Example: Show that $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix}$ is linear and find the standard matrix.

Ans: Check linearity:

$$T(u+kv) = T(u) + kT(v) \quad \forall\, u, v \in \mathbb{R}^2 \ \&\ k \in \mathbb{R}$$

Let $u, v \in \mathbb{R}^2$, say $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Then $u + kv = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + k\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + kv_1 \\ u_2 + kv_2 \end{bmatrix}$.

Hence

$$T(u+kv) = \begin{bmatrix} u_1 + kv_1 + u_2 + kv_2 \\ u_1 + kv_1 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + u_2 + k(v_1 + v_2) \\ u_1 + kv_1 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} k(v_1 + v_2) \\ kv_1 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + u_2 \\ u_1 \end{bmatrix} + k\begin{bmatrix} v_1 + v_2 \\ v_1 \end{bmatrix}$$

$$= T(u) + kT(v).$$

$\Rightarrow$ T is linear.

To find the standard matrix A:

Consider $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

Lecture 6

Sect. 2.3 – Properties of determinants and Cramer's rule

Properties: Let A be an $n \times n$ matrix.

① If two rows/columns of A are same, then $\det(A) = 0$

② $\det(kA) = k^n \det(A)$.

③ If $A = \begin{pmatrix} — R_1 — \\ \vdots \\ — A_i — \\ \vdots \\ — R_n — \end{pmatrix}$, $B = \begin{pmatrix} — R_1 — \\ \vdots \\ — B_i — \\ \vdots \\ — R_n — \end{pmatrix}$,

and $C = \begin{pmatrix} — R_1 — \\ \vdots \\ — A_i + B_i — \\ \vdots \\ — R_n — \end{pmatrix}$,

then $\det(C) = \det(A) + \det(B)$

④ $\det(AB) = \det(A)\det(B)$ (B is $n \times n$)

⑤ A is invertible $\Longleftrightarrow$ $\det(A) \neq 0$, and $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

(Proof: Take $B = A^{-1}$ in ④).

Ex. Find the determinant of

$$\begin{pmatrix} 1 & & c_1 & & \\ & 1 & c_2 & & 0 \\ & & \ddots\vdots & & \\ & & c_j & & \\ 0 & & \vdots & \ddots & \\ & & c_n & & 1 \end{pmatrix}$$

Ans. Let $c_j \neq 0$

$R_1 \to R_1 - \frac{c_1}{c_j} R_j$, $R_2 \to R_2 - \frac{c_2}{c_j} R_j$

& so on for $R_n \to R_n - \frac{c_n}{c_j} R_j$

(except for the j^{th} row)

These EROs convert the original matrix into the diagonal matrix $\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & c_j & \\ 0 & & & & \ddots \ 1 \end{pmatrix}$.

& the determinant is unchanged.

But the determinant of $\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & c_j & \\ 0 & & & & \ddots \ 1 \end{pmatrix}$

$= 1 \cdot 1 \cdot \ldots \cdot c_j \cdots 1 \cdot 1 \cdot 1 = c_j$.

What if $c_j = 0$?

Do the cofactor expansion along j^{th} row.

Then $\det(A) = c_j C_{jj}$

$= c_j (-1)^{j+j} M_{jj}$

$= c_j (1) \cdot 1$ ← diag. matrix of $(n-1) \times (n-1)$ size with 1's on its main diagonal

$= c_j$.

Third proof

Cofactor expansion along 1^{st} column

$$= 1 \cdot \det \begin{pmatrix} 1 & 0 & \cdots & c_2 & \cdots & 0 \\ & & & c_3 & & \\ & & & c_j & \ddots & \\ & & & & & 1 \end{pmatrix}_{(n-1)\times(n-1)}$$

$$\vdots$$

$$= 1 \cdot \det \begin{pmatrix} c_j & & 0 \\ c_{j+1} & \ddots & \\ \vdots & & \\ c_n & & 1 \end{pmatrix}$$

$$= 1 \cdot c_j \cdot \det \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$= 1 \cdot c_j \cdot 1 = c_j .$$

Eg. $\det \begin{pmatrix} 1 & 5 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix} = 3$

Def. Let $A = (a_{ij})$ be an $n \times n$ matrix.
Let C_{ij} be the cofactor of a_{ij}.

Recall: Cofactor of $a_{ij} = (-1)^{i+j} M_{ij}$.

The matrix of cofactor of A is

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & & & \vdots \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

The transpose of this matrix is called the adjoint of A. It is denoted by adj(A). So

$$\text{adj}(A) = C^T.$$

EXAMPLE 6 Adjoint of a 3 × 3 Matrix

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

As noted in Example 5, the cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of A is

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Important property of adj(A)

$$A\ \mathrm{adj}(A) = \det(A)\cdot I = \begin{pmatrix} \det(A) & & 0 \\ & \ddots & \\ 0 & & \det(A) \end{pmatrix}$$

WATCH

Proof:

$$\underbrace{A\cdot \mathrm{adj}(A)}_{\text{call this } B} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & & & \vdots \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}$$

$B_{11} = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$

= cofactor expansion along 1^{st} row of A

$= \det(A)$.

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

[cofactor expansion along the *i*th row]

$B_{12} = a_{11}C_{21} + a_{12}C_{22} + \dots a_{1n}C_{2n}$

= determinant of the following matrix $= 0$ ($\because R_1$ & R_2 are the same).

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ a_{31} & \dots & \dots & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix}$$

Similarly, $B_{ij} = \begin{cases} 0 & \text{, if } i \neq j \\ \det(A) & \text{, if } i = j \end{cases}$.

$$\Rightarrow B = \begin{pmatrix} \det(A) & & 0 \\ & \ddots & \\ 0 & & \det(A) \end{pmatrix}$$

$$= \det(A) \cdot I$$

$$\Rightarrow A \operatorname{adj}(A) = \det(A) \cdot I.$$

THEOREM 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Using this, we can derive an extremely useful result for solving a linear system with its coefficient matrix having non-zero determinant:

THEOREM 2.3.7 Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Proof: $\quad A x = B$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Let us find x_1,

We consider the following product

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 & & & 0 \\ x_2 & 1 & & \\ \vdots & & \ddots & \\ x_n & & & 1 \end{pmatrix} \leftarrow$$

$I_{n \times n}$ matrix with 1^{st} column replaced by $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

$$= \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 + \cdots + a_{2n}x_n & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n & a_{n2} & & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & & a_{nn} \end{pmatrix}$$

$$= A_1 .$$

Take determinants on both sides so that

$\det(A) \cdot x_1 = \det(A_1)$ (Why?)

If $\det(A) \neq 0$, then $x_1 = \dfrac{\det(A_1)}{\det(A)}$.

Similarly, for $2 \leq j \leq n$,

$$x_j = \frac{\det(A_j)}{\det(A)}.$$

Ex.

Use Cramer's rule to solve

$$\begin{aligned} x_1 + \qquad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Ans. $Ax = b$, where

$$\begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} + 2 \begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix}$$

$$= 1(12+12) + 2(6+4)$$

$$= 24 + 20 = 44 \neq 0$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$$

$$\det(A_1) = 6\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} 30 & 4 \\ 8 & -2 \end{vmatrix}$$

$$= 6(12+12) + 2(-60-32)$$

$$= 144 - 184$$

$$= -40.$$

$$\Rightarrow x_1 = \frac{-40}{44} = -\frac{10}{11},$$

Similarly, find x_2 & x_3

Ex Let $A = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & & & & \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}$,

where $c_0, c_1, \ldots, c_n$ are distinct real numbers. Show that $\det(A) = \prod_{0 \le i < j \le n} (c_j - c_i)$.

This determinant is called Vandermonde determinant.