MA 103 - SVC Lecture 5

Sect. 4.8 - Antiderivatives

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Examples: (1) $f(x) = x^3$; $F(x) = \frac{x^4}{4}$ since $F'(x) = \frac{4x^3}{4} = x^3$.

(2) $f(x) = x^4 - \sin x$; $F(x) = \frac{x^5}{5} + \cos x$

The most general antiderivative of f on an interval I is the function $F(x) + c$, where $F'(x) = f(x)$ $\forall x \in I$ and c is an arbitrary constant

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1$, n rational
2.	$\sin kx$	$-\frac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

Example Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin(2x)$.

Ans. $F(x) = G(x) + H(x) + c$, $G'(x) = 3/\sqrt{x} = 3x^{-1/2}$

$H'(x) = \sin(2x)$

$$G(x) = \frac{3\sqrt{x}}{(1/2)} = 6\sqrt{x}, \quad H(x) = \frac{-\cos(2x)}{2}.$$

- Initial value problems and differential equations

Finding antiderivative of a function is eqvt. to solving $\frac{dy}{dx} = f(x)$. This is known as differential equation. If we are given $y(x_0) = y_0$, then this is known as initial condition. The diff. eqn. along with the initial condition is called the initial value problem.

Ex. 1 A balloon ascending at the rate of 12ft./sec is at a height 80 ft. above the ground when a package is dropped. How long does it take the package to reach the ground?

Ans.

Let $v(t)$ denote the velocity of the package at time t. Let $s(t)$ denote its height above the ground.

We assume that except for acceleration due to gravity, no other force acts on the package;

Hence $$\frac{dv}{dt} = -32$$ (acc. due to gravity is $9.8 m/sec^2$ $\approx 32 ft./sec^2$)

in the direction of decreasing s.

Initial value problem for $v(t)$

Diff. eqn. $\frac{dv}{dt} = -32$

Initial condition $v(0) = 12$

$$v(t) = -32t + c$$

$v(0) = 12$ implies

$$12 = -32(0) + c$$

$$\Rightarrow c = 12.$$

So $v(t) = -32t + 12$.

Initial value problem for $s(t)$:

$$\frac{ds}{dt} = -32t + 12$$

$$s(0) = 80$$

$$s(t) = -16t^2 + 12t + c_1$$

$s(0) = 80$ implies

$$80 = -16(0)^2 + 12(0) + c_1$$

$$\Rightarrow c_1 = 80$$

$$\Rightarrow s(t) = -16t^2 + 12t + 80$$

We want to find t s.t. $s(t) = 0$.

Hence we solve the quadratic

$$-16t^2+12t+80=0$$
$$\Rightarrow 16t^2-12t-80=0$$
$$\Rightarrow 4t^2-3t-20=0$$
$$\Rightarrow t=\frac{3+\sqrt{9+320}}{8}=\frac{3+\sqrt{329}}{8}$$
$$\approx 2.64 \text{ sec.}$$

DEFINITION **Indefinite Integral, Integrand**

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

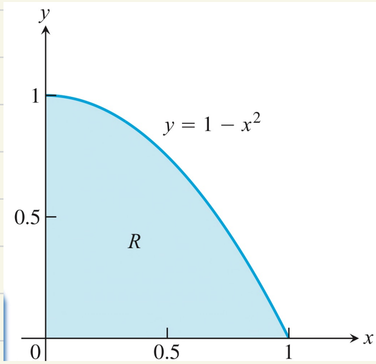
Example Find the general antiderivative of $\int \cot^2 x\,dx$.

Use $\cot^2 x = \operatorname{cosec}^2 x - 1$. Hence

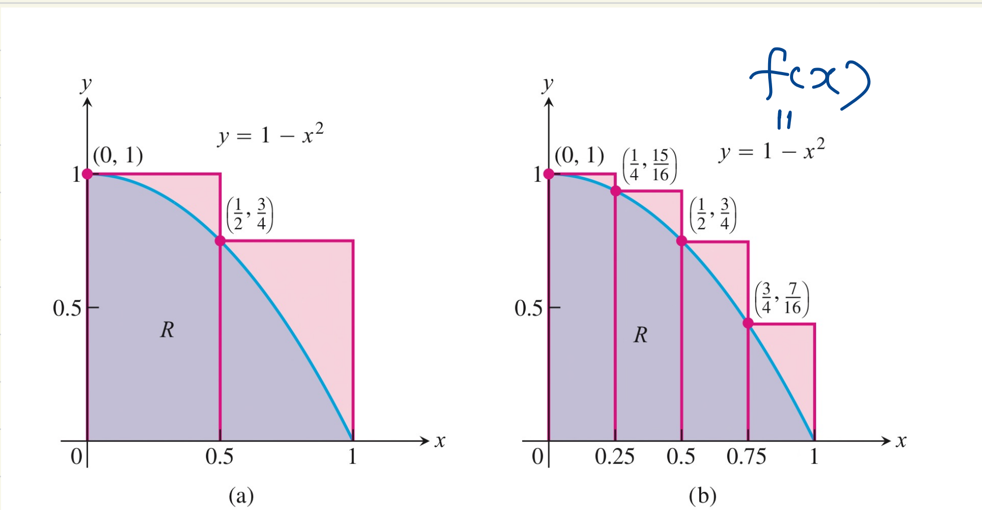
$$\int \cot^2 x\,dx = \int(\operatorname{cosec}^2 x-1)\,dx = -\cot x - x + c.$$

Towards the concept of a definite integral

Sect. 5.1 Estimating with finite sums



Let A denote the sum of areas of rectangles in all of the pictures below,

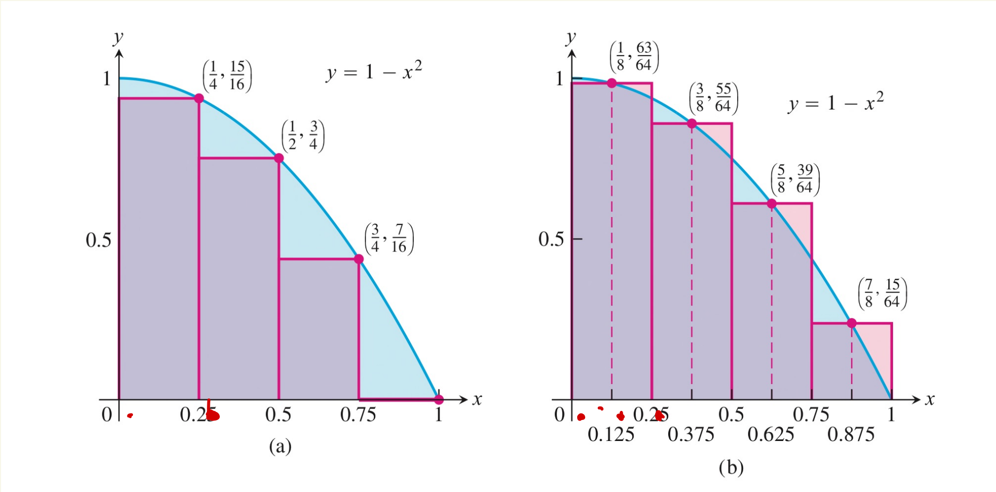


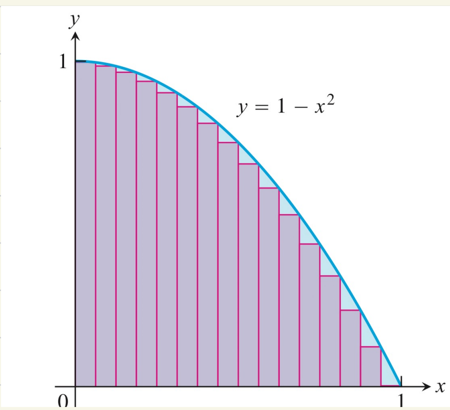
$A = \frac{1}{2}(1) + \frac{1}{2} \cdot \frac{3}{4} = \frac{7}{8}$

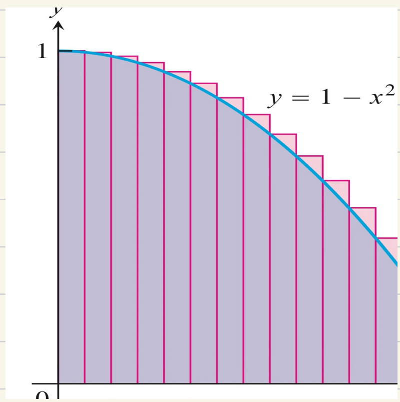
$= 0.8$

$A = \frac{1}{4} f(0) + \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right)$

$= 0.78125$

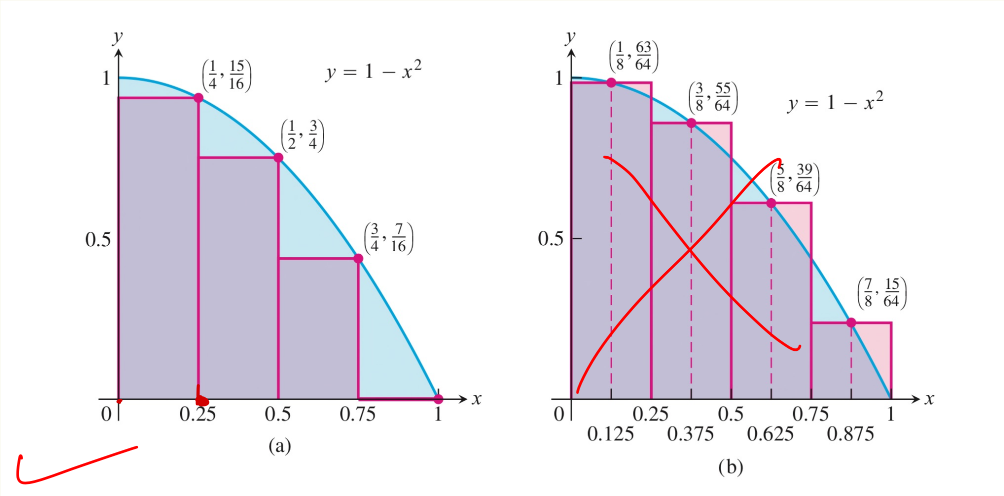






Ex. 2 Find the limiting value of lower sum approximations to the area of the region R below the graph of $y=1-x^2$ and above the interval $[0,1]$ on the x-axis using equal width rectangles whose widths approach zero and whose number approaches infinity.

Ans. $y = 1 - x^2$



Let $f(x) = 1 - x^2$.

Take the interval $[0, 1]$ and divide into 'n' equal sub-intervals by means of the partition $\left\{ \underset{"x_0"}{0}, \underset{"x_1"}{\frac{1}{n}}, \underset{"x_2"}{\frac{2}{n}}, \frac{3}{n}, \ldots, \frac{n-1}{n}, \underset{"x_n"}{1} \right\}$ of $[0, 1]$. The width of each sub-interval is $\Delta x = \frac{1-0}{n} = \frac{1}{n}$.

Hence the Riemann sum is

$$\sum_{k=1}^{n} f(c_k) \Delta x_k .$$

Here for the interval $[x_{k-1}, x_k]$, i.e. $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, we have $c_k = \frac{k}{n}$ (so as to get the lower sum approx.; this follows from the fact that f is decreasing on $[0,1]$.)

Hence the Riemann sum is

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \sum_{k=1}^{n}\left\{1-\left(\frac{k}{n}\right)^2\right\}\cdot\frac{1}{n}$$

$$= \sum_{k=1}^{n}\left(\frac{1}{n}-\frac{k^2}{n^3}\right) = \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3}$$

$$= \frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{1}{n}\cdot n - \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= 1 - \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

As $n\to\infty$, $1-\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \longrightarrow 1-\frac{1}{6}(1)(2)$

$$= 1-\frac{1}{3}$$

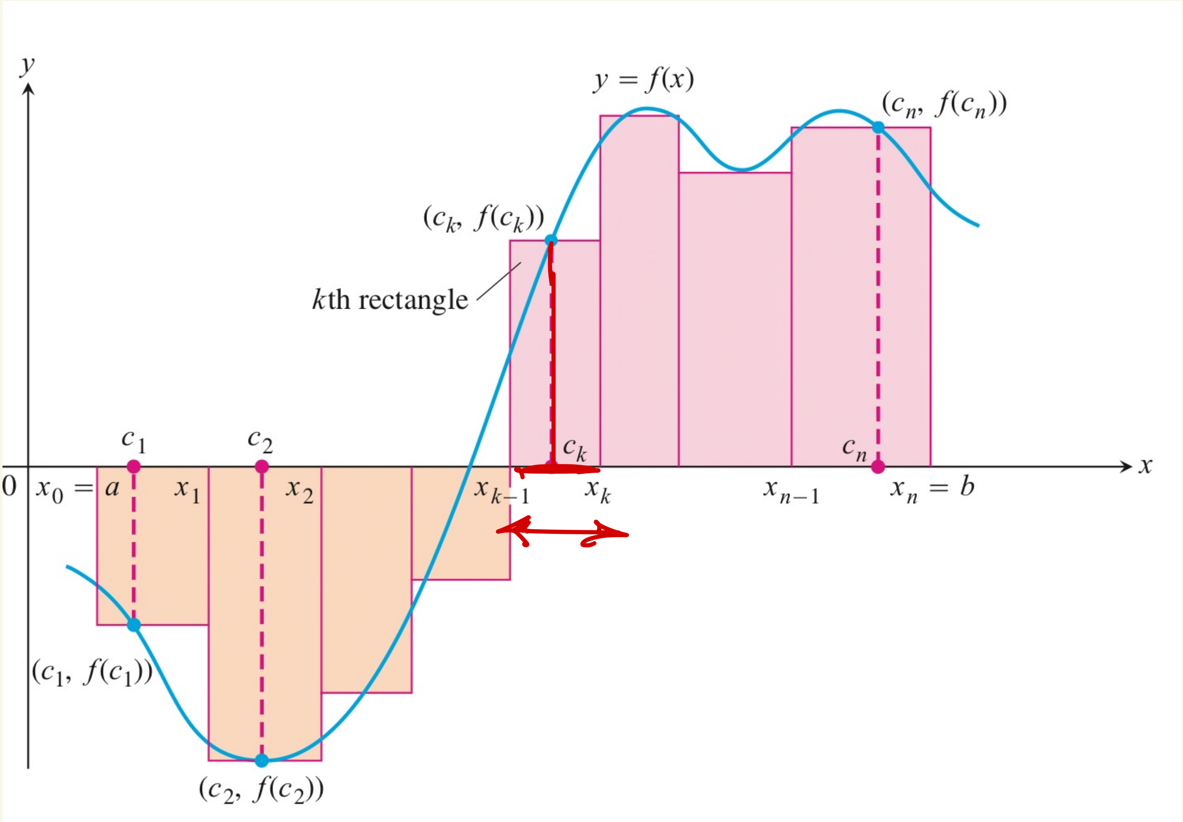
$$= \frac{2}{3}.$$

Check:

$$\int_0^1 f(x)\,dx = \int_0^1 1-x^2\,dx$$

$$= \left[x-\frac{x^3}{3}\right]_0^1 = \left(1-\frac{1}{3}\right)-(0-0) = \frac{2}{3}.$$

Sect. 5.2 – Riemann sums



Let f be an arbitrary function defined on the interval $[a, b]$.

We subdivide $[a, b]$ into subintervals, not necessarily of the same width, by choosing $n-1$ points $\{x_1, x_2, \ldots, x_{n-1}\}$ between a & b which satisfy

$$x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < b = x_n$$

These numbers $x_0, x_1, \ldots, x_{n-1}, x_n$ form a partition $\{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$.

In each sub-interval, we select some point, say c_k, in the sub-interval $[x_{k-1}, x_k]$. On each such subinterval, we build a rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$.

Then form the product $\underbrace{f(c_k)\cdot\Delta x_k}_{\text{area of the rectangle built on } [x_{k-1}, x_k]}$, where $\Delta x_k = x_k - x_{k-1}$.

We sum all such product, say,

$$S_p = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

$$= f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_j)\Delta x_j + \cdots + f(c_n)\Delta x_n.$$

This sum S_p is called a Riemann sum for f on the interval $[a,b]$.

Equal widths: $\Delta x = \dfrac{b-a}{n}$.

<u>Varying widths</u>: When a partition has sub-intervals of varying widths, then we can control all widths by controlling the width of the longest subinterval.

The norm of a partition P, denoted by $\|P\|$, to be the largest of all subinterval widths. If $\|P\| \to 0$, then of course, the widths of each of the subintervals approach zero.

Sect. 5.3 – The definite integral (Definition)

Equal widths : $\Delta x = \frac{b-a}{n}$, $n \to \infty$.

Arbitrary widths : $\|P\| \to 0$

DEFINITION The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon.$$

If $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \Delta x_k$ exists, then this is called the definite integral of f from a to b, and is denoted by $\int_a^b f(x)\,dx$.

upper limit of integration

lower limit of integration

$\lim_{x \to c} f(x) = L$ $\quad \varepsilon > 0, \exists\, \delta > 0 \ni |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$

Sect. 5.3 – Definite integrals (Examples)

DEFINITION **The Definite Integral as a Limit of Riemann Sums**

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon.$$

When this defn. is satisfied, we say the Riemann sums of f over [a,b] converge to the number I denoted by $\int_a^b f(x)\,dx$, and we say f is integrable over [a,b].

We have a number of choices for partition P whose norm $\|P\|$ goes to zero, and a number of choices for c_k for each partition. We say the definite integral exists when we get the same limit I, irrespective of the choices of the partitions P & points c_k. When the limit exists, we write it as

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = I = \int_a^b f(x)\,dx .$$

The limit is always taken as the norm of the partitions approach zero and the number of subintervals goes to infinity.

THEOREM 1 **The Existence of Definite Integrals**

A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

Remarks w.r.t. the above theorem

① f is cont. on $[a,b]$, so it attains its maximum value and minimum value on each of $[x_{k-1}, x_k]$ where $1 \le k \le n$.

So if we choose c_k so that $f(c_k)$ is the max. value of f on $[x_{k-1}, x_k]$, we get an upper sum. Similarly, if we choose c_k so that $f(c_k)$ is the min. value of f on $[x_{k-1}, x_k]$, we get a lower sum.

* A discontinuous function on an interval $[a,b]$ may or may not be integrable.

* Discontinuous functions that are increasing on $[a,b]$, or which are piecewise continuous (finitely many discontinuities) are integrable on $[a,b]$.

Example of a non-integrable function on $[0,1]$

Show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

has no Riemann integral over $[0,1]$.

If we pick a partition P of $[0,1]$ & let c_k be that point in $[x_{k-1}, x_k]$ which gives the max. value of f on $[x_{k-1}, x_k]$.

So let us a choose a rational number in $[x_{k-1}, x_k]$ to be our c_k so that $f(c_k) = 1$.

Hence the Riemann sum

$$U = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} (1)\,\Delta x_k = \sum_{k=1}^{n} (x_k - x_{k-1})$$

$= (x_n - \cancel{x_{n-1}}) + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + \cdots + (\cancel{x_2} - \cancel{x_1}) + (\cancel{x_1} - x_0)$

$= b - a = 1 - 0 = 1.$

So the limit of Riemann sums with these choices is 1

Now choose c_k to be that number in $[x_{k-1}, x_k]$ which gives you minimum value of f on $[x_{k-1}, x_k]$, i.e., choose c_k to be an irrational number in $[x_{k-1}, x_k]$. Then the Riemann sum is

$$L = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} (0)\Delta x_k = 0.$$

Hence limit of Riemann sums with these choices is 0.

The limit thus depends on choices of c_k, hence not integrable on $[0, 1]$.

Properties of definite integrals

TABLE 5.3 Rules satisfied by definite integrals

Suppose f and g are two integrable functions.

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$	
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)

Ex. 1 Show that the value of $\int_0^1 \sqrt{1+\cos x}\,dx$ is less than $3/2$.

Ans. $\sqrt{1+\cos x} = \sqrt{2\cos^2(x/2)} = \sqrt{2}\cos(x/2)$

$$\int_0^1 \sqrt{1+\cos x}\,dx = \sqrt{2}\int_0^1 \cos(x/2)\,dx = \sqrt{2}\left[\frac{\sin(x/2)}{1/2}\right]_0^1$$

$$= 2\sqrt{2}\sin(1/2) < 3/2.$$

2nd proof:- $-1 \le \cos x \le 1$

$\Rightarrow 0 \le 1+\cos x \le 2 \Rightarrow 0 \le \sqrt{1+\cos x} \le \sqrt{2}$

$\Rightarrow \int_0^1 \sqrt{1+\cos x}\,dx \le \int_0^1 \sqrt{2}\,dx = \sqrt{2}(1-0) = \sqrt{2}.$

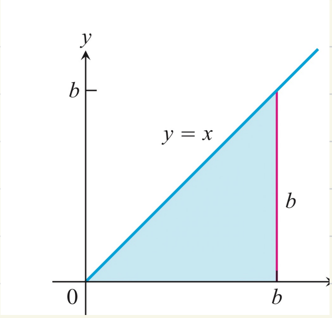
DEFINITION **Area Under a Curve as a Definite Integral**

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Example Area under the line $y = x$

Compute $\int_0^b x\,dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.



