MA 103 - Lecture 12 (SVC)

Sect. 11.8 - Taylor and Maclaurin series

- Within its interval of convergence, the sum of a power series is a continuous function with derivatives of all orders.

- Now if a function has derivatives of all orders on an interval I, can it be expressed as a power series on I? And if it can, how do its coefficients look like?

DEFINITIONS **Taylor Series, Maclaurin Series**

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

Example: Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

$f(x) = x^{-1}$ $\qquad f(2) = 1/2$

$f'(x) = -x^{-2}$ $\qquad f'(2) = -1/4$

$f''(x) = 2x^{-3} = 2!\,x^{-3}$, $\qquad f''(2) = \frac{2!}{2^3} \Rightarrow \frac{f''(2)}{2!} = \frac{1}{2^3}$.

$f'''(x) = -6x^{-4} = -3!\,x^{-4}$ $\qquad f'''(2) = -\frac{3!}{2^4} \Rightarrow \frac{f'''(2)}{3!} = \frac{-1}{2^4}$

$\vdots$

$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$ $\qquad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$

Hence the Taylor series of f at $x = 2$ is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots$$

Geometric series with first term $= 1/2$ & $r = -\frac{(x-2)}{2}$

It converges absolutely for $|x-2|<2$ (or $0<x<4$)

$$\frac{1/2}{1-\left(-\frac{(x-2)}{2}\right)} = \frac{\frac{1}{2}}{1+\frac{x-2}{2}} = \frac{1}{x}, \quad \text{for } 0<x<4.$$

Taylor polynomials

DEFINITION Taylor Polynomial of Order n

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x=0$.

$f(x) = \cos x \qquad f'(x) = -\sin x$

$f''(x) = -\cos x \qquad f'''(x) = \sin x$

$\vdots \qquad \vdots$

$f^{(2n)}(x) = (-1)^n \cos x \qquad f^{(2n+1)}(x) = (-1)^{n+1}\sin x.$

$f^{(2n)}(0) = (-1)^n \cos(0) = (-1)^n, \quad f^{(2n+1)}(0) = 0$

$\Rightarrow$ Taylor series generated by f at 0 is

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots$$

$$= 1 + (0)x + (-1)^1 \frac{x^2}{2!} + 0 \cdot x^3 + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Since $f^{(2n+1)}(0) = 0$, the Taylor polynomials of orders $2n$ and $2n+1$ are same:

$$P_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!}.$$

THEOREM 22 Taylor's Theorem

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \quad (1)$$

Taylor's formula

where

Remainder of order n

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

If $R_n(x) \to 0$ as $n \to \infty$ $\forall\, x \in I$, we say that the Taylor series generated by f at $x=a$ converges to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)\,(x-a)^k}{k!}.$$

Ex. 1 Show that the Taylor series generated by $f(x) = e^x$ at $x=0$ converges to $f(x)$ for every real value of x.

Ans. e^x has derivatives of all orders through-out the interval $(-\infty, \infty)$. So

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x) \quad \left(\because f^{(n)}(0) = e^0 = 1\right)$$

& $R_n(x) = \dfrac{e^c}{(n+1)!}\, x^{n+1}$, for some c between 0 & x.

e^x increasing on $(-\infty, \infty)$ implies

$$1 = e^0 \le e^c \le e^x.$$

If $x = 0$, $R_n(x) = 0$

If $x < 0$, $c < 0$ & $e^c < 1$

If $x > 0$, $c > 0$ & $e^c < e^x$.

$\Rightarrow |R_n(x)| \le \dfrac{|x|^{n+1}}{(n+1)!}$ when $x \le 0$

& $|R_n(x)| \le \dfrac{e^x |x|^{n+1}}{(n+1)!}$ when $x > 0$.

But $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \forall x$

Proof: ① $x = 0$ ✓

② $x > 0$. For $n \in \mathbb{N}$, $a_n > 0$.

for a suff. large $n \geq x$,

$$a_{n+1} = \frac{x^{n+1}}{(n+1)!} = \frac{x}{n+1} \cdot \frac{x^n}{n!} = \frac{x}{n+1} a_n < a_n$$

$\Rightarrow \{a_n\}$ bdd monotonically decreasing seq.

Hence converges, say, to a.

$$\Rightarrow a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{x}{n+1} \cdot \lim_{n\to\infty} a_n$$

$$= 0 \cdot a \quad \Rightarrow a = 0.$$

③ $x < 0$, Let $x = -y$, $y > 0$.

We know that $\lim_{n\to\infty} \frac{y^n}{n!} = 0$

$$\lim_{n\to\infty} \left| \frac{x^n}{n!} \right| = 0 \quad \text{iff} \quad \lim_{n\to\infty} \frac{x^n}{n!} = 0,$$

$$\lim_{n\to\infty} \left| \frac{(-y)^n}{n!} \right| = \lim_{n\to\infty} \frac{y^n}{n!} = 0,$$

Another simple proof that $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \forall x \in \mathbb{R}$

Consider $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Let $a_n = \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \quad (\text{for every } x \in \mathbb{R})$$

$\Rightarrow \sum a_n$ converges absolutely & hence converges

$\Rightarrow \lim_{n\to\infty} a_n = 0$, i.e., $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$ & so,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

THEOREM 23 The Remainder Estimation Theorem

If there is a positive constant M such that $|f^{(n+1)}(t)| \le M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Proof: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, $a < c < x$

$\Rightarrow |R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}$

Ex. 2 Show that the Taylor series for $\sin x$ at $x=0$ converges for all x.

Proof: $f(x) = \sin x$

$$f^{(2k)}(x) = (-1)^k \sin x, \quad f^{(2k+1)}(x) = (-1)^k \cos x$$

$$\Rightarrow f^{(2k)}(0) = 0, \quad f^{(2k+1)}(x) = (-1)^k$$

Taylor series of $f(x)$ at $x=0$:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

- only odd-powered terms.

By Taylor's theorem,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x)$$

Note that $|f^{(n)}(x)| \leq 1 \quad \forall n \in \mathbb{N}$.

$$\Rightarrow |R_{2k+1}(x)| \leq \frac{|x|^{2k+2}}{(2k+2)!}$$

$$\Rightarrow \lim_{k \to \infty} R_{2k+1}(x) = 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ The Maclaurin series for $\sin x$ converges to $\sin x$ for every real x.

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Similarly, we can show

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

RIGOROUS PROOF OF $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \ |x| \leq 1$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}$$

$$\Rightarrow \int_0^x \frac{dt}{1+t^2} = \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots + (-1)^n \frac{t^{2n+1}}{2n+1}\right]_0^x + \overbrace{\int_0^x \frac{(-1)^{n+1} t^{2n+2}\, dt}{1+t^2}}^{R_n(x)}$$

$$\Rightarrow \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + R_n(x).$$

Since $1+t^2 \geq 1$,

$$|R_n(x)| \leq \int_0^{|x|} t^{2n+2}\, dt = \frac{|x|^{2n+3}}{2n+3}.$$

For $|x| \leq 1$, $\displaystyle\lim_{n\to\infty} \frac{|x|^{2n+3}}{2n+3} = 0$

so that $\displaystyle\lim_{n\to\infty} R_n(x) = 0$. This implies

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{for } |x| \leq 1.$$

APPLICATIONS

① $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$

$$= \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\cancel{x^3}\left(\frac{1}{6} - \frac{x^2}{5!} + \cdots\right)}{\cancel{x^3}}$$

$$= \frac{1}{6}.$$

② $\lim_{y\to 0} \dfrac{\tan^{-1}(y) - \sin y}{y^3 \cos y}$

$= \lim_{y\to 0} \dfrac{\left(\cancel{y} - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right) - \left(\cancel{y} - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)}{y^3\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right)}$

$= \lim_{y\to 0} \dfrac{y^3\left(\frac{1}{6} - \frac{1}{3}\right) + y^5\left(\left(\frac{1}{5} - \frac{1}{20}\right) - \cdots\right)}{y^3\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right)}$

$= \lim_{y\to 0} \dfrac{\cancel{y^3}\left(\frac{1}{6} - \frac{1}{3}\right)}{\cancel{y^3}\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right)} + \lim_{y\to 0} \dfrac{y^2\left(\left(\frac{1}{5} - \frac{1}{20}\right) - \cdots\right)}{1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots}$

$= \dfrac{-1}{6}$

BINOMIAL SERIES

Consider $f(x) = (1+x)^m$, $m \in \mathbb{R}$.

$f(0) = 1$

$f'(x) = m(1+x)^{m-1}$, $f'(0) = m$

$f''(x) = m(m-1)(1+x)^{m-2}$, $f''(0) = m(m-1)$

$\vdots$

$f^{(k)}(x) = m(m-1) \cdots (m-k+1)(1+x)^{m-k}$,

$f^{(k)}(0) = m(m-1) \cdots (m-k+1)$.

$\Rightarrow$ Taylor series of f at $x=0$ is

$$1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3$$
$$+\dots+\frac{m(m-1)\cdots(m-k+1)}{k!}x^k+\dots = \sum_{k=0}^{\infty} u_k x^k$$

It converges for $|x|<1$, since

$$\left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{m-k}{k+1}x\right| \longrightarrow |x| \text{ as } k\to\infty$$

So by ratio test, the series converges for $|x|<1$.

- It can be shown that the series converges to $(1+x)^m$ for $|x|<1$, using the same approach as done for e^x, $\sin x$, $\tan^{-1}x$. Thus, for $-1<x<1$,

$$(1+x)^m = 1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3$$
$$+\dots+\frac{m(m-1)\cdots(m-k+1)}{k!}x^k+\dots\dots$$

- The series terminates if $m\in\mathbb{N}$.

$$\lim_{x\to 0} \frac{(1+x)^{1/2} - 1 - \frac{x}{2}}{\sin^2 x}$$

$$= \lim_{x\to 0} \frac{\left(\cancel{1}+\cancel{\frac{x}{2}}+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}x^2 + \cdots\right) - \cancel{1} - \cancel{\frac{x}{2}}}{\sin^2 x}$$

$$= \lim_{x\to 0} \frac{x^2\left(\frac{-1}{8} + cx + \cdots\right)}{\sin^2 x}$$

$$= \frac{-1}{8}$$

$$f(x) = |x|^3$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{|x|^3 - 0}{x - 0}$$

$$= \lim_{x\to 0^+} \frac{|x|^3}{x} = \lim_{x\to 0} \frac{x^3}{x} = 0$$

$$\lim_{x\to 0^-} = \lim_{x\to 0^-} \frac{|x|^3}{x} = -\lim_{x\to 0} \frac{x^3}{x} = 0.$$

WEIRD THINGS IN SOME EXAMPLES ON TAYLOR SERIES THAT ONE SHOULD BE AWARE OF

① Not every function has an infinite Taylor series:

Consider $f(x) = |x|^3$.

Then $f(0) = 0$

$f'(0) = 0$

$f''(0) = 0$

$f'''(0)$ DNE

② Derivatives grow very fast may pose a problem:

Suppose f is such that

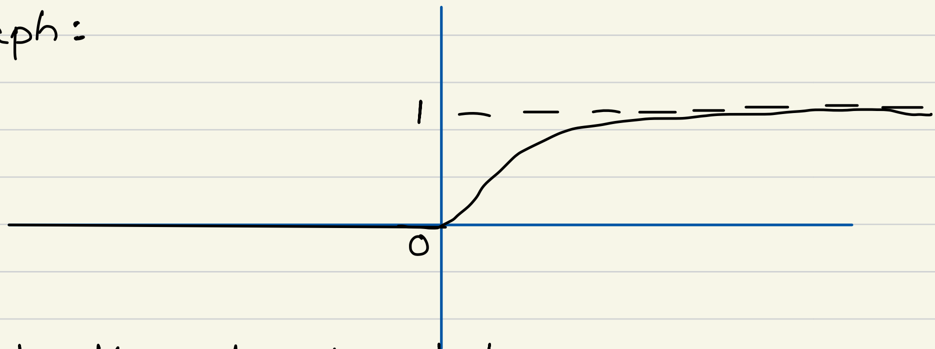
$$f^{(n)}(0) = (n!)^2.$$

Then $c_n = \dfrac{f^{(n)}(0)}{n!} = n!$, so the Taylor series of f is $\sum_{n=0}^{\infty} n!\, x^n$. It has radius of convergence 0.

③ There are functions whose Taylor series converge but not to $f(x)$ at all pts, but to something else!

Consider $f(x) = \begin{cases} 0, & x \le 0, \\ e^{-1/x}, & x > 0 \end{cases}$

Graph:



- Continuity at all points:

$$\lim_{x\to 0^-} f(x) = 0 = \lim_{x\to 0^+} f(x) = f(0).$$

- Derivatives of all orders exist at $x=0$:

$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{1}{x} e^{-1/x}$$

$$= \lim_{y\to\infty} \frac{y}{e^y} = \lim_{y\to\infty} \frac{1}{e^y} = 0.$$

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{0-0}{x-0} = 0.$$

$\Rightarrow$ $f'(0)$ exists and is equal to 0.

- Moreover, $f^{(n)}(0) = 0 \quad \forall\, n \in \mathbb{N}$

$\Rightarrow$ Taylor series of f is

$$0 + \frac{0x}{1!} + \frac{0x^2}{2!} + \cdots = 0.$$

In any neighborhood of $x=0$, the Taylor series of f converges to 0, but not to f!