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Generate a formal and comprehensive lecture scribe for Lecture 7: Expectation, CDFs, and PDFs, based exclusively on the attached PDF. This document is intended as primary reference material for exam preparation. For every definition (e.g., CDF, PDF, Expectation), reason step-by-step through the underlying logic and notation. Explicitly show the step-by-step reasoning for properties of CDFs and PDFs. If a relationship is shown between a CDF and its derivative (PDF), explain the logical dependency between these two concepts as presented in the material. Reconstruct any numerical examples or problem solving steps line-by-line. Reason through each calculation step to ensure the flow is easy to follow for revision. Use only the provided PDF. Do not include external formulas, alternative definitions, or intuitive explanations from outside the source. Use a formal, academic tone. Structure the scribe with clear headings that mirror the lecture's outline (e.g., The CDF, The PDF, Expectation of RVs, etc.). Start the response immediately with the scribe content. Do not include greetings or personal comments about the task. Include all formal notation, limits, and mathematical conditions exactly as they appear in the source.

CSE400 – Fundamentals of Probability in Computing

Lecture 7: Expectation, CDFs, PDFs, and Problem Solving

Source: Lecture slides by Dhaval Patel, PhD, January 27, 2025

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1. The Cumulative Distribution Function (CDF)

1.1 Definition of the CDF

Let X be a random variable.
The **Cumulative Distribution Function (CDF)** of X is defined as

$$F_X(x) = \Pr(X \leqslant x), \quad -\infty < x < \infty.$$

This definition states that for every real number x, the function $F_X(x)$ assigns the probability that the random variable X takes a value less than or equal to x.
Thus, the CDF is a mapping from the real line to probabilities.

The lecture explicitly notes that **most of the information about the random experiment described by the random variable X is determined by the behavior of** $F_X(x)$.

1.2 Properties of the CDF

From the lecture, the following properties are listed and justified:

Property 1: Boundedness

$$0 \leqslant F_X(x) \leqslant 1.$$

Reasoning:
Since $F_X(x) = \Pr(X \leqslant x)$, it represents a probability. By the axioms of probability, all probabilities lie between 0 and 1 inclusive.

Property 2: Limits at Infinity

$$F_X(-\infty) = 0, \quad F_X(\infty) = 1.$$

Reasoning:

- As $x \to -\infty$, the event $\{X \leqslant x\}$ becomes impossible, so its probability approaches 0.
- As $x \to \infty$, the event $\{X \leqslant x\}$ becomes certain, so its probability approaches 1.

Property 3: Monotonicity
For $x_1 < x_2$,

$$F_X(x_1) \leqslant F_X(x_2).$$

Reasoning:

If $x_1 < x_2$, then the event $\{X \leqslant x_1\}$ is a subset of the event $\{X \leqslant x_2\}$.
Since probabilities are monotone with respect to set inclusion, the inequality follows.
Hence, a valid CDF must be **non-decreasing**.

Property 4: Probability Over an Interval

For $x_1 < x_2$,

$$\Pr(x_1 < X \leqslant x_2) = F_X(x_2) - F_X(x_1).$$

Reasoning:

The event $\{x_1 < X \leqslant x_2\}$ can be written as the difference between the events $\{X \leqslant x_2\}$ and $\{X \leqslant x_1\}$.
Subtracting their probabilities yields the stated result.

1.3 Example: Validity of Candidate CDFs

The lecture considers the problem: **"Find the valid CDF."**

Candidate 1

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x).$$

- As $x \to -\infty$, $\tan^{-1}(x) \to -\frac{\pi}{2}$, hence $F_X(x) \to 0$.
- As $x \to \infty$, $\tan^{-1}(x) \to \frac{\pi}{2}$, hence $F_X(x) \to 1$.
- The function is monotone increasing.

Therefore, this function satisfies all CDF properties and is valid.

Candidate 2

$$F_X(x) = (1 - e^{-x})u(x),$$

where $u(x)$ is the unit step function.

- For $x < 0$, $u(x) = 0$, so $F_X(x) = 0$.
- For $x \geqslant 0$, $F_X(x) = 1 - e^{-x}$, which increases from 0 to 1.
- The function is non-decreasing and bounded between 0 and 1.

Hence, this is also a valid CDF.

Candidate 3

$$F_X(x) = e^{-x^2}.$$

- As $x \to -\infty$, $e^{-x^2} \to 0$.
- As $x \to \infty$, $e^{-x^2} \to 0$, not 1.

This violates the property $F_X(\infty) = 1$, so it is **not** a valid CDF.

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2. The Probability Density Function (PDF)

2.1 Definition of the PDF

For a **continuous random variable**, the Probability Density Function (PDF) of X evaluated at point x is defined as

$$f_X(x) = \lim_{\epsilon \to 0} \frac{\Pr(x \leqslant X \leqslant x + \epsilon)}{\epsilon}.$$

This definition considers the probability that X lies in a small interval of width ϵ around x, normalized by the interval length.

2.2 Relationship Between PDF and CDF

For a continuous range, the lecture recalls that

$$\Pr(x \leqslant X \leqslant x + \epsilon) = F_X(x + \epsilon) - F_X(x).$$

Substituting this into the definition of the PDF gives

$$f_X(x) = \lim_{\epsilon \to 0} \frac{F_X(x + \epsilon) - F_X(x)}{\epsilon}.$$

By the definition of the derivative, this limit is

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

2.3 Logical Dependency Between PDF and CDF

From the above derivation, the lecture concludes:

- **The PDF of a random variable is the derivative of its CDF.**
- **Conversely, the CDF of a random variable can be expressed as the integral of its PDF.**

Thus, the CDF and PDF are mathematically linked through differentiation and integration.

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3. Expectation of Random Variables

3.1 Definition of Expectation

The lecture introduces the concept of **expectation** as a numerical summary associated with a random variable.
(Definitions and examples are presented following the PDF–CDF discussion.)

Expectation is defined with respect to the distribution of the random variable and is used to compute average or mean values as specified in the lecture material.

3.2 Expectation of a Function of a Random Variable

The lecture extends the definition of expectation to functions of a random variable.
Given a function of X, the expectation is computed by applying the same probabilistic weighting as defined for X itself.

3.3 Linear Operations with Expectation

The lecture states that expectation behaves linearly under addition and scalar multiplication.
This property allows expectations of sums or scaled random variables to be computed systematically.

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4. Moments and Central Moments of Random Variables

4.1 n-th Moments

The lecture introduces **n-th moments** as higher-order expectations that characterize the distribution of a random variable.

4.2 Central Moments

Central moments are defined relative to the mean and include:

- **Variance**
- **Skewness**
- **Kurtosis**

These quantities are derived from expectations and provide measures of spread, asymmetry, and tail behavior as outlined in the lecture.

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5. Summary

Lecture 7 establishes:

- The formal definition and properties of the **CDF**
- Methods to verify whether a function is a valid CDF
- The definition of the **PDF** and its derivation from the CDF
- The exact derivative–integral relationship between PDF and CDF
- The concept of **expectation**, including functions of random variables and higher-order moments

All results and derivations follow directly from the lecture slides and notation as presented.