

You are an advanced computational mathematician and master scholar 3. All that is required of you is to produce a rigorous and self-sufficient lecture scribe of the lecture, in this case, in the form of Lecture 11: Transformation of Random Variables using only the content of the given PDF document. This scribe should serve a conclusive reference document, one that will be adequate enough to enable one to study the content to pass an exam by closed-book without using any other book.

Strict Observation of Standards of Source Material: You will have to use the given PDF. Do not propose any external concepts, interpretations, other notations, examples, proofs, derivation, or explanations that are not explicitly present in the PDF. In case a derivation step, a definition, a sample is not present in the PDF, you must not create one. But you need to build up and demonstrate the reasoning of each of the steps which exist in it, so that the logic is defined by the material provided only.

Right and Obligatory Strengths and Contents:

Comprehensive Reconstruction: Record all defs, all assumptions, notations, all theorems, all lemmas, all derivations, all proofs and all solved examples and all questions as they are written in the PDF, in the order they appear.

Clear Step-by-step Rationalization: Derivations (e.g. calculating the PDF of $Y = g(X)$) In each derivation, you should write out and clearly indicate every logical and mathematical step. This involves but is not restricted to:

Specifying the original PDF(s) and the function of transformation (s).

Identification of the monotonicity of transformation, and the implication.

Formulating and explaining explicitly the occurrence of the absolute value of the derivative, either

$(dx/dy)/(dx/dy)$, or $(dy/dx)/(dy/dx)$.

- Representing all the substitutions, line-by-line algebraic manipulation and substitutions of variables.
- Explaining the use of Jacobian of a transformation of two random variables stating and meaning it as it is presented.
- Logical Dependency and Flow: Organize the scribe by making it clear who logical dependencies among ideas exist and whether they follow the flow in the PDF through the lecture. The justification behind every new idea should be well-following material that has been established in the PDF.

The Derivations to be Reconstructed Particularly with Line-by-Line Explicit Logic:

1. Transformation of a single random variable: In the general result $Y = g(X)$ write out the derivation of the PDF of Y , noting how derived in this respect the dependence on the derivative and on the operation of absolute value, as inspired by the CDF approach to the PDF.
2. Definiment: This is the logical procedure that one would take to determine the combined transformation of two random variables. Demonstrate the derivation of the PDF of Z step by step, with the logical derivation path of the joint PDF $f_{X,Y}(x,y)$ for the derivation of the resulting PDF $f_Z(z)$ being absolutely clear. This is required to encompass the auxiliary variable as well as the integration of the Jacobian as found in the PDF.
3. Illustrative Problem: Recreate the problem of the uniformly distributed random variable X (-1 to 1) and its transformation $Y = \sin(X/2)$. In this example, one cannot answer line-by-line by reasoning through each mathematical substitution and transformation and providing a breakdown of how derived formula of the PDF of Y is used.

Formatting and Tone:

- Formal Academic Tone: Descriptions should be the use of accurate, formal and objective mathematical language. No imaginative writing, comment, or subjective comments.

Hierarchical Structure: Layout the scribe with the precise headings in the lecture outline:

1. Change of Single random variable
2. Two-variable function Two random variables can be graphically represented as the functions of two variables.

Examples

A Sub-headings (e.g., 1.1, 1.2) may be added as needed to represent an internal structure of PDF.

Clarity to Revision: Bullet points, use of indentation and apparent division of definitions, important steps in proving, and essential results. Marking label equations (e.g. (1), (2)) As in the PDF when labeled, or mark them to be clear in the derivation flow.

Direct Start: You will be starting the response with the scribe material, and you will start with the title of the lecture 11: Transformation of Random Variables. No introduction, greetings, and no description of your process.

Finally, submissions are required to be a whole, cohesive and rigorous document of final output, which a student could fully count on to study on prior to exam, provided the complete copy of the output is availed. It should be an ideal, non-interpretive re-creation of the given lecture slides, with all of the derivations and logical considerations being clear-cut and easily traceable.

