MA 103 – Quiz 2 (2024) Solutions

1 (i) Full points to all

(ii) (a)

(iii) (b)

(iv) (a)

(v) (c)

2 (i) $\lim_{x \to 0} \dfrac{\sqrt{9-x} - 3}{3 - \sqrt{9+x}}$

$$= \lim_{x \to 0} \frac{(\sqrt{9-x} - 3)(\sqrt{9-x} + 3)(3 + \sqrt{9+x})}{(\sqrt{9-x} + 3)(3 - \sqrt{9+x})(3 + \sqrt{9+x})}$$

$$= \lim_{x \to 0} \frac{((9-x) - 9)(3 + \sqrt{9+x})}{(\sqrt{9-x} + 3)(9 - (9+x))}$$

$$= \lim_{x \to 0} \frac{\cancel{-x}(3 + \sqrt{9+x})}{\cancel{-x}(\sqrt{9-x} + 3)}$$

$$= \frac{3 + \sqrt{9+0}}{\sqrt{9-0} + 3} = \frac{3+3}{3+3} = 1.$$

(iii) $x^3y^3 + y^2 = x + y$

Differentiate both sides w.r.t. y to get

$$3x^2y^3 + x^3\left(3y^2\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow 3x^2y^3 - 1 = (1 - 3x^3y^2 - 2y)\frac{dy}{dx}$$

So that $\dfrac{dy}{dx} = \dfrac{3x^2y^3 - 1}{1 - 3x^3y^2 - 2y}$

Now slope of the tangent to the curve at $(1, -1)$ is $\dfrac{3(1)^2(-1)^3 - 1}{1 - 3(1)^3(-1)^2 - 2(-1)}$

$$= \frac{-4}{0} = -\infty$$

Therefore slope of the normal to the curve at $(1, -1)$ is 0.

$\Rightarrow$ eqn. of the normal at $(1, -1)$ is

$$y + 1 = 0(x - 1)$$

$$\Rightarrow \boxed{y = -1.}$$

③ (Proof by contradiction)

Suppose $\exists$ a value of $k \ni x^3 - 3x + k = 0$ has 2 distinct roots in $[0, 1]$.

We first show that these two roots cannot be 0 & 1.

If both 0 & 1 are the roots, then

$$\left.\begin{array}{l} 0^3 - 3(0) + k = 0 \implies k = 0 \\ \& \quad 1^3 - 3(1) + k = 0 \implies k = 2 \end{array}\right\} \text{this is a contradiction.}$$

So 0 & 1 cannot be the 2 distinct roots.

So if there are 2 distinct roots x_1 & x_2 in $[0, 1]$, we must have

$$0 \leq x_1 < x_2 \leq 1,$$

Let $f(x) = x^3 - 3x + k$. Then $f(x_1) = 0$
$f(x_2) = 0$

It is continuous on $[x_1, x_2]$ & differentiable on (x_1, x_2).

Moreover, $f(x_1) = f(x_2)$ $(= 0)$.

By Rolle's theorem, $\exists\, c \in (x_1, x_2) \ni$
$f'(c) = 0$

Now $f'(x) = 3x^2 - 3$

So $f'(c) = 0$ implies

$$3c^2 - 3 = 0$$
$$\Rightarrow c = \pm 1.$$

But $0 \leq x_1 < c < x_2 \leq 1$

So c cannot be 1 (and obviously it cannot be -1).

Therefore there does not exist any k $\ni$ $x^3 - 3x + k = 0$ has 2 distinct roots in $[0, 1]$.

(4) $f(x) = x^3 - 3x^2 + 3x = x(x^2 - 3x + 3)$

f is continuous on $(-\infty, \infty)$.
f is also differentiable on $(-\infty, \infty)$ (being a polynomial) & we have
$f'(x) = 3x^2 - 6x + 3$

f does not have absolute maximum or minimum since $\lim_{x\to\infty} f(x) = +\infty$ & $\lim_{x\to -\infty} f(x) = -\infty$.

Also, since f is well-defined on $(-\infty, \infty)$, the critical point can occur when
$$f'(x) = 0.$$
$$\Rightarrow 3(x-1)^2 = 0$$
$$\Rightarrow x = 1.$$

1 is the only critical point of f.

Consider the 2 intervals:

Intervals	$(-\infty, 1)$	$(1, \infty)$
sign of f'	$+$	$+$
behavior of f	increasing	increasing

Now $f''(x) = 6x - 6 = 6(x-1)$.

$f''(x) < 0$ for $x \in (-\infty, 1)$ &

$f''(x) > 0$ for $x \in (1, \infty)$.

Thus f' is decreasing on $(-\infty, 1)$ & increasing on $(1, \infty)$

$\Rightarrow f$ is concave down on $(-\infty, 1)$ & concave up on $(1, \infty)$.

Note ~~that~~ the point $(1, 1)$ lies on the curve.

Also, at the pt. $(1,1)$, the graph has a tangent line, which is the line $y = 1$:

$$f'(x)\Big|_{x=1} = 0$$

So the eqn. of the tangent line to f at $(1,1)$ is $y-1 = 0(x-1)$
$\Rightarrow y = 1$.

So $(1,1)$ is the inflection point (by defn.).

Also, note that the graph passes through the origin.

With the help of all of these, we can sketch the graph of f as follows:-