MA 103 Quiz 1 solutions (Total 40 points)

1.
   a. (ii)
   b. (iii)
   c. (iii)
   d. (i) (True)
   e. (i) (True)

2.

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

First $\det(A) = 2(-1-1) - 3(-3) + 5(3)$
$= -4 + 9 + 15 = 20 \neq 0$

So Cramer's rule can be applied.

The matrices A_i, $1 \leq i \leq 3$, are obtained by replacing the i^{th} column of A by $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ &

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad 1 \leq i \leq 3.$$

$$\det(A_1) = \begin{vmatrix} 2 & 3 & 5 \\ 1 & -1 & 1 \\ 5 & 1 & 1 \end{vmatrix} = 2(-1-1) - 3(1-5) + 5(1+5)$$
$$= -4 + 12 + 30$$
$$= 38$$

$$\Rightarrow \boxed{x_1 = \frac{38}{20} = \frac{19}{10}}$$

$$\det(A_2) = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 1 & 1 \\ 3 & 5 & 1 \end{bmatrix} \begin{aligned} &= 2(1-5) - 2(-3) + 5(-3) \\ &= -8+6-15 \\ &= -17 \end{aligned}$$

$$\Rightarrow \boxed{x_2 = \frac{-17}{20}}$$

$$\det(A_3) = \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \begin{aligned} &= 2(-5-1) - 3(-3) + 2(3) \\ &= -12+9+6 \\ &= 3 \end{aligned}$$

$$\Rightarrow \boxed{x_3 = \frac{3}{20}}$$

3

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Augment the given matrix with the identity matrix :

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_1 \longleftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 0 & 1 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 0 & 1 \\ 0 & 3 & -10 & 0 & 1 & -2 \\ 0 & 1 & -13 & 1 & 0 & -3 \end{array}\right]$$

$$\Big\downarrow \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 0 & 1 \\ 0 & 1 & -13 & 1 & 0 & -3 \\ 0 & 3 & -10 & 0 & 1 & -2 \end{array}\right]$$

$$\Big\downarrow \quad R_3 \rightarrow R_3 - 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 0 & 1 \\ 0 & 1 & -13 & 1 & 0 & -3 \\ 0 & 0 & 29 & -3 & 1 & 7 \end{array}\right]$$

$$\Big\downarrow \quad R_3 \rightarrow \frac{R_3}{29}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 0 & 1 \\ 0 & 1 & -13 & 1 & 0 & -3 \\ 0 & 0 & 1 & \frac{-3}{29} & \frac{1}{29} & \frac{7}{29} \end{array}\right]$$

$$\Big\downarrow \quad \begin{array}{l} R_1 \rightarrow R_1 - 5R_3 \\ R_2 \rightarrow R_2 + 13R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{15}{29} & \frac{-5}{29} & \frac{-6}{29} \\ 0 & 1 & 0 & \frac{-10}{29} & \frac{13}{29} & \frac{4}{29} \\ 0 & 0 & 1 & \frac{-3}{29} & \frac{1}{29} & \frac{7}{29} \end{array}\right]$$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{29}\begin{bmatrix} 15 & -5 & -6 \\ -10 & 13 & 4 \\ -3 & 1 & 7 \end{bmatrix}}$$

4. First, suppose $Ax = 0$ has only the trivial solution,
Then by a theorem done in class, A is invertible, that is, A^{-1} exists.
Note that A is an $n \times n$ matrix (since we have n eqns. in n unknowns.)

Also Q is an invertible $n \times n$ matrix, that is, Q^{-1} exists.

Then consider $(QA)x = 0$ & pre-multiply both sides by $A^{-1}Q^{-1}$. This implies

$$A^{-1}(Q^{-1}Q)(Ax) = 0$$
$$\Rightarrow (A^{-1}A)x = 0$$
$$\Rightarrow x = 0$$

$\Rightarrow (QA)x = 0$ has only the trivial solution.

Now suppose

$(QA)x = 0$ has only the trivial solution

$\Rightarrow QA$ is invertible. Let $B = (QA)^{-1}$.

If we show A is invertible, then $Ax = 0$ will have only the trivial solution, and we will be done.

Q is invertible, so Q^{-1} exists; also $QAB = I$

Pre-multiply both sides by Q^{-1} so that

$$Q^{-1}(QAB) = Q^{-1} \cdot I$$

$$\Rightarrow (Q^{-1}Q)AB = Q^{-1}$$

$$\Rightarrow AB = Q^{-1}$$

Now post-multiply both sides by Q so that

$$ABQ = Q^{-1}Q$$

$$\Rightarrow A(BQ) = I.$$

Similarly

$$(BQ)A = (QA)^{-1}QA$$

$$= (QA)^{-1}(QA) = I$$

By defn., BQ is A^{-1}, that is, A is invertible

$\Rightarrow Ax = 0$ has only the trivial solution,