Tutorial 1 (SVC) – Solutions sketch

Q1. (a) $\lim_{x \to 1^-} g(x) = 1 \neq 0 = \lim_{x \to 1^+} g(x)$, so does not exist.

(b) $\lim_{x \to 2} g(x) = 1$

(c) $\lim_{x \to 3} g(x) = 0$.

Q2.
$$\lim_{t \to -1} \frac{t^2+3t+2}{t^2-t-2} = \lim_{t \to -1} \frac{(t+1)(t+2)}{(t+1)(t-2)}$$
$$= \lim_{t \to -1} \frac{t+2}{t-2}$$
$$= \frac{-1}{3}$$

Q3.
$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$
$$\Rightarrow \quad -x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Now use squeeze theorem.

Q4. The rigorous proof using $\varepsilon - \delta$ is not expected here. We can just say that

$$\sin\left(\frac{1}{x}\right) = \begin{cases} \sin\left(\frac{n\pi}{2}\right) = 1, \text{ if } x = \frac{2}{n\pi}, & n = 4k+1, \ k \in \mathbb{Z} \\ \sin\left(\frac{n\pi}{2}\right) = -1, \text{ if } x = \frac{2}{n\pi}, & n = 4k+3, \ k \in \mathbb{Z} \end{cases}$$

& since $\sin(\frac{1}{x})$ is continuous $\forall\, x \in \mathbb{R}\setminus\{0\}$ & the sequence $\left\{\frac{2}{n\pi}\right\}_{n=1}^{\infty} \to 0$, we see that

$\sin(\frac{1}{x})$ oscillates infinitely as x becomes closer & closer to zero (both from left and from right).

Hence $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

Q5.

$$\lim_{r\to\infty} \frac{r+\sin r}{2r+7-5\sin r} = \lim_{r\to\infty} \frac{1+\frac{\sin r}{r}}{2+\frac{7}{r}-5\frac{\sin r}{r}}$$

$$= \frac{1+0}{2+0-0} = \frac{1}{2}.$$

Q6. (a) $\lim_{x\to 0^+}\left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right)$

$= +\infty$ (since $\lim_{x\to 0^+} \frac{1}{(x-1)^{4/3}} = \frac{1}{((-1)^{1/3})^4} = 1$)

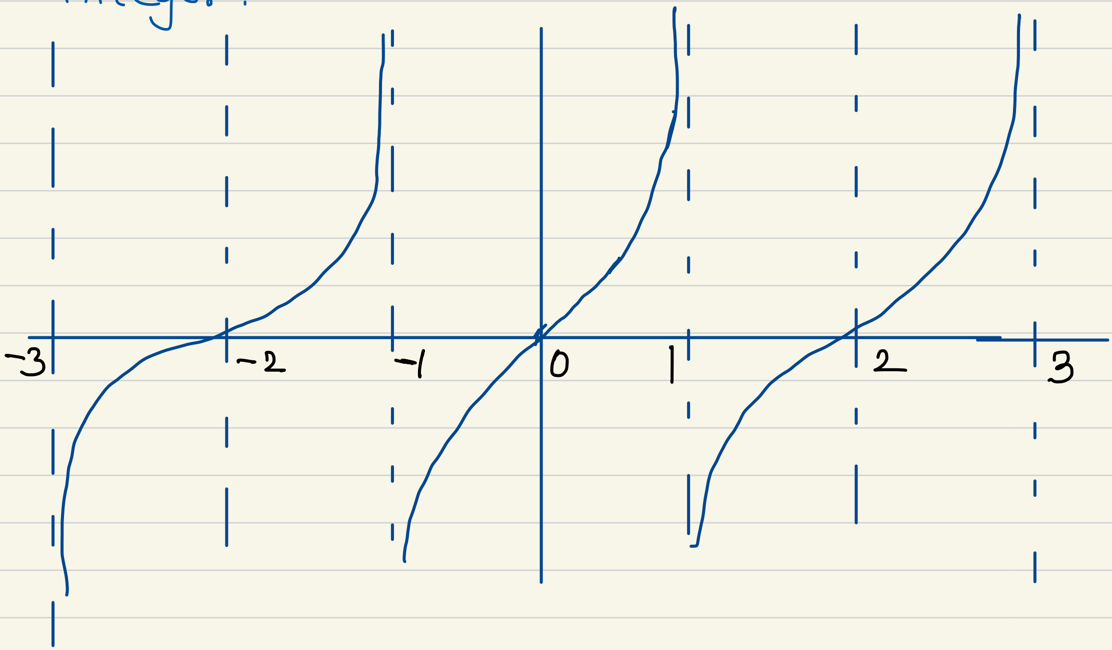
(b) $\lim_{x\to 0^-}\left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right) = -\infty$

(c) $\lim_{x\to 1^+}\left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right) = -\infty$

(d) $\lim_{x\to 1^-}\left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right) = \lim_{\varepsilon\to 0^+}\left(\frac{1}{(1-\varepsilon)^{1/3}} - \frac{1}{(-1)^{4/3}\varepsilon^{4/3}}\right)$

$= 1-\infty = -\infty.$

Q 7. ① $y = \tan\left(\frac{\pi x}{2}\right)$ is continuous at any real x which is not an odd integer.



② $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$ is continuous $\forall x \in \mathbb{R}$ since $1+\sin^2 x \geq 1$ and $x^4+1 \geq 1 \quad \forall x \in \mathbb{R}$.

Q 8] It is straightforward to see g is continuous for any $x \neq -2$.

Now $g(-2) = bx^2$; also

$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} x = -2$ &

$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} bx^2 = 4b$.

For $\lim_{x \to -2} g(x)$ to exist, we must have

$-2 = 4b$ so that $b = \frac{-1}{2}$, and then

$\lim_{x \to -2} g(x) = g(-2) = \frac{-1}{2}(-2)^2 = -2.$

$\Rightarrow$ g is continuous $\forall x \in \mathbb{R}$ provided $b = -1/2$.

Q 9] See example 2 on p.137 of the textbook.

Solution We let $f(x) = mx + b$ and organize the work into three steps.

1. *Find $f(x_0)$ and $f(x_0 + h)$.*

$$f(x_0) = mx_0 + b$$
$$f(x_0 + h) = m(x_0 + h) + b = mx_0 + mh + b$$

2. *Find the slope $\lim_{h\to 0}(f(x_0 + h) - f(x_0))/h$.*

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(mx_0 + mh + b) - (mx_0 + b)}{h}$$
$$= \lim_{h\to 0} \frac{mh}{h} = m$$

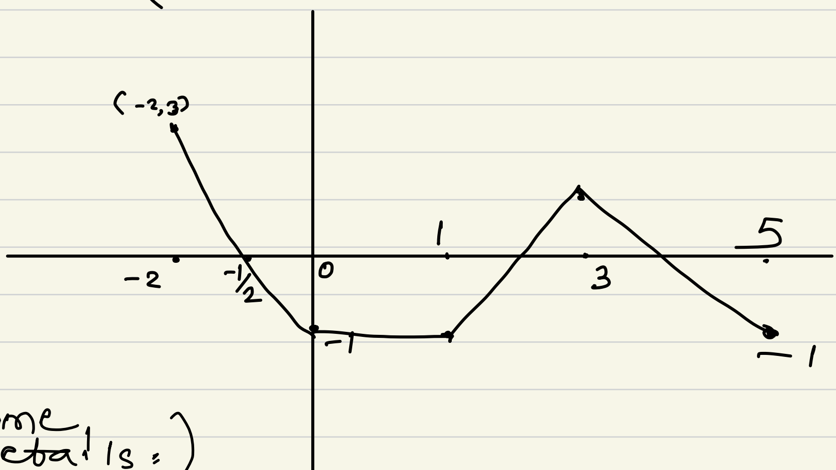
3. *Find the tangent line using the point-slope equation.* The tangent line at the point $(x_0, mx_0 + b)$ is

$$y = (mx_0 + b) + m(x - x_0)$$
$$y = mx_0 + b + mx - mx_0$$
$$y = mx + b. \quad \blacksquare$$

(1) $$y' = -2, \quad -2 \le x < 0$$
$$y = -2x + c$$
$$3 = 4 + c$$
$$c = -1$$
$$\Rightarrow y = -2x - 1, \quad -2 \le x < 0.$$

Proceeding similarly, we see that

$$y = \begin{cases} -2x - 1, & -2 \le x \le 0 \\ -1, & 0 \le x \le 1 \\ x - 2, & 1 \le x \le 3 \\ -x + 4, & 3 \le x \le 5 \end{cases}$$



(Some details:)

Now (i) implies f is continuous on $[-2, 5]$.
In particular f is continuous at 0,
So $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$ —— $(*)$

Now $\lim_{x \to 0^-} f(x) = -2(0) - 1 = -1.$

Also $y' = 0$ on $0 < x < 1$

$\Rightarrow y = c_1$ on $0 < x < 1$

i.e. $f(x) = c_1$ on $0 < x < 1$

$\Rightarrow \lim_{x \to 0^+} f(x) = c_1$

So $(*)$ implies $c_1 = -1$ & $f(0) = -1$.

The other part can now be completed by proceeding along the similar lines.

a

2. $|f(x)| \le x^2 \ \forall x \in [-1,1]$ implies, in particular, $f(0)=0$.

Now if $x \in [-1,0)$, then $x \le \frac{f(x)}{x} \le -x$

& for $x \in (0,1]$, $-x \le \frac{f(x)}{x} \le x$.

Thus, for $x \in [-1,1] \setminus \{0\}$, $-|x| \le \frac{f(x)}{x} \le |x|$

Hence $-|x| \le \frac{f(x)-f(0)}{x-0} \le |x|$

By sandwich thm., $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = 0$.

Hence $f'(0) = 0$.

b $|x^2 \sin(\frac{1}{x})| \le |x^2| = x^2 \ \forall x \in \mathbb{R}$ Now use part a with $x \in [-1,1]$, in particular.

3. $s = 24t - 0.8t^2$

a
$$\begin{cases} \frac{ds}{dt} = 24 - 1.6t & \rightarrow \text{vel.} \\ \frac{d^2s}{dt^2} = -1.6 & \rightarrow \text{acc.} \end{cases}$$

b Highest pt: $24 - 1.6t = 0$

$\Rightarrow t = \frac{24}{1.6} = \frac{24}{16} \times 10 = 15$ sec.

c $s = 24(15) - 0.8(15)^2 = 180$ m.

d $90 = 24t - 0.8t^2$

$\Rightarrow t \sim 4.39$ sec.

e Rock is aloft (above the ground) for $(15)(2) = 30$ sec.

(4) (C): graph of position s
(B): graph of velocity v
(A): graph of acceleration a

Reasoning:

(i) (A) cannot be the graph of position, for, the slopes of the tgt. lines to (A) are positive throughout, i.e. $v = \frac{ds}{dt} > 0$, which is not seen in (B) or (C).

(ii) (B) cannot be the graph of position s, because then from the figure, $v = \frac{ds}{dt} < 0$ for $t \in (0, t_1)$ for some $t_1 > 0$, & $v = \frac{ds}{dt} > 0$ for (t_1, ∞).

So then (A) must be the graph of velocity (as (C) is always below t-axis).

But then using the same reasoning as in i), we see that (C) cannot be the graph of acc.

Hence (C) is the graph of position s.
Then the rest is clear.

(5) $$\lim_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right) = \cos\left(\lim_{\theta \to 0} \frac{\pi\theta}{\sin\theta}\right)$$

(by cont. of cosine)

$$= \cos\left(\frac{\pi}{\lim_{\theta \to 0} \frac{\sin\theta}{\theta}}\right) = \cos(\pi) = -1.$$

$$\lim_{x\to 0} \sec\left[\cos x + \pi \tan\left(\frac{\pi}{4\sec x}\right) - 1\right]$$

using cont. of the functions involved

$$= \sec\left[\cos(0) + \pi \tan\left(\frac{\pi}{4\sec(0)}\right) - 1\right]$$

$$= \sec\left[1 + \pi \tan\left(\frac{\pi}{4}\right) - 1\right]$$

$$= \sec(\pi) = \frac{1}{\cos(\pi)} = -1.$$

6 (a) $$\frac{dy}{dt} = -\operatorname{cosec}^2\left(\frac{\sin t}{t}\right) \cdot \left(\frac{t\cos t - \sin t}{t^2}\right).$$

(b) $$\frac{dy}{dt} = 4\cos\left(\sqrt{1+\sqrt{t}}\right) \cdot \frac{1}{2\sqrt{1+\sqrt{t}}} \cdot \frac{1}{2\sqrt{t}}$$

(c) $$\frac{dy}{dt} = 3\left(1+\tan^4\left(\frac{t}{12}\right)\right)^2 \cdot 4\tan^3\left(\frac{t}{12}\right)$$

$$\times \sec^2\left(\frac{t}{12}\right) \cdot \frac{1}{12}$$

(d) $$\frac{dy}{dt} = -2\left(1+\cot\left(\frac{t}{2}\right)\right)^{-3} \cdot \left(-\operatorname{cosec}^2\left(\frac{t}{2}\right)\right) \cdot \frac{1}{2}.$$

7 $T = 2\pi\sqrt{\frac{L}{g}}$, $\frac{dL}{du} = kL$.

