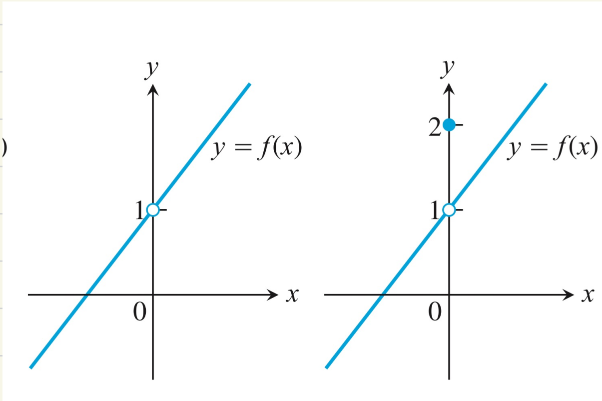
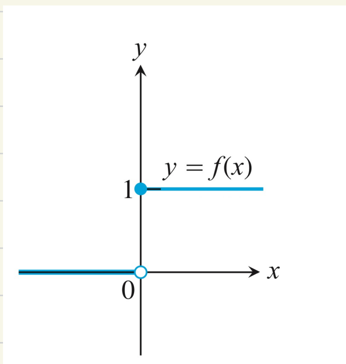
Types of discontinuities

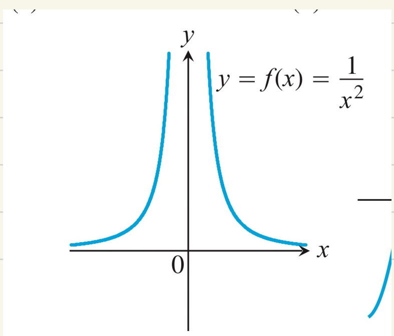
① Removable



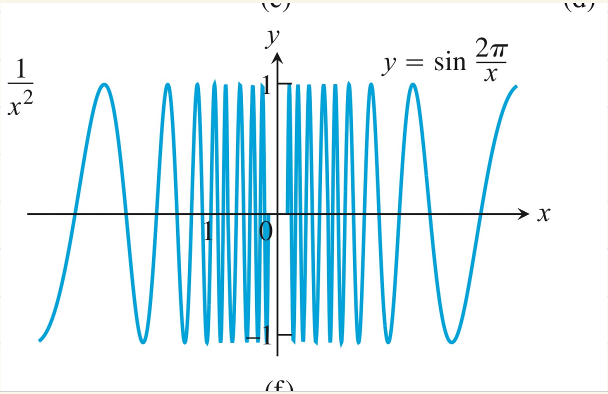
② Jump discontinuity



③ Infinite discontinuity



④ Oscillating discontinuity



Continuity on an interval

f is said to be continuous on an interval if it is continuous at every point of that interval.

Examples of continuous functions:

$f(x) = x$ is cont. on $(-\infty, \infty)$

- $\lim_{x \to c} f(x) = \lim_{x \to c} x = c = f(c)$.

- $f(x) \equiv a \quad \forall\, x \in (-\infty, \infty)$.

More examples of continuous functions

- Every polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is continuous $\forall\, x \in \mathbb{R}$ & $\forall\, n \in \mathbb{N}$.

- $f(x) = \dfrac{x^2 - 6}{x^2 + 1}$ is continuous $\forall\, x \in \mathbb{R}$

Ex. 2 Show that the following functions are continuous on their respective domains D:

(a) $y = \sqrt{x^2 - 2x - 5}$; D: $[0, \infty)$

(b) $y = \dfrac{x^{2/3}}{1 + x^6}$; D: $(-\infty, \infty)$

(c) $y = \left| \dfrac{x - 2}{x^2 - 2} \right|$; D: $\mathbb{R} \setminus \{\pm\sqrt{2}\}$

(d) $y = \left| \dfrac{x \sin x}{x^2 + 2} \right|$; D: $\mathbb{R}$

(a) $x^2 - 2x - 5 \geq 0$

$x^2 - 2x - 5 = 0$

$\Longleftrightarrow \quad x = \dfrac{2 \pm \sqrt{4+20}}{2} = \dfrac{2 \pm 2\sqrt{6}}{2}$

$= 1 \pm \sqrt{6}$,

So the fn. is continuous on

$\mathbb{R} \setminus (1-\sqrt{6}, 1+\sqrt{6})$

(b) $y = \dfrac{x^{2/3}}{1+x^6}$; $D: (-\infty, \infty)$

Note that $y = \dfrac{f(x)}{g(x)}$, where both f & g are cont. on $(-\infty, \infty)$, & $g(x) \neq 0$ on $(-\infty,$
Hence y is cont. on $(-\infty, \infty)$.

(c) $y = \left| \dfrac{x-2}{x^2-2} \right|$: $D: \mathbb{R} \setminus \{\pm\sqrt{2}\}$.

$y = (a \circ b)(x)$, where $a(x) = |x|$ &
$b(x) = \dfrac{x-2}{x^2-2}$
Since $x \neq \pm\sqrt{2}$, b is cont. Also, a is cont. on D. Hence y is cont. on D

(d) $y = \left| \dfrac{x\sin x}{x^2+2} \right|$ $D: \mathbb{R}$

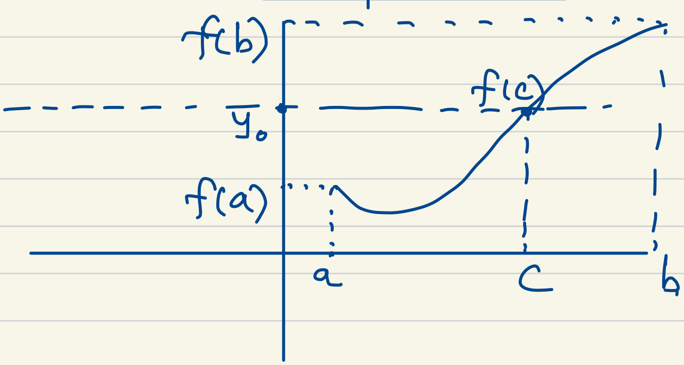
$|x|$ is cont. & $x\sin x$ as well as x^2+2 is cont. on $\mathbb{R}$, & $x^2+2 \neq 0$ on $\mathbb{R}$,

Hence cont. on $\mathbb{R}$,

THEOREM 11 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Geometric interpretation



Example: Show that $f(x) = x^3 - x - 1$ has a root in $[1, 2]$.

Proof:

$f(1) = 1^3 - 1 - 1 = -1 < 0$

$f(2) = 2^3 - 2 - 1 = 5 > 0$

$f(1) < 0 < f(2)$

& $f(x)$ is cont. on $[1, 2]$.

Then by the Intermediate value theorem,

$\exists\, c \in (1, 2) \ni f(c) = 0.$

such that

$\Rightarrow c$ is root of f lying in $[1, 2]$.

Derivative of a function

Defn. f is said to have the derivative at $x = x_0$ if $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. This limit is denoted by $f'(x_0)$.

Alternate formula: $f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Let $x = x_0 + h$

One-sided derivatives

$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$ left-hand derivative at x_0

$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$ right-hand derivative of f at x_0.

$\lim_{x \to c} f(x) = f(c) \Leftrightarrow \lim_{x \to c} \left(f(x) - f(c)\right) = 0$

Differentiability implies continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof: $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

$\lim_{x \to c} f(x) - f(c) = \lim_{x \to c} \frac{f(x) - f(c)}{(x - c)} \cdot (x - c)$

$$= \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right) \cdot \lim_{x \to c} (x - c) = f'(c) \cdot 0 = 0.$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c) \Rightarrow f \text{ is continuous at } x = c.$$

The intermediate value property of derivatives

If a and b are 2 points in an interval on which f is differentiable, then f' assumes every value between $f'(a)$ and $f'(b)$.

Differentiation rules:

- If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

- If n is a positive integer, then

$$\frac{d}{dx} x^n = nx^{n-1}.$$

(Later, it can be shown to be true for negative integer n too)

- If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Ex. 3 Find the equation for the tangent to the curve $y = x + \frac{2}{x}$ at the point $(1, 3)$.

Derivatives of trigonometric functions

① $\frac{d}{dx}(\sin x) = \cos x$ ② $\frac{d}{dx}(\cos x) = -\sin x$

③ $\frac{d}{dx}(\tan x) = \sec^2 x$ ④ $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

④ $\frac{d}{dx}(\sec x) = \sec x \tan x$

⑤ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec}(x)\cot(x)$.

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Example: Let $h(x) = \sin^2 x$.
Find $h'(x)$.

Note that $h(x) = (f \circ g)(x) \; (= f(g(x)))$,
where $f(x) = x^2$
and $g(x) = \sin(x)$

By chain rule,

$h'(x) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

$f'(x) = 2x, \quad g'(x) = \cos(x)$.

$\Rightarrow f'(g(x)) = f'(\sin x) = 2\sin(x)$.

Hence $h'(x) = 2\sin x \cdot \cos x$.

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Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Parametric curve: $x = f(t)$, $y = g(t)$.

Eg. Circle: $x^2 + y^2 = 9$; parametric form is $x = 3\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$.

Example If $x = 2t+3$ & $y = t^2 - 1$, find the value of dy/dx at $t = 6$.

$\frac{dx}{dt} = \frac{d}{dt}(2t+3) = 2$ & $\frac{dy}{dt} = \frac{d}{dt}(t^2-1) = 2t$

Hence $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t$. Hence $\left.\frac{dy}{dx}\right|_{t=6} = 6$.

Implicit Differentiation:

* **Concept**: It may not always be possible to explicitly represent y as a function of x. For example, consider $y^2 = x^2 + \sin(xy)$. Yet, it may be possible to differentiate y w.r.t. x, i.e. $\frac{dy}{dx}$. When this is the case, the process of differentiation is called as 'implicit differentiation'.

* **Example**: Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin(xy)$.

We have $y^2 = x^2 + \sin(xy)$.
Differentiate both sides w.r.t. x to get

$$2y\frac{dy}{dx} = 2x + \frac{d}{dx}\sin(xy)$$

$$\Rightarrow 2y\frac{dy}{dx} = 2x + \cos(xy)\cdot\frac{d(xy)}{dx}$$

$$= 2x + \cos(xy)\left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow (2y - x\cos(xy))\frac{dy}{dx} = 2x + y\cos(xy)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}}.$$

Ex.1 Show that the point $(2,4)$ lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve at $(2,4)$.

$$x^3+y^3-9xy=0$$ (Foliate of Descartes)

- $(2,4)$ lies on the curve because

$$(2)^3+(4)^3-9(2)(4)$$
$$= 8+64-72$$
$$= 0.$$

- Tangent to the curve at $(2,4)$:

$$3x^2+3y^2\frac{dy}{dx}-9\left(x\frac{dy}{dx}+y\right)=0$$

$$\Rightarrow 3x^2-9y=(9x-3y^2)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3x^2-9y}{9x-3y^2}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(2,4)}=\frac{3(2)^2-9(4)}{9(2)-3(4)^2}=\frac{12-36}{18-48}$$

$$=\frac{-24}{-30}=\frac{4}{5}.$$

$\Rightarrow$ Eqn. of the tangent at $(2,4)$:

$$y-4=\frac{4}{5}(x-2)$$

$$\Rightarrow y=\frac{4x}{5}+4-\frac{8}{5}$$

$$\Rightarrow \boxed{y=\frac{4x}{5}+\frac{12}{5}}$$

Normal to the curve at (2, 4).

slope $= -\frac{5}{4}$

$\Rightarrow$ Eqn. is $y - 4 = -\frac{5}{4}(x-2)$

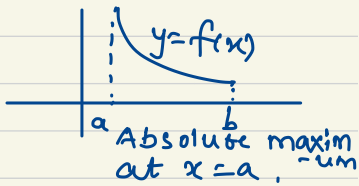
$\Rightarrow y - 4 = -\frac{5x}{4} + \frac{10}{4}$

$\Rightarrow \boxed{y = -\frac{5x}{4} + \frac{26}{4}}$

Local Maxima and Minima (Applications of derivatives)

- **Absolute maximum and minimum**

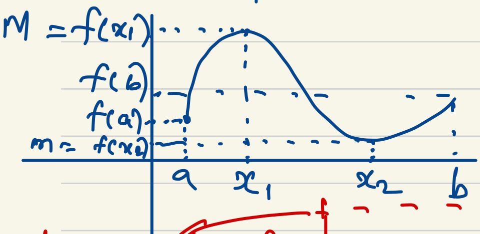
If c is a point in the domain of f such that $f(x) \le f(c)$ for all values of x in the domain, then f is said to have the absolute maximum at c.
Similarly, if $f(x) \ge f(c)$, then f is said to have the absolute minimum at c.

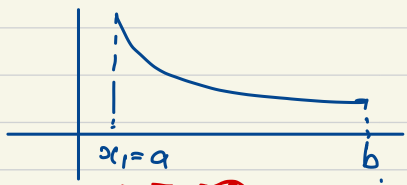


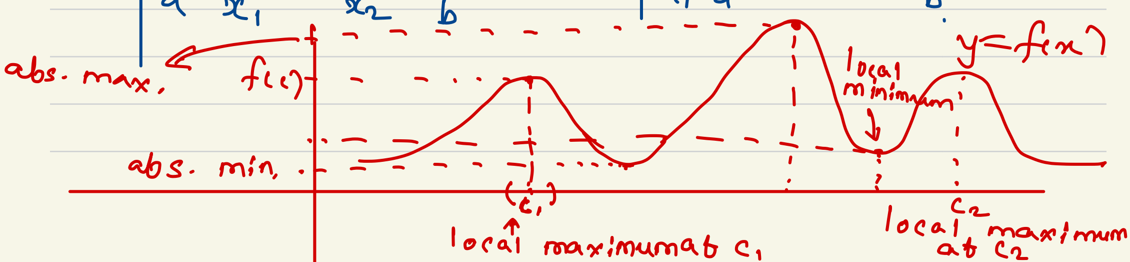
- **The extreme value theorem**

If f is continuous on the closed interval $[a, b]$, then f attains its maximum and minimum values at some points in the interval $[a, b]$, i.e; there are points x_1 and x_2 in $[a, b]$ such that $f(x_1) = M$, $f(x_2) = m$ &

$$m \le f(x) \le M \quad \forall\, x \in [a, b].$$







Defn. Local maximum, Local minimum

A function f is said to have a local maximum at an interior point c of its domain if $f(x) \leq f(c)$ for all x in some open interval containing c.

A function f is said to have a local minimum at an interior point c of its domain if $f(x) \geq f(c)$ for all x in some open interval containing c.

Thm. The first derivative theorem for local extreme values

- If f has a local maximum or minimum at an interior point c of its domain and if $f'(c)$ exist, then $f'(c) = 0$.

Proof: Suppose f has a local max. at int. pt. c, i.e. $f(x) \leq f(c)$ for all x in some open int. around c.

$f(x) - f(c) \leq 0$.

Since f is diff. at c, $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists,

and moreover, $f'(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$ — (1)

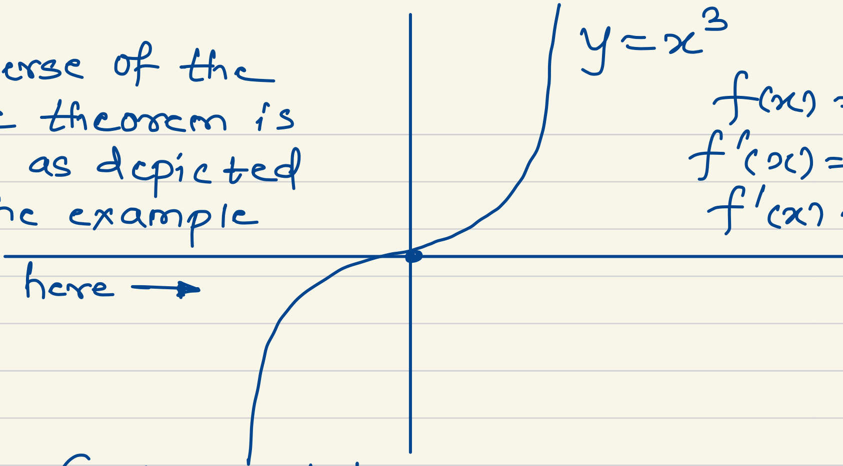
Similarly, $f'(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$. — (2)

From (1) & (2), we have

$f'(c) = 0$.

Similarly, one can prove the result for local minimum

Converse of the above theorem is false as depicted by the example given here →



$f(x) = x^3$

$f'(x) = 3x^2$

$f'(x) = 0 \Leftrightarrow$

$x = 0$.

Defn. Critical point

A critical point of a function f is that interior point of its domain where f' is either zero or undefined.

To find the absolute extrema of a continuous function f on a finite closed interval

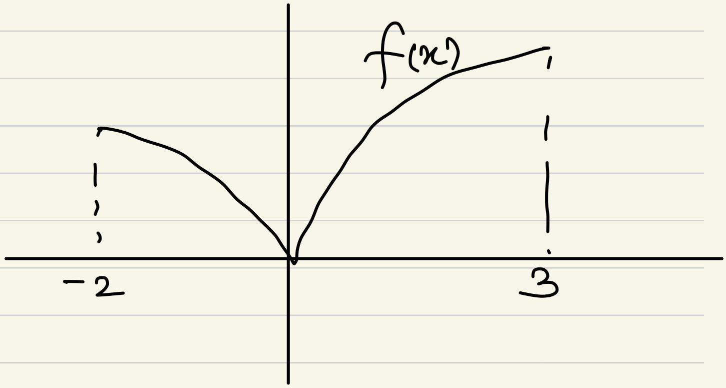
1. Evaluate f at all critical points and at end points.

2. Take the largest and smallest of these values.

Ex. 2 Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Ans.

$f(x) = x^{2/3}$, $[-2, 3]$



$f'(x) = \frac{2}{3} x^{2/3 - 1} = \frac{2}{3x^{1/3}}$.

f' is undefined at $x = 0$.

So the only critical point of f is $x = 0$.

$f(0) = 0$.

$f(-2) = (-2)^{2/3} = 2^{2/3}$,

$f(3) = 3^{2/3}$

$\Rightarrow$ Absolute max. of f is $f(3) = 3^{2/3}$ & it occurs at $x = 3$.

Abs. min. of f is $f(0) = 0$ and it occurs at $x = 0$.