CSE400 – Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

1 Transformation of One Random Variable

Let X be a continuous random variable with known probability density function (PDF) $f_X(x)$. Let a new random variable Y be defined as a function of X:

$$Y = g(X).$$

The objective is to derive the PDF of Y, denoted $f_Y(y)$, given the PDF of X.

1.1 Step 1: Start from the Cumulative Distribution Function (CDF)

By definition, the CDF of Y is

$$F_Y(y) = \Pr(Y \leq y).$$

Substituting $Y = g(X)$,

$$F_Y(y) = \Pr(g(X) \leq y).$$

1.2 Step 2: Use Monotonicity of the Transformation

The lecture considers monotonic transformations $g(x)$, which may be:

- Monotonically increasing, or
- Monotonically decreasing.

This assumption is essential because it allows inversion of the function $g(x)$.

Case 1: $g(x)$ is Monotonically Increasing

If $g(x)$ is increasing, then

$$g(X) \leq y \iff X \leq g^{-1}(y).$$

Hence,

$$F_Y(y) = \Pr(X \leq g^{-1}(y)) = F_X(g^{-1}(y)).$$

Case 2: $g(x)$ is Monotonically Decreasing

If $g(x)$ is decreasing, then

$$g(X) \leq y \iff X \geq g^{-1}(y).$$

Thus,

$$F_Y(y) = \Pr(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y)).$$

1.3 Step 3: Differentiate the CDF to Obtain the PDF

The PDF of Y is obtained by differentiating the CDF with respect to y:

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

For the increasing case,

$$f_Y(y) = \frac{d}{dy}\left[F_X(g^{-1}(y))\right].$$

Applying the chain rule,

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}\left[g^{-1}(y)\right].$$

1.4 Step 4: Role of the Absolute Derivative

Both increasing and decreasing cases can be combined using the absolute value of the derivative:

$$\boxed{f_Y(y) = \frac{f_X(x)}{\left|\dfrac{dy}{dx}\right|} \quad \text{evaluated at } x = g^{-1}(y)}$$

The absolute value accounts for:

- Positive slope (increasing transformation),
- Negative slope (decreasing transformation).

Thus, the magnitude of stretching or compression of the transformation directly affects the density.

2 Function of Two Random Variables

Let X and Y be two continuous random variables with joint PDF $f_{X,Y}(x, y)$. Define a new random variable

$$Z = X + Y.$$

The objective is to derive the PDF $f_Z(z)$.

2.1 Step 1: Define the CDF of Z

By definition,

$$F_Z(z) = \Pr(Z \leq z).$$

Substituting $Z = X + Y$,

$$F_Z(z) = \Pr(X + Y \leq z).$$

2.2 Step 2: Geometric Interpretation

The condition $X + Y \leq z$ represents a region in the (x, y)-plane bounded by:

- The line $x + y = z$,
- The support of the joint PDF $f_{X,Y}(x, y)$.

The probability is computed by integrating the joint PDF over this region.

2.3 Step 3: Express the CDF as a Double Integral

From the lecture setup,

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{X,Y}(x, y)\, dy\, dx.$$

2.4 Step 4: Differentiate to Obtain the PDF

Differentiating with respect to z,

$$f_Z(z) = \frac{d}{dz} F_Z(z).$$

Differentiation with respect to the upper limit yields

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x)\, dx}.$$

3 Illustrative Examples

3.1 Example 1: Transformation of a Uniform Random Variable

Let

- $X \sim \text{Uniform}(-1, 1)$,
- $Y = \sin\left(\frac{\pi X}{2}\right)$.

Step 1: PDF of X

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Step 2: Transformation and Inverse

$$y = \sin\left(\frac{\pi x}{2}\right) \quad \Rightarrow \quad x = \frac{2}{\pi} \sin^{-1}(y).$$

Step 3: Derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1 - y^2}}.$$

Step 4: Apply the Transformation Formula

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = \frac{1}{2} \cdot \frac{2}{\pi\sqrt{1 - y^2}} = \frac{1}{\pi\sqrt{1 - y^2}}.$$

Step 5: Support of Y

Since $x \in (-1, 1)$ implies $y \in (-1, 1)$,

$$f_Y(y) = \begin{cases} \frac{1}{\pi\sqrt{1 - y^2}}, & -1 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

3.2 Example 2: Derivation for $Z = X + Y$

The lecture follows these steps:

1. Define $Z = X + Y$,
2. Write $F_Z(z) = \Pr(X + Y \leq z)$,
3. Identify the region under the line $x + y = z$,
4. Integrate the joint PDF over this region,
5. Differentiate with respect to z.

This yields

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x)\, dx}.$$