By chain rule, $\frac{dT}{du} = \frac{dT}{dL} \cdot \frac{dL}{du}$

$$= \frac{2\pi}{2\sqrt{Lg}} \cdot kL = \pi k \sqrt{\frac{L}{g}} = \frac{kT}{2}.$$

8) (1) Let $0 \le t \le \pi/2$. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos(2t)}{\cos t},$$

$\Rightarrow \frac{dy}{dx} = 0$ when $t = \frac{\pi}{4}$. Hence reqd. pt. is $\left(\frac{1}{\sqrt{2}}, 1\right)$.

Also,

$$\left.\frac{dy}{dx}\right|_{(0,0)} = \left.\frac{dy}{dx}\right|_{t=0} = 2$$

Hence eqn. of one of the two tangents at the origin is

$$y - 0 = 2(x-0), \text{ i.e. } y = 2x$$

The other is $y = -2x$.

(2) Can be similarly done.

9) (a)

$$y\sin\left(\frac{1}{y}\right) = 1 - xy$$

$$\Rightarrow y\cos\left(\frac{1}{y}\right)\cdot\left(\frac{-1}{y^2}\right)\frac{dy}{dx} + \sin\left(\frac{1}{y}\right)\frac{dy}{dx}$$

$$= -\left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow \left(-\frac{1}{y}\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x\right)\frac{dy}{dx} = -y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{\left(-\frac{1}{y}\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x\right)}.$$

(b) $xy = \cot(xy)$

$$\Rightarrow x\frac{dy}{dx} + y = -\operatorname{cosec}^2(xy) \cdot \left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow \left(x + x\operatorname{cosec}^2(xy)\right)\frac{dy}{dx} = -y\operatorname{cosec}^2(xy) - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y(1+\operatorname{cosec}^2(xy))}{x(1+\operatorname{cosec}^2(xy))} = -\frac{y}{x}.$$

(10) $x^2 + 2xy - 3y^2 = 0.$

$$\Rightarrow 2x + 2\left(x\frac{dy}{dx} + y\right) - 6y\frac{dy}{dx} = 0$$

$$\Rightarrow (6y - 2x)\frac{dy}{dx} = 2x + 2y$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{3y-x}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2}{2} = 1$$

Thus, slope of the normal to the curve at $(1,1)$ is -1.

$\Rightarrow$ Eqn. of the normal is

$$y - 1 = -1(x-1)$$

$$\Rightarrow x + y = 2$$

Points of intersection of $y = 2 - x$ with the curve can be found by substituting y as $2-x$ in $x^2 + 2xy - 3y^2 = 0$.

$$\Rightarrow x^2 + 2x(2-x) - 3(2-x)^2 = 0$$

$$\Rightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow (x-1)(x-3) = 0$$

$$\Rightarrow x = 1 \text{ or } 3.$$

$\Rightarrow$ points of intersection are $(1, 1)$ & $(3, -1)$.

Hence the second point is $(3, -1)$.

11) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ with R, R_1 & R_2 being functions of time t.

By chain rule,

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt} \quad \text{—— (1)}$$

Now $\frac{dR_1}{dt} = -1$ & $\frac{dR_2}{dt} = 0.5$

$$\Rightarrow \frac{dR}{dt} = R^2\left(\frac{-1}{R_1^2} + \frac{0.5}{R_2^2}\right).$$

when $R_1 = 75$ & $R_2 = 50$,

$$\frac{dR}{dt} = R^2\left(\frac{-1}{75^2} + \frac{0.5}{50^2}\right)$$

But $\frac{1}{R} = \frac{1}{75} + \frac{1}{50}$ implies $R = 30$.

Hence $\frac{dR}{dt} = 30^2 \left(\frac{-1}{75^2} + \frac{0.5}{50^2} \right)$

$= 0.02$ ohms/sec.

(12) $g(x) = \sqrt{4-x^2}, \quad -2 \le x \le 1$

$g'(x) = \frac{1}{2\sqrt{4-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{4-x^2}}$

$g'(x) = 0$ when $x = 0$, and $g'(x)$ is undefined when $x = -2$.

Hence critical points of g are 0 & -2.

