Tutorial 3 (SVC) – Solutions sketch

(1) $f(x) = \sin x$ is continuous and differentiable on $\mathbb{R}$, hence, in particular, it is continuous on $[a, b]$ and differentiable on (a, b) for any $a, b \in \mathbb{R}$ with $a<b$.
Thus, by MVT, $\exists\, c \in (a, b) \ni$

$$f'(c) = \frac{f(b) - f(a)}{b-a}, \quad \text{i.e.}$$

$$\cos c = \frac{\sin b - \sin a}{b - a}$$

Since $|\cos c| \le 1$, the result follows.

(2) Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $n \geq 2$.
If b & c are any 2 roots of $f(x)$.
Then $f(b) = f(c)$. Of course, f is continuous on $[b, c]$ & differentiable on (b, c) whence $\exists\, d$ with $b<d<c \ni f'(d) = 0$.

[After discussing this problem, you can tell them about 'interlacing property'.]

(3) Let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ $(a_3 \neq 0)$ denote the cubic polynomial.
Then $p'(x) = 3a_3x^2 + 2a_2x + a_1$
& $p''(x) = 6a_3x + 2a_2$
Now $p''(x)$ has only one root, namely, $x = -\dfrac{a_2}{3a_3}$ (Note that $a_3 \neq 0$)

By prob. (2) p' can have at most 2 real zeros, & again by prob. (2), we see that p can have at most 3 real zeros.

(4) $f(x) = \frac{x^2-3}{x-2}$, $x \neq 2$

$f'(x) = \frac{(x-1)(x-3)}{(x-2)^2}$, $(x \neq 2)$.

Intervals	$(-\infty, 1)$	$(1, 2)$	$(2, 3)$	$(3, \infty)$
Sign of f'	$+$	$-$	$-$	$+$
Behavior of f	increasing	decreasing	decreasing	increasing

(a) f is increasing on $(-\infty, 1)$ & $(3, \infty)$, and decreasing on $(1, 2)$ & $(2, 3)$.

(b) Since f changes sign from $+$ to $-$ when we move from the left of 1 to its right, by the first der. test for local extrema, f has a local max. at $x = 1$. It is $f(1) = \frac{(1)^2-3}{1-2} = 1$

Similarly, f has a loc. min. at $x = 3$, & it is $f(3) = 6$.

- 2 is not in the domain of f, however, $\lim_{x \to 2^-} \frac{x^2-3}{x-2} = -\infty$, whereas $\lim_{x \to 2^+} \frac{x^2-3}{x-2} = +\infty$.

(c) f does not have any absolute extrema since $\lim_{x \to -\infty} \frac{x^2-3}{x-2} = -\infty$ & $\lim_{x \to +\infty} \frac{x^2-3}{x-2} = +\infty$.

⑤ To graph the $f(x) = \dfrac{x^2-3}{x-2}$, $x \neq 2$, we first find $f''(x)$:

$$f''(x) = \frac{d}{dx}\left(\frac{x^2-4x+3}{x^2-4x+4}\right)$$

$$= \frac{(x^2-4x+4)(2x-4) - (x^2-4x+3)(2x-4)}{(x-2)^4}$$

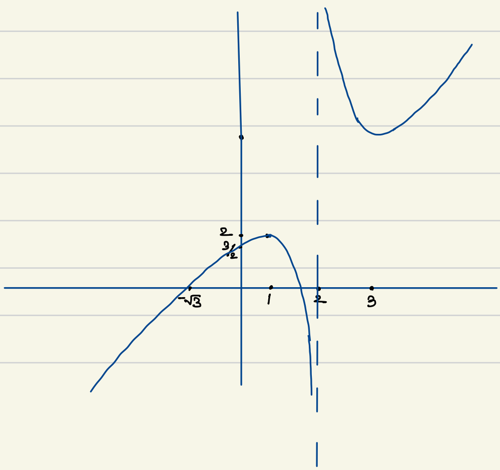
$$= \frac{2}{(x-2)^3}, \text{ which exists } \forall x \in \mathbb{R}\setminus\{2\}.$$

By the 2^{nd} derivative test for concavity,

$f'' < 0$ on $(-\infty, 2)$ implies f is concave down on $(-\infty, 2)$.
Similarly, f is concave up on $(2, \infty)$.

Note also that $f(\pm\sqrt{3}) = 0$ & $f(0) = 3/2$
Hence with the help of this information as well as that in ④, we graph f as follows:

$y = \dfrac{x^2-3}{x-2}$



6 (i) Let b and c with $b<c$ denote the 2 roots of $f'(x)$.
Then

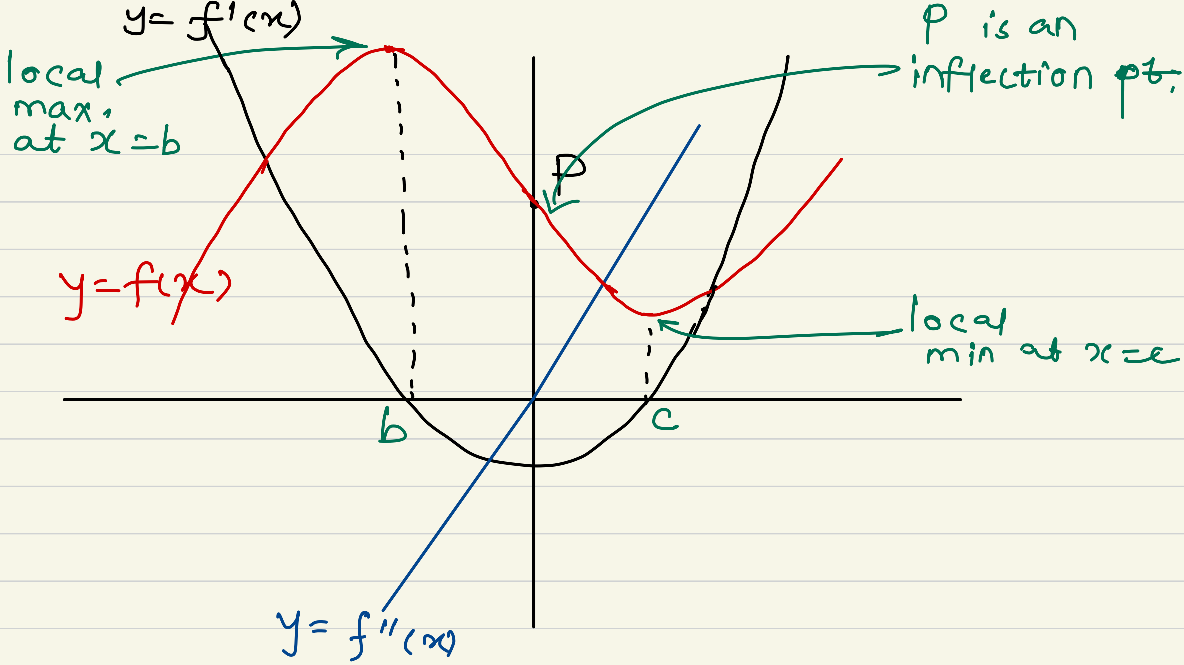
Intervals	$(-\infty, b)$	(b, c)	(c, ∞)
sign of f'	$+$	$-$	$+$
Behavior of f	$\nearrow$	$\searrow$	$\nearrow$

(ii) The sign of f' changes from $+$ to $-$ when we move from the left of b to its right. Hence f has local max. at $x=b$.

Similarly f has a loc. min. at $x=c$.

(iii) $f''<0$ on $(-\infty, 0)$, so f is concave down there whereas $f''>0$ on $(0,\infty)$, hence f is concave up there.
Also, as f changes its concavity at P, f has an inflection point at $x=0$, which is P.

With this, we graph $y=f(x)$ as follows:



Q 7] The thermometer took 14 sec. to rise from $-19°c$ to $100°c$.

So if $y = f(t)$ denotes the temperature (in celcius) of the thermometer at time t, then we are given, with $a < b$,

$$f(a) = -19, \quad f(b) = 100 \;\&\; b - a = 14.$$

Assuming the continuity of f on $[a, b]$ & its differentiability on (a, b), we see that there exists c with $a < c < b$ ∋

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad (\text{by MVT})$$

$$= \frac{100 + 19}{14} = 8.5$$

Hence somewhere along the way the mercury was rising at $8.5°c/sec$.

Q8] $\displaystyle\int \frac{\csc(\theta)\, d\theta}{\csc(\theta)-\sin(\theta)} = \int \frac{1}{\sin\theta\left(\frac{1-\sin^2\theta}{\sin\theta}\right)} d\theta$

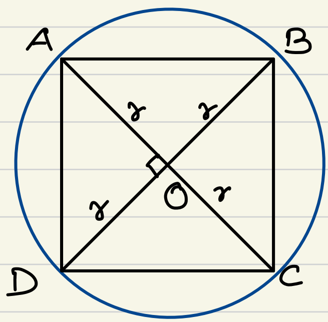
$= \displaystyle\int \sec^2\theta \, d\theta = \tan\theta + c.$

Q9]

It would be nice to have students visualize what happens to the area of the regular n-gon inscribed in a circle of radius r as $n \to \infty$.

(This takes care of problems 21 & 22 of the exercises at the end of Sect. 5.1)

(i) ⓐ $n = 4$:

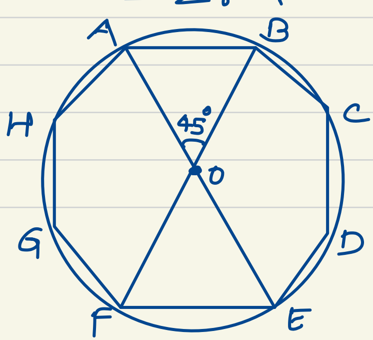


Area of the square $= 4 \cdot$ Area of ΔAOB

$= 4 \cdot \frac{1}{2}(r)(r)$

$= 2r^2$.

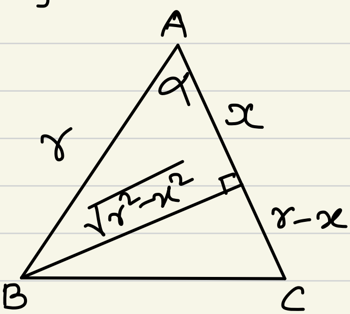
ⓑ $n = 8$:



Area of the octagon $= 8 \cdot$ Area of ΔAOB

$$= 8 \cdot \frac{1}{2} r^2 \sin(45^\circ) = 2\sqrt{2}\, r^2$$

Here we have used the fact that if we have an isosceles Δ with its congruent sides of length r forming an angle of α degrees, then its area $= \frac{1}{2} r^2 \sin(\alpha)$. It can be easily derived as follows:



$A(\Delta ABC)$

$$= \frac{1}{2} \cdot r \cdot \sqrt{r^2-x^2}$$

$$= \frac{1}{2} r \cdot (r \sin\alpha) \quad \left(\because \sin\alpha = \frac{\sqrt{r^2-x^2}}{r} \right)$$

$$= \frac{1}{2} r^2 \sin(\alpha)$$

Thus, in general, the area of the general n-gon inscribed in a circle of radius r

$$= n \cdot \frac{1}{2} r^2 \sin\left(\frac{2\pi}{n}\right).$$

Thus, we have

$$\lim_{n\to\infty} \text{Area of the } n\text{-gon} = \lim_{n\to\infty} n \cdot \frac{1}{2} r^2 \sin\left(\frac{2\pi}{n}\right)$$

$$= \pi r^2 \cdot \lim_{n\to\infty} \frac{\sin(2\pi/n)}{(2\pi/n)}$$

$$= \pi r^2 \lim_{\frac{n}{2\pi} \to \infty} \frac{\sin(2\pi/n)}{(2\pi/n)}$$

$$= \pi r^2 \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$$

$$= \pi r^2 (1)$$

$$= \pi r^2 .$$

Q 10] Yes $\int_a^b av(f)\,dx = \int_a^b f(x)\,dx$

Proof: Since $av(f) = \int_a^b f(t)\,dt$, we have

$$\int_a^b av(f)\,dx = \int_a^b \frac{1}{(b-a)} \left(\int_a^b f(t)\,dt \right) dx$$

$$= \frac{1}{(b-a)} \left(\int_a^b f(t)\,dt \right) \left(\int_a^b dx \right)$$ ($\because \int_a^b f(t)\,dt$ is a number independent of x)

$$= \frac{1}{(b-a)} \cdot \int_a^b f(t)\,dt \; (b-a)$$

$$= \int_a^b f(x)\,dx .$$

Q 11] (i) $f(x) = \frac{1}{1+x^2}$ is decreasing on $[0,1]$.

So on $[0,1]$, $\max f = f(0) = 1$
& $\min f = f(1) = \frac{1}{2}$

Thus, by max-min inequality,

$$\frac{1}{2}(1-0) \le \int_0^1 \frac{1}{1+x^2}\,dx \le 1(1-0)$$

i.e., $$\frac{1}{2} \le \int_0^1 \frac{1}{1+x^2}\,dx \le 1. \quad \text{———} \quad (*)$$

(ii) On $[0, \frac{1}{2}]$ $\max f = f(0) = 1$
& $\min f = f(\frac{1}{2}) = 4/5$

$$\Rightarrow \frac{4}{5}\cdot\frac{1}{2} \le \int_0^{1/2} \frac{1}{1+x^2}\,dx \le 1\left(\frac{1}{2}\right)$$

i.e. $$\frac{2}{5} \le \int_0^{1/2} \frac{dx}{1+x^2} \le \frac{1}{2} \quad \text{———} \quad (A)$$

Moreover, on $[\frac{1}{2}, 1]$, $\max f = f(\frac{1}{2}) = 4/5$
& $\min f = f(1) = \frac{1}{2}$

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{2} \le \int_{1/2}^{1} \frac{dx}{1+x^2} \le \frac{4}{5}\cdot\frac{1}{2}$$

i.e. $$\frac{1}{4} \le \int_{1/2}^{1} \frac{dx}{1+x^2} \le \frac{2}{5} \quad \text{———} \quad (B)$$

From (A) & (B)

$$\frac{2}{5}+\frac{1}{4} \le \int_0^1 \frac{dx}{1+x^2} \le \frac{2}{5}+\frac{1}{2}$$

i.e. $$\underset{\substack{\| \\ 0.65}}{\frac{13}{20}} \le \int_0^1 \frac{dx}{1+x^2} \le \frac{18}{20} = 0.9$$

0.65 ——— improved estimate over the one in (*)!

Ex. 2 Solve the integral $\int_0^b x\,dx$ by taking c_k to be the left endpoint of $[x_{k-1}, x_k]$ and show that you get the same answer as before.

Ans. to Ex. 2

Take $\left\{0, \frac{b}{n}, \frac{2b}{n}, \ldots, \frac{(n-1)b}{n}, b\right\}$ as a partition of $[a, b]$.

$$\Delta x_k = \Delta x = \frac{b-0}{n} = \frac{b}{n}.$$

$$c_k = \frac{(k-1)b}{n}, \quad 1 \le k \le n.$$

Hence the Riemann sum

$$= \sum_{k=1}^{n} f(c_k) \cdot \Delta x$$

$$= \sum_{k=1}^{n} \frac{(k-1)b}{n} \cdot \frac{b}{n}$$

$$= \frac{b^2}{n^2} \sum_{k=1}^{n} (k-1)$$

$$= \frac{b^2}{n^2} \left(\sum_{k=0}^{n-1} k \right)$$

$$= \frac{b^2}{n^2} \left(\sum_{k=1}^{n-1} k \right) = \frac{b^2}{n^2} \frac{n(n-1)}{2} = \frac{b^2}{2}\left(1 - \frac{1}{n}\right)$$

$$\longrightarrow \frac{b^2}{2} \text{ as } n \to \infty.$$

Ex. 3 Using a Riemann sum calculation similar to the one in the above example, show that $\int_a^b c\,dx = c(b-a)$ where c is any constant and $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$, $a<b$.

Ans. (i) Take the partition

$$\left\{a,\ a+\frac{b-a}{n},\ a+\frac{2(b-a)}{n},\ \dots,\ a+\frac{(n-1)(b-a)}{n},\ b\right\}$$

of $[a,b]$ which divides $[a,b]$ into n sub-intervals of equal width $\Delta x = \frac{b-a}{n}$.

Hence Riemann sum

$$= \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

$$= \sum_{k=1}^{n} c\left(\frac{b-a}{n}\right)$$

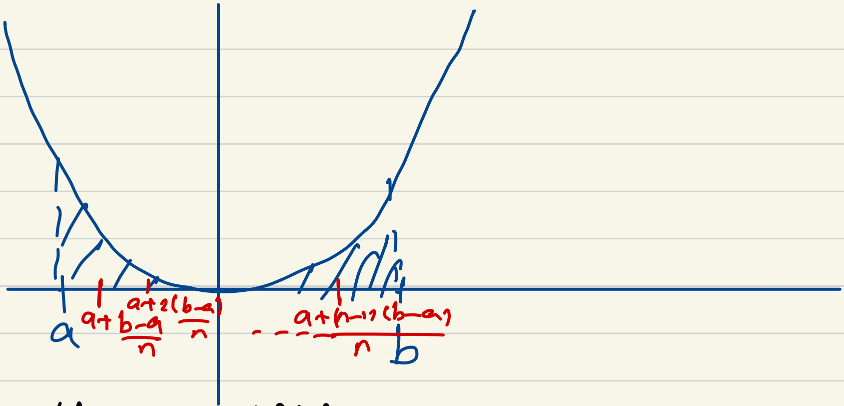
$$= \frac{(b-a)}{n}\,c\sum_{k=1}^{n}(1)$$

$$= c(b-a).$$

Hence $\int_a^b c\,dx = \lim_{n\to\infty} c(b-a) = c(b-a)$.

(ii) $\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$, $a<b$.

(ii)



Take the partition

$$\left\{a,\ a+\frac{b-a}{n},\ a+\frac{2(b-a)}{n},\ \dots,\ a+\frac{(n-1)(b-a)}{n},\ b\right\}$$

of $[a, b]$ which divides $[a, b]$ into n sub-intervals of equal width $\Delta x = \dfrac{b-a}{n}$.

Riemann sum

$$= \sum_{k=1}^{n} c_k^2 \, \Delta x .$$

But let c_k denote the right end-point of the sub-interval $[x_{k-1}, x_k]$.

Then $c_k = a + \dfrac{k(b-a)}{n}$.

Hence, the Riemann sum

$$= \sum_{k=1}^{n} \left(a + \frac{k(b-a)}{n}\right)^2 \cdot \frac{(b-a)}{n}$$

$$= \sum_{k=1}^{n} \left(a^2 + \frac{2a(b-a)k}{n} + \frac{k^2(b-a)^2}{n^2} \right) \left(\frac{b-a}{n}\right)$$

$$= \frac{a^2(b-a)}{n} \sum_{k=1}^{n} (1) + \frac{2a(b-a)^2}{n^2} \sum_{k=1}^{n} k$$

$$+ \frac{(b-a)^3}{n^3} \sum_{k=1}^{n} k^2$$

$$= a^2(b-a) + \frac{2a(b-a)^2}{n^2} \cdot \frac{n(n+1)}{2}$$

$$+ \frac{(b-a)^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$\left.\begin{aligned} &= a^2(b-a) + a(b-a)^2 \left(1+\frac{1}{n}\right) \\ &\quad + \frac{(b-a)^3}{6} \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \end{aligned}\right\} A$$

Thus, $\displaystyle\int_a^b x^2\,dx = \lim_{n\to\infty} A$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3}$$

$$= (b-a)\left\{ \cancel{a^2} + a(b-\cancel{a}) + \frac{(b-a)^2}{3} \right\}$$

$$= (b-a)\left\{ab + \frac{(b-a)^2}{3}\right\}$$

$$= (b-a)\left\{\frac{3ab + b^2 - 2ab + a^2}{3}\right\}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3}$$

$$= \frac{b^3 - a^3}{3}.$$

since $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$.