CSE400: Fundamentals of Probability in Computing

Lecture 7: Expectation, CDFs, PDFs and Problem Solving

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1 Cumulative Distribution Function (CDF)

1.1 Definition

Let X be a random variable. The **Cumulative Distribution Function (CDF)** of X is defined as:

$$F_X(x) = \Pr(X \leq x), \quad -\infty < x < \infty$$

The behavior of $F_X(x)$ determines most of the information about the random experiment described by the random variable X.

1.2 Properties of the CDF

The CDF $F_X(x)$ satisfies the following properties:

1. **Bounds:**
$$0 \leq F_X(x) \leq 1$$

2. **Limits at Infinity:**
$$F_X(-\infty) = 0, \quad F_X(\infty) = 1$$

3. **Monotonicity:** For $x_1 < x_2$,
$$F_X(x_1) \leq F_X(x_2)$$

4. **Probability Over an Interval:** For $x_1 < x_2$,
$$\Pr(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$

These properties are necessary for a function to be a valid CDF.

1.3 Example 1: Validity of CDFs

Determine whether each of the following functions is a valid CDF.

(a)

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x)$$

Verification:

- $\tan^{-1}(x)$ is monotone increasing.

- $\tan^{-1}(-\infty) = -\frac{\pi}{2}$ and $\tan^{-1}(\infty) = \frac{\pi}{2}$.

Thus,

$$F_X(-\infty) = 0, \quad F_X(\infty) = 1$$

Hence, this is a valid CDF.

(b)

$$F_X(x) = \left[1 - e^{-x}\right] u(x)$$

where $u(x)$ is the unit step function.
Verification:

- $F_X(x) = 0$ for $x < 0$.
- $F_X(x)$ is non-decreasing.
- $\lim_{x\to\infty} F_X(x) = 1$.

Hence, this is a valid CDF.

(c)

$$F_X(x) = e^{-x^2}$$

This function does not satisfy the limit condition:

$$\lim_{x\to\infty} F_X(x) \neq 1$$

Hence, it is **not** a valid CDF.

1.4 Example 2: Probability Computation Using a CDF

Suppose a random variable has CDF:

$$F_X(x) = (1 - e^{-x})u(x)$$

Compute the following probabilities.

(a) $\Pr(X > 5)$

$$\Pr(X > 5) = 1 - \Pr(X \leq 5) = 1 - F_X(5)$$

$$= 1 - (1 - e^{-5}) = e^{-5}$$

(b) $\Pr(X < 5)$

$$\Pr(X < 5) = F_X(5) = 1 - e^{-5}$$

(c) $\Pr(3 < X < 7)$

$$\Pr(3 < X < 7) = F_X(7) - F_X(3)$$

$$= (1 - e^{-7}) - (1 - e^{-3}) = e^{-3} - e^{-7}$$

(d) $\Pr(X > 5 \mid X < 7)$

Using conditional probability:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(X > 5 \mid X < 7) = \frac{\Pr(5 < X < 7)}{\Pr(X < 7)}$$

$$= \frac{F_X(7) - F_X(5)}{F_X(7)} = \frac{(1 - e^{-7}) - (1 - e^{-5})}{1 - e^{-7}}$$

2 Probability Density Function (PDF)

2.1 Definition

For a continuous random variable X, the **Probability Density Function (PDF)** is defined as:

$$f_X(x) = \lim_{\epsilon \to 0} \frac{\Pr(x \leq X < x + \epsilon)}{\epsilon}$$

2.2 PDF–CDF Relationship

For a continuous random variable:

$$\Pr(x \leq X < x + \epsilon) = F_X(x + \epsilon) - F_X(x)$$

Substituting into the definition:

$$f_X(x) = \lim_{\epsilon \to 0} \frac{F_X(x + \epsilon) - F_X(x)}{\epsilon}$$

Thus,

$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

2.3 Inverse Relationship

Conversely, the CDF can be obtained from the PDF by integration:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

Hence:

- The PDF is the derivative of the CDF.
- The CDF is the integral of the PDF.

3 Expectation of Random Variables

The lecture outline introduces the following topics:

- Expectation of random variables
- Expectation of a function of a random variable
- Linear operation with expectation
- n^{th} moments and central moments (variance, skewness, kurtosis)

Note: The provided lecture slides do not include definitions, formulas, derivations, or examples for these topics. Therefore, no additional content is reconstructed here.