APPENDIX A

THREE-PHASE CIRCUITS

Almost all electric power generation and most of the power transmission in the world today is in the form of three-phase ac circuits. A three-phase ac power system consists of three-phase generators, transmission lines, and loads. Ac power systems have a great advantage over dc systems in that their voltage levels can be changed with transformers to reduce transmission losses, as described in Chapter 2. *Three-phase* ac power systems have two major advantages over single-phase ac power systems: (1) it is possible to get more power per kilogram of metal from a three-phase machine and (2) the power delivered to a three-phase load is constant at all times, instead of pulsing as it does in single-phase systems. Three-phase systems also make the use of induction motors easier by allowing them to start without special auxiliary starting windings.

A.1 GENERATION OF THREE-PHASE VOLTAGES AND CURRENTS

A three-phase generator consists of three single-phase generators, with voltages equal in magnitude but differing in phase angle from the others by 120°. Each of these three generators could be connected to one of three identical loads by a pair of wires, and the resulting power system would be as shown in Figure A–1c. Such a system consists of three single-phase circuits that happen to differ in phase angle by 120°. The current flowing to each load can be found from the equation

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} \tag{A–1}$$

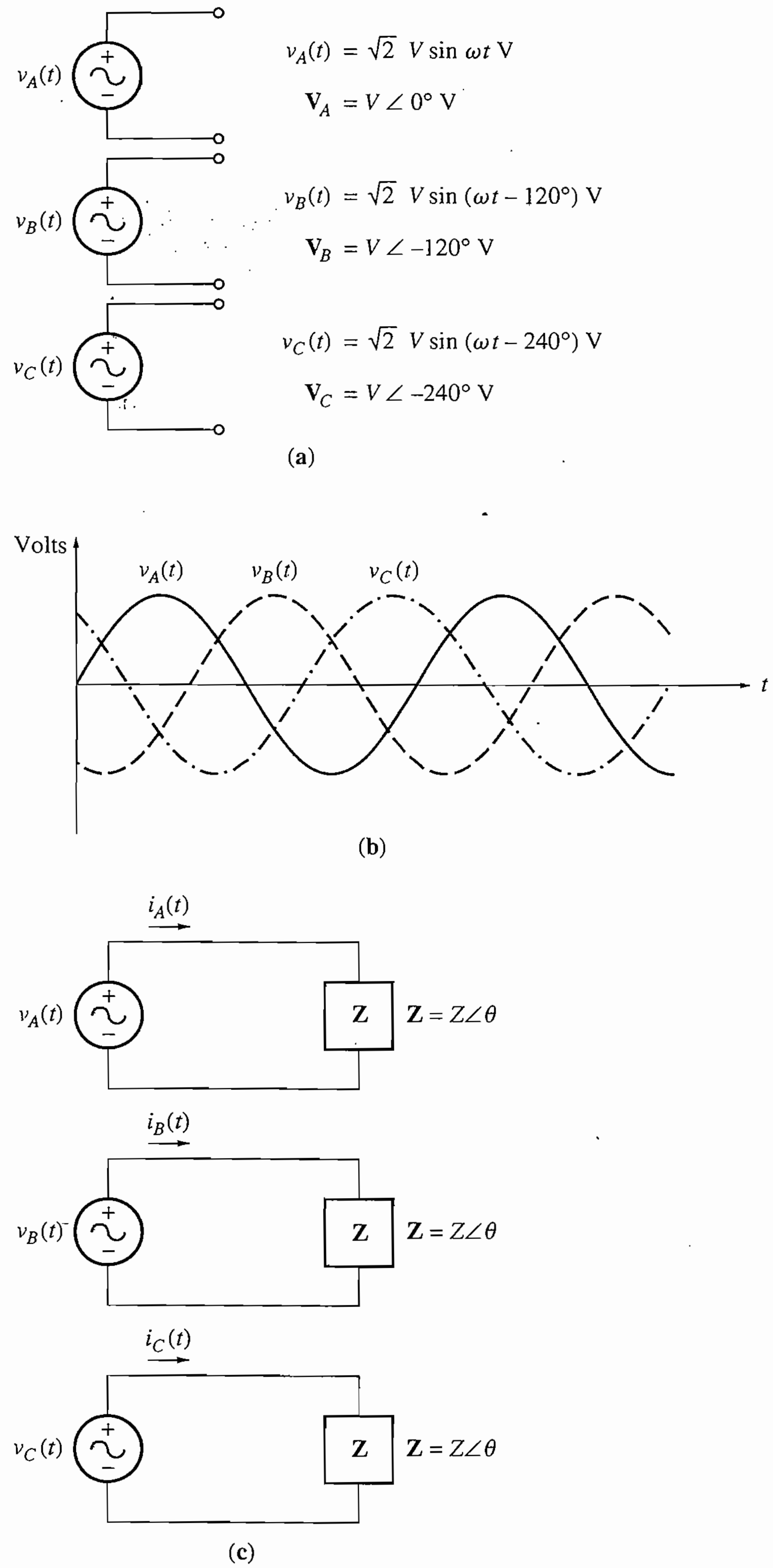

FIGURE A–1
(a) A three-phase generator, consisting of three single-phase sources equal in magnitude and 120° apart in phase. (b) The voltages in each phase of the generator. (c) The three phases of the generator connected to three identical loads.

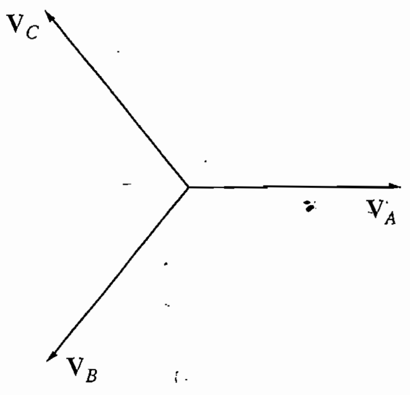

FIGURE A–1 (concluded)
(d) Phasor diagram showing the voltages in each phase.

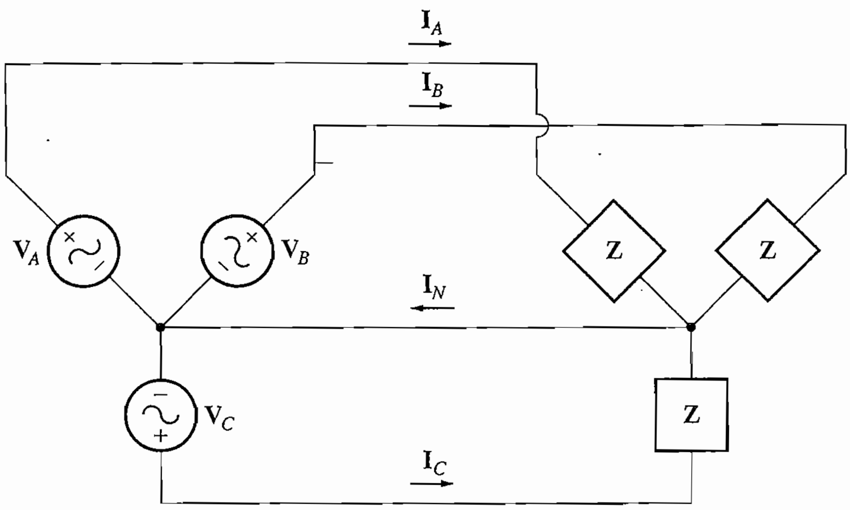

FIGURE A–2
The three circuits connected together with a common neutral.

Therefore, the currents flowing in the three phases are

$$\mathbf{I}_A = \frac{V\angle 0°}{Z\angle\theta} = I\angle -\theta \tag{A–2}$$

$$\mathbf{I}_B = \frac{V\angle -120°}{Z\angle\theta} = I\angle -120° - \theta \tag{A–3}$$

$$\mathbf{I}_C = \frac{V\angle -240°}{Z\angle\theta} = I\angle -240° - \theta \tag{A–4}$$

It is possible to connect the negative ends of these three single-phase generators and loads together, so that they share a common return line (called the *neutral*). The resulting system is shown in Figure A–2; note that now only *four* wires are required to supply power from the three generators to the three loads.

How much current is flowing in the single neutral wire shown in Figure A–2? The return current will be the sum of the currents flowing to each individual load in the power system. This current is given by

$$
\begin{aligned}
\mathbf{I}_N &= \mathbf{I}_A + \mathbf{I}_B + \mathbf{I}_C \qquad\qquad (A\text{–}5)\\
&= I\angle -\theta + I\angle -\theta - 120^\circ + I\angle -\theta - 240^\circ\\
&= I\cos(-\theta) + jI\sin(-\theta)\\
&\quad + I\cos(-\theta - 120^\circ) + jI\sin(-\theta - 120^\circ)\\
&\quad + I\cos(-\theta - 240^\circ) + jI\sin(-\theta - 240^\circ)\\
&= I\,[\cos(-\theta) + \cos(-\theta - 120^\circ) + \cos(-\theta - 240^\circ)]\\
&\quad + jI\,[\sin(-\theta) + \sin(-\theta - 120^\circ) + \sin(-\theta - 240^\circ)]
\end{aligned}
$$

Recall the elementary trigonometric identities:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad (A\text{–}6)$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad (A\text{–}7)$$

Applying these trigonometric identities yields

$$
\begin{aligned}
\mathbf{I}_N = {} & I[\cos(-\theta) + \cos(-\theta)\cos 120^\circ + \sin(-\theta)\sin 120^\circ + \cos(-\theta)\cos 240^\circ\\
& + \sin(-\theta)\sin 240^\circ]\\
& + jI[\sin(-\theta) + \sin(-\theta)\cos 120^\circ - \cos(-\theta)\sin 120^\circ\\
& + \sin(-\theta)\cos 240^\circ - \cos(-\theta)\sin 240^\circ]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_N = {} & I\left[\cos(-\theta) - \frac{1}{2}\cos(-\theta) + \frac{\sqrt{3}}{2}\sin(-\theta) - \frac{1}{2}\cos(-\theta) - \frac{\sqrt{3}}{2}\sin(-\theta)\right]\\
& + jI\left[\sin(-\theta) - \frac{1}{2}\sin(-\theta) - \frac{\sqrt{3}}{2}\cos(-\theta) - \frac{1}{2}\sin(-\theta) + \frac{\sqrt{3}}{2}\cos(-\theta)\right]
\end{aligned}
$$

$$\mathbf{I}_N = 0\ \text{A}$$

As long as the three loads are equal, the return current in the neutral is zero! A three-phase power system in which the three generators have voltages that are exactly equal in magnitude and 120° different in phase, and in which all three loads are identical, is called a *balanced three-phase system*. In such a system, the neutral is actually unnecessary, and we could get by with only *three* wires instead of the original six.

PHASE SEQUENCE. The *phase sequence* of a three-phase power system is the order in which the voltages in the individual phases peak. The three-phase power system illustrated in Figure A–1 is said to have phase sequence *abc*, since the voltages in the three phases peak in the order *a*, *b*, *c* (see Figure A–1b). The phasor diagram of a power system with an *abc* phase sequence is shown in Figure A–3a.

It is also possible to connect the three phases of a power system so that the voltages in the phases peak in the order *a*, *c*, *b*. This type of power system is said to have phase sequence *acb*. The phasor diagram of a power system with an *acb* phase sequence is shown in Figure A–3b.

The result derived above is equally valid for both *abc* and *acb* phase sequences. In either case, if the power system is balanced, the current flowing in the neutral will be 0.

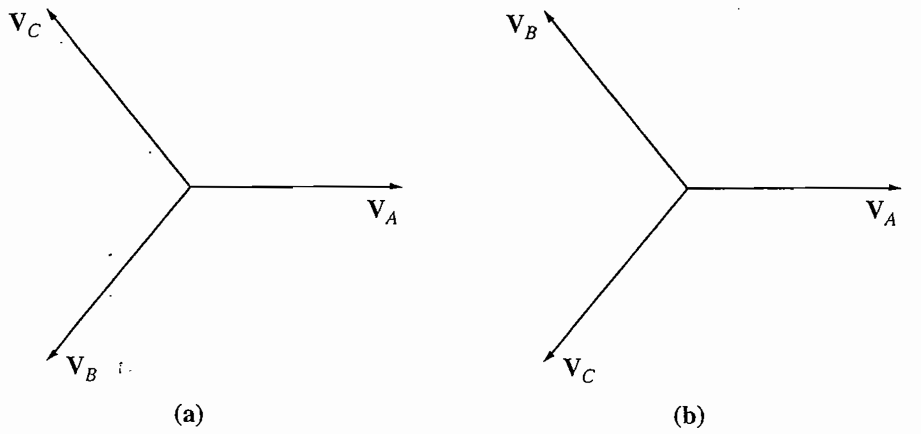

FIGURE A–3
(a) The phase voltages in a power system with an *abc* phase sequence. (b) The phase voltages in a power system with an *acb* phase sequence.

A.2 VOLTAGES AND CURRENTS IN A THREE-PHASE CIRCUIT

A connection of the sort shown in Figure A–2 is called a wye (Y) connection because it looks like the letter Y. Another possible connection is the delta (Δ) connection, in which the three generators are connected head to tail. The Δ connection is possible because the sum of the three voltages $\mathbf{V}_A + \mathbf{V}_B + \mathbf{V}_C = 0$, so that no short-circuit currents will flow when the three sources are connected head to tail.

Each generator and each load in a three-phase power system may be either Y- or Δ-connected. Any number of Y- and Δ-connected generators and loads may be mixed on a power system.

Figure A–4 shows three-phase generators connected in Y and in Δ. The voltages and currents in a given phase are called *phase quantities,* and the voltages between lines and currents in the lines connected to the generators are called *line quantities.* The relationship between the line quantities and phase quantities for a given generator or load depends on the type of connection used for that generator or load. These relationships will now be explored for each of the Y and Δ connections.

Voltages and Currents in the Wye (Y) Connection

A Y-connected three-phase generator with an *abc* phase sequence connected to a resistive load is shown in Figure A–5. The phase voltages in this generator are given by

$$\begin{aligned}\mathbf{V}_{an} &= V_\phi\angle 0^\circ \\ \mathbf{V}_{bn} &= V_\phi\angle -120^\circ \\ \mathbf{V}_{cn} &= V_\phi\angle -240^\circ\end{aligned} \tag{A–8}$$

