MA 103 Lecture 11

Sect. 11.7 – Power series

DEFINITIONS **Power Series, Center, Coefficients**

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \quad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \quad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Examples: (1) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ — converges for $-1 < x < 1$ (or, in other words, for $|x| < 1$)

(2) $1 - \frac{1}{2}(x-2) + \frac{(x-2)^2}{4} - \cdots + \left(\frac{-1}{2}\right)^n (x-2)^n + \cdots$

Here, comparing with the defn. in (2), we see that $a = 2$, $c_0 = -\frac{1}{2}$, $c_1 = \frac{1}{4}$, $c_2 = \frac{-1}{8}$, $\ldots$ $c_n = \left(\frac{-1}{2}\right)^n$. Hence this is a power series centered at 2 with n^{th} coeff. $c_n = \left(\frac{-1}{2}\right)^n$.

It is a geometric series with $r = \frac{-1}{2}(x-2)$.
It converges for $\left|\frac{-1}{2}(x-2)\right| < 1$, i.e., when $|x-2| < 2$ or, in other words, when $-2 < x-2 < 2$, i.e. $0 < x < 4$.
& it converges to $\dfrac{1}{1-\left(\frac{-1}{2}(x-2)\right)} = \dfrac{1}{1+\frac{x-2}{2}}$

$= \dfrac{2}{x}$.

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ $\quad -1 < x \le 1$

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ $\quad -1 \le x \le 1$

Ex 3 **(c)** $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad \frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n} = \frac{x}{n+1} \xrightarrow{n\to\infty} 0 < 1$

(d) $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

$\frac{(n+1)!x^{n+1}}{n!\,x^n} = (n+1)x \to \infty$ for any x.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$ is a power series centered at $x=0$ with the n^{th} coefficient $c_n = (-1)^{n-1}/n$. Regard the above series as $\sum u_n$ with $u_n = \frac{(-1)^{n-1}}{n}x^n$. It converges absolutely for $|x|<1$ because $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{(-1)^n}{n+1}x^{n+1}}{\frac{(-1)^{n-1}}{n}x^n}\right|$

$= \frac{n}{n+1}|x|$, so $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x|$.

So if $|x|<1$ then the series conv. abs. by ratio test (and hence $\sum u_n$ also conv.)

If $|x|>1$, then $\sum u_n$ diverges since n^{th} term does not go to zero. To prove $\lim_{n\to\infty} u_n \neq 0$, it suffices to show $\lim_{n\to\infty}|u_n| \neq 0$

$\lim_{n\to\infty} u_n = 0$ if and only if $\lim_{n\to\infty}|u_n| = 0$.

$-|u_n| \le u_n \le |u_n|$

$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \left|\frac{(-1)^{n-1}x^n}{n}\right| = \lim_{n\to\infty} \frac{|x|^n}{n} \neq 0$

Now when $x=1$, then $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, the alternating harmonic series, which converges.

Now let $x=-1$. Then $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n}$

$= -\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

Conclusion: $\sum_{n=1}^{\infty} u_n$ converges for $-1 < x \leq 1$.

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\cdots$

Here $u_n = \frac{(-1)^{n-1} x^{2n-1}}{2n-1}$

$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1} \cdot |x^2|$

$= \left(\frac{2n-1}{2n+1}\right) x^2 \longrightarrow x^2$ as $n \to \infty$.

By ratio test, $\sum u_n$ conv. abs. for $x^2 < 1$ and hence $\sum u_n$ converges for $x^2 < 1$. & diverges for $x^2 > 1$ because the n^{th} term does not converge to zero.

For $x=1$, $\sum_{n=1}^{\infty} \frac{(-1)^{n-N}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$
(Mādhava–Gregory series

For $x=-1$, we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^{2n-1}}{2n-1}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$, again an alternating series which conv. by alt. ser. test.

Hence $\sum_{n=1}^{\infty} u_n$ conv. for $-1 \le x \le 1$,

THEOREM 18 **The Convergence Theorem for Power Series**

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

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COROLLARY TO THEOREM 18

The convergence of the series $\sum c_n (x - a)^n$ is described by one of the following three possibilities:

1. There is a positive number R such that the series diverges for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

Defns. The number R in the above corollary is called the radius of convergence of the power series $\sum c_n (x-a)^n$ & the interval $(a-R, a+R)$ is called the interval of convergence of the power series. Sometimes the interval of convergence can also be $[a-R, a+R)$ or $(a-R, a+R]$ or $[a-R, a+R]$.