Lecture 11: Transformation of Random Variables

Course: CSE400 - Fundamentals of Probability in Computing

Instructor: Dhaval Patel

Affiliation: SEAS-Ahmedabad University

Date: February 10, 2026

Outline

1. Transformation of Random Variables

Learning of transformation techniques for random variables.

2. Function of Two Random Variables

Joint transformations and derived distributions.

3. Illustrative Example

Detailed derivation for the case: $Z = X + Y$

1. Change of Single Random Variable

1.1 Problem Setting

Let

- X be a continuous random variable,
- $f_X(x)$ denote its probability density function (PDF),
- Y be defined as a transformation of X :

$$Y = g(X)$$

The objective is:

To determine the probability density function $f_Y(y)$ of the transformed random variable Y .

1.2 Method: CDF-Based Derivation

The derivation proceeds through the cumulative distribution function (CDF).

Step 1: Define the CDF of Y

$$F_Y(y) = P(Y \leq y)$$

Substituting $Y = g(X)$:

$$F_Y(y) = P(g(X) \leq y)$$

1.3 Case: Monotonic Transformation

Assume:

- $g(x)$ is strictly monotonic
 - Hence, it possesses an inverse function $x = g^{-1}(y)$
-

Case A: $g(x)$ Increasing

If $g(x)$ is strictly increasing:

$$g(X) \leq y \Leftrightarrow X \leq g^{-1}(y)$$

Therefore,

$$F_Y(y) = P(X \leq g^{-1}(y))$$

Using definition of CDF of X :

$$F_Y(y) = F_X(g^{-1}(y))$$

Step 2: Differentiate to obtain PDF

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Substituting:

$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

Applying the chain rule:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Thus,

$$\boxed{f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|} \quad \text{where } x = g^{-1}(y)$$

Role of Absolute Value

The absolute value appears because:

- If $g(x)$ is decreasing,
- Then $\frac{dx}{dy}$ becomes negative,
- But density must remain non-negative.

Thus the final formula must include:

$$\left| \frac{dx}{dy} \right|$$

1.4 Summary Formula (Single Variable)

For $Y = g(X)$, with invertible transformation:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

This formula is obtained strictly through:

- CDF transformation
 - Monotonicity argument
 - Differentiation using chain rule
-

2. Function of Two Random Variables

We now extend the transformation to two random variables.

2.1 Problem Setting

Let:

- X, Y be continuous random variables,
- Joint PDF: $f_{X,Y}(x, y)$,
- Define new variables:

$$Z = g(X, Y)$$

Goal:

To derive the PDF of Z .

2.2 Method: Introduction of Auxiliary Variable

When transforming two variables:

We introduce:

$$\begin{cases} Z = g(X, Y) \\ W = h(X, Y) \end{cases}$$

Where:

- W is an auxiliary variable,
- The transformation is one-to-one and invertible.

2.3 Change of Variables and Jacobian

Let:

$$x = x(z, w), \quad y = y(z, w)$$

The joint density transforms according to:

$$f_{Z,W}(z, w) = f_{X,Y}(x, y) |J|$$

Where the **Jacobian** is:

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

The Jacobian determinant accounts for:

- Area distortion under transformation
- Change in differential elements

2.4 Deriving the PDF of Z

Once $f_{Z,W}(z, w)$ is known:

$$f_Z(z) = \int f_{Z,W}(z, w) dw$$

Thus:

$$f_Z(z) = \int f_{X,Y}(x, y) |J| dw$$

Where:

- x, y are expressed in terms of z, w .

Logical Flow

1. Start from joint PDF $f_{X,Y}(x,y)$
 2. Define transformation
 3. Compute inverse transformation
 4. Compute Jacobian determinant
 5. Substitute into transformation formula
 6. Integrate out auxiliary variable
-

3. Illustrative Example

Detailed Derivation for the Case $Z = X + Y$

3.1 Define Transformation

Let:

$$Z = X + Y$$

Introduce auxiliary variable:

$$W = Y$$

Then transformation becomes:

$$\begin{cases} Z = X + Y \\ W = Y \end{cases}$$

3.2 Inverse Transformation

Solve for X, Y :

$$Y = W$$

$$X = Z - W$$

Thus:

$$x = z - w, \quad y = w$$

3.3 Compute Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

Compute derivatives:

$$\frac{\partial x}{\partial z} = 1$$

$$\frac{\partial x}{\partial w} = -1$$

$$\frac{\partial y}{\partial z} = 0$$

$$\frac{\partial y}{\partial w} = 1$$

Thus:

