MA 103 – SVC Lecture 1 – Limits

Overview

* Genesis
* Defn. of limit of a function
* Examples

• **Average and instantaneous speed**

A rock falls from the top of a cliff. What is its average speed

(a) during the first two seconds of fall?
(b) during the 1-sec. time interval between second 1 & second 2?

$y = 16t^2$

(a) First 2 seconds:

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2-0} = 32 \text{ ft/sec.}$$

(b) From sec. 1 to sec. 2:

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2-1} = 48 \text{ ft/sec.}$$

Defn. Average rate of change over an interval

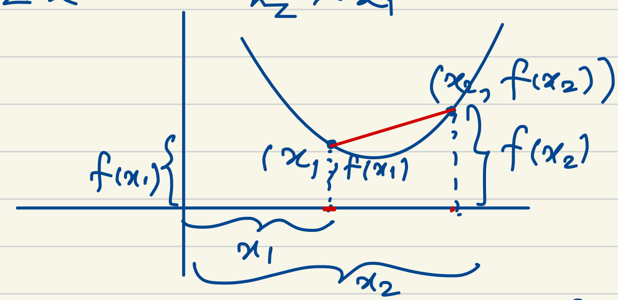
Average rate of change of $y = 16t^2$ over the interval $[x_0, x_0 + h]$ is given by

$$\frac{16(x_0+h)^2 - 16x_0^2}{(x_0+h) - x_0} = \frac{16(x_0^2 + 2x_0h + h^2) - 16x_0^2}{h}$$

$$= 16(2x_0 + h) = 32x_0 + 16h$$

Average rate of change of $y = f(x)$ w.r.t. x over the Interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1+h) - f(x_1)}{h}, \ h \neq 0.$$



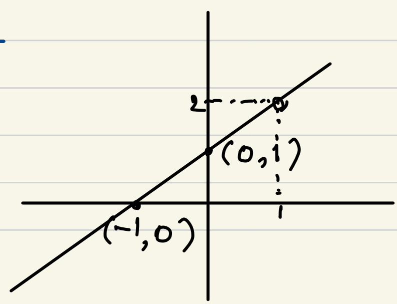
What do we mean by $\lim_{x \to x_0} f(x) = L$?

Let f be defined on an open interval containing x_0, except possibly at x_0 itself. Then by $\lim_{x \to x_0} f(x) = L$, we mean that as x becomes closer and closer to x_0, $f(x)$ becomes closer & closer to L.

Eg. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)}$

$= \lim_{x \to 1} (x+1)$

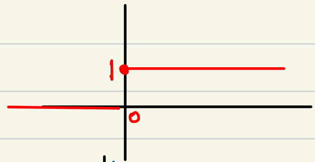
$= 1 + 1 = 2.$



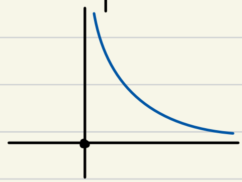
A function may not have a limit at a point in its domain

Ex. Discuss the behaviour of the following functions as $x \to 0$:

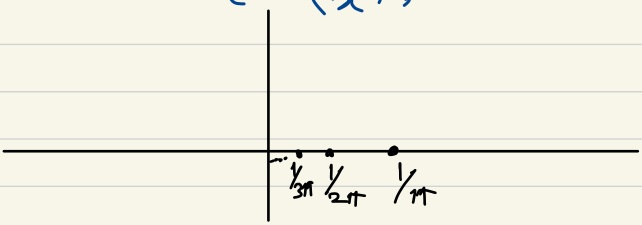
(a) $U(x) = \begin{cases} 0 & , x < 0 \\ 1 & , x \geq 0 \end{cases}$



(b) $g(x) = \begin{cases} \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$



(c) $f(x) = \begin{cases} 0 & , x \leq 0 \\ \sin\left(\frac{1}{x}\right) & , x > 0 \end{cases}$



Calculating limits using the limit laws

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $$\lim_{x \to c} (f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $$\lim_{x \to c} (f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $$\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

<u>Example</u> $\lim_{x \to 0} \frac{4 - \sqrt{16+x}}{x}$.

$$= \lim_{x \to 0} \frac{(4 - \sqrt{16+x})(4 + \sqrt{16+x})}{x(4 + \sqrt{16+x})}$$

$$= \lim_{x \to 0} \frac{16 - (16+x)}{x(4 + \sqrt{16+x})}$$

$$= \lim_{x \to 0} \frac{-\cancel{x}}{\cancel{x}(4 + \sqrt{16+x})}$$

$$= -\frac{\lim_{x \to 0} (1)}{\lim_{x \to 0} (4 + \sqrt{16+x})}$$

$$= \frac{-1}{(4 + \sqrt{16+0})} = \frac{-1}{(4+4)} = \frac{-1}{8}.$$

$$L = \lim_{x \to 0} \frac{x - \sin(x)}{x^3}.$$

$$x = 3\theta$$

$$\frac{x - \sin x}{x^3} = \frac{3\theta - \sin(3\theta)}{(3\theta)^3}$$

$$= \frac{3\theta - (3\sin\theta - 4\sin^3\theta)}{27\theta^3}$$

$$= \frac{3(\theta - \sin\theta)}{27\theta^3} + \frac{4}{27}\underbrace{\frac{\sin^3\theta}{\theta^3}}.$$

$$L = \frac{L}{9} + \frac{4}{27}$$

$$f(x) = \frac{1}{x} - \frac{1}{x^2}$$

$$g(x) = \frac{1}{x}$$

$x > 0$ $\quad f(x) < g(x)$ on $(0, \infty)$

But

$$\lim_{x \to \infty} f(x) = 0 = \lim_{x \to \infty} g(x)$$

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Example: $\displaystyle\lim_{x \to \infty} \frac{\sin x}{x}$

$$-1 \leq \sin x \leq 1$$

$$\Rightarrow \frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

Since $\displaystyle\lim_{x \to \infty} \frac{-1}{x} = 0 = \lim_{x \to \infty} \frac{1}{x}$, we must have $\displaystyle\lim_{x \to \infty} \frac{\sin x}{x} = 0$.

