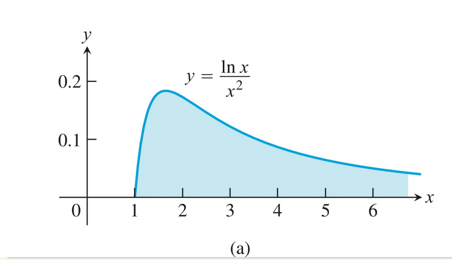
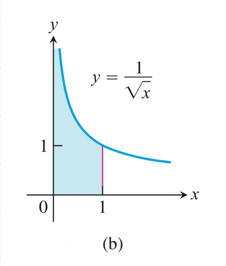
MA 103- SVC Lecture 8

Sect. 8.8 – Improper integrals

Background:



(a)



(b)

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example 1: $\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx$

$= \lim_{b\to\infty}\left[\frac{e^{-x/2}}{-1/2}\right]_0^b = \lim_{b\to\infty}\left(-2\cdot\left[e^{-x/2}\right]_0^b\right)$

$= -2\lim_{b\to\infty}\left(e^{-b/2}-1\right) = -2(0-1) = 2$

Example 2: $\int_1^\infty \frac{\ln x}{x^2}\,dx$ $\quad u = \ln x$, $dv = \frac{1}{x^2}dx$

$\Rightarrow du = \frac{1}{x}dx, \quad v = \frac{-1}{x}$

$\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\left(\left[\frac{-\ln x}{x}\right]_1^b - \int_1^b\left(\frac{-1}{x}\right)\frac{1}{x}dx\right)$

$= \lim_{b\to\infty}\left(-\frac{\ln(b)}{b}+0-\left[\frac{1}{x}\right]_1^b\right) = \lim_{b\to\infty}\left(-\frac{\ln(b)}{b}-\frac{1}{b}+1\right)$

$= 0-0+1 = 1.$

Example 3 $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$

$= \int_{-\infty}^0 \frac{1}{1+x^2}dx + \int_0^\infty \frac{1}{1+x^2}dx$

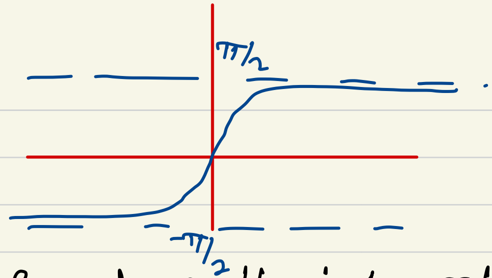
$= \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}dx + \lim_{b\to\infty}\int_0^b\frac{1}{1+x^2}dx$

$= \lim_{a\to-\infty}\left[\tan^{-1}x\right]_a^0 + \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b$

$= \lim_{a\to-\infty}\left(\tan^{-1}(0)-\tan^{-1}(a)\right) + \lim_{b\to\infty}\left(\tan^{-1}(b)-\tan^{-1}(0)\right)$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$= \pi$$



Ex. 4 For what values of p does the integral $\int_1^\infty \frac{1}{x^p}$ converge? Find the value of the integral when it converges.

Ans. $$\int_1^\infty \frac{1}{x^p} = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$

Case 1 $\quad p \neq 1$:

$$\int_1^b \frac{1}{x^p}\,dx = \int_1^b x^{-p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

$$\Rightarrow \int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left(\frac{b^{-p+1}-1}{-p+1}\right)$$

$$= \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \\ \infty, & \text{if } p<1 \end{cases}$$

$\Rightarrow \int_1^\infty \frac{1}{x^p}\,dx$ converges if $p>1$ & is equal to $\frac{1}{p-1}$ there, whereas it diverges if $p<1$.

