MA 103 – End-sem (2024) solutions

1. (a) (ii)
   (b) (ii)
   (c) (i)
   (d) (i)
   (e) (ii)

2. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{n^{1/n}}{n^2}$.

For $n \geq 2$, $\frac{1}{n} \leq \frac{1}{2} \Rightarrow n^{1/n} \leq n^{1/2}$

$$\Rightarrow \frac{n^{1/n}}{n^2} \leq \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}.$$

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges (by p-test with $p = \frac{3}{2} > 1$), by comparison test, $\displaystyle\sum_{n=2}^{\infty} \frac{n^{1/n}}{n^2}$ converges too, and thus $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2}$ converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^2}$

For a sufficiently large N, $\log n \leq c\sqrt{n}$ for $\forall n \geq N$ & some constant $c > 0$. Then

$$\frac{\log n}{n^2} \leq \frac{c\sqrt{n}}{n^2} = \frac{c}{n^{3/2}} \quad \text{for } n \geq N$$

$\Rightarrow$ As in (a), $\displaystyle\sum_{n=N}^{\infty} \frac{1}{n^{3/2}}$ (and hence $\displaystyle c\sum_{n=N}^{\infty} \frac{1}{n^{3/2}}$ converges),

& thus by comparison test, $\sum_{n=N}^{\infty} \frac{\log n}{n^2}$ (and hence $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$) converges.

Students may do this problem by integral test by comparing the series with $\int_1^{\infty} \frac{\log x}{x^2}\,dx$.

As long as they prove the hypotheses (f cont; positive & decreasing on $[1, \infty)$) & do the calculations correctly, you can give them full points.

(c) $\sum_{n=1}^{\infty} \frac{1}{1+\log^2 n}$; $a_n = \frac{1}{1+\log^2 n}$

Let $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{1+\log^2 n}\right)}{\left(1/n\right)} = \lim_{n\to\infty} \frac{n}{1+\log^2 n}.$$

Now $\lim_{x\to\infty} \frac{x}{1+\log^2 x}$ ($\frac{\infty}{\infty}$ form)

$= \lim_{x\to\infty} \frac{1}{\frac{2\log x}{x}}$ (L'Hospital's rule)

$= \lim_{x\to\infty} \frac{x}{2\log x}$ ($\frac{\infty}{\infty}$ form)

$= \frac{1}{2}\lim_{x\to\infty} \frac{1}{1/x}$ (L'Hospital's rule)

$= \frac{1}{2}\lim_{x\to\infty} x$

$= \infty.$

By limit comparison test, if $\sum b_n$ diverges, so does $\sum a_n$.

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\log^2 n}$ diverges.

③ Consider $\displaystyle\sum_{n=1}^{\infty} |q|^{n^2}$

We use root test with $a_n = |q|^{n^2}$

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(|q|^{n^2}\right)^{1/n}$$

$$= \lim_{n\to\infty} |q|^n$$

$$= \begin{cases} 0, & \text{if } |q| < 1 \\ \infty, & \text{if } |q| > 1 \\ 1, & \text{if } |q| = 1. \end{cases}$$

Thus by root test, $\displaystyle\sum_{n=1}^{\infty} q^{n^2}$ converges absolutely for $|q|<1$, that is, $-1<q<1$, and diverges for $|q|>1$, that is, for $q>1$ & $q<-1$.

Since abs. conv. $\Rightarrow$ conv., the series $\displaystyle\sum_{n=1}^{\infty} q^{n^2}$ converges for $-1<q<1$, & diverges for $q>1$ & $q<-1$.

Moreover, we know that $\displaystyle\sum_{n=1}^{\infty} 1$ as well as $\displaystyle\sum_{n=1}^{\infty} (-1)^{n^2} = \sum_{n=1}^{\infty} (-1)^n$ diverge, so the series diverges for $q=1$ & -1 as well.

(4) The equation of the larger circle with radius a is $x^2+y^2=a^2$. So the eqn. of the quarter circle in the first quadrant is $y=\sqrt{a^2-x^2}$.