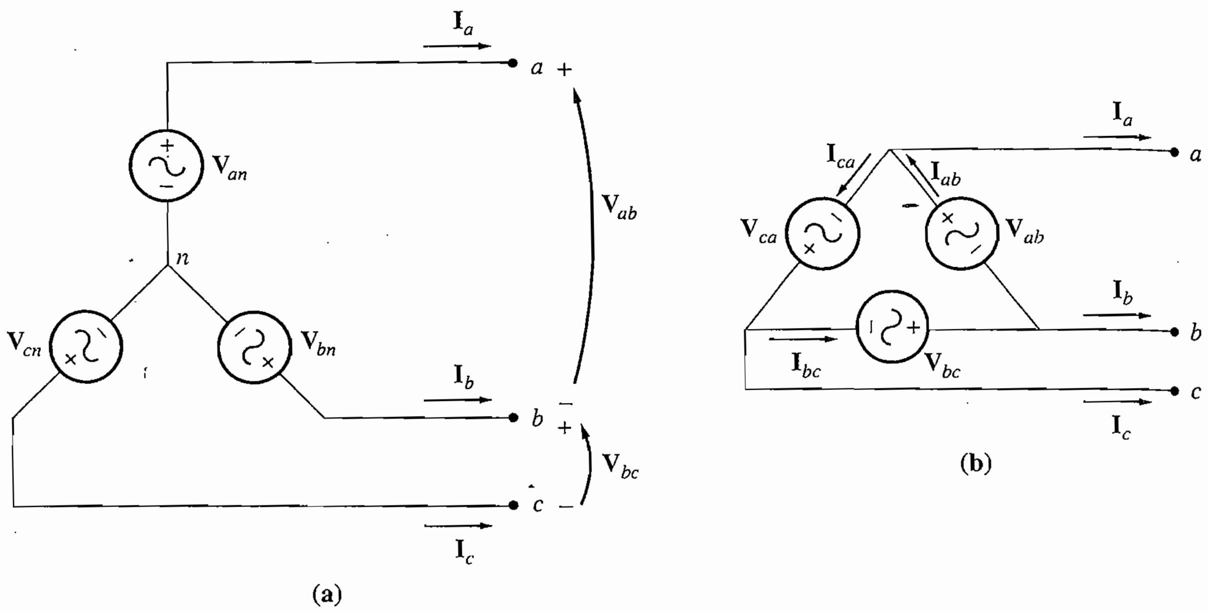

FIGURE A–4
(a) Y connection. (b) Δ connection.

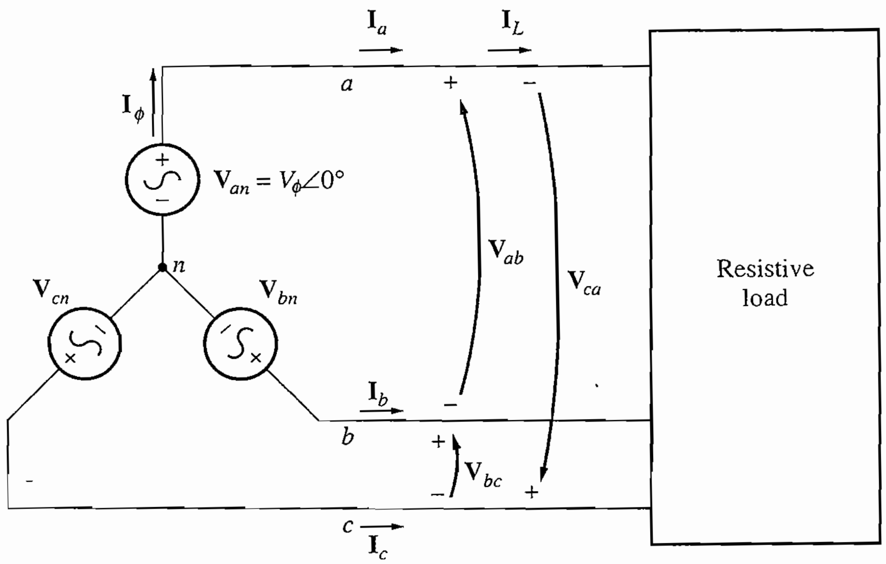

FIGURE A–5
Y-connected generator with a resistive load.

Since the load connected to this generator is assumed to be resistive, the current in each phase of the generator will be at the same angle as the voltage. Therefore, the current in each phase will be given by

$$
\begin{aligned}
\mathbf{I}_a &= I_\phi\angle 0° \\
\mathbf{I}_b &= I_\phi\angle -120° \\
\mathbf{I}_c &= I_\phi\angle -240°
\end{aligned}
\qquad \text{(A–9)}
$$

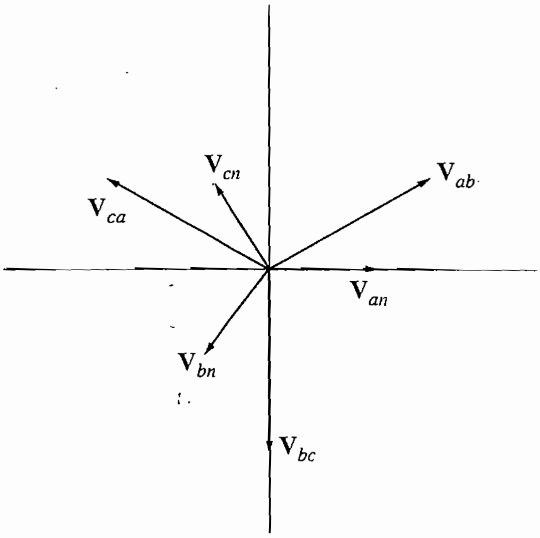

FIGURE A–6
Line-to-line and phase (line-to-neutral) voltages for the Y connection in Figure A–5.

From Figure A–5, it is obvious that the current in any line is the same as the current in the corresponding phase. Therefore, for a Y connection,

$$\boxed{I_L = I_\phi \qquad \text{Y connection}} \tag{A–10}$$

The relationship between line voltage and phase voltage is a bit more complex. By Kirchhoff's voltage law, the line-to-line voltage $\mathbf{V}_{ab}$ is given by

$$\begin{aligned}
\mathbf{V}_{ab} &= \mathbf{V}_a - \mathbf{V}_b \\
&= V_\phi \angle 0^\circ - V_\phi \angle -120^\circ \\
&= V_\phi - \left(-\frac{1}{2}V_\phi - j\frac{\sqrt{3}}{2}V_\phi\right) = \frac{3}{2}V_\phi + j\frac{\sqrt{3}}{2}V_\phi \\
&= \sqrt{3}V_\phi\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) \\
&= \sqrt{3}V_\phi \angle 30^\circ
\end{aligned}$$

Therefore, the relationship between the magnitudes of the line-to-line voltage and the line-to-neutral (phase) voltage in a Y-connected generator or load is

$$\boxed{V_{LL} = \sqrt{3}V_\phi \qquad \text{Y connection}} \tag{A–11}$$

In addition, the line voltages are shifted 30° with respect to the phase voltages. A phasor diagram of the line and phase voltages for the Y connection in Figure A–5 is shown in Figure A–6.

