MA 103- SVC Lecture 6

Section 4.6 – Indeterminate forms and L'Hôpital's rule

THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof: $\text{RHS} = \dfrac{f'(a)}{g'(a)} = \dfrac{\lim_{x \to a} \dfrac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \dfrac{g(x)-g(a)}{x-a}}$

$$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \frac{x-a}{g(x)-g(a)}$$

$$= \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}, \quad \Big\} \text{ since } f(a)=g(a)=0$$

Example: Evaluate $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$

Soln: (i) without using L'Hôpital's rule:

Let $L = \lim_{x \to 0} \dfrac{x - \sin x}{x^3}$. Let $x = 3\theta$. Then

$$L = \lim_{3\theta \to 0} \frac{3\theta - \sin(3\theta)}{(3\theta)^3} = \lim_{\theta \to 0} \frac{3\theta - \overbrace{(3\sin\theta - 4\sin^3\theta)}^{\text{triple-angle formula}}}{27\theta^3}$$

$$= \frac{1}{9} \lim_{\theta \to 0} \frac{\theta - \sin\theta}{\theta^3} + \frac{4}{27} \lim_{\theta \to 0} \frac{\sin^3\theta}{\theta^3}$$

$$\Rightarrow L = \frac{L}{9} + \frac{4}{27} \quad \Rightarrow L = \frac{4}{27} \cdot \frac{9}{8} = \frac{1}{6}.$$

(ii) using L'Hôpital's rule: will be discussed soon.

THEOREM 7 L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Example $\displaystyle\lim_{x \to 0} \frac{x - \sin x}{x^3}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \left(\frac{0}{0} \text{ form}\right) = \lim_{x \to 0} \frac{\sin x}{6x} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}. \quad (\because \cos(0) = 1)$$

Ex.1 Use L'Hôpital's rule to evaluate $\displaystyle\lim_{x \to 0} \frac{x(\cos x - 1)}{\sin x - x}$.

THEOREM 8 Cauchy's Mean Value Theorem

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Spl. case: $g(x) = x$ gives the usual mean-value theorem.

Using L'Hôpital's Rule

To find

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example: $\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \lim_{x\to 0} \frac{\sin x}{1+2x} = \frac{0}{1} = 0$

What about $\lim_{x\to 0} \frac{1+\cos x}{x+x^2}$?

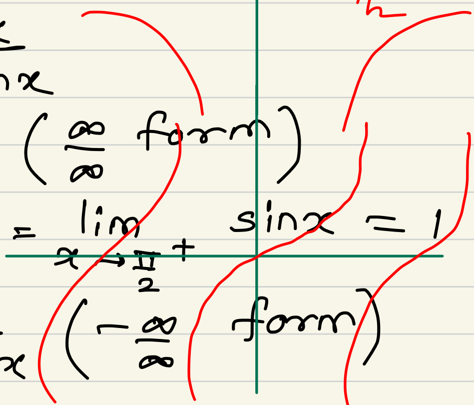
L'Hôpital's rule for one-sided limits:

Ex. $\lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \frac{\cos x}{2x} = +\infty$

$\lim_{x\to 0^-} \frac{\sin x}{x^2} = \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty$

Indeterminate forms $\frac{\infty}{\infty}$

Example: $\lim_{x\to \frac{\pi}{2}} \frac{\sec x}{1+\tan x}$



First, $\lim_{x\to \frac{\pi}{2}^+} \frac{\sec x}{1+\tan x}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to \frac{\pi}{2}^+} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to \frac{\pi}{2}^+} \sin x = 1$

Similarly, $\lim_{x\to \frac{\pi}{2}^-} \frac{\sec x}{1+\tan x}$ ($-\frac{\infty}{\infty}$ form)

$= \lim_{x\to \frac{\pi}{2}^-} \sin x = 1.$

Hence, $\lim_{x\to \frac{\pi}{2}} \frac{\sec x}{1+\tan x} = 1.$

Indeterminate form $\infty \cdot 0$

Convert into $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

Example $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$ ($\infty \cdot 0$ form)

$$= \lim_{x \to \infty} \frac{\sin(1/x)}{(1/x)} \quad \left(\frac{0}{0} \text{ form}\right) \quad = \lim_{y \to 0} \frac{\sin y}{y} = 1.$$

