MA 103: Mid-sem (2024)

(Note: Except for the first problem, justify all steps that are relevant.)

(1) **(2 marks each)** Pick all of the correct answers out of the choices given for each of the questions below. **No justification required. But note that in some of them, there may be more than one answer that is correct.**

(a) The system of equations

$$\begin{aligned} x + 2y + 3z &= 4 \\ x + ky + 4z &= 6 \\ x + 2y + (k+2)z &= 6 \end{aligned}$$

is consistent when

(i) $k = 2$
(ii) $k = 1$
(iii) k is any real number other than 1 and 2
(iv) $k \neq 1$.

(b) $\mathbb{R}^3$ has no subspace of dimension 0.

(i) True
(ii) False.

(c) If $B = S^{-1}AS$ and x is an eigenvector of B corresponding to an eigenvalue λ, then Sx is an eigenvector of A corresponding to λ.

(i) True
(ii) False.

(d) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + 4y, 3x - y)$. If $B = \{(1, 0), (0, 1)\}$ and $C = \{(2, 1), (1, 2)\}$ are two different basis for $\mathbb{R}^2$, then the transition matrix $P_{B \to C}$ from B to C is

$$\text{(i)} \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 6 & 9 \\ 5 & 1 \end{bmatrix}$$

(e) The vectors $1 + 3x + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2$

(i) are linearly independent in P_2
(ii) form a basis for P_2
(iii) are linearly dependent in P_2.

(2) **(10 marks)** Apply the Gram-Schmidt orthogonalization process to the subset $\{(1,0,1),(0,1,1),(1,3,3)\}$ of the inner product space $\mathbb{R}^3$ (having usual dot product as inner product). Then find an orthonormal basis for $\mathbb{R}^3$.

(3) **(13 marks)** Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 + 2a_3 \\ a_1 + 4a_3 \end{bmatrix}.$$

(a) **(3 marks)** Determine the eigenvalues of the standard matrix corresponding to the above linear transformation.

(b) **(6 marks)** Find the eigenspace of the matrix corresponding to each eigenvalue.

(c) **(2 marks)** Determine the algebraic and geometric multiplicities of each eigenvalue.

(d) **(2 marks)** Is the standard matrix diagonalizable? If so, why?

(4) **(7 marks)** Let A be a 3×3 matrix and let X_1, X_2 and X_3 be vectors in $\mathbb{R}^3$. Show that if the vectors

$$Y_1 = AX_1, \quad Y_2 = AX_2, \quad Y_3 = AX_3.$$

are linearly independent, then the matrix A must be nonsingular and the vectors X_1, X_2 and X_3 must be linearly independent.

(5) **(5 marks)** An $m \times n$ matrix is said to have a *right inverse* if there exists an $n \times m$ matrix B such that $AB = I_m$. Show that if A has a right inverse, then the column vectors of A span $\mathbb{R}^m$.

(6) **(5 marks)** Let A and B be two $n \times n$ matrices. Show that if λ is a non-zero eigenvalue of AB, then it is also an eigenvalue of BA.