Note that for Y connections with the *abc* phase sequence such as the one in Figure A–5, the voltage of a line *leads* the corresponding phase voltage by 30°. For Y connections with the *acb* phase sequence, the voltage of a line *lags* the corresponding phase voltage by 30°, as you will be asked to demonstrate in a problem at the end of the appendix.

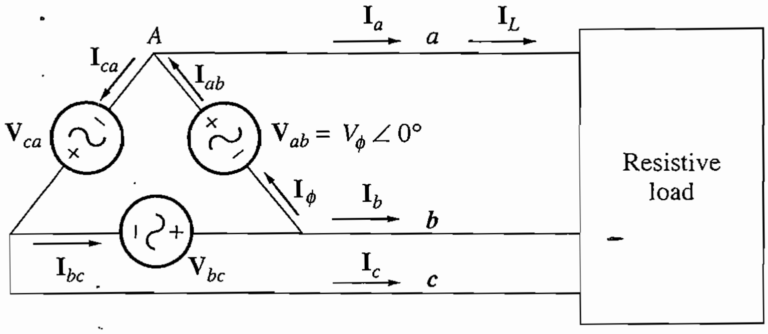

FIGURE A–7
Δ-connected generator with a resistive load.

Although the relationships between line and phase voltages and currents for the Y connection were derived for the assumption of a unity power factor, they are in fact valid for any power factor. The assumption of unity-power-factor loads simply made the mathematics slightly easier in this development.

Voltages and Currents in the Delta (Δ) Connection

A Δ-connected three-phase generator connected to a resistive load is shown in Figure A–7. The phase voltages in this generator are given by

$$\begin{aligned} \mathbf{V}_{ab} &= V_\phi \angle 0° \\ \mathbf{V}_{bc} &= V_\phi \angle -120° \\ \mathbf{V}_{ca} &= V_\phi \angle -240° \end{aligned} \tag{A–12}$$

Because the load is resistive, the phase currents are given by

$$\begin{aligned} \mathbf{I}_{ab} &= I_\phi \angle 0° \\ \mathbf{I}_{bc} &= I_\phi \angle -120° \\ \mathbf{I}_{ca} &= I_\phi \angle -240° \end{aligned} \tag{A–13}$$

In the case of the Δ connection, it is obvious that the line-to-line voltage between any two lines will be the same as the voltage in the corresponding phase. *In a Δ connection,*

$$\boxed{V_{LL} = V_\phi \qquad \Delta \text{ connection}} \tag{A–14}$$

The relationship between line current and phase current is more complex. It can be found by applying Kirchhoff's current law at a node of the Δ. Applying Kirchhoff's current law to node A yields the equation

$$\begin{aligned} \mathbf{I}_a &= \mathbf{I}_{ab} - \mathbf{I}_{ca} \\ &= I_\phi \angle 0° - I_\phi \angle -240° \\ &= I_\phi - \left(-\frac{1}{2} I_\phi + j\frac{\sqrt{3}}{2} I_\phi\right) = \frac{3}{2} I_\phi - j\frac{\sqrt{3}}{2} I_\phi \end{aligned}$$

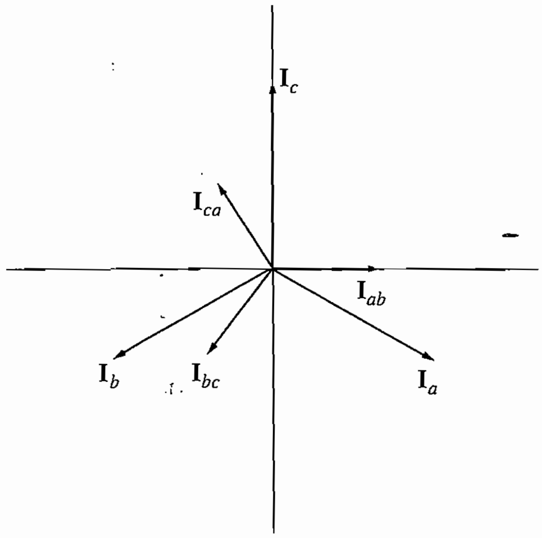

FIGURE A–8
Line and phase currents for the Δ connection in Figure A–7.

