# CSE400 – Fundamentals of Probability in Computing

## Lecture 11: Transformation of Random Variables

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## Outline

1. **Transformation of Random Variables** Learning of transformation techniques for random variables.
2. **Function of Two Random Variables** Joint transformations and derived distributions.
3. **Illustrative Example** Detailed derivation for the case: $Z = X + Y$

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## 1. Transformation of Random Variables

**Assumption**

The PDF of random variable $X$, denoted $f_X(x)$, is known *a priori*.

Objective: To find the PDF of a new random variable

$$Y = g(X)$$

### Case 1: Monotonic Transformation

Assume $g(x)$ is monotonic (either strictly increasing or strictly decreasing).

**Step S1: Find the CDF of $Y$**

$$F_Y(y) = \Pr(Y \leq y)$$

Since $Y = g(X)$,

$$F_Y(y) = \Pr(g(X) \leq y)$$

If $g$ is monotonic increasing,

$$g(X) \le y \iff X \le g^{-1}(y)$$

$$F_Y(y) = F_X(g^{-1}(y))$$

**Step S2: Differentiate to obtain PDF**

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

$$= f_X(x) \left| \frac{dx}{dy} \right|_{x=g^{-1}(y)}$$

### Case 2: Monotonic Decreasing Function

$$g(X) \le y \iff X \ge g^{-1}(y)$$

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|_{x=g^{-1}(y)}$$

### Final Result (Monotonic Case)

$$\boxed{f_Y(y) = \left. \frac{f_X(x)}{\left| \frac{dy}{dx} \right|} \right|_{x=g^{-1}(y)}}$$

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## 2. Example: Transformation

Given

$$X \sim \text{Uniform}(-1, 1)$$

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Transformation:

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

### Step 1: Inverse Transformation

$$X = \frac{2}{\pi}\sin^{-1}(Y)$$

### Step 2: Derivative

$$\frac{dx}{dy} = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}}$$

### Step 3: Apply Formula

$$f_Y(y) = \frac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}} = \frac{1}{\pi\sqrt{1-y^2}}$$

### Step 4: Limits

$$f_Y(y) = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}}, & -1 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

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## 3. Function of Two Random Variables

Consider

$$Z = X + Y$$

We are required to find:

1. PDF of $Z$, $f_Z(z)$
2. $f_Z(z)$ if $X$ and $Y$ are independent
3. If $X \sim N(0,1)$ and $Y \sim N(0,1)$, prove $Z \sim N(0,2)$
4. If $X$ and $Y$ are exponential with parameter $\lambda$, find $f_Z(z)$

## Illustrative Example: Derivation for $Z = X + Y$

### Step 1: CDF

$$F_Z(z) = \Pr(X + Y \leq z)$$

## Step 2: Double Integral

$$F_Z(z) = \iint_{x+y\leq z} f_{X,Y}(x,y)\,dx\,dy$$

## Step 3: Horizontal Strip

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x,y)\,dx\,dy$$

## Step 4: Vertical Strip

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy\,dx$$

## Step 5: Differentiate

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

*This completes the lecture material exactly as presented in Lecture 11.*