The equation of circle with radius $a/2$ is
$$\left(x-\frac{a}{2}\right)^2+(y-0)^2=\left(\frac{a}{2}\right)^2$$
$$\Rightarrow x^2-ax+y^2=0$$
$\Rightarrow$ Eqn. of the semi-circle is
$$y=\sqrt{ax-x^2}.$$

Thus area of the shaded region is
$$\int_0^a\left(\sqrt{a^2-x^2}-\sqrt{ax-x^2}\right)dx$$

Now $\displaystyle\int_0^a\sqrt{a^2-x^2}\,dx=\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$
$$=\frac{a^2}{2}\sin^{-1}(1)-0=\frac{\pi a^2}{4}.$$

Now $\sqrt{ax-x^2}=\sqrt{\frac{a^2}{4}-\left(\frac{a}{2}-x\right)^2}$
$$\Rightarrow \int_0^a\sqrt{\frac{a^2}{4}-\left(\frac{a}{2}-x\right)^2}\,dx$$

Use change of variable $\frac{a}{2}-x=t$
$\Rightarrow x=a/2-t$
$dx=-dt$
when $x=0$, $t=a/2$ & when $x=a$, $t=-\frac{a}{2}$

$$\Rightarrow \int_0^a \sqrt{\frac{a^2}{4} - \left(\frac{a}{2} - x\right)^2}\, dx$$

$$= -\int_{a/2}^{-a/2} \sqrt{\left(\frac{a}{2}\right)^2 - t^2}\, dt = \int_{-a/2}^{a/2} \sqrt{\left(\frac{a}{2}\right)^2 - t^2}\, dt$$

$$= \left[\frac{t}{2}\sqrt{\left(\frac{a}{2}\right)^2 - t^2} + \frac{a^2}{8}\sin^{-1}\left(\frac{2t}{a}\right)\right]_{-a/2}^{a/2}$$

$$= \frac{\pi a^2}{16} - \left(-\frac{\pi a^2}{16}\right) = \frac{\pi a^2}{8}.$$

$\Rightarrow$ Area of the shaded region

$$= \int_0^a \left(\sqrt{a^2 - x^2} - \sqrt{ax - x^2}\right) dx$$

$$= \frac{\pi a^2}{4} - \frac{\pi a^2}{8} = \frac{\pi a^2}{8}.$$

5 a

$f(x) = \cos x$	$f(0) = \cos(0) = 1$
$f'(x) = -\sin x$	$f'(0) = -\sin(0) = 0$
$f''(x) = -\cos x$	$f''(0) = -\cos(0) = -1$
$f'''(x) = \sin x$	$f'''(0) = \sin(0) = 0$
$f^{(iv)}(x) = \cos x$	$f^{(iv)}(0) = \cos(0) = 1$

Hence $f^{(4k)}(0) = 1$ & $f^{(4k+2)}(0) = -1$.

In other words, $f^{(2k)}(0) = (-1)^k$

Thus by Taylor's formula, we have for all $x \in \mathbb{R}$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \frac{(-1)^k x^{2k}}{(2k)!} + R_{2k}(x),$$

$$= \sum_{n=0}^{2k} \frac{(-1)^n x^{2n}}{(2n)!} + R_{2k}(x)$$

where $R_{2k}(x) = \frac{f^{(2k+1)}(c)}{(2k+1)!} x^{2k+1}$ for $0 < c < x$.

Now since $|f^{(n)}(x)| \leq 1$ $\forall n \in \mathbb{N}$ since $\sin x$ as well as $\cos x$ lie between -1 & 1.

$$\Rightarrow |R_{2k}(x)| \leq \frac{|x|^{2k+1}}{(2k+1)!},$$

But we know that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ $\forall x \in \mathbb{R}$.

$$\Rightarrow \lim_{k\to\infty} |R_{2k}(x)| = 0$$

← the students may skip this step & directly write the next. It's okay if they do that.

$$\Rightarrow \lim_{k\to\infty} R_{2k}(x) = 0 \quad \forall x \in \mathbb{R}.$$

Hence for every $x \in \mathbb{R}$

$$\cos x = \lim_{k\to\infty} \left(\sum_{n=0}^{2k} \frac{(-1)^n x^{2n}}{(2n)!} + R_{2k}(x) \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

(b) $\cos^2 x = \frac{1 + \cos(2x)}{2}$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \quad (\text{by part ⓐ})$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \cdot x^{2n}}{(2n)!}.$$

⑥ ⓐ $\sqrt{\tan x}$ is a continuous function on $[-\pi/4, \pi/4]$, say, & is also differentiable on an open interval around any point $t \in (-\pi/4, \pi/4)$, in particular, at $t=0$.

$\Rightarrow$ By the fundamental theorem of calculus,

$$\frac{d}{dx} \int_0^x \sqrt{\tan t}\, dt = \sqrt{\tan x}.$$

Since $\dfrac{\int_0^x \sqrt{\tan t}\, dt}{x^{3/2}}$ is of $\frac{0}{0}$ form, by L'Hospital's rule,

$$\lim_{x \to 0} \frac{\int_0^x \sqrt{\tan t}\, dt}{x^{3/2}} = \lim_{x \to 0} \frac{\sqrt{\tan x}}{\frac{3}{2} x^{1/2}} = \frac{2}{3} \lim_{x \to \infty} \sqrt{\frac{\tan x}{x}}$$

$$= \frac{2}{3} \sqrt{\lim_{x \to 0} \frac{\tan x}{x}} \quad (\text{by continuity of the sq. root fn.})$$

$$= \frac{2}{3} \sqrt{\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x}}$$

$$= \frac{2}{3} \sqrt{1 \cdot \frac{1}{\cos(0)}} = \frac{2}{3} \sqrt{1} = \frac{2}{3}.$$

(b) $$\lim_{x\to 0} \frac{2^x - 2^{\sin x}}{x(1-\cos x)}$$

$$= \lim_{x\to 0} \left(\frac{2^x - 2^{\sin x}}{x^3} \cdot \frac{x^2}{1-\cos x} \right)$$

$$= L \cdot \lim_{x\to 0} \frac{x^2}{1-\cos x}, \text{ where}$$

$$L = \lim_{x\to 0} \frac{2^x - 2^{\sin x}}{x^3}.$$

Clearly, $$\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{x^2}{2\sin^2(x/2)}$$

$$= 2\lim_{x\to 0} \frac{1}{\left(\frac{\sin(x/2)}{x/2}\right)^2}.$$

$$= 2 \quad \text{———} \quad (1)$$

$\left(\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \right.$ & sq. fn. is cont.

(Alternate method using power series:

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} = \lim_{x\to 0} \frac{x^2}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!} - \cdots\right)}$$

$$= \lim_{x\to 0} \frac{x^2}{\frac{x^2}{2} - \frac{x^4}{4!} + \cdots}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2\left(\frac{1}{2} - \frac{x^2}{4!} + \cdots\right)}$$

$$= \frac{1}{\frac{1}{2} - 0} = 2.)$$

$$L = \lim_{x \to 0} \frac{2^x - 2^{\sin x}}{x^3}$$

$$= \lim_{x \to 0} \frac{e^{x \log 2} - e^{\sin x \log 2}}{x^3}$$

$$= \lim_{x \to 0} \frac{1}{x^3} \left[\left(1 + x \log 2 + \frac{(x \log 2)^2}{2!} + \cdots \right) - \left(1 + \sin x \log 2 + \frac{(\sin x \log 2)^2}{2!} + \cdots \right) \right]$$

$$= \lim_{x \to 0} \frac{1}{x^3} \sum_{n=1}^{\infty} \left(x^n - \sin^n(x) \right) \frac{\log^n(2)}{n!} \quad \text{—— } (2)$$

Now consider $\lim_{x \to 0} \frac{x^n - \sin^n(x)}{x^3}$ for $n \geq 1$

$$= \lim_{x \to 0} \frac{x^n - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)^n}{x^3}$$

$$= \lim_{x \to 0} \frac{x^n - \left(a^n + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \cdots + n a b^{n-1} + b^n \right)}{x^3},$$

where $a = x$, $b = -\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

by binomial theorem

$$= \lim_{x \to 0} \frac{x^n - \left\{ x^n + n x^{n-1} \left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) + \cdots + \left(-\frac{x^3}{3} + \frac{x^5}{5!} - \cdots \right)^n \right\}}{x^3}$$

$$= \frac{+\frac{n}{3!}x^{n+2} + \text{constant times higher powers of } x}{x^3}$$

$$= \begin{cases} \frac{1}{6}, & \text{if } n=1 \\ 0, & \text{else} \end{cases}$$

$\Rightarrow L = \frac{1}{6}\log 2$ (from 2) —— 3

From 1 & 3,

the reqd. limit $= \frac{2}{6}\log 2 = \frac{1}{3}\log 2$.

ANOTHER METHOD:

Apply L'Hospital's rule 3 times.