

CSE400 – Fundamentals of Probability in Computing

Lecture 7: Expectation, CDFs, PDFs and Problem Solving Skills

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1 Outline of the Lecture

- The Cumulative Distribution Function (CDF)
 - Definition
 - Properties
 - Examples
- The Probability Density Function (PDF)
 - Definition
 - PDF–CDF relationship
 - Properties
 - Examples
- Expectation of Random Variables
 - Definition
 - Expectation of a function of a random variable
 - Linear operations with expectation
- n^{th} moments and central moments of random variables
 - Variance
 - Skewness
 - Kurtosis

2 CDF and PDF: Intuition (Water Tank Analogy)

- The cumulative distribution function (CDF) is explained using a water tank analogy.
- The height (h) of water in the tank represents the value of a random variable.
- The volume of water up to height (h), denoted by $V(h)$, corresponds to the cumulative probability up to that value.

Mathematical Representation

$$V(h) = \int_0^h \pi R^2 dh = (\pi R^2)h$$

- Here, πR^2 is analogous to the probability density function (PDF) of a uniform distribution.

The maximum volume of the tank is:

$$V(H) = \pi R^2 H$$

This maximum volume is analogous to total probability equal to 1.

3 Cumulative Distribution Function (CDF)

3.1 Definition

The cumulative distribution function (CDF) of a random variable X is defined as:

$$F_X(x) = \Pr(X < x), \quad -\infty < x < \infty$$

Most of the information about the random experiment described by the random variable X is determined by the behavior of $F_X(x)$.

3.2 Properties of the CDF

The CDF satisfies the following properties:

- $0 \leq F_X(x) \leq 1$
- $F_X(-\infty) = 0$
- $F_X(\infty) = 1$
- If $x_1 < x_2$, then:

$$F_X(x_1) \leq F_X(x_2)$$

- For $x_1 < x_2$:

$$\Pr(x_1 < X < x_2) = F_X(x_2) - F_X(x_1)$$

3.3 CDF Example #1: Validity of CDFs

Question: Find the valid CDF.

Given candidate functions:

1. $F_X(x) = 3 + 2 \tan^{-1}(x)$
2. $F_X(x) = [1 - e^{-x}]u(x)$
3. $F_X(x) = e^{x^2}$
4. Other given expressions as shown in slides

Validity is checked using CDF properties:

- Range between 0 and 1
- Proper limits at $-\infty$ and $+\infty$
- Monotonic non-decreasing behavior

3.4 CDF Example #2

Given:

$$F_X(x) = (1 - e^{-x})u(x)$$

Find the following probabilities:

- $\Pr(X > 5)$
- $\Pr(X < 5)$
- $\Pr(3 < X < 7)$
- $\Pr(X > 5 \mid X < 7)$

Each probability is computed directly using:

$$\Pr(a < X < b) = F_X(b) - F_X(a)$$

4 Probability Density Function (PDF)

4.1 Definition and PDF–CDF Relationship

The PDF of a continuous random variable X at point x is defined as:

$$f_X(x) = \lim_{\epsilon \rightarrow 0} \frac{\Pr(x < X < x + \epsilon)}{\epsilon}$$

For a continuous range:

$$\Pr(x < X < x + \epsilon) = F_X(x + \epsilon) - F_X(x)$$

Substituting and taking the limit:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Hence:

- The PDF is the derivative of the CDF.
- The CDF is the integral of the PDF:

$$F_X(x) = \int_{-\infty}^x f_X(y) dy$$

4.2 Properties of the PDF

- $f_X(x) \geq 0$
- $f_X(x) = \frac{dF_X(x)}{dx}$
- $F_X(x) = \int_{-\infty}^x f_X(y) dy$
- Total probability:

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$

- Probability over an interval:

$$\Pr(a < X < b) = \int_a^b f_X(x) dx$$

4.3 PDF Example #1: Validity of PDFs

Question: Which of the following are valid PDFs?

Candidate functions include:

- $f_X(x) = e^{-x}u(x)$
- Piecewise-defined functions over bounded intervals
- $f_X(x) = 2xe^{-x}u(x)$

Validity is determined by checking:

- Non-negativity
- Integral over entire range equals 1

4.4 PDF Example #2

Given CDF:

$$F_X(x) = (1 - e^{-x})u(x)$$

The PDF is obtained by differentiation:

$$f_X(x) = \frac{dF_X(x)}{dx} = e^{-x}u(x)$$

Conversely, if:

$$f_X(x) = 2xe^{-x^2}u(x)$$

Then the CDF is:

$$F_X(x) = \int_0^x 2ye^{-y^2} dy = (1 - e^{-x^2})u(x)$$

5 Expectation of Random Variables

5.1 Expectation

Expectation of a random variable X is denoted by:

$$\mathbb{E}[X]$$

The expectation operator is linear.

5.2 Expectation of a Function of a Random Variable

For a function $g(X)$:

$$\mathbb{E}[g(X)]$$

The definition follows directly from the PDF-based formulation as presented in the slides.

5.3 Linear Operations with Expectation

Expectation satisfies linearity:

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

6 Moments and Central Moments of Random Variables

- n^{th} moments and central moments are introduced.
- These include:
 - Variance

- Skewness
 - Kurtosis
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End of Lecture 7 Scribe