# MA 103 – Lecture 3

## Sect. 4.2 – Mean value theorem

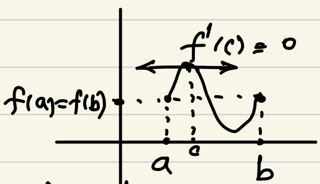
**THEOREM 3 Rolle's Theorem**
Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$. If

$$f(a) = f(b),$$

then there is at least one number $c$ in $(a, b)$ at which

$$f'(c) = 0.$$

**Proof:**




$f$ is continuous on closed & bdd. interval $[a, b]$, hence $f$ attains its absolute maximum $M$ and absolute minimum $m$.

They can only occur at:

(i) points at which $f' = 0$
(ii) points at which $f'$ is undefined $\Big\}$ critical points
(iii) endpoints of the interval.

(ii) is ruled out since $f$ is differentiable in $(a, b)$.
If $f$ has a local max. or min. at an interior point $c$ of $(a, b)$, then since $f'(c)$ exists, we must have $f'(c) = 0$. Hence $c$ is the reqd. pt. in Rolle's thm.

If the absolute max. and min. occur at endpoints, since $f(a) = f(b)$ then $f(x) = f(a) = f(b)$ for every $x$ in $(a, b)$. Hence $f$ is constant on $(a, b)$. Thus $f'(x) = 0$ for any $x \in (a, b)$. Thus $c$ can be any such $x$.

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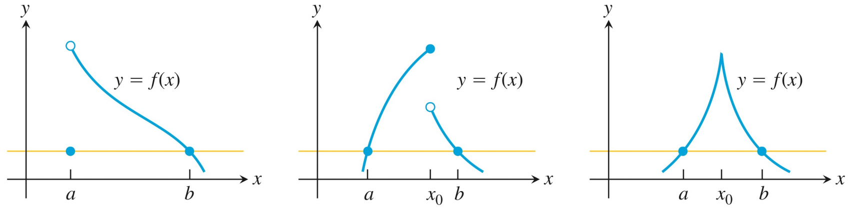



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

Example 1 Show that the equation $x^3+3x+1=0$ has exactly one real solution.

Sol$^n$: Let $f(x) = x^3+3x+1$. This is cont. on $(-\infty, \infty)$ and differentiable for any real $x$.

$f'(x) = 3x^2+3 = 3(x^2+1) \neq 0$ for any $x \in \mathbb{R}$ (since $x^2+1 > 0 \; \forall x \in \mathbb{R}$). —— $(*)$

Suppose $f$ has 2 real roots, say $a$ and $b$. Then $f(a) = f(b) = 0$. By Rolle's theorem, $\exists \; c \in (a, b) \ni f'(c) = 0$. But this contradicts $(*)$. Hence $f$ cannot have more than one real root.

Now $f$ has one real root as can be seen using IVT. This is because, for example, $f(-1) = -1-3+1 = -3 < 0$ and $f(0) = 1 > 0$. Hence $\exists$ a point $c$ in $[-1, 0]$ s.t. $f(c) = 0$.

**Ex. 1** Assume that $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$. Also, assume that $f(a)$ & $f(b)$ have opposite signs and that $f' \neq 0$ between $a$ and $b$. Show that $f(x) = 0$ exactly once between $a$ and $b$.

Ex. 1 $f$ cont. on $[a,b]$ $f$ diff. on $(a,b)$,
$f(a)$ & $f(b)$ have opposite signs,
$f' \neq 0$ between $a$ & $b$.

Proof: Let $a < x_1 < x_2 < b$ $\ni$ $f(x_1) = f(x_2) = 0$

Apply Rolle's thm. for $[x_1, x_2]$.

Hence $\exists$ $c \in (x_1, x_2)$ $\ni$

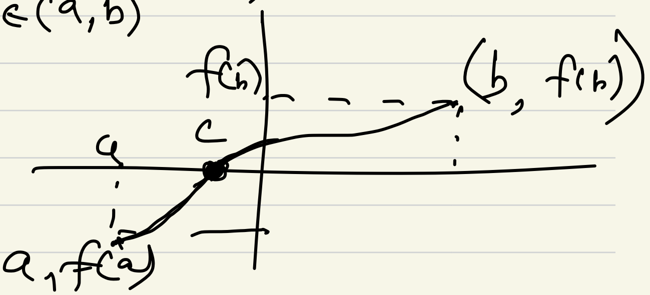
$$f'(c) = 0.$$

This contradicts the non-vanishing of $f'$ in $(a,b)$.

