Lecture 11: Transformation of Random Variables

CSE400 - Fundamentals of Probability in Computing
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Outline

1. Transformation of Random Variables
2. Function of Two Random Variables
3. Illustrative Example: $Z = X + Y$

1 Change of Single Random Variable

1.1 Problem Setting

Let X be a continuous random variable with probability density function (PDF) $f_X(x)$. Let Y be defined as

$$Y = g(X)$$

The objective is to determine the PDF of Y, denoted $f_Y(y)$.

1.2 CDF-Based Derivation

We begin with the cumulative distribution function (CDF):

$$F_Y(y) = P(Y \leq y)$$

Substituting $Y = g(X)$:

$$F_Y(y) = P(g(X) \leq y)$$

Assume $g(x)$ is strictly monotonic and invertible.

Case 1: $g(x)$ is Increasing

If $g(x)$ is strictly increasing:

$$g(X) \leq y \quad \Longleftrightarrow \quad X \leq g^{-1}(y)$$

Thus,

$$F_Y(y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

Differentiate with respect to y:

$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

Using the chain rule:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Therefore,

$$\boxed{f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|} \quad \text{where } x = g^{-1}(y)$$

Role of Absolute Value

If $g(x)$ is decreasing, then $\frac{dx}{dy}$ is negative. Since density must be non-negative, we take absolute value:

$$\boxed{f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|}$$

2 Function of Two Random Variables

2.1 Problem Setting

Let X and Y be continuous random variables with joint PDF:

$$f_{X,Y}(x, y)$$

Define new variables:

$$Z = g(X, Y)$$

Our goal is to determine the PDF of Z.

2.2 Auxiliary Variable Method

Introduce an auxiliary variable W such that:

$$\begin{cases} Z = g(X, Y) \\ W = h(X, Y) \end{cases}$$

Assume transformation is one-to-one and invertible:

$$x = x(z, w), \qquad y = y(z, w)$$

2.3 Jacobian of Transformation

The joint PDF transforms as:

$$\boxed{f_{Z,W}(z, w) = f_{X,Y}(x, y)\,|J|}$$

where the Jacobian determinant is:

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

2.4 Obtaining the PDF of Z

Marginalize over W:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Z,W}(z, w)\, dw$$

Thus,

$$\boxed{f_Z(z) = \int f_{X,Y}(x, y)\, |J|\, dw}$$

3 Illustrative Example: $Z = X + Y$

3.1 Step 1: Define Transformation

$$Z = X + Y$$

Choose auxiliary variable:

$$W = Y$$

Thus,

$$\begin{cases} Z = X + Y \\ W = Y \end{cases}$$

3.2 Step 2: Inverse Transformation

Solve for X and Y:

$$Y = W$$

$$X = Z - W$$

Thus,

$$x = z - w, \qquad y = w$$

3.3 Step 3: Compute Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

Compute derivatives:

$$\frac{\partial x}{\partial z} = 1, \quad \frac{\partial x}{\partial w} = -1,$$
$$\frac{\partial y}{\partial z} = 0, \quad \frac{\partial y}{\partial w} = 1$$

Thus,

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Hence,

$$|J| = 1$$

3.4 Step 4: Joint Density Transformation

$$f_{Z,W}(z,w) = f_{X,Y}(z-w,w)$$

3.5 Step 5: Obtain PDF of Z

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z-w,w)\,dw$$

This gives the derived distribution of the sum.

Final Results

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|}$$

$$\boxed{f_{Z,W}(z,w) = f_{X,Y}(x,y)\,|J|}$$

$$\boxed{f_Z(z) = \int f_{X,Y}(x,y)\,|J|\ dw}$$