MA 101 – Tutorial 6 (Solutions)

Q1] Find coordinate vector of $\bar{w}$ w.r.t. basis $S = \{\bar{u}_1, \bar{u}_2\}$ for $\mathbb{R}^2$.

(a) $\bar{u}_1 = (2, -4)$, $\bar{u}_2 = (3, 8)$; $\bar{w} = (1, 1)$.

Let $\bar{w} = c_1\bar{u}_1 + c_2\bar{u}_2$

$(1, 1) = c_1(2, -4) + c_2(3, 8)$

$1 = 2c_1 + 3c_2$

$1 = -4c_1 + 8c_2$

Solve for c_1, c_2 to get the coordinate vector (c_1, c_2)

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ -4 & 8 & 1 \end{array}\right] \quad R_1 \to \frac{R_1}{2}$$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 1/2 \\ -4 & 8 & 1 \end{array}\right]$$

$R_2 \to R_2 + 4R_1$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 1/2 \\ 0 & 14 & 3 \end{array}\right]$$

$R_2 \to R_2/14$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 1/2 \\ 0 & 1 & 3/14 \end{array}\right]$$

$\Rightarrow c_2 = 3/14$

$c_1 = \frac{1}{2} - \frac{3}{2}\left(\frac{3}{14}\right) = \frac{5}{28}$

$\Rightarrow$ coor. vect. of w w.r.t. S is

$\left(\frac{5}{28}, \frac{3}{14}\right) = [W]_S$

(b) Similar

trans. matrix

Q 2] : $[V]_{B'} = P_{B \to B'} [V]_B$

$B = \{\bar{u}_1, \bar{u}_2\}$, $B' = \{\bar{u}_1', \bar{u}_2'\}$ for R^2,

$\bar{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $u_1' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $u_2' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

(a) $P_{B' \to B} = \left[\begin{array}{c|c} B & B' \end{array}\right]$

$$= \left[\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right]$$

Convert to RREF form

$R_1 \to R_1/2$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1/2 & -1/2 \\ 2 & -1 & 3 & -1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1/2 & -1/2 \\ 0 & -5 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2/-5$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1/2 & -1/2 \\ 0 & 1 & -\frac{2}{5} & 0 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 13/10 & -1/2 \\ 0 & 1 & -2/5 & 0 \end{array}\right]$$

Hence $P_{B' \to B} = \begin{bmatrix} 13/10 & -1/2 \\ -2/5 & 0 \end{bmatrix}$

Another method: Express the basis vectors of B' w.r.t. those in B.

$[\bar{u}_1']_B$: $\begin{pmatrix} 1 \\ 3 \end{pmatrix} = c_1 \bar{u}_1 + c_2 \bar{u}_2$

$\Rightarrow c_1 = \frac{13}{10}, \quad c_2 = \frac{-2}{5}$

$[\bar{u}_1']_B = \begin{pmatrix} 13/10 \\ -2/5 \end{pmatrix}$

Similarly $[\bar{u}_2']_B = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$.

$\Rightarrow P_{B' \to B} = \begin{bmatrix} 13/10 & -1/2 \\ -2/5 & 0 \end{bmatrix}$

(b) $P_{B \to B'} = \left(P_{B' \to B}\right)^{-1}$

$= \begin{bmatrix} 13/10 & -1/2 \\ -2/5 & 0 \end{bmatrix}^{-1} = \frac{1}{\left(-1/5\right)} \begin{bmatrix} 0 & 1/2 \\ 2/5 & 13/10 \end{bmatrix}$

$= \begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix}$

since

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(c) $[\bar{w}]_B$ where $\bar{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ & use

$[\bar{v}]_{B'} = P_{B \to B'} [\bar{v}]_B$ to get $[w]_{B'}$

(d) Find $[\bar{w}]_{B'}$ directly

similar to previous problems

3] Let S be st. basis for $\mathbb{R}^2$

$$S = \{\bar{e}_1, \bar{e}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$B = \{\bar{v}_1, \bar{v}_2\}$, where $\bar{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\bar{v}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

(a) Consider $\left[\begin{array}{cc:cc} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & 4 \end{array}\right]$ with $\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I}$

$$\Rightarrow P_{B \to S} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

[new basis | old basis] = [I | transition from old to new]

(b), (c) similar to previous problems

(d) $[w]_S = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$$[w]_B = \underbrace{P_{S \to B}}\ [w]_S$$

$$= \frac{1}{11}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -3/11 \\ -13/11 \end{bmatrix}.$$

4] Find a basis for
(i) rowspace & (ii) nullspace of A,
where

(a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

Rowspace = span $\{(1,-1,3), (5,-4,-4), (7,-6,2)\}$

If we find the l.i. rows, then they will form a basis for rowspace since they will not only span the space but also be l.i.

Perform ERO on A:

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 7R_1$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_1 \to R_1 + R_2$

$$\begin{bmatrix} \boxed{1} & 0 & -16 \\ 0 & \boxed{1} & -19 \\ 0 & 0 & 0 \end{bmatrix}$$ RREF form.

Note: we did not do any row interchanges.

$\therefore$ R_3 is a lin. comb. of R_1 & R_2.

$\Longrightarrow$ $\{R_1, R_2\}$ forms a basis for the rowspace.

$\quad\ \ \| $

$(1,-1,3), (5,-4,-4)$

$$\text{Nullspace} = \{\bar{x} \in \mathbb{R}^3 : A\bar{x} = 0\}$$

$= $ solution set of $A\bar{x} = 0$.

Find the general solution to $A\bar{x} = 0$,

i.e.

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow \text{ERO}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 - 16x_3 = 0$

$x_2 - 19x_3 = 0$

$\Rightarrow$ Put $x_3 = t$

Sol^n set, i.e. the null space

$$= \left\{ \begin{pmatrix} 16t \\ 19t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix} t : t \in \mathbb{R} \right\}$$

(b) Similar

Q 5] Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a lin. transf. defined by $T(x_1, x_2) = (x_1 + 3x_2, x_1 - x_2, x_1)$

(a) Rank of std. matrix for T

(b) Nullity —— " ——.

(a) Standard matrix for T:

$$A = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix}$$

$$T(e_1) = T((1,0)) = (1+3(0), 1-0, 1)$$
$$= (1, 1, 1)$$

$$T(e_2) = T((0,1)) = (0+3(1), 0-1, 0)$$
$$= (3, -1, 0)$$

$$\Rightarrow A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

rank(A) = max. # of lin. ind. columns of A

$c_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ & $c_2 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ will be lin. Indep. if they are not scalar mult. of each other.

Let $\lambda_1 c_1 + \lambda_2 c_2 = 0$

$$\begin{pmatrix} \lambda_1 + 3\lambda_2 \\ \cancel{0}\ \lambda_1 - \lambda_2 \\ \lambda_1 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 0. \end{matrix}$$

They are lin. ind.

$\Rightarrow$ rank = 2.

By
rank-Nullity theorem,

rank(A) + nullity(A) = # of columns of A.

$\Rightarrow$ nullity $= 2-2 = 0$,

6) ⓐ $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Ax = \lambda x$

$Ax = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4x$

Hence x is an eigenvector of A.
eigenvalue = 4.

ⓑ Similar.

⑦ ⓐ $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Char. eqn : $\det(A - \lambda I) = 0$.

$A - \lambda I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

Char. eqn : $\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0$

$\Rightarrow (1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right] = 0$

$\Rightarrow (1-\lambda)\left[4-5\lambda+\lambda^2+2\right] = 0$

$(1-\lambda)(\lambda^2-5\lambda+6) = 0$

$\Rightarrow \lambda = 1, 2, 3$

Alg. mult.: (AM) of $\lambda_1 = 1$

of $\lambda_2 = 1$

of $\lambda_3 = 1$

exponent of the factor corr. to the eigenvalue in the characteristic eqn

Geometric multiplicity (GM) of λ = dimension of eigenspace E_λ.

