Please refer apt321_ss11381_ML_HW3_Part2_Source_Code.ipynb file for Source Code

Question 2:

a. By hand, find the characteristic polynomial of the matrix A

Solution 2

(a) $A = \begin{bmatrix} 0 & 14 \\ 6 & 9 \end{bmatrix}$

The characteristic polynomial of Matrix A is defined as-

$|A - kI|$, where $I \rightarrow$ Identity matrix of the same dimension as A.

$\Rightarrow A - kI$

$\Rightarrow \begin{bmatrix} 0 & 14 \\ 6 & 9 \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 14 \\ 6 & 9 \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -k & 14 \\ 6 & 9-k \end{bmatrix}$

Calculating determinant,

$\Rightarrow \det\left(\begin{bmatrix} -k & 14 \\ 6 & 9-k \end{bmatrix}\right) = -k(9-k) - 14 \times 6 = 0$

$\Rightarrow -9k + k^2 - 84 = 0$

$\Rightarrow \boxed{k^2 - 9k - 84 = 0} \rightarrow$ Characteristic Polynomial

b. By hand, solve for the eigenvalues of A using the characteristic polynomial you just computed.

(b) From question (a),

$$k^2 - 9k - 84 = 0$$

$$k = \left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)$$

$$= \left(\frac{+9 \pm \sqrt{81-(4\times1\times-84)}}{2\times1}\right)$$

$$k = \left(\frac{+9 \pm \sqrt{417}}{2}\right)$$

$$\therefore k = \frac{+9+\sqrt{417}}{2} \quad \text{or} \quad \frac{+9-\sqrt{417}}{2}$$

$$\boxed{k = 14.710289 \quad \text{or} \quad -5.7102889} \Rightarrow \text{Eigenvalues of A.}$$

c. Solve for the eigenvectors of A using those eigenvalues. The L2 norm of your eigenvectors should be equal to 1.

(c) From question (b),

Eigenvectors for $\lambda = \frac{9+\sqrt{417}}{2}$,

Solve $\Rightarrow A - \lambda I$

$$\Rightarrow \begin{bmatrix} 0 & 14 \\ 6 & 9 \end{bmatrix} - \frac{9+\sqrt{417}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 14 \\ 6 & 9 \end{bmatrix} - \begin{bmatrix} 9+\sqrt{417}/2 & 0 \\ 0 & 9+417/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -(9+\sqrt{417})/2 & 14 \\ 6 & (9-\sqrt{417})/2 \end{bmatrix}$$

This can be expressed as -

$$(A - \lambda I)\,\vec{v} = 0$$

$$\underbrace{\begin{bmatrix} -(9+\sqrt{417})/2 & 14 \\ 6 & (9-\sqrt{417})/2 \end{bmatrix}}\vec{v} = 0$$

$\hookrightarrow$ Reduce the matrix to echelon form $\begin{bmatrix} a & \cdots & b \\ 0 & \ddots & \vdots \\ 0 & 0 & c \end{bmatrix}$

Cancel leading coefficient in row R_2 by performing

$$R_2 \leftarrow R_2 - \frac{9-\sqrt{417}}{28} \times R_1$$

$$= \begin{bmatrix} -(9+\sqrt{417})/2 & 14 \\ 0 & 0 \end{bmatrix}$$

Reduce matrix to reduced row echelon form $\begin{pmatrix} 1 & \cdots & b \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix}$

Multiply matrix row by constant $R_1 \leftarrow \frac{9-\sqrt{417}}{168} \times R_1$

$$= \begin{bmatrix} 1 & (9-\sqrt{417})/12 \\ 0 & 0 \end{bmatrix}$$

Now, $(A - kI)\vec{v} = 0$

$$\begin{bmatrix} 1 & (9-\sqrt{417})/12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduces to the equation,

$$x + \left(\frac{9-\sqrt{417}}{12}\right) y = 0$$

Isolate,

$$x = -\left(\frac{9-\sqrt{417}}{12}\right) y$$

Plug into $\begin{pmatrix} x \\ y \end{pmatrix}$,

$$v = \begin{pmatrix} \frac{-(9-\sqrt{417})}{12} y \\ y \end{pmatrix}, \quad y \neq 0$$

