CHALLENGING PROBLEMS FROM THE TUTORIAL

C1 (a) $6, 3\sin^2 x, 2\cos^2 x$

$$2\cos^2 x = 2 - 2\sin^2 x \quad (\sin^2\theta + \cos^2\theta = 1)$$
$$= \frac{1}{3}(6) + \left(\frac{-2}{3}\right)(3\sin^2 x)$$

l.d.

(b) $x, \cos x$

by contradiction; suppose they are l.d.

$\exists\, c \in \mathbb{R} \ni \cos x = cx \ \forall x \in \mathbb{R}$

$$\underbrace{\cos(x+2\pi) = \cos x}$$
$$\Rightarrow c(x+2\pi) = cx$$
$$\Rightarrow 2\pi c = 0$$
$$\Rightarrow c = 0$$

$\rightarrow\leftarrow$ Hence $x, \cos x$ are l.i.

C2

Let the n distinct points be (x_i, y_i) $\quad 1 \le i \le n$

Let $p(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a poly. of deg. $n-1$ or less (so some or all of the a_i's could be zero).

Then $y_i = p(x_i) \ \forall i \ni 1 \le i \le n$.

Thus, we get the following linear system of equations:

$$a_{n-1}x_1^{n-1} + a_{n-2}x_1^{n-2} + \cdots + a_1x_1 + a_0 = y_1$$
$$a_{n-1}x_2^{n-1} + \cdots + a_1x_2 + a_0 = y_2$$
$$\vdots$$
$$a_{n-1}x_n^{n-1} + \cdots + a_1x_n + a_0 = y_n$$

$$\underbrace{\begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & & x_2 & 1 \\ \vdots & \vdots & & \vdots & 1 \\ x_n^{n-1} & x_n^{n-2} & & x_n & 1 \end{bmatrix}_{n\times n}}_{A} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Now the matrix A is the Vandermonde matrix whose determinant we know how to compute

Moreover, $x_i \neq x_j$ for $i \neq j$. So the determinant is non-zero which means the matrix is invertible.

This implies that the above system has the unique solution, i.e. a_i's are unique. So there is a unique poly. of deg. $n-1$ or less.

(The condition $n-1$ or less comes into place because say $n=3$, but the three points are collinear. Then instead of circle, there's a line (deg. 1 poly.) which passes through them.

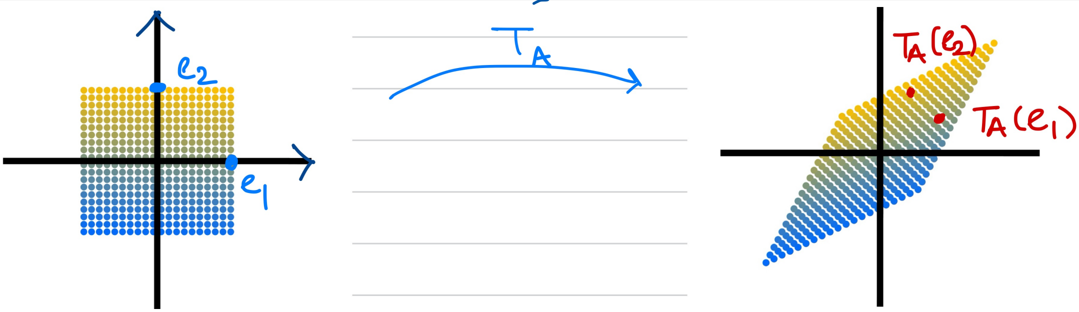
Chapter 5 – Eigenvectors and eigenvalues

Motivation :

Consider $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, $A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} \to A\begin{pmatrix} x \\ y \end{pmatrix}$

$T_A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$, $T_A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$



Grid of points forming a unit square of area 1

Image of grid forms a parallelogram of area ?

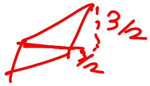
It is useful to view T_A using a different set of axes (different basis) on LHS.

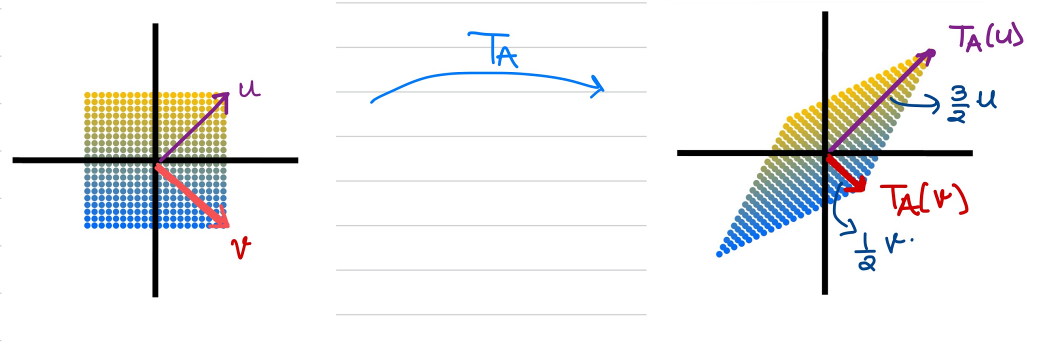
Eg. Take $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as a basis for $\mathbb{R}^2$,

$T_A(u) = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} = \frac{3}{2}u$

$T_A(v) = A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} = \frac{1}{2}v$

$\frac{3}{2}, \frac{3}{2}$





square of area 1

Parallelogram of area $3/4$.

What was special about u, v?

$T_A u = 1.5\,u$ $\qquad$ $T_A v = 0.5\,v$

Def. Let A be an $n \times n$ matrix. A vector $x \in \mathbb{R}^n$ is called an <u>eigenvector</u> of A if

- $x \neq 0$

& • Ax is a scalar multiple of x, i.e., $Ax = \lambda x$ for some $\lambda \in \mathbb{R}$

called eigenvector corresponding to λ

called an eigenvalue of A.

In the previous example, $A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ has eigenvalues 0.5 & 1.5.

An eigenvector corresponding to $\lambda = 0.5$ is $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

An eigenvector corresponding to $\lambda = 1.5$ is $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Ex. 3 λ is an eigenvalue of A, and x is an eigenvector of A corresponding to λ, then prove that kx, for any $k \in \mathbb{R}$, is also an eigenvector of A corresponding to λ.

Proof: $Ax = \lambda x$

$$A(kx) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} kx_1 \\ \vdots \\ kx_n \end{bmatrix}$$

$$= \begin{bmatrix} (ka_{11})x_1 + (ka_{12})x_2 + \cdots + (ka_{1n})x_n \\ \\ (ka_{m1})x_1 + ka_{m2}x_2 + \cdots + (ka_{mn})x_n \end{bmatrix}$$

$$= \begin{bmatrix} k(a_{11}x_1 + \cdots + a_{1n}x_n) \\ \vdots \\ k(a_{m1}x_1 + \cdots\cdots a_{mn}x_n) \end{bmatrix}$$

$$= k \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots\cdots + a_{mn}x_n \end{bmatrix}$$

$$= k(Ax)$$

$$= k(\lambda x)$$

$$= \lambda(kx)$$

$\Rightarrow kx$ is an eigenvector of A corr. to λ.

How to find eigenvectors/eigenvalues ?

Eg. For $A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, we want to get x & λ such that $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Leftrightarrow \underbrace{\begin{pmatrix} 1-\lambda & 1/2 \\ 1/2 & 1-\lambda \end{pmatrix}}_{A-\lambda I,\ \text{say},\ B.}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{——— } (*)$$

$(*)$ $Bx = 0$

If $\det(B) \neq 0$, we get a unique sol^n. $x = B^{-1}0 = 0$
$\Rightarrow$ no eigenvector

If $\det(B) = 0$, we get inf. many sol^{n}s.
$\Rightarrow$ we get eigenvectors

$\Rightarrow$ Following is the procedure:

① To get eigenvalues of A, find roots of the poly. eqn. $\underbrace{\det(A - \lambda I) = 0}_{\text{called the characteristic eqn.}}$

② To get the eigenvectors corresponding to λ, find solutions of $(A - \lambda I)x = 0$, i.e; find nullspace of $A - \lambda I$.

Ex. 4 : Find eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Ans. Solve $\underbrace{\det(A-\lambda I)=0}_{\text{characteristic eqn.}}$ to get the eigenvalues λ.

$$\left| \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(-\lambda(8-\lambda)+17) - 1(-4) = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 8\lambda + 17) + 4 = 0$$

$$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \leftarrow \text{charac. eqn.}$$

$$(\lambda^3 - 8\lambda^2 + 16\lambda) + (\lambda - 4) = 0$$

$$\Rightarrow \lambda(\lambda^2 - 8\lambda + 16) + (\lambda - 4) = 0$$

$$\Rightarrow \lambda(\lambda-4)^2 + (\lambda-4) = 0$$

$$\Rightarrow (\lambda-4)(\lambda^2 - 4\lambda + 1) = 0$$

$$\Rightarrow \lambda = 4 \text{ or } \lambda^2 - 4\lambda + 1 = 0$$

$$\Rightarrow \lambda = 4 \text{ or } \lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$$\therefore \underline{\lambda = 4} \text{ or } = \underline{2 \pm \sqrt{3}}$$

Sect. 5.1 – Diagonalization

Let A be an $n \times n$ matrix.

- We say $\lambda \in \mathbb{R}$ is an eigenvalue of A if $\exists \underset{x \neq 0}{x \in \mathbb{R}^n}$ s.t. $Ax = \lambda x$

- Such $x \in \mathbb{R}^n$ is called an eigenvector of A corresponding to λ.

Def. Let $\lambda \in \mathbb{R}$ be an eigenvalue of A. Th set $\{$all eigenvectors of A corresponding to $\lambda\}$ is called the eigenspace of A corresponding to λ & is denoted by E_λ. That is,

$$E_\lambda = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \setminus \{0\} : A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\}.$$

We know that if λ is an eigenvalue of A, then it satisfies $\det(A - \lambda I) = 0$.

Def: The number of times λ appears as a root of $\det(A - \lambda I) = 0$ is called the algebraic multiplicity of λ.

e.g. $A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$

Eigenvalues: Solve $\det(A - xI) = 0$

$$\Rightarrow \begin{vmatrix} -1-x & 3 \\ 2 & 0-x \end{vmatrix} = 0$$

$$\Rightarrow x(1+x) - 6 = 0$$

$$\Rightarrow x^2 + x - 6 = 0$$

$$\Rightarrow (x+3)(x-2) = 0$$

Roots: $-3, 2$ are the eigenvalues

Both have algebraic multiplicity 1.

Eigenspace

Consider E_{-3} (eigenspace corresponding to -3)

$$E_{-3} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} : A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}$$

we want $(A + 3I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\Rightarrow \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 2x_1 + 3x_2 = 0 \\ 2x_1 + 3x_2 = 0 \end{array} \qquad \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$=$

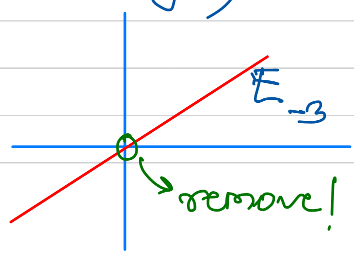
Thus,

$$E_{-3} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{array}{c} 2x_1 + 3x_2 = 0 \\ (x_1, x_2) \neq (0,0) \end{array} \right\} \to \text{line in } \mathbb{R}^2$$

(without the origin)

Let $x_2 = t$, then $x_1 = -\frac{3}{2}t$.

$$\Rightarrow E_{-3} = \left\{ \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} t : t \in \mathbb{R} \setminus \{0\} \right\}$$



Here, $\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ is said to be a basis for the eigenspace E_{-3} since

① $\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ is l.i.

② any vector in E_{-3} is a l.c. of $\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$.

• Let A be an $n \times n$ matrix with eigenvalue λ.

Def: We say that $\{v_1, v_2, \ldots, v_r\} \subseteq \mathbb{R}^n$ is a basis for the eigenspace E_λ if

(i) $v_1, v_2, \ldots, v_r$ are l.i. eigenvectors of A of A corresponding to λ.

(ii) every eigenvector of A corresponding to λ can be written as a l.c. of $v_1, \ldots, v_r$.

• Here, r is called the $\underbrace{\text{geometric multiplicity}}_{GM}$ of λ.

In the previous eg., GM of $\lambda = -3$ is 1.

Eg. 1 Find a basis for each eigenspace of $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

Ans. (i) Eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda((2-\lambda)(3-\lambda)) - 2(-(2-\lambda)) = 0$$

$$\Rightarrow \quad \lambda((\lambda-2)(\lambda-3)) + 2(\lambda-2) = 0$$

$$\Rightarrow \quad (\lambda-2)(\lambda^2 - 3\lambda + 2) = 0$$

$$\Rightarrow \quad (\lambda-2)^2(\lambda-1) = 0$$

$$\Rightarrow \quad \lambda = 1, 2$$

Let

$\lambda_1 = 1, \quad \lambda_2 = 2.$

(ii) Eigenspace E_{λ_1} :

$$E_{\lambda_1} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \setminus \{0\} : Ax = \lambda_1 x \right\}$$

Solve : $(A - I)x = 0$ $\quad \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \Leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Longrightarrow \quad x_2 - x_3 = 0, \qquad x_1 + 2x_3 = 0$

Let $x_3 = t$, $x_2 = t$, $x_1 = -2t$

(free variable)

$$\Rightarrow E_{\lambda_1} = \left\{ \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} : t \in \mathbb{R} \setminus \{0\} \right\}$$

$$= \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} t : t \in \mathbb{R} \setminus \{0\} \right\}$$

basis vector

$\Longrightarrow \quad GM(\lambda_1) = 1,$

$$E_{\lambda_2} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \setminus \{0\} : Ax = \lambda_2 x \right\}$$

Solve $(A - 2I)x = 0$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Aug. mat. : $\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 + 2R_1, \quad R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$, $\quad x_2$ is free, say t

$x_1 = -x_3$

Let $x_3 = r$, then $x_1 = -r$

$$E_{\lambda_2} = \left\{ \begin{pmatrix} -r \\ t \\ r \end{pmatrix} : (r, t) \in \mathbb{R}^2 \setminus \{0, 0\} \right\}$$

$$= \left\{ r\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : (r, t) \in \mathbb{R}^2 \setminus \{0, 0\} \right\}$$

2 basis vectors

$GM(\lambda_2) = 2$

Sect. 5.2 — Diagonalization

AIM: Given an $n \times n$ matrix A, we want a diagonal matrix D s.t. A is "similar" to D, that is, $A = PDP^{-1}$ for some invertible matrix P.

If this happens, we say A is diagonalizable.

Why do we care?

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Easy to get A^{1000}:

$$\begin{aligned} A^{1000} &= (PDP^{-1})^{1000} \\ &= (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1}) \\ &= \underbrace{PD^{1000}P^{-1}}_{\text{easy to compute!}} \end{aligned}$$

$$\begin{aligned} \det(A) &= \det(P)\det(D)\det(P^{-1}) \\ &= \det(D) \longrightarrow \text{again, easy to compute} \end{aligned}$$

CONNECTION BETWEEN DIAGONALIZABLE MATRICES AND EIGENVECTORS

Thm. 5.2.1

$A_{n \times n}$ is diagonalizable $\Longleftrightarrow$ A has n l.i. eigen vectors

Proof: A is diagonalizable

$\Rightarrow A = PDP^{-1}$ for some P (invertible)

$\Rightarrow AP = PD$

Let columns of $P_{n\times n}$ be $v_1, v_2, \ldots, v_n$.

$$\Rightarrow A\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

By 3rd rule of multiplying two matrices,

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} a\lambda_1 \\ d\lambda_1 \\ g\lambda_1 \end{pmatrix}$$

$$LHS = AP = \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix}$$

& $RHS = P\begin{pmatrix} \lambda_1 & & 0 \\ \vdots & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ has its j^{th} column as

$$P\begin{pmatrix} 0 \\ \vdots \\ \lambda_j \\ \vdots \\ 0 \end{pmatrix} = \lambda_j v_j \quad \text{(check!)}$$

$\Rightarrow Av_j = \lambda_j v_j \ \forall j$.

$\Rightarrow$ each v_j is an eigenvector of A corresponding to the eigenvalue λ_j.

$\Rightarrow$ columns of P are eigenvectors of A.

$\exists$ n columns & they are l.i. because P is invertible (why?)

$\Rightarrow$ A has n l.i. eigenvectors.

"$\Leftarrow$" can be similarly proved

A Procedure for Diagonalizing an $n \times n$ Matrix

Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n, then it is not.

Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$ whose column vectors are the n basis vectors you obtained in Step 1.

Step 3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that correspond to the successive columns of P.

Eg:

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$