Let us find the eigenspaces:

$$E_{\lambda_1} = \{x \in \mathbb{R}^3, x \neq 0, Ax = x\}$$

$$= \{x \in \mathbb{R}^3 \setminus \{0\}, (A-I)x = 0\}$$

want the sol^n set of

$$\left(\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -2x_1 = 0$

$3x_1 + x_3 = 0$

$x_1 = 0, \ x_3 = 0$

x_2 is a free variable, say t.

$$E_{\lambda_1} = \left\{ \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R}, t \neq 0 \right\}$$

Since we omit the zero vector.

$$= \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t : t \in \mathbb{R}, t \neq 0 \right\}$$

Basis for $E_{\lambda_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Dimension of $E_{\lambda_1} = 1$.

$\Rightarrow$ GM of $\lambda_1 = 1$.

Similarly, show that GM of $\lambda_2 = 1$

& GM of $\lambda_3 = 1$.

A useful inequality:

$1 \leq GM \leq AM$

this is because $E_\lambda \cup \{0\}$ is a subspace of $\mathbb{R}^n$ containing at least one non-zero vector.

(c) $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 4-\lambda & 0 & -1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$(4-\lambda)(3-\lambda)(2-\lambda) + 1(3-\lambda) = 0$

$\Rightarrow (3-\lambda)\left[(4-\lambda)(2-\lambda)+1\right] = 0$

$\Rightarrow (3-\lambda)(8-6\lambda+\lambda^2+1) = 0$

$\Rightarrow (3-\lambda)(\lambda^2-6\lambda+9) = 0$

$\Rightarrow \lambda = 3$

AM of $\lambda = 3$

Hence GM can be 1, 2 or 3.

Eigenspace solve:

$$\left\{ x \in \mathbb{R}^3 \setminus \{0\} : \underbrace{Ax = 3x}_{\Rightarrow (A-3I)x = 0} \right\}$$

$(A - 3I)x = 0$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 - x_3 = 0$

x_2 is free, say t.

x_3 is free too, say r.

Eigenspace E_λ $= \left\{ \begin{pmatrix} r \\ t \\ r \end{pmatrix} : (r, t) \in \mathbb{R}^2 \setminus \{0, 0\} \right\}$

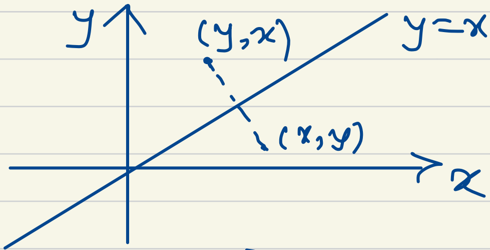
$$= \left\{ \underbrace{t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{basis vectors}} : (r, t) \in \mathbb{R}^2 \setminus \{0, 0\} \right\}.$$

$\Rightarrow$ GM of $\lambda = 2$

Q1] $T_A : \mathbb{R}^n \to \mathbb{R}^n$ given by $T_A(x) = cx$, where x is an eigenvector of A corresponding to the eigenvalue c.

(a) Reflection about the line $y = x$

$T(x, y) = (y, x)$



(Standard matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$)

$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$

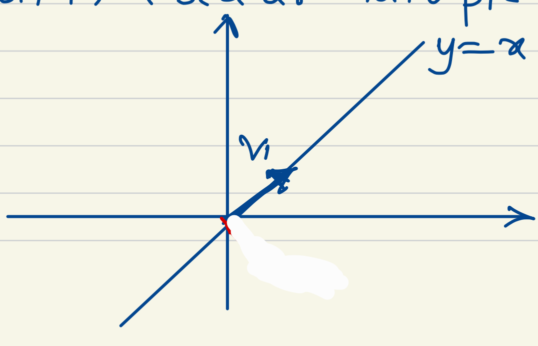
$\Rightarrow \lambda^2 - 1 = 0$

$\Rightarrow \lambda = \pm 1$)

Geometric reasoning:

As the std. matrix is 2×2, there are at most 2 linear independent eigenvectors.

An eigenvector is a vector whose image after reflection is a scalar multiple of itself.



By observation one such vector can be taken along $y = x$, say $v_1 = (1, 1)$

Reflection gives

$$T_A v_1 = 1 \cdot v_1$$

Another can be taken to be $(1, -1)$, say v_2. Then $T_A v_2 = -v_2$ as the reflection simply reverses the direction of v_2.

So $(1, 1)$ & $(1, -1)$ are 2 lin. ind. eigen-vectors, and all others are $c_1 v_2, c_2 v_2$ for some $c_i \in \mathbb{R}$, $i = 1, 2$.

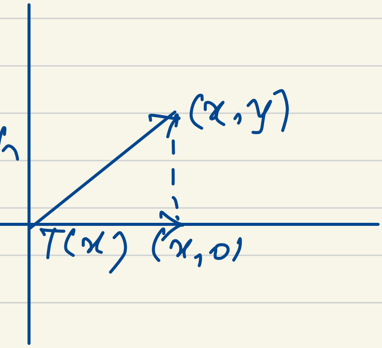
$$\text{Eigenvalues} = \{1, -1\}$$

Eigenspaces: $E_1 = \left\{ t\begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \setminus \{0\} \right\}$

$$E_{-1} = \left\{ t\begin{pmatrix} 1 \\ -1 \end{pmatrix} : t \in \mathbb{R} \setminus \{0\} \right\}.$$

(b) $T(x, y) = (x, 0)$

The only vectors which are fixed or reversed by this map are the ones on the x-axis.



e.g. $v_1 = (1, 0)$, $T_A v_1 = v_1$

Eigenvalue $= \{1\}$

Eigenspace $E_1 = \left\{ c\begin{pmatrix}1\\0\end{pmatrix} : c \in \mathbb{R}\setminus\{0\} \right\}$

(c) Rotation about the origin thro' a pos. angle of 90°.

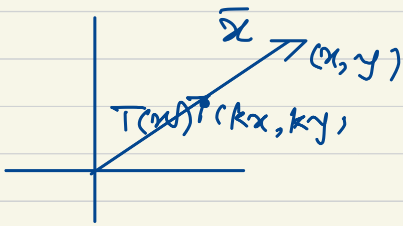
For any $v \in \mathbb{R}^2$, Tv is not a scalar multiple of v as its direction changes by 90°

$\therefore$ No eigenvectors (we consider only real eigenvalues).

Eigenvalues (in $\mathbb{R}$) $= \phi$

we cannot get eigenspaces.

(d) contraction with a factor k in $\mathbb{R}^2$ $(0 \le k < 1)$



Let $k \neq 0$:

Any vector's direction is preserved by the contraction map.

$\Rightarrow$ Every non-zero vector in $\mathbb{R}^2$ is an eigenvector.

$$T(x,y) = k(x,y)$$

$\Rightarrow$ k is the eigenvalue.

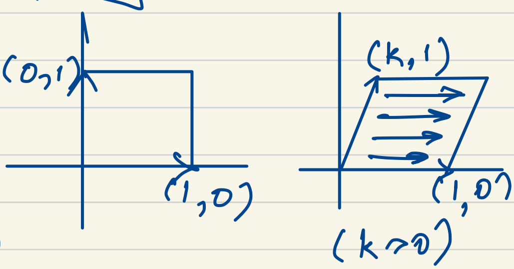
Eigenspace $= \mathbb{R}^2 \setminus \{0,0\}$

Note: $GM = 2$
$AM = 2$.

$k=0$:
The map is $T(x,y) = (0,0)$
Every non-zero vector in $\mathbb{R}^2$ is an eigenvector with eigenvalue 0.

Eigenvalue $= \{0\}$
Eigenspace $= \mathbb{R}^2 \setminus \{0,0\}$

(c) Shear in the x-direction by factor k in $\mathbb{R}^2$
$T(x,y) = (x+ky, y)$



From the fig; $(1,0)$ is fixed by T (same for any vector on x-axis)
$\therefore V_1 = (1,0)$ $\quad TV_1 = 1 \cdot V_1$

Any other vector where y-component $\neq 0$ will have its direction changed due to shear.
Eigenvalue $= \{1\}$
Eigenspace $= \left\{ t \begin{pmatrix} 1 \\ 0 \end{pmatrix} : t \in \mathbb{R}^2 \setminus \{0,0\} \right\}$.

Q 2] ⓐ $\underbrace{(\lambda-1)(\lambda+3)(\lambda-5)}_{p(\lambda)}=0$

ⓐ $\deg p(\lambda) = 3.$

$\Rightarrow$ matrix is 3×3.

(Note that $p(\lambda) = \det(\lambda I - A)$)

Let $\lambda_1 = 1,\ \lambda_2 = -3,\ \lambda_3 = 5$

$AM(\lambda_1) = AM(\lambda_2) = AM(\lambda_3) = 1$

Since $1 \le GM \le AM$,

$GM(\lambda_1) = GM(\lambda_2) = GM(\lambda_3) = 1,$

ⓑ $\underbrace{\lambda^2(\lambda-1)(\lambda-2)^3}_{p(\lambda)} = 0$

Since $\deg(\lambda) = 6$, A is 6×6 matrix.

$\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 2$

$AM(\lambda_1) = 2$ $\quad AM(\lambda_2) = 1$ $\quad AM(\lambda_3) = 3$

$GM(\lambda_1) = 1$ or 2 $\quad GM(\lambda_2) = 1$ $\quad GM(\lambda_3) = 1$ or 2 or 3.

Q 3] Show A and B are not similar matrices.

Defn: Two matrices A and B are said to be similar if $\exists$ matrix $P \ni$

$$B = PAP^{-1}$$

(a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$

$\det(A) = 4(4) - (-1)(2) = 16 + 2 = 18$

$\det(B) = 16 - 2 = 14$

$\det(A) \neq \det(B)$

Hence A and B are not similar

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(A) = 1(1-0) = 1$

$\det(B) = 1(1-1) = 0$

$$\det(A - xI) = \det\begin{pmatrix} 1-x & 2 & 3 \\ 0 & 1-x & 2 \\ 0 & 0 & 1-x \end{pmatrix}$$

$$= (1-x)^3$$

$$\det(B - xI) = \det\begin{pmatrix} 1-x & 2 & 0 \\ \frac{1}{2} & 1-x & 0 \\ 0 & 0 & 1-x \end{pmatrix}$$

$$= (1-x)^3 - 2\left(\frac{1}{2}\right)(1-x)$$

$$= (1-x)^3 - (1-x)$$

$$= (1-x)\left[(1-x)^2 - 1\right]$$

$$= (1-x)\left[x^2 - 2x\right]$$

$$= (1-x)\,x\,(x-2)$$

This implies that A and B are not similar. This is because, if they were similar, we must have had $\det(A - xI) = \det(B - xI)$ as proved below.

<u>Proposition</u>: If A & B are similar then $\det(B - xI) = \det(A - xI)$

<u>Proof</u>: $\det(B - xI) = \det(PAP^{-1} - xI)$

$= \det(PAP^{-1} - x(PIP^{-1}))$

$= \det(PAP^{-1} - P(xI)P^{-1})$ since x is a variable acting as a scalar

$= \det(P(A - xI)P^{-1})$

$= \det(P)\det(A - xI)\det(P^{-1})$

$= \det(A - xI)$ ▨

Q5] $\mathbb{R}^4$ with Euclidean inner product

$\bar{u} = (-1, 1, 0, 2)$

$\bar{w}_1 = (1, -1, 3, 0)$, $\bar{w}_2 = (4, 0, 9, 2)$

Ans. $\bar{u} \cdot \bar{w}_1 = (-1, 1, 0, 2) \cdot (1, -1, 3, 0)$

$= (-1)(1) + (1)(-1) + 0(3) + 2(0)$

$= -1 - 1 = -2 \neq 0$

$\bar{u} \cdot \bar{w}_2 = (-1, 1, 0, 2) \cdot (4, 0, 9, 2)$

$= -4 + 4 = 0$.