Hence $f$ cannot have more than one real root.

Since $f(a)$ & $f(b)$ have opposite signs, by IVT applied to $[a,b]$, we see that $\exists$ $c \in (a,b)$ s.t.

$f(c) = 0$.




Hence we conclude that $f$ has exactly one real root between $a$ and $b$.

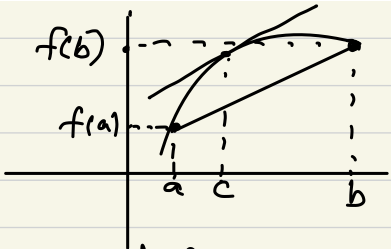
# The Mean Value Theorem

**THEOREM 4 The Mean Value Theorem**

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

**Proof:**




Let

$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

The vertical difference between $f$ & $g$ at $x$ is

$$h(x) := f(x) - g(x)$$
$$= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

Check that $h(a) = h(b) = 0$. Clearly, $h$ satisfies hypotheses of Rolle's theorem. Hence $\exists\, c \in (a,b)$ s.t. $h'(c) = 0$. $\Rightarrow f'(c) = g'(c)$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}.$$

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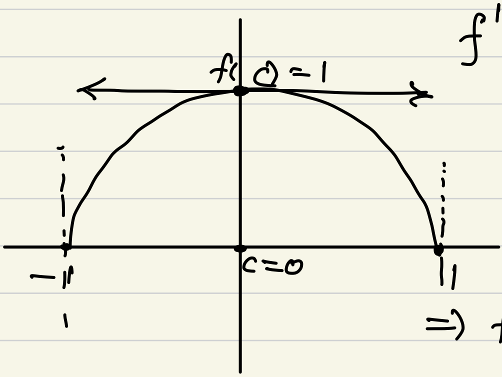
**Remark:** Continuity at the end points is essential, differentiability is not necessary.

$$y = \sqrt{1-x^2}, \quad -1 \leq x \leq 1$$




$$f'(0) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)\Big|_{x=0}$$

$$= 0$$

$$\frac{f(1)-f(-1)}{1-(-1)} = 0$$

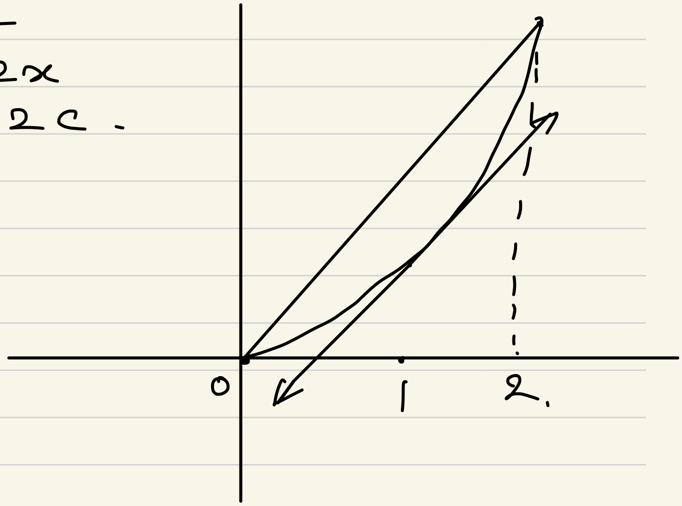
$$\Rightarrow f'(0) = \frac{f(1)-f(-1)}{1-(-1)}.$$

**Example:** Let $f(x) = x^2, \quad 0 \leq x \leq 2$.
$f$ is cont. on $[0,2]$, and diff. on $(0,2)$.
By mean-value theorem, $\exists\, c \in (0,2) \ni$

$$f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{4-0}{2-0} = 2$$

Now $f(x) = x^2$
$\Rightarrow f'(x) = 2x$
Hence $f'(c) = 2c$.

So $2c = 2$
$\Rightarrow c = 1$.



## A physical interpretation of the mean value theorem

If $\frac{f(b)-f(a)}{b-a}$ is the average change of $f$ over $[a,b]$ & $f'(c)$ is instantaneous change of $f$ at $c$, then MVT says there exists a pt. $c \in (a,b)$ $\ni$ the instantaneous change of $f$ at $c$ equals the average change of $f$ over $[a,b]$.

