Q2. Solved example 6 on p. 624 of the text. (It's important to mention that the natural logs have to combined before letting $b \to \infty$; it's initially in the $\infty - \infty$ form)

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx = \lim_{b\to\infty}\int_2^b \frac{x+3}{(x-1)(x^2+1)}\,dx$$

$$= \lim_{b\to\infty}\int_2^b \left(\frac{2}{x-1} - \frac{2x+1}{x^2+1}\right)dx \qquad \text{Partial fractions}$$

$$= \lim_{b\to\infty}\Big[2\ln(x-1) - \ln(x^2+1) - \tan^{-1}x\Big]_2^b$$

$$= \lim_{b\to\infty}\left[\ln\frac{(x-1)^2}{x^2+1} - \tan^{-1}x\right]_2^b \qquad \text{Combine the logarithms.}$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{(b-1)^2}{b^2+1}\right) - \tan^{-1}b\right] - \ln\left(\frac{1}{5}\right) + \tan^{-1}2$$

$$= 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1}2 \approx 1.1458$$

Q 5. (a) $$\int_1^2 \frac{dx}{x(\ln x)^p} = \lim_{a\to 1^+} \int_a^2 \frac{dx}{x(\ln x)^p}.$$

Case $p \neq 1$: Let $\ln x = t$.

Then $$\lim_{a\to 1^+} \int_a^2 \frac{dx}{x(\ln x)^p} = \begin{cases} \frac{(\ln(2))^{1-p}}{1-p}, & \text{if } p<1 \\ \infty, & \text{if } p>1. \end{cases}$$

Case $p=1$: $$\lim_{a\to 1^+} \int_a^2 \frac{dx}{x(\ln x)}$$

$$= \lim_{a\to 1^+} \Big[\ln|\ln(x)|\Big]_a^2$$

$$= \infty.$$

Hence the integral converges when $p<1$ & diverges for $p \geq 1$.

(b) converges for $p>1$ & diverges for $p \leq 1$.

Q 6. $$\int_0^\infty \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{2x\,dx}{x^2+1}$$

Let $x^2+1 = t$ so that $2x\,dx = dt$

$$= \lim_{b\to\infty} \int_1^{b^2+1} \frac{dt}{t} = \lim_{b\to\infty} \left[\ln|t|\right]_1^{b^2+1}$$

$$= \lim_{b\to\infty} \ln(b^2+1) = \infty$$

Hence $\int_0^\infty \frac{2x\,dx}{x^2+1}$ diverges.

Since $\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1} = \int_{-\infty}^{0} \frac{2x\,dx}{x^2+1} + \int_0^\infty \frac{2x\,dx}{x^2+1}$,

$\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\,dx$ also diverges.

But $\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left\{ \int_{-b}^{0} \frac{2x\,dx}{1+x^2} + \int_0^b \frac{2x\,dx}{1+x^2} \right\}$

$$= \lim_{b\to\infty} \left\{ \left[\ln|t|\right]_{b^2+1}^{1} + \left[\ln|t|\right]_1^{b^2+1} \right\}$$

$$= \lim_{b\to\infty} (0) = 0.$$

Q7.

$$\int_0^1 \frac{dt}{t-\sin t}$$

Let $f(t) = \frac{1}{t}$, $g(t) = \frac{1}{t-\sin t}$

Since $\sin t > 0$ for $0 < a < 1$,

$$t - \sin t < t$$

$$\Rightarrow \frac{1}{t} < \frac{1}{t - \sin t}$$

$$\Rightarrow \int_a^1 \frac{1}{t}\,dt < \int_a^1 \frac{dt}{t - \sin t}$$

$$\Rightarrow \lim_{a \to 0^+} \int_a^1 \frac{1}{t}\,dt \leq \lim_{a \to 0^+} \int_a^1 \frac{dt}{t - \sin t}.$$

But $\displaystyle\lim_{a \to 0^+} \int_a^1 \frac{1}{t}\,dt$ diverges,

so $\displaystyle\lim_{a \to 0^+} \int_a^1 \frac{dt}{t - \sin t} = \int_0^1 \frac{dt}{t - \sin t}$ diverges too.

MA 103 Tutorial 5 (SVC)

Q 1] Theorem 6 tells that a nondecreasing seq. of real numbers converges iff it is bounded from above.

Let $\{a_n\}$ be a non-increasing sequence that is bounded from below, that is $\exists\, m \in \mathbb{R} \ni$

$$m \le a_n \quad \forall\, n \in \mathbb{N}.$$

Then $\{-a_n\}$ is a non-decreasing seq. bdd. from above, hence converges to L, say.

Then $\lim\limits_{n\to\infty} a_n = -L$.

If $\{a_n\}$ is a non-increasing seq., not bounded below, then for any $M \in \mathbb{N}$, $\exists\, N \in \mathbb{N}$ (N depending on M) $\ni \forall\, n > N$,

$$a_n < -M$$

This, along with the fact that $a_{n+1} \le a_n$ $\forall\, n \in \mathbb{N}$ (in particular, for $n > N$) implies that $\{a_n\}$ diverges.

Q2 (a) To show
$$\ln(n+1) \leq 1+\frac{1}{2}+\cdots+\frac{1}{n} \leq 1+\ln(n).$$

Proof: $\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,dx$

$$= \int_1^{n} \frac{1}{x}\,dx + \int_n^{n+1} \frac{1}{x}\,dx \quad \text{—— (1)}$$

$$\leq \log n + \frac{1}{n} \quad \left(\because n \leq x \Rightarrow \frac{1}{x} \leq \frac{1}{n} \Rightarrow \int_n^{n+1} \frac{1}{x}\,dx \leq \frac{1}{n}\right)$$

Applying inductively, we see that
$$\ln(n+1) \leq \log(n-1) + \frac{1}{n-1} + \frac{1}{n}$$
$$\vdots$$
$$\leq 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$\Rightarrow \boxed{\ln(n+1) \leq 1+\frac{1}{2}+\cdots+\frac{1}{n}} \quad \text{—— (2)}$$

Also, from (1),
$$\ln(n+1) \geq \log n + \frac{1}{n+1} \quad \left(\because x \leq n+1 \Rightarrow \frac{1}{x} \geq \frac{1}{n+1} \Rightarrow \int_n^{n+1} \frac{1}{x}\,dx \geq \frac{1}{n+1}\right)$$
$$\geq \log(n-1) + \frac{1}{n} + \frac{1}{n+1}$$
$$\vdots$$
$$\geq \frac{1}{2} + \cdots + \frac{1}{n+1}$$

Replacing n by $n-1$, we have
$$\boxed{\frac{1}{2}+\cdots+\frac{1}{n} \leq \ln(n)} \quad \text{—— (3)}$$

From (2) & (3),
$$\boxed{\ln(n+1) \leq 1+\frac{1}{2}+\cdots+\frac{1}{n} \leq 1+\ln(n)} \quad \text{—— (4)}$$

Another proof. From Fig. 11.8 (a) & $f(x) = 1/x$,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{dx}{x} = \ln(n+1). \longrightarrow (5)$$

Also, from fig. 11.8 (b) & $f(x) = 1/x$,

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^{n} dx/x = \ln(n)$$

$$\Rightarrow 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln(n) \longrightarrow (6)$$

From (5) & (6), $\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln(n)$.

$$\Rightarrow \ln(n+1) - \ln(n) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n) \leq 1 \longrightarrow (3)$$

Also, it's clear that $0 < \ln(n+1) - \ln(n)$ for $n \geq 1$.
(or take $f(x) = \ln(x+1) - \ln(x)$, & show $f'(x) < 0$ on $[1, \infty)$).
Hence the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n),$$ is bounded from

both below & above.

(b) Now $$\int_n^{n+1} \frac{1}{x}\,dx = \ln|x| \Big|_n^{n+1} = \ln(n+1) - \ln(n)$$

But on $(n, n+1)$, $x < n+1$ implies

$$\frac{1}{n+1} < \frac{1}{x}, \text{ and hence } \int_n^{n+1} \frac{1}{n+1}\,dx < \int_n^{n+1} \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx = \ln(n+1) - \ln(n). \longrightarrow (4)$$

<u>Claim</u>: $\{a_n\}$ is a decreasing sequence.

Consider $a_n - a_{n+1}$

$$= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} - \ln(n)\right) - \left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1} - \ln(n+1)\right)$$

$$= \left(\ln(n+1) - \ln(n)\right) - \frac{1}{n+1}$$

$$> 0.$$

Thus $\{a_n\}$ is a decreasing seq. bounded from below. Hence by Q 1], it converges.

Q 3] (a) Consider $\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{1} = 0$

$$\Rightarrow \lim_{y\to\infty} y\left(1-\cos\left(\frac{1}{y}\right)\right) = 0.$$

(Note: it's important that the students convert the limit problem on discrete variable n to a continuous var. x or y <u>before</u> using L'Hôpital's rule.)

In particular, $\lim_{n\to\infty} n\left(1-\cos\left(\frac{1}{n}\right)\right) = 0.$

(b) $a_n = \left(\frac{x^n}{2n+1}\right)^{1/n}, \quad x > 0$

Thus $a_n = \frac{x}{(2n+1)^{1/n}}$

$$\Rightarrow \ln(a_n) = \ln(x) - \frac{1}{n}\ln(2n+1)$$

$$\lim_{n\to\infty} \ln(a_n) = \ln(x) - \lim_{n\to\infty} \frac{\ln(2n+1)}{n}.$$

Now consider $\lim_{y\to\infty} \frac{\ln(2y+1)}{y}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \lim_{y\to\infty} \frac{\frac{1}{2y+1}}{1} = 0.$$

Thus $\lim_{n\to\infty} \frac{\ln(2n+1)}{n} = 0$ as well,

and so $\lim_{n\to\infty} \ln(a_n) = \ln(x) - 0$

Since e^x is continuous on $\mathbb{R}$,

$$a_n = \lim_{n\to\infty} e^{\ln(a_n)} = e^{\lim_{n\to\infty} \ln(a_n)} = e^{\ln(x)} = x.$$

ⓒ $\lim_{n\to\infty} \sinh(\ln(n)) = \lim_{n\to\infty} \frac{e^{\ln(n)} - e^{-\ln(n)}}{2}$

$$= \lim_{n\to\infty} \frac{n - 1/n}{2} = +\infty.$$

So $\{\sinh(\ln(n))\}_{n=1}^{\infty}$ diverges.

Q 4] Using $\tan^{-1}\left(\frac{a-b}{1+ab}\right) = \tan^{-1}(a) - \tan^{-1}(b)$, we get

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{-1}{n^2+n+1}\right) = \sum_{n=1}^{\infty} \left(\tan^{-1}(n) - \tan^{-1}(n+1)\right)$$

Now $S_k = \sum_{n=1}^{k} \left(\tan^{-1}(n) - \tan^{-1}(n+1)\right)$

$$= \tan^{-1}(1) - \tan^{-1}(k+1)$$

$$\Rightarrow \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{-1}{n^2+n+1}\right) = \lim_{k\to\infty} S_k$$

$$= \frac{\pi}{4} - \tan^{-1}(\infty) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$

Q5. Note that $L_1 = 3$

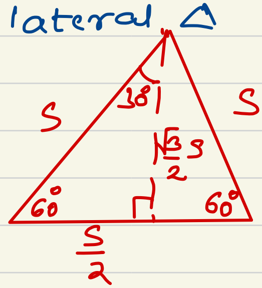
$$L_2 = 3\left(1+\frac{1}{3}\right) = 3.\left(\frac{4}{3}\right)^{2-1}$$

$$L_3 = \frac{16}{3} = 3\left(\frac{4}{3}\right)^{3-1}$$

In general, $L_n = 3\left(\frac{4}{3}\right)^{n-1}$

Hence $\lim_{n\to\infty} L_n = \infty \quad \left(\because \frac{4}{3} > 1\right)$

However, since the area of the equilateral Δ of side 's' is $\frac{\sqrt{3}}{4}s^2$, (Note that Area $= \frac{1}{2}.s.\frac{\sqrt{3}}{2}.s$ $= \frac{\sqrt{3}}{4}s^2$)



we see that

$$A_1 = \frac{\sqrt{3}}{4}, \quad A_2 = A_1 + 3\frac{\sqrt{3}}{4}\cdot\left(\frac{1}{3}\right)^2$$

$$A_3 = A_2 + 3\cdot(4)\cdot\frac{\sqrt{3}}{4}\cdot\left(\frac{1}{3^2}\right)^2$$

$$A_4 = A_3 + 3(4^2)\cdot\frac{\sqrt{3}}{4}\left(\frac{1}{3^3}\right)^2$$

$$\vdots$$

$$A_n = A_{n-1} + 3(4^{n-2})\,\frac{\sqrt{3}}{4}\left(\frac{1}{3^{n-1}}\right)^2$$

$$\Rightarrow A_n = \frac{\sqrt{3}}{4} + \sum_{k=2}^{n} 3\cdot 4^{k-2}\,\frac{\sqrt{3}}{4}\left(\frac{1}{3^{k-1}}\right)^2$$

$$= \frac{\sqrt{3}}{4} + \frac{27\sqrt{3}}{64}\sum_{k=2}^{n}\left(\frac{4}{9}\right)^k$$

$$= \frac{\sqrt{3}}{4} + \frac{27\sqrt{3}}{64}\cdot\frac{4}{9}\sum_{k=1}^{n-1}\left(\frac{4}{9}\right)^k$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\sum_{k=1}^{n-1}\left(\frac{4}{9}\right)^k$$

Now let $n\to\infty$. Then

$$\lim_{n\to\infty} A_n = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\sum_{k=1}^{\infty}\left(\frac{4}{9}\right)^k$$

$$= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\cdot\frac{4/9}{1-4/9} = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\cdot\frac{4}{5}$$

$$= \frac{\sqrt{3}}{4}\cdot\left(1+\frac{3}{5}\right) = \frac{8}{5}\cdot\frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{5},$$

which is finite.

Divergence of the harmonic series

THEOREM 6 The Nondecreasing Sequence Theorem

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

Corollary of Theorem 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Example Divergence of harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

$$\geq 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{=\frac{1}{2}} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{\frac{1}{2}} + \underbrace{\left(\underbrace{\frac{1}{16} + \frac{1}{16} + \cdots}_{8 \text{ terms}}\right)}_{\frac{1}{2}} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \cdots$$

If $n = 2^k$, then $S_n > \frac{k}{2} \Rightarrow \lim_{k \to \infty} S_{2^k}$ is unbounded

Hence $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges