Tutorial 4 (SVC) – Solutions sketch

Q1. $f(x) = \begin{cases} 3, & x = 0 \\ -x^2+3x+a, & 0<x<1 \\ mx+b, & 1 \leq x \leq 2 \end{cases}$

For f to satisfy the hypotheses of the MVT on $[0,2]$, it should be continuous on $[0,2]$ & differentiable on $(0,2)$.

(i) Continuity on $(0,2)$ implies, in particular, continuity at 1. We must have $\lim_{x\to 1} f(x) = f(1)$, so we require

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad -(1)^2+3(1)+a = m(1)+b$$

$$\Rightarrow \quad \boxed{m+b = 2+a} \quad \text{———} \quad (1)$$

But continuity at 0 implies

$$\lim_{x\to 0^+} f(x) = f(0)$$

i.e. $\quad -(0)^2+3(0)+a = 3$

$$\Rightarrow \quad \boxed{a=3} \quad \text{———} \quad (2)$$

(ii) Differentiability on $(0,2)$ implies, in particular, differentiability at $x=1$.

So

$$f'(1-) = f'(1+)$$

$\Rightarrow \quad -2(1)+3 = m$

$\Rightarrow \quad m = 1$ ———————————— (3)

From (1), (2) & (3),

$m=1, \quad a=3$ and $b=4$

Q2. $f(x) = \begin{cases} x+2, & x \neq 0 \\ 0, & x=0 \end{cases}$, $g(x) = \begin{cases} x+1, & x \neq 0 \\ 0, & x=0 \end{cases}$

(a) $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} \dfrac{x+2}{x+1} = 2$

$\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to 0} \dfrac{(x+2)'}{(x+1)'} = \lim\limits_{x\to 0} \dfrac{1}{1} = 1.$

(b) L'Hôpital's rule requires not only $f(0)=g(0)=0$ & $g'(x) \neq 0$ for $x \neq 0$ (which, indeed, is satisfied here) but also that f & g be differentiable on an open interval containing 0. The latter is not satisfied here since $f'(0)$ & $g'(0)$ do not exist: For example,

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{(x+2)-0}{x} = \lim_{x\to 0}\left(1+\frac{2}{x}\right)$$

Thus, L'Hôpital's does not get contradicted.

Q3]. We must have $\lim\limits_{x\to 0} \dfrac{9x-3\sin(3x)}{5x^3} = c$

Use L'Hôpital's rule twice (or 3 times if you don't want to use $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$)

to get $c = 27/10$.

Q4. 1. $\lim_{x \to 0} x \csc^2(\sqrt{2x})$

$$= \lim_{x \to 0} \frac{\frac{x}{(\sqrt{2x})^2}}{\frac{\sin^2(\sqrt{2x})}{(\sqrt{2x})^2}} = \frac{\lim_{x \to 0} \frac{1}{2}}{\left(\lim_{\sqrt{2x} \to 0} \frac{\sin\sqrt{2x}}{\sqrt{2x}}\right)^2}$$

(using continuity of x^2)

$= 1/2$.

OR $\lim_{x \to 0} x \csc^2(\sqrt{2x}) = \lim_{x \to 0} \frac{x}{\sin^2(\sqrt{2x})}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{1}{2\sin(\sqrt{2x})\cos(2\sqrt{x}) \cdot \frac{\sqrt{2}}{2\sqrt{x}}}$$

$$= \sqrt{2} \lim_{x \to 0} \frac{\sqrt{x}}{\sin(2\sqrt{2x})} = \frac{1}{2} \lim_{2\sqrt{2}x \to 0} \frac{2\sqrt{2x}}{\sin(2\sqrt{2x})}$$

$$= \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

2. L'Hôpital's rule is not useful if we apply right in the beginning,

$$\lim_{x \to \infty} \frac{\sqrt{x+5}}{\sqrt{x}+5} = \lim_{x \to \infty} \frac{\sqrt{x+5}\,(\sqrt{x}-5)}{x-25} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

L'Hôpital's rule

$$\downarrow$$

$$= \lim_{x\to\infty} \frac{\left(\frac{\sqrt{x+5}}{2\sqrt{x}} + \frac{\sqrt{x}-5}{2\sqrt{x+5}}\right)}{1}$$

$$= \lim_{x\to\infty} \frac{2(x+5)+2x-10\sqrt{x}}{4\sqrt{x}\sqrt{x+5}}$$

$$= \lim_{x\to\infty} \frac{4x-10\sqrt{x}+10}{4\sqrt{x}\sqrt{x+5}}$$

$$= \lim_{x\to\infty} \frac{4+\frac{10}{\sqrt{x}}+\frac{10}{x}}{4\sqrt{1+\frac{5}{x}}}$$

$$= 1.$$

Q5. Note that the intersection of $y=\sqrt{2}$ with $y=\sec\theta\tan\theta$ in the first quadrant is $\left(\frac{\pi}{4}, \sqrt{2}\right)$

since $\sqrt{2}=\sec\theta\tan\theta$ $\left(0<\theta<\pi/2\right)$
implies $\frac{\sin\theta}{\cos^2\theta} = \sqrt{2}$

$$\Rightarrow 1-\sin^2\theta - \frac{1}{\sqrt{2}}\sin\theta = 0$$

$$\Rightarrow \sin^2\theta + \frac{1}{\sqrt{2}}\sin\theta - 1 = 0$$

$$\Rightarrow \sin\theta = \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{\frac{1}{2}-4(-1)}}{2}$$

$$= \frac{1}{\sqrt{2}} \text{ or } -\sqrt{2}$$

However $-\sqrt{2}$ is an extraneous root since $|\sin\theta| \le 1$.

Hence $\sin\theta = \frac{1}{\sqrt{2}}$, which along with $0<\theta<\frac{\pi}{2}$, implies $\theta = \pi/4$.

Hence the area of the shaded region

$$= \int_{-\pi/4}^{\pi/4} (\sqrt{2} - \sec\theta\tan\theta)\, d\theta$$

$$= \left[\sqrt{2}\theta - \sec\theta\right]_{-\pi/4}^{\pi/4} \quad (\text{by FTC})$$

$$= \sqrt{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{\sqrt{2}}$$

Q 6. (a) velocity at time $t=5$ is 2.

(b) negative, since the slope of tangent to $f(x)$ at $(5,2)$ is negative.

(c) $s(3) = \int_0^3 f(x)\,dx = \int_0^3 x\,dx = \left[\frac{x^2}{2}\right]_0^3 = \frac{9}{2}$.

(d) • $\frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)\,dx = f(t)$

• $f(t) = 0$ at $t = 0$ & 6 for $0 \le t \le 9$.

Since $s(0) = \int_0^0 f(t)\,dt = 0$, let's check whether $s''(6) < 0$, whence s would have a loc. max. at $t = 6$.

Note that the tangent to $f(x)$ at $x = 6$ is negative, hence

$$s''(6) = f'(6) < 0.$$

Hence the answer is 'at $t = 6$ sec.'.

(e) $t = 4$ & 7 sec. (Horizontal tangents)

(f) towards the origin — when velocity is negative, i.e., when $6 < t < 9$.

away from the origin — when the velocity is positive, i.e. when $0 < t < 6$

(g) To the right of origin since

$\int_0^9 f(x)\,dx > 0$. (from observing that the area under the curve above x-axis $>$ than that below the x-axis.)

Q 7. Note that $f(t) = \dfrac{t^2}{t^4+1}$ is continuous on $\mathbb{R}$, in particular on $(0, x)$ for any $x \in \mathbb{R}$. Hence, by the Fund. Thm. Calc.,

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4+1}\, dt \quad \left(\frac{0}{0} \text{ form, so use L'Hôpital's rule}\right)$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{(x^4+1)}}{3x^2} = \lim_{x \to 0} \frac{1}{3(x^4+1)} = \frac{1}{3}$$

Q 8. Let $I = \displaystyle\int \frac{\sin(\sqrt{\theta})}{\sqrt{\theta \cos^3(\sqrt{\theta})}}\, d\theta$

Let $\sqrt{\theta} = u$; $\theta = u^2 \Rightarrow d\theta = 2u\,du$

$$\Rightarrow I = \int \frac{\sin(u)\ (2u\,du)}{u\sqrt{\cos^3 u}}$$

$$= -2 \int \frac{(-\sin u)\,du}{\sqrt{\cos^3(u)}}$$

$$= -2 \int \frac{dt}{t^{3/2}} = -2\,\frac{t^{-1/2}}{(-1/2)} = \frac{4}{\sqrt{t}}$$

(Letting $t = \cos u$)

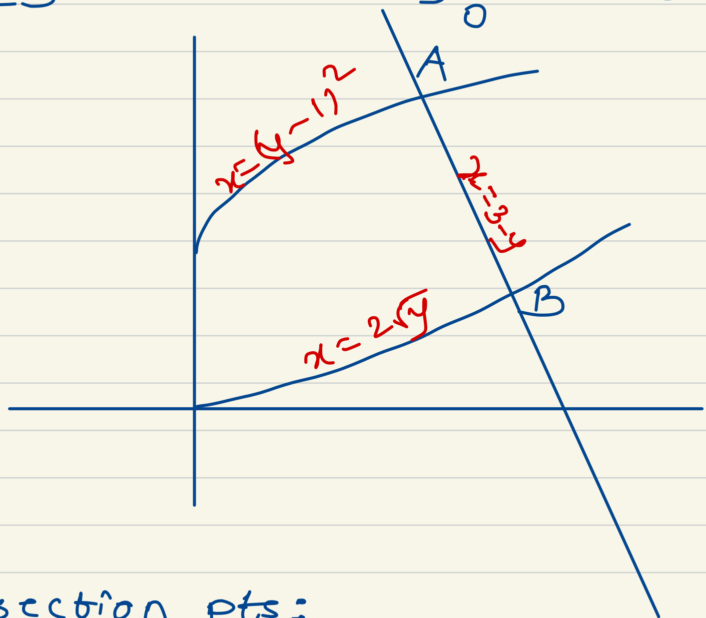
$$= \frac{4}{\sqrt{\cos u}} = \frac{4}{\sqrt{\cos\sqrt{\theta}}}.$$

Q9. Area $= \int_0^1 (12y^2 - 12y^3) - (2y^2 - 2y)\, dy$

$= \int_0^1 (10y^2 - 12y^3 + 2y)\, dy$

$= \left[\frac{10}{3}y^3 - 3y^4 + y^2\right]_0^1 = \frac{10}{3} - 3 + 1 = \frac{4}{3}.$

Q10.



Intersection pts:

$A = (1, 2)$ & $B = (2, 1)$.

Hence reqd. area

$= \int_0^1 \left(1 + \sqrt{x} - \frac{x^2}{4}\right) dx + \int_1^2 \left(3 - x - \frac{x^2}{4}\right) dx$

$= \left[x + \frac{2x^{3/2}}{3} - \frac{x^3}{12}\right]_0^1 + \left[3x - \frac{x^2}{2} - \frac{x^3}{12}\right]_1^2$

$$= 1 + \cancel{\frac{2}{3}} - \cancel{\frac{1}{12}} + 6 - 2 - \cancel{\frac{8}{12}} - 3 + \frac{1}{2} + \cancel{\frac{1}{12}}$$

$$= 2 + \frac{1}{2} = \frac{5}{2}.$$

PART II

Q1. Since $|x|<1$, if we let $y = \tanh^{-1} x$,
then $x = \tanh y$

$$\Rightarrow \frac{e^y - e^{-y}}{e^y + e^{-y}} = x,$$

$$\Rightarrow \frac{(e^y - \cancel{e^{-y}}) + (e^y + \cancel{e^{-y}})}{(\cancel{e^y} - e^{-y}) - (\cancel{e^y} + e^{-y})} = \frac{x+1}{x-1}$$

(Componendo – dividendo, or use any of your favorite method)

$$\Rightarrow -e^{2y} = \frac{x+1}{x-1}, \text{ or } e^{2y} = \frac{1+x}{1-x}$$

Take log on both sides so that

$$2y = \ln\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$\blacksquare$

Q2. Solved example 6 on p. 624 of the text. (It's important to mention that the natural logs have to combined before letting $b \to \infty$; it's initially in the $\infty - \infty$ form)

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx = \lim_{b\to\infty}\int_2^b \frac{x+3}{(x-1)(x^2+1)}\,dx$$

$$= \lim_{b\to\infty}\int_2^b \left(\frac{2}{x-1} - \frac{2x+1}{x^2+1}\right)dx \quad \text{Partial fractions}$$

$$= \lim_{b \to \infty} \left[2 \ln (x - 1) - \ln (x^2 + 1) - \tan^{-1} x \right]_2^b$$

$$= \lim_{b \to \infty} \left[\ln \frac{(x - 1)^2}{x^2 + 1} - \tan^{-1} x \right]_2^b \qquad \text{Combine the logarithms.}$$

$$= \lim_{b \to \infty} \left[\ln \left(\frac{(b - 1)^2}{b^2 + 1} \right) - \tan^{-1} b \right] - \ln \left(\frac{1}{5} \right) + \tan^{-1} 2$$

$$= 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1} 2 \approx 1.1458$$

Q3. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta \, d\theta}{\sin\theta(\cos\theta + 1)}$

Let $\tan\left(\frac{\theta}{2}\right) = t$, $-\pi < \theta < \pi$

$\Rightarrow \theta = 2\tan^{-1}(t)$ so that

$$d\theta = \frac{2}{1+t^2}\, dt .$$

Also, $\cos(\theta) = \dfrac{1-t^2}{1+t^2}$ & $\sin(\theta) = \dfrac{2t}{1+t^2}$.

