MA 103 - SVC Lecture 4

Sect. 4.4 Concavity, convexity and curve-sketching

- At a critical point of a differentiable function, the First Derivative Test tells us whether there is a local maximum or a local minimum or whether the function continues to just rise or fall from there.

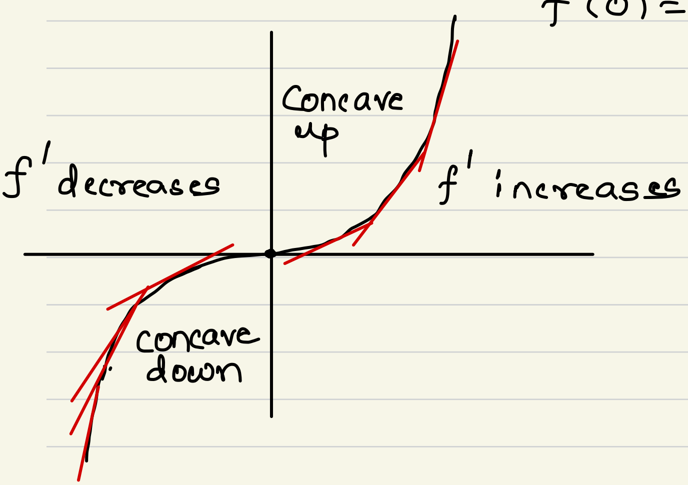
That is, if $f'(c) \neq 0$, then f cannot have a local extremum at c.

If $f'(c) = 0$, then f may or may not have a local extremum at c.

CONCAVITY

Example: $y = x^3$ $\qquad f(x) = x^3,\ f'(x) = 3x^2$

$f'(0) = 0$.



DEFINITION **Concave Up, Concave Down**

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

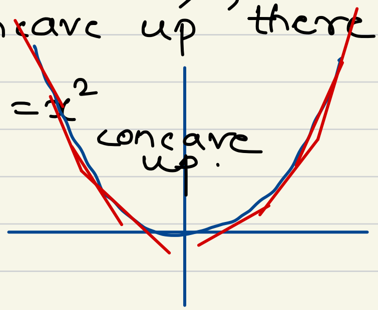
1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Example 1 (a) $y = x^3$ (b) $y = x^2$

(a) $y = x^3$, $y' = 3x^2$, $y'' = 6x$

$y'' < 0$ if $x \in (-\infty, 0)$ hence concave down on $(-\infty, 0)$; and $y'' > 0$ if $x \in (0, \infty)$, hence concave up there.

(b) $y = x^2$



$y' = 2x$, $y'' = 2 > 0$

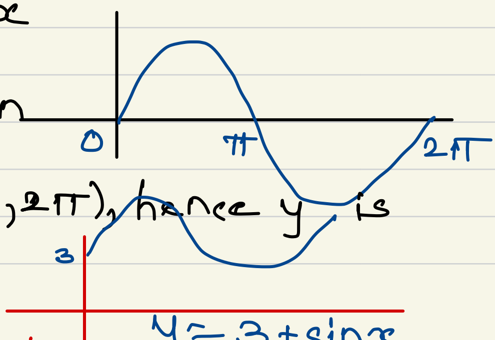
Hence concave up on $(-\infty, \infty)$.

Example 2 $y = 3 + \sin x$, $0 \le x \le 2\pi$.

$y' = \cos x$, $y'' = -\sin x$

$y'' < 0$ if $x \in (0, \pi)$

& hence concave down on $(0, \pi)$.



Similarly, $y'' > 0$ on $(\pi, 2\pi)$, hence y is concave up $(\pi, 2\pi)$.

$(\pi, 3)$ is a point of inflection. See the defn. below.

Points of inflection

DEFINITION **Point of Inflection**

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

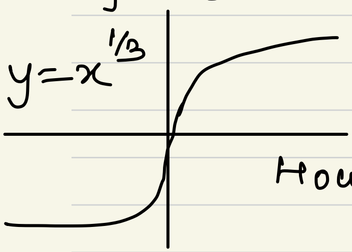
Key remarks:

1. A point on the curve where y'' is +ve on one side and −ve on the other is a pt. of inflection

2. At the point of inflection, $y''=0$ or undefined.

3. If y is a twice-differentiable fn, $y''=0$ at the point of inflection, and y' has a local max. or min.

Example An inflection point may occur where y'' does not exist.

