MA 101 Tutorial 3 (2024)

Ex. 1 (a) $EA = B$, where

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}$$

B is obtained from A by $R_1 \leftrightarrow R_3$.
Hence E, s.t. $EA = B$, can be obtained by performing the same operation on I, that is,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(c) $\begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$

$EA = C$, where $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Q 2] a $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1/\sqrt{2}$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/\sqrt{2} & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + 4\sqrt{2}R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2/13\sqrt{2}$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 4/13\sqrt{2} & 1/13\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 3R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/13\sqrt{2} & \frac{-3}{13\sqrt{2}} & 0 \\ 0 & 1 & 0 & 4/13\sqrt{2} & 1/13\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\underbrace{}_{I}$ $\underbrace{}_{A^{-1}}$

Q 3]
$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2/4$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - \frac{3}{4} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & -3/4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 + 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1/4 & -3/4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Repeat these EROs on the identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2/4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow E_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \frac{3}{4} R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow E_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow E_3$$

$$(E_3 E_2 E_1) A = I$$

$$\Rightarrow A^{-1} = E_3 E_2 E_1$$

$$\Rightarrow A = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}.$$

This way of representing A is not unique.
(for example we could have first done $R_1 \to R_1 + 2R_3$ followed by $R_2 \to R_2 - \frac{3}{4} R_3$)

Q 4]

$$x + y + z = 5$$
$$x + y - 4z = 10$$
$$-4x + y + z = 0$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -4 \\ -4 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}}_{B}$$

$$\Rightarrow AX = B$$

$$\Rightarrow X = A^{-1} B$$

To find A^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + 4R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2/5$, $R_3 \to R_3/{-5}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4/5 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -1/5 \\ 0 & 1 & 1 & 4/5 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -1/5 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{bmatrix}}_{A^{-1}}$$

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

$\Rightarrow x = 1, \; y = 5, \; z = -1.$

Q5] $A = LU$

$Ax = B$ (Linear system)

$(LU)X = B$

Let $UX = Y$

& solve $LY = B$ to get Y

Then solve $UX = Y$ to get X.

(a) $$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{B}.$$

Let $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ such that $UX = Y$,
i.e.;

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \text{---} \quad (*)$$

We will first solve $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$\Rightarrow y_1 = 1$

& $-2y_1 + 3y_2 = -2 \Rightarrow y_2 = 0$

$2y_1 + 4y_2 + y_3 = 0$,

$\Rightarrow y_3 = -2$.

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Hence from (*), $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$\Rightarrow 4x_3 = -2 \Rightarrow \boxed{x_3 = -1/2}$

$x_2 + 2x_3 = 0 \Rightarrow \boxed{x_2 = 1}$

$2x_1 - x_2 + 3x_3 = 1$

$\Rightarrow \boxed{x_1 = 7/4}$.

Q 6] Find domain, codomain & standard matrix of the given matrix transf.

(a) $T_A x = Ax$, A has size $\overset{m}{1} \times \overset{n}{6}$

$$\underset{\text{domain}}{T_A : \mathbb{R}^6} \longrightarrow \underset{\text{co-domain}}{\mathbb{R}}$$

We cannot find the standard matrix here.

(b)
$$w_1 = 5x_1 - 7x_2$$
$$w_2 = 6x_1 + x_2$$
$$w_3 = 2x_1 + 3x_2$$

$$T_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix},$$ where $A =$

$m = 3,\ n = 2$

Domain : $\mathbb{R}^2$

Co-domain : $\mathbb{R}^3$

Standard matrix : $\begin{bmatrix} 5 & -7 \\ 6 & 1 \\ 2 & 3 \end{bmatrix}$.

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -7 \\ 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ 6 & 1 \\ 2 & 3 \end{bmatrix}.$$

(c) $$\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix}_{3\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$$

Domain: $\mathbb{R}^2$

Co-domain: $\mathbb{R}^3$.

Standard matrix: $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix}$.

(d) $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$

Domain: $\mathbb{R}^4$

Co-domain: $\mathbb{R}^2$

$\mathbb{R}^4 \longrightarrow \mathbb{R}^2$

Standard matrix $A = \begin{bmatrix} | & | & | & | \\ T(e_1) & T(e_2) & T(e_3) & T(e_4) \\ | & | & | & | \end{bmatrix}$.