* If $f(x) \leq g(x)$ $\forall x$ in an open interval containing c except possibly at c itself. If $\displaystyle\lim_{x \to c} f(x)$ & $\displaystyle\lim_{x \to c} g(x)$ both exist, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

The precise definition of a limit

DEFINITION Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Ex. Explain the concept in detail with the help of a diagram.

One-sided limits

$\lim_{x \to x_0^+} f(x) = L$: f is said to have a right-hand limit at x_0 if, given an $\varepsilon > 0$, $\exists\ \delta > 0$ such that for all x satisfying $x_0 < x < x_0 + \delta$, we have $|f(x) - L| < \varepsilon$.

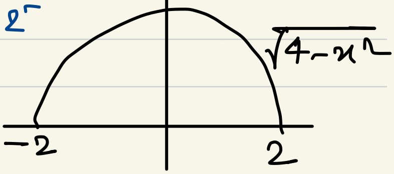
Similarly, we can define $\lim_{x \to x_0^-} f(x) = L$.

$x_0 - \delta < x < x_0$

Example Let $f: [-2, 2] \to \mathbb{R}$ be defined by $f(x) = \sqrt{4 - x^2}$.

Find $\lim_{x \to -2^+} \sqrt{4 - x^2}$ & $\lim_{x \to 2^-} \sqrt{4 - x^2}$.

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0$$



Important formula:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Limits at infinity

$\lim_{x \to \infty} f(x) = L$ if, given $\varepsilon > 0$, $\exists$ a number M s.t. $x > M$, then $|f(x) - L| < \varepsilon$.

Similarly, define $\lim_{x \to -\infty} f(x) = L$.

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Defn. $\lim_{x \to x_0} f(x) = \infty$. This means given a large positive real number B, $\exists$ a $\delta > 0$ $\ni$ for all x, if $0 < |x - x_0| < \delta$, then $f(x) > B$.

Similarly define

- $\lim_{x \to x_0} f(x) = -\infty$.

Continuity & Derivatives

Defn. A function f is said to be continuous at an interior point c of its domain if

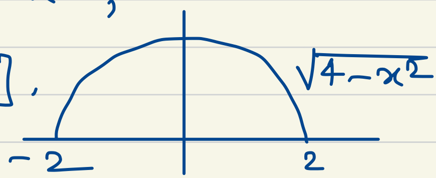
$$\boxed{\lim_{x \to c} f(x) = f(c).}$$

If c is the left-end point of an interval and $\lim_{x \to c^+} f(x) = f(c)$, then we say f is right-continuous at c.

Similarly, if $\lim_{x \to c^-} f(x) = f(c)$, where c is the right-end point of an interval, we say that f is left-continuous at c.

Examples: ① $f(x) = \sqrt{4-x^2}$, $-2 \le x \le 2$

f is continuous on $[-2, 2]$.



② Step function

$$f(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

This function is continuous everywhere except $x = 0$

Test for checking continuity of a function at a point

① $f(c)$ must exist

② $\lim_{x \to c} f(x)$ must exist

③ $\lim_{x \to c} f(x) = f(c)$

Ex. 1 Show that the greatest integer function $f(x) = \lfloor x \rfloor$ is continuous on $\mathbb{R} \setminus \mathbb{Z}$, and that while it is discontinuous at every integer, it is right-continuous there.

Case 1

Let $c \in \mathbb{R} \setminus \mathbb{Z} \leftarrow$ "$\mathbb{Z}$" $\rightarrow$ Zahlen

To show $\lim_{x \to c} f(x) = f(c)$

i.e; $\lim_{x \to c} \lfloor x \rfloor = \lfloor c \rfloor$

Since $c \notin \mathbb{Z}$, $\exists\, n \in \mathbb{Z} \ni$

$$n - 1 < c < n$$

Now $\lfloor c \rfloor = n - 1$

Also $\lim_{x \to c} \lfloor x \rfloor = n - 1$

$\Rightarrow \lim_{x \to c} \lfloor x \rfloor = \lfloor c \rfloor$, i.e; f is continuous on $\mathbb{R} \setminus \mathbb{Z}$.

Case 2: Let $c \in \mathbb{Z}$.

$$\left.\begin{aligned} \lim_{x \to c^-} \lfloor x \rfloor &= c - 1 \\ \lim_{x \to c^+} \lfloor x \rfloor &= c \end{aligned}\right\} \Rightarrow \lim_{x \to c} \lfloor x \rfloor \text{ does not exist, hence } f \text{ is discontinuous at integers.}$$

Also $\lfloor c \rfloor = c$

$$\lim_{x \to c^+} \lfloor x \rfloor = \lfloor c \rfloor$$

$\Downarrow$

f is right-continuous at integers