Eg. 3 $P_n = \{$ polynomials with real coefficients of degree $\leq n \}$
$\subseteq F(-\infty, \infty)$.

Then S_1, S_2, S_3 hold.
$\Rightarrow P_n$ is a subspace of $F(-\infty, \infty)$.

Q. What about {polynomials of degree n} ?
Not a subspace since the zero function (additive inverse in $F(-\infty,\infty)$ is NOT in this set.

Eg. 4 set of diagonal matrices $\subseteq M_{mn}$
— " — triangular — " —
— " — symmetric — " —

— All are subspaces

THEOREM 4.2.2 *If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.*

Proof: Let $W = W_1 \cap W_2 \cap \cdots \cap W_r$. Each W_i contains the additive idty 0_V, hence $0_V \in W$.
Let $u, v \in W \Rightarrow u, v \in W_i \ \forall i \Rightarrow u+v \in W_i \ \forall i$
$\Rightarrow u+v \in W$; Also, if $u \in W$ & $k \in \mathbb{R}$, then $u \in W_i \ \forall i$
$\Rightarrow ku \in W_i \ \forall i \Rightarrow ku \in W \Rightarrow W$ is a subspace of V.

Q Let $S = \{w_1, w_2, \ldots, w_r\} \subseteq V$.
If S is not a subspace, how to make it a subspace ?

Put 0_V, w_1+w_2, $\dots$, $w_1+\dots+w_r$, c_1w_1, $\dots$ c_rw_r if $c_1, c_2 \in \mathbb{R}$.

So we have to include $c_1w_1+\dots+c_rw_r$ $\forall\, c_1,\dots,c_r \in \mathbb{R}$

Need $W = \{$All linear combinations of $w_1,\dots,w_r\}$
W: smallest subspace containing S
i.e, if W_0 is a subspace of V containing S then

$$W_0 \supseteq W.$$

Def. This W is called span(S) or subspace generated by S.

Def. S is called a spanning set for W.

Eg. 1 $S = \{e_1,\dots,e_n\}$, where $e_i \in \mathbb{R}^n$ $\forall\, 1 \le i \le n$.
What is span(S)?

Span(S) $= \mathbb{R}^n$ since any vector $x \in \mathbb{R}^n$ can be written as a lin. comb. of e_i's.

Eg. 2 $S = \{(1,1)\} \subseteq \mathbb{R}^2$ spans the line $y = x$ in $\mathbb{R}^2$.

Eg 3 $\{1, x, \dots, x^n\}$ in a spanning set for $P_n = \{$all polynomials with real coefficients of deg. $\le n\}$.

Eg. 4 A spanning set of M_{22} is
$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Eg. 5 Take a homogeneous linear system $Ax=0$.
m equations, n unknowns.

$$S = \text{Sol}^n \text{ set} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\}$$

① S_1: $\quad 0_V = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in S$

② S_2: $\bar{x}, \bar{y}$ are sol^{n}s to $Ax = 0$

$\Rightarrow A\bar{x} = 0, \quad A\bar{y} = 0$

$A(\bar{x} + \bar{y}) = A\bar{x} + A\bar{y} = 0 \in \mathbb{R}^m$

③ S_3: $A(k \cdot \bar{x}) = k \cdot (A\bar{x}) = k \cdot 0 = 0.$

$\in \mathbb{R}^m$.

$\Rightarrow$ Sol^n set is a subspace of $\mathbb{R}^n$.

4.3 Linear independence

Defn. $S = \{v_1, \dots, v_r\} \subset V$ is a linearly independent (l.i.) set if the only coefficients $k_1, k_2, \dots, k_r$ satisfying the relation

$$k_1 v_1 + k_2 v_2 + \dots + k_r v_r = 0_V \quad \text{———} \quad (*)$$

are $k_1 = k_2 = \dots = k_r = 0$.

Cor. If S is l.i. then no element of S can be expressed as a l.c. of the others.

e.g. if $v_1 = 2v_2 - 5v_3$, then
$0 = -v_1 + 2v_2 - 5v_3$ which contradicts $(*)$.

Eg. ① $S = \phi \subseteq V$ is l.i.

② $S = \{0_V\} \subseteq V$ is l.d. because
$k \cdot 0_V = 0_V$

③ $S = \{v\} \subseteq V$ is l.i. if $v \neq 0_V$

④ $S = \{1, x, \dots, x^n\} \subseteq P_n$ is l.i.

⑤ $S = \{e_1, \dots, e_n\} \subseteq \mathbb{R}^n$ is l.i.

⑥ $S = \{v_1, v_2\} \subseteq V$ is l.i. only if v_2 is not a scalar multiple of v_1.

"$\Rightarrow$" Suppose S is l.i. & $v_2 = cv_1$ for some $c \in \mathbb{R}$.
Then $c \cdot v_1 + (-1)v_2 = 0$ contradicts the fact that S is l.i.

"$\Leftarrow$" Spse. S is l.d. Then $\exists\, c_1, c_2$ not all zero $\ni$ $c_1v_1 + c_2v_2 = 0$
w.l.g. say $c_2 \neq 0 \Rightarrow v_2 = -\frac{c_1}{c_2} v_1$ $\rightarrow\leftarrow$ to that fact that v_2 is not a s.m.

⑦ $\left\{ \underset{M_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{M_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{M_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{M_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$ is a l.i. set.

BASIS

The set $\{M_1, M_2, M_3, M_4\} \subseteq M_{22}$
- spans M_{22}
- is l.i.

Def. $S \subseteq V$ is called a basis for V if
(a) S spans V.
(b) S is a l.i. set.

Examples

V	Basis
$\mathbb{R}^n$	$\{e_1, \dots, e_n\}$
P_n	$\{1, x, \dots, x^n\}$
M_{22}	M_1, M_2, M_3, M_4
M_{mn}	$M_1, \dots, M_{mn}$

finite dimensional vector spaces

$P_\infty = \{$ all polynomials with coefficients in $\mathbb{R}\}$

Eg. Let $V = \mathbb{R}^3$, $S = \{(1,0,0), (0,1,0)\}$.
$\mathrm{span}(S) = \{c_1(1,0,0) + c_2(0,1,0) : c_1, c_2 \in \mathbb{R}\}$
$= xy$-plane in $\mathbb{R}^3$
$\Rightarrow \mathrm{span}(S) \neq V$.

Is S l.i. ?

Yes!

② $S \subseteq \mathbb{R}^3$, $S = \{\overset{v_1}{(1,1,0)}, \overset{v_2}{(1,2,0)}, \overset{v_3}{(2,3,0)}\}$

Not l.i since $v_3 = v_1 + v_2$
So not a basis.

Recall: If $S \subseteq V$ spans V and is l.i, then S is called a basis of V.

In the above examples, S was finite. If it is infinite, the same definition works, keeping in mind that the linear combination means a linear combination of finitely many elements.

Recall: $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$.

Ex. Check that $S = \{(1,2,1), (2,9,0), (3,3,4)\}$ is a basis for $\mathbb{R}^3$.

① To show: spanning:

Given $(b_1, b_2, b_3) \in \mathbb{R}^3$, $\exists$ c_1, c_2, c_3 $\ni$

$$(b_1, b_2, b_3) = c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3$$

$$\Longleftrightarrow \quad \begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 + 0c_2 + 4c_3 &= b_3 \end{aligned}$$

$$A \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This means we should show that the above system is consistent.

② Linear independence:

To show that the homogeneous system

$$A \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the trivial solution

In order to show ① & ② hold, it suffices to show that $\det(A) \neq 0$.

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = 1(9\times4) - 2(8-3) + 3(-9)$$
$$= 36 - 10 - 27$$
$$= -1$$
$$\neq 0.$$

Hence $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ is a basis for $\mathbb{R}^3$.

Use of a basis

Thm. 4.4.1 If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then every $v \in V$ can be expressed as a linear combination

(*) — $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ $(c_i \in \mathbb{R})$

in a unique way.

Proof: ① Existence of (*)

Take any $v \in V$. Then $v \in \text{span}(S)$ ($\because \text{span}(S) = V$)

$\Rightarrow$ v can be written as a linear combination of elements of S, that is,

$v = c_1 v_1 + \cdots + c_n v_n$ for some $c_i \in \mathbb{R}$.

② Uniqueness of (*)

Suppose $\exists$ two ways to write v as a l.c. of

v_i's, namely,

$$v = c_1 v_1 + \cdots + c_n v_n \qquad c_i \in \mathbb{R}$$
$$v = k_1 v_1 + \cdots + k_n v_n \qquad k_i \in \mathbb{R}$$
$$\Rightarrow 0 = \underbrace{(c_1 - k_1) v_1 + \cdots + (c_n - k_n) v_n}_{\text{l.c. of } v_1, \ldots, v_n \text{ giving } 0}$$

But S is l.i. $\Rightarrow$ the above l.c. must be trivial.

$\Rightarrow c_i - k_i = 0 \quad \forall i$

$\Rightarrow c_i = k_i \quad \forall i$.

Def. Let $B = \{v_1, \ldots, v_n\}$ be a basis of a v.s. V. If $v \in V$ has the unique expression,

$$v = c_1 v_1 + \cdots + c_n v_n$$

in terms of basis elements, then $c_1, \ldots, c_n$ are called coordinates of v relative to B.

$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is called the coordinate vector of v relative to B, denoted by $[v]_B$.

V	B	coordinate vector of
	$c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\11\\1\end{pmatrix}$	$(3, 11, 1) = v$
$\mathbb{R}^3$	$\{e_1, e_2, e_3\} = B$	$\begin{pmatrix}3\\11\\1\end{pmatrix} = [v]_B$
$\mathbb{R}^3$	$\{\overset{v_1}{(1,2,1)}, \overset{v_2}{(2,9,0)}, \overset{v_3}{(3,3,4)}\} = B'$	$\begin{pmatrix}1\\1\\0\end{pmatrix} = [v]_{B'}$

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix} + c_2\begin{pmatrix}2\\9\\0\end{pmatrix} + c_3\begin{pmatrix}3\\3\\4\end{pmatrix} = \begin{pmatrix}3\\11\\1\end{pmatrix}$$

Note: $(3,11,1) = 3e_1 + 11e_2 + 1e_3$
$(3,11,1) = 1v_1 + 1v_2 + 0v_3$

Sect. 4.5 - DIMENSION

Thm. 4.5.1 Given a v.s. V, all basis sets for V have the same cardinality (same number of elements)

This number is called the dimension of V, denoted by $\dim(V)$.

e.g. $\dim(\mathbb{R}^3) = 3$
$\dim(\{0\}) = 0$ (by convention)
$\dim(P_n) = n+1$
$\dim(M_{mn}) = mn$
$\dim(\mathbb{R}^n) = n$

Thm. 4.5.2 Suppose V has dimension n. Then
(a) If $S \subseteq V$ and $|S| > n$, then S is linearly dependent.

e.g. $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3, (1,1,1)\}$ fails lin. indep.

(b) If $S \subseteq V$ and $|S| < n$, then S does not span V.