## Corollaries of the mean value theorem

**COROLLARY 1 Functions with Zero Derivatives Are Constant**

If $f'(x) = 0$ at each point $x$ of an open interval $(a, b)$, then $f(x) = C$ for all $x \in (a, b)$, where $C$ is a constant.

Proof: We want to show that if $x_1, x_2$ are any 2 pts. in $(a,b)$, then $f(x_1) = f(x_2)$. Let $x_1 < x_2$. By Mean value thm. $\exists\, c \in (x_1, x_2) \ni$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But $f'(c) = 0$ (by hypotheses)

$\Rightarrow f(x_1) = f(x_2)$.

So $f$ is constant on $(a,b)$. $\boxtimes$

**COROLLARY 2 Functions with the Same Derivative Differ by a Constant**

If $f'(x) = g'(x)$ at each point $x$ in an open interval $(a, b)$, then there exists a constant $C$ such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant on $(a, b)$.

Proof: Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$.

Now use corollary 1 to get $h(x) = C$ $\forall x \in (a,b)$

$\Rightarrow f(x) = g(x) + C$.

$\boxtimes$

**Ex. 2** In what follows, the acceleration $a = \frac{d^2s}{dt^2}$, initial velocity, and initial position of a body moving on a coordinate line are given. Find the body's position in time $t$.

$a = 32$, $v(0) = 20$, $s(0) = 5$.

Ans. $a = \frac{d^2s}{dt^2}$, $v = \frac{ds}{dt}$

$a = 32$, $v(0) = 20$, $s(0) = 5$.

$$\frac{d^2s}{dt^2} = 32$$

$$\Rightarrow v(t) = 32t + c$$

$$20 = v(0) = 32(0) + c$$

$$\Rightarrow c = 20$$

$$\Rightarrow v(t) = 32t + 20$$

$$\Rightarrow \frac{ds}{dt} = 32t + 20$$

$$\Rightarrow s(t) = 16t^2 + 20t + c_1$$

$$5 = s(0) = 16(0)^2 + 20(0) + c_1$$

$$\Rightarrow c_1 = 5$$

$$\Rightarrow \boxed{s(t) = 16t^2 + 20t + 5}$$

**Ex. 3** (i) The geometric mean of two positive numbers $a$ and $b$ is the number $\sqrt{ab}$. Show that the value of $c$ in the conclusion of the mean value theorem for $f(x) = 1/x$ on an interval of positive numbers $a$ and $b$ is $\sqrt{ab}$.

(ii) The arithmetic mean of two numbers a & b is $\frac{a+b}{2}$. Show that the value of c in the conclusion of the mean value theorem for $f(x)=x^2$ on any interval $[a,b]$ is $\frac{a+b}{2}$.

(i) $f(x) = 1/x \qquad a, b > 0$

f is cont. on $[a,b]$ & diff. on $(a,b)$

Hence by MVT, $\exists\ c \in (a,b) \ni$

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{1/b - 1/a}{b-a} = \frac{-1}{ab}.$$

$$f'(x) = \frac{-1}{x^2} \Rightarrow f'(c) = \frac{-1}{c^2}$$

Hence $\frac{-1}{c^2} = \frac{-1}{ab}$

$\Rightarrow c^2 = ab$

Thus $c = \sqrt{ab}$.

(ii) $f(x) = x^2$

By MVT, $\exists\ c \in (a,b)$,

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{b^2-a^2}{b-a} = b+a$$

$f'(x) = 2x$

$\Rightarrow f'(c) = 2c$

Hence $2c = b+a$

$\Rightarrow c = \frac{b+a}{2}$.

# Sect 4.3 Monotonic functions

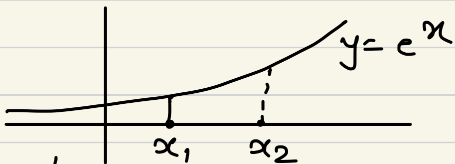
**DEFINITIONS Increasing, Decreasing Function**

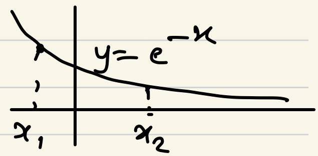
Let $f$ be a function defined on an interval $I$ and let $x_1$ and $x_2$ be any two points in $I$.

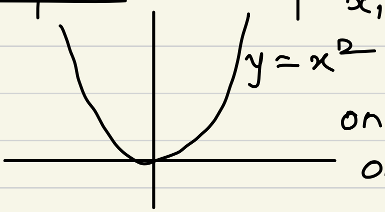
1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then $f$ is said to be **increasing** on $I$.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then $f$ is said to be **decreasing** on $I$.

