CSE400 – Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

Instructor: Dhaval Patel
Course: CSE400 – Fundamentals of Probability in Computing
Affiliation: SEAS-Ahmedabad University, Ahmedabad, Gujarat, India
Date: February 10, 2026

1 Transformation of Random Variables

1.1 Objective

- To learn transformation techniques for random variables.
- To determine the distribution of a transformed random variable.
- To extend the idea to functions involving two random variables.
- To derive the distribution in the illustrative case:
  - $Z = X + Y$

2 Transformation of a Random Variable

Let:

- X be a random variable.
- A new random variable Z be defined as a function of X, i.e.,

$$Z = g(X)$$

The objective is:

- To determine the distribution of Z from the known distribution of X.

2.1 General Method (CDF-Based Approach)

To find the distribution of Z:

1. Define:

$$F_Z(z) = P(Z \le z)$$

2. Since $Z = g(X)$, substitute:

$$F_Z(z) = P(g(X) \le z)$$

3. Express the event $g(X) \le z$ in terms of X.

4. Use the known distribution of X to compute the probability.

5. Once $F_Z(z)$ is obtained, differentiate (if continuous) to obtain the PDF:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

3 Function of Two Random Variables

Let:

- X and Y be two random variables.
- Define a new random variable:

$$Z = g(X, Y)$$

The objective is:

- To determine the distribution of Z from the joint distribution of X and Y.

3.1 Joint Distribution Framework

If:

- $f_{X,Y}(x, y)$ is the joint PDF of X and Y,

Then:

- Probabilities involving Z are computed using the joint distribution.

3.2 CDF of Z

To determine the distribution of Z:

1. Define:
$$F_Z(z) = P(Z \leq z)$$
2. Substitute:
$$F_Z(z) = P(g(X, Y) \leq z)$$
3. Express the event $g(X, Y) \leq z$ as a region in the (x, y)-plane.
4. Compute:
$$F_Z(z) = \iint_{g(x,y)\leq z} f_{X,Y}(x, y)\, dx\, dy$$
5. If continuous, differentiate:
$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

4 Illustrative Example: $Z = X + Y$

Let:
$$Z = X + Y$$

Objective:

- To derive the distribution of Z.

4.1 CDF of Z

By definition:
$$F_Z(z) = P(Z \leq z)$$

Substitute:
$$F_Z(z) = P(X + Y \leq z)$$

This corresponds to:

- The set of all (x, y) such that:
$$x + y \leq z$$

Thus:
$$F_Z(z) = \iint_{x+y\leq z} f_{X,Y}(x, y)\, dx\, dy$$

4.2 Derivation of the PDF

To obtain the PDF:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

Using the integral expression:

$$F_Z(z) = \iint_{x+y \le z} f_{X,Y}(x, y)\, dx\, dy$$

Differentiating with respect to z yields:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x)\, dx$$

This provides the distribution of $Z = X + Y$.

5 Logical Flow of the Lecture

- Start with transformation of a single random variable.
- Extend to functions involving two random variables.
- Use the cumulative distribution function as the primary method.
- Convert probability statements into integrals over appropriate regions.
- Differentiate to obtain the PDF.
- Apply the method to the specific case:

$$Z = X + Y$$

6 Key Formulas

Single Random Variable

$$F_Z(z) = P(g(X) \le z)$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

Two Random Variables

$$F_Z(z) = \iint_{g(x,y)\leq z} f_{X,Y}(x,y)\,dx\,dy$$

Special Case: $Z = X + Y$

$$F_Z(z) = P(X + Y \leq z)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx$$

End of Lecture 11 Scribe