$$\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Neither REF nor RREF

Solve the system.

$x_1 + 3x_3 = -1$

$x_2 - 4x_3 = 2$

a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

inconsistent

leading variables

free variable

We will now illustrate Gaussian elimination method as well as Gauss-Jordan elimination method through an example.

Solve:

$$-2x_3 + 7x_5 = 12$$
$$2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 = 28$$
$$2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 = -1$$

Augmented matrix:

$$\left[\begin{array}{ccccc|c} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right]$$

Step 1: Locate the left-most non-zero column & the top entry of that column.

Step 2: If needed, interchange the top row with another row to bring a non-zero entry to the top of column found in step 1.

$R_1 \leftrightarrow R_2$
$$\left[\begin{array}{ccccc|c} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right]$$

Step 3: If the entry at the top of the leftmost non-zero column is a, perform $R_1 \to \frac{1}{a} R_1$ to get a leading 1.

leading 1

$$R_1 \to \frac{1}{2} R_1 \quad \left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right]$$

Step 4: Add suitable multiples of R_1 to other rows to make all entries below the leading 1 as zero.

$$R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right]$$

Step 5: Freeze the first row and repeat the procedure with sub-matrix.

Step 6: Locate the left-most non-zero column of the sub-matrix.

$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right]$$

left-most non-zero column

Step 7: $R_2 \rightarrow \frac{R_2}{(-2)}$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right]$$

Step 8: $R_3 \rightarrow R_3 - 5R_2$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{array}\right]$$

Step 9: $R_3 \rightarrow 2R_3$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

REF form

To convert the above matrix into RREF we need to perform some additional steps.

- Start with the bottom row.

- $R_1 \rightarrow R_1 - 6R_3$, $R_2 \rightarrow R_2 + \frac{7}{2}R_3$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

• $R_1 \to R_1 + 5R_2$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \leftarrow \text{RREF form}$$

Solutions: $x_5 = 2$, $x_3 = 1$, $x_1 + 2x_2 + 3x_4 = 7$

Let $x_2 = s$, $x_4 = t$ $\Rightarrow$ $x_1 = 7 - 2s - 3t$

• Steps 1–5 : called Gaussian elimination
(Forward phase) – gives REF form

• Steps 1–6 : called Gauss-Jordan elimination
(Forward + Backward phase) – gives RREF form

HOMOGENEOUS LINEAR SYSTEMS

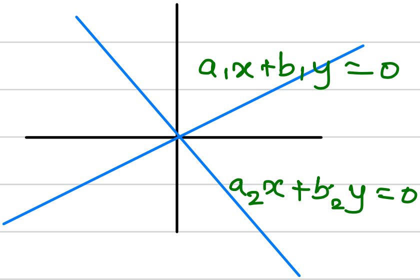
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

• Always consistent since
$x_1 = x_2 = \cdots = x_n = 0$ is a solution
called the trivial solution.

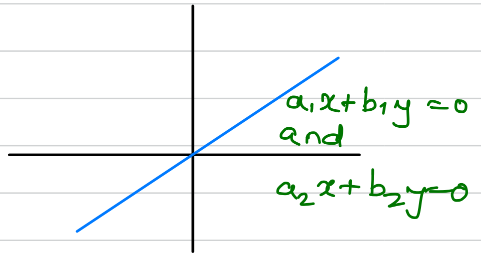
• Other solutions, if any, are called non-trivial solutions.

Thus, there is either just the trivial solution or there are infinitely many solutions, in addition to the trivial solution.

• Homogeneous linear system of 2 equations in 2 unknowns:



only the trivial solution



Infinitely many solutions

Consider a homogeneous linear system with n unknowns.
Suppose the RREF of the augmented matrix has r non-zero rows.

each Non-zero row $\longrightarrow$ leading 1 $\longrightarrow$ leading variable

So the augmented matrix has r leading variables and $n-r$ free variables. The system takes the form

$$
\begin{aligned}
x_{k_1} \quad &+ \Sigma(\ ) = 0 \\
x_{k_2} \quad &+ \Sigma(\ ) = 0 \\
&\ \ \vdots \\
x_{k_r} \quad &+ \Sigma(\ ) = 0
\end{aligned}
$$

($\Sigma(\ )$ denotes a sum involving free variables)

Thm. 1.2.1 (Free variable theorem for homogeneous systems)

If a homogeneous linear system has 'n' unknowns, and the RREF of the augmented matrix has 'r' non-zero rows, then the system has $n-r$ free variable

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$n=3$, $r=2$ — Free variable $= 1$

$n=3$, $r=1$ — $n-r=2$.

Cor. 1.2.2

(i) If $n-r>0$, then the homogeneous linear system has infinitely many solutions.

(ii) If no. of equations (m) < no. of unknowns (n), then the system has infinitely many solutions.

Proof: (i) is clear (since there is at least one free variable.)

(ii) Note that $\underbrace{r \le m}$ $< n \Rightarrow n-r>0$ because there cannot be more non-zero rows than the number of equations. Hence (ii) $\Rightarrow$ (i).

ASSIGNMENT PROBLEM 1

Are the results of Thm. 1.2.1 & Cor. 1.2.2 valid for a non-homogeneous system as well. Justify your answer. In particular, can you always say that for a linear system, if no. of equations is less than the number of unknowns, then the system must have infinitely many solutions?

SOME FACTS ON ECHELON FORMS

① Every matrix has a unique RREF but not a unique REF since different sequence of EROS will lead to different REFs.

② Although REFs aren't unique, RREF & REFs of a matrix A have same number of zero rows & the leading 1's always occur at the same positions, called the pivot positions of A.

A column containing a pivot position is called a pivot column of A.

Discuss the existence and uniqueness of solutions to the following systems:

(a) $\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

inconsistent

$x_3 + 6x_4 = 9$

$x_3 = 9 - 6x_4$

$x_2 + 2x_3 - 4x_4 = 1$

$x_2 = 1 - 2(9 - 6x_4) + 4x_4 = -17 + 16x_4$

$x_1 = 5 + 3(-17 + 16x_4) - 7(9 - 6x_4) - 2x_4$

$= 5 - 51 - 63 + 48x_4 + 42x_4 - 2x_4$

$= -119 + 88x_4$

Unique solution

$x_4 = 0$

$x_3 = 9 - 6x_4 = 9$

$x_2 = 1 - 2(9) = -17$

$x_1 = 5 + 3(-17) - 7(9) = -109$