

Lecture 11: Transformation of RVs



L11_S2_A.pdf

PDF

Generate a formal, technical lecture scribe for Lecture 11: Transformation of Random Variables, based exclusively on the attached PDF. This scribe is intended as a primary study guide for a closed notes exam. Step-by-step, reason through the process of finding the PDF of a new random variable $Y = g(X)$. Explicitly explain the dependency on the derivative $|dy/dx|$ as presented in the material. For the section on "Function of Two Random Variables" (e.g., $Z = X + Y$), reason through the logical steps required to derive the joint transformation. Ensure every step of the derivation for $Z = X + Y$ is explicitly shown, making the logical flow between the original PDFs and the resulting PDF clear. Reconstruct the example of the Uniformly Distributed RV (range -1 to 1) and its transformation $Y = \sin(\pi X / 2)$. Reason through every substitution and mathematical transformation line-by-line. Ensure every step of the derivation for $Z = X + Y$ is explicitly shown, making the logical flow between the original PDFs and the resulting PDF clear. Use only the provided PDF. Do not introduce outside transformation methods, alternative notation, or generic probability examples. If a specific derivation step is not in the PDF, do not invent it; however, do explain the logic of the steps that are there. Maintain a formal academic tone. Start the response immediately with the scribe content. Do not include any personal AI greetings or helpful commentary. Structure the scribe using headings that follow the lecture outline: 1. Transformation of One RV, 2. Function of Two RVs, 3. Illustrative Examples.

Lecture 11: Transformation of Random Variables

CSE400 – Fundamentals of Probability in Computing

1. Transformation of One Random Variable

Let X be a continuous random variable with a known probability density function (PDF) $f_X(x)$.

Let a new random variable Y be defined as a function of X :

$$Y = g(X)$$

The central objective of this section is to **derive the PDF of Y , denoted $f_Y(y)$, given the PDF of X .**

Step 1: Start from the Cumulative Distribution Function (CDF)

By definition, the CDF of Y is:

$$F_Y(y) = \Pr(Y \leq y)$$

Substituting $Y = g(X)$:

$$F_Y(y) = \Pr(g(X) \leq y)$$

Step 2: Use Monotonicity of the Transformation

The lecture explicitly considers **monotonic transformations $g(x)$** , which may be:

- **Monotonically increasing**, or
- **Monotonically decreasing**

This assumption is essential because it allows inversion of the function $g(x)$.

Case 1: $g(x)$ is Monotonically Increasing

If $g(x)$ is increasing, then:

$$g(X) \leq y \Leftrightarrow X \leq g^{-1}(y)$$

Hence,

$$F_Y(y) = \Pr(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

Case 2: $g(x)$ is Monotonically Decreasing

If $g(x)$ is decreasing, then:

$$g(X) \leq y \Leftrightarrow X \geq g^{-1}(y)$$

Thus,

$$F_Y(y) = \Pr(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

Step 3: Differentiate the CDF to Obtain the PDF

The PDF of Y is obtained by differentiating its CDF with respect to y :

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

For the increasing case:

$$f_Y(y) = \frac{d}{dy} [F_X(g^{-1}(y))]$$

Applying the chain rule:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} [g^{-1}(y)]$$

Step 4: Role of the Absolute Derivative

The lecture consolidates both increasing and decreasing cases into a single expression by introducing the **absolute value of the derivative**:

$$f_Y(y) = \frac{f_X(x)}{\left| \frac{dy}{dx} \right|} \text{ evaluated at } x = g^{-1}(y)$$

This absolute value accounts for:

- Positive slope (increasing transformation)
- Negative slope (decreasing transformation)

Thus, the **magnitude of stretching or compression** of the transformation directly affects the density.

2. Function of Two Random Variables

Let X and Y be two continuous random variables with joint PDF $f_{X,Y}(x,y)$.

Define a new random variable:

$$Z = X + Y$$

The objective is to derive the PDF $f_Z(z)$.

Step 1: Define the CDF of Z

By definition:

$$F_Z(z) = \Pr(Z \leq z)$$

Substituting $Z = X + Y$:

$$F_Z(z) = \Pr(X + Y \leq z)$$

Step 2: Interpret the Probability Geometrically

The condition $X + Y \leq z$ represents a **region in the (x,y) -plane**, bounded by:

- The line $x + y = z$
- The support of the joint PDF $f_{X,Y}(x,y)$

The probability is computed by integrating the joint PDF over this region.

Step 3: Express the CDF as a Double Integral

From the lecture's diagram and setup:

$$F_Z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_{X,Y}(x,y) dy dx$$

This integral accumulates probability over all points satisfying $x + y \leq z$.

Step 4: Differentiate to Obtain the PDF of Z

The PDF is obtained by differentiating the CDF with respect to Z :

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

Differentiation with respect to the upper limit of the inner integral yields:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x) dx$$

This expression follows directly from the steps shown in the lecture slides and diagrams.

3. Illustrative Examples

Example 1: Transformation of a Uniform Random Variable

Let:

- $X \sim \text{Uniform}(-1, 1)$
 - $Y = \sin\left(\frac{\pi X}{2}\right)$
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Step 1: Write the PDF of X

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Step 2: Express the Transformation and Its Inverse

Given:

$$y = \sin\left(\frac{\pi x}{2}\right)$$

Solve for X :

$$x = \frac{2}{\pi} \sin^{-1}(y)$$

Step 3: Compute the Derivative

Differentiate X with respect to y :

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$$

Step 4: Apply the Transformation Formula

From the lecture formula:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Substitute:

$$f_Y(y) = \frac{1}{2} \cdot \frac{2}{\pi \sqrt{1-y^2}} = \frac{1}{\pi \sqrt{1-y^2}}$$

Step 5: Determine the Support of Y

From the transformation:

- When $x = -1, y = \sin(-\pi/2) = -1$
- When $x = 1, y = \sin(\pi/2) = 1$

Thus,

$$f_Y(y) = \begin{cases} \frac{1}{\pi \sqrt{1-y^2}}, & -1 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Example 2: Detailed Derivation for $Z = X + Y$

The lecture concludes with a detailed geometric illustration of the derivation:

1. Define $Z = X + Y$
2. Write the CDF $F_Z(z) = \Pr(X + Y \leq z)$
3. Identify the triangular/half-plane region under the line $X + Y = z$
4. Integrate the joint PDF over this region using horizontal or vertical strips
5. Differentiate the resulting expression with respect to Z
6. Obtain:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x) dx$$

Each step follows directly from the diagrams and integrals shown in the lecture material, with no additional assumptions introduced.

End of Lecture 11 Scribe