Tutorial 6 - to be discussed on 23rd September 2024

Q1. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.

(a) $\mathbf{u}_1 = (2, -4)$, $\mathbf{u}_2 = (3, 8)$; $\mathbf{w} = (1, 1)$

(b) $\mathbf{u}_1 = (1, 1)$, $\mathbf{u}_2 = (0, 2)$; $\mathbf{w} = (a, b)$

Q2. Recall: $[\mathbf{v}]_{B'} = P_{B\to B'}[\mathbf{v}]_B$ (12)

Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \mathbf{u}'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

(a) Find the transition matrix from B' to B.

(b) Find the transition matrix from B to B'.

(c) Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and use (12) to compute $[\mathbf{w}]_{B'}$.

(d) Check your work by computing $[\mathbf{w}]_{B'}$ directly.

Q3. Recall: $[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}]$ (14)

Let S be the standard basis for R^2, and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis in which $\mathbf{v}_1 = (2, 1)$ and $\mathbf{v}_2 = (-3, 4)$.

(a) Find the transition matrix $P_{B\to S}$ by inspection.

(b) Use Formula (14) to find the transition matrix $P_{S\to B}$.

(c) Confirm that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another.

(d) Let $\mathbf{w} = (3, -5)$. Find $[\mathbf{w}]_S$ and then use Formula (12) to compute $[\mathbf{w}]_B$.

Q4. Find a basis for i) the rowspace and ii) the nullspace of A

(a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

Q5. Let $T: R^2 \to R^3$ be the linear transformation defined by the formula

$$T(x_1, x_2) = (x_1 + 3x_2, x_1 - x_2, x_1)$$

(a) Find the rank of the standard matrix for T.

(b) Find the nullity of the standard matrix for T.

Q6. Confirm by multiplication that **x** is an eigenvector of A and find the corresponding eigen value.

a) $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}; \ \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}; \ \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Q7. Find the characteristic equation, the eigenvalues, and bases for the eigen spaces of the matrix. State the algebraic multiplicity and geometric multiplicity for each eigenvalue obtained.

a) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ c) $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

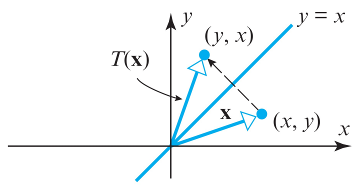
Q1. Suppose x is an eigenvector of a matrix A corresponding to the eigenvalue c, i.e. Ax=cx. If we consider the matrix transformation

$$T_A : R^n \to R^n$$

then $T_A(\mathrm{x}) = \mathrm{cx}$. This means that the image of x under the transformation T_A is simply a scalar multiple of x by the factor c.
In each of the following, a matrix transformation from R^2 to R^2 is given to you. Find the eigenvalues and corresponding eigenspaces of the standard matrix of the transformation by using only geometric reasoning. No computations are needed.

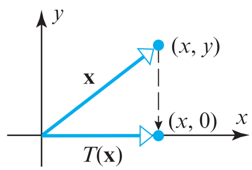
(a) Reflection about the line $y = x$

$T(x, y) = (y, x)$



(b) Orthogonal projection onto the x-axis

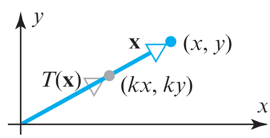
$T(x, y) = (x, 0)$



(c) Rotation about the origin through a positive angle of 90°.

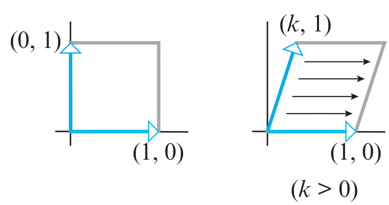
(d) Contraction with factor k in R^2

$(0 \le k < 1)$



(e) Shear in the x-direction by a factor k in R^2

$T(x, y) = (x + ky, y)$



Q2. In each of the following, the characteristic equation of a matrix A is given. Find the size of the matrix and the possible dimensions of its eigenspaces (geometric multiplicities).

(a) $(\lambda - 1)(\lambda + 3)(\lambda - 5) = 0$

(b) $\lambda^2(\lambda - 1)(\lambda - 2)^3 = 0$

Q3. Show that A and B are not similar matrices.

a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q4. In each of the following, find the geometric and algebraic multiplicity of each eigenvalue of the matrix A and determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P which diagonalizes A, and find $P^{-1}AP$.

a) $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$ b) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Q5. Let R^4 have the Euclidean inner product, and let $\mathbf{u} = (-1, 1, 0, 2)$. Determine whether the vector $\mathbf{u}$ is orthogonal to the subspace spanned by the vectors $\mathbf{w}_1 = (1, -1, 3, 0)$ and $\mathbf{w}_2 = (4, 0, 9, 2)$.

Q6. Let R^4 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.

$$\mathbf{u}_1 = (0, 2, 1, 0), \quad \mathbf{u}_2 = (1, -1, 0, 0),$$
$$\mathbf{u}_3 = (1, 2, 0, -1), \quad \mathbf{u}_4 = (1, 0, 0, 1)$$