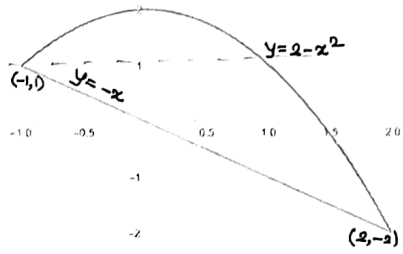
MA 103: End-sem (2024)

(1) **(2 points each)** Pick the correct answer out of the choices given for each of the questions below, or say whether True or False. **No justification required. No partial marking.**

(a) The average value of $f(x) = \tan(x)$ on $[0, \pi/4]$ is ....

(i) $4/\pi$.
(ii) $\frac{2}{\pi}\log(2)$.
(iii) $\frac{1}{2}\log(2)$.
(iv) 1.

(b) The region enclosed by the parabola $y = 2 - x^2$ and $y = -x$ is shown in the figure below. The area of this region is given by



(i) $\int_{-1}^{2}(\sqrt{2-x}+x)\,dx$.
(ii) $\int_{-2}^{1}(\sqrt{2-y}+y)\,dy + \int_{-1}^{1}(1-x^2)\,dx$.
(iii) $\int_{-2}^{2}(\sqrt{2-y}+y)\,dy$.
(iv) None of these.

(c) If $a_n \not\to 0$, then the series $\sum_{n=1}^{\infty} a_n$ does not converge absolutely.

(i) True
(ii) False

(d) The Lagrange mean value theorem is a special case of Taylor's theorem.

(i) True
(ii)False

(e) Let u_n be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{99}{100}$ Then $\lim_{n\to\infty} u_n = \ldots$.

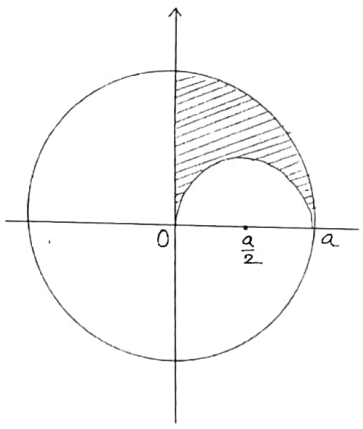
(i) does not exist.
(ii) 0.
(iii) 1.

(2) **(4 points each)** Which of the following series converge or diverge? **Justify your answer in detail.**

(a) $\sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2}$ (b) $\sum_{n=1}^{\infty} \frac{\log(n)}{n^2}$ (c) $\sum_{n=1}^{\infty} \frac{1}{1+\log^2(n)}$.

(3) **(7 points)** Let $q \in \mathbb{R}$. For which values of q does the series $\sum_{n=1}^{\infty} q^{n^2}$ converge? For which values does it diverge? **Justify your answer.**

(4) **(8 points)** Find the area of the shaded region bounded by the circle, semi-circle and the y-axis as shown in the figure below.



(5) (a) **(7 points)** Show that the Taylor series for $\cos(x)$ at $x = 0$ converges to $\cos(x)$ for every value of x.

(b) **(3 points)** Give the explicit Taylor series representation of $\cos^2(x)$ at the point $x = 0$.

(6) (a) **(5 points)** Evaluate the limit

$$\lim_{x \to 0} \frac{\int_0^x \sqrt{\tan(t)}\, dt}{x^{3/2}}.$$

(b) **(8 points)** Evaluate the limit

$$\lim_{x \to 0} \frac{2^x - 2^{\sin(x)}}{x(1 - \cos(x))}.$$

(**Hint:** You may want to do it using power series.)