Vandermonde determinant

let $A_{n+1} = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}_{n+1 \times n+1}$

To show that:

$$\det A_{n+1} = \prod_{0 \le i < j \le n} (c_j - c_i) \quad — \circledast$$

<u>Sol^n</u> Let's put $n=1$

Then $A_2 = \begin{pmatrix} 1 & c_0 \\ 1 & c_1 \end{pmatrix}$ has

$\det A_2 = (c_1 - c_0)$

$\therefore$ $\circledast$ works for $n=1$

This is the base step for Induction.

<u>Induction Hypothesis</u>:

Let us assume $\circledast$ holds for n. i.e.

$$\det A_n = \det \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^{n-1} \\ 1 & \vdots & \vdots & & \vdots \\ \vdots & & & & \\ 1 & c_{n-1} & & \cdots & c_{n-1}^{n-1} \end{pmatrix} = \prod_{0 \le i < j \le n-1} (c_j - c_i)$$

We can also relabel & write this as

$$\det \begin{pmatrix} 1 & d_1 & d_1^2 & \cdots & d_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & d_n & d_n^2 & \cdots & d_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (d_j - d_i) \quad — \circledast$$

We will now try to prove ⊛ for $n+1$.

$$\det A_{n+1} = \det \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & c_0 & c_0^2 & & c_0^n \\ 0 & c_1 - c_0 & c_1^2 - c_0^2 & \cdots & c_1^n - c_0^n \\ \vdots & & & & \\ 0 & c_{n-1} - c_0 & & \cdots & c_{n-1}^n - c_0^n \\ 0 & c_n - c_0 & c_n^2 - c_0^2 & \cdots & c_n^n - c_0^n \end{pmatrix}$$

(we do $R_2 - \cancel{c_0}R_1$ ⋮ $R_n - \cancel{c_0}R_1$)

$$= \det \begin{pmatrix} c_1 - c_0 & c_1^2 - c_0^2 & \cdots & c_1^n - c_0^n \\ c_{n-1} - c_0 & & \cdots & c_{n-1}^n - c_0^n \\ c_n - c_0 & c_n^2 - c_0^2 & \cdots & c_n^n - c_0^n \end{pmatrix}$$

(by cofactor exp^n along Col 1)

$$= \det \begin{pmatrix} c_1 - c_0 & \cdots & c_n - c_0 \\ c_1^2 - c_0^2 & \cdots & c_n^2 - c_0^2 \\ \vdots & & \vdots \\ c_1^n - c_0^n & \cdots & c_n^n - c_0^n \end{pmatrix}$$

(transpose)

$c_1^n - c_0^n - c_0(c_1^{n-1} - c_0^{n-1})$

$= c_1^n - \cancel{c_0^n} - c_0 c_1^{n-1} + \cancel{c_0^n}$

$= c_1^{n-1}(c_1 - c_0)$

$$= \det \begin{pmatrix} c_1 - c_0 & \cdots & c_n - c_0 \\ c_1(c_1 - c_0) & & c_n(c_n - c_0) \\ \vdots & & \vdots \\ c_1^{n-2}(c_1 - c_0) & \cdots & c_n^{n-2}(c_n - c_0) \\ c_1^{n-1}(c_1 - c_0) & \cdots & c_n^{n-1}(c_n - c_0) \end{pmatrix}$$

($R_n \to R_n - c_0 R_{n-1}$
$R_{n-1} \to R_{n-1} - c_0 R_{n-2}$
⋮
$R_2 \to R_2 - c_0 R_1$)

in this order!

$$= (c_1-c_0)\cdots(c_n-c_0)\ \det\begin{pmatrix} 1 & \cdots & 1 \\ c_1 & & c_n \\ c_1^2 & & \vdots \\ \vdots & & \vdots \\ c_1^{n-2} & & c_n^{n-2} \\ c_1^{n-1} & \cdots & c_n^{n-1} \end{pmatrix}$$

(Taking $c_j - c_0$ factor out from j^{th} row for each j)

$$= (c_1-c_0)\cdots(c_n-c_0)\ \underbrace{\det\begin{pmatrix} 1 & c_1 & \cdots & c_1^{n-1} \\ 1 & c_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & c_n & & c_n^{n-1} \end{pmatrix}_{n\times n}}$$

$$= \prod_{1\le i<j\le n}(c_j-c_i) \quad \text{(by ⊛ above)}$$

$\therefore \det A_n$

$$= (c_1-c_0)\cdots(c_n-c_0)\cdot\underbrace{\prod_{1\le i<j\le n}(c_j-c_i)}_{\substack{(c_2-c_1)(c_3-c_1)\cdots(c_n-c_1) \\ (c_3-c_2)\cdots\cdots}}$$

$$= \prod_{0\le i<j\le n}(c_j-c_i) \qquad \square$$

Lecture 7 — Vector spaces

Consider $\mathbb{R}^n$: the elements are vectors / column matrices.

- Any two vectors can be added to get another vector in $\mathbb{R}^n$
- we can scalar multiply a vector with any $k \in \mathbb{R}$ to get a new vector.

$+, \cdot$ satisfy certain properties:

Eg: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$1 \cdot \vec{u} = \vec{u}$

$k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}$.

Q: Do there exist sets of other objects that have $+$ and $\cdot$ operations satisfying similar properties as in $\mathbb{R}^n$?

Ans. Yes!

Eg: ① Take the set

$F(-\infty, \infty) = \{ \text{functions } f \mid f : (-\infty, \infty) \to \mathbb{R} \}$.

$+$: $f+g$ is a function defined by
$(f+g)(x) = f(x) + g(x)$

$\cdot$ kf is a function defined by
$(kf)(x) = k(f(x))$

Check that $f+g = g+f \Rightarrow (f+g)(x) = (g+f)(x)$

$\qquad\qquad\qquad\qquad f(x)+g(x) \qquad g(x)+f(x)$

$1 \cdot f = f \quad (\because (1 \cdot f)(x) = 1 \cdot f(x) = f(x))$

$(k\cdot(f+g))(x) = k\cdot(f+g)(x) = k(f(x)+g(x))$

$= kf(x) + kg(x) = (kf)(x) + (kg)(x)$

$= (k\cdot f + k\cdot g)(x).$

$\Rightarrow k\cdot(f+g) = k\cdot f + k\cdot g.$

② Set of all 2×2 matrices

$M_{22} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}.$

$+$: usual matrix addition

$\cdot$ $k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \in M_{22}.$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Def: Real vector space

We say V is a vector space (v.s.) over $\mathbb{R}$ if there are operations $+$ and $\cdot$ defined on V such that $\forall\, u,v \in V$ and $\forall\, k \in \mathbb{R}$, we have

A0 $u+v \in V$ (closed under $+$)

A1 $u+v = v+u$ (commutative)

A2 $u+(v+w) = (u+v)+w$ (associative)

A3 $\exists$ an element in V, called 0_V s.t. $0_V + u = u \;\; \forall\, u \in V$ (additive identity)

A4 For any $u \in V$, $\exists$ an element called $-u$ s.t. $u+(-u) = 0_V$ (additive inverse)

M0 $k\cdot u \in V$ (closed under $\cdot$)

M1 $1\cdot u = u$ (multiplicative identity)

M2 $(k\cdot m)u = k\cdot(mu)$

D1 $k\cdot(u+v) = k\cdot u + k\cdot v$

D2 $(k+m)u = ku + mu$

$\Big\}$ Distributivity

If the above axioms hold for scalars $k, m \in \mathbb{C}$, we say V is a vector space over $\mathbb{C}$.

Trivial examples: $V = \phi$, $V = \{0\}$.

Eg. ① $\mathbb{R}^n$ $(n \in \mathbb{N})$ is a v.s. over $\mathbb{R}$.
$= \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$

② $M_{mn} = \{$ all $m \times n$ matrices with entries in $\mathbb{R}\}$

$+$ on M_{mn} : $A + B = (a_{ij} + b_{ij}) \in M_{m \times n}$

$\cdot$ on M_{mn} : $kA = (k a_{ij}) \in M_{mn}$.

A_0 $A, B \in M_{mn}$
$A + B \in M_{mn}$
$A + B = (a_{ij} + b_{ij}) = (b_{ij} + a_{ij}) = B + A$
$A + (B + C) = (A + B) + C$
$0_V = (0_{ij})$ ← add. idty.
$-A = (-a_{ij})$ ← add. inverse

$k.A = (k a_{ij}) \in M_{mn}$
$1.A = (1.a_{ij}) = (a_{ij}) = A$.

Check that M_2, D_1, D_2 are satisfied.

③ Prove that
$V = \{ f \mid f : (-\infty, \infty) \to \mathbb{R} \}$ is a vector space over $\mathbb{R}$.