In the interval of convergence $(a-R, a+R)$, the series converges absolutely & for $|x-a| > R$, it diverges. (If it converges at one or both the endpoints, the convergence may not be absolute.)

- If the series converges $\forall x$, we say $R = \infty$
- If it converges only at $x = a$, we say $R = 0$.

How to Test a Power Series for Convergence

1. *Use the Ratio Test (or nth-Root Test) to find the interval where the series converges absolutely*. Ordinarily, this is an open interval

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint*, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. *If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$* (it does not even converge conditionally), because the nth term does not approach zero for those values of x.

Term-by-term differentiation theorem

If $\sum c_n (x-a)^n$ converges for $a-R<x<a+R$, where $R>0$, then it defines a fn. f : $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, in $a-R<x<a+R$.

• Such a function has derivatives of all order in $(a-R, a+R)$. These derivatives can be found by term-of-term differentiation of $\sum c_n (x-a)^n$:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

derived series & so on.

These derived power series have the same radius of convergence R.

Example: Find $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

Here $a=0$, $R=1$, By the term-by-term diff. thm.;

$c_n = 1 \ \forall n \in \mathbb{N}$

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n \cdot 1 \cdot x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$$
$$= 1 + 2x + 3x^2 + \cdots, \quad |x| < 1$$

$$f''(x) = \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2} = 2 + 6x + 12x^2 + \cdots, \quad |x| < 1$$

THEOREM 20 The Term-by-Term Integration Theorem

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R \ (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

Example:

Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad -1 < x < 1.$$

$$\underset{\frac{1}{1+x^2}}{f'(x)} = \underbrace{1 - x^2 + x^4 - x^6 + \cdots}_{\text{Geom. ser. with 1st term } = 1 \ \& \ r = -x^2}, \quad -1 < x < 1$$

Also $|-x^2| = |x|^2 < 1$

Hence $f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$

Applying the term-by-term integration thm.,

$$\int f'(x)\,dx = \int \frac{1}{1+x^2}\,dx + c$$

$$\Rightarrow f(x) = \tan^{-1}(x) + c$$

Now that $f(0)=0$. Hence $C=0$.

$$\Rightarrow f(x) = \tan^{-1}(x). \Rightarrow \lim_{x\to 1^-}\tan^{-1}(x) = \lim_{x\to 1^-} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$-1 < x < 1$. $\quad -1 < x \le 1$

It is also true that the above representation is valid at $x = \pm 1$, (*)

Ex.1 Obtain a power series representation for $\ln(1+x)$ for $-1<x<1$. What can we say for $x=1$?

Multiplication of power series

THEOREM 21 The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Example Multiply the power series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$, $-1<x<1$, to get a power series for $\frac{1}{(1-x)^2}$ in $-1<x<1$.

Abel's test for unif. conv.

(*) Let $f_n(x)$ be a non-increasing seq. of functions s.t. $0 \le f_n(x) \le M$ $\forall x \in [a,b]$. If $\sum a_n$ conv., then $\sum a_n f_n(x)$ conv. unif. in $[a,b]$.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

valid for $-1 < x < 1$

Integrating term-by-term, we have

$$\int \frac{1}{1+x}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$+c$ ~~for~~ $-1 < x < 1$

$$\Rightarrow \ln(1+x) + c = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Let $x = 0$

$\Rightarrow 0 + c = 0$

$\Rightarrow c = 0$

$$\Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

in $-1 < x < 1$,

It is true that this formula holds for $x = 1$ as well.
(Note that the series conv. for $x = 1$ by Alt. ser. test)

$A(x)$

$$= \frac{1}{1-x} = 1+x+x^2+\cdots = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$B''(x)$

$$\Rightarrow c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} (1)(1) = n+1$$

Then by the above result, $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for $-1<x<1$, and.

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1)x^n$$

$$= \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x}$$

$$= \frac{1}{(1-x)^2}.$$

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=1}^{\infty} n x^{n-1}, \quad |x|<1$$