Let, $y = 1$ (L_2 norm is 1)

$$\Rightarrow \vec{v} = \begin{pmatrix} \frac{-9-\sqrt{417}}{12} \\ 1 \end{pmatrix}$$

Eigenvectors for $\lambda = \frac{9-\sqrt{417}}{2}$,

Solve $\Rightarrow A - \lambda I$

$$\Rightarrow \begin{bmatrix} 0 & 14 \\ 6 & 9 \end{bmatrix} - \frac{9-\sqrt{417}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 14 \\ 6 & 9 \end{bmatrix} - \begin{bmatrix} 9-\sqrt{417}/2 & 0 \\ 0 & 9+\sqrt{417}/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -(9-\sqrt{417})/2 & 14 \\ 6 & 9+\sqrt{417}/2 \end{bmatrix}$$

This can be expressed as,

$$(A - KI)\,\vec{v} = 0$$

$$\underbrace{\begin{bmatrix} -(9-\sqrt{417})/2 & 14 \\ 6 & (9+\sqrt{417})/2 \end{bmatrix}}\vec{v} = 0$$

↳ Reduce the matrix to echelon form $\begin{bmatrix} a & \cdots & b \\ 0 & \ddots & \vdots \\ 0 & 0 & c \end{bmatrix}$

~~Cancel leading co-efficient in row~~

Swap matrix rows : $R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 6 & (9+\sqrt{417})/2 \\ -(9-\sqrt{417})/2 & 14 \end{bmatrix}$$

Cancel leading co-efficient in row R_2 by performing,

$$R_2 \leftarrow R_2 + \frac{9-\sqrt{417}}{12} \times R_1$$

$$\Rightarrow \begin{bmatrix} 6 & (9+\sqrt{417})/2 \\ 0 & 0 \end{bmatrix}$$

Reduce the matrix to reduced row echelon form $\begin{bmatrix} 1 & \cdots & b \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{bmatrix}$

Multiply matrix row by constant, $R_1 \leftarrow \frac{1}{6} \times R_1$

$$\Rightarrow \begin{bmatrix} 1 & (9+\sqrt{417})/12 \\ 0 & 0 \end{bmatrix}$$

Now, $(A - kI)\vec{v} = 0$

$$\begin{bmatrix} 1 & (9+\sqrt{417})/12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduces to the equation,

$$x + \frac{9+\sqrt{417}}{12} y = 0$$

Isolate,

$$x = -\frac{(9+\sqrt{417})}{12} y$$

Plug into $\begin{pmatrix} x \\ y \end{pmatrix}$,

$$v = \begin{bmatrix} \left(-(9+\sqrt{417})/12\right) y \\ y \end{bmatrix}, \quad y \neq 0$$

Let, $y=1$ (L_2 norm is 1)

$$\Rightarrow \vec{v} = \begin{bmatrix} -\frac{(9+\sqrt{417})}{12} \\ 1 \end{bmatrix}$$

$\therefore$ The eigenvectors for $\begin{bmatrix} 0 & 14 \\ 6 & 9 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -(9-\sqrt{417})/12 \\ 1 \end{bmatrix}, \begin{bmatrix} \cancel{9} -(9+\sqrt{417})/12 \\ 1 \end{bmatrix}$$

d. Validate your results in Python (numpy) using linalg.eig(A) using the command [V,D]=eig(A). Give the returned eigenvalues and eigenvectors.

```
Eigen values are:  [-5.71028893 14.71028893]

Eigen vectors are:
[[-0.9259401  -0.6894021 ]
 [ 0.37767039 -0.72437887]]
```

Question 3:
a. Calculate the sample means for each column [s1, s2, s3] by hand, and then subtract these values from the entries in each column (e.g. s3 is subtracted from all values in column 3). Call the resulting training set (matrix) B. Show the matrix B.

Solution 3

(a) $$X = \begin{bmatrix} 5 & 2 & 4 \\ 9 & 6 & 4 \\ 7 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$

$$\text{Mean}(X_1) = S_1 = \frac{5+9+7+2}{4} = 5.75$$

$$\text{Mean}(X_2) = S_2 = \frac{2+6+1+5}{4} = 3.5$$

$$\text{Mean}(X_3) = S_3 = \frac{4+4+0+6}{4} = 3.5$$

(b) ~~The Resulting Co-Variance~~

The Resulting training set (matrix) B is,

$$B = \begin{bmatrix} -0.75 & -1.5 & 0.5 \\ 3.25 & 2.5 & 0.5 \\ 1.25 & -2.5 & -3.5 \\ -3.75 & 1.5 & 2.5 \end{bmatrix} \Rightarrow$$ By subtracting each S_i from the entries in each column.

b. By hand, calculate $s_{1,3}$ the sample covariance of x1 and x3 in B. Show your result.