Indeterminate form $\infty - \infty$

Convert into $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

Ex. 2 $\lim_{x \to \infty} \left(x - \sqrt{x^2 + x}\right)$

$$= \lim_{x \to \infty} \frac{\left(1 - \sqrt{1 + \frac{1}{x}}\right)}{1/x}$$

$$= \lim_{x \to \infty} \frac{\frac{-1}{2\sqrt{1 + \frac{1}{x}}} \cdot \cancel{\frac{-1}{x^2}}}{\cancel{-1/x^2}}$$

$$= \frac{-1}{2}.$$

Sect. 5.4 - The Fundamental Theorem of Calculus

THEOREM 3 **The Mean Value Theorem for Definite Integrals**
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = av(f)$$

Proof:

By max-min inequality,

$$\min f \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \max f$$

$\Rightarrow$ By intermediate value thm., $\exists\, c \in (a,b) \ni$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$\blacksquare$

Example Show that if f is continuous on $[a,b]$, $a \neq b$ & $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ at least once on $[a, b]$.

Soln: By the above mean value thm, $\exists\, c \in (a,b)$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{b-a}(0)$$

$$= 0.$$

$\blacksquare$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

THEOREM 4 The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

Proof: $\dfrac{F(x+h)-F(x)}{h} = \dfrac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$

$= \dfrac{1}{h}\left(\int_a^{x+h} f(t)\,dt + \int_{x+h}^{a} f(t)\,dt\right) = \dfrac{1}{h}\int_x^{x+h} f(t)\,dt.$

By MVT for definite integrals, $\exists\, c \in (x, x+h)$ s.t. $f(c) = \dfrac{1}{h}\int_x^{x+h} f(t)\,dt$

Now let $h \to 0$. Then $x+h \to x$ & hence $c \to x$. But f is continuous at x, thus

$\lim_{h\to 0} f(c) = f(x)$. This implies $\lim_{h\to 0} \dfrac{F(x+h)-F(x)}{h} = f(x)$.

Hence, F is differentiable on (a, b) & $F'(x) = f(x)$. Since differentiability implies continuity, F is continuous on (a, b). By taking right- and left-hand limits at a & b resp., one can prove continuity at the end points. ∎

Example (i) $\dfrac{d}{dx}\int_0^x \dfrac{dt}{1+t^2} = \dfrac{1}{1+x^2}$.

(ii) $\dfrac{dy}{dx}$ if $y = \int_1^{x^2} \cos(t)\,dt$ | $\dfrac{dy}{dx} \overset{\text{chain rule}}{=} \cos(x^2)\cdot\dfrac{d}{dx}(x^2)$

$= 2x\cos(x^2)$.

THEOREM 4 (Continued) **The Fundamental Theorem of Calculus Part 2**

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof: By continuity of f, we know that antiderivative of f exists, namely,

$$G(x) = \int_a^x f(t)\,dt \text{ for } a < x \leq b.$$

If F is any antiderivative of f, $F(x) = G(x) + c$ for some constant c for $a < x < b$. Since both F & G are continuous on $[a, b]$, we see that $F(x) = G(x) + c$ also holds for $x = a$ & $x = b$ by taking $\lim_{x \to a^+}$ & $\lim_{x \to b^-}$ respectively.

Now $F(b) - F(a) = (G(b) + c) - (G(a) + c)$

$$= G(b) - G(a) = \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{= 0}$$

$$= \int_a^b f(t)\,dt.$$

Differentiation and integration as inverse processes

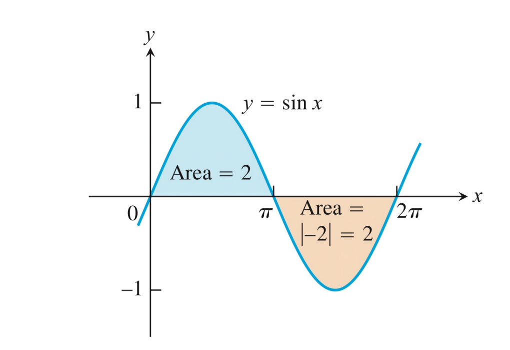
(1) $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = \dfrac{dF}{dx} = f(x).$

(2) $\displaystyle\int_a^x \dfrac{dF}{dt}\,dt = \int_a^x f(t)\,dt = F(x) - F(a)$

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Example :



$$\int_0^{2\pi} \sin x \, dx$$

$$= \Big[-\cos x\Big]_0^{2\pi}$$

$$= -1 - (-1)$$

$$= -1 + 1 = 0$$

not the area under the curve though!

