Lecture 4 — Finding inverses

THEOREM 1.4.3 *If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n.*

Proof: Suppose the RREF form of A is

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

Proof: The last row is either a zero row or a non-zero row. If it's a non-zero row, then none of the rows before this row can be a zero row (otherwise, the RREF defn. is violated).

But then every row of R has a leading 1 and because it's an RREF form, the leading 1 of a row below a previous one is to the right of the leading 1 of that row.

This implies all leading 1's are on the main diagonal.

Since R is RREF, the non-diagonal entries must be zero.

$\Rightarrow R = I_n$.

Let A be an $m \times n$ matrix.

$$A_{m\times n} \xrightarrow{\text{ERO}} B_{m\times n}$$

$\left(\underset{m\times n}{B} = E \underset{m\times n}{A} \text{ for some "elementary" matrix } \underset{m\times m}{E} \right)$

ERO	Elementary matrix (= matrix obtained by applying some ERO to I)
① $R_i \leftrightarrow R_j$	$E = E_1$ is obtained by applying $R_i \leftrightarrow R_j$ on $I_{m\times m}$.
② $R_i \to cR_i \ (c \neq 0)$	$E = E_2$ is obtained by applying $R_i \to cR_i$ to $I_{m\times m}$.
③ $R_i \to R_i + cR_j$	$E = E_3$ is obtained by applying $R_i \to R_i + cR_j$ to $I_{m\times m}$

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}_{3\times 4}$$

and consider the elementary matrix

$R_3 \to R_3 + 3R_1$ in $I_{3\times 3}$ gives $\rightarrow$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}_{3\times 3}$$

which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is precisely the matrix that results when we add 3 times the first row of A to the third row. ◀

Thm. 1.5.1 :

If we do an ERO on $I_{m\times m}$, call the corresponding elem. mat. E, and then pre-multiply $A_{m\times n}$ by E, then the resulting matrix $(EA)_{m\times n}$ has the same effect, that is, it is the matrix obtained by performing the same ERO on A.

Reverse operations	Elementary matrices
① $R_i \leftrightarrow R_j$	E_1^{-1}
② $R_i \to \frac{1}{c} R_i$ $(c \neq 0)$	E_2^{-1}
③ $R_i \to R_i - cR_j$	E_3^{-1}

Thm. 1.5.2 Every elementary matrix is invertible & the inverse is also elementary

Proof: Suppose E is the matrix obtained by doing an ERO on I. Now if we do the reverse operation corresponding to the previous ERO on E and say the corresponding elementary matrix is E_0, then $E_0 E$ must be I (because the reverse ERO cancels the effect of previous ERO.
Similarly, $EE_0 = I \Rightarrow E$ is invertible.

If we have a system $\underset{n\times n}{A}\ \underset{n\times 1}{x} = \underset{n\times 1}{b}$

& A is invertible, then

$$x = A^{-1} b \qquad \text{(left multiplication by } A^{-1})$$

Finding inverses

Find inverse (if it exists) of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}.$$

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Ans. Write the matrix $(A \mid I)$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

② Apply EROs to $(A|I)$ & try to get RREF form of A or try to get $(A|I)$ to be $(I|B)$ for some B.

③ Then $B = A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

← We added −2 times the first row to the second and −1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

← We added 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We multiplied the third row by −1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We added 3 times the third row to the second and −3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We added −2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$ ◀

Another example:

Find the inverse (if it exists) of

$$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

$R_2 \to \frac{-1}{8} R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & 9/8 & 1/4 & -1/8 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 8R_2$, $R_1 \to R_1 - 6R_2$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 9/8 & 1/4 & -1/8 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Since there is a zero-row, we conclude that A is not invertible.

THEOREM 1.5.3 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.

(a) *A is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row echelon form of A is I_n.*

(d) *A is expressible as a product of elementary matrices.*

Proof: (a) → (b).

$Ax = 0 \Rightarrow A^{-1}(Ax) = A^{-1} \cdot 0$

$\Rightarrow (A^{-1}A)x = 0 \Rightarrow Ix = 0$

$\Rightarrow x = 0$.

(b) → (c)

Apply Gauss-Jordan to get →

$x_1 = 0$

$x_2 = 0$

$\vdots$

$x_n = 0$

The coeff. matrix in the resulting system is I_n.

(c) → (d)

(d) → (a) $A = E_1 E_2 \cdots E_k$.

$A^{-1} = (E_1 E_2 \cdots E_k)^{-1} = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1}$

Prove that

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof: We want to show

$$(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$$

By matrix associativity,

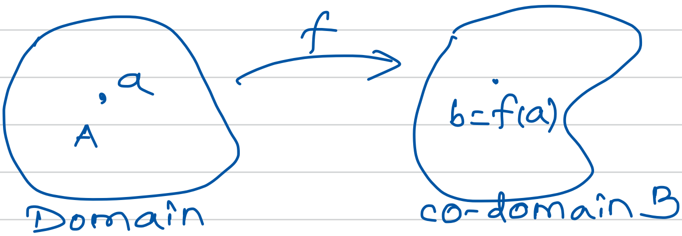
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1}$$
$$= AA^{-1}$$
$$= I.$$

Similarly, $(B^{-1}A^{-1})(AB) = I$.

$\boxtimes$

MATRIX TRANSFORMATIONS

Function / transformation



We are concerned with

$$A = \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}.$$

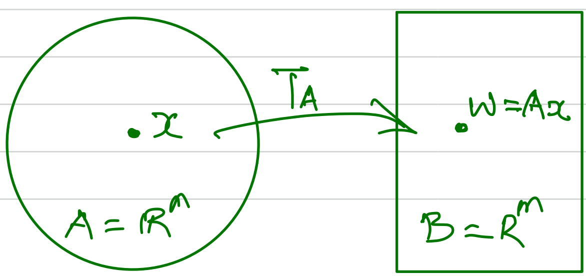
$$B = \mathbb{R}^m$$

Recall a standard linear system:

$$\begin{aligned}
w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\ \ \vdots \qquad\quad \vdots \qquad\qquad\qquad\quad \vdots \\
w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}$$

In other words

$$\underbrace{\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}}_{w \in \mathbb{R}^m} = A \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{x \in \mathbb{R}^n}$$



T_A is called a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

because we multiply the elements in $\mathbb{R}^n$ by a matrix A.

$T_A : \mathbb{R}^n \to \mathbb{R}^m$

$$\underset{}{x} \longrightarrow \underset{m\times 1}{w} = \underset{m\times n}{A}\,\underset{n\times 1}{x} \implies T_A(x) = Ax$$

$\Rightarrow$ A must be of size $m \times n$.

Eg. 1 Zero transformation

$T_0(x)$: matrix transformation corresponding to the zero matrix.

$$T_0(x) = \underset{m\times n}{0}\,\underset{n\times 1}{x} = \underset{m\times 1}{0} \qquad \forall x \in \mathbb{R}^n$$

2 Identity transformation

$$T_I(x) = Ix = x \quad \forall x \in \mathbb{R}^n.$$

$$T_I(x) = \underset{n\times n}{I}\,\underset{n\times 1}{x} = \underset{n\times 1}{x} \quad \forall x \in \mathbb{R}^n.$$

Note that $T_I : \mathbb{R}^n \to \mathbb{R}^n$

Properties of matrix transformation

Let A be an $m \times n$ matrix so that $T_A : \mathbb{R}^n \to \mathbb{R}^m$
Then for any $u, v \in \mathbb{R}^n$ and $k \in \mathbb{R}$,

① $T_A(u+v) = T_A(u) + T_A(v)$

Since
$A(u+v) = Au + Av$, the result follows.

② $T_A(ku) = k\,T_A(u)$

$A(ku) = k(Au)$

③ $T_A(0) = 0$ $\quad$ 1^{st} proof: Let $u = v = 0$ in ①

$T_A(0) = T_A(0) + T_A(0) \Rightarrow T_A(0) = 0$

2^{nd}: Let $u = -v$ in ①. Then $T_A(0) = T_A(-v) + T_A(v)$

$= -T_A(v) + T_A(v)$

$= 0$.

④ $T_A(u-v) = T_A(u) - T_A(v)$.

Use ① & ②.

Are the maps below matrix transformations?

① $T : \mathbb{R}^2 \to \mathbb{R}^2$

$x \to 2x$

Matrix transformation

② $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1^2 \\ x_2 \\ x_3 \end{pmatrix}$$

Not a matrix transformation

Section 1.8

$$\mathbb{R}^n = \left\{ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \ \forall i \right\}$$

special members of $\mathbb{R}^n$:

$$\vec{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e_n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

They are called standard basis vectors for $\mathbb{R}^n$.

Any $\vec{x} \in \mathbb{R}^n$ can be uniquely written as a linear combination of $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$, that is,

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ since } \vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$= x_1 \bar{e}_1 + x_2 \bar{e}_2 + \cdots + x_n \bar{e}_n .$$

<u>Defn.</u> A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be a linear transformation if $\forall\, u, v \in \mathbb{R}^n$ & $k \in \mathbb{R}$, we have

(1) $T(u+v) = T(u) + T(v)$ (<u>Additivity</u>)

(2) $T(ku) = kT(u)$ (<u>Homogeneity</u>)