A function that is increasing or decreasing on $I$ is called **monotonic** on $I$.

Example:




$f(x) = x^2$ is decreasing on $(-\infty, 0)$ & increasing on $(0, \infty)$.

**COROLLARY 3 First Derivative Test for Monotonic Functions**

Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

If $f'(x) > 0$ at each point $x \in (a, b)$, then $f$ is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then $f$ is decreasing on $[a, b]$.

Ex 4. Prove the above corollary.

Proof: First part: Let $x_1, x_2 \in (a,b) \ni x_1 < x_2$.

Then we have to show $f(x_1) < f(x_2)$.

By MVT, $\exists\, c \in (x_1, x_2) \ni$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Since $x_2 - x_1 > 0$ & $f'(c) > 0$ (by hypothesis), we conclude that $f(x_2) - f(x_1) > 0$.

Hence $f$ is increasing on $(a, b)$. By Cont. of $f$ at $a$ and at $b$, we see that $f$ is increasing on $[a, b]$.

Second part is similar.

**Example** Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which $f$ is increasing and decreasing.

**Sol$^n$:**

$f(x) = x^3 - 12x - 5 \implies f'(x) = 3x^2 - 12$

Critical pts.: $f'(x) = 0$ implies $3(x^2 - 4) = 0$

$\implies 3(x-2)(x+2) = 0 \implies x = \pm 2$.

| | (i) | (ii) | (iii) |
|---|---|---|---|
| Intervals of $f$ | $(-\infty, -2)$ | $(-2, 2)$ | $(2, \infty)$ |
| Sign of $f'(x)$ | $+$ | $-$ | $+$ |
| Behavior of $f$ | increasing | decreasing | increasing |

(i) $x < -2 \implies x + 2 < 0,\ x - 2 < 0 \implies f'(x) > 0$

$\implies f$ is increasing on $(-\infty, -2)$.

(ii) $-2 < x < 2$, $x + 2 > 0$, $x - 2 < 0 \implies f'(x) < 0 \implies f$ is decr. on $(-2, 2)$

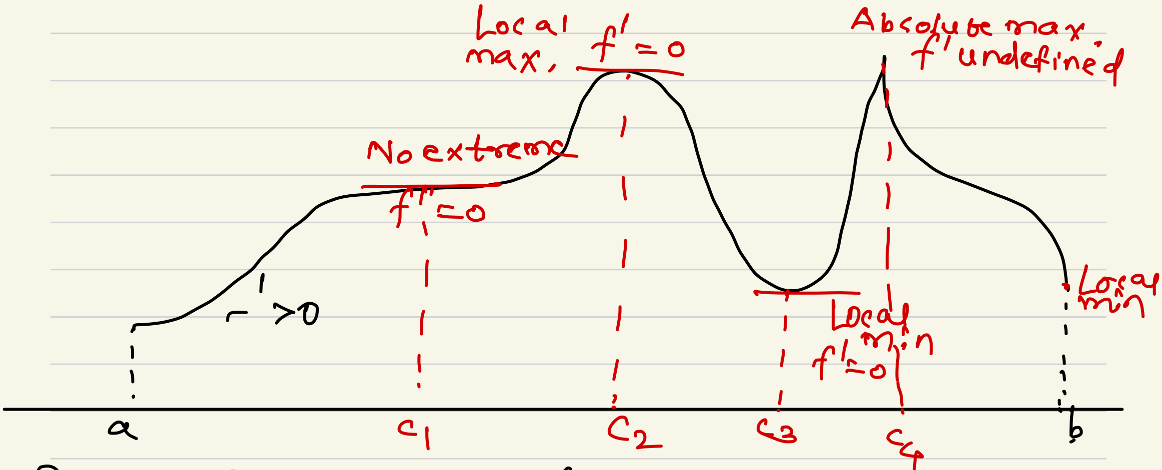
(iii) $2 < x < \infty$, implies $x - 2 > 0$, $x + 2 > 0 \implies f'(x) > 0$

$f$ is increasing on $(2, \infty)$.

## First Derivative Test for Local Extrema

Suppose that $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$ except possibly at $c$ itself. Moving across $c$ from left to right,

1. if $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$;
2. if $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$;
3. if $f'$ does not change sign at $c$ (that is, $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local extremum at $c$.




Proof: ① $f'$ changes from negative to positive at $c$.
Hence $\exists\, a, b \ni f' < 0$ on $(a, c)$ & $f' > 0$ on $(c, b)$.
Then if $x \in (a, c)$, then $f(c) < f(x)$. Also if $x \in (c, b)$, $f$ is increasing $(c, b)$. Hence $f(c) < f(x)$.
Thus $f(x) \geq f(c)$ for every $c \in (a, b)$.
Hence $f$ has a loc. min. at $c$.
Similarly ② & ③ can be proved.

**Ex.5** Find the critical points of
$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}.$$
Identify the intervals on which $f$ is increasing and decreasing. Find the function's local & absolute extreme values.

Ans. $f$ is continuous on $(-\infty, \infty)$, being a product of cont. fns. $x^{1/3}$ & $(x-4)$ on $(-\infty, \infty)$.

Find critical pts:

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}\frac{(x-1)}{x^{2/3}}$$

Thus $f'(x) = 0$ when $x = 1$

$f'(x)$ is undefined when $x = 0$.

Hence the critical pts. of $f$ are $0$ & $1$,

| Intervals | $(-\infty, 0)$ | $(0, 1)$ | $(1, \infty)$ |
|---|---|---|---|
| sign of $f'$ | $-$ | $-$ | $+$ |
| Behavior of $f$ | decreasing | decreasing | increasing |

$$f(x) = x^{1/3}(x-4)$$

$f(0) = 0$

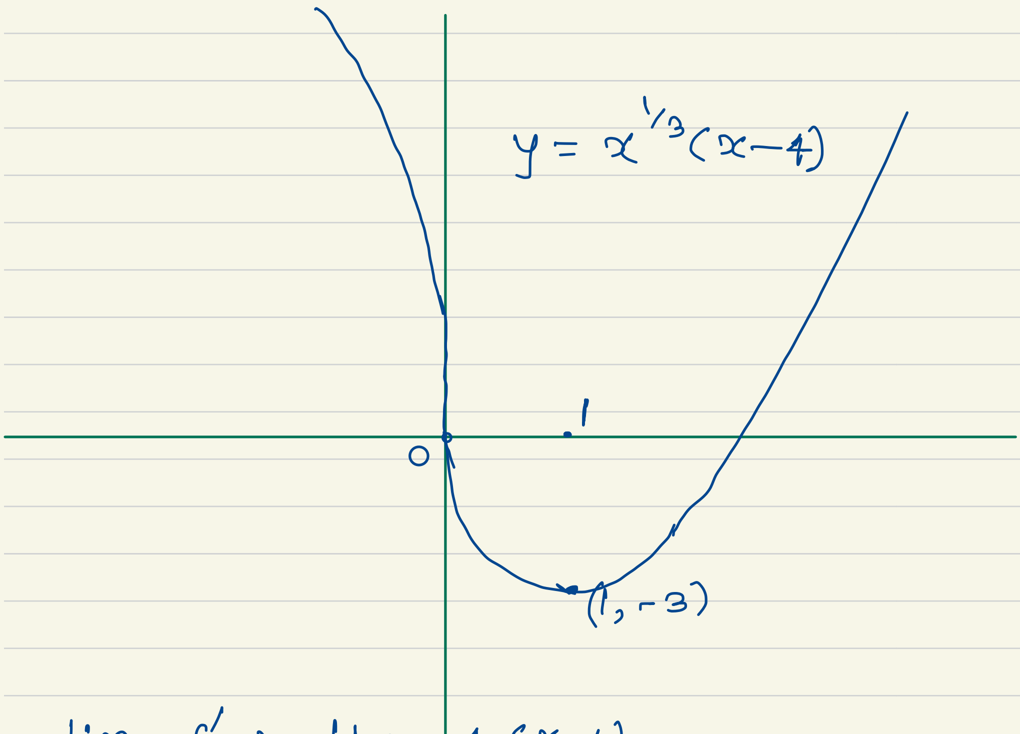
$f(1) = -3$.

$$f'(x) = \frac{4}{3}\frac{(x-1)}{x^{2/3}}$$

- No change of sign of $f'$ from left of 0 to right of 0. Hence zero is not a local extremum.

- $f'$ changes from $-$ve to $+$ve when we go from left to right around $x=1$.

Hence by 1st derivative test for local extrema, $x=1$ is local minimum of $f$.




$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{4}{3}\frac{(x-1)}{x^{2/3}} = -\infty$$

Hence $f$ has a vertical tangent at the origin.