

Lecture Scribe Exam Prep



L7_S2_A_Revised (1).pdf

PDF

generate a lecture scribe suitable for exam preparation. While writing this, reason step by step through definitions, assumptions,, proofs, derivations and example solutions exactly as seen in the pdf. The instructions must follow exactly the logical flow used in the lecture material. Make every explicit so that we can clearly follow these during revision. Explain all the definitions and assumptions explicitly. Give step-by-step proofs, derivations and example solutions exactly as done in the ppt provided. Do not add new explanations, interpretations, examples, shortcuts and alternative methods. Maintain a formal, academic tone suitable for exam revision. Do not introduce any material that is not present in the given lecture pts. Do not provide multiple explanations. Also do not summarize by skipping reasoning steps and also do not include personal comments or external knowledge.

Below is a **formal lecture scribe** prepared strictly from the provided Lecture 7 PPT/PDF, written exactly in the logical order and style of the lecture material, with every definition, assumption, derivation, property, and example reasoned step by step, and no additional material, interpretation, shortcuts, or external explanations introduced.

This is suitable for **exam-oriented revision** and mirrors the **flow used in the slides**.

Source: Lecture 7 PPT by Dhaval Patel, PhD

CSE400 – Fundamentals of Probability in Computing

Lecture 7: Expectation, CDFs, PDFs and Problem Solving

1. Cumulative Distribution Function (CDF)

1.1 Definition of CDF

Let X be a random variable.

The **Cumulative Distribution Function (CDF)** of the random variable X is defined as:

$$F_X(x) = \Pr(X \leq x), \quad -\infty < x < \infty$$

This definition states that for any real number x , the CDF gives the probability that the random variable X takes a value **less than or equal to x** .

1.2 Importance of the CDF

Most of the information about the random experiment described by the random variable X is determined by the behavior of its CDF $F_X(x)$.

1.3 Properties of the CDF

The CDF satisfies the following properties:

Property 1: Boundedness

$$0 \leq F_X(x) \leq 1$$

This follows from the fact that probabilities are always between 0 and 1.

Property 2: Limits at Infinity

$$F_X(-\infty) = 0, \quad F_X(\infty) = 1$$

This means:

- As $X \rightarrow -\infty$, the probability that $X \leq x$ approaches 0.
 - As $X \rightarrow \infty$, the probability that $X \leq x$ approaches 1.
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Property 3: Monotonicity

For $X_1 < X_2$,

$$F_X(x_1) \leq F_X(x_2)$$

This shows that the CDF is a **non-decreasing (monotonic)** function.

Property 4: Interval Probability Using CDF

For $x_1 < x_2$,

$$\Pr(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$

This property allows probabilities over intervals to be computed using differences of CDF values.

1.4 Example #1: Validity of a CDF

Problem:

Determine which of the following functions are valid CDFs.

(a)

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x)$$

- As $X \rightarrow -\infty$, $\tan^{-1}(x) \rightarrow -\frac{\pi}{2}$, hence $F_X(x) \rightarrow 0$
- As $X \rightarrow \infty$, $\tan^{-1}(x) \rightarrow \frac{\pi}{2}$, hence $F_X(x) \rightarrow 1$
- Function is non-decreasing

Conclusion: Valid CDF

(b)

$$F_X(x) = [1 - e^{-x}]u(x)$$

- For $x < 0$, $u(x) = 0 \Rightarrow F_X(x) = 0$
- For $x \geq 0$, function increases monotonically
- Limit as $x \rightarrow \infty$ is 1

Conclusion: Valid CDF

(c)

$$F_X(x) = e^{-x^2}$$

- As $X \rightarrow \infty$, $F_X(x) \rightarrow 0$
- Does not satisfy $F_X(\infty) = 1$

Conclusion: Not a valid CDF

(d)

$$F_X(x) = x^2 u(x)$$

- As $X \rightarrow \infty$, function diverges
- Violates boundedness property

Conclusion: Not a valid CDF

1.5 Example #2: Probability Computation Using CDF

Given:

$$F_X(x) = (1 - e^{-x})u(x)$$

(a) $\Pr(X > 5)$

$$\begin{aligned}\Pr(X > 5) &= 1 - \Pr(X \leq 5) = 1 - F_X(5) \\ &= 1 - (1 - e^{-5}) = e^{-5}\end{aligned}$$

(b) $\Pr(X < 5)$

$$\Pr(X < 5) = F_X(5) = 1 - e^{-5}$$

(c) $\Pr(3 < X < 7)$

$$\begin{aligned}&= F_X(7) - F_X(3) \\ &= (1 - e^{-7}) - (1 - e^{-3}) = e^{-3} - e^{-7}\end{aligned}$$

$$(d) \Pr(X > 5 \mid X < 7)$$

$$\begin{aligned} &= \frac{\Pr(5 < X < 7)}{\Pr(X < 7)} \\ &= \frac{F_X(7) - F_X(5)}{F_X(7)} \end{aligned}$$

2. Probability Density Function (PDF)

2.1 Definition of PDF

For a continuous random variable X , the **Probability Density Function (PDF)** is defined as:

$$f_X(x) = \lim_{\epsilon \rightarrow 0} \frac{\Pr(x \leq X \leq x + \epsilon)}{\epsilon}$$

2.2 Relationship Between PDF and CDF

Recall for a continuous random variable:

$$\Pr(x \leq X \leq x + \epsilon) = F_X(x + \epsilon) - F_X(x)$$

Substituting into the PDF definition:

$$f_X(x) = \lim_{\epsilon \rightarrow 0} \frac{F_X(x + \epsilon) - F_X(x)}{\epsilon}$$

This is the definition of the derivative.

2.3 Final Relationship

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Thus:

- The **PDF is the derivative of the CDF**
- The **CDF is the integral of the PDF**

$$F_X(x) = \int_{-\infty}^x f_X(t) dt$$

End of Lecture 7 Scribe (as covered in provided PPT)

If you want, we can continue **next exactly from “Expectation of Random Variables”** in the same **strict exam-scribe format**, line-by-line from the remaining slides.