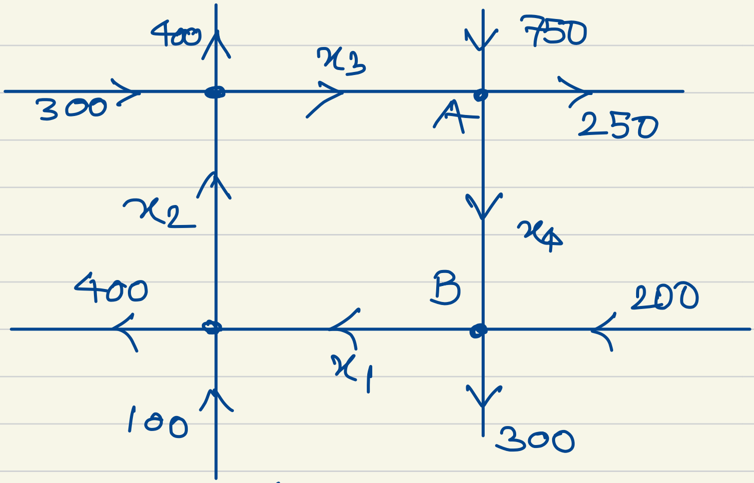
MA 101 - Tutorial 4 (2024)

Ex. 1



(a)

$$300 + x_2 = 400 + x_3$$
$$x_3 + 750 = x_4 + 250$$
$$x_1 + 100 = x_2 + 400$$
$$x_4 + 200 = x_1 + 300$$

(b) Augmented matrix :

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 1 & -1 & 0 & 0 & 300 \\ 1 & 0 & 0 & -1 & -100 \end{array}\right]$$

Convert the augmented matrix into RREF form:

$R_1 \leftrightarrow R_4$,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & -500 \\ 1 & -1 & 0 & 0 & 300 \\ 0 & 1 & -1 & 0 & 100 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & -1 & 0 & 1 & 400 \\ 0 & 1 & -1 & 0 & 100 \end{array}\right]$$

$R_2 \leftrightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & -1 & 0 & 1 & 400 \\ 0 & 0 & 1 & -1 & -500 \end{array}\right]$$

$R_3 \rightarrow R_3 + R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & -1 & 1 & 500 \\ 0 & 0 & 1 & -1 & -500 \end{array}\right]$$

$R_3 \rightarrow -R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 1 & -1 & -500 \end{array}\right]$$

$R_4 \rightarrow R_4 - R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 + R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & -1 & -400 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - x_4 = -100$$
$$x_2 - x_4 = -400$$
$$x_3 - x_4 = -500$$

Let $x_4 = t$, then $x_1 = t - 100$
$x_2 = t - 400$
$x_3 = t - 500$

(c) Flow along $AB = x_4 = t$.
We want to minimize $t > 0$ s.t.
$x_1, x_2, x_3 > 0$
$t > 100$, $t > 400$, $t > 500$

Hence we must have $t > 500$
Thus, the minimum flow along AB is 500.

Ex. 2 $(-1,-1)$, $(0,1)$, $(1,3)$, $(4,-1)$

Let $p(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

$p(x)$ passes through $(-1,-1)$, $(0,1)$, $(1,3)$, and $(4,-1)$.
$\Rightarrow p(-1) = -1$, $p(0) = 1$, $p(1) = 3$,
$p(4) = -1$.

$$a_0 - a_1 + a_2 - a_3 = -1$$
$$a_0 = 1$$
$$a_0 + a_1 + a_2 + a_3 = 3$$
$$a_0 + 4a_1 + 16a_2 + 64a_3 = -1$$

Augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & 4 & 4^2 & 4^3 & -1 \end{array}\right]$$

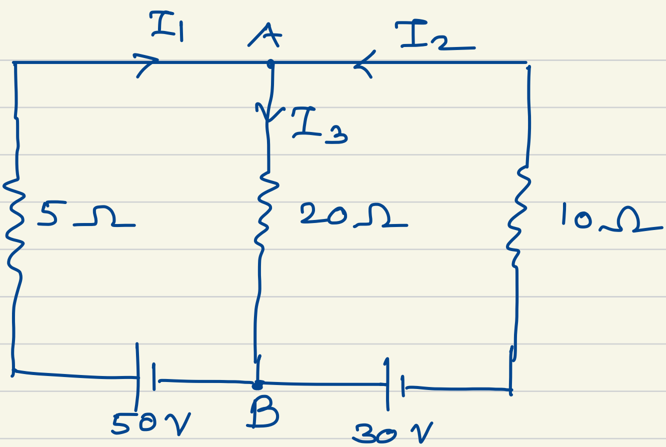
Apply ERO's & convert into REF form

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 13/6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{array}\right]$$

