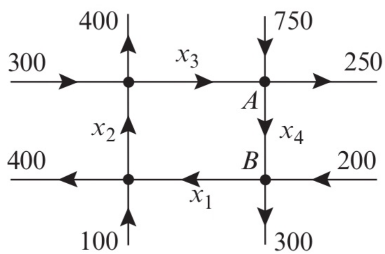
Tutorial 4 - to be discussed on 18th September 2024

Q1. The accompanying figure shows a network of one-way streets with traffic flowing in the directions indicated. The flow rates along the streets are measured as the average number of vehicles per hour.

(a) Set up a linear system whose solution provides the unknown flow rates.

(b) Solve the system for the unknown flow rates.

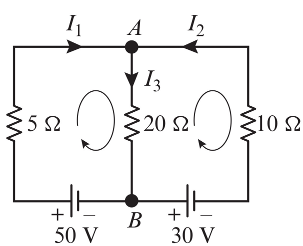
(c) If the flow along the road from A to B must be reduced for construction, what is the minimum flow that is required to keep traffic flowing on all roads?



◀ Figure Ex-3

Q2. Find the cubic polynomial whose graph passes through the points $(-1, -1)$, $(0, 1)$, $(1, 3)$, $(4, -1)$.
Use an augmented matrix to represent the linear system and then solve.

Q3. Determine the currents I_1, I_2, and I_3 in the circuit shown using the augmented matrix method for the linear system involved.



Q4. Evaluate the determinant of the matrix A below by cofactor expansion along: (a) the second row (b) the second column

(i) $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}$ (ii) $\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix}$

Q5. Confirm the identities without evaluating the determinant directly.

(a) $$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(b) $$\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Q6. Show that the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is invertible for all values of θ; then find A^{-1} using adjoint formula.

Q7. Prove that a square matrix A is invertible if and only if A^TA is invertible.

Q8. Find the values of k for which the matrix A is invertible

$$A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{bmatrix}$$

Q9. Solve by Cramer's rule, if it applies

(a) $$\begin{aligned} 3x_1 - x_2 + x_3 &= 4 \\ -x_1 + 7x_2 - 2x_3 &= 1 \\ 2x_1 + 6x_2 - x_3 &= 5 \end{aligned}$$

(b) $$\begin{aligned} -x_1 - 4x_2 + 2x_3 + x_4 &= -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 &= 14 \\ -x_1 + x_2 + 3x_3 + x_4 &= 11 \\ x_1 - 2x_2 + x_3 - 4x_4 &= -4 \end{aligned}$$

Recall the vector space axioms:

DEFINITION 1 Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by numbers called ***scalars***. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

Q1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u}=(u_1, u_2)$ and $\mathbf{v}=(v_1, v_2)$:

$$\mathbf{u}+\mathbf{v}=(u_1+v_1+1,\, u_2+v_2+1), \quad k\mathbf{u}=(ku_1, ku_2)$$

(a) Compute $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u}=(0,4)$, $\mathbf{v}=(1,-3)$, and $k=2$.

(b) Show that $(0,0)\neq\mathbf{0}$.

(c) Show that $(-1,-1)=\mathbf{0}$.

(d) Show that Axiom 5 holds by producing an ordered pair $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ for $\mathbf{u}=(u_1, u_2)$.

(e) Find two vector space axioms that fail to hold.

Q2. In each problem given below, identify whether each set equipped with the given operations is a vector spacer it is not, identify the axioms that fail.

(a) The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2.

(b) The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on R^2.

(c) The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y') \quad \text{and} \quad k(1, y) = (1, ky)$$

(d) The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with the standard matrix addition and scalar multiplication.

Q3. Determine which of the following are subspaces of R^3.

(a) All vectors of the form $(a, 0, 0)$.

(b) All vectors of the form $(a, 1, 1)$.

(c) All vectors of the form (a, b, c), where $b = a + c$.

(d) All vectors of the form (a, b, c), where $b = a + c + 1$.

(e) All vectors of the form $(a, b, 0)$.

Q4. Determine which of the following are subspaces of M_{nn}.

(a) The set of all diagonal $n \times n$ matrices.

(b) The set of all $n \times n$ matrices A such that $\det(A) = 0$.

(c) The set of all $n \times n$ matrices A such that $\mathrm{tr}(A) = 0$.

(d) The set of all symmetric $n \times n$ matrices.

(e) The set of all $n \times n$ matrices A such that $A^T = -A$.