Note: $(3, 11, 1) = 3e_1 + 11e_2 + 1e_3$
$(3, 11, 1) = 1v_1 + 1v_2 + 0v_3$

Sect. 4.5 - DIMENSION

Thm. 4.5.1 Given a v.s. V, all basis sets for V have the same cardinality (same number of elements)

This number is called the dimension of V, denoted by $\dim(V)$.

e.g. $\dim(\mathbb{R}^3) = 3$
$\dim(\{0\}) = 0$ (by convention)
$\dim(P_n) = n+1$
$\dim(M_{mn}) = mn$
$\dim(\mathbb{R}^n) = n$

Thm. 4.5.2 Suppose V has dimension n. Then
(a) If $S \subseteq V$ and $|S| > n$, then S is linearly dependent.

e.g. $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3, (1,1,1)\}$ fails lin. indep.

(b) If $S \subseteq V$ and $|S| < n$, then S does not span V.

e.g. $V = \mathbb{R}^3$, $S = \{e_1, e_2\}$. Then $\text{span}(S) \neq V$.

An easier way of checking if a set is a basis

Thm. 4.5.4 If $\dim(V) = n$ and $S \subseteq V$ with $|S| = n$, then S is a basis for V if

(i) S spans V

or (ii) S is l.i.

(No need to check both (i) & (ii) if S is of the right size.)

Sect. 4.6 — Change of basis

Consider:

V	Basis	coordinate vector of $v = (3, 2)$
① $\mathbb{R}^2$	$\{\underset{e_1}{(1,0)}, \underset{e_2}{(0,1)}\} = B$	$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = [v]_B$
② $\mathbb{R}^2$	$\{\underset{u_1}{(1,1)}, \underset{u_2}{(2,1)}\} = B'$	$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [v]_{B'}$ x_1, x_2 unknown

Q: Is there a general formula to go from a co-ordinate vector ① to ②, that is, from $[v]_B$ to $[v]_{B'}$ for a general v?

Old basis B

$B = \{e_1, e_2\}$

$[v]_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

New basis B'

$B' = \{u_1, u_2\}$

$[v]_{B'} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$v = 3e_1 + 2e_2 \quad \text{—— (A)}$$

with $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$$v = x_1 u_1 + x_2 u_2 \quad \text{—— (1)}$$

with $u_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $u_2 = \begin{pmatrix}2\\1\end{pmatrix}$

(old basis vectors)

Is there a way to write e_1, e_2 in terms of u_1, u_2?

(New basis vectors)

$$\left.\begin{aligned} e_1 &= -1 \cdot u_1 + 1 \cdot u_2 \\ e_2 &= 2 \cdot u_1 - 1 \cdot u_2 \end{aligned}\right\} \text{(B)} \quad \Rightarrow \quad [e_1]_{B'} = \begin{pmatrix}-1\\1\end{pmatrix}, \quad [e_2]_{B'} = \begin{pmatrix}2\\-1\end{pmatrix}$$

Substituting (B) in (A), we get

$$v = 3(-1 \cdot u_1 + 1 \cdot u_2) + 2(2 \cdot u_1 - 1 \cdot u_2) \quad \text{—— (2)}$$

Compare (1) & (2) to get

$$x_1 = (-1) \cdot (3) + (2)\,(2)$$
$$x_2 = (1)\,(3) + (-1)\,(2)$$

$$\Rightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-1 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}$$

$$[v]_{B'} = P_{B \to B'} \quad [v]_B$$

coordinate vector of v w.r.t. new basis

"Transition matrix" from $B \to B'$

coordinate vector of v w.r.t. old basis

Note: $P_{B \to B'}$ is the matrix whose columns are coordinate vectors of the old basis vectors (in B) w.r.t. the new basis B'.

That is,

$$P_{B \to B'} = \begin{pmatrix} | & | \\ [e_1]_{B'} & [e_2]_{B'} \\ | & | \end{pmatrix}.$$

Similarly, we can talk about $P_{B' \to B}$:

$$[v]_B = P_{B' \to B} [v]_{B'}$$

FACT: $P_{B \to B'}$ & $P_{B' \to B}$ are inverses of each other!

THEOREM 4.6.1 *If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B to B'.*

An efficient way to compute transition matrix

Let B' = matrix with columns as new basis vectors.

B = Matrix with columns as old basis vectors.

A Procedure for Computing $P_{B\to B'}$

Step 1. Form the matrix $[B' \mid B]$.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be $[I \mid P_{B\to B'}]$.

Step 4. Extract the matrix $P_{B\to B'}$ from the right side of the matrix in Step 3.

transition matrix from old to new

Eg. In the previous example,

$B = \{e_1, e_2\}$, $B' = \{\underset{u_1}{(1,1)}, \underset{u_2}{(2,1)}\}$

$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad B' = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$$[B' \mid B] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

$\downarrow R_2 \to -R_2$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$\downarrow R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \qquad \underbrace{\begin{matrix} -1 & 2 \\ 1 & -1 \end{matrix}}_{P_{B\to B'}}$$

Sect. 4.7 Row space, column space, null space

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & a_{2n} \\ a_{31} & & & \\ a_{41} & & & \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix}_{m \times n}$$

Row vectors of A are

$$r_1 = [a_{11} \; \ldots \; a_{1n}]$$
$$r_2 = [a_{21} \; \ldots \; a_{2n}]$$
$$\vdots$$
$$r_m = [a_{m1} \; \ldots \; a_{mn}].$$

- $r_i \in \mathbb{R}^n \quad \forall \; 1 \leq i \leq m$.

Column vectors of A are

$$c_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \ldots, \quad c_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix},$$

where $c_i \in \mathbb{R}^m$, $1 \leq i \leq n$.

Def: <u>Rowspace of A</u> is the subspace of $\mathbb{R}^n$ spanned by its rows

$$\text{Rowspace} = \text{span}\{r_1, r_2, \ldots, r_m\} \subseteq \mathbb{R}^n.$$

Def. <u>Columnspace of A</u> is the subspace of $\mathbb{R}^m$ spanned by its columns.

$$\text{Columnspace} = \text{span}\{c_1, c_2, \ldots, c_n\} \subseteq \mathbb{R}^m$$

<u>Columnspace turns up in a natural way:</u>

Eg ① Show that $Ax = b$ has a solution iff b is in the columnspace of A, that is, b can be written as a l.c. of columns of A.

Proof: "$\Leftarrow$" b is in the column space of A.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m \times n}$

Then let $S :=$ Column space of A

$$= \left\{ t_1 c_1 + \cdots + t_n c_n : c_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}, \begin{array}{l} 1 \le i \le n \\ t_i \in \mathbb{R} \end{array} \right\}$$

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in S$ implies. $\exists\, t_i \in \mathbb{R}, 1 \le i \le n$, s.t.

$$t_1 a_{11} + t_2 a_{12} + \cdots + t_n a_{1n} = b_1$$

$$\vdots$$

$$t_1 a_{m1} + t_2 a_{m2} + \cdots + t_n a_{mn} = b_m$$

Thus $\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$ is a solution to $Ax = b$.

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = t_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + t_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots t_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

"$\Longrightarrow$" Now suppose $Ax = b$ has a solution, say $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

By column-sum rule,

$$\underbrace{c_1 x_1 + c_2 x_2 + \cdots + c_n x_n}_{\in \operatorname{span}\{c_1, \ldots, c_n\}} = b$$

So $b \in$ column space of A.

Recall: The solution space of $Ax=0$ forms a subspace of $\mathbb{R}^n$.
It is known as the nullspace of A.

Def: ① Nullspace of A

$$= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n : A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \right\}.$$

② Nullity of A = dimension of the nullspace of A.

Nullspace turns up in a natural way:

THEOREM 4.7.2 *If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x}=\mathbf{b}$, and if $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ is a basis for the null space of A, then every solution of $A\mathbf{x}=\mathbf{b}$ can be expressed in the form*

particular solution of $Ax=b$ free variables or parameters

$$\mathbf{x} = \mathbf{x}_0 + \underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k}_{\text{general solution of } Ax=0} \tag{3}$$

Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x}=\mathbf{b}$.

Eg. ② Find a basis for the nullspace of A.
Find nullity(A).
Then find a general solution for $Ax=b$, where $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$.

(i) Basis for the nullspace of A & Nullity of A

Consider $Ax = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right]$$

Convert into RREF form

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 + x_3 = 0.$$

(*) If $x_3 = t$, $x_2 = s$, then $x_1 = -s-t$.

solution space of $Ax = 0$

$=$ Null space of A

$$= \left\{ \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{array}{l} x_1 + x_2 + x_3 = 0, \\ x_i \in \mathbb{R}, \ 1 \le i \le 3 \end{array} \right\}.$$

From (*)

$$(x_1, x_2, x_3) = (-s-t, s, t)$$

$$= s(-1, 1, 0) + t(-1, 0, 1)$$

Since $(-1,1,0)$ & $(-1,0,0)$ are l.i. (and that is because they are not scalar multiples of each other), we see that

nullity of A = dim (nullspace of A)
$= 2$.

(ii) General solution of $Ax=b$

Since $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution of $Ax=b$.

Hence the general solution of $Ax=b$ is

$$\left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix}}_{\begin{bmatrix} 4-s-t \\ s \\ t \end{bmatrix}} : s,t \in \mathbb{R} \right\}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 4-s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4-s-t+s+t \\ 8-2s-2t+2s+2t \\ 12-3s-3t+3s+3t \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix},$$

Sect. 4.8 – RANK

Consider the equations of Eg. ② :

$$(*)\begin{cases} x_1 + x_2 + x_3 = 4 \\ 2x_1 + 2x_2 + 2x_3 = 8 \\ 3x_1 + 3x_2 + 3x_3 = 12 \end{cases}$$

Each equation represents a plane in $\mathbb{R}^3$.
So solution set is intersection of 3 planes.

However, here each equation is the same plane.
Therefore, the number of planes that intersect for the solution set is just 1.

Is there a way to get this info directly from the matrix A?
Ans. Rank

Def. Rank of a matrix A is the maximum number of linearly independent rows of A.
= ———————— " ———————— columns of A.

Eg. $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$ $R_2 = 2R_1,\ R_3 = 3R_1$
$\Rightarrow$ Max. number of l.i. rows is 1
$\Rightarrow$ rank(A) = 1.

Note: Nullity of A = 2 (from Eg. ②)
Here, rank + nullity = 3 = number of unknowns

Is this always true?
Yes!

Dimension Theorem for Matrices/ Rank - Nullity theorem

For any matrix $A_{m\times n}$,

$$\text{Rank}(A) + \text{Nullity}(A) = \text{number of columns of } A.$$

Proof: Observe that

rank = # leading ones in RREF of A

= # leading variables (for $Ax = b$).

(When we do RREF, linearly dependent rows will reduce to zero rows in RREF)

Nullity = # free variables

(because the general solution for $Ax = b$ looks like $x_0 + c_1 v_1 + c_2 v_2 + \cdots + c_r v_r$, where $c_1, c_2, \ldots, c_r \in \mathbb{R}$

free variables

leading variables + # free variables

= total number of variables

= # columns of A.

$$x_1 + x_2 + x_3 = 0,$$

$x_2 = s$, $x_3 = t$

$$x_1 = -s - t$$

Table 1

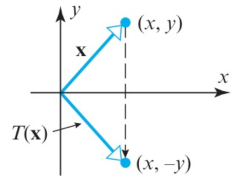
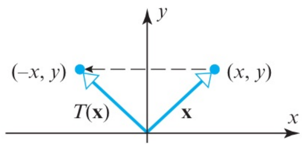
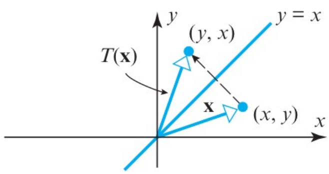
Operator	**Illustration**	**Images of $\mathbf{e}_1$ and $\mathbf{e}_2$**	**Standard Matrix**
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Table 2

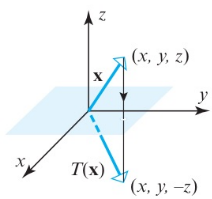
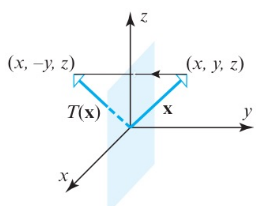
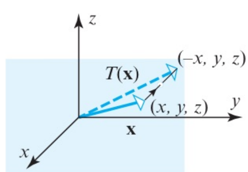
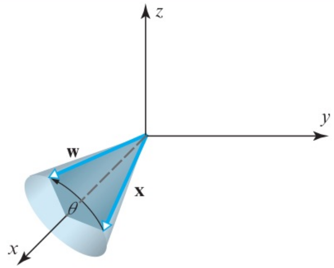
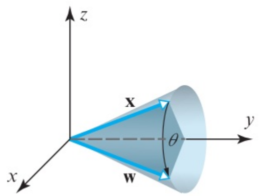
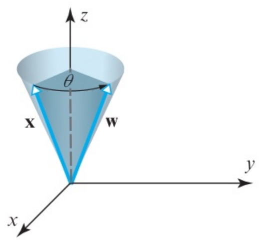
Operator	**Illustration**	**Images of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$**	**Standard Matrix**
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Table 6

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$