Case 2: $p = 1$

$$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x}\, dx$$

$$= \lim_{b \to \infty} \Big[\ln |x| \Big]_1^b$$

$$= \lim_{b \to \infty} \big[\ln b - \ln 1 \big]$$

$$= \infty.$$

$$\Rightarrow \int_1^{\infty} \frac{1}{x^p}\, dx \text{ diverges when } p = 1$$

DEFINITION **Type II Improper Integrals**

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx.$$

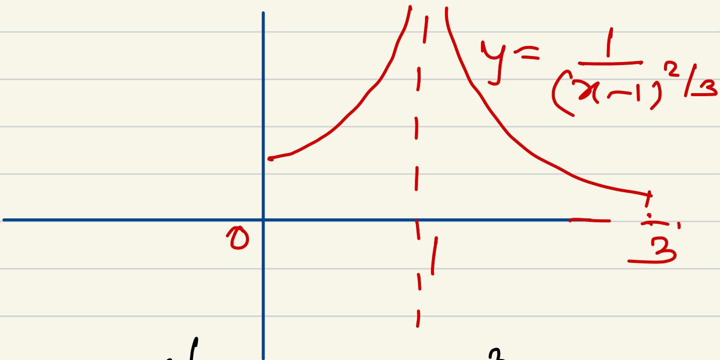
3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Example (Vertical asymptote at an interior pt.)

Evaluate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$.



- $$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

- $$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \left[\frac{(x-1)^{\frac{1}{3}}}{1/3}\right]_0^b$$

$$= 3 \lim_{b\to 1^-} \left[(x-1)^{1/3}\right]_0^b = 3 \lim_{b\to 1^-} \left((b-1)^{1/3} - (0-1)^{1/3}\right)$$

$$= 3 \lim_{b\to 1^-} \left((b-1)^{1/3} + 1\right) = 3$$

- $$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} \int_a^3 \frac{dx}{(x-1)^{2/3}} = 3\cdot 2^{1/3}$$

$$\Rightarrow \int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\times 2^{1/3}$$

Evaluate $\int_0^3 \frac{dx}{x-1}$

WARNING! INCORRECT EVALUATION

$$\int_0^3 \frac{dx}{x-1} = \Big[\ln|x-1|\Big]_0^3 = \ln 2 - 0, = \ln 2.$$

THIS IS WRONG!

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

$$\int_0^1 \frac{dx}{x-1} = \lim_{b\to 1^-} \int_0^b \frac{dx}{x-1} = \lim_{b\to 1^-}\Big[\ln|x-1|\Big]_0^b$$

$$= \lim_{b\to 1^-}\left(\ln|b-1|\right) = -\infty.$$

$\Rightarrow \int_0^1 \frac{dx}{x-1}$ does not exist & hence $\int_0^3 \frac{dx}{x-1}$ does not exist.

Tests for convergence and divergence

THEOREM 1 Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\, dx$ converges if $\int_a^\infty g(x)\, dx$ converges

2. $\int_a^\infty g(x)\, dx$ diverges if $\int_a^\infty f(x)\, dx$ diverges.

Example ① $\int_1^\infty e^{-x^2}\,dx$ ② $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

$f(x) = e^{-x^2}$, $g(x) = e^{-x}$ On $[1,\infty)$,

$e^{-x^2} \le e^{-x}$

$$\Rightarrow \int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_1^\infty$$

$$= 0 + \frac{1}{e}.$$

converges

Hence by direct comparison test, $\int_1^\infty e^{-x^2}\,dx$ also converges.

THEOREM 2 Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

Example ① $\int_1^\infty \frac{dx}{1+x^2}$ ② $\int_1^\infty \frac{3}{e^x+5}\,dx$

① Let $g(x) = \frac{1}{1+x^2}$, $f(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1+x^2}{x^2} = 1 > 0$$

Both $\int_1^\infty f(x)\,dx$ & $\int_1^\infty g(x)\,dx$ both converge or both diverge

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[\frac{-1}{x}\right]_1^b$$

$$= 1$$

$$\Rightarrow \int_1^\infty \frac{dx}{1+x^2} \text{ also converges.}$$

(2) Take $f(x) = e^{-x}$, $g(x) = \dfrac{3}{e^x+5}$

$$\frac{3}{e^x+5} = \frac{3e^{-x}}{1+5e^{-x}} \xrightarrow{x\to\infty} 3e^{-x}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^{-x}}{\frac{3}{e^x+5}}$$

$$= \lim_{x\to\infty} \frac{\cancel{e^{-x}}}{3\cancel{e^{-x}}} \cdot (1+5e^{-x})$$

$$= \frac{1}{3} \leftarrow L$$

$\int_1^\infty e^{-x}\,dx$ & $\int_1^\infty \dfrac{3\,dx}{e^x+5}$

either both converge or both diverge

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{e^{-x}}{-1}\right]_1^b = \lim_{b\to\infty} \left(e^{-b} + e^{-1}\right)$$

$$= \frac{1}{e}$$

Sect. 8.3 – Integration of rational functions by partial fractions

- Method to rewrite rational functions as a sum of simple fractions is called the method of partial fractions.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example: Evaluate $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$$\Rightarrow x^2+4x+1 = A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)$$

$$= x^2(A+B+C) + x(4A+2B) + (3A-3B-C)$$

$$\Rightarrow A+B+C=1$$

$$4A+2B=4$$

$$3A-3B-C=1$$

$$\Rightarrow A=\frac{3}{4},\ B=\frac{1}{2},\ C=\frac{-1}{4}.$$

$$\Rightarrow \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + c$$

<u>Example</u> (Repeated linear factor)

Evaluate $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Rightarrow x^2 = A(x^2+2x+1) + B(x^2-1) + C(x-1)$$

$$\Rightarrow x^2 = (A+B)x^2 + (2A+C)x + (A-B-C)$$

$$\Rightarrow \begin{array}{l} A+B=1 \\ 2A+C=0 \\ A-B-C=0 \end{array} \quad \Rightarrow \begin{array}{l} B=1-A \\ C=-2A \end{array}$$

$$\Rightarrow A-(1-A)-(-2A)=0$$

$$\Rightarrow 4A-1=0 \Rightarrow A=1/4$$

$B=3/4$, $C=-1/2$

$$\Rightarrow \int \frac{x^2\,dx}{(x-1)(x^2+2x+1)} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + c$$

Example Evaluate $\displaystyle\int \frac{(-2x+4)\,dx}{(x^2+1)(x-1)^2}$

$$\frac{(-2x+4)}{(x^2+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow -2x+4 = A(x^2+1)(x-1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$$\Rightarrow A=-2,\ B=1,\ C=2,\ D=1$$

$$\Rightarrow \int \frac{(-2x+4)\,dx}{(x^2+1)(x-1)^2}$$

$$= -2\ln|x-1| - \frac{1}{x-1} + \int \frac{2x+1}{x^2+1}\,dx$$

$$= -2\ln|x-1| - \frac{1}{x-1} + \ln|x^2+1| + \tan^{-1}(x) + c$$

Heaviside Method

1. *Write the quotient with* $g(x)$ *factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$

2. *Cover the factors* $(x - r_i)$ *of* $g(x)$ *one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.$$

3. *Write the partial-fraction expansion of* $f(x)/g(x)$ as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

Example

$$\int \frac{(x^2 + 7x - 1)\,dx}{(x-4)(x+2)(x-3)}$$

$$\frac{x^2+7x-1}{(x-4)(x+2)(x-3)} = \frac{A}{x-4} + \frac{B}{x+2} + \frac{C}{x-3}.$$

Then $A = \dfrac{(4)^2+7(4)-1}{(4+2)(4-3)} = \dfrac{43}{6}$

$B = \left.\dfrac{x^2+7x-1}{(x-4)(x-3)}\right|_{x=-2} =$

$C = \left.\dfrac{x^2+7x-1}{(x-4)(x+2)}\right|_{x=3} =$

Complete the problem!

Sect. 8.4 – Trigonometric integrals

Integral: $\int \sin^m x \cos^n x \, dx$

Case 1: m is odd, say, $m = 2k+1$

$\sin^m x = (\sin x)^{2k+1} = (\sin^2 x)^k \cdot \sin x$

$= (1-\cos^2 x)^k \cdot \sin x$

Let $\cos x = u$. Then $-\sin x \, dx = du$

Case 2: m is even & n is odd; write $n = 2k+1$ & so

$(\cos x)^{2k+1} = (1-\sin^2 x)^k \cdot \cos x$

Let $\sin x = u$, then $\cos x \, dx = du$,

Case 3: m & n both are even, say $m = 2k$, $n = 2j$

$\sin^{2k}(x) = (\sin^2 x)^k = \left(\dfrac{1-\cos(2x)}{2}\right)^k$

$\cos^{2j}(x) = (\cos^2 x)^j = \left(\dfrac{1+\cos(2x)}{2}\right)^j.$

Example: $\int \cos^7 x \, dx$

$\cos^7 x = (\cos x)^6 \cdot \cos x = (\cos^2 x)^3 \cos x$

$= (1-\sin^2 x)^3 \cos x.$

Let $\sin x = t$, $dt = \cos x \, dx$

$\int \cos^7 x \, dx = \int (1-t^2)^3 \, dt$

$= \int [1 - t^6 - 3t^2(1-t^2)] \, dt$

$= \int (1 - 3t^2 + 3t^4 - t^6) \, dt$

$= t - t^3 + \frac{3}{5} t^5 - \frac{t^7}{7} + c$

$= \sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{\sin^7 x}{7} + c.$

Example: $\int \sin^2 x \cos^4 x \, dx$

$= \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)^2 dx$

$= \frac{1}{8} \int (1-\cos(2x))(1+2\cos(2x)+\cos^2(2x)) \, dx$

$= \frac{1}{8} \int (1 + 2\cos(2x) + \cos^2(2x) - \cos(2x) - 2\cos^2(2x) - \cos^3(2x)) \, dx$

$= \frac{1}{8} \int (1 + \cos(2x) - \cos^2(2x) - \cos^3(2x)) \, dx$

$= \frac{1}{8} \left[x + \frac{\sin(2x)}{2} \right] - \frac{1}{8} \int (\cos^2(2x) + \cos^3(2x)) \, dx$

$$\int \cos^2(2x)\,dx = \int \frac{1+\cos(4x)}{2}\,dx$$

$$= \frac{1}{2}\left(x + \frac{\sin(4x)}{4}\right)$$

$$\int \cos^3(2x)\,dx = \int (1-\sin^2(2x))\cos(2x)\,dx$$

$$\left[\sin(2x) = t \qquad \cos(2x)\,dx = \frac{dt}{2}\right]$$

$$= \int (1-t^2)\,\frac{dt}{2} = \frac{1}{2}\left(t - \frac{t^3}{3}\right) = \frac{1}{2}\left(\sin(2x) - \sin^3(2x)\right),$$

Now complete the problem.

Example $\int \tan^4 x\,dx$

$$= \int \tan^2 x \cdot \tan^2 x\,dx$$

$$= \int \tan^2 x\,(\sec^2 x - 1)\,dx$$

$$= \underbrace{\int \tan^2 x \sec^2 x\,dx}_{\substack{\tan x = u \\ \sec^2 x\,dx = du}} - \int \underbrace{\tan^2 x}_{= \sec^2 x - 1}\,dx$$

$$= \int u^2\,du - \int (\sec^2 x - 1)\,dx$$

$$= \frac{u^3}{3} - \tan x + x + c$$

$$= \frac{\tan^3 x}{3} - \tan x + x + c.$$

Example $\int \sec^3 x \, dx$

(Write $\sec^3 x \, dx = (\sec x)(\sec^2 x \, dx)$

& use integration by parts.

Example $\int \cos 5x \cos 4x \, dx$

$$= \frac{1}{2} \int [\cos(9x) + \cos(x)] \, dx$$

$$= \frac{1}{2}\left(\frac{\sin(9x)}{9} + \sin x\right) + c.$$

Sect. 8.5 – Trigonometric substitutions

- Useful in evaluating integrals which contain $\sqrt{a^2 - x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$.

$\sqrt{a^2 - x^2}$ Let $\boxed{x = a \sin\theta}$

Then $\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$

$= \sqrt{a^2\cos^2\theta} = a|\cos\theta|.$

$\sqrt{a^2+x^2}$ Let $\boxed{x = a\tan\theta}$.

$\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)}$

$= \sqrt{a^2\sec^2\theta} = a|\sec(\theta)|$

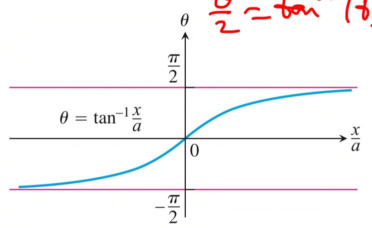
$\sqrt{x^2-a^2}$ Let $\boxed{x = a\sec\theta}$

$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$

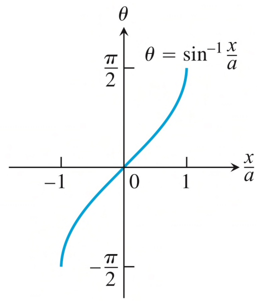
$= \sqrt{a^2\tan^2\theta} = a|\tan\theta|.$

$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ $\quad \tan(\theta)=t \downarrow \frac{\theta}{2}=\tan^{-1}(t)$

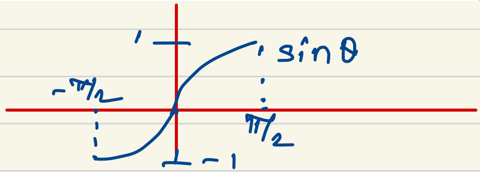
$$\int_{\pi/2}^{2\pi/3}\frac{\cos\theta\, d\theta}{\sin\theta(1+\cos\theta)}$$

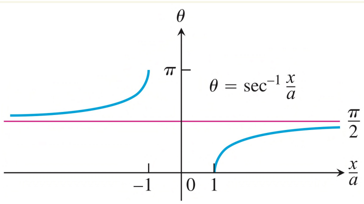


$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$



$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$





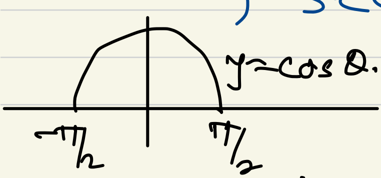
$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \ge 1, \\ \frac{\pi}{2} < \theta \le \pi & \text{if } \frac{x}{a} \le -1. \end{cases}$$

Example $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$ $\quad$ Let $x = 2\tan\theta$; $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$dx = 2\sec^2\theta\, d\theta$.

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{2|\sec\theta|}$$

$$= \int \frac{\sec^2\theta}{\sec\theta}\, d\theta \quad \left(\because \sec\theta > 0 \text{ on } -\pi/2 < \theta < \pi/2\right)$$



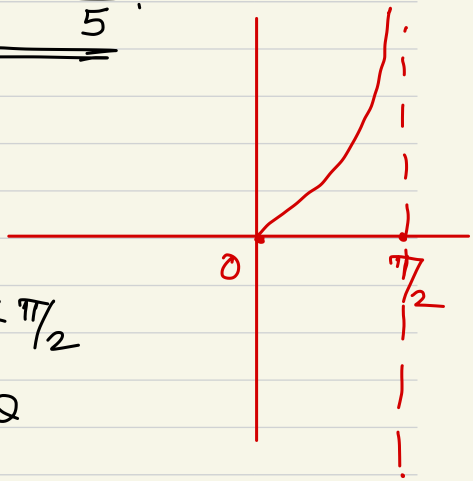
$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + c .$$

since $\sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{\left(\frac{x}{2}\right)^2+1} = \frac{\sqrt{x^2+4}}{2}$

* Example $\int \frac{dx}{\sqrt{25x^2-4}}$, $x > \frac{2}{5}$.



$$= \int \frac{dx}{5\sqrt{x^2-\left(\frac{2}{5}\right)^2}}$$

Let $x = \frac{2}{5}\sec\theta$, $0<\theta<\pi/2$

$dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta$

$x^2-\left(\frac{2}{5}\right)^2 = \left(\frac{2}{5}\right)^2\tan^2$.

$$\int \frac{dx}{\sqrt{25x^2-4}} = \frac{1}{5}\int \frac{\frac{2}{5}\sec\theta\tan\theta\, d\theta}{\frac{2}{5}|\tan\theta|}$$

$$= \frac{1}{5}\int \sec\theta\, d\theta \quad (\because \tan\theta > 0 \text{ if } 0<\theta<\pi/2)$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + c$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \sqrt{\frac{25x^2}{4}-1}\right| + c$$

Chapter 11 - Infinite sequences and series

Sect. 11.1 - Sequences

DEFINITION Infinite Sequence

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

Examples ① $\{1, 2, 3, 4, 5, \ldots\}$

$\{a_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$; $f(n) = n \quad \forall n \in \mathbb{N}$.

② $\{1, 1, 1, 1, \ldots\}$: $\{a_n\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty}$; $f(n) = 1 \quad \forall n \in \mathbb{N}$.

③ $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$; $a_n = \frac{1}{n}$, $f(n) = \frac{1}{n} \quad \forall n \in \mathbb{N}$.

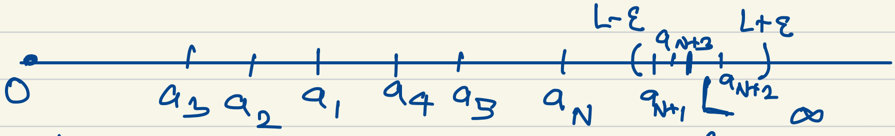
④ $\{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots\}$ $a_n = \frac{(-1)^{n+1}}{n}$;

DEFINITIONS Converges, Diverges, Limit

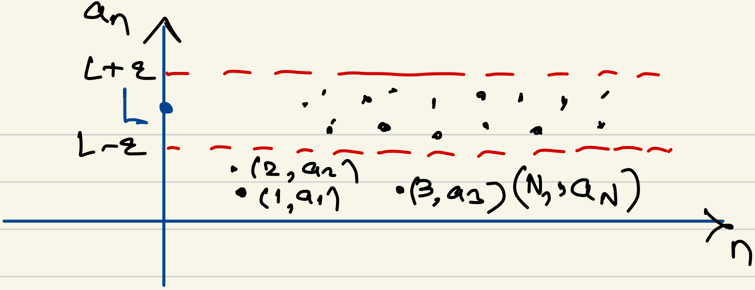
The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.



$\lim_{n \to \infty} a_n = L$, and we say $\{a_n\}_{n=1}$ converges to L.



Example Show that $\lim_{n\to\infty} \frac{1}{n} = 0$ by defn.

Proof: Let $\varepsilon > 0$ be given. We have to find an $N \in \mathbb{N}$ s.t. $\forall\, n > N$, $\left|\frac{1}{n} - 0\right| < \varepsilon$, i.e., $\frac{1}{n} < \varepsilon$, i.e. $n > \frac{1}{\varepsilon}$. Hence choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$ (Archimedean property) so that $n > \frac{1}{\varepsilon}$ $\forall\, n > N$.

Hence by defn, $\lim_{n\to\infty} \frac{1}{n} = 0$.

ASSIGNMENT PROBLEM

Ex. 1 Prove that the sequence $\{(-1)^n\}_{n=0}^{\infty}$ diverges.