Table A–1
Summary of relationships in Y and Δ connections

	Y connection	Δ connection
Voltage magnitudes	$V_{LL} = \sqrt{3}\, V_\phi$	$V_{LL} = V_\phi$
Current magnitudes	$I_L = I_\phi$	$I_L = \sqrt{3}\, I_\phi$
abc phase sequence	$\mathbf{V}_{ab}$ leads $\mathbf{V}_a$ by 30°	$\mathbf{I}_a$ lags $\mathbf{I}_{ab}$ by 30°
acb phase sequence	$\mathbf{V}_{ab}$ lags $\mathbf{V}_a$ by 30°	$\mathbf{I}_a$ leads $\mathbf{I}_{ab}$ by 30°

$$= \sqrt{3}I_\phi\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)$$
$$= \sqrt{3}I_\phi\angle -30^\circ$$

Therefore, the relationship between the magnitudes of the line and phase currents in a Δ-connected generator or load is

$$\boxed{I_L = \sqrt{3}I_\phi \qquad \Delta \text{ connection}} \tag{A–15}$$

and the line currents are shifted 30° relative to the corresponding phase currents.

Note that for Δ connections with the *abc* phase sequence such as the one shown in Figure A–7, the current of a line *lags* the corresponding phase current by 30° (see Figure A–8). For Δ connections with the *acb* phase sequence, the current of a line *leads* the corresponding phase current by 30°.

The voltage and current relationships for Y- and Δ-connected sources and loads are summarized in Table A–1.

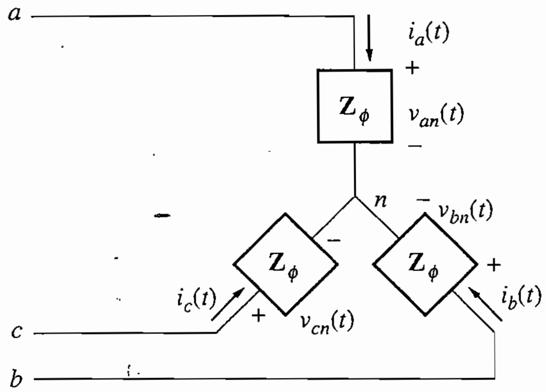

FIGURE A–9
A balanced Y-connected load.

A.3 POWER RELATIONSHIPS IN THREE-PHASE CIRCUITS

Figure A–9 shows a balanced Y-connected load whose phase impedance is $\mathbf{Z}_\phi = Z\angle\theta°$. If the three-phase voltages applied to this load are given by

$$\begin{aligned} v_{an}(t) &= \sqrt{2}V \sin \omega t \\ v_{bn}(t) &= \sqrt{2}V \sin(\omega t - 120°) \\ v_{cn}(t) &= \sqrt{2}V \sin(\omega t - 240°) \end{aligned} \tag{A–16}$$

then the three-phase currents flowing in the load are given by

$$\begin{aligned} i_a(t) &= \sqrt{2}I \sin(\omega t - \theta) \\ i_b(t) &= \sqrt{2}I \sin(\omega t - 120° - \theta) \\ i_c(t) &= \sqrt{2}I \sin(\omega t - 240° - \theta) \end{aligned} \tag{A–17}$$

where $I = V/Z$. How much power is being supplied to this load from the source?

The instantaneous power supplied to one phase of the load is given by the equation

$$\boxed{p(t) = v(t)i(t)} \tag{A–18}$$

Therefore, the instantaneous power supplied to each of the three phases is

$$\begin{aligned} p_a(t) &= v_{an}(t)i_a(t) = 2VI \sin(\omega t) \sin(\omega t - \theta) \\ p_b(t) &= v_{bn}(t)i_b(t) = 2VI \sin(\omega t - 120°) \sin(\omega t - 120° - \theta) \\ p_c(t) &= v_{cn}(t)i_c(t) = 2VI \sin(\omega t - 240°) \sin(\omega t - 240° - \theta) \end{aligned} \tag{A–19}$$

A trigonometric identity states that

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha - \beta)] \tag{A–20}$$

Applying this identity to Equations (A–19) yields new expressions for the power in each phase of the load:

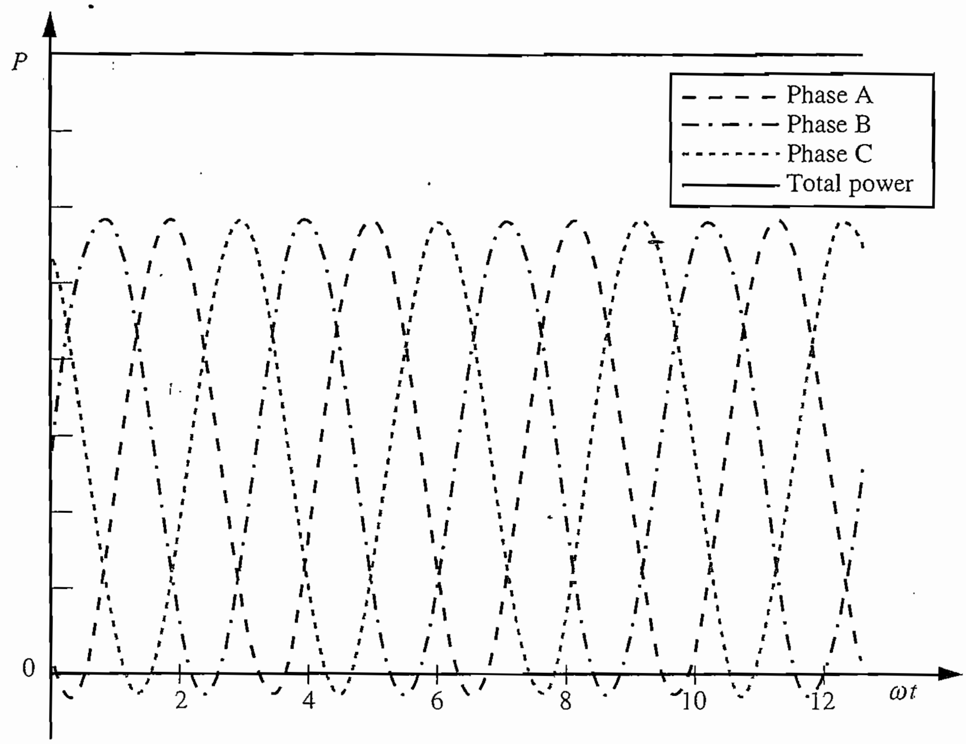

FIGURE A–10
Instantaneous power in phases *a, b,* and *c,* plus the total power supplied to the load.

$$
\begin{aligned}
p_a(t) &= VI[\cos\theta - \cos(2\omega t - \theta)] \\
p_b(t) &= VI[\cos\theta - \cos(2\omega t - 240^\circ - \theta)] \\
p_c(t) &= VI[\cos\theta - \cos(2\omega t - 480^\circ - \theta)]
\end{aligned}
\tag{A–21}
$$

The total power supplied to the entire three-phase load is the sum of the power supplied to each of the individual phases. The power supplied by each phase consists of a constant component plus a pulsing component. However, *the pulsing components in the three phases cancel each other out since they are 120° out of phase with each other*, and the final power supplied by the three-phase power system is constant. This power is given by the equation:

$$p_{\text{tot}}(t) = p_A(t) + p_B(t) + p_C(t) = 3VI\cos\theta \tag{A–22}$$

The instantaneous power in phases a, b, and c are shown as a function of time in Figure A–10. Note that *the total power supplied to a balanced three-phase load is constant at all times*. The fact that a constant power is supplied by a three-phase power system is one of its major advantages compared to single-phase sources.

Three-Phase Power Equations Involving Phase Quantities

The single-phase power Equations (1–60) to (1–66) apply to *each phase* of a Y- or Δ-connected three-phase load, so the real, reactive, and apparent powers supplied to a balanced three-phase load are given by

$$\boxed{P = 3V_\phi I_\phi \cos\theta} \tag{A–23}$$

$$\boxed{Q = 3V_\phi I_\phi \sin\theta} \tag{A–24}$$

$$\boxed{S = 3V_\phi I_\phi} \tag{A–25}$$

$$\boxed{P = 3I_\phi^2 Z \cos\theta} \tag{A–26}$$

$$\boxed{Q = 3I_\phi^2 Z \sin\theta} \tag{A–27}$$

$$\boxed{S = 3I_\phi^2 Z} \tag{A–28}$$

The angle θ is again the angle between the voltage and the current in any phase of the load (it is the same in all phases), and the power factor of the load is the cosine of the impedance angle θ. The power-triangle relationships apply as well.

Three-Phase Power Equations Involving Line Quantities

It is also possible to derive expressions for the power in a balanced three-phase load in terms of line quantities. This derivation must be done separately for Y- and Δ-connected loads, since the relationships between the line and phase quantities are different for each type of connection.

For a Y-connected load, the power consumed by a load is given by

$$P = 3V_\phi I_\phi \cos\theta \tag{A–23}$$

For this type of load, $I_L = I_\phi$ and $V_{LL} = \sqrt{3}V_\phi$, so the power consumed by the load can also be expressed as

$$P = 3\left(\frac{V_{LL}}{\sqrt{3}}\right) I_L \cos\theta$$

$$\boxed{P = \sqrt{3}V_{LL} I_L \cos\theta} \tag{A–29}$$

For a Δ-connected load, the power consumed by a load is given by

$$P = 3V_\phi I_\phi \cos\theta \tag{A–23}$$

For this type of load, $I_L = \sqrt{3}I_\phi$ and $V_{LL} = V_\phi$, so the power consumed by the load can also be expressed in terms of line quantities as

$$\begin{aligned} P &= 3V_{LL}\left(\frac{I_L}{\sqrt{3}}\right)\cos\theta \\ &= \sqrt{3}V_{LL}I_L \cos\theta \end{aligned} \tag{A–29}$$

This is exactly the same equation that was derived for a Y-connected load, so Equation (A–29) gives the power of a balanced three-phase load in terms of line quantities *regardless of the connection of the load.* The reactive and apparent powers of the load in terms of line quantities are

$$\boxed{Q = \sqrt{3}V_{LL}\,I_L \sin\theta} \tag{A–30}$$

$$\boxed{S = \sqrt{3}V_{LL}\,I_L} \tag{A–31}$$

It is important to realize that the $\cos\theta$ and $\sin\theta$ terms in Equations (A–29) and (A–30) are the cosine and sine of the angle between the *phase* voltage and the *phase* current, not the angle between the line-to-line voltage and the line current. Remember that there is a 30° phase shift between the line-to-line and phase voltage for a Y connection, and between the line and phase current for a Δ connection, so it is important not to take the cosine of the angle between the line-to-line voltage and line current.

A.4 ANALYSIS OF BALANCED THREE-PHASE SYSTEMS

If a three-phase power system is balanced, it is possible to determine the voltages, currents, and powers at various points in the circuit with a *per-phase equivalent circuit.* This idea is illustrated in Figure A–11. Figure A–11a shows a Y-connected generator supplying power to a Y-connected load through a three-phase transmission line.

