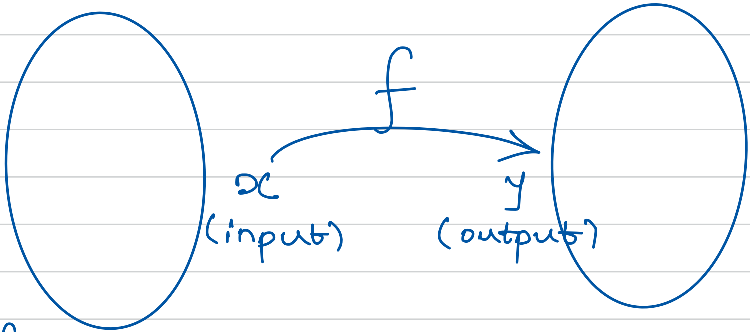
19/8/24

MA 103 – Lecture 1 (LA)

INTRODUCTION



- f: function of one variable, say, $f(x) = ax+b$.

- What about multiple inputs? Multiple outputs?

Input	Output
t (time)	$x(t)$, $y(t)$
x, y, z (coordinates)	$T = T(x,y,z)$
$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$

Input

$x_1, x_2, \ldots, x_n$

Output

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\vdots$$
$$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

Question

How to represent the function f taking $x_1, x_2, \ldots, x_n$ to $y_1, y_2, \ldots, y_m$ in the prescribed form mentioned above?

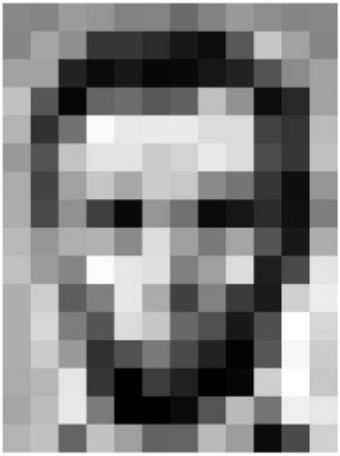
f can be written in the form of a matrix

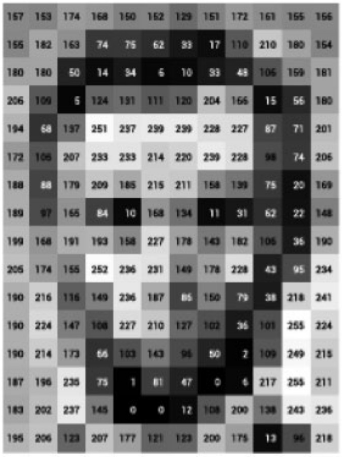
$$f = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & \cdots & & & a_{mn} \end{pmatrix}$$

An application

① Sending picture through email / Whatsapp (say)

- When we send the picture, the computer/device converts it into a matrix.





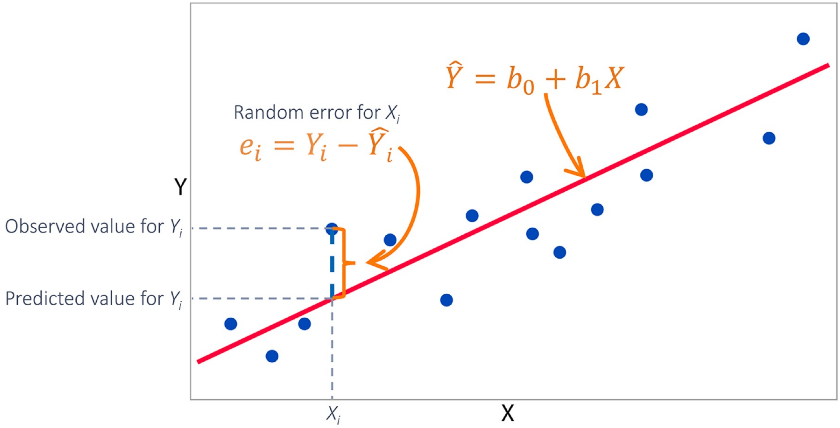
157	153	174	168	150	152	129	151	172	161	155	156
155	182	163	74	75	62	33	17	110	210	180	154
180	180	50	14	34	6	10	33	48	106	159	181
206	109	5	124	131	111	120	204	166	15	56	180
194	68	137	251	237	239	239	228	227	87	71	201
172	105	207	233	233	214	220	239	228	98	74	206
188	88	179	209	185	215	211	158	139	75	20	169
189	97	165	84	10	168	134	11	31	62	22	148
199	168	191	193	158	227	178	143	182	106	36	190
205	174	155	252	236	231	149	178	228	43	95	234
190	216	116	149	236	187	86	150	79	38	218	241
190	224	147	108	227	210	127	102	36	101	255	224
190	214	173	66	103	143	96	50	2	109	249	215
187	196	235	75	1	81	47	0	6	217	255	211
183	202	237	145	0	0	12	108	200	138	243	236
195	206	123	207	177	121	123	200	175	13	96	218

Image processing / Data compression
— resize, rotate, transmit

— Singular Value Decomposition (SVD)

② <u>Least squares fitting</u> —

Given a set of data points (x_i, y_i), how to get the best linear fit for this data.



