# CSE400 – Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

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## Overview / Outline (as in lecture)

1. **Transformation of Random Variables**
   Learning of transformation techniques for random variables.

2. **Function of Two Random Variables**
   Joint transformations and derived distributions.

3. **Illustrative Example**
   Detailed derivation for the case: $Z = X + Y$.

## 1 Definitions & Notation

- Random variable (RV): $X$, $Y$, $Z$.
- Transformation (single RV): $Y = g(X)$.
- CDF of a RV $X$: $F_X(x) = \Pr(X \leq x)$.
- PDF of a continuous RV $X$: $f_X(x)$, with the relationship

$$f_X(x) = \frac{d}{dx} F_X(x) \quad \text{(when differentiable).}$$

- Inverse mapping (when it exists): $x = g^{-1}(y)$.

## 2 Assumptions / Constraints (as used in the lecture derivations)

- The PDF of the original RV (e.g., $f_X(x)$) is assumed to be known *a priori*.
- For the single-variable transformation method shown:
  - $g(\cdot)$ is **monotonic** (lecture separates cases: $+ve$ / increasing vs $-ve$ / decreasing).
  - $g^{-1}(\cdot)$ exists on the relevant range.
  - Differentiation steps assume the needed derivatives exist.
- If a detail/condition is not explicitly provided beyond the above, it is **Not specified in the provided context.**

## 3 Main Results / Theorems (with conditions)

### 3.1 Single RV transformation: $Y = g(X)$ (monotonic $g$)

**Step S1 (CDF method).**

**Case 1: $g$ is increasing (*"+ve" monotone*)**

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) && (1)\\ &= \Pr(g(X) \le y) && (2)\\ &= \Pr\big(X \le g^{-1}(y)\big) && (3)\\ &= F_X\big(g^{-1}(y)\big). && (4) \end{aligned}$$

**Case 2: $g$ is decreasing (*"-ve" monotone*)**

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) && (5)\\ &= \Pr(g(X) \le y) && (6)\\ &= \Pr\big(X \ge g^{-1}(y)\big) && (7)\\ &= 1 - F_X\big(g^{-1}(y)\big). && (8) \end{aligned}$$

**Step S2 (Differentiate to obtain PDF).**

**Increasing case** Differentiate $F_Y(y) = F_X(g^{-1}(y))$ w.r.t. $y$:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) && (9)\\ &= \frac{d}{dy}\Big[F_X\big(g^{-1}(y)\big)\Big] && (10)\\ &= f_X\big(g^{-1}(y)\big) \cdot \frac{d}{dy}\big[g^{-1}(y)\big]. && (11) \end{aligned}$$

Let $x = g^{-1}(y)$. Then $\dfrac{dx}{dy} = \dfrac{d}{dy} g^{-1}(y)$, so

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| \quad \text{with } x = g^{-1}(y).$$

**Decreasing case** Starting from $F_Y(y) = 1 - F_X(g^{-1}(y))$:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) && (12)\\ &= -f_X\big(g^{-1}(y)\big) \cdot \frac{d}{dy}\big[g^{-1}(y)\big]. && (13) \end{aligned}$$

Taking absolute value (as emphasized in the lecture's boxed formula), the unified form is

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| \quad \text{with } x = g^{-1}(y).$$

**Step S3 (Change the limits / support).**
The valid range of $y$ is obtained by mapping the support of $x$ through $y = g(x)$ (lecture notes: "change the limits for $Y$").

### 3.2 Equivalent boxed formula shown in the lecture

Using $\left|\frac{dx}{dy}\right| = \frac{1}{\left|\frac{dy}{dx}\right|}$ (when derivatives exist),

$$\boxed{f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\Bigg|_{x=g^{-1}(y)}}$$

This is the boxed relationship displayed in the lecture.

# 4 Proofs / Derivations (step-by-step, as in lecture)

### 4.1 Derivation structure for $Y = g(X)$

The lecture's derivation proceeds in the following explicit sequence:

**S1:** Compute $F_Y(y) = \Pr(Y \leq y)$ and rewrite the event in terms of $X$ using monotonicity and the inverse $g^{-1}$.

**S2:** Differentiate $F_Y(y)$ w.r.t. $y$ to obtain $f_Y(y)$, applying the chain rule.

**S3:** Determine the correct support/range for $y$ by transforming the original $x$-limits through $y = g(x)$.

### 4.2 Two-RV function setup used for the example $Z = X + Y$

The lecture sets up the CDF of $Z$ via a region in the $(x, y)$-plane:

$$\begin{aligned} F_Z(z) &= \Pr(Z \leq z) &(14)\\ &= \Pr(X + Y \leq z). &(15)\end{aligned}$$

Using the joint PDF $f_{X,Y}(x, y)$, the probability is written as a double integral over the region

$$\{(x, y) : x + y \leq z\}.$$

One explicit "vertical strip" form written in the lecture is:

$$F_Z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_{X,Y}(x, y)\, dy\, dx.$$

(An equivalent "horizontal strip" order is implied by the lecture's annotation about vertical/horizontal swapping, but any further simplification is **Not specified in the provided context**.)

Then, to obtain the PDF:

$$f_Z(z) = \frac{d}{dz} F_Z(z).$$

Any closed-form $f_Z(z)$ beyond this setup depends on additional assumptions/distributions, which are listed as sub-questions in the lecture (see Worked Examples).

# 5 Worked Examples (step-by-step)

## 5.1 Example 1: $X \sim$ Uniform$(-1,1)$, $Y = g(X) = \sin\left(\frac{\pi X}{2}\right)$

**Given (as written in the lecture):**

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases} \qquad Y = \sin\left(\frac{\pi X}{2}\right).$$

**Step 1: Invert the transformation.**
From $y = \sin\left(\frac{\pi x}{2}\right)$,

$$x = \frac{2}{\pi}\sin^{-1}(y).$$

**Step 2: Compute the derivative factor.**

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}.$$

**Step 3: Apply the transformation formula.** Using $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$ with $x = \frac{2}{\pi}\sin^{-1}(y)$:

$$f_Y(y) = \frac{1}{2} \cdot \left|\frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}\right| \tag{16}$$

$$= \frac{1}{\pi\sqrt{1-y^2}}. \tag{17}$$

**Step 4: Determine the support (change limits).**
Lecture maps endpoints:

$$x = -1 \Rightarrow y = \sin\left(-\frac{\pi}{2}\right) = -1, \qquad x = 1 \Rightarrow y = \sin\left(\frac{\pi}{2}\right) = 1.$$

Hence,

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}}, & -1 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

## 5.2 Example 2 (illustrative setup): $Z = X + Y$

The lecture explicitly states the following items to find/prove (no full solutions shown beyond the CDF-region setup):

(i) Find the PDF of $Z$, $f_Z(z)$.

(ii) Find $f_Z(z)$ if $X$ and $Y$ are independent.

(iii) Let $X \sim N(0,1)$ and $Y \sim N(0,1)$. Prove that $Z \sim N(0,2)$.

(iv) If $X$ and $Y$ are exponential distributed RVs with parameter $\lambda$, find $f_Z(z)$.

**Derivation shown (CDF setup):**

$$F_Z(z) = \Pr(Z \le z) \tag{18}$$

$$= \Pr(X + Y \le z) \tag{19}$$

$$= \iint_{x+y\le z} f_{X,Y}(x,y)\,dx\,dy \tag{20}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy\,dx. \tag{21}$$

Further evaluation for cases (ii)–(iv) is **Not specified in the provided context** beyond listing these tasks.

## Summary of Key Takeaways (only from context)

- To find the distribution of a transformed RV $Y = g(X)$ (with monotonic $g$), the lecture uses the **CDF method** followed by differentiation.
- For monotonic transformations, the resulting PDF uses the absolute derivative factor:

$$f_Y(y) = f_X\big(g^{-1}(y)\big)\left|\frac{d}{dy}g^{-1}(y)\right| \quad \text{equivalently} \quad f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\Bigg|_{x=g^{-1}(y)}.$$

- The support of the new RV must be obtained by **changing limits** (mapping $x$-range through $y = g(x)$).
- For $Z = X + Y$, the lecture sets up $F_Z(z) = \Pr(X + Y \le z)$ as a **double integral over the half-plane** $x + y \le z$ using $f_{X,Y}(x, y)$, and then uses $f_Z(z) = \dfrac{d}{dz}F_Z(z)$.