Boundary Value Problems for the Magnetic Laplacian in Semiclassical Analysis

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Abstract

This paper is concerned with the magnetic Laplacian $P^h(\mathbf{A}) = (hD + \mathbf{A})^2$ in semiclassical analysis, where h is a semiclassical parameter. We study the L^2 Neumann and Dirichlet problems for the equation $P^h(\mathbf{A})u = 0$ in a bounded Lipschitz domain Ω . Under the assumption that the magnetic field $\nabla \times \mathbf{A}$ is of finite type on $\overline{\Omega}$, we establish the nontangential maximal function estimates for $(hD + \mathbf{A})u$, which are uniform for $0 < h < h_0$. This extends a well-known result due to D. Jerison and C. Kenig for the Laplacian in Lipschitz domains to the magnetic Laplacian in the semiclassical setting. Our results are new even for smooth domains.

Keywords. Schrödinger Operator; Magnetic Field; Semiclassical Analysis; Boundary Value Problem; Lipschitz Domain.

MR (2020) Subject Classification: 35P25.

1 Introduction

Let $D = -i\nabla$ and $\mathbf{A} = (A_1, A_2, \dots, A_d) \in C^1(\mathbb{R}^d; \mathbb{R}^d)$. Consider the Schrödinger operator with a magnetic potential in semiclassical analysis,

$$P^{h}(\mathbf{A}) = (hD + \mathbf{A})^{2},\tag{1.1}$$

also called the magnetic Laplacian, where $h \in (0,1)$ is a semiclassical parameter. In this paper, we are interested in boundary regularities of $P^h(\mathbf{A})$ that are uniform in h. More precisely, consider the Neumann problem in a bounded Lipschitz domain Ω ,

$$\begin{cases} P^{h}(\mathbf{A})u = 0 & \text{in } \Omega, \\ n \cdot (hD + \mathbf{A})u = g & \text{on } \partial\Omega, \end{cases}$$
 (1.2)

where $n = (n_1, n_2, \dots, n_d)$ denotes the outward unit normal to $\partial \Omega$, and the Dirichlet problem,

$$\begin{cases} P^{h}(\mathbf{A})u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$
 (1.3)

Let $\mathbf{B} = \nabla \times \mathbf{A}$ denote the magnetic field. In the main results of this paper, we shall assume that \mathbf{B} is of finite type on $\overline{\Omega}$; i.e., there exist an integer $\kappa \geq 0$ and $c_0 > 0$ such that

$$\sum_{|\alpha| < \kappa} |\partial^{\alpha} \mathbf{B}(x)| \ge c_0 \tag{1.4}$$

for any $x \in \overline{\Omega}$. Recall that the nontangential maximal function is defined by

$$(u)^*(x) = \sup \{|u(y)| : y \in \Omega \text{ and } |y - x| < M_0 \operatorname{dist}(y, \partial\Omega)\}$$
(1.5)

for $x \in \partial \Omega$, where $M_0 > 1$ is a large (fixed) constant depending on Ω . The following two theorems are the main results of this paper.

Theorem 1.1. Let Ω be a bounded Lipschitz domain in \mathbb{R}^d , $d \geq 2$. Let $\mathbf{A} \in C^{\infty}(\overline{\Omega}; \mathbb{R}^d)$. Suppose \mathbf{B} is of finite type on $\overline{\Omega}$. Then for any $g \in L^2(\partial\Omega, \mathbb{C})$, a solution $u \in H^1(\Omega; \mathbb{C})$ to the Neumann problem (1.2) satisfies the nontangential maximal function estimate,

$$\|(v_h)^*\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}$$
 (1.6)

for $h \in (0, h_0)$, where $v_h(x) = |(hD + \mathbf{A})u(x)| + h m(x, h^{-1}\mathbf{B})|u(x)|$. Moreover, we have

$$\int_{\Omega} m(x, h^{-1}\mathbf{B}) |v_h(x)|^2 dx \le C \int_{\partial \Omega} |g|^2.$$
(1.7)

The constants C in (1.6)-(1.7) and h_0 depend only on Ω , (κ, c_0) in (1.4), and $\|\mathbf{B}\|_{C^{\kappa+1}(\overline{\Omega})}$.

Let $T^h = (T_{ik}^h)$, where $1 \leq j, k \leq d$ and

$$T_{jk}^{h} = n_{j}(hD_{k} + A_{k}) - n_{k}(hD_{j} + A_{j})$$

is a (tangential) differential operator on $\partial\Omega$.

Theorem 1.2. Assume that Ω and \mathbf{B} satisfy the same conditions as in Theorem 1.1. Then for any $f \in H^1(\partial\Omega;\mathbb{C})$, a solution $u \in H^1(\Omega;\mathbb{C})$ to the Dirichlet problem (1.3) satisfies the estimate,

$$\|(v_h)^*\|_{L^2(\partial\Omega)} \le C \Big\{ \|T^h f\|_{L^2(\partial\Omega)} + h \|m(x, h^{-1}\mathbf{B})f\|_{L^2(\partial\Omega)} \Big\}$$
(1.8)

for $h \in (0, h_0)$, where $v_h(x) = |(hD + \mathbf{A})u(x)| + h m(x, h^{-1}\mathbf{B})|u(x)|$. Moreover, we have

$$\int_{\Omega} m(x, h^{-1}\mathbf{B}) |v_h(x)|^2 dx \le C \Big\{ \|T^h f\|_{L^2(\partial\Omega)}^2 + h^2 \|m(x, h^{-1}\mathbf{B}) f\|_{L^2(\partial\Omega)}^2 \Big\}.$$
 (1.9)

The constants C in (1.8)-(1.9) and h_0 depend only on Ω , (κ, c_0) in (1.4), and $\|\mathbf{B}\|_{C^{\kappa+1}(\overline{\Omega})}$.

A few remarks are in order.

Remark 1.3. In the case $\mathbf{A} \equiv 0$ in Ω and h = 1, the Neumann and Dirichlet problems (1.2) and (1.3) reduce to

$$\Delta u = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} = g \quad \text{on } \partial \Omega,$$
 (1.10)

and

$$\Delta u = 0 \quad \text{in } \Omega \quad \text{and} \quad u = f \quad \text{on } \partial \Omega,$$
 (1.11)

respectively. In this case, it was proved by D. Jerison and C. Kenig [4, 5] that the weak solutions in $H^1(\Omega)$ to (1.10) and (1.11) satisfy the nontangential maximal function estimates,

$$\|(\nabla u)^*\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}$$
 and $\|(\nabla u)^*\|_{L^2(\partial\Omega)} \le C\|f\|_{H^1(\partial\Omega)}$,

respectively, assuming that Ω is a bounded Lipschitz domain. Our main results extend this classical work to the magnetic Laplacian $P^h(\mathbf{A})$ in the semiclassical setting, where the estimates are uniform in $h \in (0, h_0)$. To the best of the author's knowledge, Theorems 1.1 and 1.2 are new even for smooth domains.

Remark 1.4. The auxiliary function $m(x, \mathbf{B})$ in Theorems 1.1 and 1.2 is defined by

$$\frac{1}{m(x, \mathbf{B})} = \sup \left\{ r > 0 : \sup_{\mathcal{B}(x, r)} |\mathbf{B}| \le \frac{1}{r^2} \right\},\tag{1.12}$$

where $\mathcal{B}(x,r)$ denotes the ball centered at x with radius r. This function, which was introduced by the present author in [9, 10], plays the role of a critical scaling in the study of Schrödinger operators with electrical and magnetic potentials. See earlier work in [2, 12, 14] as well as references for more recent work in [7, 8]. Note that if \mathbf{B} is a (matrix-valued) polynomial of degree κ , then

$$\sup_{\mathcal{B}(x,r)} |\mathbf{B}| \approx \sum_{|\alpha| \le \kappa} |\partial^{\alpha} \mathbf{B}(x)| r^{|\alpha|}.$$

It follows that

$$c\sum_{|\alpha| \le \kappa} |\partial^{\alpha} \mathbf{B}(x)|^{\frac{1}{|\alpha|+2}} \le m(x, \mathbf{B}) \le C\sum_{|\alpha| \le \kappa} |\partial^{\alpha} \mathbf{B}(x)|^{\frac{1}{|\alpha|+2}},$$

where C, c > 0 depend only on d and κ . Under the finite-type condition on **B** in Theorem 1.1, we have

$$m(x, h^{-1}\mathbf{B}) \ge c \sum_{|\alpha| \le \kappa} h^{-\frac{1}{|\alpha|+2}} |\partial^{\alpha}\mathbf{B}(x)|^{\frac{1}{|\alpha|+2}}$$

for $h \in (0, h_0)$. See Remark 10.2.

Remark 1.5. Consider the Dirichlet-to-Neumann map associated with the operator $P^h(\mathbf{A})$,

$$\Lambda^h: f \to n \cdot (hD + \mathbf{A})u$$
,

where u is the solution of the Dirichlet problem (1.3). It follows from Theorems 1.1 and 1.2 that

$$\|\Lambda^h f\|_{L^2(\partial\Omega)} \approx \|T^h f\|_{L^2(\partial\Omega)} + h\|m(x, h^{-1}\mathbf{B})f\|_{L^2(\partial\Omega)}$$
 (1.13)

for $h \in (0, h_0)$. Note that $T^h f$ represents the tangential component of $(hD + \mathbf{A})u$ on $\partial\Omega$. In fact, our proof yields the weighted estimates,

$$||m(x, h^{-1}\mathbf{B})^{\ell} \Lambda^{h} f||_{L^{2}(\partial\Omega)} \approx ||m(x, h^{-1}\mathbf{B})^{\ell} T^{h} f||_{L^{2}(\partial\Omega)} + h||m(x, h^{-1}\mathbf{B})^{\ell+1} f||_{L^{2}(\partial\Omega)}$$

for any $\ell \in \mathbb{R}$ (the bounding constants depend on ℓ). See Theorem 7.9.

Remark 1.6. Let u be a solution of the Dirichlet problem (1.3) with $f \in C(\partial\Omega; \mathbb{C})$. Then

$$||(u)^*||_{L^2(\partial\Omega)} \le C||f||_{L^2(\partial\Omega)}.$$
 (1.14)

This follows readily from the maximum principle for subharmonic functions. No addition condition beyond $\mathbf{A} \in C^1(\overline{\Omega}; \mathbb{R}^d)$ is not needed. See Theorem 2.5.

Remark 1.7. Suppose **B** is of finite type on $\partial\Omega$; i.e., the inequality (1.4) holds for any $x \in \partial\Omega$. By compactness it follows that (1.4) holds in a neighborhood of $\partial\Omega$. In this case, the estimates (1.6) and (1.8) hold if one modifies the definition of the nontangential maximal function $(u)^*$ by considering only points near $\partial\Omega$,

$$(u)^*(x) = \sup \{|u(y)|: y \in \Omega, |y - x| < M_0 \operatorname{dist}(y, \partial \Omega) \text{ and } \operatorname{dist}(y, \partial \Omega) < c\}.$$

This follows directly from the proofs of Theorems 1.1 and 1.2.

We now describe the main ingredients in the proofs of Theorems 1.1 and 1.2. First, we rewrite the equation $P^h(\mathbf{A})u = 0$ as $(D + \beta \mathbf{A})^2 u = 0$, where $\beta = h^{-1} > 1$. By a localization argument, we reduce the problem to the equation $(D + \mathbf{A})^2 u = 0$ in a bounded Lipschitz domain Ω , assuming that $\Omega \subset \mathcal{B}(0, R_0)$ for some $R_0 > 0$ and the magnetic field \mathbf{B} satisfies the following conditions. There exists $C_0 > 0$ such that

$$\sup_{\mathcal{B}(x,r)} |\nabla \mathbf{B}| \le \frac{C_0}{r} \oint_{\mathcal{B}(x,r)} |\mathbf{B}| \tag{1.15}$$

for any $\mathcal{B}(x,r) \subset \mathcal{B}(0,4R_0)$, and

$$\sup_{\mathcal{B}(x_0, R_0)} |\mathbf{B}| \ge 4R_0^{-2} \tag{1.16}$$

for any $x_0 \in \overline{\Omega}$. Note that when **B** is replaced by β **B**, the condition (1.15) remains invariant, while (1.16) is satisfied for β sufficiently large. Moreover, under the conditions (1.15)-(1.16), it is proved in [8] that

$$\int_{\Omega} m(x, \mathbf{B})^2 |\psi|^2 dx \le C \int_{\Omega} |(D + \mathbf{A})\psi|^2 dx \tag{1.17}$$

for any $\psi \in C^1(\overline{\Omega}; \mathbb{C})$.

Next, let $u \in H^1(\Omega; \mathbb{C})$ be a weak solution of $(D + \mathbf{A})^2 u = 0$ in Ω . By an approximation argument, we may assume that Ω and thus u are smooth. To relate the L^2 norm of the normal component of $(D + \mathbf{A})u$ on $\partial\Omega$ to the L^2 norm of its tangential components, we use two Rellich-type identities, adapted to the operator $(D + \mathbf{A})^2$. See Lemmas 7.1 and 7.2. As a result, the proof for (1.13) with h = 1 is reduced to the estimates,

$$\begin{cases}
\int_{\Omega} m(x, \mathbf{B})^{3} |u|^{2} dx \leq C \int_{\partial \Omega} m(x, \mathbf{B})^{2} |u|^{2} dx, \\
\int_{\Omega} m(x, \mathbf{B})^{3} |u|^{2} dx \leq C \int_{\partial \Omega} |n \cdot (D + \mathbf{A})u|^{2} dx.
\end{cases} (1.18)$$

To establish (1.18), we study the Green function $G_{\mathbf{A}}(x,y)$ and the Neumann function $N_{\mathbf{A}}(x,y)$ for the operator $(D+\mathbf{A})^2$ in Ω . Suppose $d \geq 3$ (the case d=2 is handled by the method of descending). Using (1.17), we are able to show that under the assumptions (1.15)-(1.16),

$$\begin{cases}
|G_{\mathbf{A}}(x,y)| \leq \frac{C_{\ell}}{\{1 + |x - y|m(x, \mathbf{B})\}^{\ell}} \cdot \frac{1}{|x - y|^{d-2}}, \\
|N_{\mathbf{A}}(x,y)| \leq \frac{C_{\ell}}{\{1 + |x - y|m(x, \mathbf{B})\}^{\ell}} \cdot \frac{1}{|x - y|^{d-2}}
\end{cases}$$
(1.19)

for any $\ell \geq 0$ and $x, y \in \Omega$. See Sections 5 and 6. The estimates in (1.18) follow from (1.19) by using the Rellich identities mentioned above and a duality argument.

Finally, with the Rellich estimates (1.13) at our disposal, we establish the nontangential maximal function estimates (1.6) and (1.8) with the help of (1.14). A key observation here is that although the commutator $[(D + \mathbf{A})^2, D_k + A_k]$ may not be zero in Ω , the function $v(x) = |(D + \mathbf{A})u(x)| + m(x, \mathbf{B})|u(x)|$ nevertheless satisfies the interior estimate,

$$v(x) \le C \int_{\mathcal{B}(x,r)} v,$$

if $\mathcal{B}(x,r) \subset \Omega$. See Section 4. This allows us to use the techniques developed for (1.18) to control the additional error terms.

2 Preliminaries

Throughout the section we assume that Ω is a bounded Lipschitz domain and $\mathbf{A} \in C^1(\Omega; \mathbb{R}^d)$.

Lemma 2.1. Let $\varepsilon > 0$ and $u_{\varepsilon} = \sqrt{|u|^2 + \varepsilon^2}$. Then for $u \in C^1(\Omega; \mathbb{C})$,

$$|\nabla u_{\varepsilon}| \le |(\nabla + i\mathbf{A})u|,\tag{2.1}$$

and for $u \in C^2(\Omega; \mathbb{C})$,

$$u_{\varepsilon} \Delta u_{\varepsilon} \ge \Re \left\{ (\nabla + i\mathbf{A})^2 u \cdot \overline{u} \right\}.$$
 (2.2)

Proof. By differentiating $u_{\varepsilon}^2 = |u|^2 + \varepsilon^2$, we obtain

$$u_{\varepsilon}\partial_{i}u_{\varepsilon} = \Re\{(\partial_{i} + iA_{i})u \cdot \overline{u}\}$$
(2.3)

for $1 \leq j \leq d$. Hence, $u_{\varepsilon} |\nabla u_{\varepsilon}| \leq |(\nabla + i\mathbf{A})u||u|$. Since $|u| \leq u_{\varepsilon}$, this yields (2.1).

Next, by differentiating (2.3), we see that

$$|\nabla u_{\varepsilon}|^{2} + u_{\varepsilon} \Delta u_{\varepsilon} = \Re\{(\nabla + i\mathbf{A})^{2} u \cdot \overline{u}\} + |(\nabla + i\mathbf{A})u|^{2}, \tag{2.4}$$

which, together with (2.1) and $|u| \leq u_{\varepsilon}$, leads to (2.2).

Lemma 2.2. Let $u \in H_0^1(\Omega; \mathbb{C})$. Then

$$\left(\int_{\Omega} |u|^{p_d}\right)^{1/p_d} \le C\left(\int_{\Omega} |(D+\mathbf{A})u|^2\right)^{1/2},\tag{2.5}$$

where $p_d = \frac{2d}{d-2}$ for $d \geq 3$, $2 < p_d < \infty$ for d = 2, and C depends only on Ω and p_d . Furthermore, if $u \in H^1(\Omega; \mathbb{C})$, then

$$\left(\int_{\Omega} |u|^{p_d}\right)^{1/p_d} \le C \left(\int_{\Omega} |(D+\mathbf{A})u|^2\right)^{1/2} + C \left(\int_{\Omega} |u|^2\right)^{1/2},\tag{2.6}$$

and

$$\int_{\partial\Omega} |u|^2 \le C \int_{\Omega} |(D+\mathbf{A})u|^2 + C \int_{\Omega} |u|^2. \tag{2.7}$$

Proof. Let $v_{\varepsilon} = \sqrt{|u|^2 + \varepsilon^2} - \varepsilon$, where $\varepsilon > 0$. It follows from (2.1) that $|\nabla v_{\varepsilon}| \leq |(D + \mathbf{A})u|$. Since $v_{\varepsilon} \in H_0^1(\Omega; \mathbb{R})$, by Sobolev inequality,

$$\left(\int_{\Omega} |v_{\varepsilon}|^{p_d}\right)^{1/p_d} \le C \left(\int_{\Omega} |\nabla v_{\varepsilon}|^2\right)^{1/2} \le C \left(\int_{\Omega} |(D+\mathbf{A})u|^2\right)^{1/2}.$$

By letting $\varepsilon \to 0$, we obtain (2.5) for $u \in C_0^1(\Omega, \mathbb{C})$. A density argument gives (2.5) for $u \in H_0^1(\Omega; \mathbb{C})$. The proofs for (2.6) and (2.7) are similar.