(b)

$$\sigma_{13} = \left[(-0.75)(0.5) + (3.25)(0.5) + (1.25)(-3.5) + (-3.75)(2.5)\right] / (4-1)$$

$$= \frac{-0.375 + 1.625 - 4.375 - 9.375}{3}$$

$$= \frac{-12.5}{3}$$

$$\therefore \boxed{\sigma_{13} = -4.1666667}$$

c. For the same B as in the previous question, use the Python command np.cov(B) to produce the sample covariance matrix of X. What is the largest eigenvalue of the sample covariance matrix of B? Use Python to compute this answer.

```
Matrix B:
[[-0.75 -1.5   0.5 ]
 [ 3.25  2.5   0.5 ]
 [ 1.25 -2.5  -3.5 ]
 [-3.75  1.5   2.5 ]]

Covariance of Matrix B:
[[ 8.91666667  0.16666667 -4.16666667]
 [ 0.16666667  5.66666667  4.33333333]
 [-4.16666667  4.33333333  6.33333333]]

Eigen values of the sample covariance matrix of B are:
[ 0.49703343 12.97881887  7.44081437]

Largest eigenvalue of the sample covariance matrix of B:  12.978818870145858
```

d. Perform PCA on Matrix B and show the first two columns of Z.

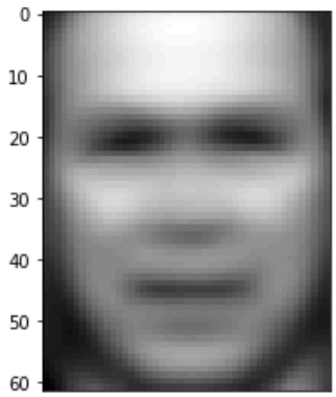
```
Matrix Z:
[[ 0.26018674 -1.41900435  0.99057029]
 [-0.87353472  4.03721245  0.01878129]
 [-4.04749635 -1.8486773  -0.51250906]
 [ 4.66084433 -0.7695308  -0.49684253]]

Showing the first two columns of Z:
[[ 0.26018674 -1.41900435]
 [-0.87353472  4.03721245]
 [-4.04749635 -1.8486773 ]
 [ 4.66084433 -0.7695308 ]]
```

Question 5:

a. Display the fourth face in the dataset using the appropriate command above. Print the resulting display.

b. Compute the mean of all the examples in the dataset fea. (That is, compute an image such that each pixel i of the image is the mean of pixel i in all the images in fea.) Display it using a modification of the above command. Give the Python commands you used and show the resulting display.

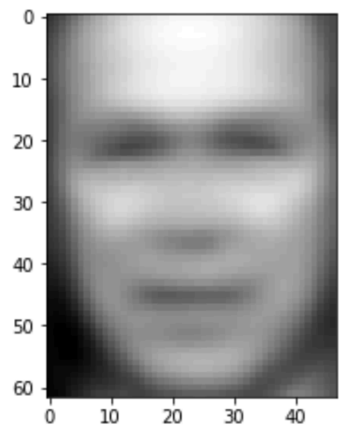


c. Let's do dimensionality reduction with pca. Using Python, compute the 5 top principal components of the data matrix fea. Give the Python commands you used. What are the values of the associated 5 attributes of the fourth image in the dataset?

```
Z:
 [ 202.54253  -261.4768    418.9738    -29.3992     39.785084]
```

d. Using the reconstruction equation X = WTZ + M described above, but with just the first 5 columns of Z and W (the attributes associated with the first 5 principal components), approximate the fourth image in the dataset. Give the Python commands you used. Display the resulting approximate image and print the resulting display. Repeat with the first 50 columns of Z and W. (These are representations of the fourth image, based on 5 or 50 features instead of the original 2914 pixel features.)

Reconstructing Image 4 with Number of Components = 5



Reconstructing Image 4 with Number of Components = 50