Sect. 1.1 – SYSTEMS OF LINEAR EQUATIONS

Defn. A linear equation in 'n' variables $x_1, x_2, \ldots, x_n$ is an equation which can be put in the form

$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$, where a_i, $1 \le i \le n$, and b are constants.
(Not all a_i's are zero)

If $b = 0$, we call it a homogeneous linear equation in the variables $x_1, \ldots, x_n$.

System of linear equations : A finite set of linear equations.

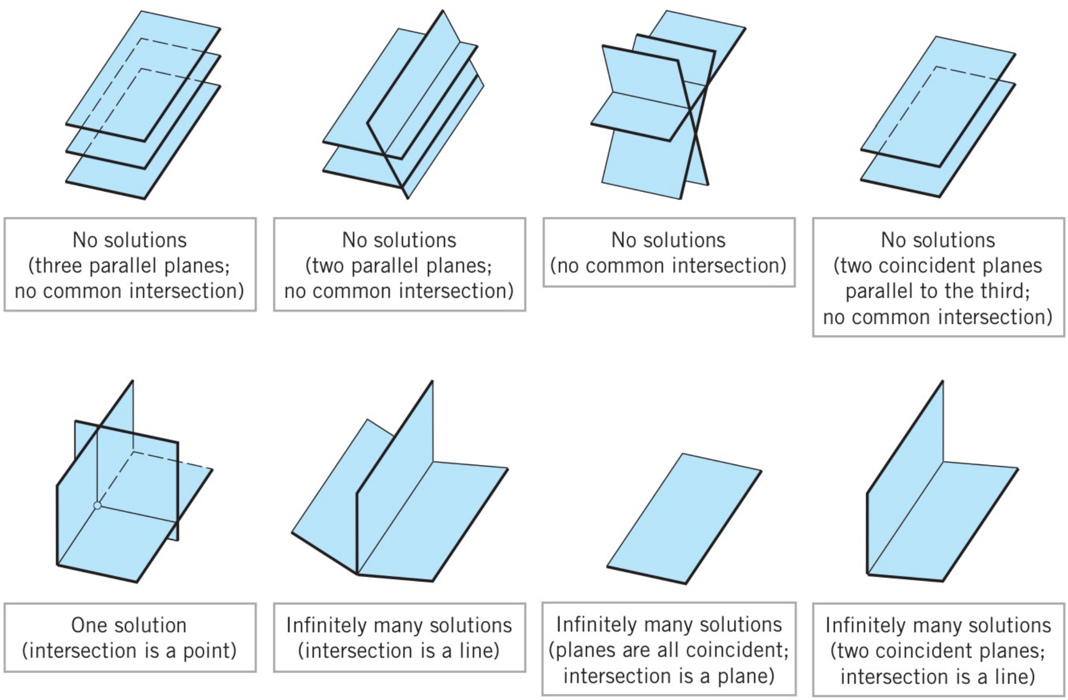
Eg. $3x + 4y = 1$
$2x - y = 0$ (x & y are called unknowns)

Three possibilities in general :

- intersect at a single point (above eg.)
- represent the same line
  eg. $2x - 7y = 3$
  $10x - 35y = 15$
- parallel lines (no intersection)
  $x - y = 3$
  $2x - 2y = 5$

Even if we go with m equations in n unknowns, still we have only 3 possibilities! That is,

Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.



▲ Figure 1.1.2

An arbitrary linear system of m equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Homogenous provided $b_1 = b_2 = \cdots = b_m = 0$, otherwise inhomogeneous / non-homogeneous.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

(Coefficient matrix) (Augmented matrix)

BASIC METHOD FOR SOLVING A LINEAR SYSTEM

— perform algebraic operations on it that do not alter the solution and produce simpler and simpler systems to the point where it can be easily whether a system is consistent or not, and where the solutions can be found in the former case.

Algebraic operations (Elementary row operations) (EROS)

① Multiply an equation by a non-zero constant

$$R_i \to cR_i \quad (c \neq 0)$$

② Interchange two equations

$$R_i \leftrightarrow R_j$$

③ Add a constant times one equation to another

$$R_i \to R_i + bR_j$$

Eg. Suppose we are given the following system:

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \xrightarrow{\text{Augmented matrix}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right] \qquad R_2 \to R_2/2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$R_1 \to R_1 - R_2$; $R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 11/2 & 35/2 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{array}\right]$$

Complete the problem hereafter.

Sect. 1.2 – GAUSSIAN ELIMINATION

- Systematic method for solving a linear system.

- Need to know
  - Reduced Row Echelon Form (RREF)
  - Row Echelon Form (REF)

Rows

zero row (all entries are zero)

Non-zero row (at least one non-zero entry)

Defn. A matrix is said to be in **Reduced Row Echelon Form (RREF)** if

(i) The first non-zero number of a non-zero row is 1 (called a leading 1)

(ii) Zero rows are flushed to the bottom.

(iii) In any 2 successive non-zero rows, the leading 1 of the lower row occurs farther to the right of the leading 1 in the higher row.

(iv) Each column containing a leading 1 has zeros everywhere in that column.

Defn: A matrix satisfying (i), (ii) & (iii) but not necessarily (iv) is said to be in Reduced Echelon Form (REF).

Important: RREF $\Longrightarrow$ REF
(converse may not be true)

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

RREF
REF

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

REF

$$\begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF