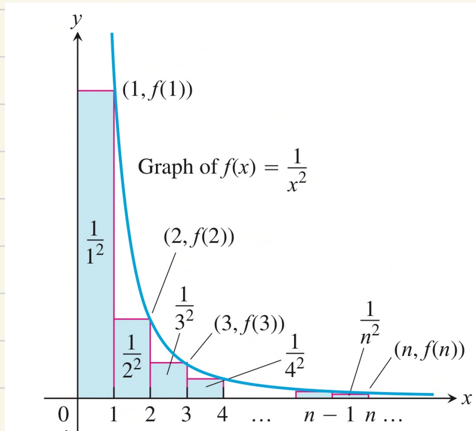
The integral test

Explanation of the concept thro' an example

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$

& compare it with

$\int_1^{\infty} \frac{1}{x^2}\, dx$.



Note that

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}$$

$$= f(1) + f(2) + f(3) + \ldots + f(n), \quad \text{where } f(x) = \frac{1}{x^2}$$

$$< f(1) + \int_1^n \frac{dx}{x^2}$$

$$< f(1) + \int_1^{\infty} \frac{dx}{x^2}$$

$$< f(1) + 1$$

$$= 1 + 1$$

$$= 2.$$

Hence the partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are bounded from above.

Hence from above Cor; we see that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

THEOREM 9 The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\, dx$ both converge or both diverge.

Ex.1 Show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p \in \mathbb{R}$, converges if $p > 1$ and diverges if $p \leq 1$.

Case 1 : $p = 1$: Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Case 2 : $p > 1$. Let $f(x) = \frac{1}{x^p}$.

Then f is continuous, positive & decreasing fn. on $[1, \infty)$. Hence by integral test.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ & $\int_1^{\infty} \frac{1}{x^p}\, dx$ both converge or diverge.

But

$\int_1^{\infty} \frac{1}{x^p}\, dx$ converges for $p > 1$.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

Case 3 : $p < 1$.

Since $\int_1^{\infty} \frac{1}{x^p}\, dx$ diverges for $p < 1$, so does

$\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Example Discuss the convergence/divergence of $\sum_{n=1}^{\infty} e^{-n^2}$.

Soln: Let $f(x) = e^{-x^2}$. Then f is continuous, positive and decreasing function of x for $x \geq 1$. The latter follows from the fact that $f'(x) = -2xe^{-x^2} < 0$.

$\sum_{n=1}^{\infty} a_n$, where $a_n = e^{-n^2}$ can now be investigated using the integral test, by comparing it with $\int_1^{\infty} e^{-x^2}\,dx$.

Note that for $x \geq 1$, $x^2 \geq x$ & hence $-x^2 \leq -x$ & hence, $e^{-x^2} \leq e^{-x}$

So by comparison test, since $0 \leq e^{-x^2} < e^{-x}$, & since $\int_1^{\infty} e^{-x^2}\,dx \leq \int_1^{\infty} e^{-x}\,dx$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_1^b = e^{-1} < \infty$$

Hence $\int_1^{\infty} e^{-x^2}\,dx$ converges

& thus $\sum_{n=1}^{\infty} e^{-n^2}$ converges by integral test.

Sect. 11.4 – Comparison tests

THEOREM 10 The Comparison Test
Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.

(b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer N.

Example (a) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$. (b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

(a) $$\sum_{n=1}^{\infty} \frac{5}{5n-1} = \sum_{n=1}^{\infty} \frac{1}{n - 1/5}$$

Observe that $n - \frac{1}{5} \leq n \implies \frac{1}{n-5} \geq \frac{1}{n}$

But since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by comparison test, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$ also diverges.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$: Note that $n! \geq 2^{n-1}$ for $n \geq 1$

(Proof: $1! \geq 2^{1-1}$. ✓ $n+1 \geq 2$
Assume $n! \geq 2^{n-1}$. Then
$(n+1)! = (n+1) \cdot n! \geq (n+1)\, 2^{n-1} \geq 2 \cdot 2^{n-1} = 2^n$)

Hence $$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \leq 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$
$$= 1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1 - 1/2} = 1 + 2 = 3.$$

Hence $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} \leq 3$. Hence $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges,

The Limit comparison test

THEOREM 11 **Limit Comparison Test**

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof: We will prove only part 1.

Since $c > 0$, so is $c/2$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ implies $\exists\, N \in \mathbb{N} \ni \forall\, n > N$,

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2} \Rightarrow -\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}$$

$$\Rightarrow \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}$$

$$\Rightarrow \left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n \quad \text{———} \ (*)$$

If $\sum b_n$ converges, then so does $\frac{3c}{2} \sum b_n$ & hence $\sum a_n$ converges by 2^{nd} ineq. of $(*)$ & direct comparison test. If $\sum b_n$ diverges, so does $\left(\frac{c}{2}\right) \sum b_n$ & hence $\sum a_n$ diverges (using the 1^{st} inequality in $(*)$ & dir. comp. test).

Example: Discuss the convergence/divergence of $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$.

Sol^n:

One cannot use the n^{th} term test since $\lim_{n\to\infty} \frac{2n+1}{n^2+2n+1} = 0$.

We also show this by limit comparison test.

$\frac{2n+1}{n^2+2n+1} \sim \frac{2n}{n^2} = \frac{2}{n}$ as $n\to\infty$.

Let $b_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} \left(\frac{\frac{2n+1}{n^2+2n+1}}{\left(\frac{1}{n}\right)} \right) = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1}$

$= \lim_{n\to\infty} \frac{2+\frac{1}{n}}{1+\frac{2}{n}+\frac{1}{n^2}} = 2 > 0.$

But since $\sum b_n$ diverges (harmonic series) so does $\sum a_n$ (by limit comparison test).

Ex.2 Use limit comparison test to discuss convergence/divergence of

ⓐ $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ $\qquad$ ⓑ $\sum_{n=1}^{\infty} \frac{1+n\ln(n)}{n^2+5}$

Ex.2 Use limit comparison test to discuss convergence/divergence of

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ (b) $\sum_{n=1}^{\infty} \frac{1+n\ln(n)}{n^2+5}$

(a) $a_n = \frac{1}{2^n - 1}$, $b_n = \frac{1}{2^n}$ (for large enough n, $\frac{1}{2^n-1}$ is almost equal to $\frac{1}{2^n}$)

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges $\left(\because \frac{1}{2} < 1\right)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^n - 1} \cdot 2^n = \lim_{n\to\infty} \frac{1}{\left(1 - \frac{1}{2^n}\right)}$$
$$= 1$$

By limit comparison test, since $\sum b_n$ converges so does $\sum a_n$.

(b) $\sum_{n=1}^{\infty} \frac{1+n\ln n}{n^2+5}$ $a_n = \frac{1+n\ln(n)}{n^2+5}$

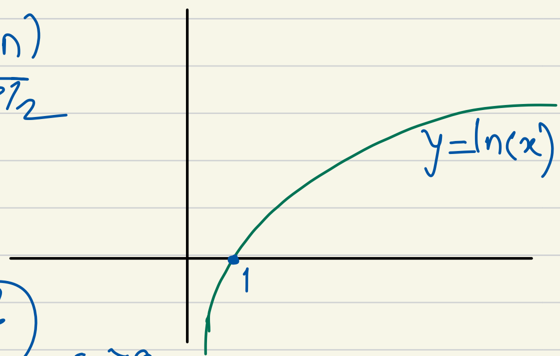
$b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+n\ln(n)}{n^2+5} \cdot n = \lim_{n\to\infty} \frac{n^2\ln(n)+n}{n^2+5}$$

$$= \lim_{n\to\infty} \frac{\ln(n) + \frac{1}{n}}{1 + 5/n^2} = \infty$$

So since $\sum b_n$ diverges, by comparison test, $\sum_{n=1}^{\infty} a_n$ also diverges.

$$\frac{\ln(n)}{n^{3/2}}$$



x^c

$c > 0$

$$\lim_{x \to \infty} \frac{\log x}{x^c} = \lim_{x \to \infty} \frac{1/x}{c x^{c-1}}$$

$$= 0$$

$c = 0.01$

$x^{c-1} = x^{-0.99}$

$$\frac{\frac{1}{x}}{x^{c-1}} = \frac{\frac{1}{x}}{\frac{1}{x^{0.99}}}$$

$$= \frac{1}{x^{0.01}} \to 0$$

Example: Does $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$ converge?

Sol^n: $\ln(n)$ grows slower than n^c, any $c>0$

Hence for a sufficiently large n, (i.e. $n > N$ for N very large),

$$\frac{\ln(n)}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}.$$

Take $a_n = \frac{\ln(n)}{n^{3/2}}$, $b_n = \frac{1}{n^{5/4}}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n)}{n^{3/2}} \cdot n^{5/4}$$

$$= \lim_{n\to\infty} \frac{\ln(n)}{n^{1/4}}$$

Now $\lim_{x\to\infty} \frac{\ln(x)}{x^{1/4}}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{x\to\infty} \frac{1/x}{\frac{1}{4}x^{-3/4}} = 4 \lim_{x\to\infty} \frac{1}{x^{1/4}} = 0$$

Hence $\lim_{n\to\infty} \frac{\ln(n)}{n^{1/4}} = 0$.

Thus, by limit comparison test & the fact that $\sum \frac{1}{n^{5/4}}$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$ also converges.

Sect. 11.5 – The ratio and root tests

- Nature of ratio test : This is a powerful technique which determines the growth or decline of a series by looking at $\frac{a_{n+1}}{a_n}$; it's an extension of the result on convergence or divergence of a geometric series.

THEOREM 12 The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

Example: Use the ratio test to determine convergence/divergence of

① $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ ② $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

③ $\sum_{n=1}^{\infty} \frac{4^n \cdot (n!)^2}{(2n)!}$

① $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ Let $a_n = \frac{n^{10}}{10^n}$.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^{10}}{10^{n+1}}\cdot\frac{10^n}{n^{10}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^{10}}{10\,n^{10}}$$

$$= \frac{1}{10}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{10}$$

$$= \frac{1}{10}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{10}$$

$$= \frac{1}{10}\cdot 1 = \frac{1}{10} < 1$$

Hence by ratio test, $\displaystyle\sum_{n=1}^{\infty}\frac{n^{10}}{10^n}$ converges.

(2)

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+2)!}{((n+1)!)^2}\cdot\frac{(n!)^2}{(2n)!}$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2} = \lim_{n\to\infty}\frac{\cancel{n^2}\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}$$

$$= 4 > 1$$

By ratio test, $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

③ $\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4^{n+1}((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{4^n (n!)^2}$

$= 4 \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = 1.$

The ratio test is inconclusive.

But

$\frac{a_{n+1}}{a_n} = \frac{2(n+1)}{2n+1} = \frac{2n+2}{2n+1} > 1$

$a_{n+1} > a_n > a_{n-1} > a_{n-2} \cdots\cdots > a_1 = 2$

this implies that $a_n \not\to 0$ as $n\to\infty$.

Hence $\sum a_n$ diverges.

Failure of ratio test in certain cases

Let $a_n = \begin{cases} n/2^n, & n \text{ odd}, \\ 1/2^n, & n \text{ even}. \end{cases}$

Does $\sum a_n$ converge?

$$\frac{a_{n+1}}{a_n} = \begin{cases} \left(\dfrac{1/2^{n+1}}{n/2^n}\right), & n \text{ odd} \\ \dfrac{(n+1)/2^{n+1}}{1/2^n}, & n \text{ even} \end{cases}$$

$$= \begin{cases} \dfrac{1}{2n}, & n \text{ odd} \\ \dfrac{n+1}{2}, & n \text{ even} \end{cases}$$

Note that $\lim\limits_{\substack{n\to\infty \\ n \text{ odd}}} \frac{1}{2n} = 0$ & $\lim\limits_{\substack{n\to\infty \\ n \text{ even}}} \frac{n+1}{2} = \infty$.

Hence we cannot use ratio test.