Tutorial 1 - to be discussed on 2nd September 2024

Q1. Find a linear system in the unknowns $x_1, x_2, x_3, \ldots,$ that corresponds to the given augmented matrix.

(a) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

Q2. Find the augmented matrix for the linear system

(a) $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$

(b) $\begin{aligned} 2x_1 \qquad + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$

(c) $\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_2 \qquad &= 2 \\ x_3 &= 3 \end{aligned}$

Q3. Under what conditions on a and b will the following linear system have no solutions, one solution, infinitely many solutions?

$$\begin{aligned} 2x - 3y &= a \\ 4x - 6y &= b \end{aligned}$$

Q4. The solutions of the linear equation : $4x - 2y = 1$ can be expressed by the pair of equations

$$x = \tfrac{1}{4} + \tfrac{1}{2}t, \quad y = t$$

These are called parametric equations, since y is assigned an arbitrary value t. Each linear system given below has infinitely many solutions. Use parametric equations to describe its solution set.

(a) $\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned}$

(b) $\begin{aligned} 2x - y + 2z &= -4 \\ 6x - 3y + 6z &= -12 \\ -4x + 2y - 4z &= 8 \end{aligned}$

(c) $4x_1 + 2x_2 + 3x_3 + x_4 = 20$

Q5. Find a single elementary row operation that will create a 1 in the upper left corner of the given augmented matrix and will not create any fractions in its first row.

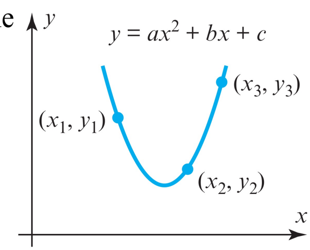
(a) $\begin{bmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$

Q6. We say that a linear system is *consistent* if it has at least one solution and *inconsistent* if it has no solutions. Below find all values of k for which the given augmented matrix corresponds to a consistent linear system.

(a) $\begin{bmatrix} 1 & k & -4 \\ 4 & 8 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & k & -1 \\ 4 & 8 & -4 \end{bmatrix}$

Q7. The curve $y = ax^2 + bx + c$ shown in the accompanying figure passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that the coefficients a, b, and c form a solution of the system of linear equations whose augmented matrix is

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$



Note: this exercise plays a key role when trying to model a given set of data points by a least squares quadratic fit, as we will see later.

Q8. Find values of a, b and c for which the curve $y = ax^2 + bx + c$ passes through the points $(1, 1)$, $(2, 4)$, and $(-1, 1)$.
How many values of each unknown do you expect to find and why?

Q9. Suppose that a certain diet calls for 7 units of fat, 9 units of protein, and 16 units of carbohydrates for the main meal, and suppose that an individual has three possible foods to choose from to meet these requirements:

Food 1: Each ounce contains 2 units of fat, 2 units of protein, and 4 units of carbohydrates.

Food 2: Each ounce contains 3 units of fat, 1 unit of protein, and 2 units of carbohydrates.

Food 3: Each ounce contains 1 unit of fat, 3 units of protein, and 5 units of carbohydrates.

Let x, y, and z denote the number of ounces of the first, second, and third foods that the dieter will consume at the main meal. Find (but do not solve) a linear system in x, y, and z whose solution tells how many ounces of each food must be consumed to meet the diet requirements.

Q10. Determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

Tutorial 2 - to be discussed on 2nd September 2024

Q1. Solve the linear system by Gaussian elimination.

a)
$$\begin{aligned} x_1 + \ \ x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

b)
$$\begin{aligned} - 2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$$

c)
$$\begin{aligned} x - \ \ y + 2z - \ \ w &= -1 \\ 2x + \ \ y - 2z - 2w &= -2 \\ -x + 2y - 4z + \ \ w &= 1 \\ 3x \qquad\qquad\quad - 3w &= -3 \end{aligned}$$

Q2. Solve the above systems using Gauss-Jordan elimination.

Q3. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.

Q4. Find the coefficients a, b, c, and d so that the circle shown in the accompanying figure is given by the equation $ax^2 + ay^2 + bx + cy + d = 0$.

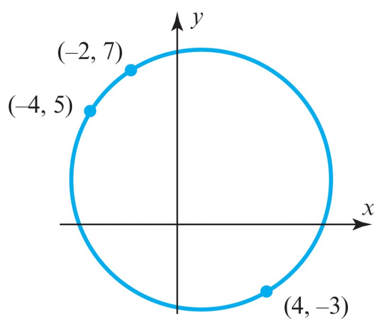


Figure Ex-38

Q5. Let $A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$

(a) Express each column of AB as a linear combination of the columns of A. What are the coefficients of this linear combination?

(b) Express each row of AB as a linear combination of the rows of B. What are the coefficients of this linear combination?

(c) Show that if we interchange two columns of A and also interchange the same two corresponding rows of B, then the product AB remains unchanged.

Q6. A matrix B is said to be a ***square root*** of a matrix A if $BB = A$.

(a) Find two square roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) How many different square roots can you find of
$A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$?

(c) Do you think that every 2×2 matrix has at least one square root? Explain your reasoning.

Q7. (a) Give an example of two 2×2 matrices such that

$$(A + B)(A - B) \neq A^2 - B^2$$

(b) State a valid formula for multiplying out

$$(A + B)(A - B)$$

Q8. Show that if $p(x)$ is a polynomial with a nonzero constant term, and if A is a square matrix for which $p(A) = 0$, then A is invertible.