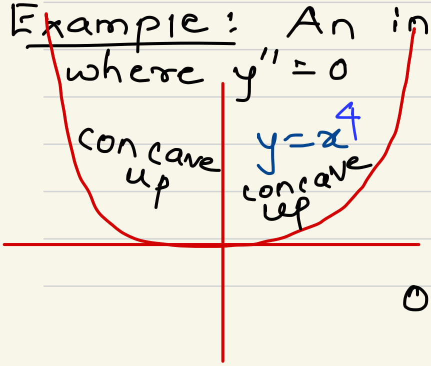


$y' = \frac{1}{3}x^{-2/3}$, $y'' = -\frac{2}{9}x^{-5/3}$

y'' does not exist at $x=0$

However $x=0$ is a pt. of inflection.

Example: An inflection point may not exist where $y''=0$



$y' = 4x^3$

$y'' = 12x^2 > 0$ for $x \in (-\infty, \infty)\setminus\{0\}$

$y''(0) = 0.$

0 is not a point of inflection.

Ex.1 A particle is moving along a horizontal line with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

<u>Ans</u>

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0$$

$$v(t) = \frac{ds}{dt} = 6t^2 - 28t + 22$$
$$= 2(3t^2 - 14t + 11)$$
$$= 2(t-1)(3t-11)$$

$$a(t) = \frac{dv}{dt} = 12t - 28$$
$$= 4(3t - 7)$$

Intervals	$(0, 1)$	$(1, 11/3)$	$(11/3, \infty)$
Sign of $s' = v$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
particle motion	right	left	right

- particle moves to the right in $(0, 1)$ & $(\frac{11}{3}, \infty)$ & to the left in $(1, 11/3)$
- The particle is momentarily stationary (at rest) when $t = 1$, and $t = 11/3$.

Intervals	$(0, 7/3)$	$(7/3, \infty)$
sign of s''	$-$	$+$
Graph of s	concave down	concave up

The acceleration is directed towards the left when $t \in [0, 7/3]$ & momentarily zero at $t = 7/3$, and then directed to the right in $(7/3, \infty)$.

THEOREM 5 **Second Derivative Test for Local Extrema**

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Pros & cons of the second derivative test

Pros : The test requires us to know f'' only at c itself and not in an interval about c. Hence it's easier to apply.

Cons : The test is inconclusive if $f''=0$ or if f'' does not exist at $x=c$. In this case, we can use the First Derivative Test for local extremum values.

Example Sketch the graph of the function $f(x) = x^4 - 4x^3 + 10$ using the following:

a) Identify where the extrema of f occur.

b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

ⓒ Find where the graph of f is concave up and where it is concave down.

ⓓ Sketch the general shape for f.

Ans. ⓐ f is continuous

$f'(x) = 4x^3 - 12x^2$, exists for any $x \in (-\infty, \infty)$.

Domain of f : $(-\infty, \infty)$

Domain of f' : $(-\infty, \infty)$

Because of the domain of f' being $(-\infty, \infty)$, we see that the critical pts. of f can occur only at the zeros of f'.

$$f'(x) = 0 \iff 4x^3 - 12x^2 = 0$$
$$\iff 4x^2(x-3) = 0$$
$$\iff x = 0 \text{ or } 3.$$

Intervals	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

ⓐ Using the 1st der. test for local extrema, we see that there is no extremum at $x=0$, but a local minimum at $x=3$

(b) f is decreasing on $(-\infty, 0]$ & on $(0, 3)$, and increasing on $(3, \infty)$

(c) $f''(x) = 12x^2 - 24x = 12x(x-2)$.

$f''(x) = 0$ when $x = 0$ or 2.

$x<0$

$0<x<2$

$x-2<0$

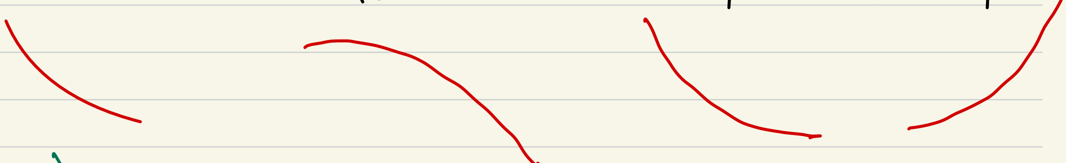
$x>2$

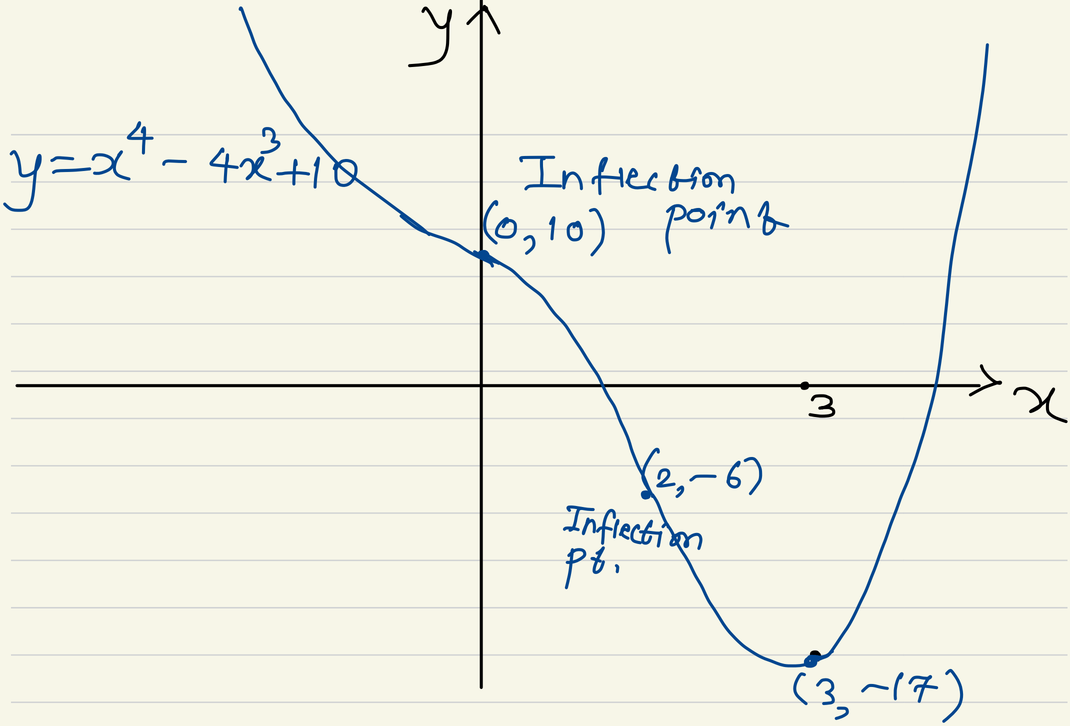
Intervals	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
sign of f''	$+$	$-$	$+$
Behavior of f	Concave up	concave down	concave up

Hence f is concave up on $(-\infty, 0)$ & $(2, \infty)$, and concave down on $(0, 2)$.

(d) Summary:

$x<0$	$0<x<2$	$2<x<3$	$x>3$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up





Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

Ex. 2 Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

$$f(x) = \frac{(x+1)^2}{1+x^2}.$$

- f is continuous on $(-\infty, \infty)$.
- No symmetry
- $f'(x) = \dfrac{(1+x^2)(2(x+1)) - (x+1)^2(2x)}{(1+x^2)^2}$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

- $f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3}$

* Critical points: are at $x = \pm 1$ (There are no points where f' is undefined.)

At $x = -1$, $f''(-1) > 0$
By the 2^{nd} derivative test, we see that f has a local minimum at $x = -1$.

At $x = 1$, $f''(1) < 0$, so local max at $x = 1$

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Sect. 4.8 - Antiderivatives

DEFINITION Antiderivative
A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Examples: 1 $f(x) = x^3$; $F(x) = \frac{x^4}{4}$ since $F'(x) = \frac{4x^3}{4} = x^3$.

2 $f(x) = x^4 - \sin x$; $F(x) = \frac{x^5}{5} + \cos x$

The most general antiderivative of f on an interval I is the function $F(x) + c$, where $F'(x) = f(x)$ $\forall x \in I$ and c is an arbitrary constant

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\frac{x^{n+1}}{n+1} + C$, $n \neq -1$, n rational
2.	$\sin kx$	$-\frac{\cos kx}{k} + C$, k a constant, $k \neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k} + C$, k a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

Example Find the general antiderivative of $f(x) = \frac{3}{\sqrt{x}} + \sin(2x)$.

Ans. $F(x) = G(x) + H(x) + c$, $G'(x) = 3/\sqrt{x} = 3x^{-1/2}$

$H'(x) = \sin(2x)$

$G(x) = \dfrac{3\sqrt{x}}{(1/2)} = 6\sqrt{x}$, $H(x) = \dfrac{-\cos(2x)}{2}$.

- Initial value problems and differential equations

Finding antiderivative of a function is eqvt. to solving $\dfrac{dy}{dx} = f(x)$. This is known as differential equation. If we are given $y(x_0) = y_0$, then this is known as initial condition. The diff. eqn. along with the initial condition is called the initial value problem.

Ex. 1 A balloon ascending at the rate of 12ft./sec is at a height 80 ft. above the ground when a package is dropped. How long does it take the package to reach the ground?

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Example Find the general antiderivative of $\int \cot^2 x\,dx$.

Use $\cot^2 x = \operatorname{cosec}^2 x - 1$. Hence

$\int \cot^2 x\,dx = \int(\operatorname{cosec}^2 x - 1)\,dx = -\cot x - x + c$,