$$\Rightarrow a_3 = -1/6$$
$$a_2 = 0$$
$$a_1 = 13/6$$
$$a_0 = 1$$

$$\Rightarrow p(x) = \frac{-1}{6}x^3 + \frac{13}{6}x + 1$$

Q. 3]



Kirchoff's current law ensures

$$I_1 + I_2 = I_3 \quad \text{—— (1)}$$

Kirchoff's voltage law gives

$$50 - 5I_1 - 20I_3 = 0 \quad \text{—— (2)}$$

$$30 + 20I_3 + 10I_2 = 0 \quad \text{—— (3)}$$

$$30 + 50 - 5I_1 + 10I_2 = 0$$

$$\Rightarrow \quad 5I_1 - 10I_2 = 80$$

$$\Rightarrow \quad I_1 - 2I_2 = 16 \quad \text{—— (4)}$$

The 4^{th} eqn is superfluous since (2) + (3) = (4).

Hence the linear system is

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 5I_1 + 0I_2 + 20I_3 = 50 \\ 0I_1 - 10I_2 - 20I_3 = 30 \end{cases}$$

$$\Rightarrow \boxed{I_1 = 6,\ I_2 = -5,\ I_3 = 1}$$

4 a the second row

$$A = \begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}$$

$$\det(A) = -2 \begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 5 \begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix}$$

$$= 0 - (-12) - 5(27)$$

$$= 12 - 135$$

$$= -123$$

b the second column

$$A = \begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}$$

$$\det(A) = -0 \begin{vmatrix} 2 & 5 \\ 1 & -4 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 9 \begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix}$$

$$= -(-12) - 9(15) = 12 - 135$$

$$= -123.$$

Q 5] (a)

$$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{LHS} = \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & b_1+c_1 \\ a_2 & b_2 & b_2+c_2 \\ a_3 & b_3 & b_3+c_3 \end{vmatrix}$$

$$= \overset{0}{\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}} + \overset{0}{\begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix}} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= \text{RHS}.$

(b)

$$\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{LHS}: \begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$= \begin{vmatrix} (1+t)(a_1+b_1) & (1+t)(a_2+b_2) & (1+t)(a_3+b_3) \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (1+t)\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$$= (1+t)\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ (t-1)a_1 & (t-1)a_2 & (t-1)a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (t^2-1)\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (t^2-1)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ (Splitting the determinant into two)

$$= (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$ (Interchanging the first 2 rows)

Q 6] $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det(A) = 1\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$= \cos^2\theta + \sin^2\theta = 1 \neq 0$$

Hence A is invertible for all values of θ.

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) = \operatorname{adj}(A)$$

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

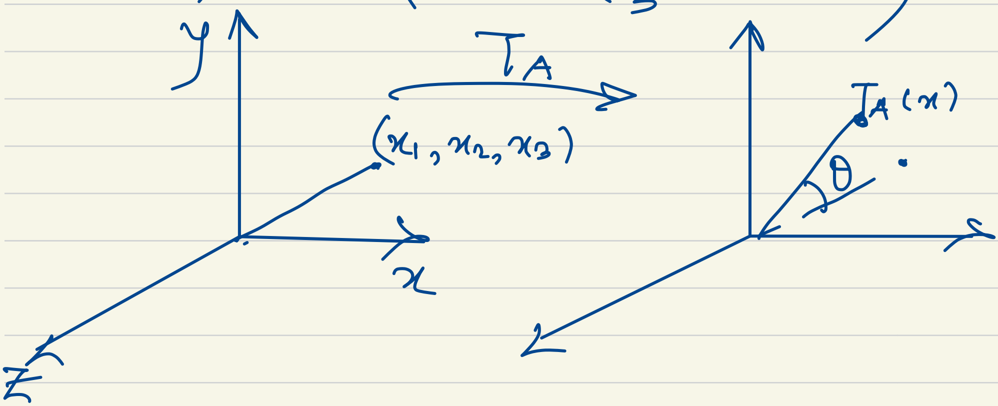
$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$= A^{-1}$.

Consider $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta\, x_1 + \sin\theta\, x_2 \\ -\sin\theta\, x_1 + \cos\theta\, x_2 \\ x_3 \end{pmatrix}$$



T_A is the map that rotates any vector in $\mathbb{R}^3$ by an angle θ anti-clockwise along z-axis.

$T_{A^{-1}}$ does the same as above, but with angle $(-\theta)$

Q 7] "$\Longrightarrow$" A is invertible

$\Rightarrow \det(A) \neq 0$

$$\det(A^T A) = \det(A^T)\det(A) = \det(A) \cdot \det(A) \neq 0.$$

$\Rightarrow A^T A$ is invertible

"$\Longleftarrow$"

$A^T A$ is invertible

$\Rightarrow \det(A^T A) \neq 0$

$\Rightarrow 0 \neq \det(A^T) \cdot \det(A) = \det(A)^2$

$\Rightarrow \det(A) \neq 0$

$\Rightarrow A$ is invertible.

Q 8] $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

In order to be invertible, $\det(A) \neq 0$

$\Rightarrow (k-3)(k-2) - 4 \neq 0$

$\Rightarrow k^2 - 5k + 2 \neq 0$

So we need to omit those k for which $k^2 - 5k + 2 = 0$, i.e. when

$$k = \frac{+5 \pm \sqrt{25 - 4(2)}}{2} = \frac{5 \pm \sqrt{17}}{2}.$$

Hence $k \in \mathbb{R} \setminus \left\{ \frac{5 \pm \sqrt{17}}{2} \right\}$.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{bmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 1 & k \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} k & k \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} k & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 1 - 2k - 2k \neq 0$$

$$\Rightarrow k \neq 1/4$$

Hence $k \in \mathbb{R} \setminus \{1/4\}$.

9] (a) $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 7 & -2 \\ 2 & 6 & -1 \end{bmatrix}$

$$\det(A) = 3 \begin{vmatrix} 7 & -2 \\ 6 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 7 \\ 2 & 6 \end{vmatrix}$$

$$= 3(-7+12) + (1+4) + (-6-14)$$

$$= 15 + 5 - 20$$

$$= 0$$

Hence Cramer's rule is not applicable.

(b)

$$-x_1 - 4x_2 + 2x_3 + x_4 = -32$$
$$2x_1 - x_2 + 7x_3 + 9x_4 = 14$$
$$-x_1 + x_2 + 3x_3 + x_4 = 11$$
$$x_1 - 2x_2 + x_3 - 4x_4 = -4$$

$$A = \begin{bmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ 1 & -2 & 1 & -4 \end{bmatrix}$$

$$\det(A) = -\begin{vmatrix} -1 & 7 & 9 \\ 1 & 3 & 1 \\ -2 & 1 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 7 & 9 \\ -1 & 3 & 1 \\ 1 & 1 & -4 \end{vmatrix}$$
$$+ 2\begin{vmatrix} 2 & -1 & 9 \\ -1 & 1 & 1 \\ 1 & -2 & -4 \end{vmatrix} - \begin{vmatrix} 2 & -1 & 7 \\ -1 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= -(13 + 14 + 63) + 4(-26 - 21 - 36)$$
$$+ 2(-4 + 3 + 9) - (14 - 4 + 7)$$
$$= -90 - 4(83) + 16 - 17$$
$$= -90 - 332 + 16 - 17$$
$$= -91 - 332$$
$$= -423 \neq 0$$

$$A_1 = \begin{bmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ 1 & -4 & 1 & -4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & -4 & -32 & 1 \\ 2 & -1 & 14 & 9 \\ -1 & 1 & 11 & 1 \\ 1 & -2 & -4 & -4 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -1 & -4 & 2 & -32 \\ 2 & -1 & 7 & 14 \\ -1 & 1 & 3 & 11 \\ 1 & -2 & 1 & -4 \end{bmatrix}$$

$$\det(A_1) = -2115$$
$$\det(A_2) = -3384$$
$$\det(A_3) = -1269$$
$$\det(A_4) = 423$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 3 \\ -1 \end{pmatrix}.$$

$\blacksquare$

Vector spaces related problems

Q1. $V = \{ \bar{u} = (u_1, u_2) : u_1, u_2 \in \mathbb{R} \}$

$\bar{u} + \bar{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1)$

$k\bar{u} = (ku_1, ku_2)$

(a) $\bar{u} = (0, 4)$, $\bar{v} = (1, -3)$, $k = 2$

$\bar{u} + \bar{v} = (0 + 1 + 1, 4 - 3 + 1)$
$= (2, 2)$

$k\bar{u} = 2 \cdot (0, 4) = (0, 8)$.

(b) $(0, 0) \neq \bar{0}$

If $(0, 0) = \bar{0}$, then

$(0, 0) + (u_1, u_2) = (u_1, u_2)$

But $(0, 0) + (u_1, u_2) = (u_1 + 1, u_2 + 1)$
$\neq (u_1, u_2)$.

Hence $(0, 0) \neq \bar{0}$.

(c) $(-1, -1) = \bar{0}$

$(-1, 1) + (u_1, u_2) = (-1 + u_1 + 1, -1 + u_2 + 1)$
$= (u_1, u_2)$

Hence $(-1, -1) = \bar{0}$.

(d) Find $-\bar{u}$.

Let $-\bar{u} = (x, y)$.

Then $\bar{u} + (-\bar{u}) = \bar{0}$ implies

$(u_1 + x + 1, u_2 + y + 1) = (-1, -1)$

$\Rightarrow u_1 + x + 1 = -1, \; u_2 + y + 1 = -1$

$\Rightarrow x = -u_1 - 2, \; y = -u_2 - 2$

Hence $-\bar{u} = (-u_1 - 2, -u_2, -2)$

(e) (i) Is $k(\bar{u} + \bar{v}) = k\bar{u} + k\bar{v}$?

$k(\bar{u} + \bar{v}) = k(u_1 + v_1 + 1, u_2 + v_2 + 1)$
$= (ku_1 + kv_1 + k, ku_2 + kv_2 + k)$

$k\bar{u} = (ku_1, ku_2), \; k\bar{v} = (kv_1, kv_2)$

Then $k\bar{u} + k\bar{v} = (ku_1 + kv_1 + 1, ku_2 + kv_2 + 1)$

Hence $k(\bar{u} + \bar{v}) \neq k\bar{u} + k\bar{v}$ in general.

(ii) Is $(k+m)\bar{u} = k\bar{u} + m\bar{u}$?

$(k+m)\bar{u} = ((k+m)u_1, (k+m)u_2)$

$k\bar{u} + m\bar{u} = (ku_1, ku_2) + (mu_1, mu_2)$
$= (ku_1 + mu_1 + 1, ku_2 + mu_2 + 1)$

Hence $(k+m)\bar{u} \neq k\bar{u} + m\bar{u}$.

Q 2] (a) $V = \{(x,0) : x \in \mathbb{R}\}$, $\underbrace{+, \cdot}_{\text{standard}}$

$+$: $(x,0) + (y,0) = (x+y, 0)$

$\cdot$: $k(x,0) = (kx, 0)$

Easy to verify that V is a subspace of $\mathbb{R}^2$ by verifying S_1, S_2, S_3, or directly.

(b) $V = \{(x,y) : x,y \in \mathbb{R},\ x \geq 0\}$

$+, \cdot$ standard operations

- Additive inverse is $(-x,-y)$, with $-x \leq 0$ (A4 does not hold)
- Also, for $k<0$, $k(x,y) = (kx, ky)$ with $kx < 0$, and hence M_0 does not hold.

Hence V is not a vector space.

(c) $V = \{(1,x) : x \in \mathbb{R}\}$, where

$+$: $(1,y) + (1,y') = (1, y+y')$

$\cdot$: $k(1,y) = (1, ky)$

Closure : ✓ Commutative : ✓

Associativity :

$((1,y)+(1,y')) + (1,y'') = (1,y+y') + (1,y'')$

$= (1,(y+y')+y'') = (1, y+(y'+y''))$

$= (1,y) + ((1,y') + (1,y''))$ ✓

Additive idty. is $(1,0)$ since

$(1,x) + (1,0) = (1, x+0) = (1,x)$ ✓

Additive inverse: $(1, -x)$ since

$$(1,x)+(1,-x) = (1, x+(-x)) = (1,0)$$ ✓

M_0: $k(1,y) = (1, ky) \in V$ ✓

M_1: $1\cdot(1,y) = (1, 1\cdot y) = (1,y)$ ✓

M_2: $(km)\cdot(1,y) = (1, (km)y) = k(1, my)$ ✓

$D1$:
$$\begin{aligned} k(\bar{u}+\bar{v}) &= k((1,x)+(1,y)) \\ &= k(1, x+y) \\ &= (1, k(x+y)) \\ &= (1, kx+ky) \\ &= (1, kx) + (1, ky) \\ &= k(1,x) + k(1,y) \\ &= k\bar{u} + k\bar{v}. \end{aligned}$$