Theorem 2.3. Let $u \in C^2(\mathcal{B}(x_0, 2r); \mathbb{C})$ be a solution of $(D + \mathbf{A})^2 u = F$ in $\mathcal{B}(x_0, 2r)$. Then

$$\left(\oint_{\mathcal{B}(x_0,r)} |u|^q \right)^{1/q} \le C \oint_{\mathcal{B}(x_0,2r)} |u| + Cr^2 \left(\oint_{\mathcal{B}(x_0,2r)} |F|^p \right)^{1/p}, \tag{2.8}$$

where $2 \le p < q \le \infty$ and $\frac{1}{p} - \frac{1}{q} < \frac{2}{d}$. The constant C depends only on d, p and q,

Proof. We give the proof for $d \geq 3$. The case d = 2 is similar. Let $u_{\varepsilon} = \sqrt{|u|^2 + \varepsilon^2}$, where $\varepsilon > 0$. By (2.2), we have $\Delta u_{\varepsilon} \geq -|F|$ in $\mathcal{B}(x_0, 2r)$. By differentiating $\int_{\partial \mathcal{B}(y,t)} u_{\varepsilon}$ in t, as in the proof of the mean value property for harmonic functions, it follows that if $\mathcal{B}(y,t) \subset \mathcal{B}(x_0, 2r)$,

$$u_{\varepsilon}(y) \le \int_{\partial \mathcal{B}(y,t)} u_{\varepsilon} + C \int_{\mathcal{B}(y,t)} \frac{|F(z)|}{|z-y|^{d-2}} dz.$$

Hence, for $y \in \mathcal{B}(x_0, r)$,

$$u_{\varepsilon}(y) \le \int_{\mathcal{B}(u,r)} u_{\varepsilon} + C \int_{\mathcal{B}(u,r)} \frac{|F(z)|}{|z-y|^{d-2}} dz.$$

By letting $\varepsilon \to 0$, we obtain

$$|u(y)| \le \int_{B(y,r)} |u| + C \int_{B(y,r)} \frac{|F(z)|}{|z - y|^{d-2}} dz$$

for any $y \in \mathcal{B}(x_0, r)$. This yields (2.8) by using Young's inequality.

Lemma 2.4. Let $u \in C^2(\Omega; \mathbb{C}) \cap C(\overline{\Omega}; \mathbb{C})$ be a solution of the Dirichlet problem,

$$(D + \mathbf{A})^2 u = 0 \quad \text{in } \Omega \quad \text{and} \quad u = f \quad \text{on } \partial\Omega, \tag{2.9}$$

where $f \in C(\partial\Omega; \mathbb{C})$. Then $|u| \leq v$ in Ω , where v is harmonic in Ω and v = |f| on $\partial\Omega$.

Proof. Let $\varepsilon > 0$ and $w_{\varepsilon} = u_{\varepsilon} - v$, where $u_{\varepsilon} = \sqrt{|u|^2 + \varepsilon^2}$. It follows from (2.2) that $\Delta w_{\varepsilon} = \Delta u_{\varepsilon} \geq 0$ in Ω . Thus, w_{ε} is subharmonic in Ω . Note that

$$w_{\varepsilon} = \sqrt{|f|^2 + \varepsilon^2} - |f| \le \varepsilon$$
 in $\partial \Omega$.

By the maximum principle for subharmonic functions, this implies that

$$\max_{\overline{\Omega}} w_{\varepsilon} = \max_{\partial \Omega} w_{\varepsilon} \le \varepsilon.$$

Hence, $u_{\varepsilon} \leq v + \varepsilon$ in Ω . By letting $\varepsilon \to 0$, we obtain $|u| \leq v$ in Ω .

Theorem 2.5. Let Ω be a bounded Lipschitz domain and $\mathbf{A} \in C^1(\Omega; \mathbb{R}^d)$. Let $u \in C^2(\Omega; \mathbb{C}) \cap C(\overline{\Omega}; \mathbb{C})$ be a solution of the Dirichlet problem (2.9). Then

$$||(u)^*||_{L^p(\partial\Omega)} \le C||f||_{L^p(\partial\Omega)}$$
(2.10)

for $2-\delta , where <math>\delta > 0$ depends on Ω . The constant C in (2.10) depends only on d, p, the Lipschitz character of Ω . If Ω is C^1 , the estimate (2.10) holds for 1 .

Proof. By Lemma 2.4, we see that $(u)^* \leq (v)^*$ on $\partial\Omega$, where v is a harmonic function in Ω such that v = |f| on $\partial\Omega$. As a consequence, the estimate (2.10) follows from the well-known results for harmonic functions in Lipschitz and C^1 domains [6].

3 The function $m(x, \mathbf{B})$

Let Ω be a bounded Lipschitz domain in \mathbb{R}^d , $d \geq 2$. It follows that there exist $r_0 > 0$ and $M_0 > 0$ such that for any $x_0 \in \partial \Omega$,

$$\Omega \cap \mathcal{B}(x_0, r_0) = \left\{ (x', x_d) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \phi(x') \right\} \cap \mathcal{B}(x_0, r_0)$$
(3.1)

in a new coordinate system, obtained from the standard one through translation and rotation, where $\phi: \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function with $\|\nabla \phi\|_{\infty} \leq M_0$ and $\phi(x_0) = 0$. A constant C is said to depend on the Lipschitz character of Ω if C depends on M_0 and the number of balls $\mathcal{B}(x_0, r_0)$, centered on $\partial\Omega$, which are needed to cover $\partial\Omega$. By translation, without the loss of generality, we assume that $\Omega \subset \mathcal{B}(0, R_0)$ for some $R_0 = Cr_0 > 0$, where C depends on the Lipschitz character of Ω .

We will impose the following conditions on **B** in this and next few sections: $\mathbf{B} \in C^1(\mathcal{B}(0, 4R_0); \mathbb{R}^{d \times d})$ and there exists $C_0 > 0$ such that

$$\sup_{\mathcal{B}(x,r)} |\nabla \mathbf{B}| \le \frac{C_0}{r} \oint_{\mathcal{B}(x,r)} |\mathbf{B}| \tag{3.2}$$

for any ball $\mathcal{B}(x,r) \subset \mathcal{B}(0,4R_0)$. It follows from (3.2) that

$$\sup_{\mathcal{B}} |\mathbf{B}| \le C_1 \oint_{\mathcal{B}} |\mathbf{B}| \tag{3.3}$$

for any ball $\mathcal{B} \subset \mathcal{B}(0, 4R_0)$; i.e., $|\mathbf{B}|$ is a B_{∞} weight in $\mathcal{B}(0, 4R_0)$. In particular, $|\mathbf{B}|$ satisfies the doubling condition,

$$\int_{2\mathcal{B}} |\mathbf{B}| \le C \int_{\mathcal{B}} |\mathbf{B}| \tag{3.4}$$

for any $2\mathcal{B} \subset \mathcal{B}(0,4R_0)$, where C depends only on d and C_1 in (3.3). As a consequence, we also have

$$\sup_{Q(x,r)} |\mathbf{B}| \le C \oint_{Q(x,r)} |\mathbf{B}| \tag{3.5}$$

if $Q(x,r) \subset \mathcal{B}(0,2R_0)$, where Q(x,r) denotes the cube center at x with side length r.

For $x \in \mathcal{B}(0, R_0)$, let $m(x, \mathbf{B})$ be defined by (1.12). Under the assumption that for any $x \in \mathcal{B}(0, R_0)$,

$$\sup_{\mathcal{B}(x,R_0/2)} |\mathbf{B}| > 4R_0^{-2},\tag{3.6}$$

it follows by definition that $m(x, \mathbf{B}) \geq 2R_0^{-1}$ for any $x \in \mathcal{B}(0, R_0)$.

Lemma 3.1. Suppose **B** satisfies (3.2) for any ball $\mathcal{B}(x,r) \subset \mathcal{B}(0,4R_0)$. Also assume that (3.6) holds for any $x \in \mathcal{B}(0,R_0)$. Then,

$$|\mathbf{B}(x)|^{1/2} + |\nabla \mathbf{B}(x)|^{1/3} \le Cm(x, \mathbf{B})$$
 (3.7)

for any $x \in \mathcal{B}(0, R_0)$. Moreover, for any $x, y \in \mathcal{B}(0, R_0)$,

$$m(y, \mathbf{B}) \le C \{1 + |x - y| m(x, \mathbf{B})\}^{\kappa_0} m(x, \mathbf{B}),$$
 (3.8)

$$m(y, \mathbf{B}) \ge \frac{c \, m(x, \mathbf{B})}{\{1 + |x - y| m(x, \mathbf{B})\}^{\frac{\kappa_0}{\kappa_0 + 1}}}.$$
 (3.9)

The constants $C, c, \kappa_0 > 0$ depend only on C_0 in (3.2).

Proof. Let $r = \{m(x, \mathbf{B})\}^{-1}$. Using (3.6), it is not hard to see that

$$\sup_{\mathcal{B}(x,r)} |\mathbf{B}| = r^{-2}.\tag{3.10}$$

This implies that $|\mathbf{B}(x)| \le r^{-2} = \{m(x, \mathbf{B})\}^2$. Also, by (3.2) and (3.10),

$$|\nabla \mathbf{B}(x)| \le C_0 r^{-3} = C_0 \{ m(x, \mathbf{B}) \}^3.$$

As a result, we have proved (3.7).

The inequalities (3.8)-(3.9) were proved by the present author in [10] for any $x, y \in \mathbb{R}^d$, under the assumption that $|\mathbf{B}|$ is a B_q weight in \mathbb{R}^d for some q > (d/2). With the conditions in the lemma, the same argument gives (3.8)-(3.9) for any $x, y \in \mathcal{B}(0, R_0)$. Note that the estimates (3.7)-(3.8) are scaling invariant; the constants C, c, κ_0 do not depend on R_0 .

It follows readily from (3.8)-(3.9) that

$$c\left\{1 + |x - y| m(y, \mathbf{B})\right\}^{\frac{1}{\kappa_0 + 1}} \le 1 + |x - y| m(x, \mathbf{B})$$

$$\le C\left\{1 + |x - y| m(y, \mathbf{B})\right\}^{\kappa_0 + 1}$$
(3.11)

for any $x, y \in \mathcal{B}(0, R_0)$.

Theorem 3.2. Suppose **B** satisfies (3.2) for any ball $\mathcal{B}(x,r) \subset \mathcal{B}(0,4R_0)$. Also assume that (3.6) holds for any $x \in \mathcal{B}(0,R_0)$. Then

$$c \int_{\Omega} \{m(x, \mathbf{B})\}^2 |\psi|^2 dx \le \int_{\Omega} |(D + \mathbf{A})\psi|^2 dx$$
 (3.12)

for any $\psi \in C^1(\overline{\Omega}; \mathbb{C})$, where c > 0 depends only on C_0 in (3.2) and the Lipschitz character of Ω .

Proof. See [8, Theorem 3.8]. We point out that the proof in [8] only uses the fact $\mathbf{B} = \nabla \times \mathbf{A}$ in Ω . Outside of Ω , there is no need to assume that \mathbf{B} is a curl of some vector field. This observation allows us to extend \mathbf{B} to a neighborhood of Ω without extending \mathbf{A} in the proof of our main results.

The next lemma gives a Caccioppoli inequality for the operator $(D + \mathbf{A})^2$. The conditions on **B** are not needed.

Lemma 3.3. Let $x_0 \in \overline{\Omega}$ and $0 < r < r_0$, where either $x_0 \in \partial\Omega$ or $\mathcal{B}(x_0, r) \subset \Omega$. Suppose that $u \in H^1(\mathcal{B}(x_0, r) \cap \Omega; \mathbb{C})$ and $(D + \mathbf{A})^2 u = F$ in $\mathcal{B}(x_0, r) \cap \Omega$, where $F \in L^2(\mathcal{B}(x_0, r) \cap \Omega; \mathbb{C})$. If $x_0 \in \partial\Omega$, we also assume that either u = 0 or $n \cdot (D + \mathbf{A})u = 0$ on $\mathcal{B}(x_0, r) \cap \partial\Omega$, where n denotes the outward unit normal to $\partial\Omega$. Then for 0 < s < t < 1,

$$\int_{\mathcal{B}(x_0, sr) \cap \Omega} |(D + \mathbf{A})u|^2 \le \frac{C}{(t - s)^2 r^2} \int_{\mathcal{B}(x_0, tr) \cap \Omega} |u|^2 + Cr^2 \int_{\mathcal{B}(x_0, tr) \cap \Omega} |F|^2, \tag{3.13}$$

where C depends only on d.

Proof. The proof is similar to the case of $\mathbf{A} \equiv 0$. Using

$$\int_{\Omega} (D + \mathbf{A}) u \cdot \overline{(D + \mathbf{A})(u\varphi^2)} = \int_{\Omega} F \cdot \overline{u\varphi^2}$$

where $\varphi \in C_0^{\infty}(\mathcal{B}(x_0, r); \mathbb{R})$, we obtain

$$\int_{\Omega} |(D+\mathbf{A})u|^2 |\varphi|^2 \le C \int_{\Omega} |u|^2 |\nabla \varphi|^2 + C \int_{\Omega} |F||u||\varphi|^2.$$
(3.14)

To finish the proof, we choose $\varphi \in C_0^{\infty}(\mathcal{B}(x_0, tr); \mathbb{R})$ such that $\varphi = 1$ in $\mathcal{B}(x_0, sr)$ and $|\nabla \varphi| \leq C(t-s)^{-1}r^{-1}$.

Remark 3.4. Let $x_0 \in \partial \Omega$ and $0 < r < r_0$. Suppose that $u \in C^2(\overline{\mathcal{B}(x_0, r) \cap \Omega}; \mathbb{C})$ and $(D + \mathbf{A})^2 u = F$ in $\mathcal{B}(x_0, r) \cap \Omega$. It follows from the proof of Lemma 3.3 that

$$\int_{\mathcal{B}(x_0,r/2)\cap\Omega} |(D+\mathbf{A})u|^2 \le \frac{C}{r^2} \int_{\mathcal{B}(x_0,r)\cap\Omega} |u|^2 + Cr^2 \int_{\mathcal{B}(x_0,r)\cap\Omega} |F|^2 + C \int_{\mathcal{B}(x_0,r)\cap\partial\Omega} |n\cdot(D+\mathbf{A})u||u|, \tag{3.15}$$

where C depends only on d.

Theorem 3.5. Suppose B satisfies the same conditions as in Theorem 3.2. Let $x_0 \in \overline{\Omega}$ and $0 < r < r_0$, where either $x_0 \in \partial \Omega$ or $\mathcal{B}(x_0, r) \subset \Omega$. Suppose $u \in H^1(\mathcal{B}(x_0, r) \cap \Omega; \mathbb{C})$ and $(D + \mathbf{A})^2 u = 0$ in $\mathcal{B}(x_0, r) \cap \Omega$. If $x_0 \in \partial \Omega$, we also assume that either u = 0 on $\mathcal{B}(x_0, r) \cap \partial \Omega$ or $n \cdot (D + \mathbf{A})u = 0$ on $\mathcal{B}(x_0, r) \cap \partial \Omega$. Then

$$\int_{\mathcal{B}(x_0, r/2) \cap \Omega} |u|^2 \le \frac{C_{\ell}}{\{1 + rm(x_0, \mathbf{B})\}^{\ell}} \int_{\mathcal{B}(x_0, r) \cap \Omega} |u|^2$$
(3.16)

for any $\ell \geq 1$, where C_{ℓ} depends on ℓ , the Lipschitz character of Ω , and C_0 in (3.2).

Proof. We may assume $rm(x_0, \mathbf{B}) > 10$ for otherwise the estimate is trivial. Let $\varphi \in C_0^{\infty}(\mathcal{B}(x_0, r) \cap \Omega; \mathbb{R})$. It follows from (3.12) that

$$\int_{\Omega} \{m(x, \mathbf{B})\}^{2} |u\varphi|^{2} dx \le C \int_{\Omega} |(D + \mathbf{A})(u\varphi)|^{2}
\le C \int_{\Omega} |u|^{2} |\nabla \varphi|^{2} dx,$$
(3.17)

where we have used (3.14) for the last inequality. Let (1/2) < s < t < 1. Choose $\varphi \in C_0^{\infty}(\mathcal{B}(x_0, tr); \mathbb{R})$ so that $\varphi = 1$ in $\mathcal{B}(x_0, sr)$ and $|\nabla \varphi| \leq C(t - s)^{-2}r^{-2}$. By (3.17) and (3.9), this gives

$$\frac{\{m(x_0, \mathbf{B})\}^2}{\{1 + rm(x_0, \mathbf{B})\}^{\frac{2\kappa_0}{\kappa_0 + 1}}} \int_{\mathcal{B}(x_0, sr) \cap \Omega} |u|^2 \le \frac{C}{(t - s)^2 r^2} \int_{\mathcal{B}(x_0, tr) \cap \Omega} |u|^2,$$

which leads to

$$\int_{\mathcal{B}(x_0, sr) \cap \Omega} |u|^2 \le \frac{C}{(t-s)^2 \{1 + rm(x_0, \mathbf{B})\}^{\frac{2}{\kappa_0 + 1}}} \int_{\mathcal{B}(x_0, tr) \cap \Omega} |u|^2.$$
(3.18)

The estimate (3.16) now follows by iterating (3.18).

4 Interior estimates

Suppose $(D + \mathbf{A})^2 u = 0$ in $\mathcal{B}(x_0, r)$. It follows by Theorem 2.3 that

$$|u(x_0)| \le C \int_{\mathcal{B}(x_0,r)} |u|,$$
 (4.1)

where C depends only on d. In this section we establish pointwise estimates for $|(D + \mathbf{A})u|$ and $m(\cdot, \mathbf{B})|u|$ under the conditions (3.2) and (3.6) on the magnetic field \mathbf{B} .

Theorem 4.1. Assume **B** satisfies the condition (3.2) for any ball $\mathcal{B}(x,r) \subset \mathcal{B}(0,4R_0)$. Also assume that (3.6) holds for any $x \in \mathcal{B}(0,R_0)$. Suppose $(D+\mathbf{A})^2u = 0$ in $\mathcal{B}(x_0,r) \subset \mathcal{B}(0,R_0)$. Then for any $\ell \geq 0$,

$$|u(x_0)| \le \frac{C_\ell}{\{1 + rm(x_0, \mathbf{B})\}^\ell} \int_{\mathcal{B}(x_0, r)} |u|,$$
 (4.2)

$$m(x_0, \mathbf{B})|u(x_0)| \le \frac{C_\ell}{\{1 + rm(x_0, \mathbf{B})\}^\ell} \int_{\mathcal{B}(x_0, r)} m(x, \mathbf{B})|u(x)| dx,$$
 (4.3)

where C_{ℓ} depends on ℓ and C_0 in (3.2).

Proof. It follows from (4.1) and (3.16) that

$$|u(x_0)| \le C \left(f_{\mathcal{B}(x_0, r/4)} |u|^2 \right)^{1/2} \le \frac{C_{\ell}}{\{1 + rm(x_0, \mathbf{B})\}^{\ell}} \left(f_{\mathcal{B}(x_0, r/2)} |u|^2 \right)^{1/2}$$

$$\le \frac{C_{\ell}}{\{1 + rm(x_0, \mathbf{B})\}^{\ell}} f_{\mathcal{B}(x_0, r)} |u|$$

$$(4.4)$$

for any $\ell \geq 1$. To see (4.3), note that by (3.9),

$$m(x_0, \mathbf{B}) \le C\{1 + rm(x_0, \mathbf{B})\}^{\frac{\kappa_0}{\kappa_0 + 1}} m(x, \mathbf{B}) \tag{4.5}$$

for any $x \in \mathcal{B}(x_0, r)$. This, together with (4.2), gives (4.3).

Theorem 4.2. Under the same assumptions as in Theorem 4.1, we have

$$|(D+\mathbf{A})u(x_0)| \le \frac{C}{r} \oint_{\mathcal{B}(x_0,r)} |u|,\tag{4.6}$$

where C depends on C_0 in (3.2).

Proof. Note that

$$(D + \mathbf{A})^{2}(D_{k} + A_{k})u$$

$$= -[D_{k} + A_{k}, D_{j} + A_{j}](D_{j} + A_{j})u - (D_{j} + A_{j})[D_{k} + A_{k}, D_{j} + A_{j}]u$$

$$= -2[D_{k} + A_{k}, D_{j} + A_{j}](D_{j} + A_{j})u - [[D_{j} + A_{j}, [D_{k} + A_{k}, D_{j} + A_{j}]]u$$

where the repeated index j is summed from 1 to d. It follows that $(D + \mathbf{A})^2(D_k + A_k)u = F$, where

$$|F| \le 2|\mathbf{B}||(D+\mathbf{A})u| + |\nabla \mathbf{B}||u|.$$

In view of (2.8), this implies that if $\mathcal{B}(y,2t) \subset \mathcal{B}(x_0,r/4)$,

$$\left(\oint_{\mathcal{B}(y,t)} |(D+\mathbf{A})u|^q \right)^{1/q} \leq C \oint_{\mathcal{B}(y,2t)} |(D+\mathbf{A})u| + Ct^2 \left(\oint_{\mathcal{B}(y,2t)} |\mathbf{B}|^p |(D+\mathbf{A})u|^p \right)^{1/p} + Ct^2 \left(\oint_{\mathcal{B}(y,2t)} |\nabla \mathbf{B}|^p |u|^p \right)^{1/p}, \tag{4.7}$$

where $1 \le p < q \le \infty$ and $\frac{1}{p} - \frac{1}{q} < \frac{2}{d}$. Using $|\mathbf{B}(x)| \le Cm(x, \mathbf{B})^2$ and $|\nabla \mathbf{B}(x)| \le Cm(x, \mathbf{B})^3$, we obtain

$$\left(\oint_{\mathcal{B}(y,t)} |(D+\mathbf{A})u|^q \right)^{1/q} \leq C \{ 1 + r \sup_{\mathcal{B}(x_0,r)} m(\cdot, \mathbf{B}) \}^2 \left(\oint_{\mathcal{B}(y,2t)} |(D+\mathbf{A})u|^p \right)^{1/p} + C \{ 1 + r \sup_{\mathcal{B}(x_0,r)} m(\cdot, \mathbf{B}) \}^3 \frac{1}{r} \oint_{\mathcal{B}(x_0,r/2)} |u|,$$

if $\mathcal{B}(y,2t) \subset \mathcal{B}(x_0,r/4)$. It then follows by an iteration argument that

$$|(D + \mathbf{A})u(x_0)| \le C\{1 + r \sup_{\mathcal{B}(x_0, r)} m(\cdot, \mathbf{B})\}^N \left(\int_{\mathcal{B}(x_0, r/4)} |(D + \mathbf{A})u|^2 \right)^{1/2}$$

$$+ C\{1 + r \sup_{\mathcal{B}(x_0, r)} m(\cdot, \mathbf{B})\}^N \frac{1}{r} \int_{\mathcal{B}(x_0, r/2)} |u|$$

$$\le C\{1 + r m(x_0, \mathbf{B})\}^{N_1} \frac{1}{r} \int_{\mathcal{B}(x_0, r/2)} |u|,$$

for some $N, N_1 \ge 1$, where we have used (3.13) and (3.8) for the last step. The desired estimate now follows readily from (4.4) by choosing $\ell = N_1$.

Theorem 4.3. Under the same assumptions as in Theorem 4.1, we have

$$|(D + \mathbf{A})u(x_0)| \le C \int_{\mathcal{B}(x_0, r)} |(D + \mathbf{A})u| \, dx + C \int_{\mathcal{B}(x_0, r)} m(x, \mathbf{B})|u(x)| \, dx, \tag{4.8}$$

where C depends on C_0 in (3.2).

Proof. We consider two cases.

Case 1. Assume that

$$r^2 \sup_{\mathcal{B}(x_0, r)} |\mathbf{B}| \le 1. \tag{4.9}$$

It follows from (4.7) that if $\mathcal{B}(y, 2t) \subset \mathcal{B}(x_0, r/4)$,

$$\left(\oint_{\mathcal{B}(y,t)} |(D+\mathbf{A})u|^{q} \right)^{1/q} \leq C \left(1 + t^{2} \sup_{\mathcal{B}(y,2t)} |\mathbf{B}| \right) \left(\oint_{\mathcal{B}(y,2t)} |(D+\mathbf{A})u|^{p} \right)^{1/p}
+ Ct^{2} \sup_{\mathcal{B}(y,2t)} |\nabla \mathbf{B}| \oint_{\mathcal{B}(y,3t)} |u|$$

$$\leq C \left(\oint_{\mathcal{B}(y,2t)} |(D+\mathbf{A})u|^{p} \right)^{1/p} + Cr \sup_{\mathcal{B}(x_{0},r)} |\mathbf{B}| \oint_{\mathcal{B}(x_{0},r)} |u|,$$
(4.10)

where $1 \le p < q \le \infty$, $\frac{1}{p} - \frac{1}{q} < \frac{2}{d}$, and the fact $\sup_{\mathcal{B}(y,2t)} |\nabla \mathbf{B}| \le Ct^{-1} \sup_{\mathcal{B}(y,2t)} |\mathbf{B}|$ is used for the last step. By iterating the estimate (4.10), we obtain

$$|(D + \mathbf{A})u(x_0)| \le C \int_{\mathcal{B}(x_0,r)} |(D + \mathbf{A})u| + Cr \sup_{\mathcal{B}(x_0,r)} |\mathbf{B}| \int_{\mathcal{B}(x_0,r)} |u|.$$
 (4.11)

Note that under the assumption (4.9), we have $r < Cm(x_0, \mathbf{B})^{-1}$ and

$$r \sup_{\mathcal{B}(x_0,r)} |\mathbf{B}| \le Cr \sup_{\mathcal{B}(x_0,r)} m(\cdot, \mathbf{B})^2$$
$$\le Crm(x_0, \mathbf{B})^2$$
$$\le Cm(x_0, \mathbf{B}) \le Cm(y, \mathbf{B}),$$

for any $y \in \mathcal{B}(x_0, r)$. This, together with (4.11), yields (4.8).

Case 2. Assume that

$$r^2 \sup_{\mathcal{B}(x_0, r)} |\mathbf{B}| > 1. \tag{4.12}$$

Then $r \geq cm(x_0, \mathbf{B})^{-1}$. It follows from (4.6) that

$$|(D + \mathbf{A})u(x_0)| \le Cm(x_0, \mathbf{B}) \int_{\mathcal{B}(x_0, r/2)} |u|$$

$$\le \frac{C_{\ell}m(x_0, \mathbf{B})}{\{1 + rm(x_0, \mathbf{B})\}^{\ell}} \int_{\mathcal{B}(x_0, r)} |u|$$

for any $\ell \geq 1$, where we have used (4.4) for the last inequality. By (4.5), we obtain

$$|(D+\mathbf{A})u(x_0)| \le C \int_{\mathcal{B}(x_0,r)} m(x,\mathbf{B})|u(x)| dx,$$

which completes the proof.

We introduce a modified nontangential maximal function. For $u \in L^1_{loc}(\Omega; \mathbb{C})$, define

$$\mathcal{M}(u)(x) = \sup \left\{ \int_{\mathcal{B}(y,\delta(y)/2)} |u|: \ y \in \Omega \text{ and } |y - x| < M_0 \operatorname{dist}(y,\partial\Omega) \right\}$$
(4.13)

for $x \in \partial \Omega$.

Corollary 4.4. Assume **B** satisfies the same conditions as in Theorem 4.1. Suppose $u \in C^2(\overline{\Omega}; \mathbb{C})$ and $(D + \mathbf{A})^2 u = 0$ in Ω . Let $v(x) = |(D + \mathbf{A})u(x)| + m(x, \mathbf{B})|u(x)|$. Then

$$v(y) \le C \int_{\mathcal{B}(y,r)} v \tag{4.14}$$

for any $\mathcal{B}(y,r) \subset \Omega$. Consequently,

$$(v)^*(x) \le C\mathcal{M}(v)(x) \tag{4.15}$$

for any $x \in \partial \Omega$, where C depends on the Lipschitz character of Ω and C_0 in (3.2).

Proof. This follows readily from
$$(4.3)$$
 and (4.8) .

5 The Green function

In this section we establish decay estimates for the Green function $G_{\mathbf{A}}(x,y)$ in a bounded Lipschitz domain Ω .

Lemma 5.1. Let $x_0 \in \partial\Omega$ and $0 < r < r_0$. Let $u \in C^2(\mathcal{B}(x_0, r) \cap \Omega; \mathbb{C}) \cap C(\mathcal{B}(x_0, r) \cap \overline{\Omega}; \mathbb{C})$ be a solution of

$$\begin{cases} (D+\mathbf{A})^2 u = 0 & \text{in } \mathcal{B}(x_0, r) \cap \Omega, \\ u = 0 & \text{on } \mathcal{B}(x_0, r) \cap \partial \Omega. \end{cases}$$

$$(5.1)$$

Then

$$\sup_{\mathcal{B}(x_0, r/2) \cap \Omega} |u| \le C \int_{\mathcal{B}(x_0, r) \cap \Omega} |u|, \tag{5.2}$$

where C depends on the Lipschitz character of Ω .

Proof. Let $\varepsilon > 0$ and $v_{\varepsilon} = \sqrt{|u|^2 + \varepsilon^2} - \varepsilon \ge 0$. It follows from (2.2) and (5.1) that $\Delta v_{\varepsilon} \ge 0$. Thus, v_{ε} is subharmonic in $\mathcal{B}(x_0, r) \cap \Omega$ and $v_{\varepsilon} = 0$ on $\mathcal{B}(x_0, r) \cap \partial \Omega$. Hence,

$$\sup_{\mathcal{B}(x_0,r/2)\cap\Omega} v_{\varepsilon} \le C \int_{\mathcal{B}(x_0,r)\cap\Omega} v_{\varepsilon},$$

where C depends only on Ω . By letting $\varepsilon \to 0$, we obtain (5.2).

Consider the Dirichlet problem,

$$\begin{cases} (D+\mathbf{A})^2 u = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$
 (5.3)

Lemma 5.2. For $F \in H^{-1}(\Omega; \mathbb{C})$, the Dirichlet problem (5.3) has a unique weak solution in $H_0^1(\Omega; \mathbb{C})$. Moreover, if $F \in L^2(\Omega; \mathbb{C})$, the solution satisfies

$$||u||_{L^{p_d}(\Omega)} + ||(D + \mathbf{A})u||_{L^2(\Omega)} \le C||F||_{L^{p'_d}(\Omega)},$$
 (5.4)

where $p_d = \frac{2d}{d-2}$ for $d \ge 3$ and $2 \le p_d < \infty$ for d = 2. The constant C in (5.4) depends at most on d, p_d and Ω .

Proof. Consider the bilinear form

$$B[u,v] = \int_{\Omega} (D + \mathbf{A})u \cdot \overline{(D + \mathbf{A})v}$$
(5.5)

for $u, v \in H_0^1(\Omega; \mathbb{C})$. Since **A** is bounded in Ω , we have

$$B[u, u] \ge c \int_{\Omega} |\nabla u|^2 - C(\mathbf{A}) \int_{\Omega} |u|^2$$
$$\ge c \int_{\Omega} |\nabla u|^2 - C(\mathbf{A}) B[u, u],$$

where c > 0, $C(\mathbf{A})$ depends on \mathbf{A} , and we have used (2.5) for the second inequality. It follows that

$$B[u, u] \ge \frac{c}{1 + C(\mathbf{A})} \int_{\Omega} |\nabla u|^2.$$

Thus, by the Lax-Milgram Theorem, (5.3) has a unique weak solution in $H_0^1(\Omega; \mathbb{C})$ for any $F \in H^{-1}(\Omega; \mathbb{C})$. To see (5.4), we note that

$$\int_{\Omega} |(D + \mathbf{A})u|^2 \le \int_{\Omega} |u||F| \le ||u||_{L^{p_d}(\Omega)} ||F||_{L^{p'_d}(\Omega)}.$$
 (5.6)

This, together with (2.5), gives (5.4).

Remark 5.3. Suppose $F \in L^{\infty}(\Omega; \mathbb{C})$. Since

$$(D+\mathbf{A})^2 u = -\Delta u + 2\mathbf{A} \cdot Du + (D \cdot \mathbf{A} + |\mathbf{A}|^2)u, \tag{5.7}$$

under the assumption $\mathbf{A} \in C^1(\overline{\Omega}; \mathbb{R}^d)$ and Ω is Lipschitz, the weak solution of (5.3) is Hölder continuous in $\overline{\Omega}$.

Remark 5.4. Let $f \in H^{1/2}(\partial\Omega; \mathbb{C})$. Choose $\Phi \in H^1(\Omega; \mathbb{C})$ such that $\Phi = f$ on $\partial\Omega$ and $\|\Phi\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)}$. By considering $u - \Phi$ and applying Lemma 5.2, one obtains a unique week solution in $H^1(\Omega; \mathbb{C})$ for the Dirichlet problem (1.3).

Theorem 5.5. There is a continuous function $G_{\mathbf{A}}:\{(x,y)\in\overline{\Omega}\times\overline{\Omega}:x\neq y\}\to\mathbb{C}$ with the following properties.

1. For any $x, y \in \Omega$, $x \neq y$, and $\sigma \in (0, 1)$,

$$|G_{\mathbf{A}}(x,y)| \le \begin{cases} C|x-y|^{2-d} & \text{if } d \ge 3, \\ C_{\sigma}|x-y|^{-\sigma} & \text{if } d = 2, \end{cases}$$
 (5.8)

where C depends only on r_0 and the Lipschitz character of Ω , and C_{σ} depends only on σ and Ω .

2. For any $x, y \in \Omega$, $x \neq y$,

$$G_{\mathbf{A}}(x,y) = \overline{G_{\mathbf{A}}(y,x)}. (5.9)$$

3. Fix $y \in \Omega$. Then $G_{\mathbf{A}}(\cdot, y) \in W^{1,2}_{loc}(\Omega \setminus \{y\}; \mathbb{C})$, $G_{\mathbf{A}}(\cdot, y) = 0$ on $\partial\Omega$, and

$$(D + \mathbf{A})^2 G_{\mathbf{A}}(\cdot, y) = 0 \quad \text{in } \Omega \setminus \{y\}.$$
 (5.10)

4. For $F \in L^{\infty}(\Omega; \mathbb{C})$, the weak solution of (5.3) is given by

$$u(x) = \int_{\Omega} G_{\mathbf{A}}(x, y) F(y) \, dy. \tag{5.11}$$

Proof. The proof is similar to that in the case of second-order elliptic operators in divergence form [1, 3]. Fix $y \in \Omega$ and $\rho \in (0,1)$. Let $G^{\rho}_{\mathbf{A}}(\cdot,y)$ denote the weak solution of (5.3) with $F = \frac{1}{|\mathcal{B}(y,\rho)\cap\Omega|}\chi_{\mathcal{B}(y,\rho)\cap\Omega}$, given by Lemma 5.2. Thus,

$$\int_{\Omega} (D + \mathbf{A}) G_{\mathbf{A}}^{\rho}(\cdot, y) \cdot \overline{(D + \mathbf{A})\psi} = \int_{\mathcal{B}(y, \rho) \cap \Omega} \overline{\psi}$$
(5.12)

for any $\psi \in H_0^1(\Omega; \mathbb{C})$. It follows from (5.4) that

$$||G_{\mathbf{A}}^{\rho}(\cdot,y)||_{L^{p_{d}}(\Omega)} + ||(D+\mathbf{A})G_{\mathbf{A}}^{\rho}(\cdot,y)||_{L^{2}(\Omega)} \le C\rho^{-\frac{d}{p_{d}}},\tag{5.13}$$

where $p_d = \frac{2d}{d-2}$ for $d \geq 3$ and $2 \leq p_d < \infty$ for d = 2.