Ex.3 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2)$$
$$= x(x+1)(x-2)$$

Area of the shaded region

$$= \int_{-1}^{0} f(x)\,dx + \left| \int_{0}^{2} f(x)\,dx \right|$$

$$= \int_{-1}^{0} x^3 - x^2 - 2x \,dx + \left| \int_{0}^{2} x^3 - x^2 - 2x \,dx \right|$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^{0} + \left| \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{0}^{2} \right|$$

$$= -\left(\frac{1}{4} + \frac{1}{3} - 1 \right) + \left(\frac{16}{4} + \frac{8}{3} - 4 \right)$$

$$= -\left(\frac{7}{12} - 1 \right) + \frac{8}{3}$$

$$= \frac{5}{12} + \frac{32}{12}$$

$$= \frac{37}{12}$$

MA 103 - SVC Lecture 7

Sect. 5.5 - Indefinite integrals and the substitution rule

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof: If F is any antiderivative of f, then $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ because

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Then if $u = g(x)$, we have

$$\int f(g(x))g'(x)\,dx = \int \frac{d}{dx}F(g(x))\,dx$$

$$\underset{\uparrow\,\text{FTC}}{=} F(g(x)) + C = F(u) + C \underset{\uparrow\,\text{FTC}}{=} \int F'(u)\,du$$

$$= \int f(u)\,du.$$

Example: ① $\int x^2 \sin(x^3)\,dx$

Let $u = x^3$ $\quad du = 3x^2\,dx \Rightarrow \frac{du}{3} = x^2\,dx$

$$\Rightarrow \int x^2 \sin(x^3)\,dx = \int \sin(u)\,\frac{du}{3}$$

$$= \frac{1}{3}\{-\cos(u) + c\} = \frac{1}{3}\{-\cos(x^3) + c\}.$$

② $\int \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} \, dx$

$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + c.$

Ex. 1 $\int \frac{dx}{x^{1/2}+x^{1/3}}$

Ans.

Let $x = t^6 \quad dx = 6t^5 \, dt$

$= \int \frac{1}{t^3+t^2} \cdot 6t^5 \, dt$

$= 6 \int \frac{t^3 \, dt}{t+1}$

$= 6 \int \frac{(t^3+1)-1}{t+1} \, dt$

$= 6 \left\{ \int \frac{t^3+1}{t+1} \, dt - \int \frac{1}{t+1} \, dt \right\}$

$= 6 \left\{ \int \frac{\cancel{(t+1)}(t^2-t+1)}{\cancel{t+1}} \, dt - \int \frac{1}{t+1} \, dt \right\}$

$= 6 \left\{ \left(\frac{t^3}{3} - \frac{t^2}{2} + t \right) - \ln|t+1| \right\} + c.$

Sect. 5.6 — Substitution and area between curves

THEOREM 6 Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof: If F is any antiderivative of f, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_a^b \frac{d}{dx} F(g(x))\, dx \underset{\text{FTC part 2}}{=} \Big[F(g(x))\Big]_a^b$$

$$= F(g(b)) - F(g(a)) = \Big[F(u)\Big]_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f(u)\, du.$$

Example: $I = \displaystyle\int_{\pi/4}^{\pi/2} \cot\theta \operatorname{cosec}^2\theta \, d\theta$

Let $u = \cot\theta$ $\quad du = -\operatorname{cosec}^2\theta \, d\theta$

when $\theta = \pi/4$, $u = 1$; when $\theta = \frac{\pi}{2}$, $u = 0$

$$\Rightarrow I = \int_1^0 u(-1)\, du = \int_0^1 u\, du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\displaystyle\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$.

(b) If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\, dx = 0$.

Example: Evaluate $\displaystyle\int_{-1/2}^{1/2} \left(\frac{x^3+1}{x^3-1} - \frac{x^3-1}{x^3+1}\right) dx$