$\boxed{1}$ Tutorial − 1

ⓐ $\left[\begin{array}{cccc|c} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{array}\right]$

$$0x_1 + 3x_2 - x_3 - x_4 = -1$$
$$5x_1 + 2x_2 - 0x_3 - 3x_4 = -6$$

ⓑ $\left[\begin{array}{cccc|c} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$

$$3x_1 + x_3 - 4x_4 = 3$$
$$-4x_1 + 4x_3 + x_4 = -3$$
$$-x_1 + 3x_2 - 2x_4 = -9$$
$$-x_4 = -2$$

$\boxed{2}$ ⓐ

$$3x_1 - 2x_2 = -1$$
$$4x_1 + 5x_2 = 3$$
$$7x_1 + 3x_2 = 2$$

$\left[\begin{array}{cc|c} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{array}\right]$

$\boxed{3}$

$$2x - 3y = a$$
$$4x - 6y = b$$

If $2a = b$, then we have 2 coincident lines, hence infinitely many solutions, otherwise no solutions.

(4) $4x - 2y = 1$

$\boxed{y = t}$ $\quad \boxed{x = \frac{2y+1}{4} = \frac{2t+1}{4}}$

Solution set : $\left\{ \left(\frac{2t+1}{4}, t\right) : \text{where } t \in \mathbb{R} \right\}$.

(b) $2x - y + 2z = -4$

$6x - 3y + 6z = -12$

$-4x + 2y - 4z = 8$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & -4 \\ 6 & -3 & 6 & -12 \\ -4 & 2 & -4 & 8 \end{array}\right]$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 + 2R_1$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2x - y + 2z = -4$

$x = r, \quad z = t$

$y = 2x + 2z + 4 = 2r + 2t + 4$

Solution set $= \{ (r, 2r+2t+4, t\}, r, t \in \mathbb{R} \}$

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(a) $\left[\begin{array}{ccc|c} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{array}\right]$

$R_1 \to R_1/2$

(b) $\left[\begin{array}{ccc|c} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{array}\right]$

$R_1 \to R_1 + R_3$

or $R_1 \to R_1 - 2R_2$

but NOT $R_1 \to \underbrace{R_1 + 2R_2 + 2R_3}_{\text{since}}$

not a single ERO.

6 $\left[\begin{array}{cc|c} 1 & k & -4 \\ 4 & 8 & 2 \end{array}\right]$

$R_2 \to R_2 - 4R_1$

$\left[\begin{array}{cc|c} 1 & k & -4 \\ 0 & 8-4k & 18 \end{array}\right]$

$R_2 \to \frac{1}{8-4k} R_2$ $(k \neq 2)$

$\left[\begin{array}{cc|c} 1 & k & -4 \\ 0 & 1 & \frac{18}{8-4k} \end{array}\right]$

$R_1 \to R_1 - kR_2$

$\left[\begin{array}{cc|c} 1 & 0 & -4 - \frac{18k}{8-4k} \\ 0 & 1 & \frac{18}{8-4k} \end{array}\right]$

$\Rightarrow$ unique sol^n, hence consistent for $k \neq 2$.

- inconsistent for $k = 2$

(b) $\left[\begin{array}{cc|c} 1 & k & -1 \\ 4 & 8 & -4 \end{array}\right]$

$R_2 \to R_2 - 4R_1$

$\left[\begin{array}{cc|c} 1 & k & -1 \\ 0 & 8-4k & 0 \end{array}\right]$

$R_2 \to \frac{1}{8-4k} R_2$, $k \neq 2$

$\left[\begin{array}{cc|c} 1 & k & -1 \\ 0 & 1 & 0 \end{array}\right]$

$R_1 \to R_1 - kR_2$

$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right] \implies \begin{array}{l} x_1 = -1 \\ x_2 = 0 \end{array}$

When $k \neq 2$: unique sol^n
when $k = 2$: infinitely many sol^{n}s

$\implies$ always consistent.

(7) (x_1, y_1) lies on $y = ax^2 + bx + c$

$\implies x_1^2 a + x_1 b + c = y_1$

Similarly, $x_2^2 a + x_2 b + c = y_2$
$x_3^2 a + x_3 b + c = y_3$

$\left[\begin{array}{ccc|c} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{array}\right]$

⑧ $y = ax^2 + bx + c$

pass thro' pts. $(1,1), (2,4)$ and $(-1,1)$.

$$\begin{aligned} a+b+c &= 1 \\ 4a+2b+c &= 4 \\ a-b+c &= 1 \end{aligned}$$

$\xRightarrow{\text{Aug. matrix}}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & 1 \end{array}\right]$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 4 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

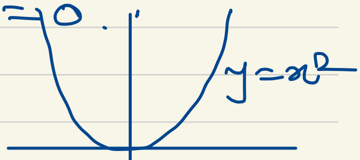
$R_2 \to R_2 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

$\Rightarrow -2b = 0 \Rightarrow b = 0$

Also, $-2b - 3c = 0$

$\Rightarrow c = 0$.



Hence $a + 0 + 0 = 1$

$\Rightarrow a = 1$

$\Rightarrow y = x^2$ (unique value for a, b, c)

⑨ 7 units of fat

9 units of proteins

16 units of carbs

<u>Food 1</u>: every ounce contains

2 F 2 P 4 C

Food 2: 3 F, 1 P, 2 C

<u>Food 3</u>: 1 F, 3 P, 5 C

x, y, z denote the number of ounces of 1^{st}, 2^{nd} & 3^{rd} foods.

$$\begin{cases} 2x + 3y + z = 7 \\ 2x + y + 3z = 9 \\ 4x + 2y + 5z = 16 \end{cases}$$

10 a $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ All are RREF & hence REF as well

$\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$ → REF but not RREF

Tutorial 2

①
$$-2b+3c=1$$
$$3a+6b-3c=-2$$
$$6a+6b+3c=5$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$R_1 \to R_1/3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$R_3 \to R_3 - 6R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$R_2 \to \frac{R_2}{-2}$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$R_3 \to R_3 + 6R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

$R_3 \to R_3/6$

$$\left.\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 1 \end{array}\right]\right\}\text{REF form of the aug. matrix.}$$

$$\Rightarrow \left.\begin{array}{l} 0 = 1 \\ x_2 - \frac{3x_3}{2} = -1/2 \\ x_1 + 2x_2 - x_3 = -2/3 \end{array}\right\} \text{inconsistent system; no sol}^n\text{.}$$

2 b REF form of aug. matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + \frac{1}{2}R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 + \frac{2}{3} R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \leftarrow \text{RREF form.}$$

G-J: $0 = 1$

$x_2 - \frac{3}{2}x_3 = 0, \quad x_1 + 2x_3 = 0.$

③ $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \leftarrow \text{REF}$$

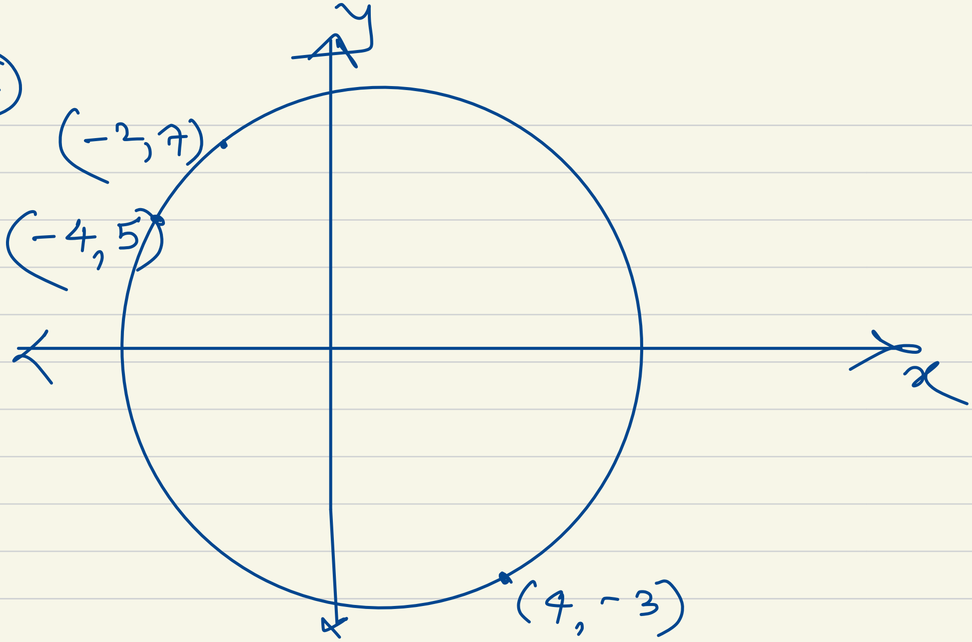
$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

More generally, $R_1 \to R_1 + kR_2$, ($k \in \mathbb{R}$, $k \neq 0$) gives

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

4



$$ax^2+ay^2+bx+cy+d=0$$

We have $4a+49a-2b+7c+d=0$

$\Rightarrow 53a-2b+7c+d=0$ —— (1)

Similarly,

$41a-4b+5c+d=0$ —— (2)

$25a+4b-3c+d=0$ —— (3)

Aug. matrix:

$$\left[\begin{array}{cccc|c} 53 & -2 & 7 & 1 & 0 \\ 41 & -4 & 5 & 1 & 0 \\ 25 & 4 & -3 & 1 & 0 \end{array}\right]$$

This is a homogeneous sys. of 3 lin. eqns. in 4 unknowns.

$\Rightarrow$ infinitely many solutions.

One can check that
$b = -2a$, $c = -4a$, $d = -29a$, where $a \in \mathbb{R}$.

Note: Though we get infinitely many solutions for (a, b, c, d), they all give us the same circle. (since 3 non-collinear points determine a unique circle).

$\Rightarrow a(x^2+y^2) - 2ax - 4ay - 29a = 0$
$\Rightarrow x^2+y^2 - 2x - 4y - 29 = 0 \quad (\because a \neq 0)$

5

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

Let $C = AB$

(a) Using rule for colums,

j^{th} column of $C = A \cdot (j^{th}$ column of $B)$

Let $C = \begin{pmatrix} | & | & | \\ C_1 & C_2 & C_3 \\ | & | & | \end{pmatrix}$.

Then $C_1 = A \cdot (1^{st}$ column of $B)$

$$= \underbrace{\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix}}_{3\times 1}$$

$$= \begin{bmatrix} 3(6)+(-2)(0)+(7)(7) \\ 6(6)+5(0)+4(7) \\ 0(6)+4(0)+9(7) \end{bmatrix}$$

$$= 6\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 7\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}.$$

Similarly write C_2 & C_3 in terms of linear combinations of the columns of A.

(b) Use rule for rows, i.e.

i^{th} row of $C = (i^{th}$ row of $A) \cdot B$

Let $C = \begin{pmatrix} \text{---} R_1 \text{---} \\ \text{---} R_2 \text{---} \\ \text{---} R_3 \text{---} \end{pmatrix}$

Then $R_1 = \begin{bmatrix} 3 & -2 & 7 \end{bmatrix}_{1\times 3} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}_{3\times 3}$

$$= \begin{bmatrix} 3(6)+(-2)(0)+7(7) & 3(-2)+(-2)(1)+7(7) & 3(4)+(-2)(3)+7(5) \end{bmatrix}$$

$$= 3\begin{bmatrix} 6 & -2 & 4 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 1 & 3 \end{bmatrix} + 7\begin{bmatrix} 7 & 7 & 5 \end{bmatrix}.$$

(c) $A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & A_3 \\ | & | & | \end{bmatrix}$, $B = \begin{bmatrix} — & B_1 & — \\ — & B_2 & — \\ — & B_3 & — \end{bmatrix}$.

$$A \xrightarrow{A_i \leftrightarrow A_j} A' \text{ (say)} \qquad B \xrightarrow{B_i \leftrightarrow B_j} B' \text{ (say)}.$$

Then we want to show $AB = A'B'$.

By column-sum rule of lec. 3,

$$AB = A_1B_1 + A_2B_2 + A_3B_3$$

Now note that RHS is unchanged if we perform the operations $A_i \leftrightarrow A_j$, $B_i \leftrightarrow B_j$.

■

(6) B is said to be a square root of a matrix A if $BB = A$.

(a) Find 2 sq. roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$\Rightarrow \quad a^2+bc = 2 = bc+d^2$

$b(a+d) = c(a+d) = 2$

$\Rightarrow a^2 = d^2$ & $b = c$

Since $a+d \neq 0$, $a \neq -d$

$\Rightarrow \underline{a = d \ \& \ b = c}$.

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2 & ab+ba \\ ab+ba & a^2+b^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$\Rightarrow \quad a^2+b^2 = 2$

$2ab = 2 \Rightarrow ab = 1.$

$(a \pm b)^2 = a^2+b^2 \pm 2ab$

$(a+b)^2 = 2+2 = 4$

$a+b = \pm 2$.

$a-b = 0$

Either $a+b = 2$ & $a-b = 0$

or $a+b = -2$ & $a-b = 0$.

$\Rightarrow \quad a = b = 1$ or $a = b = -1$,

Hence the 2 sq. roots that we get are

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

(b) $A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix} \quad (*)$$

$$\Rightarrow a^2+bc=5 \quad b(a+d) = c(a+d) = 0.$$
$$bc+d^2 = 9$$

We have $(bc+d^2) - (a^2+bc) = 9-5 = 4$

$$\Rightarrow d^2 - a^2 = 4$$
$$\Rightarrow (d-a)(d+a) = 4$$

Hence $d+a \neq 0$.

Thus, $b = c = 0$.

Also then equating the 1^{st} entries of (*), we have,

$$a^2 + 0 = 5$$
$$\Rightarrow a = \pm\sqrt{5}.$$

Similarly, $d = \pm 3$.

The possible squareroots are 4, given by

$$\begin{bmatrix} \pm\sqrt{5} & 0 \\ 0 & \pm 3 \end{bmatrix}$$

c Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Take $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose $BB = A$.

Then an argument similar to the one in b gives $a^2 = -1$.

7 a $(A+B)(A-B) \neq A^2 - B^2$

$A = \begin{bmatrix} 2 & 6 \\ 0 & 8 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$(A+B)(A-B) = A^2 - AB + BA - B^2$
$\neq A^2 - B^2$

Any pair $(A, B) \ni AB \neq BA$

such that (s.t.)

b $(A+B)(A-B) = A^2 - AB + BA - B^2$

8 Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
where $a_0 \neq 0$.

A is a sq. matrix with $p(A) = 0$.
Show that A is invertible.

important

$$p(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

Zero (null) matrix

$$\Rightarrow a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A = -a_0 I$$

Divide both sides by $-a_0$ ($\because a_0 \neq 0$ by the hypothesis)

$$-\frac{a_n}{a_0} A^n - \frac{a_{n-1}}{a_0} A^{n-1} + \cdots - \frac{a_1}{a_0} A = I$$

$$A\left(-\frac{a_n}{a_0} A^{n-1} - \frac{a_{n-1}}{a_0} A^{n-2} - \cdots - \frac{a_1}{a_0} I\right)$$

$$= \left(-\frac{a_n}{a_0} A^{n-1} - \frac{a_{n-1}}{a_0} A^{n-2} - \cdots - \frac{a_1}{a_0} I\right) A$$

$$= I.$$

$\Rightarrow A$ is invertible, with

$$A^{-1} = \left(-\frac{a_n}{a_0} A^{n-1} - \frac{a_{n-1}}{a_0} A^{n-2} - \cdots - \frac{a_1}{a_0} I\right),$$