Next, we fix $z \in \Omega$ and $z \neq y$. Let r = (1/2)|y-z|. For $F \in C_0^{\infty}(\mathcal{B}(z,r) \cap \Omega; \mathbb{C})$, let $u \in H_0^1(\Omega; \mathbb{C})$ be the weak solution of (5.3). Since $(D + \mathbf{A})^2 u = 0$ in $\Omega \setminus \mathcal{B}(z,r)$ and u = 0 on $\partial \Omega$, it follows by (5.2), (2.8) and (5.4) that

$$\sup_{\mathcal{B}(y,r/2)\cap\Omega} |u| \le C \left(\int_{\mathcal{B}(y,r)\cap\Omega} |u|^{p_d} \right)^{1/p_d}$$

$$\le Cr^{-\frac{d}{p_d}} ||u||_{L^{p_d}(\Omega)} \le Cr^{-\frac{d}{p_d}} ||F||_{L^{p'_d}(\Omega)}.$$
(5.14)

Also, note that

$$\int_{\Omega} (D + \mathbf{A}) u \cdot \overline{(D + \mathbf{A})\psi} = \int_{\Omega} F \cdot \overline{\psi}$$
 (5.15)

for any $\psi \in H_0^1(\Omega; \mathbb{C})$. In view of (5.12) and (5.15), we have

$$\int_{\Omega} F(x) \cdot \overline{G_{\mathbf{A}}^{\rho}(x, y)} \, dx = \oint_{\mathcal{B}(y, \rho) \cap \Omega} u. \tag{5.16}$$

This, together with (5.14), gives

$$\left| \int_{\Omega} F(x) \cdot \overline{G_{\mathbf{A}}^{\rho}(x, y)} \, dx \right| \le C r^{-\frac{d}{p_d}} \|F\|_{L^{p'_d}(\Omega)},$$

provided $0 < \rho < (1/2)r$. By duality, we obtain

$$||G_{\mathbf{A}}^{\rho}(\cdot,y)||_{L^{p_d}(\mathcal{B}(z,r)\cap\Omega)} \le Cr^{-\frac{d}{p_d}}.$$
(5.17)

Since $G^{\rho}_{\mathbf{A}}(\cdot,y) \in H^1_0(\Omega;\mathbb{C})$ and

$$(D + \mathbf{A})^2 G_{\mathbf{A}}^{\rho}(\cdot, y) = 0 \quad \text{in } \Omega \setminus \mathcal{B}(y, \rho), \tag{5.18}$$

it follows from Lemma 5.1 and (5.17) that if $\rho < (1/2)r$,

$$|G_{\mathbf{A}}^{\rho}(z,y)| \leq C \left(\oint_{\mathcal{B}(z,r)\cap\Omega} |G_{\mathbf{A}}^{\rho}(x,y)|^{p_d} dx \right)^{1/p_d}$$

$$\leq Cr^{-\frac{2d}{p_d}}. \tag{5.19}$$

Recall that $p_d = \frac{2d}{d-2}$ for $d \geq 3$ and $2 \leq p_d < \infty$ for d = 2. Thus, we have proved that if $\rho < (1/4)|z-y|$,

$$|G_{\mathbf{A}}^{\rho}(z,y)| \le \begin{cases} C|z-y|^{2-d} & \text{if } d \ge 3, \\ C_{\sigma}|z-y|^{-\sigma} & \text{if } d = 2, \end{cases}$$
 (5.20)

for any $\sigma \in (0,1)$.

Using the estimates (5.20) and (5.13), one can show that $\{G_{\mathbf{A}}^{\rho}(\cdot,y):0<\rho<1\}$ is bounded in $L^p(\Omega)$ for $1< p<\frac{d}{d-2}$ if $d\geq 3$, and in $L^p(\Omega)$ for any $p<\infty$ if d=2. One can also show that $\{(D+\mathbf{A})G_{\mathbf{A}}^{\rho}(\cdot,y):0<\rho<1\}$ is bounded in $L^p(\Omega)$ for $p<\frac{d}{d-1}$. It follows that $\{G_{\mathbf{A}}^{\rho}(\cdot,y):0<\rho<1\}$ is bounded in $W^{1,p}(\Omega;\mathbb{C})$ for $1< p<\frac{d}{d-1}$ and $d\geq 2$. This implies that there exists a subsequence $\{G_{\mathbf{A}}^{\rho_j}(\cdot,y)\}$, where $\rho_j\to 0$, and $G_{\mathbf{A}}(\cdot,y)\in W^{1,p}(\Omega,\mathbb{C})$ such that

$$G_{\mathbf{A}}^{\rho_j}(\cdot,y) \to G_{\mathbf{A}}(\cdot,y)$$
 weakly in $W^{1,p}(\Omega;\mathbb{C})$,

where $1 . By Lemma 3.3, <math>\{G_{\mathbf{A}}^{\rho}(\cdot,y) : 0 < \rho < 1\}$ is bounded in $W^{1,2}(\Omega \setminus \mathcal{B}(y,t);\mathbb{C})$ for any fixed $t \in (0,1)$. It follows that $G_{\mathbf{A}}(\cdot,y) \in W^{1,2}_{loc}(\Omega \setminus \{y\};\mathbb{C})$ and (5.10) holds.

Since $\mathbf{A} \in C^1(\overline{\Omega}; \mathbb{R}^d)$, using (5.20) and the standard elliptic regularity theory, one may deduce that $\{G_{\mathbf{A}}^{\rho}(\cdot,y):0<\rho<1\}$ is equicontinuous in $\Omega\setminus\mathcal{B}(y,t)$ for any fixed $t\in(0,1)$. Thus, we may assume $G_{\mathbf{A}}^{\rho_j}(\cdot,y)\to G_{\mathbf{A}}(\cdot,y)$ uniformly in $\Omega\setminus\mathcal{B}(y,t)$ for any fixed $t\in(0,1)$. As a result, (5.8) follows from (5.20). To show (5.9), we use (5.16) with $F=\frac{1}{|\mathcal{B}(z,\rho)\cap\Omega|}\chi_{\mathcal{B}(z,\rho)\cap\Omega}$ and $u=G_{\mathbf{A}}^{\rho}(\cdot,z)$ to obtain

$$\oint_{B(z,\rho)\cap\Omega} \overline{G_{\mathbf{A}}^{\rho}(x,y)} \, dx = \int_{B(y,\rho)\cap\Omega} G_{\mathbf{A}}^{\rho}(x,z) \, dx.$$

By letting $\rho = \rho_i \to 0$, we arrive at (5.9).

Finally, let u be the weak solution of (5.3) with $F \in L^{\infty}(\Omega; \mathbb{C})$. Since $\mathbf{A} \in C^1(\overline{\Omega}; \mathbb{R}^d)$, u is Hölder continuous in $\overline{\Omega}$. By taking limits in (5.16), we obtain

$$u(y) = \int_{\Omega} F(x) \cdot \overline{G_{\mathbf{A}}(x, y)} \, dx, \tag{5.21}$$

which, together with (5.9), gives (5.11).

Theorem 5.6. Let $\Omega \subset \mathcal{B}(0, R_0)$ be a bounded Lipschitz domain. Suppose **B** satisfies the condition (3.2) for any ball $\mathcal{B}(x, r) \subset \mathcal{B}(0, 4R_0)$. Also assume that (3.6) holds for any $x \in \mathcal{B}(0, R_0)$. Let u be a solution of (5.1), where $x_0 \in \partial \Omega$ and $0 < r < r_0$. Then for any $\ell \geq 0$,

$$\sup_{\mathcal{B}(x_0, r/2) \cap \Omega} |u| \le \frac{C_{\ell}}{\{1 + rm(x_0, \mathbf{B})\}^{\ell}} \int_{\mathcal{B}(x_0, r) \cap \Omega} |u|, \tag{5.22}$$

where C_{ℓ} depends on ℓ , the Lipschitz character of Ω and C_0 in (3.2).

Proof. This follows readily from Lemma 5.1 and Theorem 3.5.

Theorem 5.7. Assume Ω and \mathbf{B} satisfy the same conditions as in Theorem 5.6. Let $G_{\mathbf{A}}(x,y)$ be the Green function for the operator $(D+\mathbf{A})^2$ in Ω , given by Theorem 5.5. Let $\ell \geq 1$. Then, for any $x,y \in \Omega$ and $x \neq y$,

$$|G_{\mathbf{A}}(x,y)| \le \frac{C_{\ell}}{\{1 + |x - y|m(x,\mathbf{B})\}^{\ell}} \cdot \frac{1}{|x - y|^{d-2}}$$
 (5.23)

for $d \geq 3$, where C_{ℓ} depends on ℓ , r_0 , the Lipschitz character of Ω and C_0 in (3.2). If d = 2, we have

$$|G_{\mathbf{A}}(x,y)| \le \frac{C_{\ell,\sigma}}{\{1 + |x - y|m(x,\mathbf{B})\}^{\ell}} \cdot \frac{1}{|x - y|^{\sigma}}$$
 (5.24)

for any $\sigma \in (0,1)$, where $C_{\ell,\sigma}$ depends on ℓ , σ , r_0 , the Lipschitz character of Ω and C_0 in (3.2).

Proof. We gives the proof for $d \ge 3$. The proof for d = 2 is the same. Fix $x_0, y_0 \in \Omega$ and $x_0 \ne y_0$. Let $u(x) = G_{\mathbf{A}}(x, y_0)$ and $r = (1/8) \min\{|x_0 - y_0|, r_0\}$. We consider two cases.

Case 1. Suppose dist $(x_0, \partial \Omega) > r$. Then $\mathcal{B}(x_0, r) \subset \Omega$. It follows from (4.2) that

$$|u(x_0)| \le \frac{C_{\ell}}{\{1 + rm(x_0, \mathbf{B})\}^{\ell}} \int_{\mathcal{B}(x_0, r)} |u|$$

$$\le \frac{C_{\ell}}{\{1 + rm(x_0, \mathbf{B})\}^{\ell}} \cdot \frac{1}{r^{d-2}},$$

where we have used (5.8) for the last inequality. This gives (5.23)

Case 2. Suppose $\operatorname{dist}(x_0, \partial\Omega) \leq r$. Choose $z_0 \in \partial\Omega$ such that $|x_0 - z_0| = \operatorname{dist}(x_0, \partial\Omega)$. By (5.22) we have

$$|u(x_0)| \le \sup_{\mathcal{B}(z_0, 2r)} |u| \le \frac{C_{\ell}}{\{1 + rm(z_0, \mathbf{B})\}^{\ell}} \int_{\mathcal{B}(z_0, 4r) \cap \Omega} |u|$$

$$\le \frac{C_{\ell}}{\{1 + rm(z_0, \mathbf{B})\}^{\ell}} \cdot \frac{1}{r^{d-2}},$$
(5.25)

where we have used (5.8) for the last inequality. Finally, note that by (3.8),

$$m(x_0, \mathbf{B}) \le C\{1 + rm(z_0, \mathbf{B})\}^{\kappa_0} m(z_0, \mathbf{B}).$$

It follows that

$$1 + rm(x_0, \mathbf{B}) \le C\{1 + rm(z_0, \mathbf{B})\}^{\kappa_0 + 1}.$$

Since $\ell \geq 1$ is arbitrary, this, together with (5.25), gives (5.23).

Theorem 5.8. Let $d \geq 3$. Assume Ω and \mathbf{B} satisfy the same conditions as in Theorem 5.6. For $F \in L^{\infty}(\Omega; \mathbb{C})$, let u be a weak solution of (5.3). Then

$$||m(x, \mathbf{B})^{\ell+2}u||_{L^{p}(\Omega)} \le C_{\ell}||m(x, \mathbf{B})^{\ell}F||_{L^{p}(\Omega)}$$
 (5.26)

for any $\ell \in \mathbb{R}$ and $1 \leq p \leq \infty$, where C_{ℓ} depends on ℓ , r_0 , the Lipschitz character of Ω , and C_0 in (3.2).

Proof. By Theorem 5.5, we may write

$$m(x, \mathbf{B})^{\ell+2}u(x) = \int_{\Omega} K_{\ell}(x, y)m(y, \mathbf{B})^{\ell}F(y) dy,$$

where

$$K_{\ell}(x,y) = m(x,\mathbf{B})^{\ell+2} G_{\mathbf{A}}(x,y) m(y,\mathbf{B})^{-\ell}.$$

To show (5.26), it suffices to show that for any $x \in \Omega$,

$$\int_{\Omega} |K_{\ell}(x,y)| \, dy \le C_{\ell},\tag{5.27}$$

and for any $y \in \Omega$,

$$\int_{\Omega} |K_{\ell}(x,y)| \, dx \le C_{\ell}. \tag{5.28}$$

To see (5.27), we use the estimate

$$m(x, \mathbf{B}) \le C\{1 + |x - y| m(y, \mathbf{B})\}^{\kappa_0} m(y, \mathbf{B})$$
 (5.29)

for any $x,y\in\Omega$ and the fact that (5.23) holds for any $\ell\geq 1$. This gives

$$|K_{\ell}(x,y)| \le \frac{C_{\ell} m(x, \mathbf{B})^2}{\{1 + |x - y| m(x, \mathbf{B})\}^{\ell} |x - y|^{d-2}}$$

for any $\ell \geq 1$, from which the inequality (5.27) follows readily. The proof for (5.28) is similar.

6 The Neumann function

In this section we establish decay estimates for the Neumann function $N_{\mathbf{A}}(x,y)$ in a bounded Lipschitz domain Ω .

Lemma 6.1. Let $x_0 \in \partial \Omega$ and $0 < r < r_0$. Let $u \in C^2(\mathcal{B}(x_0, r) \cap \Omega; \mathbb{C}) \cap C^1(\mathcal{B}(x_0, r) \cap \overline{\Omega}; \mathbb{C})$ be a solution of

$$\begin{cases} (D+\mathbf{A})^2 u = 0 & \text{in } \mathcal{B}(x_0, r) \cap \Omega, \\ n \cdot (D+\mathbf{A})u = 0 & \text{on } \mathcal{B}(x_0, r) \cap \partial \Omega, \end{cases}$$
(6.1)

where n denotes the outward unit normal to $\partial\Omega$. Then

$$\sup_{\mathcal{B}(x_0, r/2) \cap \Omega} |u| \le C \int_{\mathcal{B}(x_0, r) \cap \Omega} |u|, \tag{6.2}$$

where C depends only on Ω .

Proof. Let $\varepsilon > 0$ and $u_{\varepsilon} = \sqrt{|u|^2 + \varepsilon^2}$. Since $(D + \mathbf{A})^2 u = 0$ in $\mathcal{B}(x_0, r) \cap \Omega$, by (2.2), we have $\Delta u_{\varepsilon} \geq 0$ in $\mathcal{B}(x_0, r) \cap \Omega$. Using $n \cdot (D + \mathbf{A})u = 0$ on $\mathcal{B}(x_0, r) \cap \partial \Omega$ and (2.3), we see that $n \cdot \nabla u_{\varepsilon} = 0$ on $\mathcal{B}(x_0, r) \cap \partial \Omega$. By the even reflection, one can show that u_{ε} is a nonnegative sub-solution of a second-order elliptic equation in divergence form with bounded measurable coefficients in $\mathcal{B}(x_0, cr)$. More precisely, without the loss of generality, assume that

$$\mathcal{B}(x_0, r_0) \cap \Omega = \mathcal{B}(x_0, r_0) \cap \{(x', x_d) \in \mathbb{R}^d : x_d > \phi(x')\},$$

where $\phi: \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz continuous function with $\phi(x_0) = 0$ and $\|\nabla \phi\|_{\infty} \leq M_0$. Let Φ be the map

$$(x', x_d) \rightarrow (x', 2\phi(x') - x_d)$$

from \mathbb{R}^d to \mathbb{R}^d . Note that $\Phi^{-1} = \Phi$. For $x \in \mathcal{B}(x_0, cr)$, define

$$v_{\varepsilon}(x) = \begin{cases} u_{\varepsilon}(x) & \text{if } x_d \ge \phi(x'), \\ u_{\varepsilon}(\Phi(x)) & \text{if } x_d < \phi(x'). \end{cases}$$

Then $v_{\varepsilon} \in H^1(\mathcal{B}(x_0, cr); \mathbb{R})$ and

$$\int_{\mathcal{B}(x_0,cr)} E \nabla v_{\varepsilon} \cdot \nabla \varphi \le 0$$

for any nonnegative $\varphi \in C_0^1(\mathcal{B}(x_0, cr); \mathbb{R})$, where

$$E = \begin{cases} I & \text{if } x_d \ge \phi(x'), \\ \left(\frac{\partial \Phi}{\partial x}\right) \left(\frac{\partial \Phi}{\partial x}\right)^T (\Phi^{-1}(x)) & \text{if } x_d < \phi(x'). \end{cases}$$

Note that E is symmetric. The Lipschitz condition $\|\nabla\phi\|_{\infty} \leq M_0$ ensures that E is bounded measurable and uniformly elliptic. As a result, by the De Giorgi-Nash-Moser estimates, we obtain

$$\sup_{B(x_0,cr/2)\cap\Omega}u_{\varepsilon}\leq C\int_{B(x_0,cr)\cap\Omega}u_{\varepsilon},$$

where C depends only on the Lipschitz character of Ω . By letting $\varepsilon \to 0$, this leads to

$$\sup_{B(x_0,cr/2)\cap\Omega}|u|\leq C\int_{B(x_0,cr)\cap\Omega}|u|,$$

from which and the interior estimate (4.1), the estimate (6.2) follows by a simple covering argument.

In the remaining of this section we will assume that there exists $c_0 > 0$ such that

$$c_0 \int_{\Omega} |\psi|^2 \le \int_{\Omega} |(D + \mathbf{A})\psi|^2 \tag{6.3}$$

for any $\psi \in H^1(\Omega; \mathbb{C})$. It follows from (6.3) and (2.6) that

$$\left(\int_{\Omega} |\psi|^{p_d}\right)^{1/p_d} \le C \left(\int_{\Omega} |(D+\mathbf{A})\psi|^2\right)^{1/2} \tag{6.4}$$

for any $\psi \in H^1(\Omega; \mathbb{C})$, where $p_d = \frac{2d}{d-2}$ for $d \geq 3$ and $2 \leq p_d < \infty$ for d = 2. The constant C in (6.4) depends on p_d , Ω and c_0 in (6.3).

Consider the Neumann problem,

$$\begin{cases} (D+\mathbf{A})^2 u = F & \text{in } \Omega, \\ n \cdot (D+\mathbf{A})u = g & \text{on } \partial\Omega. \end{cases}$$
 (6.5)

Let $F \in L^2(\Omega; \mathbb{C})$ and $g \in L^2(\partial\Omega; \mathbb{C})$, we say $u \in H^1(\Omega; \mathbb{C})$ is a weak solution of (6.5) if

$$\int_{\Omega} (D + \mathbf{A}) u \cdot \overline{(D + \mathbf{A}) \psi} = \int_{\Omega} F \cdot \overline{\psi} + i \int_{\partial \Omega} g \cdot \overline{\psi}$$
(6.6)

for any $\psi \in H^1(\Omega; \mathbb{C})$.