In such a balanced system, a neutral wire may be inserted with no effect on the system, since no current flows in that wire. This system with the extra wire inserted is shown in Figure A–11b. Also, notice that each of the three phases is *identical* except for a 120° shift in phase angle. Therefore, it is possible to analyze a circuit consisting of *one phase and the neutral,* and the results of that analysis will be valid for the other two phases as well if the 120° phase shift is included. Such a per-phase circuit is shown in Figure A–11c.

There is one problem associated with this approach, however. It requires that a neutral line be available (at least conceptually) to provide a return path for current flow from the loads to the generator. This is fine for Y-connected sources and loads, but no neutral can be connected to Δ-connected sources and loads.

How can Δ-connected sources and loads be included in a power system to be analyzed? The standard approach is to transform the impedances by the Y–Δ transform of elementary circuit theory. For the special case of balanced loads, the Y–Δ transformation states that a Δ-connected load consisting of three equal impedances, each of value Z, is totally equivalent to a Y-connected load consisting of three impedances, each of value $Z/3$ (see Figure A–12). This equivalence means that the voltages, currents, and powers supplied to the two loads cannot be distinguished in any fashion by anything external to the load itself.

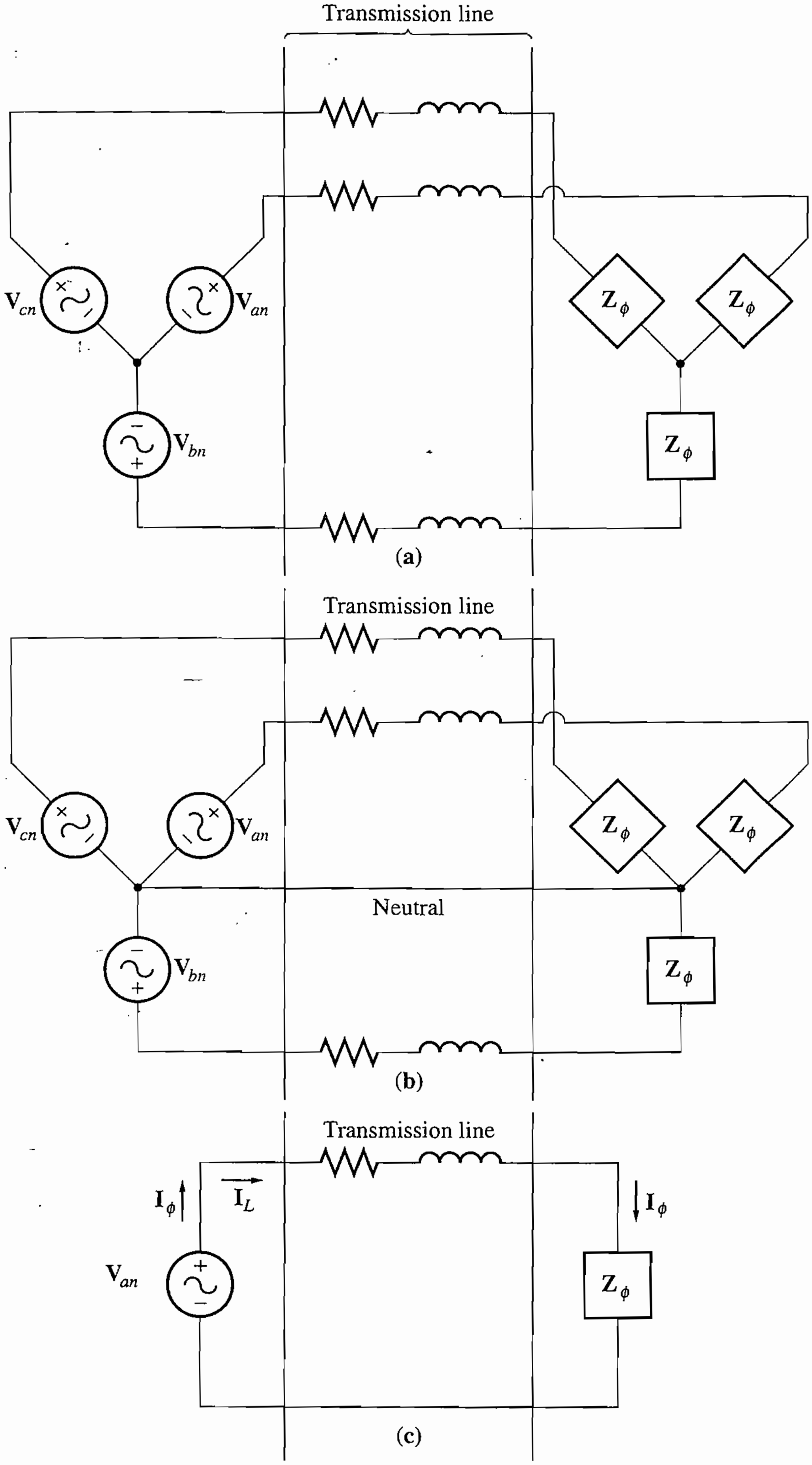

FIGURE A–11
(a) A Y-connected generator and load. (b) System with neutral inserted. (c) The per-phase equivalent circuit.