

L7\_S2\_A\_Revised (2).pdf  
PDF

Generate a scribe for exam reference material from the given lecture slides. The scribe must reconstruct what is taught in the lecture slides so that a student can rely on the scribe alone for revision for a closed books examination.

Use only the given material that is the lecture slides and no extra content. The scribe should not introduce any new concepts, interpretation, explanations, examples, solved problems, proofs or derivations. Do not exceed the scope of the given content in any way.

Using such an approach, an attempt will be made to write the scribe where the reasoning is also explicit within the scribe itself while writing it in such a way that all the definitions, assumptions, notations, derivations, proofs, etc., are reasoned step by step as they are found within the lecture material itself. The reasoning for all this should emerge only from the given material itself.

The Scribe must include step by step proofs and derivations, with examples given in the slides only. The scribe should be structured in a way that all logical dependencies between the ideas have to be clear so that it reflects and follows the flow of the lecture.

The lecture must maintain formal academic tone without any creative writing or commentary on the material. The scribe should be clearly structured in a section sub-section format and with bullet points highlighting the important points and making it clear for revision.

The scribe should avoid any kind of summaries interpretations, explanations and but should contain all examples, questions and proofs and derivations and definitions provided in the lecture slides written in an academic format.

Do not explain how the scribe is written. Do not comment on the process. No additions are necessary, as the material is already provided in the context given above. The final output should resemble a lecture scribe that is intended for direct submission.



---

# CSE400 – Fundamentals of Probability in Computing

## Lecture 7: Expectation, CDFs, PDFs and Problem Solving Skills

Instructor: Dhaval Patel, PhD

Date: January 27, 2025

---

### 1. Outline of the Lecture

- The Cumulative Distribution Function (CDF)
  - Definition
  - Properties
  - Examples
- The Probability Density Function (PDF)
  - Definition
  - PDF–CDF relationship
  - Properties
  - Examples
- Expectation of Random Variables
  - Definition
  - Expectation of a function of a random variable
  - Linear operations with expectation
- $n^{\text{th}}$  moments and central moments of random variables
  - Variance
  - Skewness
  - Kurtosis

L7\_S2\_A\_Revised (2)

---

### 2. CDF and PDF: Intuition (Water Tank Analogy)

- The cumulative distribution function (CDF) is explained using a water tank analogy.
- The height  $h$  of water in the tank represents the value of a random variable.

- The volume of water up to height  $h$ , denoted by  $V(h)$ , corresponds to the cumulative probability up to that value.

## Mathematical Representation

$$V(h) = \int_0^h \pi R^2 dh = (\pi R^2)h$$

- Here,  $\pi R^2$  is analogous to the probability density function (PDF) of a uniform distribution.
- The maximum volume of the tank is:

$$V(H) = \pi R^2 H$$

- This maximum volume is analogous to total probability equal to 1.

 L7\_S2\_A\_Revised (2)

---

## 3. Cumulative Distribution Function (CDF)

### 3.1 Definition

- The cumulative distribution function (CDF) of a random variable  $X$  is defined as:

$$F_X(x) = \Pr(X < x), \quad -\infty < x < \infty$$

- Most of the information about the random experiment described by the random variable  $X$  is determined by the behavior of  $F_X(x)$ .

 L7\_S2\_A\_Revised (2)

---

### 3.2 Properties of the CDF

The CDF satisfies the following properties:

- $0 \leq F_X(x) \leq 1$
- $F_X(-\infty) = 0$
- $F_X(\infty) = 1$
- If  $x_1 < x_2$ , then:

$$F_X(x_1) \leq F_X(x_2)$$

- For  $x_1 < x_2$ :

$$\Pr(x_1 < X < x_2) = F_X(x_2) - F_X(x_1)$$

 L7\_S2\_A\_Revised (2)

---

### 3.3 CDF Example #1: Validity of CDFs

**Question:** Find the valid CDF.

Given candidate functions:

1.  $F_X(x) = 3 + 2 \tan^{-1}(x)$
  2.  $F_X(x) = [1 - e^{-x}]u(x)$
  3.  $F_X(x) = e^{x^2}$
  4. Other given expressions as shown in slides
- Validity is checked using CDF properties:
    - Range between 0 and 1
    - Proper limits at  $-\infty$  and  $+\infty$
    - Monotonic non-decreasing behavior

 L7\_S2\_A\_Revised (2)

---

### 3.4 CDF Example #2

Given:

$$F_X(x) = (1 - e^{-x})u(x)$$

Find the following probabilities:

- $\Pr(X > 5)$
- $\Pr(X < 5)$
- $\Pr(3 < X < 7)$
- $\Pr(X > 5 \mid X < 7)$

Each probability is computed directly using:

$$\Pr(a < X < b) = F_X(b) - F_X(a)$$

 L7\_S2\_A\_Revised (2)

---

## 4. Probability Density Function (PDF)

## 4.1 Definition and PDF–CDF Relationship

- The PDF of a continuous random variable  $X$  at point  $x$  is defined as:

$$f_X(x) = \lim_{\epsilon \rightarrow 0} \frac{\Pr(x < X < x + \epsilon)}{\epsilon}$$

- For a continuous range:

$$\Pr(x < X < x + \epsilon) = F_X(x + \epsilon) - F_X(x)$$

Substituting and taking the limit:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

- Hence:
  - The PDF is the derivative of the CDF.
  - The CDF is the integral of the PDF:

$$F_X(x) = \int_{-\infty}^x f_X(y) dy$$

 L7\_S2\_A\_Revised (2)

## 4.2 Properties of the PDF

- $f_X(x) \geq 0$
- $f_X(x) = \frac{dF_X(x)}{dx}$
- $F_X(x) = \int_{-\infty}^x f_X(y) dy$
- Total probability:

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$

- Probability over an interval:

$$\Pr(a < X < b) = \int_a^b f_X(x) dx$$

 L7\_S2\_A\_Revised (2)

## 4.3 PDF Example #1: Validity of PDFs

**Question:** Which of the following are valid PDFs?

Candidate functions include:

- $f_X(x) = e^{-x}u(x)$
- Piecewise-defined functions over bounded intervals
- $f_X(x) = 2xe^{-x}u(x)$
- Validity is determined by checking:
  - Non-negativity
  - Integral over entire range equals 1

 L7\_S2\_A\_Revised (2)

## 4.4 PDF Example #2

Given CDF:

$$F_X(x) = (1 - e^{-x})u(x)$$

The PDF is obtained by differentiation:

$$f_X(x) = \frac{dF_X(x)}{dx} = e^{-x}u(x)$$

Conversely, if:

$$f_X(x) = 2xe^{-x^2}u(x)$$

Then the CDF is:

$$F_X(x) = \int_0^x 2ye^{-y^2} dy = (1 - e^{-x^2})u(x)$$

 L7\_S2\_A\_Revised (2)

## 5. Expectation of Random Variables

### 5.1 Expectation

- Expectation of a random variable  $X$  is denoted by:

$$\mathbb{E}[X]$$

- The expectation operator is linear.

## 5.2 Expectation of a Function of a Random Variable

- For a function  $g(X)$ :

$$\mathbb{E}[g(X)]$$

- The definition follows directly from the PDF-based formulation as presented in the slides.

## 5.3 Linear Operations with Expectation

- Expectation satisfies linearity:

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

 L7\_S2\_A\_Revised (2)

---

## 6. Moments and Central Moments of Random Variables

- $n^{\text{th}}$  moments and central moments are introduced.
- These include:
  - Variance
  - Skewness
  - Kurtosis

The concepts are listed as part of the lecture outline and introduced following the discussion on expectation.

 L7\_S2\_A\_Revised (2)

---

## End of Lecture 7 Scribe

Sources