When $\theta = \pi/2$, $t = \tan(\pi/4) = 1$, whereas when $\theta = 2\pi/3$, $t = \tan(\pi/3) = \sqrt{3}$

Hence $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta \, d\theta}{\sin\theta(\cos\theta+1)}$

$$= \int_1^{\sqrt{3}} \frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$$

$$= \frac{1}{2}\int_1^{\sqrt{3}} \left(\frac{1}{t} - t\right) dt$$

$$= \frac{1}{2}\left[\ln|t| - \frac{t^2}{2}\right]_1^{\sqrt{3}}$$

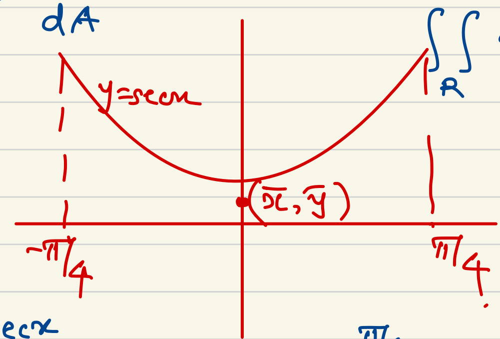
$$= \frac{1}{2}\left\{\ln(\sqrt{3}) - \frac{3}{2} - \ln(1) + \frac{1}{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2}\ln(3) - 1\right\}$$

$$= -\frac{1}{2} + \frac{1}{4}\ln(3).$$

Q4. Center of gravity $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\iint_R x \, dA}{\iint_R dA}, \quad \bar{y} = \frac{\iint_R y \, dA}{\iint_R dA}.$$



$$\bar{x} = \frac{\int_{-\pi/4}^{\pi/4}\int_0^{\sec x} x \, dy \, dx}{\int_{-\pi/4}^{\pi/4}\int_0^{\sec x} dy \, dx} = \frac{\int_{-\pi/4}^{\pi/4} x \sec x \, dx}{\int_{-\pi/4}^{\pi/4} \sec x \, dx} = 0$$

($\because$ $x \sec x$ is an odd fn. of x)

& is positive

(we'll calculate this later.)

$$\bar{y} = \frac{\int_{-\pi/4}^{\pi/4}\int_0^{\sec x} y \, dy \, dx}{\int_{-\pi/4}^{\pi/4}\int_0^{\sec x} dy \, dx} = \frac{\int_{-\pi/4}^{\pi/4} \frac{\sec^2 x}{2} \, dx}{\int_{-\pi/4}^{\pi/4} \sec x \, dx}$$

$$= \frac{\frac{1}{2}\left[\tan x\right]_{-\pi/4}^{\pi/4}}{\left[\ln\left|\sec x + \tan x\right|\right]_{-\pi/4}^{\pi/4}}$$

$$= \frac{1}{\ln\left|\sqrt{2}+1\right| - \ln\left|\sqrt{2}-1\right|}$$

$$= \frac{1}{\ln\left|\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}\right|}$$

$$= \frac{1}{2\ln(\sqrt{2}+1)}$$

Hence the center of gravity is at

$$\left(0, \frac{1}{2\ln(\sqrt{2}+1)}\right).$$

Q8. $\frac{dx}{dt} = \frac{1}{250} x(1000 - x)$, $x(0) = 2$.

Treating $\frac{dx}{dt}$ as a quotient of differentials

and rearranging, we get

$$\frac{250\,dx}{x(1000-x)} = dt$$

Integrating both sides to get

$$\int \frac{250\,dx}{x(1000-x)} = \int dt$$

$$\Rightarrow \frac{1}{4}\int \frac{x+(1000-x)}{x(1000-x)}\,dx = t+c$$

$$\Rightarrow \frac{1}{4}\left\{\int \frac{dx}{1000-x} + \int \frac{dx}{x}\right\} = t+c$$

$$\Rightarrow \frac{1}{4}\left\{\ln|x| - \ln|1000-x|\right\} = t+c$$

But it's given that $x(0)=2$ so that

$$\frac{1}{4}\left\{\ln(2) - \ln(998)\right\} = c$$

$$\Rightarrow c = \frac{-1}{4}\ln(499)$$

$$\Rightarrow \frac{1}{4}\ln\left(\frac{x}{1000-x}\right) + \frac{1}{4}\ln(499) = t$$

