CSE400 – Fundamentals of Probability in Computing
Lecture 7: Expectation, CDFs, PDFs and Problem Solving

1. Cumulative Distribution Function (CDF)

1.1 Definition of CDF

Let X be a random variable.

The Cumulative Distribution Function (CDF) of the random variable X is defined as:

$$F_X(x) = \Pr(X \leq x), \quad -\infty < x < \infty$$

This definition states that for any real number x, the CDF gives the probability that the random variable X takes a value less than or equal to x.

1.2 Importance of the CDF

Most of the information about the random experiment described by the random variable X is determined by the behavior of its CDF $F_X(x)$.

1.3 Properties of the CDF

The CDF satisfies the following properties:

Property 1: Boundedness

$$0 \leq F_X(x) \leq 1$$

This follows from the fact that probabilities are always between 0 and 1.

Property 2: Limits at Infinity

$$F_X(-\infty) = 0, \quad F_X(\infty) = 1$$

This means:

As $x \to -\infty$, the probability that $X \leq x$ approaches 0.

As $x \to \infty$, the probability that $X \leq x$ approaches 1.

Property 3: Monotonicity

For $x_1 < x_2$,

$$F_X(x_1) \le F_X(x_2)$$

This shows that the CDF is a non-decreasing (monotonic) function.

Property 4: Interval Probability Using CDF

For $x_1 < x_2$,

$$\Pr(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$$

This property allows probabilities over intervals to be computed using differences of CDF values.

1.4 Example #1: Validity of a CDF

Problem:

Determine which of the following functions are valid CDFs.

(a)

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$$

As $x \to -\infty$, $\tan^{-1}(x) \to -\frac{\pi}{2}$, hence $F_X(x) \to 0$.

As $x \to \infty$, $\tan^{-1}(x) \to \frac{\pi}{2}$, hence $F_X(x) \to 1$.

Function is non-decreasing

Conclusion: Valid CDF

(b)

$$F_X(x) = [1 - e^{-x}]u(x)$$

For $x < 0$, $u(x) = 0 \Rightarrow F_X(x) = 0$

For $x \ge 0$, function increases monotonically

Limit as $x \to \infty$ is 1

Conclusion: Valid CDF

(c)

$$F_X(x) = e^{-x^2}$$

As $x \to \infty$, $F_X(x) \to 0$

Does not satisfy $F_X(\infty) = 1$

Conclusion: Not a valid CDF

(d)

$$F_X(x) = x^2 u(x)$$

As $x \to \infty$, function diverges
Violates boundedness property
Conclusion: Not a valid CDF

1.5 Example #2: Probability Computation Using CDF

Given:

$$F_X(x) = (1 - e^{-x})u(x)$$

(a)

$$\Pr(X > 5)$$

$$\Pr(X > 5) = 1 - \Pr(X \leq 5) = 1 - F_X(5)$$

$$= 1 - (1 - e^{-5}) = e^{-5}$$

(b)

$$\Pr(X < 5)$$

$$\Pr(X < 5) = F_X(5) = 1 - e^{-5}$$

(c)

$$\Pr(3 < X < 7)$$

$$= F_X(7) - F_X(3)$$

$$= (1 - e^{-7}) - (1 - e^{-3}) = e^{-3} - e^{-7}$$

(d)

$$\Pr(X > 5 \mid X < 7)$$

$$= \frac{\Pr(5 < X < 7)}{\Pr(X < 7)}$$

$$= \frac{F_X(7) - F_X(5)}{F_X(7)}$$

2. Probability Density Function (PDF)

2.1 Definition of PDF

For a continuous random variable X, the Probability Density Function (PDF) is defined as:

$$f_X(x) = \lim_{\epsilon \to 0} \frac{\Pr(x \le X \le x + \epsilon)}{\epsilon}$$

2.2 Relationship Between PDF and CDF

Recall for a continuous random variable:

$$\Pr(x \le X \le x + \epsilon) = F_X(x + \epsilon) - F_X(x)$$

Substituting into the PDF definition:

$$f_X(x) = \lim_{\epsilon \to 0} \frac{F_X(x + \epsilon) - F_X(x)}{\epsilon}$$

This is the definition of the derivative.

2.3 Final Relationship

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Thus:
The PDF is the derivative of the CDF
The CDF is the integral of the PDF

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

End of Lecture 7 Scribe (as covered in provided PPT)