(a) Note that the shaded area is that of a Δ with base b & ht. also b. Hence its area must be $\frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \cdot b \cdot b = \frac{b^2}{2}.$$

(b) We get the above same value by calculating $\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k$.

$f(x) = x$ is cont., hence integrable. Hence the choices of P (except that $\|P\| \to 0$) and c_k's do not matter. We choose a partition P that subdivides $[0, b]$ in n subintervals of equal width $\Delta x = \frac{b-0}{n} = \frac{b}{n}$.

We choose c_k to be right end point in each sub-interval. The partition P is

$$\left\{ 0, \frac{b}{n}, \frac{2b}{n}, \cdots \frac{nb}{n} \right\}. \quad \& \quad c_k = \frac{kb}{n}.$$

Hence $\sum_{k=1}^{n} f(c_k) \Delta x = \sum_{k=1}^{n} f\left(\frac{kb}{n}\right) \cdot \frac{b}{n}$

$$= \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2} \sum_{k=1}^{n} k = \frac{b^2}{n^2} \frac{n(n+1)}{2}$$

$$= \frac{b^2}{2}\left(1 + \frac{1}{n}\right).$$

As $n \to \infty$ & $\|P\| \to 0$, we get $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \Delta x$

$$= \lim_{n \to \infty} \frac{b^2}{2}\left(1 + \frac{1}{n}\right) = \frac{b^2}{2}.$$

Hence $\int_0^b x \, dx = \frac{b^2}{2}$.

* More generally, if $0 < a < b$, $\int_a^b x \, dx = \int_a^0 x \, dx + \int_0^b x \, dx = \frac{b^2}{2} - \frac{a^2}{2}$.

Average value of a continuous function

$$\frac{f(c_1)+f(c_2)+f(c_3)+\cdots+f(c_n)}{n}$$

$$= \frac{1}{n}\sum_{k=1}^{n} f(c_k)$$

Divide $[a,b]$ into n sub-intervals of equal widths so that $\Delta x = \frac{b-a}{n}$, or in other words, $\frac{1}{n} = \frac{\Delta x}{b-a}$.

Hence $\frac{f(c_1)+\cdots+f(c_n)}{n} = \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k)$

$$= \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

$$\rightarrow \frac{1}{b-a}\int_a^b f(x)\,dx \text{ as } n \rightarrow \infty.$$

DEFINITION **The Average or Mean Value of a Function**

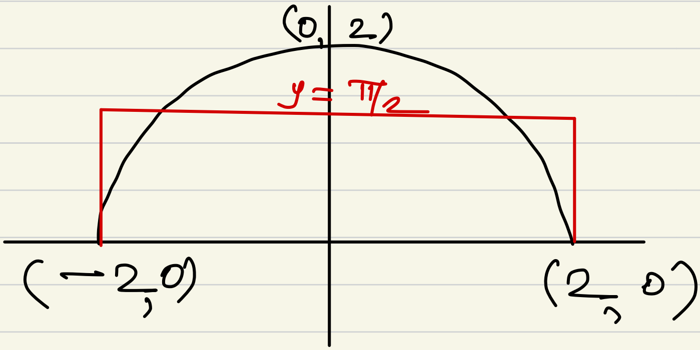
If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Ex. 4 Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$.

Ans.

$$f(x) = \sqrt{4 - x^2}, \quad \text{on } [-2, 2]$$



Average value of f is given by

$$= \frac{1}{2-(-2)} \underbrace{\int_{-2}^{2} \sqrt{4-x^2}\, dx}_{\substack{= \text{ area of the semi-circle} \\ = \frac{1}{2}\pi(2)^2 = 2\pi}}$$

$$= \frac{1}{4}(2\pi) = \pi/2.$$