$D2$:
$$\begin{aligned} (k+m)\bar{u} &= (k+m)(1,x) \\ &= (1, (k+m)x) \\ &= (1, kx+mx) \\ &= (1, kx) + (1, mx) \\ &= k(1,x) + m(1,x) \\ &= k\bar{u} + m\bar{u}. \end{aligned}$$

Hence V is a vector space over $\mathbb{R}$.

(d) $V = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

Standard $+$, $\cdot$

Verify that it is a vector space over $\mathbb{R}$.

Q 3] (a) $V = \{(a,0,0) : a \in \mathbb{R}\} \subseteq \mathbb{R}^3$

S_1 : $\underbrace{(0,0,0)}_{\text{additive idty of } \mathbb{R}^3} \in V$ ✓

S_2 : $(a,0,0)+(b,0,0) = (a+b,0,0) \in V$ ✓

S_3 : $k(a,0,0) = (ka,0,0) \in V$.

V is a subspace of $\mathbb{R}^3$.

(b) $V = \{(a,1,1) : a \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

S_1 : $(0,0,0) \notin V$.

Hence V is not a subspace of $\mathbb{R}^3$

(c) $V = \{(a,b,c) : b = a+c\} \subseteq \mathbb{R}^3$

S_1 : $(0,0,0) \in V$ ✓

S_2 : $(a,b,c)+(x,y,z)$

$= (a+x, \underline{b+y}, c+z) \in V$ ✓

where $b+y = (a+c)+(x+z) = (a+x)+(c+z)$.

S_3: $k(a,b,c) = (ka, kb, kc) \in V$
$ka + kc = kb$, since $b = a + c$,

V is a subspace of $\mathbb{R}^3$

(d) $V = \{(a,b,c) : b = a + c + 1\}$

S_1 : $(0,0,0) \notin V$

Not a subspace

(e) $V = \{(a,b,0)\} \subseteq \mathbb{R}^3$.

S_1 ✓ S_2 ✓ S_3 ✓

is a subspace.

Q4] (a) $V = \{A_{n\times n} : A \text{ is diagonal}\} \subseteq M_{nn}$

$$S_1: \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{n\times n} \in V \quad ✓$$

S_2 $A, B \in V \implies A + B \in V$

$$S_3: k\begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ka_{11} & & 0 \\ & \ddots & \\ 0 & & ka_{nn} \end{bmatrix} \in V.$$

Subspace

(b) $V = \{ A_{n\times n} : \det(A) = 0 \}$

$A, B \in V \Rightarrow \det(A) = \det(B) = 0.$

Then $\underbrace{\det(A+B)}_{} \neq \underbrace{\det(A) + \det(B)}_{=0}$ in general.

Not a subspace (S_2 is not true in general)

(c) $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$

$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$

$V = \{ A_{n\times n} : tr(A) = 0 \}$

S_1 : $\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix} \in V$

S_2 : $A, B \in V \Rightarrow A+B \in V$

S_3 : $kA = k \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

$= \begin{bmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & & \\ k_{n1} & \cdots & k_{nn} \end{bmatrix}$

$tr(kA) = k\,tr(A) = 0 \quad (\because A \in V)$

$\Rightarrow kA \in V.$

V is a subspace of M_{nn}

(d) $V = \{ A_{n\times n} : A \text{ symmetric})$

S_1 $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in V$:

S_2: $A = A^T, \quad B = B^T$

$(A+B)^T = A^T + B^T = A+B$

$\Rightarrow A+B \in V.$

S_3: $(kA)^T = kA^T = kA$

$\Rightarrow kA \in V.$

is a subspace

(e) $V = \{ A_{n\times n} : A^T = -A \}$

S_1: $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

S_2: $A^T = -A, \quad B^T = -B$

$(A+B)^T = A^T + B^T = -A - B = -(A+B)$

$\Rightarrow A+B \in V.$

S_3: $(kA)^T = kA^T = -kA \Rightarrow kA \in V.$

is a subspace.