MA 103- SVC Lecture 9

DEFINITION Diverges to Infinity

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty.$$

Examples : $\{a_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$; $\lim\limits_{n\to\infty} a_n = +\infty$.

$\{a_n\}_{n=1}^{\infty} = \{\log \frac{1}{n}\}_{n=1}^{\infty} = \{-\log n\}_{n=1}^{\infty}$ $\quad a_n \to -\infty$

THEOREM 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

<u>Example</u> Evaluate $\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3}$.

$$= \lim_{n\to\infty} \frac{\cancel{n^4}\left(\frac{1}{n^4} - 5\right)}{\cancel{n^4}\left(1+\frac{8}{n}\right)} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^4} - 5\right)}{\left(1+\frac{8}{n}\right)} = \frac{-5}{1} = -5.$$

THEOREM 2 The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

<u>Example</u> : Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$.

$\frac{-1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$ $\forall n \in \mathbb{N}$ & $\lim_{n\to\infty} \frac{1}{n} = 0$ & $\lim_{n\to\infty} \frac{-1}{n} = 0$.

Hence $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ (by Sandwich theorem)

THEOREM 3 The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$. $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

<u>Example:</u> $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} + \frac{1}{n^2}\right)$ $f(x) = \sin x$ is cont. on $\mathbb{R}$

Hence $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} + \frac{1}{n^2}\right) = \sin\left(\lim_{n\to\infty}\left(\frac{\pi}{2} + \frac{1}{n^2}\right)\right)$

$= \sin\left(\frac{\pi}{2}\right) = 1.$

THEOREM 4
Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x \to \infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n \to \infty} a_n = L.$$

Ex. 2 Does the sequence whose n^{th} term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge? If so, find $\lim_{n\to\infty} a_n$.

Let $a_n = \left(\frac{n+1}{n-1}\right)^n$

Take log on both sides to get

$$\ln(a_n) = n \ln\left(\frac{n+1}{n-1}\right) \leftarrow \infty \cdot 0 \text{ form}$$

$$= \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \leftarrow \frac{0}{0} \text{ form}$$

Hence $\lim_{n\to\infty} \ln(a_n) = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n}$

Consider $\lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{1/x}$ $\left(\frac{0}{0}\right)$ form

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}\left[\ln(x+1) - \ln(x-1)\right]}{-1/x^2} \quad \text{(L'Hôpital's rule)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x+1} - \frac{1}{x-1}}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{\frac{(x-1) - (x+1)}{x^2-1}}{-1/x^2}$$

$$- \lim_{x \to \infty} \frac{-2}{x^2-1} \cdot \frac{x^2}{-1}$$

$$= 2 \lim_{x \to \infty} \frac{x^2}{x^2-1}$$

$$= 2 \lim_{x \to \infty} \frac{1}{1 - 1/x^2}$$

$$= 2 \frac{1}{1-(0)} = 2.$$

$$\Rightarrow \lim_{n \to \infty} a_n = 2 \text{ too.}$$

COMMONLY USED LIMITS

THEOREM 5

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ $\quad \lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

DEFINITION **Nondecreasing Sequence**

A sequence $\{a_n\}$ with the property that $a_n \le a_{n+1}$ for all n is called a **nondecreasing sequence**.

DEFINITIONS Bounded, Upper Bound, Least Upper Bound

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \le M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

Example: $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots\right\}$ $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$

THEOREM 6 The Nondecreasing Sequence Theorem

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

$$\{a_n\} = \left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty}$$

$1 - \frac{1}{n} < 1 - \frac{1}{n+1} \quad \forall n \in \mathbb{N}.$

$a_n \le 1 \quad \forall n \in \mathbb{N} \Rightarrow \{a_n\}$ converges, and as can be seen, it converges to 1, the least upper bound.

Sect. 11.2 — Infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

Example

(1) $\sum_{n=1}^{\infty} n = 1+2+3+4+\dots = +\infty.$

(2) $\sum_{n=1}^{\infty} 3 = 3+3+\dots = +\infty.$

(3) $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$

Defn.

Given a sequence of numbers $\{a_n\}$, an expression of the form $a_1 + a_2 + a_3 + \dots$ is an infinite series.

- a_n is called the n^{th} term of the series.

• **Sequence of partial sums**: $S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3, \ldots S_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$

is the sequence of partial sums of series.

S_n : n^{th} partial sum.

If the sequence of partial sums of $\sum a_n$ converges to a limit L, then we say the series converges to L & that the sum is L.

$$\sum_{n=1}^{\infty} a_n = L, \quad L = \lim_{n\to\infty} S_n.$$

If the seq. of partial sums does not converge, we say that the series diverges.

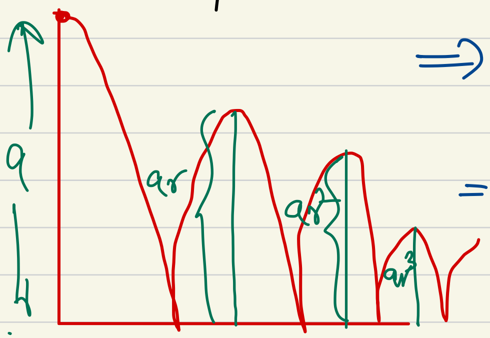
Example: Geometric series

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \quad S_n = \frac{a(1-r^n)}{1-r}.$$

Suppose $|r| < 1$. Then

$$a + ar + ar^2 + \ldots = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \quad \left(\because \lim_{n\to\infty} r^n = 0\right)$$

Example: A ball is dropped 'a' meters above a flat surface. Each time the ball hits the surface after falling a distance 'h', it rebounds a distance rh where $0<r<1$. Find the total distance the ball travels up and down.



$\Rightarrow$ Total distance is

$$a + 2ar + 2ar^2 + 2ar^3 + \ldots$$
$$= (2a + 2ar + 2ar^2 + \ldots) - a$$
$$= -a + 2\sum_{n=0}^{\infty} ar^n$$
$$= -a + \frac{2a}{1-r} = a\left(\frac{r-1+2}{1-r}\right)$$
$$= a\left(\frac{1+r}{1-r}\right)$$

Example: (Telescoping series)

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

$$\sum_{n=1}^{\infty} \frac{(n+1)-n}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$S_k = \sum_{n=1}^{k} \frac{1}{n} - \frac{1}{n+1} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{k}} - \frac{1}{k+1}$$

$$= 1 - \frac{1}{k+1}$$

$$\lim_{k \to \infty} S_k = 1 - \lim_{k \to \infty} \frac{1}{k+1} = 1 - 0 = 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

* n^{th} term test for divergence

- If $\sum_{n=1} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

Hence if $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Examples ① $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\lim_{n \to \infty} \frac{1}{n^2} = 0$; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv.

② $\sum_{n=1}^{\infty} \frac{1}{n}$ $\lim_{n \to \infty} \frac{1}{n} = 0$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

③ $\sum_{n=1} \frac{5n}{2n+7}$ $\lim_{n \to \infty} \frac{5n}{2n+7} = \lim_{n \to \infty} \frac{5}{2+\frac{7}{n}} = \frac{5}{2} \neq 0$

$\Rightarrow$ series diverges.

Ex 3 Discuss the convergence or divergence of $1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{\text{two terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4 \text{ terms}} + \cdots$

$$+ \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{2^n \text{ terms}} + \cdots$$

THEOREM 8

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (Any number k).

Corollary:

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.