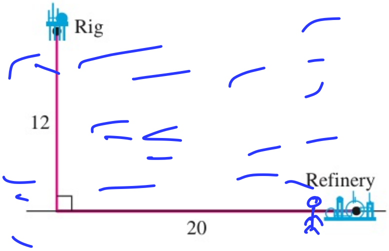
$g(0) = 2$
$g(-2) = 0$
At end point 1, we have $g(1) = \sqrt{3}$.

Hence abs. max. occurs at 0 & is 2
& abs. min. occurs at -2 & is 0.

(13) Solved example 5 on page 250 of the textbook.

Solution We try a few possibilities to get a feel for the problem:

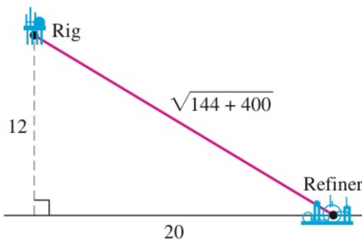
(a) *Smallest amount of underwater pipe*



Underwater pipe is more expensive, so we use as little as we can. We run straight to shore (12 mi) and use land pipe for 20 mi to the refinery.

$$\begin{aligned}\text{Dollar cost} &= 12(500{,}000) + 20(300{,}000)\\ &= 12{,}000{,}000\end{aligned}$$

(b) *All pipe underwater (most direct route)*

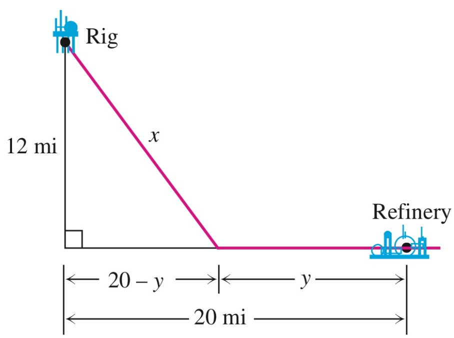


We go straight to the refinery underwater.

$$\begin{aligned}\text{Dollar cost} &= \sqrt{544}\,(500{,}000)\\ &\approx 11{,}661{,}900\end{aligned}$$

This is less expensive than plan (a).

(c) *Something in between*



Now we introduce the length x of underwater pipe and the length y of land-based pipe as variables. The right angle opposite the rig is the key to expressing the relationship between x and y, for the Pythagorean theorem gives

$$\begin{aligned} x^2 &= 12^2 + (20 - y)^2 \\ x &= \sqrt{144 + (20 - y)^2}. \end{aligned} \tag{3}$$

Only the positive root has meaning in this model.

The dollar cost of the pipeline is

$$c = 500{,}000x + 300{,}000y.$$

To express c as a function of a single variable, we can substitute for x, using Equation (3):

$$c(y) = 500{,}000\sqrt{144 + (20 - y)^2} + 300{,}000y.$$

Our goal now is to find the minimum value of $c(y)$ on the interval $0 \le y \le 20$. The first derivative of $c(y)$ with respect to y according to the Chain Rule is

$$\begin{aligned} c'(y) &= 500{,}000 \cdot \frac{1}{2} \cdot \frac{2(20 - y)(-1)}{\sqrt{144 + (20 - y)^2}} + 300{,}000 \\ &= -500{,}000 \frac{20 - y}{\sqrt{144 + (20 - y)^2}} + 300{,}000. \end{aligned}$$

Setting c' equal to zero gives

$$\begin{aligned} 500{,}000\,(20 - y) &= 300{,}000\sqrt{144 + (20 - y)^2} \\ \frac{5}{3}(20 - y) &= \sqrt{144 + (20 - y)^2} \\ \frac{25}{9}(20 - y)^2 &= 144 + (20 - y)^2 \\ \frac{16}{9}(20 - y)^2 &= 144 \\ (20 - y) &= \pm\frac{3}{4} \cdot 12 = \pm 9 \\ y &= 20 \pm 9 \\ y &= 11 \quad \text{or} \quad y = 29. \end{aligned}$$

Only $y = 11$ lies in the interval of interest. The values of c at this one critical point and at the endpoints are

$$\begin{aligned} c(11) &= 10{,}800{,}000 \\ c(0) &= 11{,}661{,}900 \\ c(20) &= 12{,}000{,}000 \end{aligned}$$

The least expensive connection costs \$10,800,000, and we achieve it by running the line underwater to the point on shore 11 mi from the refinery. ■