$$\Rightarrow \frac{499x}{1000-x} = e^{4t}$$

$$\Rightarrow 499x = 1000e^{4t} - xe^{4t}$$

$$\Rightarrow \boxed{x = \frac{1000\, e^{4t}}{e^{4t} + 499}}$$

In the second part, we have to find t so that $x = 500$

$$\Rightarrow 500\left(e^{4t} + 499\right) = 1000\, e^{4t}$$

$$\Rightarrow 500\, e^{4t} = 500 \times 499$$

$$\Rightarrow 4t = \ln(499)$$

$$\Rightarrow \underline{t = 1.553 \text{ days.}}$$

Q9.

$$x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \geq 2, \; y(2) = 0,$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2-4}}{x} \qquad (\because x \neq 0)$$

$$\Rightarrow y = \int \frac{\sqrt{x^2-4}}{x}\, dx$$

Let $x = 2\sec\theta$, $0 \leq \theta < \pi/2$; then $\theta = \sec^{-1}\left(\frac{x}{2}\right)$.

Then

$$\sqrt{x^2-4} = 2|\tan\theta|$$

$$= 2\tan\theta \qquad (\because \tan\theta \geq 0 \text{ on } 0 \leq \theta < \pi/2)$$

$$\Rightarrow \int \frac{\sqrt{x^2-4}}{x}\, dx = \int \frac{2\tan\theta}{2\sec\theta} \cdot 2\sec\theta\tan\theta\, d\theta$$

$$= 2\int \tan^2\theta\, d\theta = 2\int (\sec^2\theta - 1)\, d\theta$$

$$= 2(\tan\theta - \theta)$$

$$\Rightarrow y = 2\left[\tan\left(\sec^{-1}\left(\frac{x}{2}\right)\right) - \sec^{-1}\left(\frac{x}{2}\right)\right] + c$$

$$= 2\sqrt{\sec^2(\sec^{-1}(\tfrac{x}{2})) - 1} - 2\sec^{-1}(\tfrac{x}{2}) + c$$

$$= 2\sqrt{\tfrac{x^2}{4} - 1} - 2\sec^{-1}(\tfrac{x}{2}) + c$$

$$= \sqrt{x^2 - 4} - 2\sec^{-1}(\tfrac{x}{2}) + c$$

Now $y(2) = 0$ gives

$$0 = 0 - 2(0) + c$$

$$\Rightarrow c = 0$$

$$\Rightarrow \boxed{y = \sqrt{x^2 - 4} - 2\sec^{-1}\left(\tfrac{x}{2}\right)}$$

Q 10. Let $I = \int e^t(\sec^3(e^t - 1))\,dt$

Let $e^t - 1 = x$ so that $e^t dt = dx$

$$\Rightarrow I = \int \sec^3 x\,dx$$

Let $u = \sec x$ & $dv = \sec^2 x\,dx$
so that $du = \sec x \tan x$ & $v = \tan x$.

$$\Rightarrow I = uv - \int v\,du$$

$$= \sec x \tan x - \int \tan^2 x \sec x\,dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx$$

$$= \sec x \tan x + \int \sec x\,dx - I$$

$\Rightarrow 2I = \sec x \tan x + \ln|\sec x + \tan x|$

$\Rightarrow I = \frac{1}{2}\left\{\sec x \tan x + \ln|\sec x + \tan x|\right\} + c$

$= \frac{1}{2}\left\{\sec(e^t - 1)\tan(e^t - 1) + \ln\left|\sec(e^t-1) + \tan(e^t-1)\right|\right\} + c$

(b) $\displaystyle\int_0^{\pi/2} \frac{\sin^n x \, dx}{\sin^n x + \cos^n x}$

Let I be the above integral.

Let $x = \frac{\pi}{2} - u$ so that $dx = -du$.

When $x = 0, u = \pi/2$ & when $x = \pi/2, u = 0$

$$\Rightarrow I = \int_{\pi/2}^{0} \frac{\sin^n\left(\frac{\pi}{2} - u\right)(-du)}{\sin^n\left(\frac{\pi}{2} - u\right) + \cos^n\left(\frac{\pi}{2} - u\right)}$$

$$= \int_0^{\pi/2} \frac{\cos^n(u)\, du}{\cos^n(u) + \sin^n(u)}$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sin^n(x) + \cos^n(x)\, dx}{\sin^n(x) + \cos^n(x)}$$

$$= \pi/2$$

$$\Rightarrow I = \pi/4$$