Tutorial 4 (SVC) to be discussed on November 4, 2024.

Q 1. For what values of a, m and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Q 2. Let

$$f(x) = \begin{cases} x + 2, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x + 1, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

a. Show that

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 1 \quad \text{but} \quad \lim_{x \to 0} \frac{f(x)}{g(x)} = 2.$$

b. Explain why this does not contradict l'Hôpital's Rule.

Q 3. **Continuous extension** Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

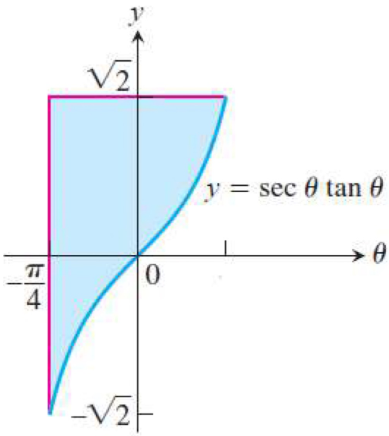
continuous at $x = 0$. Explain why your value of c works.

Q 4. Evaluate the following limits.

1. $\displaystyle\lim_{x \to 0} x \csc^2 \sqrt{2x}$

2. $\displaystyle\lim_{x \to \infty} \frac{\sqrt{x + 5}}{\sqrt{x} + 5}$

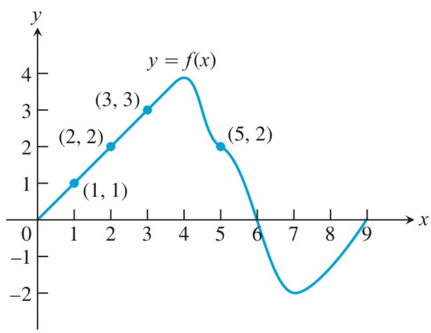
Q 5. Find the area of the shaded region.



Q 6.). Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a. What is the particle's velocity at time $t = 5$?

b. Is the acceleration of the particle at time $t = 5$ positive, or negative?

c. What is the particle's position at time $t = 3$?

d. At what time during the first 9 sec does s have its largest value?

e. Approximately when is the acceleration zero?

f. When is the particle moving toward the origin? away from the origin?

g. On which side of the origin does the particle lie at time $t = 9$?

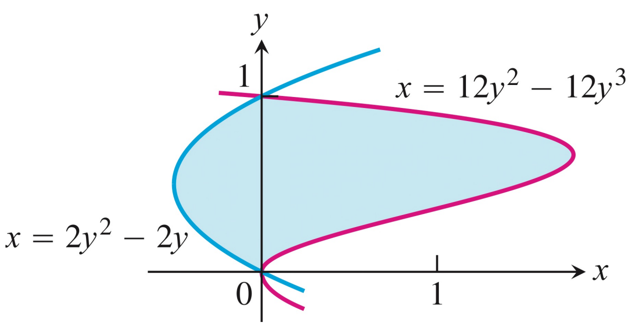
Q 7. Find

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4 + 1}\, dt.$$

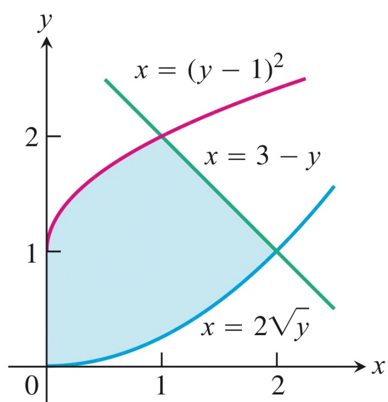
Q 8. Evaluate the following integral using substitution rule.

$$\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta \cos^3 \sqrt{\theta}}}\, d\theta$$

Q 9. Find the area of the shaded region.



Q 10. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



Part II

Q 1. Prove that for $|x| < 1$,

$$\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Q 2. Evaluate

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx.$$

Q 3. Evaluate $$\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\, d\theta}{\sin\theta\cos\theta + \sin\theta}$$

Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

Q 4. **Center of gravity** Find the center of gravity of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

Q 5. Find the values of p for which each integral converges.

a. $\displaystyle\int_1^2 \frac{dx}{x(\ln x)^p}$ **b.** $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$

Q 6. $\int_{-\infty}^{\infty} f(x)\,dx$ **may not equal** $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$ Show that

$$\int_0^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$$

Q 7. Does the following integral converge?

$$\int_0^1 \frac{dt}{t - \sin t}$$

Q 8. **Social diffusion** Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

a. Find x as a function of t.

b. When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

Q 9. Solve the initial value problen for y as a function of x.

$$x\frac{dy}{dx} = \sqrt{x^2 - 4}, \quad x \geq 2, \quad y(2) = 0$$

Q 10. Evaluate the integrals

a. $\displaystyle\int e^t \sec^3 (e^t - 1)\, dt$

b. $\displaystyle\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$