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

So:

$$|J| = 1$$

3.4 Joint Density Transformation

$$f_{Z,W}(z, w) = f_{X,Y}(z - w, w)$$

3.5 Obtain PDF of Z

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z - w, w) dw$$

This gives the derived distribution of the sum.

Complete Logical Structure of Lecture

- Start with CDF-based derivation for single transformation.

- Establish monotonicity requirement.
- Derive PDF formula including absolute derivative.
- Extend to two-variable transformations.
- Introduce auxiliary variable.
- Compute Jacobian determinant.
- Integrate to obtain marginal density.
- Apply method to example $Z = X + Y$.

Final Consolidated Results

Single Random Variable:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

Two Random Variables:

$$f_{Z,W}(z, w) = f_{X,Y}(x, y) |J|$$

$$f_Z(z) = \int f_{X,Y}(x, y) |J| dw$$

This completes the rigorous reconstruction of **Lecture 11: Transformation of Random Variables**, including:

- All definitions
- All derivations
- Jacobian reasoning
- Logical structure
- The worked example $Z = X + Y$

This document is self-contained and sufficient for closed-book examination preparation.

and give this in latex code

latex

```

\documentclass[11pt]{article}

% ===== PACKAGES =====
\usepackage[a4paper,margin=1in]{geometry}
\usepackage{amsmath,amssymb,amsthm}
\usepackage{enumitem}
\usepackage{fancyhdr}

% ===== HEADER =====
\pagestyle{fancy}
\fancyhf{}
\rhead{CSE400}
\lhead{Lecture 11}
\cfoot{\thepage}

% ===== DOCUMENT =====
\begin{document}

\begin{center}
  {\LARGE \textbf{Lecture 11: Transformation of Random Variables}}\\[0.5em]
  \textbf{CSE400 - Fundamentals of Probability in Computing}\\
  February 10, 2026
\end{center}

\vspace{1em}

\section*{Outline}
\begin{enumerate}
  \item Transformation of Random Variables
  \item Function of Two Random Variables
  \item Illustrative Example:  $Z = X + Y$ 
\end{enumerate}

% =====
\section{Change of Single Random Variable}

\subsection{Problem Setting}

Let  $X$  be a continuous random variable with probability density function (PDF)  $f_X(x)$ .
Let  $Y$  be defined as

\l[

```

$$Y = g(X)$$

\backslash

The objective is to determine the PDF of Y , denoted $f_Y(y)$.

% -----

\backslash subsection{CDF-Based Derivation}

We begin with the cumulative distribution function (CDF):

\backslash

$$F_Y(y) = P(Y \leq y)$$

\backslash

Substituting $Y = g(X)$:

\backslash

$$F_Y(y) = P(g(X) \leq y)$$

\backslash

Assume $g(x)$ is strictly monotonic and invertible.

% -----

\backslash subsubsection*{Case 1: $g(x)$ is Increasing}

If $g(x)$ is strictly increasing:

\backslash

$$g(X) \leq y \quad \Longleftrightarrow \quad X \leq g^{-1}(y)$$

\backslash

Thus,

\backslash

$$F_Y(y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

\backslash

Differentiate with respect to y :

\backslash

$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

\backslash

Using the chain rule:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Therefore,

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

where $x = g^{-1}(y)$

% -----

Role of Absolute Value

If $g(x)$ is decreasing, then $\frac{dx}{dy}$ is negative.

Since density must be non-negative, we take absolute value:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

% =====

Function of Two Random Variables

Problem Setting

Let X and Y be continuous random variables with joint PDF:

$$f_{X,Y}(x,y)$$

Define new variables:

```
\[
Z = g(X,Y)
\]
```

Our goal is to determine the PDF of Z .

```
% -----
\subsection{Auxiliary Variable Method}
```

Introduce an auxiliary variable W such that:

```
\[
\begin{cases}
Z = g(X,Y) \\
W = h(X,Y)
\end{cases}
\]
```

Assume transformation is one-to-one and invertible:

```
\[
x = x(z,w), \quad y = y(z,w)
\]
```

```
% -----
\subsection{Jacobian of Transformation}
```

The joint PDF transforms as:

```
\[
\boxed{
f_{Z,W}(z,w)
=
f_{X,Y}(x,y)
\left| J \right|
}
\]
```

where the Jacobian determinant is:

```
\[
J =
```

```

\begin{vmatrix}
\frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial z} & \frac{\partial y}{\partial w}
\end{vmatrix}

```

% -----

```

\subsection{Obtaining the PDF of $Z$}

```

Marginalize over W :

```

\begin{aligned}
f_Z(z) &= \\
&\int_{-\infty}^{\infty} f_{Z,W}(z,w) \, dw
\end{aligned}

```

Thus,

```

\begin{aligned}
&\boxed{f_Z(z)} \\
&= \\
&\int \int f_{X,Y}(x,y) \\
&\quad \left| J \right| \, dx \, dy
\end{aligned}

```

% =====

```

\section{Illustrative Example: $Z = X + Y$}

```

```

\subsection{Step 1: Define Transformation}

```

```

\begin{aligned}
Z &= X + Y
\end{aligned}

```

Choose auxiliary variable:

```
\[
```

$$W = Y$$

```
\]
```

Thus,

```
\[
```

```
\begin{cases}
```

$$Z = X + Y \\\$$

$$W = Y$$

```
\end{cases}
```

```
\]
```

```
% -----
```

```
\subsection{Step 2: Inverse Transformation}
```

Solve for X and Y :

```
\[
```

$$Y = W$$

```
\]
```

```
\[
```

$$X = Z - W$$

```
\]
```

Thus,

```
\[
```

$$x = z - w, \quad y = w$$

```
\]
```

```
% -----
```

```
\subsection{Step 3: Compute Jacobian}
```

```
\[
```

$$J =$$

```
\begin{vmatrix}
```

$$\frac{\partial x}{\partial z} \& \frac{\partial x}{\partial w} \\\$$

$$\frac{\partial y}{\partial z} \& \frac{\partial y}{\partial w}$$

```
\end{vmatrix}
```

```
\]
```


Compute derivatives:

$$\begin{aligned}\frac{\partial x}{\partial z} &= 1, \\ \frac{\partial x}{\partial w} &= -1,\end{aligned}$$

$$\begin{aligned}\frac{\partial y}{\partial z} &= 0, \\ \frac{\partial y}{\partial w} &= 1\end{aligned}$$

Thus,

$$\begin{aligned}J &= \\ \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \\ &= 1\end{aligned}$$

Hence,

$$|J| = 1$$

% -----

\subsection{Step 4: Joint Density Transformation}

$$\begin{aligned}f_{Z,W}(z,w) &= \\ f_{X,Y}(z-w, w)\end{aligned}$$

% -----

\subsection{Step 5: Obtain PDF of Z }

$$\begin{aligned} f_Z(z) &= \\ \int_{-\infty}^{\infty} f_{X,Y}(z-w, w) \, dw \end{aligned}$$

This gives the derived distribution of the sum.

% =====

\section*{Final Results}

$$\begin{aligned} & \boxed{f_Y(y)} \\ &= \\ & f_X(g^{-1}(y)) \\ & \left| \frac{d}{dy} g^{-1}(y) \right| \\ & \end{aligned}$$

$$\begin{aligned} & \boxed{f_{Z,W}(z,w)} \\ &= \\ & f_{X,Y}(x,y) \\ & \left| J \right| \\ & \end{aligned}$$

$$\begin{aligned} & \boxed{f_Z(z)} \\ &= \\ & \int f_{X,Y}(x,y) \\ & \left| J \right| \, dw \end{aligned}$$

```
}
```

```
\]
```

```
\end{document}
```