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T(e_4) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \longleftarrow \text{standard matrix.}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(e) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 - x_3 \\ 0 \end{bmatrix}$

Domain: $\mathbb{R}^3$

Co-domain: $\mathbb{R}^4$

Std. matrix has size 4×3.

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}_{4 \times 3}$$

Q 7] (a) $T(x,y) = (x, y+1)$

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Need to check:

$$\begin{cases} T(u+v) = T(u) + T(v) \\ T(ku) = k\,T(u) \quad \text{for any scalar } k \end{cases}$$

Just $T(u+kv) = T(u) + kT(v)$ is also okay!

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$T(u+v) = T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} u_1+v_1 \\ u_2+v_2+1 \end{bmatrix}$$

$$T(u)+T(v) = \begin{bmatrix} u_1 \\ u_2+1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2+1 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2+2 \end{bmatrix}$$

are not the same!

$\Rightarrow T(u+v) \neq T(u) + T(v)$

Hence it is not a linear transformation

Hence not a matrix transf. as well,

(b) $T(x_1, x_2, x_3) = (x_1, x_3, x_1 + x_2)$

Chk: $T(ku) = k\,T(u)$

$$T\left(k\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right) = T\begin{pmatrix} ku_1 \\ ku_2 \\ ku_3 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_3 \\ ku_1 + ku_2 \end{pmatrix}$$

$$kT(u) = k\,T\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_3 \\ k(u_1+u_2) \end{pmatrix}$$

<u>Chk</u>: $T(u+v) = T(u) + T(v)$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Then $T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = T\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_3+v_3 \\ u_1+v_1+u_2+v_2 \end{pmatrix}$

$$= \begin{pmatrix} u_1 \\ u_3 \\ u_1+u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_3 \\ v_1+v_2 \end{pmatrix} = T(u) + T(v)$$

Hence it is a linear transf. (& hence a matrix transf.)

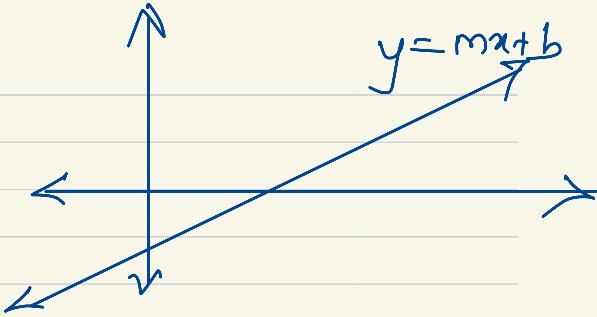
(c) $T(x, y, z) = (x, y, xz)$

<u>Chk</u>: $T(ku) \neq kT(u)$

$$\begin{pmatrix} ku_1 \\ ku_2 \\ k^2u_1u_3 \end{pmatrix} \neq k\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \\ ku_3 \end{pmatrix}$$

Hence not a linear transf (& hence not a matrix transf.)

Q 8] $f(x) = mx + b$



A transf. is a matrix transf. iff it is a linear transf.

Let $k \in \mathbb{R}$

Now $f(kx) = m(kx) + b$
$\neq k(mx+b) = k f(x)$,

unless $b = 0$.

Suppose $b = 0$.

$f(u+v) = m(u+v) = mu + mv$
$= f(u) + f(v)$.

Hence f is a linear (& hence a matrix) transformation on $\mathbb{R}$ only if $b = 0$.

Q 9] $T: \mathbb{R}^2 \to \mathbb{R}^2$ linear operator

$T(e_1) = (a, b)$, $T(e_2) = (c, d)$

Find : $T(1, 1)$

Ans. $T(e_1) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

$T(e_2) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix}$

$T\begin{pmatrix}1\\1\end{pmatrix} = T(e_1) + T(e_2) = \begin{pmatrix}a\\b\end{pmatrix} + \begin{pmatrix}c\\d\end{pmatrix}$
$= \begin{pmatrix}a+c\\b+d\end{pmatrix}$.