Sect. 1.3 – Matrices and matrix operations

- A matrix is a rectangular array of numbers. (Numbers are called entries.)

$$\begin{bmatrix} 2 & -3 & 4 & 0 & 1 \\ 1 & 2 & 4 & 0 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

A : $m\times n$ matrix is written in the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

→ an entry of matrix is of the form $(a_{ij})_{m\times n}$ or $[a_{ij}]_{m\times n}$

- M has size $m\times n$ if it has m rows and n columns

<u>Entries</u>: the entry in i^{th} row & j^{th} column is denoted by $(A)_{ij}$, that is, $(A)_{ij} = a_{ij}$.

<u>Example</u>: If $A = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix}$,

then $(A)_{32} = 0$, $(A)_{12} = -2$, $(A)_{21} = 1$.

Row matrix: $\vec{a} = (a_1, a_2, \dots, a_n)$ (size $1 \times n$)

Column matrix: $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ (size $m \times 1$)

Square matrix: has n rows & n columns

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$ Main diagonal

- A number is a 1×1 matrix.

OPERATIONS ON MATRICES

Equality: Two matrices A and B are said to be equal if

(i) they have the same size

(ii) their corresponding entries are equal, that is, $(A)_{ij} = (B)_{ij}$ $\forall\, i, j$.

Addition: $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$

Subtraction: $(A-B)_{ij} = (A)_{ij} - (B)_{ij}$.

Scalar multiplication:
Let c be a scalar, say, $c \in \mathbb{R}$ (or $c \in \mathbb{C}$).
The scalar multiple cA of A is defined by

$$(cA)_{ij} = c(A)_{ij}$$

(eg: if $A = \begin{bmatrix} 1 & 7 \\ 2 & 3 \end{bmatrix}$, then $3A = \begin{bmatrix} 3 & 21 \\ 6 & 9 \end{bmatrix}$).

Linear combination: of $A_1, A_2, \ldots, A_r$ with coefficients $c_1, \ldots, c_r$ is
$c_1 A_1 + c_2 A_2 + \cdots + c_r A_r$

Transpose: Transpose of A is denoted A^T, and is defined by $(A^T)_{ij} = A_{ji}$.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \to A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

THEOREM 1.4.8 *If the sizes of the matrices are such that the stated operations can be performed, then:*

(*a*) $(A^T)^T = A$

(*b*) $(A + B)^T = A^T + B^T$

(*c*) $(A - B)^T = A^T - B^T$

(*d*) $(kA)^T = kA^T$

(*e*) $(AB)^T = B^T A^T$

Power: Let $r \in \mathbb{N}$ and A be a square matrix.
We define $A^r = \underbrace{A \cdot A \cdot \cdots \cdot A}_{r \text{ times}}$

Trace: (defined only for a square matrix)
For an $n \times n$ matrix A, we define
$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

Product
Let A be a $m \times r$ matrix.
Let B be a $r \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

Then
① $(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ir} b_{rj}$
(Row – column rule)

Other ways of computing AB

② $$AB = \begin{pmatrix} \text{——} \; a_1 \; \text{——} \\ \text{——} \; a_2 \; \text{——} \\ \vdots \\ \text{——} \; a_m \; \text{——} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \vdots \\ b_{r1} & \cdots & \cdots & b_{rn} \end{pmatrix}$$

$$= \begin{pmatrix} \text{---} & a_1 & \text{---} \\ & \vdots & \\ \text{---} & a_m & \text{---} \end{pmatrix} B = \begin{pmatrix} \text{---} & a_1 B & \text{---} \\ & \vdots & \\ \text{---} & a_m B & \text{---} \end{pmatrix}_{m \times n}.$$

Each $\underbrace{a_i}_{1 \times r} \cdot \underbrace{B}_{r \times n}$ is of size $1 \times n$

Hence

$$\boxed{i^{th} \text{ row of } AB = (i^{th} \text{ row of } A) \cdot B}$$

③ $$AB = A \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ab_1 & \cdots & Ab_n \\ | & & | \end{pmatrix}_{m \times n}$$

Each $\underbrace{A}_{m \times r} \cdot \underbrace{b_j}_{r \times 1}$ is of size $m \times 1$.

Hence,

$$\boxed{j^{th} \text{ column of } AB = A \cdot (j^{th} \text{ column of } B)}$$

④ Column sum rule

$$\underbrace{A}_{m \times r} \underbrace{B}_{r \times n} = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_r \\ | & | & & | \end{pmatrix} \begin{pmatrix} \text{---} & r_1 & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_r & \text{---} \end{pmatrix}$$

$$= c_1 r_1 + c_2 r_2 + \cdots + c_r r_r.$$

Each $\underbrace{c_i}_{m \times 1} \underbrace{r_i}_{1 \times n}$ is of size $m \times n$

Example Find AB below using each of (1)-(4)

$$AB = \underset{2\times 2}{\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix}}$$

(1) $$AB = \begin{bmatrix} (1)(2)+(3)(-3) & (1)(0)+3(5) & (1)(4)+3(1) \\ 2(2)+(-1)(-3) & (2)(0)+(-1)(5) & (2)(4)+(-1)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

