Lecture 5 – Linear transformations and determinants

A matrix transformation satisfies (1) & (2).

Question: Are all transformations satisfying (1) & (2) coming from multiplication by a matrix, that is, are they always matrix transformations?

Ans. Yes!

Take $\vec{x} \in \mathbb{R}^n$, that is, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, say, & let

$T: \mathbb{R}^n \to \mathbb{R}^m$ satisfy (1) & (2).

Since $\vec{x} = x_1 \vec{e_1} + x_2 \vec{e_2} + \cdots x_n \vec{e_n}$,

$$\begin{aligned} T(\vec{x}) &= T(x_1 \vec{e_1} + \cdots + x_n \vec{e_n}) \\ &= T(x_1 \vec{e_1}) + \cdots + T(x_n \vec{e_n}) \quad (\text{from } (1)) \\ &= x_1 T(\vec{e_1}) + \cdots + x_n T(\vec{e_n}) \quad (\text{from } (2)) \end{aligned}$$

$(*)$

Thus it is enough to determine $T(\vec{e_1}), \ldots, T(\vec{e_n})$ to get $T(x)$ for any $x \in \mathbb{R}^n$.

We want A such that $T = T_A : \mathbb{R}^n \to \mathbb{R}^m$.

A is $m \times n$.

Each of $T(e_1), \ldots, T(e_n)$ is $m \times 1$.

Consider $A = \begin{pmatrix} | & & | \\ T(e_1) & \cdots & T(e_n) \\ | & & | \end{pmatrix}$

Then is $T_A = T$?

$$T_A(x) := Ax = \begin{pmatrix} | & & | \\ T(e_1) & \cdots & T(e_n) \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$= T(e_1)\cdot x_1 + \dots + T(e_n) x_n$ (by column-sum rule)

$= T(x)$. (by ✱)

& this is true $\forall\, x \in \mathbb{R}^n$.

$\Rightarrow T = T_A$.

So T indeed comes from a matrix

$$A = \underbrace{[\,T(e_1) \mid T(e_2) \mid \dots \mid T(e_n)\,]}_{\text{columns}}$$

Hence matrix & linear transformations are same.

THEOREM 1.8.4 *If $T_A: R^n \to R^m$ and $T_B: R^n \to R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.*

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

$$A\bar{e}_1 = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1} = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{pmatrix}_{m\times 1}$$

$$B\vec{e}_1 = \begin{pmatrix} b_{11} \\ \vdots \\ b_{m1} \end{pmatrix}$$

Since $T_A(\vec{e}_1) = T_B(\vec{e}_1)$, we have $a_{i1} = b_{i1}$ for all $1 \le i \le m$.

Similarly, $a_{ij} = b_{ij}$ for all $2 \le j \le n$ & $1 \le i \le m$

$\Rightarrow A = B$.

Ex. 1 Find the standard matrix A for the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2(1) + 0 \\ 1 - 3(0) \\ -1 + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2(0) + 1 \\ 0 - 3(1) \\ -0 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

Check: $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$

Chapter 2 – Determinants

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$;

Determinant A, denoted by $\det(A) = ad - bc$.

& $A^{-1} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, provided $\det(A) \neq 0$.

So an application of determinants is in computing inverses.

Determinant of an $n \times n$ matrix

↳ Defined in terms of det. of $(n-1)\times(n-1)$ matrices

↳ Defined in terms of det. of $(n-2)\times(n-2)$ matrices

inductively defined

Defn. • $\det(a_{11}) = a_{11}$,

• Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}_{n\times n}$

Defn. <u>Minor of a_{ij}:</u>

- Denoted by M_{ij}, determinant of the
- Defined to be the submatrix of A that remains after deleting the i^{th} row and j^{th} column from A.

$$M_{ij} = \det\begin{pmatrix} a_{11} & \cdots & & a_{1n} \\ \vdots & & a_{ij} & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

Defn. Cofactor of a_{ij}:

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 4 & 4 & -5 \end{pmatrix}$$

$$C_{11} = M_{11}, \quad C_{21} = -M_{21}, \quad C_{22} = M_{22}$$

Defn. Determinant of $n \times n$ matrix:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

[cofactor expansion along the jth column]

<u>OR</u>

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

[cofactor expansion along the ith row]

Ex. 1

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$$

<u>co-factor expansion along 1^{st} row:</u>

$$\det(A) = (1)\begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} - (-2)\begin{vmatrix} 6 & -1 \\ -3 & 4 \end{vmatrix} + 3\begin{vmatrix} 6 & 7 \\ -3 & 1 \end{vmatrix}$$

$$= (28+1) + 2(24-3) + 3(6+21)$$

$$= 29 + 42 + 81$$

$$= 152.$$

<u>co-factor expansion along 3^{rd} column:</u>

$$\det(A) = 3\begin{vmatrix} 6 & 7 \\ -3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 6 & 7 \end{vmatrix}$$

$$= 3(6+21) + 1(1-6) + 4(7+12)$$

$$= 81 - 5 + 76$$

$$= 81 + 71$$

$$= 152.$$

Ex. 2 Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

co-factor expansion along 2^{nd} column

$$= \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = (-2)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= (-2)(1+2)$$

$$= -6.$$

Ex. 3 Find det. of

$$A = \begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}\begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22}a_{33}a_{44}.$$

Note: The same can be done for any upper/lower triangular matrix.

Property: $\det(A) = \det(A^T)$

take cofactor expansion along i^{th} row

take cofactor expansion along i^{th} column

Sect. 2.2 Another way to find determinant by EROs

	Relationship	Operation
$R_1 \to kR_1$	$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	In the matrix B the first row of A was multiplied by k.
$R_i \leftrightarrow R_j$	$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	In the matrix B the first and second rows of A were interchanged.
	$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	In the matrix B a multiple of the second row of A was added to the first row.

THEOREM 2.2.4 *Let E be an $n \times n$ elementary matrix.*

(a) *If E results from multiplying a row of I_n by a nonzero number k, then* $\det(E) = k$.

(b) *If E results from interchanging two rows of I_n, then* $\det(E) = -1$.

(c) *If E results from adding a multiple of one row of I_n to another, then* $\det(E) = 1$.

Ex. 4 Find det(A) using EROs :

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix},$$

$\downarrow$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{bmatrix} \longleftrightarrow \det(A) = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$\downarrow$ $R_1 \to R_1/3$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{bmatrix} \longleftrightarrow \det(A) = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$\downarrow$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{bmatrix} \longleftrightarrow \det(A) = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$R_3 \to R_3 - 10R_2$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{bmatrix} \longleftrightarrow \det(A) = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$$= (-3)(1)(1)(-55) = 165$$

<u>Check</u>:
$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -1\begin{vmatrix} 3 & 9 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 3 & -6 \\ 2 & 6 \end{vmatrix}$$

$$= (-1)(3-18) + 5(18+12)$$

$$= 15 + 150$$

$$= 165$$