(b) Let $f(x) = \frac{1}{x(\ln(x))^p}$, $p > 0$

Then f is clearly continuous & positive on $[2, \infty)$.

Also $f'(x) = \frac{-1}{x^2(\ln(x))^{2p}} \cdot \left\{ p(\ln(x))^{p-1} + (\ln(x))^p \right\}$

< 0 on $[2, \infty)$.

Hence f is decreasing on $[2, \infty)$

By integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ converges iff $\int_2^{\infty} \frac{dx}{x(\ln(x))^p}$ converges, i.e; from part (a)

$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ converges for $p > 1$ & diverges for $p \leq 1$.

Q7] (a) $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{\sqrt{n}}{n^2+1}$

Let $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

Since $n^2 + 1 > n^2$, $\frac{1}{n^2+1} < \frac{1}{n^2} \Rightarrow \frac{\sqrt{n}}{n^2+1} \leq \frac{1}{n^{3/2}}$

Since $\sum b_n$ converges (p-series test with $p > 1$), by comparison test, $\sum a_n$ converges too.

(b) $\sum a_n$, where $a_n = \tan\left(\frac{1}{n}\right)$

Let $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n}$$

Now $\lim_{x\to\infty} \frac{\tan(1/x)}{1/x} = \lim_{x\to 0} \frac{\tan(x)}{x} = 1$.

Hence $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$.

$\Rightarrow$ By limit comparison test, since $\sum b_n$ diverges, so does $\sum a_n$.

Q8. (a) $\sum_{n=1}^{\infty} a_n$, where $a_n = n^2 e^{-n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2}$$

$$= \frac{1}{e} \lim_{n\to\infty} \frac{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{n^2}$$

$$= \frac{1}{e} < 1$$

By ratio test, $\sum a_n$ converges.

(b) $\dfrac{a_{n+1}}{a_n} = \dfrac{1+\tan^{-1}(n)}{n}$

Now $-\dfrac{\pi}{2} \le \tan^{-1}(n) \le \dfrac{\pi}{2}$

$\Rightarrow \quad \dfrac{1-\pi/2}{n} \le \dfrac{1+\tan^{-1}(n)}{n} \le \dfrac{1+\pi/2}{n}$

Letting $n \to \infty$ & using the sandwich thm; we see that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$$

By ratio test, $\sum a_n$ converges.

Q9. (a) $\sum a_n$, where $a_n = \dfrac{(n!)^n}{n^{(n^2)}}$

$n(n-1)\cdots\left(\frac{n}{2}+1\right) \le n^{n/2}$

$\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)\cdots 2\cdot 1 \le \left(\frac{n}{2}\right)^{n/2}$

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{2}{n}\cdot\frac{1}{n}$$

$$= 0 < 1$$

By root test, $\sum a_n$ converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(\ln(n))^{n/2}}$, $\quad a_n = \dfrac{n}{(\ln(n))^{n/2}}$

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{(\ln(n))^{1/2}} = 0 < 1.$$

By root test, $\sum a_n$ converges.

THEOREM 13 The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho.$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

Root test may work in cases where ratio test doesn't:

In the last lecture (on Nov. 12), we saw how the ratio test fails to determine the convergence or divergence of the following series:

$$\sum a_n, \text{ where } a_n = \begin{cases} n/2^n, & n \text{ odd}, \\ 1/2^n & n \text{ even}. \end{cases}$$

We now show that the root test does work.

$$(a_n)^{1/n} = \begin{cases} \dfrac{n^{1/n}}{2}, & n \text{ odd} \\ \dfrac{1}{2}, & n \text{ even} \end{cases}$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \frac{1}{2} < 1 \quad \left(\because \lim_{n\to\infty} n^{1/n} = 1\right)$$

$\Rightarrow \sum a_n$ converges by root test.

<u>Example</u> Discuss convergence/divergence of

(1) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (2) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ (3) $\sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n$.

Sol^n. (1) $\lim_{n\to\infty} \left(\frac{n^2}{2^n}\right)^{1/n} = \frac{\lim_{n\to\infty} (n^{1/n})^2}{\lim_{n\to\infty} (2^n)^{1/n}}$

$= \frac{1^2}{2} = \frac{1}{2} < 1$

$\Rightarrow \sum \frac{n^2}{2^n}$ conv.

(2) $\sum \frac{2^n}{n^2}$ $\lim_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{(2^n)^{\frac{1}{n}}}{n^{2/n}}$

$= \frac{2}{1} > 1$

Hence the series diverges.

(3) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ $\lim_{n\to\infty} \frac{1}{1+n} = 0 < 1$

Hence the series converges by root test

Sect. 11.6 — Alternating series, absolute and conditional convergence

Alternating series: A series which has, alternatively, positive and negative terms is called an alternating series.

Examples: ① $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

② $-1 + 2 - 3 + 4 - 5 + \cdots = \sum_{n=1}^{\infty} (-1)^n n$

③ $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n-1}}$

THEOREM 14 The Alternating Series Test (Leibniz's Theorem)

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Example: Alternating harmonic series

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Use alt. ser. test with $u_n = 1/n > 0$ $\forall n \in \mathbb{N}$

$\{u_n\}$ is decreasing seq. $\forall n \in \mathbb{N}$

$\{u_n\} \to 0$ as $n \to \infty$. Hence the series conv.

THEOREM 15 The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 14, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the numerical value of the first unused term. Furthermore, the remainder, $L - s_n$, has the same sign as the first unused term.

Example: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \cdots$

Suppose we truncate at the 8^{th} term.

$|L - S_8| < \frac{1}{256}$. To see this, $L = \frac{1}{1-(-1/2)} = \frac{2}{3}$

$\Rightarrow L = 0.666\ldots$. Also, $S_8 = 0.6640625$.

Then $L - S_8 = 0.666\ldots - 0.6640625$
$= 0.0026041666\ldots$

& $\frac{1}{256} = 0.00390625$. Hence $|L - S_8| < \frac{1}{256}$.

Also, the signs of $L - S_8$ & $\frac{1}{256}$ are same.

Absolute and conditional convergence

DEFINITION Absolutely Convergent

A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Examples: (1) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

$\Rightarrow a_n = \frac{(-1)^{n+1}}{n^2}$. Then $|a_n| = \frac{1}{n^2}$ & $\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

converges ($\because$ p-series with $p=2$). Then $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

converges absolutely.

DEFINITION **Conditionally Convergent**

A series that converges but does not converge absolutely **converges conditionally**.

Examples: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$; $a_n = \frac{(-1)^{n+1}}{n}$ $\sum a_n$ converges by alt. ser. test. But $|a_n| = \frac{1}{n}$ & $\sum |a_n|$ diverges. Hence $\sum a_n$ is a conditionally convergent series.

THEOREM 16 **The Absolute Convergence Test**

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

A series which converges absolutely also converges.

Proof: $-|a_n| \leq a_n \leq |a_n|$

$\Rightarrow \quad 0 \leq a_n + |a_n| \leq 2|a_n|$

Consider $\sum (a_n + |a_n|)$ is a series of non-negative terms, the summand of which is bounded above by $2|a_n|$.

If $\sum |a_n|$ converges, so does $\sum 2|a_n|$. Then by direct comparison test, $\sum (a_n + |a_n|)$ also converges.

$$\sum a_n = \sum \left((a_n + |a_n|) - |a_n| \right)$$
$$= \underbrace{\sum (a_n + |a_n|)}_{\text{convergent}} - \underbrace{\sum |a_n|}_{\text{convergent}}$$

Hence $\sum a_n$ converges.

Ex. 1 Do the following series converge?

① $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n^3)}$ ② $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

③ $\displaystyle\sum_{n=1}^{\infty} (-1)^n \operatorname{sech}(n)$

Alternating p-series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$

$= 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$; $a_n = \frac{(-1)^{n+1}}{n^p}$

$|a_n| = \frac{1}{n^p}$, So $\displaystyle\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ p-series

We know that p-series conv. for $p>1$.

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges absolutely for $p>1$

& hence it itself converges for $p>1$.

But $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges conditionally for $0<p\le 1$

(as can be seen by alternating series test.)

Example: $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ converges conditionally

THEOREM 17 The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n .$$

Warning: ① $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$
$= 0$

or 1 if we rearrange it, or any value that you imagine by rearrangement. But this does not contradict the above theorem because the series does not converge absolutely.

Ex. 2 Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

1. **The *n*th-Term Test:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$, since absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.