(2) $$AB = \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

(3) 1^{st} column of AB $$= \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2-9 \\ 4+3 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 7 \end{bmatrix}$$

(4) $$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} \begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

MATRIX FORM OF A LINEAR SYSTEM

Given a linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \longrightarrow (*)$$

We can write it in matrix form as

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{\text{coeff}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

$\boxed{Ax = b}$ ← matrix equation

Augmented matrix for the above system:

$[A \mid b]$

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$ **[Commutative law for matrix addition]**

(b) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(c) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(d) $A(B + C) = AB + AC$ **[Left distributive law]**

(e) $(B + C)A = BA + CA$ **[Right distributive law]**

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

Defn. A zero matrix is a matrix whose all entries are zero. (denoted by 0 or $0_{m\times n}$)

THEOREM 1.4.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be perfomed, then:

(a) $A + 0 = 0 + A = A$

(b) $A - 0 = A$

(c) $A - A = A + (-A) = 0$

(d) $0A = 0$

(e) *If* $cA = 0$, *then* $c = 0$ *or* $A = 0$.

Bottomline: We can do everything that we do with real numbers in the matrices setting.

Exceptions?

1. AB need not equal BA (even though both are defined)

2. $AB = 0$ need not imply $A = 0$ or $B = 0$

3. $AB = AC$ and $A \neq 0$ need not imply $B = C$.

Exercise: Construct an example for each of the above statements.

Real numbers	Matrices
$0: \; 0 + a = a + 0 = a$	$O_{n \times n}: \; O + A = A + O = A$
$1: \; 1 \cdot a = a \cdot 1 = a$	$I_{n \times n}$ (or I_n) $= \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$ (More generally: $I_m \cdot A_{m \times n} = A_{m \times n} = A_{m \times n} \cdot I_n$)
$\frac{1}{a}: \; \frac{1}{a} \cdot a = a \cdot \frac{1}{a} = 1$ $(a \neq 0)$	A^{-1}?

<u>Defn.</u> Let A be an $n\times n$ matrix. If $\exists$ an $n\times n$ matrix B such that

$$AB = BA = I_n, \quad \text{then}$$

① A is said to be invertible/non-singular

② B is said to be the inverse of A (denoted by $B = A^{-1}$).

If $\nexists$ such a B, we say A is singular.

- Thm. 1.4.4 If a matrix has an inverse, it is unique.

Proof: Suppose B, C are both inverses of A.

$AB = I \implies C(AB) = CI$

$BA = I \quad \implies (CA)B = C$

$(BA)C = IC \quad \implies IB = C$

$B(AC) = C \quad \implies B = C$.

$BI = C \implies B = C$.

Properties of inverse of a matrix:

Let A, B be two $n\times n$ invertible matrices.

① $(AB)^{-1} = B^{-1}A^{-1}$.

② $(A^n)^{-1} = (A^{-1})^n$ for any $n \in \mathbb{N}$.

$[\underbrace{A^{-1}\cdot A^{-1}\cdot A^{-1}\cdots A^{-1}}_{n \text{ times}}$ is denoted by $A^{-n}]$

$A^0 = I_n$.

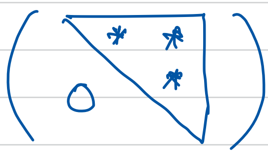
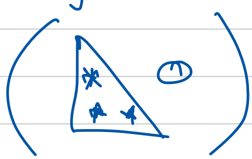
③ $(A^{-1})^{-1} = A$

④ $(kA)^{-1} = \frac{1}{k} A^{-1}$ for any $k \in \mathbb{R}$, $k \neq 0$.

⑤ $(A^T)^{-1} = (A^{-1})^T$.

SPECIAL MATRICES

Let A be a square matrix.

Name	Definition
A is diagonal	$(A)_{ij} = 0 \ \forall\ i \neq j$ $\begin{pmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & \ddots \ c_n \end{pmatrix}$
A is upper triangular	$(A)_{ij} = 0 \ \forall\ i > j$ 
A is lower triangular	$(A)_{ij} = 0 \ \forall\ i < j$ 
A is symmetric	$(A)_{ij} = (A)_{ji} \ \forall\ i,j$ $\iff A^T = A$

Inverses of above matrices are of the same type if they exist.

① Diagonal

$$\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \xrightarrow{\text{inverse}} \begin{pmatrix} 1/d_1 & & & 0 \\ & 1/d_2 & & \\ & & \ddots & \\ 0 & & & 1/d_n \end{pmatrix}$$

② Triangular

$$A = \begin{pmatrix} \diagdown & * \\ 0 & \diagdown \end{pmatrix} \xrightarrow[\text{ERO's } E_1, E_2, \dots E_k]{\text{can do}}$$

that disrupt only $*$ entries & get

$$A^{-1} = (E_k \cdots E_1)^{-1}.$$

③ Symmetric : $A^T = A$

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

$\Rightarrow A^{-1}$ is also symmetric.