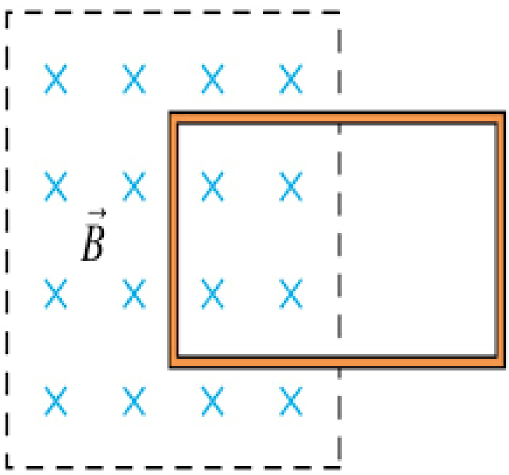

Figure 3: Conducting loop passing through a magnetic field.

We need to find an equation for P in terms of the strength of the magnetic field, and the resistance and dimensions of the loop.

Using Faraday's law, if a length x of the loop (the height of the loop being called L) is in the magnetic field then the area of the loop in the field is Lx. Using eqtn 4.2:

$$\Phi_B = BA = BLx$$

As the loop is drawn out of the field, x decreases, so the flux decreases, so an emf will be induced. Ignoring the minus sign in eqtn 4.4:

$$\mathcal{E} = \frac{d\Phi_B}{dt} = \frac{d}{dt}BLx = BL\mathcal{E}\frac{dx}{dt} = BLv$$

The direction of the current and emf can be found from the right-hand-rule. The current is

$$i = \frac{BLv}{R}$$

Because there is a current in the loop, there will be a resulting force from the magnetic field according to

$$\overrightarrow{F_B} = i\vec{L} \times \vec{B}$$

In our case, the forces at the top and bottom of the loop will cancel out, leaving only the force on the left side of the loop remaining in the field. This force is directed in opposition to the force being used to removed the loop from the field, thus:

$$F = iLB \sin 90 = iLB$$

Substituting in i from above:

$$F = \frac{B^2L^2v}{R}$$

In this scenario, B, L, v and R are all constants, so F is constant.

The rate at which work is done on the system (the power) is then:

$$P = Fv = \frac{B^2L^2v^2}{R}$$

We can also confirm this another way, by calculating the thermal energy dissipated in the loop by the resistance to the current.

$$P = i^2R = \left(\frac{BLv}{R}\right)^2 R = \frac{B^2L^2v}{R}$$

The thermal energy dissipated is equal to the work done on the loop, so the work done on the loop appears as thermal energy in the loop.

4.1. Eddy currents

Imagine replacing the loop in Figure 3 with a solid conducting plate (e.g. a sheet of copper). As the sheet is withdrawn from the field a current is induced in the conducting plate, however, the electrons do not all follow the same path. The electrons move about in paths called **eddy currents**. These eddy currents dissipate the energy of the movement of the plate as heat.

5. Induced electric fields

Consider a cooper ring of radius r placed in a unform magnetic field of radius R. Consider the scenario where the magnetic field is increasing magnitude. By Faraday's law, an induced emf and current will appear in the ring.

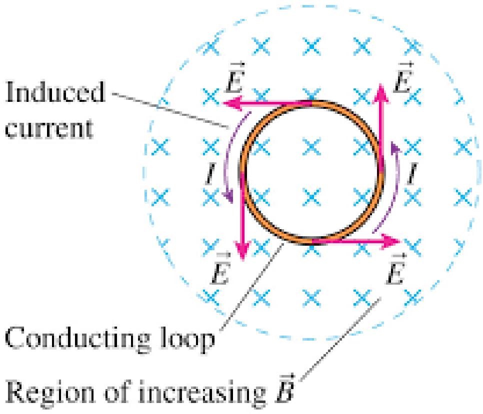

Figure 4: Induced electric field due to a changing magnetic field.

If there is current through the cooper ring, there must be an electric field present long the ring because an electric field is needed to do the work of moving the conduction electrons. This **induced electric field** has been produced by the changing magnetic flux.

This leads us to a restatement of Faraday's law:

Faraday's law:
A changing magnetic field produces an electric field.

This statement holds even if there is no conductive ring – i.e. even if the changing magnetic field were in a vacuum.

5.1. Reformulating Faraday's law

Consider a particle of charge q moving around the circular path in Figure 4. The work done on the particle in one revolution is

$$W = \mathcal{E}q$$

where $\mathcal{E}$ is the induced emf.

The work can also be expressed:

$$W = \int \vec{F} \cdot d\vec{s} = (qE)(2\pi r) \qquad (4.6)$$

where qE is the force on the particle and $2\pi r$ is the distance over which the force acts.

Equating the two formulae for the work done:

$$\mathcal{E} = 2\pi r E \tag{4.7}$$

A more general expression for the work done on a charged particle moving along any closed path is:

$$W = \oint \vec{F} \cdot d\vec{s} = q \oint \vec{E} \cdot d\vec{s} \tag{4.8}$$

Substituting $W = \mathcal{E}q$ we find:

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{s} \tag{4.9}$$

A subtly of this result is that **an induced emf can exist without the need for a current or a particle**. An induced emf is the sum (via integration) of the quantities $\vec{E} \cdot d\vec{s}$ around a closed path.

Combining eqtn 4.9 with Faraday's law, we rewrite Faraday's law as:

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{4.10}$$

which simply means that a changing magnetic field induces an electric field.

Faraday's law as expressed in eqtn 4.10 can be applied to any closed path in a changing magnetic field.

5.2. A new look at electric potential

Electric fields can be produced in two ways – 1) by a static electric charge, or 2) by a changing magnetic field. The difference between the two is that, for the case of a static charge, electric field lines start and end at points of positive and negative charge, while induced electric fields caused by a changing magnetic field must form closed loops, as shown in Figure 4.

> Electric potential only makes meaningful sense for electric fields produced by static charges. Electric potential for electric fields that are produced by induction is meaningless.

For example, a particle moving due to the emf of a battery arrives at a point of higher potential, and work has been done on the particle to move it. In the case of induced emf, such as shown in Figure 4, a particle making a complete revolution arrives back at its original position, which cannot have two different values for electric potential.

Thus, electric potential has no meaning for electric fields that are set up by changing magnetic fields.

5.3. Example – induced electric field

In Figure 4, take the radius R of the magnetic field to be 8.5 cm and dB/dt to be 0.13 T/s.

(a) Find the magnitude E of the induced electric field at the ring of radius r = 5.2 cm.

(b) Find the magnitude E of the induced electric field in a ring of radius r = 12.5 cm, i.e. a ring that is larger than the magnetic field.

We apply Faraday's law using the formulation in eqtn 4.10:

$$\oint \vec{E} \cdot d\vec{s} = \oint E \cdot ds = E \oint ds = E(2\pi r) = -\frac{d\Phi_B}{dt}$$

We then use eqtn 4.2. Since the magnetic field is uniform:

$$\Phi_B = BA = B(\pi r^2)$$

Substituting this into the above equation for Faraday's law (and ignoring the minus sign):

$$E(2\pi r) = (\pi r^2)\frac{dB}{dt}$$

which rearranges to

$$E = \frac{r}{2}\frac{dB}{dt}$$

We then substitute in the given values:

$$E = \frac{5.2 \times 10^{-2}}{2} 0.13 = 0.0034\, V/m$$

The derived formula gives the magnitude of the electric field in cases where $r < R$. We now need to find the relation for $r > R$. Equation 4.2 becomes:

$$\Phi_B = BA = B(\pi R^2)$$

Substituting this into Faraday's law and rearranging for E gives

$$E = \frac{R}{2r}\frac{dB}{dt}$$

Substituting in the given values we get

$$E = \frac{8.5 \times 10^{-2}}{2(12.5 \times 10^{-2})} 0.13 = 0.0038\, V/m$$

6. Summary of key formulae from lecture 4

Magnetic flux

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \qquad (4.1)$$

$$\Phi_B = BA \qquad (4.2)$$

Faraday's law

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \qquad (4.4)$$

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \tag{4.5}$$

Induced electric field

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{s} \tag{4.9}$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{4.10}$$