Lemma 6.2. For $F \in L^2(\Omega; \mathbb{C})$ and $g \in L^2(\partial\Omega; \mathbb{C})$, the Neumann problem (6.5) possesses a unique weak solution in $H^1(\Omega; \mathbb{C})$. Moreover, the solution satisfies

$$||u||_{L^{p_d}(\Omega)} + ||(D+\mathbf{A})u||_{L^2(\Omega)} \le C \left\{ ||F||_{L^{p'_d}(\Omega)} + ||g||_{L^2(\partial\Omega)} \right\}, \tag{6.7}$$

where $p_d = \frac{2d}{d-2}$ for $d \geq 3$ and $2 \leq p_d < \infty$ for d = 2. The constant C in (6.7) depends at most on p_d , Ω and c_0 in (6.3).

Proof. Consider the bilinear form B[u,v] in (5.5) for $u,v\in H^1(\Omega;\mathbb{C})$. Since **A** is bounded in Ω , we have

$$B[u, u] \ge \frac{1}{2} \int_{\Omega} |\nabla u|^2 - C(\mathbf{A}) \int_{\Omega} |u|^2$$
$$\ge \frac{1}{2} \int_{\Omega} |\nabla u|^2 - C(\mathbf{A}) B[u, u],$$

where $C(\mathbf{A})$ depends on \mathbf{A} and we have used the assumption (6.3) for the second inequality. It follows that

$$B[u, u] \ge c(\mathbf{A}) \|u\|_{H^1(\Omega)}^2$$

for any $u \in H^1(\Omega; \mathbb{C})$, where $c(\mathbf{A}) > 0$. By the Lax-Milgram Theorem and (2.7), this implies that (6.5) has a unique weak solution in $H^1(\Omega; \mathbb{C})$ for any $F \in L^2(\Omega; \mathbb{C})$ and $g \in L^2(\partial\Omega; \mathbb{C})$. To see (6.7), we use (6.6) with $\psi = u$, (6.4) and (2.7).

Theorem 6.3. There is a continuous function $N_{\mathbf{A}}: \{(x,y) \in \overline{\Omega} \times \overline{\Omega}: x \neq y\} \to \mathbb{C}$ with the following properties.

1. For any $x, y \in \overline{\Omega}$, $x \neq y$, and $\sigma \in (0, 1)$,

$$|N_{\mathbf{A}}(x,y)| \le \begin{cases} C|x-y|^{2-d} & \text{if } d \ge 3, \\ C_{\sigma}|x-y|^{-\sigma} & \text{if } d = 2, \end{cases}$$
 (6.8)

where C depends on d, Ω and c_0 in (6.3), and C_{σ} depends also on σ .

2. For any $x, y \in \overline{\Omega}$, $x \neq y$,

$$N_{\mathbf{A}}(x,y) = \overline{N_{\mathbf{A}}(y,x)}. (6.9)$$

3. Fix $y \in \overline{\Omega}$. Then $N_{\mathbf{A}}(\cdot, y) \in W^{1,2}_{loc}(\Omega \setminus \{y\}; \mathbb{C})$ and

$$\begin{cases} (D + \mathbf{A})^2 N_{\mathbf{A}}(\cdot, y) = 0 & in \ \Omega \setminus \{y\}, \\ n \cdot (D + \mathbf{A}) N_{\mathbf{A}}(\cdot, y) = 0 & on \ \partial \Omega. \end{cases}$$
(6.10)

4. For $F \in L^{\infty}(\Omega; \mathbb{C})$ and $g \in L^{\infty}(\partial\Omega)$, the weak solution of (6.5) is given by

$$u(x) = \int_{\Omega} N_{\mathbf{A}}(x, y) F(y) \, dy + i \int_{\partial \Omega} N_{\mathbf{A}}(x, y) g(y) dy. \tag{6.11}$$

Proof. With Lemmas 6.1 and 6.2 at our disposal, the proof is similar to that of Theorem 5.5. Indeed, fix $y \in \overline{\Omega}$ and $\rho \in (0,1)$. Let $N_{\mathbf{A}}^{\rho}(\cdot,y)$ denote the weak solution of (6.5) with $F = \frac{1}{|\mathcal{B}(y,\rho)\cap\Omega|}\chi_{\mathcal{B}(y,\rho)\cap\Omega}$ and g = 0, given by Lemma 6.2. Thus,

$$\int_{\Omega} (D + \mathbf{A}) N_{\mathbf{A}}^{\rho}(x, y) \cdot \overline{(D + \mathbf{A})\psi} \, dx = \int_{\mathcal{B}(y, \rho) \cap \Omega} \overline{\psi}$$
 (6.12)

for any $\psi \in H^1(\Omega; \mathbb{C})$. Using (6.7) and (6.2), one can show that if $\rho < (1/4)|z - y|$,

$$|N_{\mathbf{A}}(z,y)| \le \begin{cases} C|z-y|^{2-d} & \text{if } d \ge 3, \\ C_{\sigma}|z-y|^{-\sigma} & \text{if } d = 2, \end{cases}$$

for any $\sigma \in (0,1)$. It follows that $\{N_{\mathbf{A}}^{\rho}(\cdot,y): 0<\rho<1\}$ is bounded in $W^{1,p}(\Omega;\mathbb{C})$ for $p<\frac{d}{d-1}$. This implies that there exists a sequence $\{N_{\mathbf{A}}^{\rho_j}(\cdot,y)\}$, where $\rho_j\to 0$, and $N_{\mathbf{A}}^{\rho}(\cdot,y)\in W^{1,p}(\Omega;\mathbb{C})$ such that $N_{\mathbf{A}}^{\rho_j}(\cdot,y)\to N_{\mathbf{A}}(\cdot,y)$ weakly in $W^{1,p}(\Omega;\mathbb{C})$, where $1< p<\frac{d}{d-1}$. The proof for (6.8)-(6.10) is the same as in the case of the Green function. To see (6.11), let u be a weak solution of (6.5) with $F\in L^{\infty}(\Omega;\mathbb{C})$ and $g\in L^{\infty}(\partial\Omega;\mathbb{C})$. By letting $\psi=N_{\mathbf{A}}^{\rho}(\cdot,y)$ in (6.6), we obtain

$$\int_{\Omega} (D + \mathbf{A}) u(x) \cdot \overline{(D + \mathbf{A}) N_{\mathbf{A}}^{\rho}(x, y)} \, dx = \int_{\Omega} F(x) \overline{N_{\mathbf{A}}^{\rho}(x, y)} \, dx + i \int_{\partial \Omega} g(x) \overline{N_{\mathbf{A}}^{\rho}(x, y)} \, dx.$$

This, together with (6.12), yields

$$\oint_{\mathcal{B}(y,\rho)\cap\Omega} u = \int_{\Omega} F(x) \overline{N_{\mathbf{A}}^{\rho}(x,y)} \, dx + i \int_{\partial\Omega} g(x) \overline{N_{\mathbf{A}}^{\rho}(x,y)} \, dx. \tag{6.13}$$

Under the assumptions that $F \in L^{\infty}(\Omega; \mathbb{C})$ and $g \in L^{\infty}(\partial\Omega; \mathbb{C})$, using the standard regularity theory for Δ in a Lipschitz domain, one may show that u is Hölder continuous in $\overline{\Omega}$. By letting $\rho = \rho_j \to 0$ in (6.13), we see that

$$u(y) = \int_{\Omega} F(x) \overline{N_{\mathbf{A}}(x, y)} \, dx + i \int_{\partial \Omega} g(x) \overline{N_{\mathbf{A}}(x, y)} \, dx.$$

In view of (6.9), this gives the representation formula (6.11).

Theorem 6.4. Let $\Omega \subset \mathcal{B}(0, R_0)$ be a bounded Lipschitz domain. Suppose that **B** satisfies the condition (3.2) for any ball $\mathcal{B}(x,r) \subset \mathcal{B}(0,4R_0)$. Also assume that (3.6) holds for any $x \in \mathcal{B}(0,R_0)$. Let $N_{\mathbf{A}}(x,y)$ be the Neumann function for the operator $(D+\mathbf{A})^2$ in Ω , given by Theorem 6.3. Let $\ell \geq 1$. Then, for any $x,y \in \Omega$ and $x \neq y$,

$$|N_{\mathbf{A}}(x,y)| \le \frac{C_{\ell}}{\{1 + |x - y|m(x,\mathbf{B})\}^{\ell}} \cdot \frac{1}{|x - y|^{d-2}}$$
(6.14)

for $d \geq 3$, where C_{ℓ} depends on ℓ , R_0 , Ω , and C_0 in (3.2). If d = 2, we have

$$|N_{\mathbf{A}}(x,y)| \le \frac{C_{\ell,\sigma}}{\{1 + |x - y|m(x,\mathbf{B})\}^{\ell}} \cdot \frac{1}{|x - y|^{\sigma}}$$
 (6.15)

for any $\sigma \in (0,1)$, where $C_{\ell,\sigma}$ depends on ℓ , σ , R_0 , the Lipschitz character of Ω and C_0 in (3.2).

Proof. The proof is similar to that of Theorem 5.7. We only point out that under the condition 3.6, we have $m(x, \mathbf{B}) \geq 2R_0^{-1}$ for any $x \in \Omega$. In view of Theorem 3.2, this leads to

$$cR_0^{-2} \int_{\Omega} |\psi|^2 \le \int_{\Omega} |(D+\mathbf{A})\psi|^2,$$

which gives the condition (6.3) for any $\psi \in H^1(\Omega; \mathbb{C})$.

Theorem 6.5. Let $d \geq 3$. Suppose Ω and \mathbf{B} satisfy the same conditions as in Theorem 6.4. For $F \in L^{\infty}(\Omega; \mathbb{C})$, let $u \in H^1(\Omega; \mathbb{C})$ be a weak solution of (6.5) with g = 0. Then

$$||m(x, \mathbf{B})^{\ell+2}u||_{L^{p}(\Omega)} \le C_{\ell}||m(x, \mathbf{B})^{\ell}F||_{L^{p}(\Omega)}$$
 (6.16)

for any $\ell \in \mathbb{R}$ and $1 \leq p \leq \infty$, where C_{ℓ} depends on d, ℓ , R_0 , the Lipschitz character of Ω and C_0 in (3.2).

Proof. The proof is similar to that of Theorem 5.8.

7 Rellich estimates

Let Ω be a bounded Lipschitz domain in \mathbb{R}^d and $\mathbf{A} \in C^1(\overline{\Omega}; \mathbb{R}^d)$. Let $L = (L_1, \dots, L_d)$, where

$$L_j = D_j + A_j$$

for $1 \leq j \leq d$. It follows from integration by parts that if $f, g \in C^1(\overline{\Omega}; \mathbb{C})$,

$$\int_{\Omega} L_j f \cdot \overline{g} = \int_{\Omega} f \cdot \overline{L_j g} + \frac{1}{i} \int_{\partial \Omega} n_j f \cdot \overline{g}. \tag{7.1}$$

Lemma 7.1. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in C^1(\mathbb{R}^d; \mathbb{R}^d)$. Suppose $u \in C^2(\overline{\Omega}; \mathbb{C})$ and $(D + \mathbf{A})^2 u = F$ in Ω . Then

$$\int_{\partial\Omega} \alpha_{j} n_{j} |(D + \mathbf{A})u|^{2} = 2\Re \int_{\partial\Omega} \alpha_{k} n_{j} (D_{j} + A_{j}) u \overline{(D_{k} + A_{k})u}
+ 2\Im \int_{\Omega} \alpha_{k} B_{kj} u \overline{(D_{j} + A_{j})u} + \int_{\Omega} \operatorname{div}(\alpha) |(D + \mathbf{A})u|^{2}
- 2\Re \int_{\Omega} \partial_{j} \alpha_{k} (D_{j} + A_{j}) u \overline{(D_{k} + A_{k})u} + 2\Im \int_{\Omega} \alpha_{k} F \overline{(D_{k} + A_{k})u},$$
(7.2)

where the repeated indices j, k are summed from 1 to d.

Proof. Using (7.1), we obtain

$$\begin{split} \int_{\Omega} \alpha_k L_j u \overline{L_j L_k u} &= \int_{\Omega} L_k (\alpha_k L_j u) \overline{L_j u} - \frac{1}{i} \int_{\partial \Omega} n_k \alpha_k |Lu|^2 + \int_{\Omega} \alpha_k L_j u \overline{[L_j, L_k] u} \\ &= \int_{\Omega} \alpha_k L_k L_j u \overline{L_j u} + \frac{1}{i} \int_{\Omega} \operatorname{div}(\alpha) |Lu|^2 - \frac{1}{i} \int_{\partial \Omega} n_k \alpha_k |Lu|^2 \\ &+ \int_{\Omega} \alpha_k L_j u \overline{[L_j, L_k] u} \\ &= \int_{\Omega} \alpha_k L_j L_k u \overline{L_j u} + \int_{\Omega} \alpha_k [L_k, L_j] u \overline{L_j u} + \frac{1}{i} \int_{\Omega} \operatorname{div}(\alpha) |Lu|^2 \\ &- \frac{1}{i} \int_{\partial \Omega} n_k \alpha_k |Lu|^2 + \int_{\Omega} \alpha_k L_j u \overline{[L_j, L_k] u}, \end{split}$$

where $[L_j, L_k] = L_j L_k - L_k L_j$. This gives

$$2\Im \int_{\Omega} \alpha_{k} L_{j} u \overline{L_{j} L_{k} u} = 2\Im \int_{\Omega} \alpha_{k} [L_{k}, L_{j}] u \overline{L_{j} u} - \int_{\Omega} \operatorname{div}(\alpha) |Lu|^{2} + \int_{\partial \Omega} n_{k} \alpha_{k} |Lu|^{2}$$

$$= -2\Re \int_{\Omega} \alpha_{k} B_{kj} u \overline{L_{j} u} - \int_{\Omega} \operatorname{div}(\alpha) |Lu|^{2} + \int_{\partial \Omega} n_{k} \alpha_{k} |Lu|^{2},$$

$$(7.3)$$

where we have used the fact $[L_k, L_j] = -iB_{kj}$. Also note that by (7.1),

$$\int_{\Omega} L_{j}^{2} u \overline{\alpha_{k} L_{k} u} = \int_{\Omega} L_{j} u \overline{L_{j}(\alpha_{k} L_{k} u)} + \frac{1}{i} \int_{\partial \Omega} \alpha_{k} n_{j} L_{j} u \overline{L_{k} u}
= \int_{\Omega} \alpha_{k} L_{j} u \overline{L_{j} L_{k} u} + i \int_{\Omega} \partial_{j} \alpha_{k} L_{j} u \overline{L_{k} u} + \frac{1}{i} \int_{\partial \Omega} \alpha_{k} n_{j} L_{j} u \overline{L_{k} u}.$$
(7.4)

From (7.4), we deduce that

$$2\Im \int_{\Omega} \alpha_k L_j u \overline{L_j L_k u} = 2\Im \int_{\Omega} L_j^2 u \overline{\alpha_k L_k u} - 2\Re \int_{\Omega} \partial_j \alpha_k L_j u \overline{L_k u} + 2\Re \int_{\partial \Omega} \alpha_k n_j L_j u \overline{L_k u}. \tag{7.5}$$

The equation (7.2) now follows readily from (7.3) and (7.5).

Let $T = (T_{jk})$, where

$$T_{jk} = n_j L_k - n_k L_j$$

for $1 \leq j, k \leq d$.

Lemma 7.2. Let $\alpha \in C^1(\mathbb{R}^d; \mathbb{R}^d)$. Suppose $u \in C^2(\overline{\Omega}; \mathbb{C})$ and $(D + \mathbf{A})^2 u = F$ in Ω . Then

$$\int_{\partial\Omega} \alpha_{j} n_{j} |(D + \mathbf{A})u|^{2} = 2\Re \int_{\partial\Omega} \alpha_{k} T_{kj} u \overline{(D_{j} + A_{j})u}
- 2\Im \int_{\Omega} \alpha_{k} B_{kj} u \overline{(D_{j} + A_{j})u} - \int_{\Omega} \operatorname{div}(\alpha) |(D + \mathbf{A})u|^{2}
+ 2\Re \int_{\Omega} \partial_{j} \alpha_{k} (D_{j} + A_{j}) u \overline{(D_{k} + A_{k})u} - 2\Im \int_{\Omega} \alpha_{k} F \overline{(D_{k} + A_{k})u},$$
(7.6)

where the repeated indices j, k are summed from 1 to d.

Proof. Write the equation (7.2) as I = J. Then (7.6) follows from the fact I = 2I - I = 2I - J as well as the observation that

$$2\int_{\partial\Omega} \alpha_j n_j |(D+\mathbf{A})u|^2 - 2\Re \int_{\partial\Omega} \alpha_k n_j (D_j + A_j) u \overline{(D_k + A_k)u}$$
$$= 2\Re \int_{\partial\Omega} \alpha_k T_{kj} u \overline{(D_j + A_j)u}.$$

For the rest of this section, we shall further assume that $\Omega \subset \mathcal{B}(0, R_0)$ is a smooth domain for some $R_0 = Cr_0 > 0$, where C depends on the Lipschitz character of Ω . We also assume that **B** satisfies (3.2) for any $\mathcal{B}(x,r) \subset \mathcal{B}(0,4R_0)$ and (3.6) for any $x \in \mathcal{B}(0,R_0)$. We emphases that although Ω is assumed to be smooth, the dependence of constants C on Ω is through r_0 and its Lipschitz character.

Lemma 7.3. Suppose that $u \in C^2(\overline{\Omega}; \mathbb{C})$ and $(D + \mathbf{A})^2 u = F$ in Ω . Then for any $\ell \in \mathbb{R}$,

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} \le C \int_{\partial\Omega} |n\cdot(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} + C \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3}
+ C \int_{\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+1} + C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{\ell-1},$$
(7.7)

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} \le C \int_{\partial\Omega} |Tu|^2 m(x,\mathbf{B})^{\ell} + C \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3}
+ C \int_{\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+1} + C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{\ell-1},$$
(7.8)

where C depends on ℓ , Ω and C_0 in (3.2).

Proof. Fix $x_0 \in \partial \Omega$. Recall that $m(x_0, \mathbf{B}) > 2R_0^{-1}$ by (3.6). Let $r = cm(x_0, \mathbf{B})^{-1} < r_0$, where c > 0 depends on the Lipschitz character of Ω . Choose $\alpha \in C_0^1(\mathcal{B}(x_0, 2r); \mathbb{R}^d)$ such that $\alpha \cdot n \geq 0$ on $\mathcal{B}(x_0, 2r) \cap \partial \Omega$, $\alpha \cdot n \geq c_0 > 0$ on $\mathcal{B}(x_0, r) \cap \partial \Omega$, $|\alpha| \leq 1$ in $\mathcal{B}(x_0, 2r)$ and $|\nabla \alpha| \leq Cr^{-1}$ in $\mathcal{B}(x_0, 2r)$. It follows from (7.2) that

$$\int_{\mathcal{B}(x_0,r)\cap\partial\Omega} |(D+\mathbf{A})u|^2 \le C \int_{\mathcal{B}(x_0,2r)\cap\partial\Omega} |(D+\mathbf{A})u||n\cdot(D+\mathbf{A})u|
+ Cr^{-3} \int_{\mathcal{B}(x_0,2r)\cap\Omega} |u|^2 + Cr^{-1} \int_{\mathcal{B}(x_0,2r)} |(D+\mathbf{A})u|^2
+ Cr \int_{\mathcal{B}(x_0,2r)\cap\Omega} |F|^2,$$

where we have used the fact $|\mathbf{B}(x)| \leq Cr^{-2}$ for $x \in \mathcal{B}(x_0, 2r)$. We now multiply the inequality above by $m(x_0, \mathbf{B})^{\ell+d-1}$ and integrate the resulting inequality in x_0 over $\partial\Omega$. Using the fact that $m(x, \mathbf{B}) \approx m(y, \mathbf{B})$ if $|x - y| < cm(x, \mathbf{B})^{-1}$, this leads to

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} \le C \int_{\partial\Omega} |(D+\mathbf{A})u| |n\cdot(D+\mathbf{A})u| m(x,\mathbf{B})^{\ell}$$

$$+ C \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3} + C \int_{\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+1}$$

$$+ C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{\ell-1},$$

which yields (7.7) by using the Cauchy inequality. A similar argument, using (7.6), gives (7.8). \square

Lemma 7.4. Under the same assumptions as in Lemma 7.3, we have

$$\int_{\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+1} \le C \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3} + C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{\ell-1} + C \int_{\partial\Omega} |u| |n \cdot (D+\mathbf{A})u| m(x,\mathbf{B})^{\ell+1}, \tag{7.9}$$

for any $\ell \in \mathbb{R}$, where C depends on ℓ .

Proof. Fix $x_0 \in \partial \Omega$ and let $r = cm(x_0, \mathbf{B})^{-1} < r_0$. It follows from Remark 3.4 that

$$\int_{\mathcal{B}(x_0,r)\cap\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+d} \leq C \int_{\mathcal{B}(x_0,2r)\cap\Omega} |u|^2 m(x,\mathbf{B})^{\ell+d+2}$$

$$+ C \int_{\mathcal{B}(x_0,2r)\cap\Omega} |F|^2 m(x,\mathbf{B})^{\ell+d-2}$$

$$+ C \int_{\mathcal{B}(x_0,2r)\cap\partial\Omega} |u| |n \cdot (D+\mathbf{A})u| m(x,\mathbf{B})^{\ell+d}.$$

By integrating the inequality above in x_0 over $\partial \Omega$, we see that

$$\int_{\Omega_h} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+1}$$

is bounded by the right-hand side of (7.9), where

$$\Omega_b = \{ x \in \Omega : \operatorname{dist}(x, \partial \Omega) < c \, m(x, \mathbf{B})^{-1} \}.$$

To handle the region $\Omega \setminus \Omega_b$, we use the fact that if $\mathcal{B}(x_0, 2t) \subset \Omega$,

$$\int_{\mathcal{B}(x_0,t)} |(D+\mathbf{A})u|^2 \le \frac{C}{t^2} \int_{\mathcal{B}(x_0,2t)} |u|^2 + Ct^2 \int_{\mathcal{B}(x_0,2t)} |F|^2.$$

It follows that if $x_0 \in \Omega \setminus \Omega_b$, then

$$\int_{\mathcal{B}(x_0,r)} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell-d+1}
\leq C \int_{\mathcal{B}(x_0,2r)} |u|^2 m(x,\mathbf{B})^{\ell-d+3} + C \int_{\mathcal{B}(x_0,2r)} |F|^2 m(x,\mathbf{B})^{\ell-d-1},$$

where $r = cm(x_0, \mathbf{B})^{-1}$. By integrating the inequality above in x_0 over $\Omega \setminus \Omega_b$, we see that

$$\int_{\Omega \backslash \Omega_b} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+1}$$

is also bounded by the right-hand side of (7.9).

Theorem 7.5. Suppose $d \geq 3$. Let $u \in C^2(\overline{\Omega}; \mathbb{C})$. Suppose $(D + \mathbf{A})^2 u = F$ in Ω . Assume that either u = 0 on $\partial \Omega$ or $n \cdot (D + \mathbf{A})u = 0$ on $\partial \Omega$. Then

$$\int_{\Omega} |(D + \mathbf{A})u|^2 m(x, \mathbf{B})^{\ell} + \int_{\Omega} |u|^2 m(x, \mathbf{B})^{\ell+2} \le C \int_{\Omega} |F|^2 m(x, \mathbf{B})^{\ell-2}$$
 (7.10)

for any $\ell \in \mathbb{R}$, where C depends on ℓ .

Proof. It follows from (7.9) that the left-hand side of (7.10) is bounded by

$$C \int_{\Omega} |u|^2 m(x, \mathbf{B})^{\ell+2} + C \int_{\Omega} |F|^2 m(x, \mathbf{B})^{\ell-2}.$$

This yields (7.10) by using Theorems 5.8 and 6.5 with p = 2.

Lemma 7.6. Let $u \in H^1(\Omega; \mathbb{C})$. Then

$$\int_{\partial\Omega} |u|^2 m(x, \mathbf{B})^{\ell} \le C \int_{\Omega} |u|^2 m(x, \mathbf{B})^{\ell+1} + C \int_{\Omega} |(D + \mathbf{A})u|^2 m(x, \mathbf{B})^{\ell-1}, \tag{7.11}$$

where C depends on ℓ and Ω .

Proof. Let $x_0 \in \partial \Omega$ and $r = cm(x_0, \mathbf{B})^{-1} < r_0$. Then

$$\int_{\mathcal{B}(x_0,r)\cap\partial\Omega} |v|^2 \le \frac{C}{r} \int_{\mathcal{B}(x_0,2r)\cap\Omega} |v|^2 + Cr \int_{\mathcal{B}(x_0,2r)\cap\Omega} |\nabla v|^2$$

for any $v \in H^1(\Omega; \mathbb{C})$. By applying the inequality above to $v = v_{\varepsilon} = \sqrt{|u|^2 + \varepsilon^2}$ and letting $\varepsilon \to 0$, we obtain

$$\int_{\mathcal{B}(x_0,r)\cap\partial\Omega} |u|^2 \le \frac{C}{r} \int_{\mathcal{B}(x_0,2r)\cap\Omega} |u|^2 + Cr \int_{\mathcal{B}(x_0,2r)\cap\Omega} |(D+\mathbf{A})u|^2.$$

This leads to

$$\int_{\mathcal{B}(x_0,r)\cap\partial\Omega} |u|^2 m(x,\mathbf{B})^{\ell+d-1} \le C \int_{\Omega\cap\mathcal{B}(x_0,2r)} |u|^2 m(x,\mathbf{B})^{\ell+d}$$

$$+ C \int_{\Omega\cap\mathcal{B}(x_0,2r)} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+d-2}.$$

By integrating the inequality above in x_0 over $\partial\Omega$, we obtain (7.11).

Lemma 7.7. Under the same assumptions as in Lemma 7.3, we have

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} + \int_{\partial\Omega} |u|^2 m(x,\mathbf{B})^{\ell+2}
\leq C \int_{\partial\Omega} |n\cdot(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} + C \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3} + C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{\ell-1},$$
(7.12)

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} \le C \int_{\partial\Omega} |Tu|^2 m(x,\mathbf{B})^{\ell} + C \int_{\partial\Omega} |u|^2 m(x,\mathbf{B})^{\ell+2}
+ C \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3} + C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{\ell-1},$$
(7.13)

for any $\ell \in \mathbb{R}$, where C depends on ℓ , C_0 in (3.2) and the Lipschitz character of Ω .

Proof. To see (7.12), we use (7.7) and (7.11) to obtain

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} + \int_{\partial\Omega} |u|^2 m(x,\mathbf{B})^{\ell+2}
\leq C \int_{\partial\Omega} |n\cdot(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} + C \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3}
+ C \int_{\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+1} + C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{\ell-1}
\leq C \int_{\partial\Omega} |n\cdot(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} + C \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3}
+ C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{\ell-1} + C \int_{\partial\Omega} |u| |n\cdot(D+\mathbf{A})u| m(x,\mathbf{B})^{\ell+1},$$

where we have used (7.9) for the second inequality. The inequality (7.12) now follows readily by applying the Cauchy inequality to the last integral.

The proof for (7.13) is similar. By (7.8) and (7.9), we obtain

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} \le C \int_{\partial\Omega} |Tu|^2 m(x,\mathbf{B})^{\ell} + C \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3}$$

$$+ C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{\ell-1} + C \int_{\partial\Omega} |u| |n \cdot (D+\mathbf{A})u| m(x,\mathbf{B})^{\ell+1},$$

which yields (7.13) by applying the Cauchy inequality to the last integral.

Lemma 7.8. Let $d \geq 3$. Suppose $u \in C^1(\overline{\Omega}; \mathbb{C})$ and $(D + \mathbf{A})^2 u = 0$ in Ω . Then for any $\ell \in \mathbb{R}$,

$$\int_{\Omega} |u|^2 m(x, \mathbf{B})^{\ell+3} \le C \int_{\partial \Omega} |u|^2 m(x, \mathbf{B})^{\ell+2}, \tag{7.14}$$

$$\int_{\Omega} |u|^2 m(x, \mathbf{B})^{\ell+3} \le C \int_{\partial \Omega} |n \cdot (D + \mathbf{A})u|^2 m(x, \mathbf{B})^{\ell}, \tag{7.15}$$

where C depends on ℓ and Ω .

Proof. We use a duality argument. For $G \in C_0^{\infty}(\Omega; \mathbb{C})$, let $v \in H^1(\Omega; \mathbb{C})$ be the weak solution of $(D + \mathbf{A})^2 v = G$ in Ω with the Neumann condition $n \cdot (D + \mathbf{A})v = 0$ on $\partial\Omega$. Note that

$$\int_{\Omega} u \cdot \overline{G} = \int_{\Omega} (D + \mathbf{A}) u \cdot \overline{(D + \mathbf{A}) v} = i \int_{\partial \Omega} n \cdot (D + \mathbf{A}) u \cdot \overline{v}.$$

It follows by the Cauchy inequality that

$$\left| \int_{\Omega} u \cdot \overline{G} \right| \le \left(\int_{\partial \Omega} |n \cdot (D + \mathbf{A})u|^2 m(x, \mathbf{B})^{\ell} \right)^{1/2} \left(\int_{\partial \Omega} |v|^2 m(x, \mathbf{B})^{-\ell} \right)^{1/2}. \tag{7.16}$$

We will show that

$$\int_{\partial\Omega} |v|^2 m(x, \mathbf{B})^{-\ell} \le C \int_{\Omega} |G|^2 m(x, \mathbf{B})^{-\ell - 3}, \tag{7.17}$$

which, together with (7.16), yields (7.15) by duality.

Since Ω is smooth and $G \in C_0^{\infty}(\Omega; \mathbb{C})$, we have $v \in C^2(\overline{\Omega}; \mathbb{C})$. By using Lemma 7.7, we obtain

$$\int_{\partial\Omega} |v|^2 m(x,\mathbf{B})^{-\ell} \le C \int_{\Omega} |v|^2 m(x,\mathbf{B})^{-\ell+1} + C \int_{\Omega} |G|^2 m(x,\mathbf{B})^{-\ell-3}.$$

This, together with (6.16) with p = 2, gives (7.17).

The proof of (7.14) is similar. For $\Psi \in C_0^{\infty}(\Omega; \mathbb{C})$, let $w \in H_0^1(\Omega; \mathbb{C})$ be the weak solution of $(D + \mathbf{A})^2 w = \Psi$ in Ω with the Dirichlet condition w = 0 on $\partial\Omega$. Then

$$\int_{\Omega} u \cdot \overline{\Psi} = i \int_{\partial \Omega} u \overline{n \cdot (D + \mathbf{A}) w}.$$

Hence, by the Cauchy inequality,

$$\left| \int_{\Omega} u \cdot \overline{\Psi} \right| \le \left(\int_{\partial \Omega} |u|^2 m(x, \mathbf{B})^{\ell+2} \right)^{1/2} \left(\int_{\partial \Omega} |n \cdot (D + \mathbf{A}) w|^2 m(x, \mathbf{B})^{-\ell-2} \right)^{1/2}. \tag{7.18}$$

We claim that

$$\int_{\partial\Omega} |(D+\mathbf{A})w|^2 m(x,\mathbf{B})^{-\ell-2} \le C \int_{\Omega} |\Psi|^2 m(x,\mathbf{B})^{-\ell-3},\tag{7.19}$$

which, together with (7.18), leads to (7.14) by duality.

Finally, to prove (7.19), we apply the estimate (7.13). Since Ω is smooth and $\Psi \in C_0^{\infty}(\Omega; \mathbb{C})$, we have $w \in C^2(\overline{\Omega}; \mathbb{C})$. Observe that since w = 0 on $\partial\Omega$, we have Tw = 0 on $\partial\Omega$. As a result,

$$\int_{\partial\Omega} |(D+\mathbf{A})w|^2 m(x,\mathbf{B})^{-\ell-2} \le C \int_{\Omega} |w|^2 m(x,\mathbf{B})^{-\ell+1} + C \int_{\Omega} \Psi|^2 m(x,\mathbf{B})^{-\ell-3}.$$

This, together with the estimate (5.26) with p = 2, gives (7.19)

We are now ready to prove the main results of this section.

Theorem 7.9. Let $d \geq 3$. Suppose $u \in C^2(\overline{\Omega}; \mathbb{C})$ and $(D + \mathbf{A})^2 u = 0$ in Ω . Then for any $\ell \in \mathbb{R}$,

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} + \int_{\partial\Omega} |u|^2 m(x,\mathbf{B})^{\ell+2} + \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3}$$

$$+ \int_{\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+1} \le C \int_{\partial\Omega} |n \cdot (D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell},$$

$$(7.20)$$

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} + \int_{\Omega} |u|^2 m(x,\mathbf{B})^{\ell+3} + \int_{\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell+1}
\leq C \int_{\partial\Omega} |Tu|^2 m(x,\mathbf{B})^{\ell} + C \int_{\partial\Omega} |u|^2 m(x,\mathbf{B})^{\ell+2},$$
(7.21)

where C depends on ℓ and Ω .

Proof. We give the proof of (7.20). A similar argument yields (7.21).

First, by (7.12) and (7.15), we obtain

$$\int_{\partial\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell} + \int_{\partial\Omega} |u|^2 m(x,\mathbf{B})^{\ell+2} \le C \int_{\partial\Omega} |n \cdot (D+\mathbf{A})u|^2 m(x,\mathbf{B})^{\ell}. \tag{7.22}$$

Next, by (7.9) and (7.15), the third and fourth terms in the left-hand side of (7.20) are bounded by

$$C \int_{\partial\Omega} |n \cdot (D + \mathbf{A})u|^2 m(x, \mathbf{B})^{\ell} + C \int_{\partial\Omega} |u| |n \cdot (D + \mathbf{A})u|^2 m(x, \mathbf{B})^{\ell+1}.$$

In view of (7.22), the integrals above are bounded by the right-hand side of (7.20).

8 Nontangential maximal function estimates

Throughout this section, we assume that $d \geq 3$ and $\Omega \subset \mathcal{B}(0, R_0)$ is a smooth domain for some $R_0 = Cr_0 > 0$, where C depends on the Lipschitz character of Ω . We also assume that **B** satisfies (3.2) for any $\mathcal{B}(x,r) \subset \mathcal{B}(0,4R_0)$ and (3.6) for any $x \in \mathcal{B}(0,R_0)$. As before, the dependence of constants C on Ω is through r_0 and its Lipschitz character.

Lemma 8.1. Let $u \in C^1(\overline{\Omega}; \mathbb{C})$ and $w(x) = u(x)m(x, \mathbf{B})$. Let $\mathcal{M}(w)$ be defined by (4.13). Then

$$\int_{\partial\Omega} |\mathcal{M}(w)|^2 \le C \int_{\partial\Omega} |w|^2 + C \int_{\Omega} |(D+\mathbf{A})u(x)|^2 m(x,\mathbf{B}) dx + C \int_{\Omega} |u(x)|^2 m(x,\mathbf{B})^3 dx,$$
(8.1)

where C depends on Ω and C_0 in (3.2).

Proof. Fix $x_0 \in \partial \Omega$. By a change of the coordinate system, we may assume that $x_0 = 0$ and

$$\Omega \cap \mathcal{B}(0, r_0) = \left\{ (x', x_d) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \phi(x') \right\} \cap \mathcal{B}(0, r_0),$$

where $\phi: \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipchitz function with $\phi(0) = 0$ and $\|\nabla \phi\|_{\infty} \leq M_0$. For $x = (x', \phi(x')) \in \partial \Omega$ with $|x| \leq c_0 r_0$, define

$$\mathcal{M}_1(w)(x) = \sup \{ |w(x',t)| : \phi(x') < t < cr_0 \}.$$

It is not hard to see that

$$\mathcal{M}(w)(x) \leq C\mathcal{M}_{\partial\Omega}(\mathcal{M}_1(w))(x),$$

where $\mathcal{M}_{\partial\Omega}$ denotes the Hardy-Littlewood maximal operator on $\partial\Omega$, defined by

$$\mathcal{M}_{\partial\Omega}(g)(x) = \sup \left\{ \int_{\mathcal{B}(x,r) \cap \partial\Omega} |g| : 0 < r < cr_0 \right\}$$

for $x \in \partial \Omega$. As a result, by the L^2 boundedness of $\mathcal{M}_{\partial \Omega}$,

$$\int_{\mathcal{B}(0,cr_0)\cap\partial\Omega} |\mathcal{M}(w)|^2 \le C \int_{\mathcal{B}(0,c_1r_0)\cap\partial\Omega} |\mathcal{M}_1(w)|^2, \tag{8.2}$$

where $c_1 = Cc_0$.

Next, note that for $x = (x', \phi(x'))$ with $|x| \leq cr_0$,

$$\{\mathcal{M}_1(w)(x)\}^2 \le |w(x',\phi(x'))|^2 + 2\int_{\phi(x')}^{cr_0} |\partial_t w(x',t)| |w(x',t)| dt.$$

This, together with (8.2), leads to

$$\int_{\mathcal{B}(0,cr_0)\cap\partial\Omega} |\mathcal{M}(w)|^2 \leq C \int_{\mathcal{B}(0,cr_0)\cap\partial\Omega} |w|^2 + C \int_{\mathcal{B}(0,c_2r_0)\cap\Omega} |\nabla w||w|.$$

By a covering argument, we obtain

$$\int_{\partial\Omega} |\mathcal{M}(w)|^2 \le C \int_{\partial\Omega} |w|^2 + C \int_{\Omega} |\nabla w||w|
\le C \int_{\partial\Omega} |w|^2 + C \int_{\Omega} |\nabla w|^2 m(x, \mathbf{B})^{-1} + C \int_{\Omega} |w|^2 m(x, \mathbf{B})$$
(8.3)

for any $w \in C^1(\overline{\Omega}; \mathbb{C})$, where we have used the Cauchy inequality for the last step. We now apply (8.3) to the function $w_{\varepsilon} = \sqrt{|w|^2 + \varepsilon^2}$ and then let $\varepsilon \to 0$. Using $|\nabla w_{\varepsilon}| \le |(D + \mathbf{A})w|$, we deduce that

$$\int_{\partial\Omega} |\mathcal{M}(w)|^2 \le C \int_{\partial\Omega} |w|^2 + C \int_{\Omega} |(D+\mathbf{A})w|^2 m(x,\mathbf{B})^{-1} + C \int_{\Omega} |w|^2 m(x,\mathbf{B}). \tag{8.4}$$

Finally, we apply (8.4) to $w = u\widetilde{m}$, where \widetilde{m} is a function in $C^1(\mathcal{B}(0, 2R_0); \mathbb{R})$ with the properties that $\widetilde{m}(x) \approx m(x, \mathbf{B})$ for $x \in \mathcal{B}(0, 2R_0)$ and $|\nabla \widetilde{m}(x)| \leq Cm(x, \mathbf{B})^2$ [11]. It follows that

$$C \int_{\partial\Omega} |\mathcal{M}(u\widetilde{m})|^2 \le C \int_{\partial\Omega} |u\widetilde{m}|^2 + C \int_{\Omega} |(D + \mathbf{A})u|^2 m(x, \mathbf{B}) + C \int_{\Omega} |u|^2 m(x, \mathbf{B})^3$$
(8.5)

for any $u \in C^1(\overline{\Omega}; \mathbb{C})$. This gives (8.1).

Theorem 8.2. Let $u \in C^2(\overline{\Omega}; \mathbb{C})$ be a solution of the Neumann problem,

$$(D + \mathbf{A})^2 u = 0 \quad \text{in } \Omega \quad \text{and} \quad n \cdot (D + \mathbf{A}) u = g \quad \text{on } \partial\Omega.$$
 (8.6)

Let $v(x) = |(D + \mathbf{A})u(x)| + m(x, \mathbf{B})|u(x)|$. Then

$$\|(v)^*\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)},$$
 (8.7)

where C depends on C_0 in (3.2) and Ω .

Proof. In view of Corollary 4.4, it suffices to prove that

$$\|\mathcal{M}(v)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)},\tag{8.8}$$

where $\mathcal{M}(v)$ is defined by (4.13).

Let $w = (D_k + A_k)u$ for $1 \le k \le d$. Then $(D + \mathbf{A})^2w = F$, where

$$|F| \le 2|\mathbf{B}||(D+\mathbf{A})u| + |\nabla \mathbf{B}||u|. \tag{8.9}$$

Let $w = w_1 + w_2$, where $(D + \mathbf{A})^2 w_1 = 0$ in Ω and $w_1 = w$ on $\partial \Omega$. By Theorem 2.5,

$$||(w_1)^*||_{L^2(\partial\Omega)} \le C||w_1||_{L^2(\partial\Omega)} = C||(D_k + A_k)u||_{L^2(\partial\Omega)}$$

$$\le C||g||_{L^2(\partial\Omega)},$$
(8.10)

where we have used the Rellich estimate (7.20) for the last inequality.

Next, note that $(D + \mathbf{A})^2 w_2 = F$ in Ω and $w_2 = 0$ on $\partial \Omega$. It follows by Theorem 7.5 that

$$\int_{\Omega} |(D+\mathbf{A})w_2|^2 m(x,\mathbf{B})^{-1} + \int_{\Omega} |w_2|^2 m(x,\mathbf{B})$$

$$\leq C \int_{\Omega} |F|^2 m(x,\mathbf{B})^{-3}$$

$$\leq C \int_{\Omega} |\mathbf{B}|^2 |(D+\mathbf{A})u|^2 m(x,\mathbf{B})^{-3} + C \int_{\Omega} |\nabla \mathbf{B}|^2 |u|^2 m(x,\mathbf{B})^{-3},$$

where we have used (8.9) for the last inequality. Since $|\mathbf{B}(x)| \leq Cm(x, \mathbf{B})^2$ and $|\nabla \mathbf{B}(x)| \leq Cm(x, \mathbf{B})^3$, we obtain

$$\int_{\Omega} |(D+\mathbf{A})w_2|^2 m(x,\mathbf{B})^{-1} + \int_{\Omega} |w_2|^2 m(x,\mathbf{B})$$

$$\leq C \int_{\Omega} |(D+\mathbf{A})u|^2 m(x,\mathbf{B}) + C \int_{\Omega} |u|^2 m(x,\mathbf{B})^3$$

$$\leq C \int_{\partial\Omega} |g|^2,$$

where we have used the Rellich estimate (7.20) with $\ell = 0$ for the last step. In view of Lemma 8.1, this gives

$$\|\mathcal{M}(w_2)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}.\tag{8.11}$$

Finally, to prove (8.8), note that

$$\|\mathcal{M}(w)\|_{L^{2}(\partial\Omega)} \leq C\|(w_{1})^{*}\|_{L^{2}(\partial\Omega)} + C\|\mathcal{M}(w_{2})\|_{L^{2}(\partial\Omega)}$$

$$\leq C\|g\|_{L^{2}(\partial\Omega)},$$

where we have used (8.10) and (8.11) for the second inequality. Moreover, if $\tilde{v}(x) = m(x, \mathbf{B})u(x)$, then

$$\|\mathcal{M}(\widetilde{v})\|_{L^{2}(\partial\Omega)} \leq C \left\{ \|\widetilde{v}\|_{L^{2}(\partial\Omega)} + \|m(x,\mathbf{B})^{1/2}(D+\mathbf{A})u\|_{L^{2}(\Omega)} + \|m(x,\mathbf{B})^{3/2}u\|_{L^{2}(\Omega)} \right\}$$

$$\leq C \|g\|_{L^{2}(\partial\Omega)},$$

where we have used (8.1) for the first inequality and (7.20) with $\ell = 0$ for the second.

Theorem 8.3. Let $u \in C^2(\overline{\Omega}; \mathbb{C})$ be a solution of the Dirichlet problem,

$$(D + \mathbf{A})^2 u = 0 \quad \text{in } \Omega \quad \text{and} \quad u = f \quad \text{on } \partial\Omega. \tag{8.12}$$

Let $v(x) = |(D + \mathbf{A})u(x)| + m(x, \mathbf{B})|u(x)|$. Then

$$\|(v)^*\|_{L^2(\partial\Omega)} \le C \left\{ \|Tf\|_{L^2(\partial\Omega)} + \|m(x,\mathbf{B})f\|_{L^2(\partial\Omega)} \right\}, \tag{8.13}$$

where C depends on C_0 in (3.2) and Ω .

Proof. The proof is similar to that of Theorem 8.2, using (7.21) in the place of (7.20).

9 An approximation argument

In this section we use an approximation argument to extend the results in the last section to Lipschitz domains in \mathbb{R}^d , $d \geq 3$. We assume that $\Omega \subset \mathcal{B}(0, R_0)$ is a Lipschitz domain in \mathbb{R}^d , where $R_0 = Cr_0$ and C depends on the Lipschitz character of Ω . As in the last section, we also assume that **B** satisfies (3.2) for any $\mathcal{B}(x,r) \subset \mathcal{B}(0,4R_0)$ and (3.6) for any $x \in \mathcal{B}(0,R_0)$.

Lemma 9.1. Let Ω be a bounded Lipschitz domain in \mathbb{R}^d . There exists a sequence of smooth domains $\{\Omega_\ell\}$ with uniform Lipschitz characters such that $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_\ell \cdots \subset \Omega$ with the following properties:

• There exists a sequence of homeomorphisms $\Lambda_{\ell}: \partial\Omega \to \partial\Omega_{\ell}$ such that $\Lambda_{\ell}(x) \to x$ uniformly on $\partial\Omega$ as $\ell \to \infty$. Moreover,

$$|\Lambda_{\ell}(x) - x| \le C \operatorname{dist}(\Lambda_{\ell}(x), \partial \Omega)$$

for any $x \in \partial \Omega$.

- The unit outward normal to $\partial\Omega_{\ell}$, $n(\Lambda_{\ell}(x))$ converges to n(x) for a.e. $x \in \partial\Omega$.
- There are positive functions ω_{ℓ} on $\partial\Omega$ such that $0 < c \le \omega_{\ell} \le C$ uniformly in ℓ , $\omega_{\ell} \to 1$ a.e. as $\ell \to \infty$, and

$$\int_{E} \omega_{\ell} dx = \int_{\Lambda_{\ell}(E)} dx$$

for any measurable $E \subset \partial \Omega$.

• There exists $\alpha \in C_0^{\infty}(\mathbb{R}^d; \mathbb{R}^d)$ such that

$$\alpha(x) \cdot n(x) \ge c > 0$$
 for any $x \in \partial \Omega_{\ell}$ and any ℓ .

Proof. See [13]. \Box

Lemma 9.2. Let $\mathbf{A} \in C^1(\overline{\Omega}; \mathbb{R}^d)$, where Ω is a bounded Lipschitz domain in \mathbb{R}^d . Suppose that $u \in H^1(\Omega; \mathbb{C})$ is a weak solution of the Neumann problem (8.6) for some $g \in L^2(\partial\Omega; \mathbb{C})$, or a weak solution of the Dirichlet problem (8.12) for some $f \in H^1(\partial\Omega; \mathbb{C})$. Then

$$\int_{\partial\Omega} |\nabla u(\Lambda_{\ell}(x)) - \nabla u(x)|^2 dx + \int_{\partial\Omega} |u(\Lambda_{\ell}(x)) - u(x)|^2 dx \to 0$$
(9.1)

as $\ell \to \infty$, where Λ_{ℓ} is given by Lemma 9.1.

Proof. We give the proof for the Neumann problem (8.6). The proof for (8.12) is similar. Since $\mathbf{A} \in C^1(\overline{\Omega}; \mathbb{R}^d)$ and $u \in H^1(\Omega; \mathbb{C})$, we may rewrite (8.6) as

$$-\Delta u = F$$
 in Ω and $\frac{\partial u}{\partial n} = \widetilde{g}$

for some $F \in L^2(\Omega; \mathbb{C})$ and $\widetilde{g} \in L^2(\partial\Omega; \mathbb{C})$. Let u = v + w, where $w = \Gamma * (F\chi_{\Omega})$ and Γ denotes the fundamental solution for $-\Delta$ in \mathbb{R}^d . Note that $w \in H^2(\mathbb{R}^d; \mathbb{C})$. It follows that

$$\int_{\partial\Omega} |\nabla w(\Lambda_{\ell}(x)) - \nabla w(x)|^2 dx + \int_{\partial\Omega} |w(\Lambda_{\ell}(x)) - w(x)|^2 dx
\leq C \int_{\Omega \setminus \Omega_{\ell}} \left\{ |\nabla^2 w|^2 + |\nabla w|^2 \right\} \to 0,$$
(9.2)

as $\ell \to \infty$. Also observe that $v \in H^1(\Omega; \mathbb{C})$ is harmonic in Ω and $\frac{\partial v}{\partial n} = \tilde{g} - n \cdot \nabla w \in L^2(\partial \Omega; \mathbb{C})$. As a result, v and ∇v have nontangential limits a.e. on $\partial \Omega$ and $(v)^* + (\nabla v)^* \in L^2(\partial \Omega)$ [4, 5]. By the dominated convergence theorem,

$$\int_{\partial\Omega} |\nabla v(\Lambda_{\ell}(x)) - \nabla v(x)|^2 dx + \int_{\partial\Omega} |v(\Lambda_{\ell}(x)) - v(x)|^2 dx \to 0$$

as $\ell \to \infty$, which, together with (9.2), gives (9.1).

Theorem 9.3. Let $u \in H^1(\Omega; \mathbb{C})$ be a solution of the Neumann problem (8.6) for some $g \in L^2(\partial\Omega; \mathbb{C})$. Let $v(x) = |(D + \mathbf{A})u(x)| + m(x, \mathbf{B})|u(x)|$. Then

$$\|(v)^*\|_{L^2(\partial\Omega)} + \|m(x, \mathbf{B})^{1/2}v\|_{L^2(\Omega)} \le C\|g\|_{L^2(\partial\Omega)},$$
 (9.3)

where C depends on C_0 in (3.3), r_0 and the Lipschitz character of Ω .

Proof. Let $u \in H^1(\Omega; \mathbb{C})$ be a weak solution of (8.6) with $g \in L^2(\partial\Omega; \mathbb{C})$. Let $\{\Omega_\ell\}$ be a sequence of smooth domains, given by Lemma 9.1. Since $\overline{\Omega}_\ell \subset \Omega$ and $\mathbf{A} \in C^2(\overline{\Omega}; \mathbb{R}^d)$, we have $u \in C^2(\overline{\Omega}_\ell; \mathbb{C})$. This allows us to apply Theorem 8.2 and (7.20) to obtain

$$\|(v)_{\ell}^*\|_{L^2(\partial\Omega_{\ell})} + \|m(x,\mathbf{B})^{1/2}v\|_{L^2(\Omega_{\ell})} \le C\|n\cdot(D+\mathbf{A})u\|_{L^2(\partial\Omega_{\ell})},\tag{9.4}$$

where we have used $(v)_{\ell}^*$ to denote the nontangential maximal function of v for the domain Ω_{ℓ} . Thanks to Lemma 9.1, the constant C in (9.4) depends only on r_0 , the Lipschitz character of Ω , and C_0 in (3.2). In particular, the estimate (9.4) is uniform with respect to ℓ . Using the observation $\|(v)_{\ell}^*\|_{L^2(\partial\Omega_{\ell})} \leq C\|(v)_m^*\|_{L^2(\partial\Omega_m)}$ for $\ell < m$, we see that

$$\|(v)_{\ell}^*\|_{L^2(\partial\Omega_{\ell})} + \|m(x,\mathbf{B})^{1/2}v\|_{L^2(\Omega_{\ell})} \le C\|n\cdot(D+\mathbf{A})u\|_{L^2(\partial\Omega_m)}$$

for $\ell < m$. We now let $m \to \infty$ in the inequality above. Note that by Lemma 9.2,

$$||n\cdot (D+\mathbf{A})u||_{L^2(\partial\Omega_m)}\to ||g||_{L^2(\partial\Omega)}.$$

Thus,

$$\|(v)_{\ell}^*\|_{L^2(\partial\Omega_{\ell})} + \|m(x,\mathbf{B})^{1/2}v\|_{L^2(\Omega_{\ell})} \le C\|g\|_{L^2(\partial\Omega)}.$$

By letting $\ell \to \infty$ in the inequality above, we obtain (9.3).

Theorem 9.4. Let $u \in H^1(\Omega; \mathbb{C})$ be a solution of the Dirichlet problem (8.12) for some $f \in H^1(\partial\Omega; \mathbb{C})$. Let $v(x) = |(D + \mathbf{A})u(x)| + m(x, \mathbf{B})|u(x)|$. Then

$$\|(v)^*\|_{L^2(\partial\Omega)} + \|m(x,\mathbf{B})^{1/2}v\|_{L^2(\Omega)} \le C\left\{\|Tf\|_{L^2(\partial\Omega)} + \|m(x,\mathbf{B})f\|_{L^2(\partial\Omega)}\right\},\tag{9.5}$$

where C depends on C_0 in (3.3), r_0 and the Lipschitz character of Ω .

Proof. The proof is similar to that of Theorem 9.3, using Theorem 8.3 in the place of Theorem 8.2. \Box

10 Proof of Theorems 1.1 and 1.2

In this section, with Theorems 9.3 and 9.4 at our disposal, we use a localization argument to complete the proofs of our main results.

Let Ω be a bounded Lipschitz domain in \mathbb{R}^d , $d \geq 2$. Assume $\mathbf{A} \in C^{\infty}(\overline{\Omega}; \mathbb{R}^d)$ and $\mathbf{B} = \nabla \times \mathbf{A}$ is of finite type on $\overline{\Omega}$. It follows that there exist an integer $\kappa \geq 0$ and $c_0 > 0$ such that

$$\sum_{|\alpha| \le \kappa} |\partial^{\alpha} \mathbf{B}(x)| \ge c_0 \tag{10.1}$$

for any $x \in \overline{\Omega}$. By extension we may assume $\mathbf{B} \in C^{\kappa+1}(\mathbb{R}^d; \mathbb{R}^{d \times d})$ and $\|\mathbf{B}\|_{C^{\kappa+1}(\mathbb{R}^d)} \leq C\|\mathbf{B}\|_{C^{\kappa+1}(\overline{\Omega})}$ (there is no need to extend \mathbf{A}). By compactness we may further assume that (10.1) holds for any $x \in \mathbb{R}^d$ with $\operatorname{dist}(x,\Omega) < 2r_0$.