No, it is not orthogonal.

Q 6] $\mathbb{R}^4$ with Euclidean innerproduct

$\bar{u}_1 = (0, 2, 1, 0)$, $\bar{u}_2 = (1, -1, 0, 0)$

$\bar{u}_3 = (1, 2, 0, -1)$ $\bar{u}_4 = (1, 0, 0, 1)$

Sol^n. Step 1: Take $\bar{v}_1 = \bar{u}_1$

Step 2: $\bar{v}_2 = \bar{u}_2 - \text{proj}_{\bar{v}_1} \bar{u}_2$

$$= \bar{u}_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$\langle u_2, v_1 \rangle = \bar{u}_2 \cdot \bar{v}_1 = \bar{u}_2 \cdot \bar{u}_1$
$= -2$

$\|v_1\|^2 = \bar{v}_1 \cdot \bar{v}_1 = \bar{u}_1 \cdot \bar{u}_1 = 4 + 1 = 5$

$\Rightarrow \bar{v}_2 = (1, -1, 0, 0) + \frac{2}{5}(0, 2, 1, 0)$

$= \left(1, \frac{-1}{5}, \frac{2}{5}, 0\right)$.

Step 3: $\bar{v}_3 = \bar{u}_3 - \text{proj}_{\text{span}\{\bar{v}_1, \bar{v}_2\}} \bar{u}_3$

$$= \bar{u}_3 - \left(\frac{\langle \bar{u}_3, \bar{v}_1 \rangle}{\|\bar{v}_1\|^2} \bar{v}_1 + \frac{\langle \bar{u}_3, \bar{v}_2 \rangle}{\|\bar{v}_2\|^2} \bar{v}_2 \right)$$

$\langle \bar{u}_3, \bar{v}_1 \rangle = \bar{u}_3 \cdot \bar{v}_1 = \bar{u}_3 \cdot \bar{u}_1$
$= 4$

$\langle \bar{u}_3, \bar{v}_2 \rangle = \bar{u}_3 \cdot \bar{v}_2 = (1)(1) + (2)\left(\frac{-1}{5}\right)$

$= 1 - \frac{2}{5} = \frac{3}{5}$

$$\|V_2\|^2 = \bar{V}_2 \cdot \bar{V}_2 = 1 + \frac{1}{25} + \frac{4}{25} = 1 + \frac{5}{25} = \frac{6}{5}.$$

$$\Rightarrow \bar{V}_3 = (1, 2, 0, -1) - \left(\frac{4}{5}(0, 2, 1, 0) + \frac{3/5}{6/5}\left(1, \frac{-1}{5}, \frac{2}{5}, 0\right) \right)$$

$$= (1, 2, 0, -1) - \left(0, \frac{8}{5}, \frac{4}{5}, 0\right) - \left(\frac{1}{2}, \frac{-1}{10}, \frac{1}{5}, 0\right)$$

$$= \left(\frac{1}{2}, 2 - \frac{8}{5} + \frac{1}{10}, -\frac{4}{5} - \frac{1}{5}, -1 \right)$$

$$= \left(\frac{1}{2}, \frac{20 - 16 + 1}{10}, -1, -1 \right)$$

$$= \left(\frac{1}{2}, \frac{1}{2}, -1, -1 \right)$$

Step 4: $\bar{V}_4 = \bar{u}_4 - \text{proj}_{\text{span}\{v_1, v_2, v_3\}} \bar{u}_4$

$$= \bar{u}_4 - \frac{\langle \bar{u}_4, \bar{v}_1 \rangle}{\|\bar{V}_1\|^2} \bar{V}_1 - \frac{\langle \bar{u}_4, \bar{v}_2 \rangle}{\|\bar{V}_2\|^2} \bar{V}_2 - \frac{\langle \bar{u}_4, \bar{V}_3 \rangle}{\|\bar{V}_3\|^2} \bar{V}_3$$

$$\langle \bar{u}_4, \bar{v}_1 \rangle = \bar{u}_4 \cdot \bar{v}_1 = 0$$

$$\langle \bar{u}_4, \bar{v}_2 \rangle = \bar{u}_4 \cdot \bar{v}_2 = (1, 0, 0, 1) \cdot \left(1, \frac{-1}{5}, \frac{2}{5}, 0\right) = 1$$

$$\langle \bar{u}_4, \bar{v}_3 \rangle = \bar{u}_4 \cdot \bar{v}_3 = (1, 0, 0, 1) \cdot \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= \frac{1}{2} - 1 \quad -1/2$$

$$\|v_3\|^2 = \bar{v}_3 \cdot \bar{v}_3 = \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right) \cdot \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= \frac{1}{4} + \frac{1}{4} + 1 + 1 = \frac{5}{2}.$$

$$\Rightarrow \bar{v}_4 = (1, 0, 0, 1) - \frac{1}{(6/5)}\left(1, \frac{-1}{5}, \frac{2}{5}, 0\right)$$

$$- \frac{(-1/2)}{(5/2)}\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)$$

$$= (1, 0, 0, 1) - \left(\frac{5}{6}, \frac{-1}{6}, \frac{2}{6}, 0\right)$$

$$+ \left(\frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5}\right)$$

$$= \left(\frac{4}{15}, \frac{4}{15}, \frac{-8}{15}, \frac{4}{5}\right)$$

$\Rightarrow \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ will form an OG basis for $\mathbb{R}^4$.

To get an ON basis, normalize $\bar{v}_i$'s, $i = 1, 2, 3, 4$

$$\hat{v}_1 = \frac{\bar{v}_1}{\|v_1\|} = \frac{(0, 2, 1, 0)}{\sqrt{5}}$$

$$\hat{v}_2 = \frac{\left(1, \frac{-1}{5}, \frac{2}{5}, 0\right)}{\sqrt{6/5}} = \left(\sqrt{\frac{5}{6}}, \frac{-1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right)$$

$$\hat{V}_3 = \frac{\left(\frac{1}{2}, \frac{1}{2}, -1, -1\right)}{\sqrt{5/2}} = \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}\right)$$

$$\hat{V}_4 = \frac{\left(\frac{4}{15}, \frac{4}{15}, \frac{-8}{15}, \frac{4}{5}\right)}{\sqrt{\frac{4^2+4^2+(-8)^2+12^2}{15^2}}}$$

$$= \frac{1}{\sqrt{15}}(1, 1, -2, 3).$$

Q4] (a) $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$

$\det(A - \lambda I)$

$$= \begin{vmatrix} 19-\lambda & -9 & -6 \\ 25 & -11-\lambda & -9 \\ 17 & -9 & -4-\lambda \end{vmatrix}$$

$$= (19-\lambda)\left[(11+\lambda)(4+\lambda) - 81\right]$$

$$+ 9\left[25(-4-\lambda) + 17(9)\right]$$

$$-6\left(-(25)(9) + (11+\lambda)(17)\right) = 0$$

$$= (2-\lambda)(x-\lambda)^2 = 0$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 2$$

$AM(\lambda_1) = 2$ $\quad\quad AM(\lambda_2) = 1$

$GM(\lambda_2) = 1$

Eigenspace for λ_1

$$E_{\lambda_1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \setminus \{0\} : (A - I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\Rightarrow \begin{bmatrix} 18 & -9 & -6 \\ 25 & -12 & -9 \\ 17 & -9 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use EROs to get $x_1 = x_3$

$x_3 = \frac{3}{4} x_2 \Rightarrow x_2 = \frac{4}{3} x_3$

$x_3 = t, \quad x_1 = t, \quad x_2 = \frac{4}{3} t.$

$$E_{\lambda_1} = \left\{ t \begin{pmatrix} 1 \\ 4/3 \\ 1 \end{pmatrix} : t \in \mathbb{R} \setminus \{0\} \right\}.$$

$GM(\lambda_1) = 1 \neq AM$

$\Rightarrow$ A is not diagonalizable.