MA 103 — Mid-sem (2024) — Solutions

① (a) (i), (iii), (iv)

(b) (ii) (False)

(c) (i) (True)

(d) (i)

(e) (iii)

② $w_1 = (1,0,1)$
$w_2 = (0,1,1)$
$w_3 = (1,3,3)$

Note that $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 3 \end{vmatrix} \neq 0$, so w_1, w_2, w_3 are linearly independent.

Step 1 Take $v_1 = w_1 = (1,0,1)$

Step 2 (i) $v_2 = w_2 - \text{proj}_{v_1} w_2$

$$= (0,1,1) - \frac{\langle v_1, w_2 \rangle}{\|v_1\|^2} v_1$$

$$= (0,1,1) - \frac{\{(1)(0)+(0)(1)+(1)(1)\}}{(1^2+0^2+1^2)} (1,0,1)$$

$$= (0,1,1) - \frac{1}{2}(1,0,1) = \left(\frac{-1}{2}, 1, \frac{1}{2}\right).$$

(ii) $v_3 = w_3 - \text{proj}_{\text{span}\{v_1,v_2\}} w_3$

$$= w_3 - \left(\frac{\langle v_1, w_3 \rangle}{\|v_1\|^2} v_1 + \frac{\langle v_2, w_3 \rangle}{\|v_2\|^2} v_2 \right)$$

$$= (1,3,3) - \left(\frac{\{(1)(1)+(0)(3)+(1)(3)\}}{(1^2+0^2+1^2)}(1,0,1) + \frac{\{(-\frac{1}{2})(1)+(1)(3)+(\frac{1}{2})(3)\}}{(-\frac{1}{2})^2+1^2+(\frac{1}{2})^2}\left(-\frac{1}{2}, 1, \frac{1}{2}\right) \right)$$

$$= (1,3,3) - \left(\frac{4}{2}(1,0,1) + \frac{4}{(3/2)}\left(-\frac{1}{2}, 1, \frac{1}{2}\right) \right)$$

$$= (1,3,3) - \left((2,0,2) + \left(\frac{-4}{3}, \frac{8}{3}, \frac{4}{3}\right) \right)$$

$$= \left(\frac{1}{3}, \frac{1}{3}, \frac{-1}{3}\right) = \frac{1}{3}(1, 1, -1).$$

Hence orthogonal basis is $\left\{(1,0,1), \left(-\frac{1}{2}, 1, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)\right\}$

The corresponding orthonormal basis is

$$\left\{ \frac{(1,0,1)}{\sqrt{2}}, \frac{\left(-\frac{1}{2}, 1, \frac{1}{2}\right)}{\sqrt{3/2}}, \frac{\left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)}{\sqrt{1/3}} \right\}$$

$$= \left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \right\}.$$

3. The standard matrix corresponding to the transformation

$$T\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 + 2a_3 \\ a_1 + 4a_3 \end{bmatrix}$$

is $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$; call it A.

(a) Eigenvalues are the solutions of
$\det(A - \lambda I) = 0.$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)^2(3-\lambda) - (3-\lambda) = 0$$

$$\Rightarrow (3-\lambda)(16 - 8\lambda + \lambda^2 - 1) = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 - 8\lambda + 15) = 0$$

$$\Rightarrow -(\lambda-3)(\lambda-3)(\lambda-5) = 0$$

$$\Rightarrow (\lambda-3)^2(\lambda-5) = 0$$

$\Rightarrow$ Eigenvalues are $\lambda_1 = 3,\ \lambda_2 = 5.$ — (I)

(b) (i) $E_{\lambda_1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \setminus \{0\} : A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\}$

Now $(A - 3I) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ x_2 is a free variable, say t.

Moreover $x_1 + x_3 = 0$

$\Rightarrow x_1 = -x_3$.

Let $x_3 = s$, then $x_1 = -s$

Hence $E_{\lambda_1} = \left\{ \begin{pmatrix} -s \\ t \\ s \end{pmatrix} \in \mathbb{R}^3 : (s, t) \in \mathbb{R}^2 \setminus \{0, 0\} \right\}$

$= \left\{ s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : (s, t) \in \mathbb{R}^2 \setminus \{0, 0\} \right\}$ ——— I

(ii) $E_{\lambda_2} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \setminus \{0\} : A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 5\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\}$.

$(A - 5I)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$R_1 \to -R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to \frac{R_2}{-2}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{(REF form)}$$

$\Rightarrow x_1 - x_3 = 0$

$x_2 - 2x_3 = 0$

$\Rightarrow$ If $x_3 = t$, say, then $x_1 = t$, $x_2 = 2t$

$$\Rightarrow E_{\lambda_2} = \left\{ \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} \in \mathbb{R}^3 : t \neq 0 \right\}$$

$$= \left\{ t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^3 : t \neq 0 \right\}$$

III

(c) Algebraic multiplicities of λ_1 & λ_2:
From (I),

$AM(\lambda_1) = 2$, $AM(\lambda_2) = 1$.

Geometric multiplicities of λ_1 & λ_2:
From (II) & (III),

$GM(\lambda_1) = 2$, $GM(\lambda_2) = 1$
(2 basis vectors) (1 basis vector)

(d) From (c), since $AM(\lambda_i) = GM(\lambda_i)$, for $i = 1, 2$,
we conclude that A is diagonalizable.

(4) It is given that $Y_i = AX_i$, $1 \le i \le 3$ are linearly independent.

Note that since A is 3×3 matrix & X_i's are 3×1 matrices, Y_i's are 3×1 matrices.

Arrange them in 3 columns so as to get a 3×3 matrices, i.e., consider

$$\begin{bmatrix} | & | & | \\ Y_1 & Y_2 & Y_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ AX_1 & AX_2 & AX_3 \\ | & | & | \end{bmatrix}$$

Then observe that

$$\begin{bmatrix} | & | & | \\ Y_1 & Y_2 & Y_3 \\ | & | & | \end{bmatrix} = A \begin{bmatrix} | & | & | \\ X_1 & X_2 & X_3 \\ | & | & | \end{bmatrix} \longrightarrow (*)$$

Justification: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

& $X_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}$, then

$$AX_1 = \begin{bmatrix} a_{11}x_{11} + a_{12}x_{12} + a_{13}x_{13} \\ a_{21}x_{11} + a_{22}x_{12} + a_{23}x_{13} \\ a_{31}x_{11} + a_{32}x_{12} + a_{33}x_{13} \end{bmatrix},$$

which is the first column of $\begin{bmatrix} | & | & | \\ AX_1 & AX_2 & AX_3 \\ | & | & | \end{bmatrix}$.

Now $\det \begin{bmatrix} | & | & | \\ Y_1 & Y_2 & Y_3 \\ | & | & | \end{bmatrix} \neq 0$ (since Y_i's are linearly independent)

From ⊛, $\det\left(A \begin{bmatrix} | & | & | \\ X_1 & X_2 & X_3 \\ | & | & | \end{bmatrix}\right) \neq 0$

$\Rightarrow \det(A) \cdot \det\left(\begin{bmatrix} | & | & | \\ X_1 & X_2 & X_3 \\ | & | & | \end{bmatrix}\right) \neq 0$

$\Rightarrow \det(A) \neq 0$ & $\det\left(\begin{bmatrix} | & | & | \\ X_1 & X_2 & X_3 \\ | & | & | \end{bmatrix}\right) \neq 0$

Thus A is non-singular & X_i's are linearly independent.

(5) It is given that A has a right inverse. So $\exists\, B_{n\times m} \ni AB = I_m$.

Let b be any vector in $\mathbb{R}^m$.
If we show there exists a vector $x \in \mathbb{R}^n$ such that $Ax = b$, then by a theorem done in one of the lectures, b will lie in the column space of A, and hence the column vectors of A will span $\mathbb{R}^m$, and we will be done.

To that end, note that if $x = Bb$ (this makes sense since $B_{n\times m}$ & $b_{m\times 1}$, so $x \in \mathbb{R}^n$), then

$$\begin{aligned} Ax &= A(Bb) \\ &= (AB)b \\ &= I_{m\times m}\, b_{m\times 1} \\ &= b. \end{aligned}$$

$\blacksquare$

(6) We are given that λ is a non-zero eigenvalue of AB.
Let x be an eigenvector corresponding to λ, that is,

$$ABx = \lambda x$$

Pre-multiply both sides by B so that

$$B(ABx) = B(\lambda x)$$

$$\Rightarrow \underbrace{(BA)(Bx)}_{\text{Associativity of matrices}} = \lambda(Bx)$$

Thus Bx is an eigenvector corresponding to λ, and so λ is also an eigenvalue of BA.

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