Let
$$\widetilde{\Omega} = \{ x \in \mathbb{R}^d : \operatorname{dist}(x, \Omega) < r_0 \}.$$

Lemma 10.1. Suppose (10.1) holds for any $x \in \mathbb{R}^d$ with $\operatorname{dist}(x,\Omega) < 2r_0$. Then

$$\sup_{\mathcal{B}(x,r)} |\partial^{\alpha} \mathbf{B}| \le \frac{C_1}{r^{|\alpha|}} \oint_{\mathcal{B}(x,r)} |\mathbf{B}| \tag{10.2}$$

for any $\mathcal{B}(x,r) \subset \widetilde{\Omega}$ and any α with $|\alpha| \leq \kappa$. Moreover,

$$\sup_{\mathcal{B}(x,r)} |\mathbf{B}| \ge c_1 r^{\kappa} \tag{10.3}$$

for any $x \in \overline{\Omega}$ and $0 < r < r_0$. The constants $C_1, c_1 > 0$ depend on Ω , (κ, c_0) in (10.1) and $\|\mathbf{B}\|_{C^{\kappa+1}(\overline{\Omega})}$.

Proof. Let $\mathcal{B}(y,r) \subset \{x \in \mathbb{R}^d : \operatorname{dist}(x,\Omega) < 2r_0\}$. Let \mathbf{P}_y denote the κ^{th} (matrix-valued) Taylor polynomial of \mathbf{B} at y. Then, if $|\alpha| \leq \kappa$,

$$|\partial^{\alpha} \mathbf{P}_{y}(x) - \partial^{\alpha} \mathbf{B}(x)| \le C|x - y|^{\kappa + 1 - |\alpha|}$$

for any $x \in \mathcal{B}(y,r)$, where C depends on κ and $\|\mathbf{B}\|_{C^{\kappa+1}(\mathbb{R}^d)}$. It follows that

$$\sup_{\mathcal{B}(y,r)} |\partial^{\alpha} \mathbf{B}| \leq \sup_{\mathcal{B}(y,r)} |\partial^{\alpha} \mathbf{P}_{y}| + Cr^{\kappa+1-|\alpha|}$$

$$\leq \frac{C}{r^{|\alpha|}} \int_{\mathcal{B}(y,r)} |\mathbf{P}_{y}| + Cr^{\kappa+1-|\alpha|}$$

$$\leq \frac{C}{r^{|\alpha|}} \int_{\mathcal{B}(y,r)} |\mathbf{B}| + Cr^{\kappa+1-|\alpha|}$$
(10.4)

for any α with $|\alpha| \leq \kappa$. Also note that

$$\int_{\mathcal{B}(y,r)} |\mathbf{B}| \ge \int_{\mathcal{B}(y,r)} |\mathbf{P}_{y}| - Cr^{\kappa+1}$$

$$\ge c \sum_{|\alpha| \le \kappa} |\partial^{\alpha} \mathbf{B}(y)| r^{|\alpha|} - Cr^{\kappa+1}$$

$$\ge cr^{\kappa} - Cr^{\kappa+1}, \tag{10.5}$$

where we have used (10.1) for the last step. As a result, we see that

$$\oint_{\mathcal{B}(y,r)} |\mathbf{B}| \ge cr^{\kappa} \tag{10.6}$$

if 0 < r < c and c is sufficiently small. This gives (10.3). Together with (10.4), it also yields (10.2) under the additional condition 0 < r < c. Note that the inequality (10.2) with $\alpha = 0$ and 0 < r < c implies the doubling condition,

$$\int_{\mathcal{B}(x,2r)} |\mathbf{B}| \le C \int_{\mathcal{B}(x,r)} |\mathbf{B}|$$

for any $\mathcal{B}(x,4r) \subset \{x \in \mathbb{R}^d : \operatorname{dist}(x,\Omega) < 2r_0\}$. The general case for (10.2) follows by a covering argument.

Remark 10.2. Suppose (10.1) holds for any $x \in \mathbb{R}^d$ with $\operatorname{dist}(x, \partial\Omega) < 2r_0$. Also assume that

$$\sup_{\mathcal{B}(x,r_0)} |\mathbf{B}| \ge 2r_0^{-2} \qquad \text{for any } x \in \overline{\Omega}.$$
 (10.7)

It follows by definition that $m(x, \mathbf{B}) > r_0^{-1}$ for any $x \in \overline{\Omega}$. By letting $r = m(x, \mathbf{B})^{-1} < r_0$ in (10.2), we see that

$$|\partial^{\alpha} \mathbf{B}(x)| \le Cm(x, \mathbf{B})^{|\alpha|+2}$$

for any $x \in \overline{\Omega}$ and $|\alpha| \le \kappa$. As a result, we obtain

$$m(x, \mathbf{B}) \ge c \sum_{|\alpha| \le \kappa} |\partial^{\alpha} \mathbf{B}(x)|^{\frac{1}{|\alpha|+2}}$$
 (10.8)

for any $x \in \overline{\Omega}$. By definition it is not hard to see that $m(x, \mathbf{B}) \leq C \|\mathbf{B}\|_{L^{\infty}(\widetilde{\Omega})}^{1/2}$.

Proof of Theorem 1.1. We first consider the case $d \geq 3$. Rewrite (1.2) as

$$(D + \beta \mathbf{A})^2 u = 0$$
 in Ω and $n \cdot (D + \beta \mathbf{A})u = \beta g$ on $\partial \Omega$,

where $\beta = h^{-1}$. By Lemma 10.1, the magnetic field **B** satisfies the condition

$$\sup_{\mathcal{B}(x,r)} |\nabla \mathbf{B}| \le \frac{C}{r} \int_{\mathcal{B}(x,r)} |\mathbf{B}|$$

for any $\mathcal{B}(x,r) \subset \widetilde{\Omega}$. It follows that $\beta \mathbf{B}$ satisfies the same condition. Moreover, by the proof of Lemma 10.1, there exists c > 0, depending only on Ω , (κ, c_0) in (10.1) and $\|\mathbf{B}\|_{C^{\kappa+1}(\overline{\Omega})}$ such that

$$\int_{\mathcal{B}(x,r)} |\mathbf{B}| \ge cr^{\kappa}$$

for any $x \in \overline{\Omega}$ and $0 < r < r_0$. This implies that

$$\sup_{\mathcal{B}(x,r)} |\beta \mathbf{B}| \ge c\beta r^{\kappa}. \tag{10.9}$$

for any $x \in \overline{\Omega}$ and $0 < r < r_0$, where c > 0 depends only on Ω , (κ, c_0) in (10.1) and $\|\mathbf{B}\|_{C^{\kappa+1}(\overline{\Omega})}$.

Next, we fix $x_0 \in \partial\Omega$ and consider a family of Lipschitz domains $\mathcal{O}_t = \Omega \cap \mathcal{B}(x_0, tr_0)$, where $t \in (c, 2c)$ and c > 0 is sufficiently small. Let $R_0 = 2cr_0$. Note that $\mathcal{O}_t \subset \mathcal{B}(x_0, R_0)$, $\beta \mathbf{B}$ satisfies (3.2) for any $\mathcal{B}(x, r) \subset \mathcal{B}(x_0, 4R_0)$. Moreover, by (10.9),

$$\sup_{\mathcal{B}(x,R_0/2)} |\beta \mathbf{B}| \ge c\beta (R_0/2)^{\kappa} > 4R_0^{-2},$$

if $\beta > \beta_0 = CR_0^{-\kappa-2}$ and C is sufficiently large. This allows us to apply Theorem 9.3 to obtain

$$\|(v)_t^*\|_{L^2(\partial \mathcal{O}_t)} + \|m(x, \beta \mathbf{B})^{1/2}v\|_{L^2(\mathcal{O}_t)} \le C\|n \cdot (D + \beta \mathbf{A})u\|_{L^2(\partial \mathcal{O}_t)}, \tag{10.10}$$

where $v(x) = |(D+\beta \mathbf{A})u(x)| + m(x, \beta \mathbf{B})|u(x)|$ and $(v)_t^*$ denotes the nontangential maximal function of v with respect to the domain \mathcal{O}_t . We point out that the constant C in (10.10) does not depend on t, as the family of Lipschitz domains $\{\mathcal{O}_t : t \in (c, 2c)\}$ possesses uniform Lipschitz characters. It follows that

$$\int_{\mathcal{B}(x_0, cr_0) \cap \partial \Omega} |\mathcal{N}(v)|^2 + \int_{\mathcal{B}(x_0, cr_0) \cap \Omega} m(x, \beta \mathbf{B}) |v|^2 \\
\leq C \int_{\mathcal{B}(x_0, r_0) \cap \partial \Omega} |\beta g|^2 + C \int_{\Omega \cap \partial \mathcal{B}(x_0, tr_0)} |(D + \beta \mathbf{A}) u|^2$$

for $t \in (c, 2c)$, where

$$\mathcal{N}(v)(x) = \sup \{|v(y)| : y \in \Omega, |y - x| < M_0 \operatorname{dist}(y, \partial \Omega) \text{ and } \operatorname{dist}(y, \partial \Omega) < cr_0\}$$

for $x \in \partial \Omega$. By integrating the inequality above in $t \in (c, 2c)$, we obtain

$$\int_{\mathcal{B}(x_0, cr_0) \cap \partial \Omega} |\mathcal{N}(v)|^2 + \int_{\mathcal{B}(x_0, cr_0) \cap \Omega} m(x, \beta \mathbf{B}) |v|^2
\leq C \int_{\mathcal{B}(x_0, r_0) \cap \partial \Omega} |\beta g|^2 + C \int_{\Omega \cap \mathcal{B}(x_0, r_0)} |(D + \beta \mathbf{A}) u|^2.$$

By a covering argument, this gives

$$\int_{\partial\Omega} |\mathcal{N}(v)|^2 + \int_{\Omega_{cr_0}} m(x, \beta \mathbf{B}) |v|^2 \le C \int_{\partial\Omega} |\beta g|^2 + C \int_{\Omega} |(D + \beta \mathbf{A})u|^2, \tag{10.11}$$

where $\Omega_s = \{x \in \Omega : \operatorname{dist}(x, \partial \Omega) < s\}.$

To bound v(x) for x away from $\partial\Omega$, observe that if $\mathcal{B}(x,cr_0)\subset\Omega$,

$$|v(x)| \le C \left(\oint_{\mathcal{B}(x,cr_0/2)} |v|^2 \right)^{1/2}$$

$$\le C \left(\oint_{\mathcal{B}(x,cr_0/2)} |(D+\beta \mathbf{A})u|^2 \right)^{1/2},$$

where we have used the interior estimate (4.14) for the first inequality and (3.12) for the second. It follows that

$$\int_{\partial\Omega} |(v)^*|^2 \le C \int_{\partial\Omega} |\mathcal{N}(v)|^2 + C \int_{\Omega} |(D + \beta \mathbf{A})u|^2.$$

This, together with (10.11), leads to

$$\int_{\partial\Omega} |(v)^*|^2 + \int_{\Omega_{cr_0}} m(x, \beta \mathbf{B})|v|^2 \le C \int_{\partial\Omega} |\beta g|^2 + C \int_{\Omega} |(D + \beta \mathbf{A})u|^2.$$
 (10.12)

To bound $m(x, \beta \mathbf{B})|v|^2$ away from $\partial\Omega$, we let $\mathcal{B}(x_0, 2cr_0) \subset \Omega$ and apply Theorem 9.3 to the domain $\mathcal{O}_t = \mathcal{B}(x_0, tr_0)$, where $t \in (c, 2c)$. This yields

$$\int_{\mathcal{B}(x_0, cr_0)} m(x, \beta \mathbf{B}) |v|^2 \le C \int_{\partial \mathcal{B}(x_0, tr_0)} |(D + \beta \mathbf{A})u|^2.$$

By integrating the inequality above in $t \in (c, 2c)$, we obtain

$$\int_{\mathcal{B}(x_0, cr_0)} m(x, \beta \mathbf{B}) |v|^2 \le C \int_{\mathcal{B}(x_0, 2cr_0)} |(D + \beta \mathbf{A})u|^2.$$

Thus, by a covering argument,

$$\int_{\Omega \backslash \Omega_{\mathrm{Cro}}} m(x, \beta \mathbf{B}) |v|^2 \leq C \int_{\Omega} |(D + \beta \mathbf{A}) u|^2.$$

In view of (10.12), we have proved that

$$\int_{\partial\Omega} |(v)^*|^2 + \int_{\Omega} m(x, \beta \mathbf{B}) |v|^2 \le C \int_{\partial\Omega} |\beta g|^2 + C \int_{\Omega} |(D + \beta \mathbf{A})u|^2
\le C \int_{\partial\Omega} |\beta g|^2,$$
(10.13)

where we have used the energy estimate for the last step.

Finally, note that

$$\beta^{-1}v = |(hD + \mathbf{A})u(x)| + hm(x, h^{-1}\mathbf{B})|u(x)| = v_h,$$

where $h = \beta^{-1}$. Let $h_0 = \beta_0^{-1}$. The desired estimates (1.6)-(1.7) for $h \in (0, h_0)$ now follow from (10.13). Note that by Remark 10.2, if $h \in (0, h_0)$,

$$c\sum_{|\alpha| \le \kappa} h^{-\frac{1}{|\alpha|+2}} |\partial^{\alpha} \mathbf{B}(x)|^{\frac{1}{|\alpha|+2}} \le m(x, h^{-1}\mathbf{B}) \le Ch^{-\frac{1}{2}},$$

where C, c > 0 depend only on Ω , (κ, c_0) in (10.1) and $\|\mathbf{B}\|_{C^{\kappa+1}(\overline{\Omega})}$.

The case d=2 can be handled by the method of descending. Indeed, let $u \in H^1(\Omega; \mathbb{C})$ be a weak solution of the Neumann problem (1.2) with boundary data $g \in L^2(\partial\Omega; \mathbb{C})$ in a two-dimensional Lipschitz domain Ω . Define $\widetilde{\mathbf{A}} = (A_1(x_1, x_2), A_2(x_1, x_2), 0)$ and $v(x_1, x_2, x_3) = u(x_1, x_2)$. Then $(hD + \widetilde{\mathbf{A}})^2 v = 0$ in \mathcal{O} , where $\mathcal{O} = \Omega \times (0, r_0)$ is a bounded Lipschitz domain in \mathbb{R}^3 . Note that $n \cdot (hD + \widetilde{\mathbf{A}})v = 0$ on $\Omega \times \{0, r_0\}$, and that $n \cdot (hD + \widetilde{\mathbf{A}})v = (n_1, n_2) \cdot (hD + \mathbf{A})u$ on $\partial\Omega \times (0, r_0)$. It follows from the case d=3 that

$$\|((hD + \mathbf{A})u)^*\|_{L^2(\partial\Omega)} \le \|((hD + \widetilde{\mathbf{A}})v)^*\|_{L^2(\partial\mathcal{O})}$$

$$\le C\|n \cdot (hD + \widetilde{\mathbf{A}})v\|_{L^2(\partial\mathcal{O})} = C\|g\|_{L^2(\partial\Omega)}.$$

Observe that if $x=(x_1,x_2,x_3)=(x',x_3)$, then $m(x,\nabla\times\widetilde{\mathbf{A}})\approx m(x',\nabla\times\mathbf{A})$. The other estimates in Theorem 1.1 may be obtained in the same manner. We omit the details.

Proof of Theorem 1.2. The proof for the case $d \geq 3$ is similar to that of Theorem 1.2, using Theorem 9.4. The case d=2 can also be treated by the method of descending. Let $u \in H^1(\Omega; \mathbb{C})$ be a weak solution of the Dirichlet problem (1.3) with boundary data $f \in H^1(\partial\Omega; \mathbb{C})$ in a two-dimensional Lipschitz domain Ω . Let $\widetilde{\mathbf{A}}$, v and \mathcal{O} be defined as in the proof of Theorem 1.1. Then $v \in H^1(\mathcal{O}; \mathbb{C})$ is a weak solution of $(hD + \widetilde{\mathbf{A}})^2v = 0$ in \mathcal{O} with Dirichlet data F, where $F(x_1, x_2, x_3) = u(x_1, x_2)$ on $\Omega \times \{0, r_0\}$, and $F(x_1, x_2, x_3) = f(x_1, x_2)$ for $(x_1, x_2, x_3) \in \partial\Omega \times (0, r_0)$. It is not hard to check that

$$||T^{h}(F)||_{L^{2}(\partial\Omega\times(0,r_{0}))} + h||m(x,h^{-1}\nabla\times\widetilde{\mathbf{A}})F||_{L^{2}(\partial\Omega\times(0,r_{0}))}$$

= $||T^{h}(f)||_{L^{2}(\partial\Omega)} + h||m(x',h^{-1}\nabla\times\mathbf{A})f||_{L^{2}(\partial\Omega)},$

and

$$||T^{h}(F)||_{L^{2}(\Omega\times\{0,r_{0}\})} + h||m(x,h^{-1}\nabla\times\widetilde{\mathbf{A}})F||_{L^{2}(\Omega\times\{0,r_{0}\})}$$

$$\leq C||(hD+\mathbf{A})u||_{L^{2}(\Omega)} + Ch||m(x',h^{-1}\nabla\times\mathbf{A})u||_{L^{2}(\Omega)}$$

$$\leq C||(hD+\mathbf{A})u||_{L^{2}(\Omega)},$$

where we have used (3.12). Moreover,

$$\int_{\Omega} |(hD + \mathbf{A})u|^{2} \le h \int_{\partial\Omega} |n \cdot (hD + \mathbf{A})u||u|$$

$$\le C \|(hD + \mathbf{A})u\|_{L^{2}(\partial\Omega)} \|hm(x, h^{-1}\mathbf{B})u\|_{L^{2}(\partial\Omega)}$$

$$\le \varepsilon \|(hD + \mathbf{A})u\|_{L^{2}(\partial\Omega)}^{2} + C\varepsilon^{-1} \|hm(x, h^{-1}\mathbf{B})f\|_{L^{2}(\partial\Omega)}^{2},$$

where we have used the fact $m(x, h^{-1}\mathbf{B}) \ge r_0^{-1}$ for any $x \in \overline{\Omega}$, if $h \in (0, h_0)$. As a result, we have proved that

$$\| ((hD + \mathbf{A})u)^* \|_{L^2(\partial\Omega)} + h \| (m(x, h^{-1}\mathbf{B})u)^* \|_{L^2(\partial\Omega)}$$

$$\leq C \| T^h f \|_{L^2(\partial\Omega)} + C(1 + \varepsilon^{-1})h \| m(x, h^{-1}\mathbf{B})f \|_{L^2(\partial\Omega)} + C\varepsilon \| (hD + \mathbf{A})u \|_{L^2(\partial\Omega)}.$$

By choosing ε small so that $C\varepsilon \leq (1/2)$, we obtain (1.8). The estimate (1.9) follows in a similar manner.

Conflict of interest. The author declares that there is no conflict of interest.

Data availability. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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