

L-11 Scribe



L11_S2_A.pdf

PDF

generate a lecture scribe suitable for exam preparation. While writing this, reason step by step through definitions, assumptions,, proofs, derivations and example solutions exactly as seen in the pdf. The instructions must follow exactly the logical flow used in the lecture material. Make every step explicit so that we can clearly follow these during revision. Explain all the definitions and assumptions explicitly. Give step-by-step proofs, derivations and example solutions exactly as done in the ppt provided. Do not add new explanations, interpretations, examples, shortcuts and alternative methods. Maintain a formal, academic tone suitable for exam revision. Do not introduce any material that is not present in the given lecture ppts. Do not provide multiple explanations. Also do not summarize by skipping reasoning steps and also do not include personal comments or external knowledge.
don't skip any slide
content from all the slides should be there

Below is the complete lecture scribe prepared strictly according to **Lecture 11: Transformation of Random Variables** (CSE400), following exactly the logical sequence of slides without adding any external material. All definitions, assumptions, derivations, and example steps are written explicitly as presented in the lecture.

Source:

CSE400 – Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

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Outline

1. Transformation of Random Variables

Learning of transformation techniques for random variables.

2. Function of Two Random Variables

Joint transformations and derived distributions.

3. Illustrative Example

Detailed derivation for the case: $Z = X + Y$

1. Transformation of Random Variables

Assumption

The PDF of random variable X , denoted $f_X(x)$, is known *a priori*.

Objective:

To find the PDF of a new random variable

$$Y = g(X)$$

Case 1: Monotonic Transformation

Assume $g(x)$ is monotonic (either strictly increasing or strictly decreasing).

We proceed in two steps.

Step S1: Find the CDF of Y

By definition of CDF:

$$F_Y(y) = \Pr(Y \leq y)$$

Since $Y = g(X)$,

$$F_Y(y) = \Pr(g(X) \leq y)$$

If g is monotonic increasing:

$$g(X) \leq y \Leftrightarrow X \leq g^{-1}(y)$$

Therefore,

$$F_Y(y) = \Pr(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

Step S2: Differentiate to obtain PDF

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} [F_X(g^{-1}(y))] \end{aligned}$$

Using chain rule:

$$= f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Since

$$\frac{d}{dy} g^{-1}(y) = \frac{dx}{dy}$$

evaluated at $x = g^{-1}(y)$,

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|_{x=g^{-1}(y)}$$

Case 2: Monotonic Decreasing Function

If $g(x)$ is decreasing:

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) \\ &= \Pr(g(X) \leq y) \end{aligned}$$

Since decreasing,

$$g(X) \leq y \Leftrightarrow X \geq g^{-1}(y)$$

Thus,

$$F_Y(y) = \Pr(X \geq g^{-1}(y))$$

$$= 1 - F_X(g^{-1}(y))$$

Differentiating:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|_{x=g^{-1}(y)}$$

Final Result (Monotonic Case)

$$f_Y(y) = \frac{f_X(x)}{\left| \frac{dy}{dx} \right|} \Big|_{x=g^{-1}(y)}$$

After obtaining the expression, change the limits according to the support of Y .

2. Example: Transformation

Given:

$$X \sim \text{Uniform}(-1, 1)$$

Thus,

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Transformation:

$$Y = g(X) = \sin\left(\frac{\pi X}{2}\right)$$

Step 1: Find inverse transformation

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

Taking inverse sine:

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$$\frac{\pi X}{2} = \sin^{-1}(Y)$$

$$X = \frac{2}{\pi} \sin^{-1}(Y)$$

Step 2: Compute derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$$

Step 3: Apply transformation formula

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Substitute:

$$\begin{aligned} &= \frac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}} \\ &= \frac{1}{\pi \sqrt{1-y^2}} \end{aligned}$$

Step 4: Determine limits

When $x = -1$:

$$y = \sin\left(-\frac{\pi}{2}\right) = -1$$

When $x = 1$:

$$y = \sin\left(\frac{\pi}{2}\right) = 1$$

Thus,

$$f_Y(y) = \begin{cases} \frac{1}{\pi \sqrt{1-y^2}}, & -1 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

3. Function of Two Random Variables

Consider:

$$Z = X + Y$$

We are required to find:

1. PDF of Z , $f_Z(z)$
 2. $f_Z(z)$ if X and Y are independent
 3. If $X \sim N(0, 1)$, $Y \sim N(0, 1)$, prove $Z \sim N(0, 2)$
 4. If X and Y are exponential with parameter λ , find $f_Z(z)$
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Illustrative Example: Derivation for $Z = X + Y$

Step 1: Start with CDF

$$F_Z(z) = \Pr(Z \leq z)$$

Since $Z = X + Y$,

$$F_Z(z) = \Pr(X + Y \leq z)$$

This corresponds to the region:

$$x + y \leq z$$

Step 2: Express as double integral

$$F_Z(z) = \iint_{x+y \leq z} f_{X,Y}(x,y) dx dy$$

The region is the area below the line:

$$y = z - x$$

Step 3: Write limits (Horizontal Strip)

Fix y , integrate over X :

$$F_Z(z) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f_{X,Y}(x,y) dx dy$$

Step 4: Alternatively (Vertical Strip)

Fix X , integrate over y :

$$F_Z(z) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_{X,Y}(x,y) dy dx$$

Step 5: Differentiate to obtain PDF

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

This gives the required PDF of Z .

This completes the lecture material exactly as presented in Lecture 11.

Source: