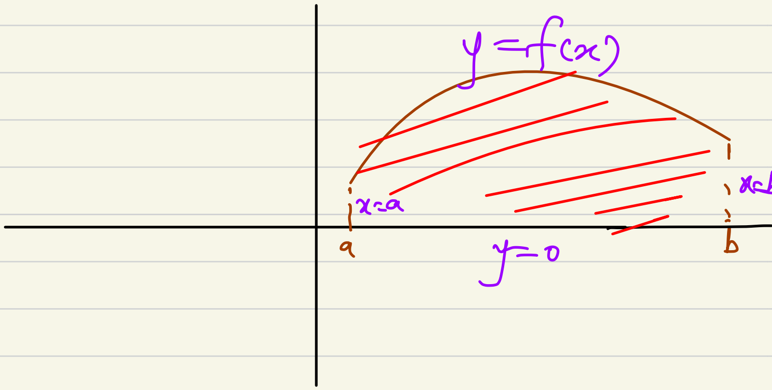
MA 103- SVC Lecture 7

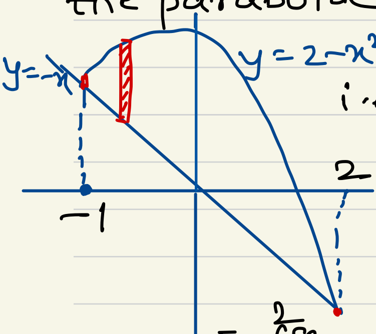


DEFINITION **Area Between Curves**

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Example: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and $y = -x$.



Find where the 2 curves intersect

i.e. solve $2 - x^2 = -x$ for x.

$$x^2 - x - 2 = 0$$

$$\Rightarrow (x+1)(x-2) = 0$$

$$\Rightarrow x = -1 \text{ or } x = 2.$$

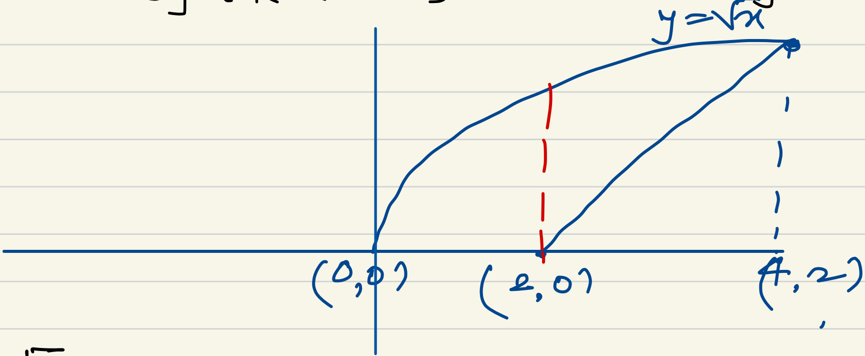
Area between the curves

$$= \int_{-1}^{2} [(2 - x^2) - (-x)]\, dx = \int_{-1}^{2} (2 + x - x^2)\, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = 4 + 2 - \frac{8}{3} - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= 8 - \frac{1}{2} - 3 = \frac{9}{2}.$$

Ex. 2 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.



$$\sqrt{x} = x - 2$$

$$\Rightarrow \quad x = (x-2)^2$$

$$\Rightarrow \quad x^2 - 4x + 4 = x$$

$$\Rightarrow \quad x^2 - 5x + 4 = 0$$

$$\Rightarrow \quad (x-1)(x-4) = 0$$

$$\Rightarrow \quad x = 1 \text{ or } 4$$

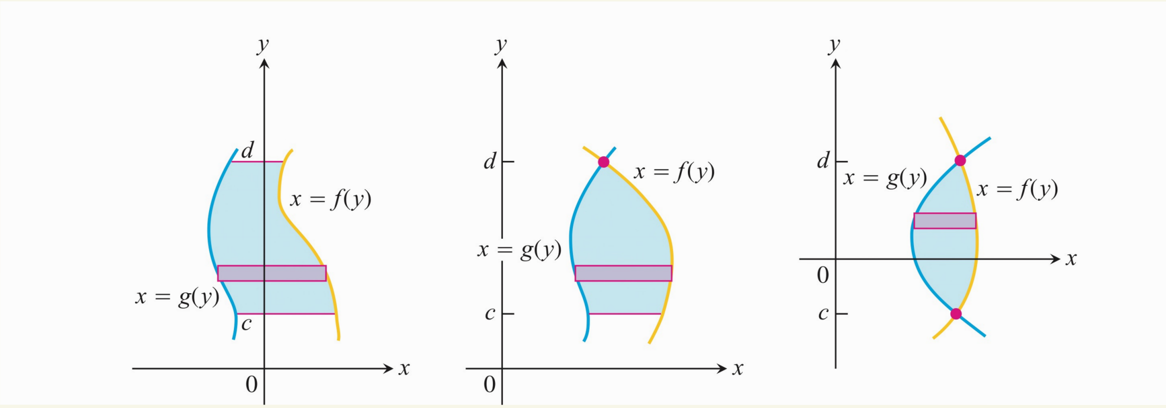
$x = 1$ is not a solution, so should be omitted.

$$\text{Area} = \int_0^2 \sqrt{x}\, dx + \int_2^4 \left(\sqrt{x} - (x-2)\right) dx$$

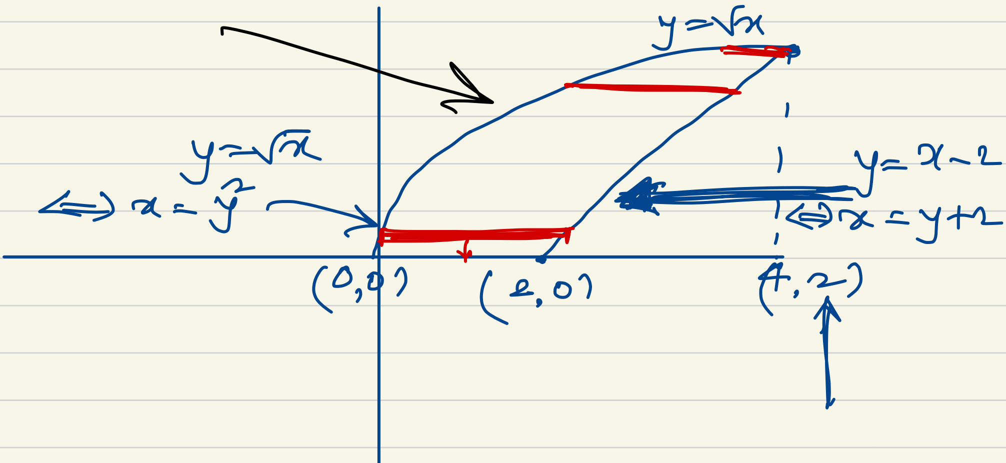
$$= \frac{2}{3}\left[x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}\cancel{(2\sqrt{2})} + \left(\frac{2}{3}\times 8 - 8 + 8\right) - \left(\cancel{\frac{4\sqrt{2}}{3}} - 2 + 4\right)$$

$$= \frac{16}{3} - 2 = \frac{10}{3}.$$



Ex. 3 Find the area of the same region above by integrating w.r.t. y.



Area

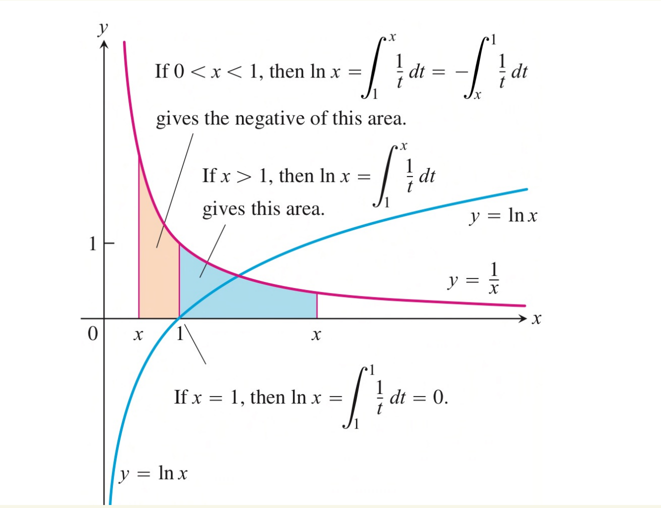
$$= \int_0^2 (y+2) - y^2 \, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2$$

$$= 2 + 4 - \frac{8}{3} = \frac{18-8}{3} = \frac{10}{3}.$$

Sect. 7.1 Transcendental functions

(I) LOGARITHM FUNCTION (Sect. 7.2)

- $\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$ (Natural logarithm)



- $\frac{d}{dx} \ln x = \frac{1}{x}$
- $\int \frac{1}{u}\,du = \ln|u| + c$
- $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$

Example Find dy/dx if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \quad x > 1.$$

Taking natural log on both sides, we see that

$$\ln(y) = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot(2x) + \frac{1}{2(x+3)} - \frac{1}{x-1}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right)$$

$$= \frac{(x^2+1)(x+3)^{1/2}}{(x-1)}\left\{\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right\}.$$

II EXPONENTIAL FUNCTION (Sect. 7.3)

Ex. 4 Solve the initial value problem

$$e^y\frac{dy}{dx} = 2x, \quad x > \sqrt{3}\ ; \ y(2) = 0.$$

- $\lim_{x\to 0}(1+x)^{1/x} = e$

$e^y = x^2 + c$ ← Differentiating this gives us the given eqn.

Ans Integrate $e^y\frac{dy}{dx} = 2x$ w.r.t. x to get

$$e^y = x^2 + c$$

Since $y(2)=0$, we have

$$e^0 = 2^2 + c \Rightarrow c = -3$$

$$\Rightarrow e^y = x^2 - 3$$

Taking log on both sides, we get

$$\boxed{y = \ln(x^2-3)}$$

Check the answer: $\frac{dy}{dx} = \frac{2x}{x^2-3}$

$$e^y\frac{dy}{dx} = e^{\ln(x^2-3)}\cdot\frac{2x}{x^2-3} = 2x$$

Sect. 7.4, 7.7 - To be read on your own

Important formulas:

- $\frac{d}{dx}(a^x) = a^x \ln(a)$

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$
2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$
3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1+u^2}$
4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1+u^2}$
5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$
6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)
2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)
3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

Hyperbolic functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cosh^2(x) - \sinh^2(x)$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}\left[e^{2x} + 2 + e^{-2x} - \left(e^{2x} - 2 + e^{-2x}\right)\right]$$

$$= \frac{1}{4} \cdot 4 = 1.$$

$$y^2 - x^2 = 1$$

$$y = \cosh(t)$$

$$x = \sinh(t)$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\operatorname{cosech}(x) = \frac{1}{\sinh x}, \quad \operatorname{sech}(x) = \frac{1}{\cosh x}.$$

Sect. 7.8 – Hyperbolic functions

Ex. 1 Discuss hyperbolic functions & prove some of the properties from the tables below.

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}} \tag{1}$$

Example:

By the derivative rule for inverses,

$$\left.\frac{d}{dx}\sinh^{-1}(x)\right|_{x=b} = \left.\frac{1}{\frac{d}{dx}\sinh(x)}\right|_{x=f^{-1}(b)}$$

$$= \left.\frac{1}{\cosh x}\right|_{x=\sinh^{-1}(b)}$$

$$= \left.\frac{1}{\sqrt{1+\sinh^2 x}}\right|_{x=\sinh^{-1}(b)}$$

$$= \frac{1}{\sqrt{1+b^2}}.$$

Ex. 2 Discuss inverse hyperbolic functions & prove some of the properties from the tables below.

TABLE 7.10 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \qquad u \neq 0$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

Ch. 8 - Techniques of integration

Sect. 8.1 - Basic integration formulas

Procedures for Matching Integrals to Basic Formulas

Procedure	Example
Making a simplifying substitution	$\dfrac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x$ $= \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x\tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\dfrac{3x^2-7x}{3x+2} = x - 3 + \dfrac{6}{3x+2}$
Separating a fraction	$\dfrac{3x+2}{\sqrt{1-x^2}} = \dfrac{3x}{\sqrt{1-x^2}} + \dfrac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}$ $= \dfrac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}$

Sect. 8.2 - Integration by parts

* Integration by parts formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

or in other words, with $u = f(x)$, $v = g(x)$,

$$\int u\,dv = uv - \int v\,du$$

* An equivalent form: If u & v are functions of x, then

$$\int uv\,dx = u\int v\,dx - \int\left(\left(\frac{du}{dx}\right)\int v\,dx\right)dx$$

Ex. 3 Prove the above equivalent form of the integration by parts formula.

Proof: $\dfrac{d(uw)}{dx} = u\dfrac{dw}{dx} + w\dfrac{du}{dx}$

Let $w = \int v\,dx$

$$\Rightarrow \frac{d}{dx}\left(u\int v\,dx\right) = uv + \left(\frac{du}{dx}\right)\left(\int v\,dx\right)$$

Integrating both sides w.r.t. x, we get

$$\Rightarrow u\int v\,dx = \int uv\,dx + \int \frac{du}{dx}\left(\int v\,dx\right)dx$$

Also

$$\int_a^b uv\,dx = \left[u\int v\,dx\right]_a^b - \int_a^b\left(\frac{du}{dx}\right)\left(\int v\,dx\right)dx.$$

LIATE rule : Choice of u :

Logarithmic	OR	Inverse trig.
Inverse trig.		Logarithmic
Algebraic		Algebraic
Trig.		Trig.
Exponential		Exponential

Example : Evaluate $\int e^x \cos x \, dx$

Let $u = \cos x$, $dv = e^x \, dx$

$du = -\sin x \, dx$, $v = e^x$

$I =$

$$\Rightarrow \int e^x \cos x \, dx = (\cos x)(e^x) - \int e^x (-\sin x) \, dx$$

$$= e^x \cos x + \int e^x \sin x \, dx$$

$$= e^x \cos x + \left[(\sin x)(e^x) - \int e^x \cos x \, dx \right]$$

$$\Rightarrow I = e^x (\sin x + \cos x) - I$$

$$\Rightarrow I = \frac{1}{2} e^x (\sin x + \cos x) + c .$$

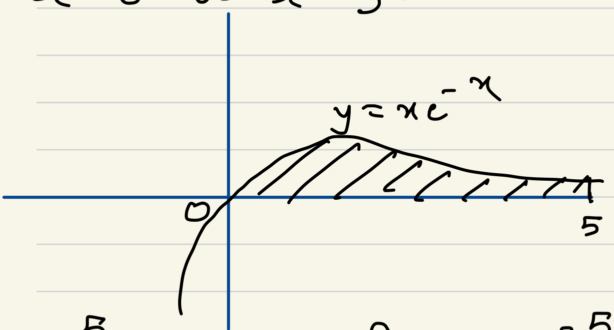
Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

Alternate form: If u & v are functions of x,

$$\int_a^b uv\,dx = \left[u\int v\,dx\right]_a^b - \int_a^b \left(\frac{du}{dx}\right)\left(\int v\,dx\right)dx \,,$$

Example Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 5$.



$$\text{Area} = \int_0^5 \underset{u}{x}\,\underbrace{e^{-x}\,dx}_{dv}$$

$u = x$, $dv = e^{-x}\,dx$

$du = dx$, $v = -e^{-x}$

$$\Rightarrow \int_0^5 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^5 - \int_0^5 \left(-e^{-x}\right)dx$$

$$= -5e^{-5} + 0 + \left[-e^{-x}\right]_0^5$$

$$= -5e^{-5} + \left(-e^{-5} - \left(-e^{-0}\right)\right)$$

$$= -5e^{-5} - e^{-5} + 1$$

$$= 1 - 6e^{-5}.$$