Proof:

A0 : $f+g \in V$ ($\because$ domain = codomain = $\mathbb{R}$)

A1 : $f+g = g+f$

A2 : $((f+g)+h)(x) = (f+g)(x) + h(x)$
$= f(x)+g(x)+h(x)$
$= f(x) + (g(x)+h(x))$
$= f(x) + (g+h)(x)$
$= (f+(g+h))(x)$

A3 : Let $0_V : (-\infty, \infty) \to \mathbb{R}$ $\ni$
$0_V(x) = 0$.

Then $(f+0_V)(x) = f(x) + 0_V(x)$
$= f(x) + 0$
$= f(x)$
$\Rightarrow f + 0_V = f$.

additive identity

A4 : Consider $(-f)(x) = -f(x)$
$-f : (-\infty, \infty) \to \mathbb{R}$
$\Rightarrow -f \in V$
& $(f+(-f))(x) = f(x) + (-f)(x)$
$= f(x) - f(x)$
$= 0$.

$\underline{M0}$: $(k\cdot f)(x) = \underline{\underline{k\cdot f(x)}} \in \mathbb{R}$ $\quad$ $kf : (-\infty,\infty) \to \mathbb{R}$

$\uparrow \in \mathbb{R}$ $\quad$ $kf \in V$.

$\underline{M1}$: $(1\cdot f)(x) = 1\cdot f(x) = f(x)$

$\Rightarrow 1\cdot f = f$.

1 is the multiplicative identity

$\underline{M2}$: $(km\cdot f)(x) = kmf(x)$

$= k(mf(x))$

$= k\,(m\cdot f)(x)$

$\underline{D1}$

$k\cdot(f+g))(x) = k\cdot(f+g)(x) = k(f(x)+g(x))$

$= kf(x) + kg(x) = (kf)(x) + (kg)(x)$

$= (k\cdot f + k\cdot g)(x)$

$\Rightarrow k\cdot(f+g) = k\cdot f + k\cdot g$.

$\underline{D2}$ $((k+m)\cdot f)(x) = (k+m)\cdot f(x)$

$= k\cdot f(x) + m\cdot f(x)$

$= ((k\cdot f) + (m\cdot f))(x)$

$\Rightarrow (k+m)\cdot f = k\cdot f + m\cdot f$.

$\Rightarrow$ V is vector space over R.

An example of a set which is not a vector space

Take $\mathbb{R}^2$ with usual $+$ but $\cdot$ defined by

$k\,u = (k u_1, 0)$

$\parallel$

$k \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$1 \cdot u = (1 \cdot u_1, 0) = (u_1, 0) \neq u$

$= (u_1, u_2)$

THEOREM 4.1.1 *Let V be a vector space,* $\mathbf{u}$ *a vector in V, and k a scalar; then:*

(*a*) $\overset{\in \mathbb{R}}{0}\mathbf{u} = \mathbf{0} \in V$

(*b*) $k\mathbf{0} = \mathbf{0}$

(*c*) $(-1)\mathbf{u} = -\mathbf{u}$

(*d*) *If* $k\mathbf{u} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{u} = \mathbf{0}$.

Proof: (a) $0 \cdot u = (0+0)\,u$ $\quad (0 = 0+0 \text{ in } \mathbb{R})$

$= 0 \cdot u + 0 \cdot u$ $\quad$ (by D2)

Now $0 \cdot u$ is a vector in V. So it has additive inverse $-0 \cdot u$ by A4.

$\Rightarrow 0 \cdot u - 0 \cdot u = 0 \cdot u + 0 \cdot u - 0 \cdot u$

$= (0 \cdot u) + (0 \cdot u - 0 \cdot u)$

$\Rightarrow \quad 0_V = 0_V + 0 \cdot u \quad$ (by A4 & A2)
$\Rightarrow \quad 0 \cdot u = 0_V \quad$ (by A3).

SUBSPACES

Def. Let V be a vector space. Then $W \subseteq V$ is called a subspace of V if W is itself a vector space under $+$ and $\cdot$ defined on V.

Criteria $W \subseteq V$ is a subspace of V iff

S1 $0_V \in W$ (here, 0_V is the additive inverse of V from A3)

S2 If $u, v \in W$ then $u + v \in W$

S3 If $u \in W$, $k \in \mathbb{R}$, then $ku \in W$

scalar multiple of u, as defined in V

Eg ① The zero subspace $W = \{0_V\} \subseteq V$ is a subspace of V.
Similarly, $V \subseteq V$ ———————— " ————————

Eg ② Which ones are subspaces of $\mathbb{R}^2$?