This parabolic arch bridge supports the deck above it.

5 Cables and Arches

Cables and arches often form the main load-carrying element in many types of structures, and in this chapter we will discuss some of the important aspects related to their structural analysis. The chapter begins with a general discussion of cables, followed by an analysis of cables subjected to a concentrated load and to a uniform distributed load. Since most arches are statically indeterminate, only the special case of a three-hinged arch will be considered. The analysis of this structure will provide some insight regarding the fundamental behavior of all arched structures.

5.1 Cables

Cables are often used in engineering structures for support and to transmit loads from one member to another. When used to support suspension roofs, bridges, and trolley wheels, cables form the main load-carrying element in the structure. In the force analysis of such systems, the weight of the cable itself may be neglected; however, when cables are used as guys for radio antennas, electrical transmission lines, and derricks, the cable weight may become important and must be included in the structural analysis. Two cases will be considered in the sections that follow: a cable subjected to concentrated loads and a cable subjected to a distributed load. Provided these loadings are coplanar with the cable, the requirements for equilibrium are formulated in an identical manner.

When deriving the necessary relations between the force in the cable and its slope, we will make the assumption that the cable is *perfectly flexible* and *inextensible*. Due to its flexibility, the cable offers no resistance to shear or bending and, therefore, the force acting in the cable is always tangent to the cable at points along its length. Being inextensible, the cable has a constant length both before and after the load is applied. As a result, once the load is applied, the geometry of the cable remains fixed, and the cable or a segment of it can be treated as a rigid body.

5.2 Cable Subjected to Concentrated Loads

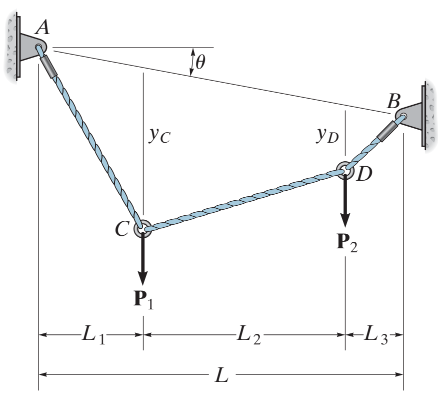


Fig. 5–1

When a cable of negligible weight supports several concentrated loads, the cable takes the form of several straight-line segments, each of which is subjected to a constant tensile force. Consider, for example, the cable shown in Fig. 5–1. Here θ specifies the angle of the cable's *cord AB*, and L is the cable's span. If the distances L_1, L_2, and L_3 and the loads $\mathbf{P}_1$ and $\mathbf{P}_2$ are known, then the problem is to determine the *nine unknowns consisting of the tension in each of the three* segments, the *four* components of reaction at A and B, and the sags y_C and y_D at the *two* points C and D. For the solution we can write *two* equations of force equilibrium at each of points A, B, C, and D. This results in a total of *eight equations*. To complete the solution, it will be necessary to know something about the geometry of the cable in order to obtain the necessary ninth equation. For example, if the cable's total *length* $\mathscr{L}$ is specified, then the Pythagorean theorem can be used to relate $\mathscr{L}$ to each of the three segmental lengths, written in terms of θ, y_C, y_D, L_1, L_2, and L_3. Unfortunately, this type of problem cannot be solved easily by hand. Another possibility, however, is to specify one of the sags, either y_C or y_D, instead of the cable length. By doing this, the equilibrium equations are then sufficient for obtaining the unknown forces and the remaining sag. Once the sag at each point of loading is obtained, $\mathscr{L}$ can then be determined by trigonometry.

When performing an equilibrium analysis for a problem of this type, the forces in the cable can also be obtained by writing the equations of equilibrium for the entire cable or any portion thereof. The following example numerically illustrates these concepts.

The deck of a cable-stayed bridge is supported by a series of cables attached at various points along the deck and pylons.

EXAMPLE 5.1

Determine the tension in each segment of the cable shown in Fig. 5–2*a*. Also, what is the dimension *h*?

SOLUTION

By inspection, there are four unknown external reactions (A_x, A_y, D_x, and D_y) and three unknown cable tensions, one in each cable segment. These seven unknowns along with the sag *h* can be determined from the eight available equilibrium equations ($\Sigma F_x = 0$, $\Sigma F_y = 0$) applied to points *A* through *D*.

A more direct approach to the solution is to recognize that the slope of cable *CD* is specified, and so a free-body diagram of the entire cable is shown in Fig. 5–2*b*. We can obtain the tension in segment *CD* as follows:

$$\circlearrowleft + \Sigma M_A = 0;$$

$$T_{CD}(3/5)(2\text{ m}) + T_{CD}(4/5)(5.5\text{ m}) - 3\text{ kN}(2\text{ m}) - 8\text{ kN}(4\text{ m}) = 0$$

$$T_{CD} = 6.79\text{ kN} \qquad \textit{Ans.}$$

Now we can analyze the equilibrium of points *C* and *B* in sequence.
Point *C* (Fig. 5–2*c*);

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 6.79\text{ kN}(3/5) - T_{BC}\cos\theta_{BC} = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad 6.79\text{ kN}(4/5) - 8\text{ kN} + T_{BC}\sin\theta_{BC} = 0$$

$$\theta_{BC} = 32.3^\circ \qquad T_{BC} = 4.82\text{ kN} \qquad \textit{Ans.}$$

Point *B* (Fig. 5–2*d*);

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad -T_{BA}\cos\theta_{BA} + 4.82\text{ kN}\cos 32.3^\circ = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad T_{BA}\sin\theta_{BA} - 4.82\text{ kN}\sin 32.3^\circ - 3\text{ kN} = 0$$

$$\theta_{BA} = 53.8^\circ \qquad T_{BA} = 6.90\text{ kN} \qquad \textit{Ans.}$$

Hence, from Fig. 5–2*a*,

$$h = (2\text{ m})\tan 53.8^\circ = 2.74\text{ m} \qquad \textit{Ans.}$$

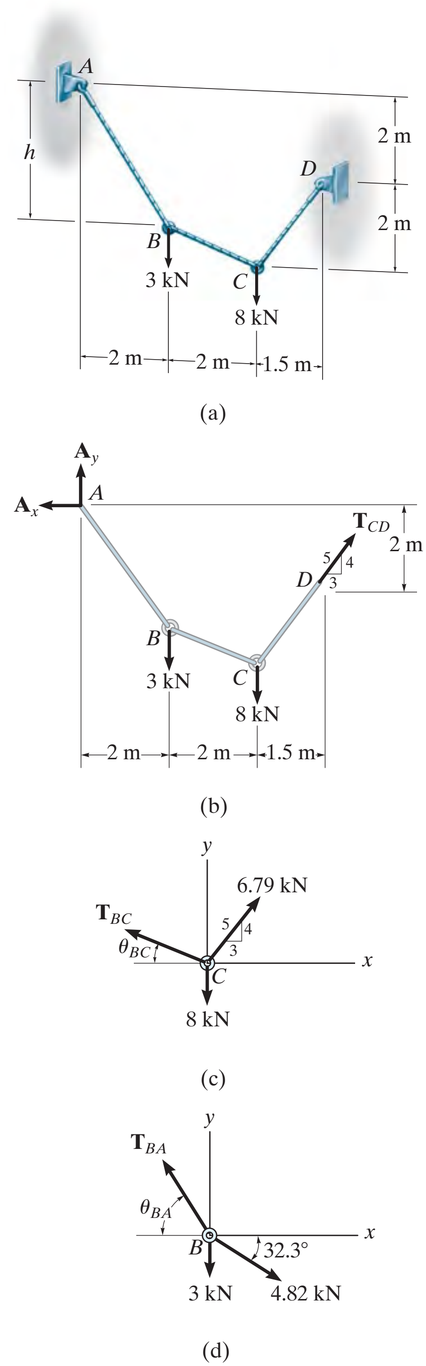


Fig. 5–2

5.3 Cable Subjected to a Uniform Distributed Load

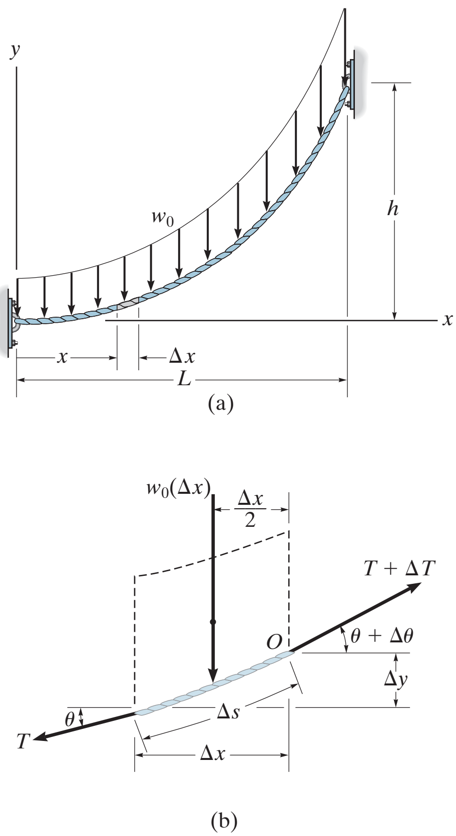


Fig. 5–3

Cables provide a very effective means of supporting the dead weight of girders or bridge decks having very long spans. A suspension bridge is a typical example, in which the deck is suspended from the cable using a series of close and equally spaced hangers.

In order to analyze this problem, we will first determine the shape of a cable subjected to a uniform *horizontally* distributed vertical load w_0, Fig. 5–3*a*. Here the x,y axes have their origin located at the lowest point on the cable, such that the slope is zero at this point. The free-body diagram of a small segment of the cable having a length Δs is shown in Fig. 5–3*b*. Since the tensile force in the cable changes continuously in both magnitude and direction along the cable's length, this change is denoted on the free-body diagram by ΔT. The distributed load is represented by its resultant force $w_0 \Delta x$, which acts at $\Delta x/2$ from point O. Applying the equations of equilibrium yields

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad -T\cos\theta + (T + \Delta T)\cos(\theta + \Delta\theta) = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad -T\sin\theta - w_0(\Delta x) + (T + \Delta T)\sin(\theta + \Delta\theta) = 0$$

$$\circlearrowleft + \Sigma M_O = 0; \qquad w_0(\Delta x)(\Delta x/2) - T\cos\theta\,\Delta y + T\sin\theta\,\Delta x = 0$$

Dividing each of these equations by Δx and taking the limit as $\Delta x \rightarrow 0$, and hence $\Delta y \rightarrow 0$, $\Delta\theta \rightarrow 0$, and $\Delta T \rightarrow 0$, we obtain

$$\frac{d(T\cos\theta)}{dx} = 0 \tag{5–1}$$

$$\frac{d(T\sin\theta)}{dx} = w_0 \tag{5–2}$$

$$\frac{dy}{dx} = \tan\theta \tag{5–3}$$

Integrating Eq. 5–1, where $T = F_H$ at $x = 0$, we have:

$$T\cos\theta = F_H \tag{5–4}$$

which indicates the horizontal component of force at *any point* along the cable remains *constant.*

Integrating Eq. 5–2, realizing that $T\sin\theta = 0$ at $x = 0$, gives

$$T\sin\theta = w_0 x \tag{5–5}$$

Dividing Eq. 5–5 by Eq. 5–4 eliminates T. Then using Eq. 5–3, we can obtain the slope at any point,

$$\tan\theta = \frac{dy}{dx} = \frac{w_0 x}{F_H} \tag{5–6}$$

Performing a second integration with $y = 0$ at $x = 0$ yields

$$y = \frac{w_0}{2F_H}x^2 \tag{5–7}$$

This is the equation of a *parabola*. The constant F_H may be obtained by using the boundary condition $y = h$ at $x = L$. Thus,

$$F_H = \frac{w_0 L^2}{2h} \tag{5–8}$$

Finally, substituting into Eq. 5–7 yields

$$y = \frac{h}{L^2}x^2 \tag{5–9}$$

From Eq. 5–4, the maximum tension in the cable occurs when θ is maximum; i.e., at $x = L$. Hence, from Eqs. 5–4 and 5–5,

$$T_{\max} = \sqrt{F_H^2 + (w_0 L)^2} \tag{5–10}$$

Or, using Eq. 5–8, we can express $T_{\max}$ in terms of w_0, i.e.,

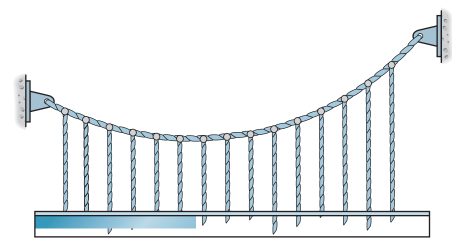
$$T_{\max} = w_0 L\sqrt{1 + (L/2h)^2} \tag{5–11}$$

Realize that we have neglected the weight of the cable, which is *uniform* along the *length* of the cable, and not along its horizontal projection. Actually, a cable subjected to its own weight and free of any other loads will take the form of a *catenary* curve. However, if the sag-to-span ratio is small, which is the case for most structural applications, this curve closely approximates a parabolic shape, as determined here.

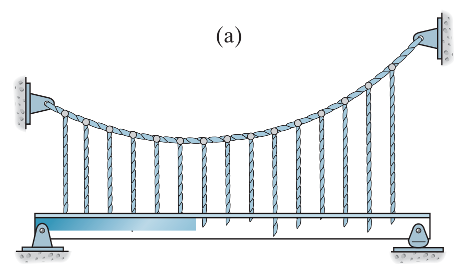
From the results of this analysis, it follows that a cable will *maintain a parabolic shape,* provided the dead load of the deck for a suspension bridge or a suspended girder will be *uniformly distributed* over the horizontal projected length of the cable. Hence, if the girder in Fig. 5–4*a* is supported by a series of hangers, which are close and uniformly spaced, the load in each hanger must be the *same* so as to ensure that the cable has a parabolic shape.

Using this assumption, we can perform the structural analysis of the girder or any other framework which is freely suspended from the cable. In particular, if the girder is simply supported as well as supported by the cable, the analysis will be statically indeterminate to the first degree, Fig. 5–4*b*. However, if the girder has an internal pin at some intermediate point along its length, Fig. 5–4*c*, then this would provide a condition of zero moment, and so a determinate structural analysis of the girder can be performed.

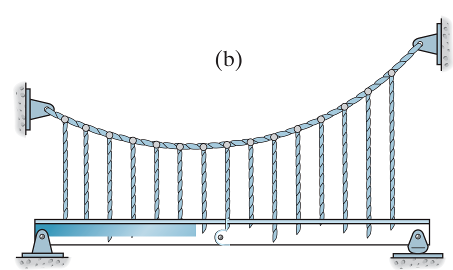
The Verrazano-Narrows Bridge at the entrance to New York Harbor has a main span of 4260 ft (1.30 km).



(a)



(b)



(c)

Fig. 5–4

EXAMPLE 5.2

The cable in Fig. 5–5*a* supports a girder which weighs 850 lb/ft. Determine the tension in the cable at points A, B, and C.

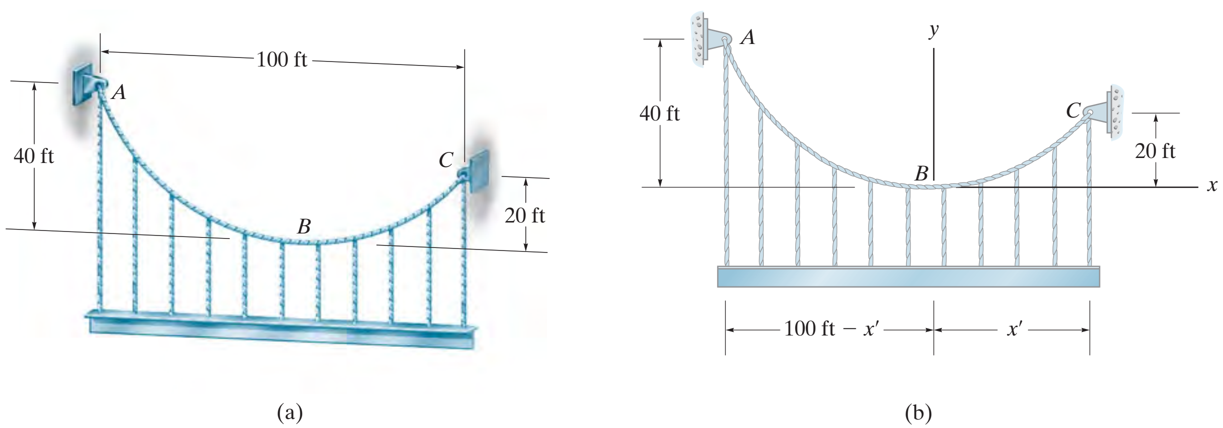


Fig. 5–5

5

SOLUTION

The origin of the coordinate axes is established at point B, the lowest point on the cable, where the slope is zero, Fig. 5–5*b*. From Eq. 5–7, the parabolic equation for the cable is:

$$y = \frac{w_0}{2F_H}x^2 = \frac{850\text{ lb/ft}}{2F_H}x^2 = \frac{425}{F_H}x^2 \qquad (1)$$

Assuming point C is located x' from B, we have

$$20 = \frac{425}{F_H}x'^2$$

$$F_H = 21.25x'^2 \qquad (2)$$

Also, for point A,

$$40 = \frac{425}{F_H}[-(100 - x')]^2$$

$$40 = \frac{425}{21.25x'^2}[-(100 - x')]^2$$

$$x'^2 + 200x' - 10\,000 = 0$$

$$x' = 41.42\text{ ft}$$

Thus, from Eqs. 2 and 1 (or Eq. 5–6) we have

$$F_H = 21.25(41.42)^2 = 36\,459.2 \text{ lb}$$

$$\frac{dy}{dx} = \frac{850}{36\,459.2}x = 0.02331x \qquad (3)$$

At point A,

$$x = -(100 - 41.42) = -58.58 \text{ ft}$$

$$\tan\theta_A = \left.\frac{dy}{dx}\right|_{x=-58.58} = 0.02331(-58.58) = -1.366$$

$$\theta_A = -53.79^\circ$$

Using Eq. 5–4,

$$T_A = \frac{F_H}{\cos\theta_A} = \frac{36\,459.2}{\cos(-53.79^\circ)} = 61.7 \text{ k} \qquad \textit{Ans.}$$

At point B, $x = 0$,

$$\tan\theta_B = \left.\frac{dy}{dx}\right|_{x=0} = 0, \qquad \theta_B = 0^\circ$$

$$T_B = \frac{F_H}{\cos\theta_B} = \frac{36\,459.2}{\cos 0^\circ} = 36.5 \text{ k} \qquad \textit{Ans.}$$

At point C,

$$x = 41.42 \text{ ft}$$

$$\tan\theta_C = \left.\frac{dy}{dx}\right|_{x=41.42} = 0.02331(41.42) = 0.9657$$

$$\theta_C = 44.0^\circ$$

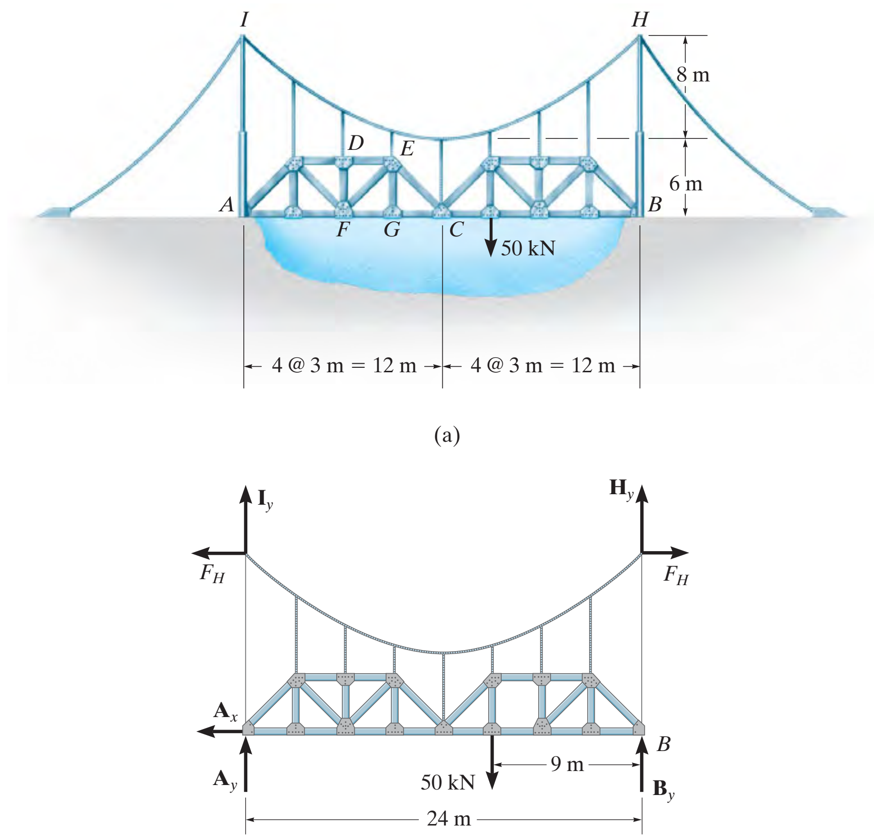
$$T_C = \frac{F_H}{\cos\theta_C} = \frac{36\,459.2}{\cos 44.0^\circ} = 50.7 \text{ k} \qquad \textit{Ans.}$$

5

5

EXAMPLE 5.3

The suspension bridge in Fig. 5–6*a* is constructed using the two stiffening trusses that are pin connected at their ends *C* and supported by a pin at *A* and a rocker at *B*. Determine the maximum tension in the cable *IH*. The cable has a parabolic shape and the bridge is subjected to the single load of 50 kN.



(a)

(b)

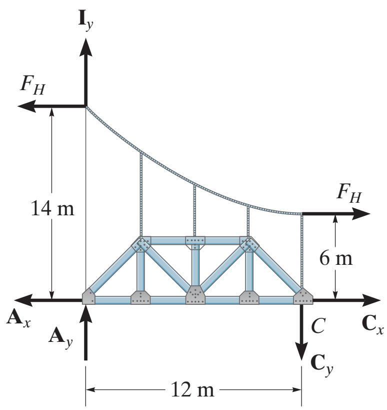
Fig. 5–6

SOLUTION

The free-body diagram of the cable-truss system is shown in Fig. 5–6*b*. According to Eq. 5–4 ($T\cos\theta = F_H$), the horizontal component of cable tension at *I* and *H* must be constant, F_H. Taking moments about *B*, we have

$$\circlearrowright + \Sigma M_B = 0; \qquad -I_y(24\text{ m}) - A_y(24\text{ m}) + 50\text{ kN}(9\text{ m}) = 0$$

$$I_y + A_y = 18.75$$



(c)

If only half the suspended structure is considered, Fig. 5–6*c*, then summing moments about the pin at C, we have

$$\circlearrowright +\Sigma M_C = 0; \quad F_H(14\text{ m}) - F_H(6\text{ m}) - I_y(12\text{ m}) - A_y(12\text{ m}) = 0$$
$$I_y + A_y = 0.667F_H$$

From these two equations,

$$18.75 = 0.667F_H$$
$$F_H = 28.125\text{ kN}$$

To obtain the maximum tension in the cable, we will use Eq. 5–11, but first it is necessary to determine the value of an assumed uniform distributed loading w_0 from Eq. 5–8:

$$w_0 = \frac{2F_H h}{L^2} = \frac{2(28.125\text{ kN})(8\text{ m})}{(12\text{ m})^2} = 3.125\text{ kN/m}$$

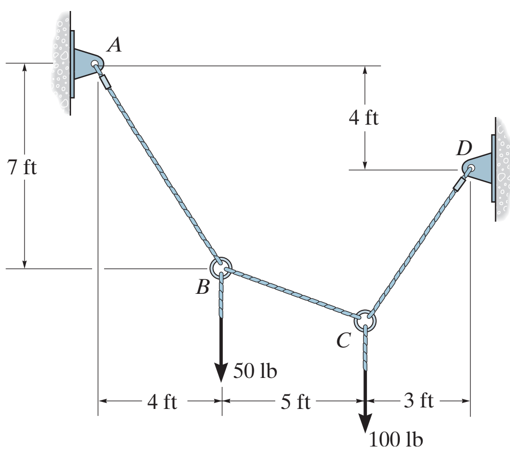
Thus, using Eq. 5–11, we have

$$\begin{aligned} T_{max} &= w_0 L\sqrt{1 + (L/2h)^2} \\ &= 3.125(12\text{ m})\sqrt{1 + (12\text{ m}/2(8\text{ m}))^2} \\ &= 46.9\text{ kN} \end{aligned}$$

Ans.

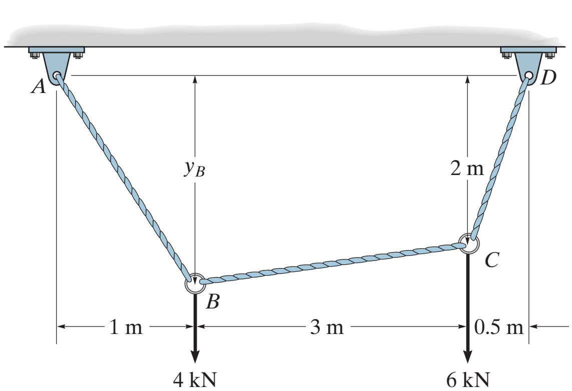
PROBLEMS

5–1. Determine the tension in each segment of the cable and the cable's total length.



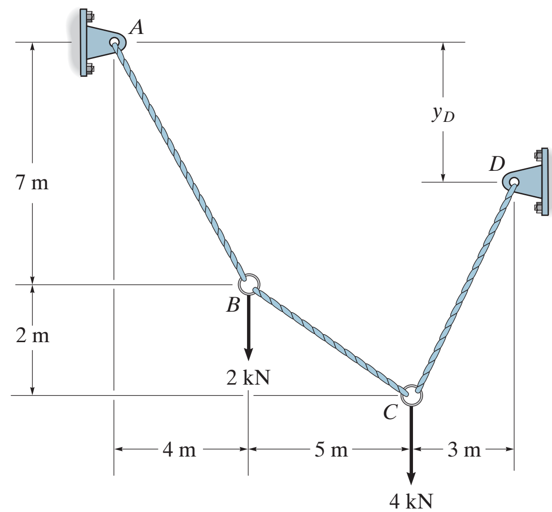
Prob. 5–1

5–2. Cable $ABCD$ supports the loading shown. Determine the maximum tension in the cable and the sag of point B.



Prob. 5–2

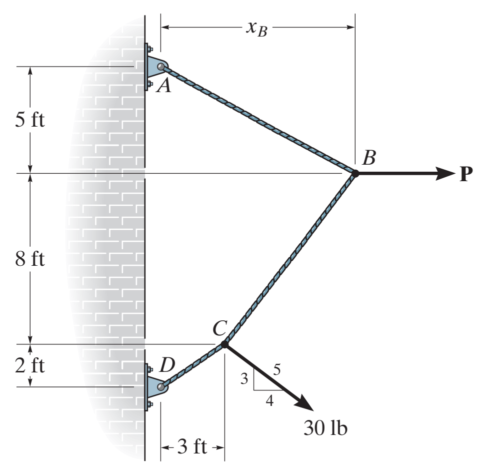
5–3. Determine the tension in each cable segment and the distance y_D.



Prob. 5–3

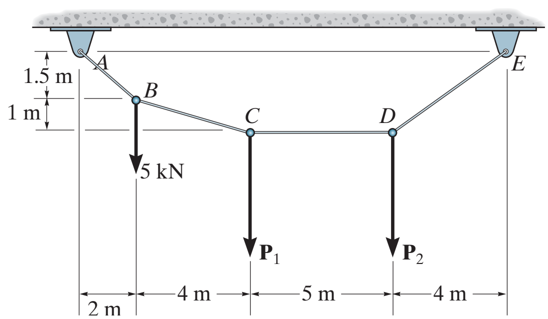
***5–4.** The cable supports the loading shown. Determine the distance x_B the force at point B acts from A. Set $P = 40$ lb.

5–5. The cable supports the loading shown. Determine the magnitude of the horizontal force $\mathbf{P}$ so that $x_B = 6$ ft.



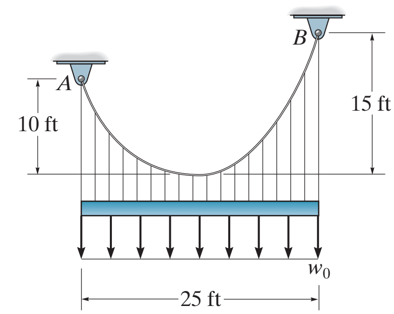
Probs. 5–4/5–5

5–6. Determine the forces P_1 and P_2 needed to hold the cable in the position shown, i.e., so segment CD remains horizontal. Also find the maximum loading in the cable.



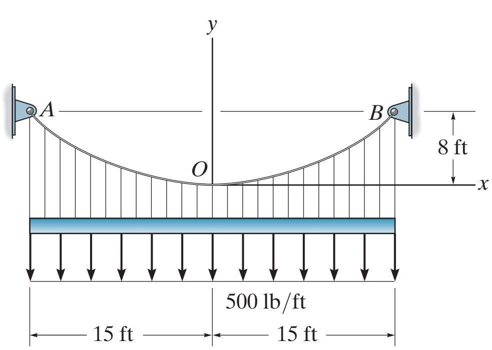
Prob. 5–6

***5–8.** The cable supports the uniform load of $w_0 = 600$ lb/ft. Determine the tension in the cable at each support A and B.



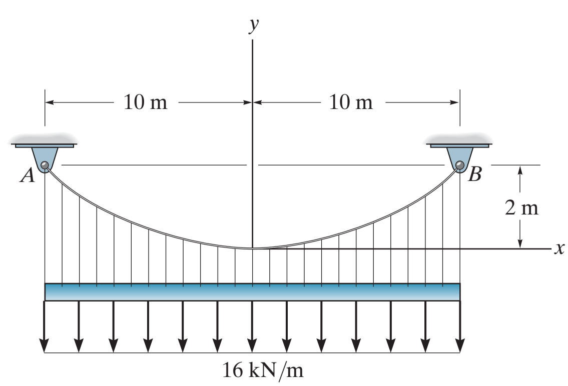
Prob. 5–8

5–7. The cable is subjected to the uniform loading. If the slope of the cable at point O is zero, determine the equation of the curve and the force in the cable at O and B.



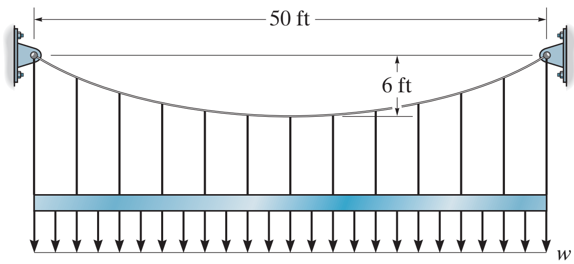
Prob. 5–7

5–9. Determine the maximum and minimum tension in the cable.



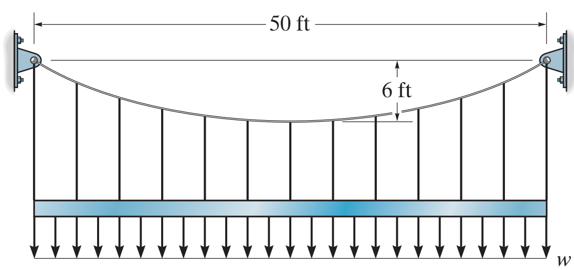
Prob. 5–9

5–10. Determine the maximum uniform loading w, measured in lb/ft, that the cable can support if it is capable of sustaining a maximum tension of 3000 lb before it will break.



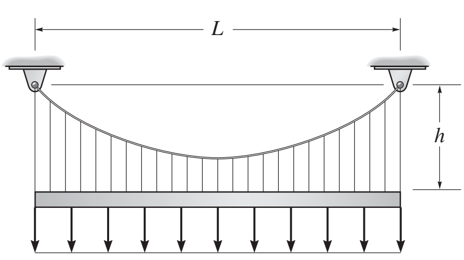
Prob. 5–10

5–11. The cable is subjected to a uniform loading of $w = 250$ lb/ft. Determine the maximum and minimum tension in the cable.



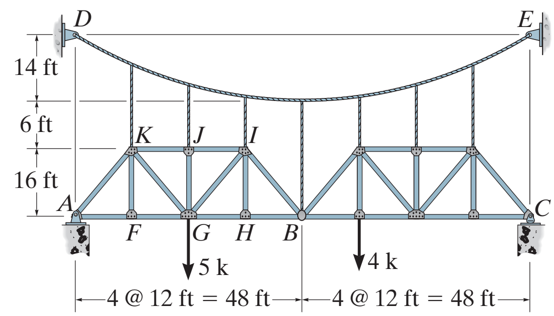
Prob. 5–11

***5–12.** The cable shown is subjected to the uniform load w_0. Determine the ratio between the rise h and the span L that will result in using the minimum amount of material for the cable.



Prob. 5–12

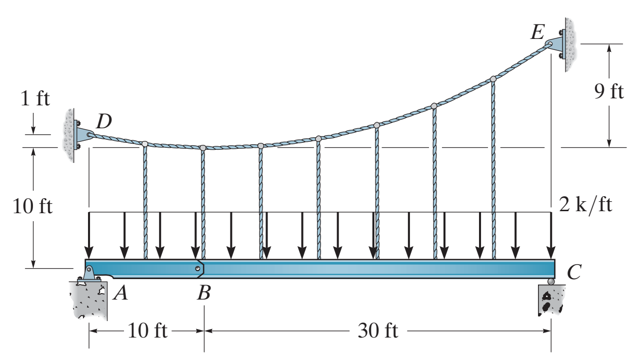
5–13. The trusses are pin connected and suspended from the parabolic cable. Determine the maximum force in the cable when the structure is subjected to the loading shown.



Prob. 5–13

5–14. Determine the maximum and minimum tension in the parabolic cable and the force in each of the hangers. The girder is subjected to the uniform load and is pin connected at B.

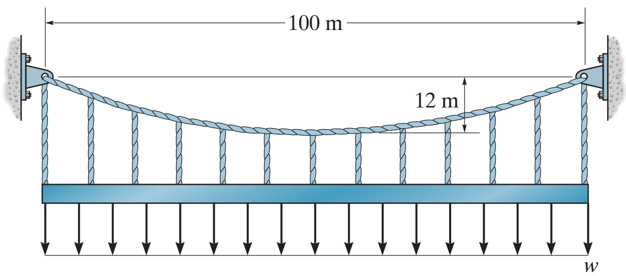
5–15. Draw the shear and moment diagrams for the pin-connected girders AB and BC. The cable has a parabolic shape.



Probs. 5–14/5–15

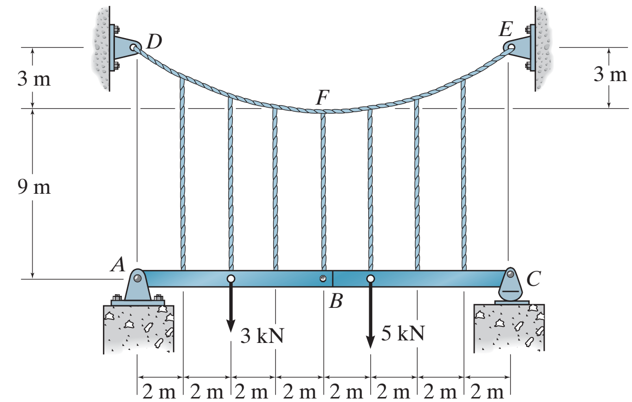
*5–16.** The cable will break when the maximum tension reaches $T_{max} = 5000$ kN. Determine the maximum uniform distributed load w required to develop this maximum tension.

5–17. The cable is subjected to a uniform loading of $w = 60$ kN/m. Determine the maximum and minimum tension in cable.



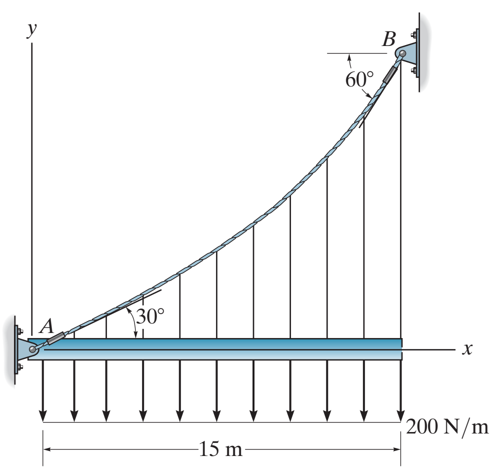
Probs. 5–16/5–17

5–19. The beams AB and BC are supported by the cable that has a parabolic shape. Determine the tension in the cable at points D, F, and E, and the force in each of the equally spaced hangers.



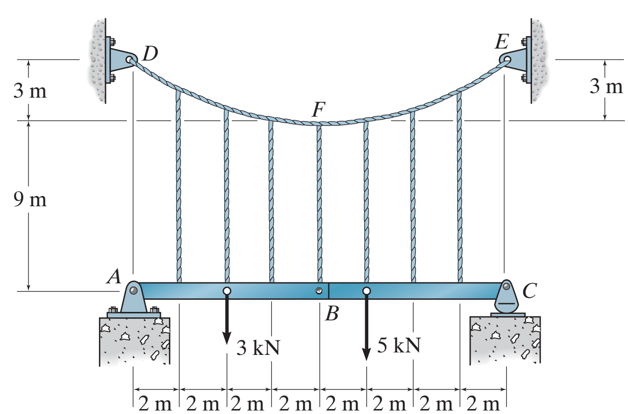
Prob. 5–19

5–18. The cable AB is subjected to a uniform loading of 200 N/m. If the weight of the cable is neglected and the slope angles at points A and B are 30° and 60°, respectively, determine the curve that defines the cable shape and the maximum tension developed in the cable.



Prob. 5–18

*5–20.** Draw the shear and moment diagrams for beams AB and BC. The cable has a parabolic shape.



Prob. 5–20

5.4 Arches

Like cables, arches can be used to reduce the bending moments in long-span structures. Essentially, an arch acts as an inverted cable, so it receives its load mainly in compression although, because of its rigidity, it must also resist some bending and shear depending upon how it is loaded and shaped. In particular, if the arch has a *parabolic shape* and it is subjected to a *uniform* horizontally distributed vertical load, then from the analysis of cables it follows that *only compressive forces* will be resisted by the arch. Under these conditions the arch shape is called a *funicular arch* because no bending or shear forces occur within the arch.

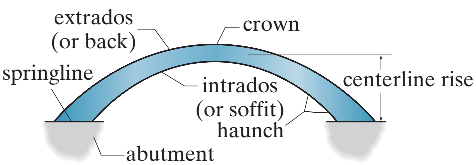
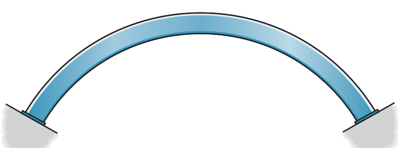
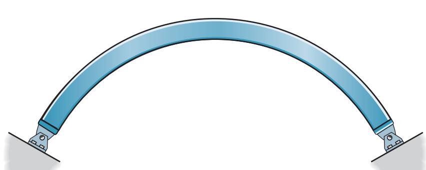


Fig. 5–7

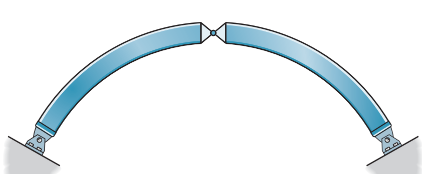
A typical arch is shown in Fig. 5–7, which specifies some of the nomenclature used to define its geometry. Depending upon the application, several types of arches can be selected to support a loading. A *fixed arch,* Fig. 5–8*a*, is often made from reinforced concrete. Although it may require less material to construct than other types of arches, it must have solid foundation abutments since it is indeterminate to the third degree and, consequently, additional stresses can be introduced into the arch due to relative settlement of its supports. A *two-hinged arch,* Fig. 5–8*b*, is commonly made from metal or timber. It is indeterminate to the first degree, and although it is not as rigid as a fixed arch, it is somewhat insensitive to settlement. We could make this structure statically determinate by replacing one of the hinges with a roller. Doing so, however, would remove the capacity of the structure to resist bending along its span, and as a result it would serve as a curved beam, and *not* as an arch. A *three-hinged arch,* Fig. 5–8*c*, which is also made from metal or timber, is statically determinate. Unlike statically indeterminate arches, it is not affected by settlement or temperature changes. Finally, if two- and three-hinged arches are to be constructed without the need for larger foundation abutments and if clearance is not a problem, then the supports can be connected with a tie rod, Fig. 5–8*d*. A *tied arch* allows the structure to behave as a rigid unit, since the tie rod carries the horizontal component of thrust at the supports. It is also unaffected by relative settlement of the supports.



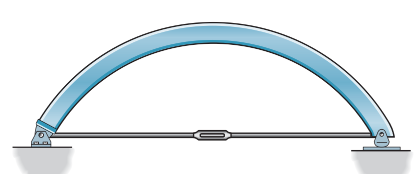
fixed arch
(a)



two-hinged arch
(b)



three-hinged arch
(c)



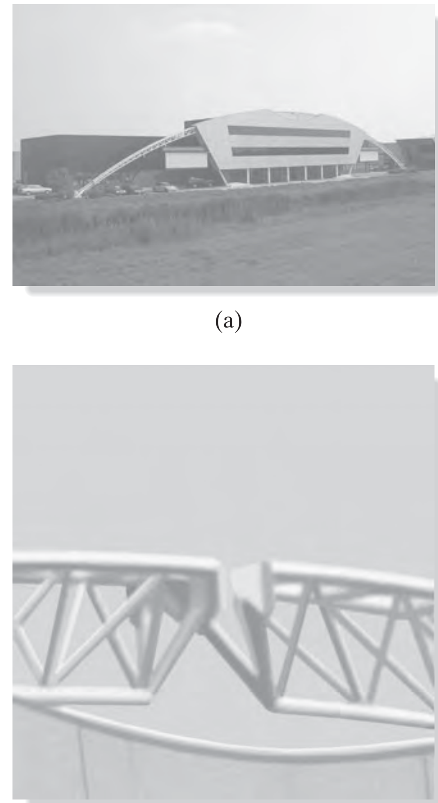
tied arch
(d)

Fig. 5–8

5.5 Three-Hinged Arch

To provide some insight as to how arches transmit loads, we will now consider the analysis of a three-hinged arch such as the one shown in Fig. 5–9*a*. In this case, the third hinge is located at the crown and the supports are located at different elevations. In order to determine the reactions at the supports, the arch is disassembled and the free-body diagram of each member is shown in Fig. 5–9*b*. Here there are six unknowns for which six equations of equilibrium are available. One method of solving this problem is to apply the moment equilibrium equations about points A and B. Simultaneous solution will yield the reactions C_x and C_y. The support reactions are then determined from the force equations of equilibrium. Once obtained, the internal normal force, shear, and moment loadings at any point along the arch can be found using the method of sections. Here, of course, the section should be taken perpendicular to the axis of the arch at the point considered. For example, the free-body diagram for segment AD is shown in Fig. 5–9*c*.

Three-hinged arches can also take the form of two pin-connected trusses, each of which would replace the arch ribs AC and CB in Fig. 5–9*a*. The analysis of this form follows the same procedure outlined above. The following examples numerically illustrate these concepts.



(a)

(b)

The three-hinge truss arch is used to support a portion of the roof loading of this building (a). The close-up photo shows the arch is pinned at its top (b).

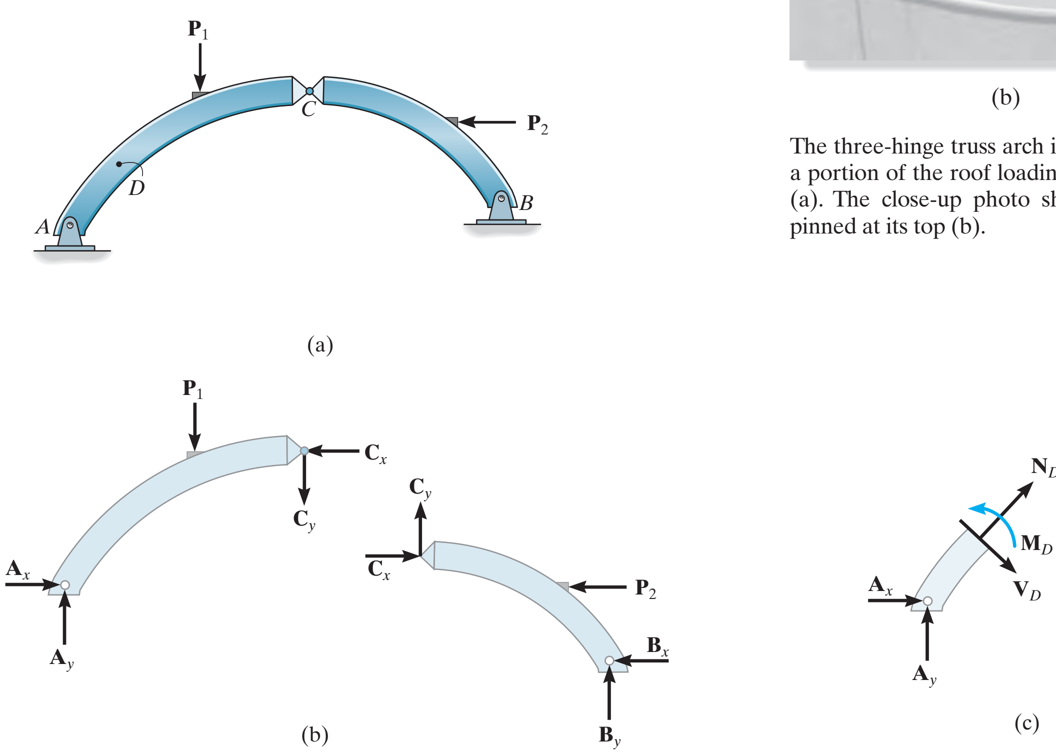
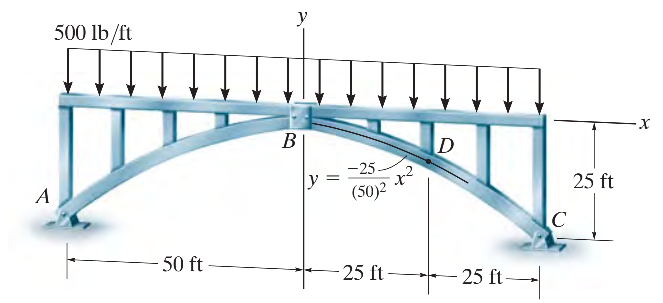


Fig. 5–9

EXAMPLE 5.4

The three-hinged open-spandrel arch bridge like the one shown in the photo has a parabolic shape. If this arch were to support a uniform load and have the dimensions shown in Fig. 5–10*a*, show that the arch is subjected *only to axial compression* at any intermediate point such as point *D*. Assume the load is uniformly transmitted to the arch ribs.

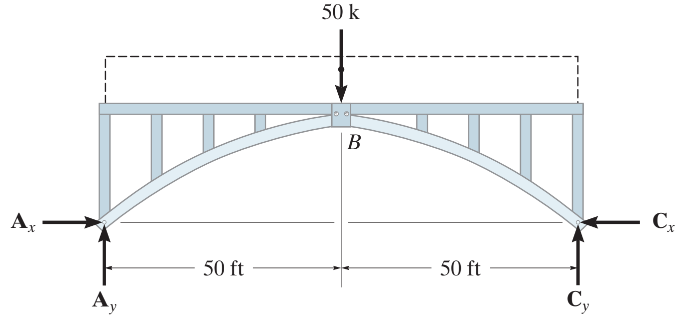


(a)

Fig. 5–10

SOLUTION

Here the supports are at the same elevation. The free-body diagrams of the entire arch and part *BC* are shown in Fig. 5–10*b* and Fig. 5–10*c*. Applying the equations of equilibrium, we have:



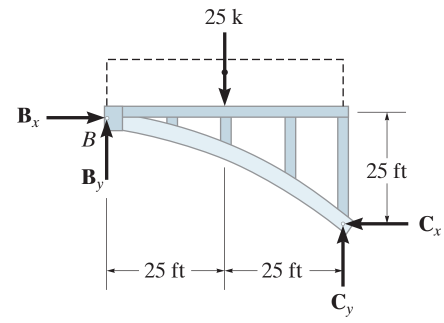
(b)

Entire arch:

$$\circlearrowleft + \Sigma M_A = 0; \qquad C_y(100\text{ ft}) - 50\text{ k}(50\text{ ft}) = 0$$

$$C_y = 25\text{ k}$$

Arch segment BC:



(c)

$$\circlearrowleft + \Sigma M_B = 0; \qquad -25\text{ k}(25\text{ ft}) + 25\text{ k}(50\text{ ft}) - C_x(25\text{ ft}) = 0$$

$$C_x = 25\text{ k}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad B_x = 25\text{ k}$$

$$+\uparrow \Sigma F_y = 0; \qquad B_y - 25\text{ k} + 25\text{ k} = 0$$

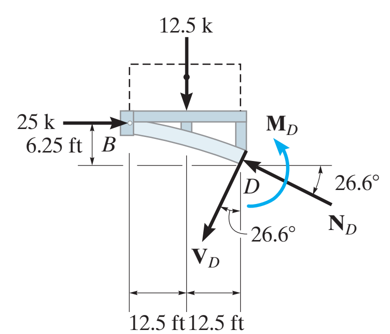
$$B_y = 0$$

A section of the arch taken through point D, $x = 25$ ft, $y = -25(25)^2/(50)^2 = -6.25$ ft, is shown in Fig. 5–10d. The slope of the segment at D is

$$\tan\theta = \frac{dy}{dx} = \frac{-50}{(50)^2}x\Big|_{x=25\text{ ft}} = -0.5$$

$$\theta = -26.6^\circ$$

Applying the equations of equilibrium, Fig. 5–10d we have



(d)

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 25\text{ k} - N_D\cos 26.6^\circ - V_D\sin 26.6^\circ = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad -12.5\text{ k} + N_D\sin 26.6^\circ - V_D\cos 26.6^\circ = 0$$

$$\circlearrowleft + \Sigma M_D = 0; \qquad M_D + 12.5\text{ k}(12.5\text{ ft}) - 25\text{ k}(6.25\text{ ft}) = 0$$

$$N_D = 28.0\text{ k} \qquad \textit{Ans.}$$

$$V_D = 0 \qquad \textit{Ans.}$$

$$M_D = 0 \qquad \textit{Ans.}$$

Note: If the arch had a different shape or if the load were nonuniform, then the internal shear and moment would be nonzero. Also, if a simply supported beam were used to support the distributed loading, it would have to resist a maximum bending moment of $M = 625$ k · ft. By comparison, it is more efficient to structurally resist the load in direct compression (although one must consider the possibility of buckling) than to resist the load by a bending moment.

5

EXAMPLE 5.5

The three-hinged tied arch is subjected to the loading shown in Fig. 5–11*a*. Determine the force in members *CH* and *CB*. The dashed member *GF* of the truss is intended to carry no force.

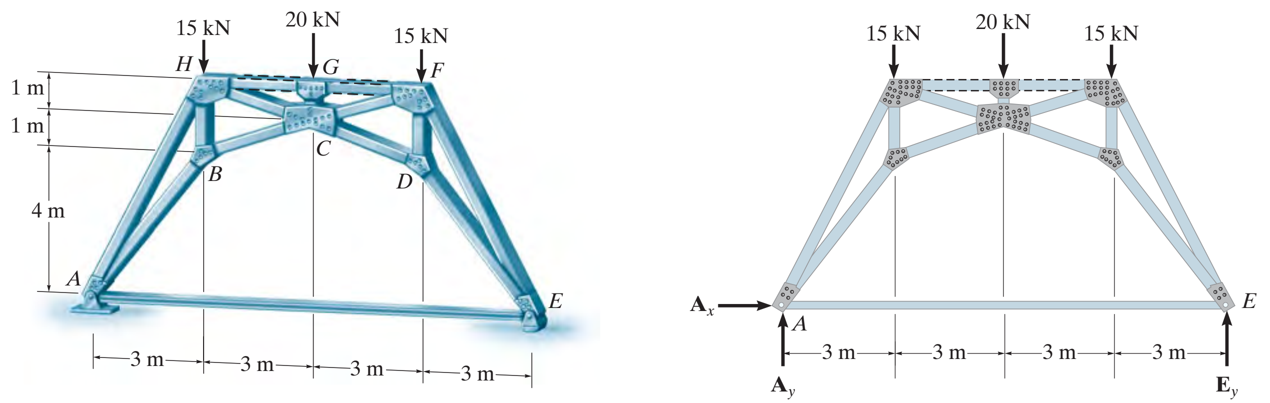


Fig. 5–11

SOLUTION

The support reactions can be obtained from a free-body diagram of the entire arch, Fig. 5–11*b*:

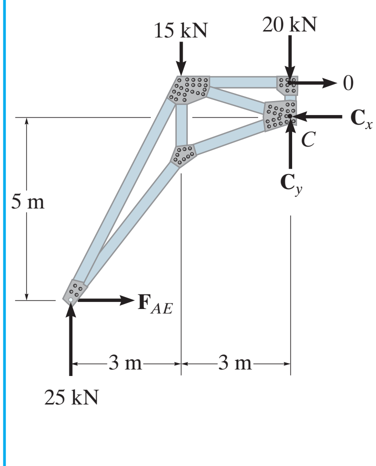
$$\circlearrowright + \Sigma M_A = 0; \quad E_y(12\text{ m}) - 15\text{ kN}(3\text{ m}) - 20\text{ kN}(6\text{ m}) - 15\text{ kN}(9\text{ m}) = 0$$

$$E_y = 25\text{ kN}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \quad A_x = 0$$

$$+\uparrow \Sigma F_y = 0; \quad A_y - 15\text{ kN} - 20\text{ kN} - 15\text{ kN} + 25\text{ kN} = 0$$

$$A_y = 25\text{ kN}$$



(c)

The force components acting at joint *C* can be determined by considering the free-body diagram of the left part of the arch, Fig. 5–11*c*. First, we determine the force:

$$\circlearrowright + \Sigma M_C = 0; \quad F_{AE}(5\text{ m}) - 25\text{ kN}(6\text{ m}) + 15\text{ kN}(3\text{ m}) = 0$$

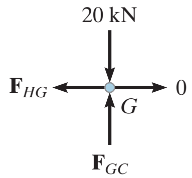
$$F_{AE} = 21.0\text{ kN}$$

Then,

$$\xrightarrow{+} \Sigma F_x = 0; \qquad -C_x + 21.0\text{ kN} = 0, \quad C_x = 21.0\text{ kN}$$

$$+\uparrow \Sigma F_y = 0; \qquad 25\text{ kN} - 15\text{ kN} - 20\text{ kN} + C_y = 0, \quad C_y = 10\text{ kN}$$

To obtain the forces in CH and CB, we can use the method of joints as follows:

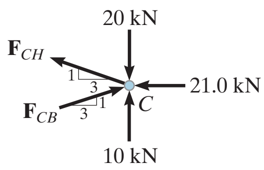


(d)

Joint G; Fig. 5–11d,

$$+\uparrow \Sigma F_y = 0; \qquad F_{GC} - 20\text{ kN} = 0$$

$$F_{GC} = 20\text{ kN (C)}$$



(e)

Joint C; Fig. 5–11e,

$$\xrightarrow{+} \Sigma F_x = 0; \qquad F_{CB}\left(\frac{3}{\sqrt{10}}\right) - 21.0\text{ kN} - F_{CH}\left(\frac{3}{\sqrt{10}}\right) = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad F_{CB}\left(\frac{1}{\sqrt{10}}\right) + F_{CH}\left(\frac{1}{\sqrt{10}}\right) - 20\text{ kN} + 10\text{ kN} = 0$$

Thus,

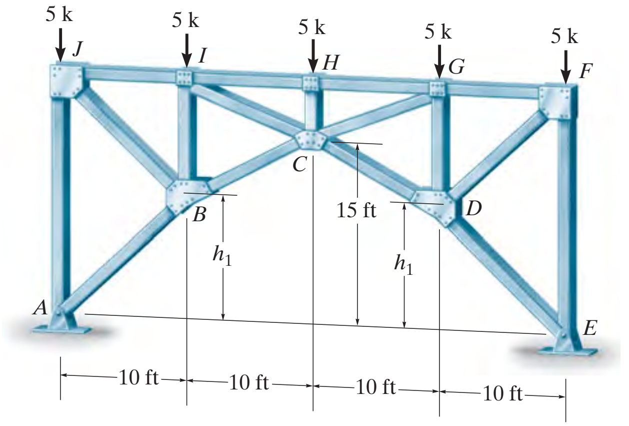
$$F_{CB} = 26.9\text{ kN (C)} \qquad \textit{Ans.}$$

$$F_{CH} = 4.74\text{ kN (T)} \qquad \textit{Ans.}$$

Note: Tied arches are sometimes used for bridges. Here the deck is supported by suspender bars that transmit their load to the arch. The deck is in tension so that it supports the actual thrust or horizontal force at the ends of the arch.

EXAMPLE 5.6

The three-hinged trussed arch shown in Fig. 5–12*a* supports the symmetric loading. Determine the required height h_1 of the joints *B* and *D*, so that the arch takes a funicular shape. Member *HG* is intended to carry no force.



(a)

SOLUTION

For a symmetric loading, the funicular shape for the arch must be *parabolic* as indicated by the dashed line (Fig. 5–12*b*). Here we must find the equation which fits this shape. With the *x*, *y* axes having an origin at *C*, the equation is of the form $y = -cx^2$. To obtain the constant *c*, we require

$$-(15 \text{ ft}) = -c(20 \text{ ft})^2$$
$$c = 0.0375/\text{ft}$$

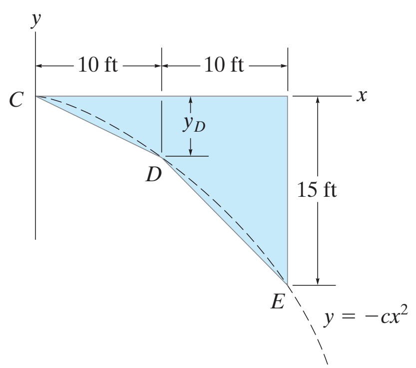
Therefore,

$$y_D = -(0.0375/\text{ft})(10 \text{ ft})^2 = -3.75 \text{ ft}$$

So that from Fig. 5–12*a*,

$$h_1 = 15 \text{ ft} - 3.75 \text{ ft} = 11.25 \text{ ft} \qquad \textit{Ans.}$$

Using this value, if the method of joints is now applied to the truss, the results will show that the top cord and diagonal members will all be zero-force members, and the symmetric loading will be supported *only by the bottom cord* members *AB*, *BC*, *CD*, and *DE* of the truss.

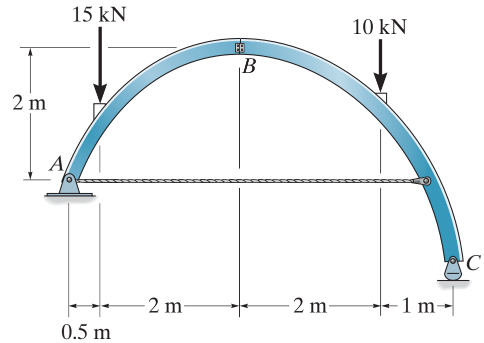


(b)

Fig. 5–12

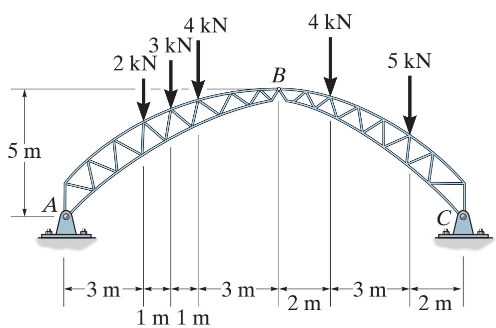
PROBLEMS

5–21. The tied three-hinged arch is subjected to the loading shown. Determine the components of reaction at A and C and the tension in the cable.



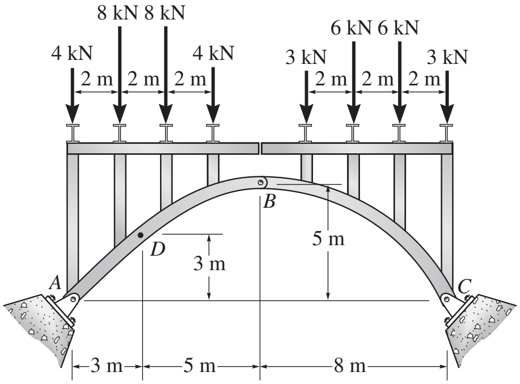
Prob. 5–21

5–22. Determine the resultant forces at the pins A, B, and C of the three-hinged arched roof truss.



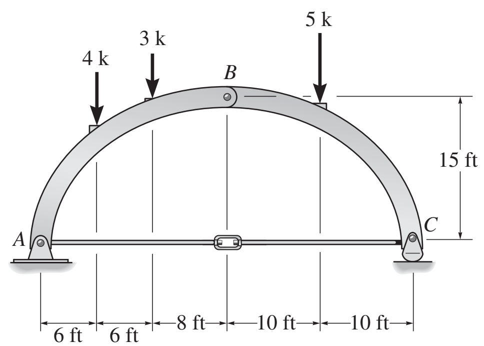
Prob. 5–22

5–23. The three-hinged spandrel arch is subjected to the loading shown. Determine the internal moment in the arch at point D.



Prob. 5–23

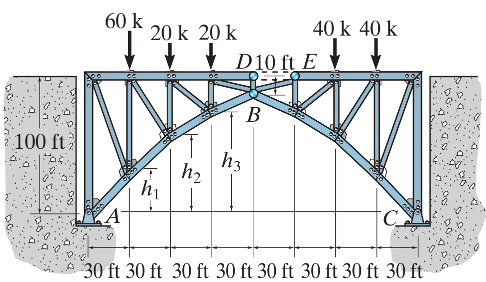
***5–24.** The tied three-hinged arch is subjected to the loading shown. Determine the components of reaction A and C, and the tension in the rod.



Prob. 5–24

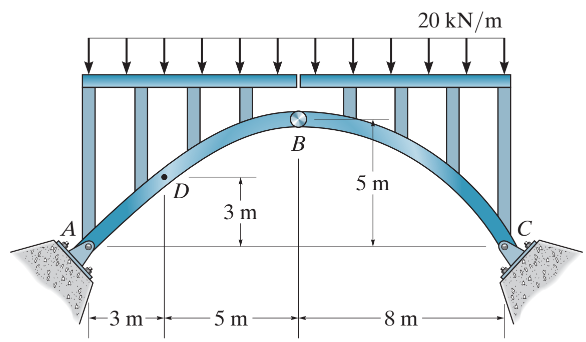
5–25. The bridge is constructed as a *three-hinged trussed arch*. Determine the horizontal and vertical components of reaction at the hinges (pins) at A, B, and C. The dashed member DE is intended to carry *no* force.

5–26. Determine the design heights h_1, h_2, and h_3 of the bottom cord of the truss so the three-hinged trussed arch responds as a funicular arch.



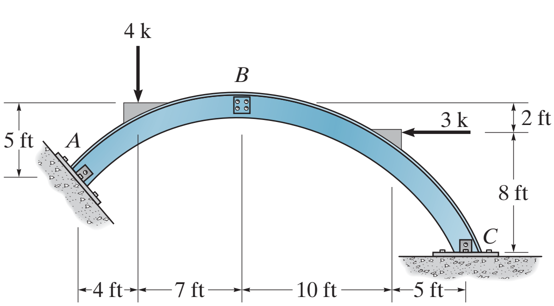
Probs. 5–25/5–26

***5–28.** The three-hinged spandrel arch is subjected to the uniform load of 20 kN/m. Determine the internal moment in the arch at point D.



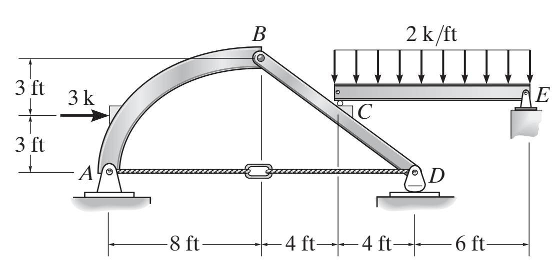
Prob. 5–28

5–27. Determine the horizontal and vertical components of reaction at A, B, and C of the three-hinged arch. Assume A, B, and C are pin connected.



Prob. 5–27

5–29. The arch structure is subjected to the loading shown. Determine the horizontal and vertical components of reaction at A and D, and the tension in the rod AD.

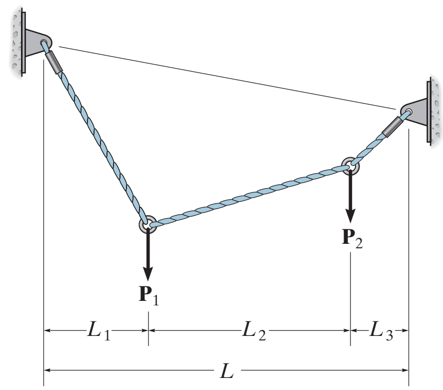


Prob. 5–29

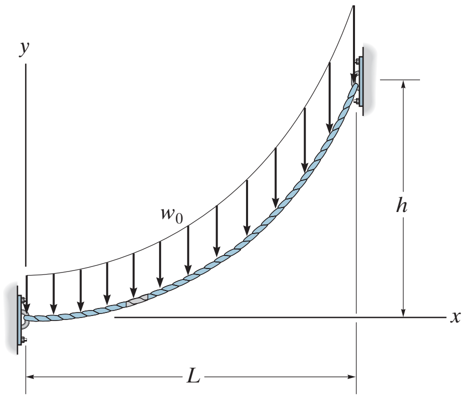
CHAPTER REVIEW

Cables support their loads in tension if we consider them perfectly flexible.

If the cable is subjected to concentrated loads then the force acting in each cable segment is determined by applying the equations of equilibrium to the free-body diagram of groups of segments of the cable or to the joints where the forces are applied.

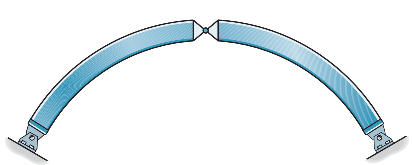


If the cable supports a uniform load over a projected horizontal distance, then the shape of the cable takes the form of a parabola.



Arches are designed primarily to carry a compressive force. A parabolic shape is required to support a uniform loading distributed over its horizontal projection.

Three-hinged arches are statically determinate and can be analyzed by separating the two members and applying the equations of equilibrium to each member.



three-hinged arch

Moving loads caused by trains must be considered when designing the members of this bridge. The influence lines for the members become an important part of the structural analysis.

Influence Lines for Statically Determinate Structures

6

Influence lines have important application for the design of structures that resist large live loads. In this chapter we will discuss how to draw the influence line for a statically determinate structure. The theory is applied to structures subjected to a distributed load or a series of concentrated forces, and specific applications to floor girders and bridge trusses are given. The determination of the absolute maximum live shear and moment in a member is discussed at the end of the chapter.

6.1 Influence Lines

In the previous chapters we developed techniques for analyzing the forces in structural members due to *dead* or *fixed loads*. It was shown that the *shear* and *moment diagrams* represent the most descriptive methods for displaying the variation of these loads in a member. If a structure is subjected to a *live* or *moving load,* however, the variation of the shear and bending moment in the member is best described using the *influence line*. An influence line represents the variation of either the reaction, shear, moment, or deflection at a *specific point* in a member as a concentrated force moves over the member. Once this line is constructed, one can tell at a glance where the moving load should be placed on the structure so that it creates the greatest influence at the specified point. Furthermore, the magnitude of the associated reaction, shear, moment, or deflection at the point can then be calculated from the ordinates of the influence-line diagram. For these reasons, influence lines play an important part in the design of bridges, industrial crane rails, conveyors, and other structures where loads move across their span.

Although the procedure for constructing an influence line is rather basic, one should clearly be aware of the *difference* between constructing an influence line and constructing a shear or moment diagram. Influence lines represent the effect of a *moving load* only at a *specified point* on a member, whereas shear and moment diagrams represent the effect of *fixed loads* at *all points* along the axis of the member.

Procedure for Analysis

Either of the following two procedures can be used to construct the influence line at a specific point P in a member for any function (reaction, shear, or moment). For both of these procedures we will choose the moving force to have a *dimensionless magnitude of unity.**

Tabulate Values

- Place a unit load at various locations, x, along the member, and at *each* location use statics to determine the value of the function (reaction, shear, or moment) at the specified point.
- If the influence line for a vertical force *reaction* at a point on a beam is to be constructed, consider the reaction to be *positive* at the point when it acts *upward* on the beam.
- If a shear or moment influence line is to be drawn for a point, take the shear or moment at the point as positive according to the same sign convention used for drawing shear and moment diagrams. (See Fig. 4–1.)
- All statically determinate beams will have influence lines that consist of straight line segments. After some practice one should be able to minimize computations and locate the unit load *only* at points representing the *end points* of each line segment.
- To avoid errors, it is recommended that one first construct a table, listing "unit load at x" versus the corresponding value of the function calculated at the specific point; that is, "reaction R," "shear V," or "moment M." Once the load has been placed at various points along the span of the member, the tabulated values can be plotted and the influence-line segments constructed.

Influence-Line Equations

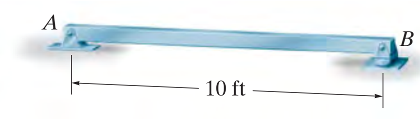
- The influence line can also be constructed by placing the unit load at a *variable* position x on the member and then computing the value of R, V, or M at the point as a function of x. In this manner, the equations of the various line segments composing the influence line can be determined and plotted.

*The reason for this choice will be explained in Sec. 6–2.

6

EXAMPLE 6.1

Construct the influence line for the vertical reaction at A of the beam in Fig. 6–1*a*.

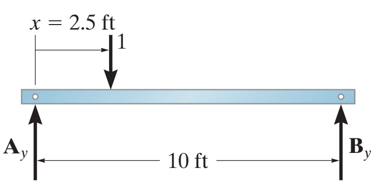


(a)

Fig. 6–1

SOLUTION

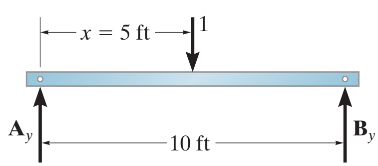
Tabulate Values. A unit load is placed on the beam at each selected point x and the value of A_y is calculated by summing moments about B. For example, when $x = 2.5$ ft and $x = 5$ ft, see Figs. 6–1*b* and 6–1*c*, respectively. The results for A_y are entered in the table, Fig. 6–1*d*. A plot of these values yields the influence line for the reaction at A, Fig. 6–1*e*.



$\circlearrowleft + \Sigma M_B = 0;\ -A_y\,(10) + 1\,(7.5) = 0$

$A_y = 0.75$

(b)



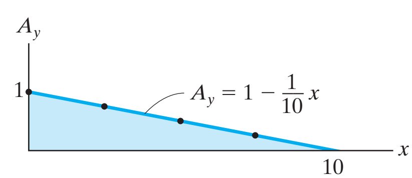
$\circlearrowleft + \Sigma M_B = 0;\ -A_y\,(10) + 1\,(5) = 0$

$A_y = 0.5$

(c)

x	A_y
0	1
2.5	0.75
5	0.5
7.5	0.25
10	0

(d)



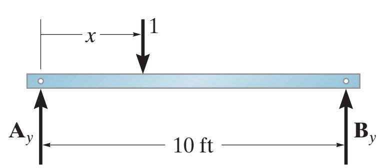
influence line for A_y

(e)

Influence-Line Equation. When the unit load is placed a variable distance x from A, Fig. 6–1*f*, the reaction A_y as a function of x can be determined from

$$\circlearrowleft + \Sigma M_B = 0; \qquad -A_y(10) + (10 - x)(1) = 0$$

$$A_y = 1 - \frac{1}{10}x$$



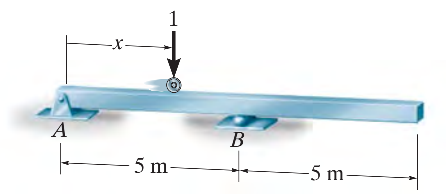
(f)

This line is plotted in Fig. 6–1*e*.

6

EXAMPLE 6.2

Construct the influence line for the vertical reaction at B of the beam in Fig. 6–2*a*.



(a)

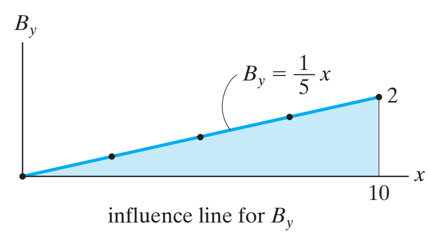
Fig. 6–2

SOLUTION

Tabulate Values. Using statics, verify that the values for the reaction B_y listed in the table, Fig. 6–2*b*, are correctly computed for each position x of the unit load. A plot of the values yields the influence line in Fig. 6–2*c*.

x	B_y
0	0
2.5	0.5
5	1
7.5	1.5
10	2

(b)



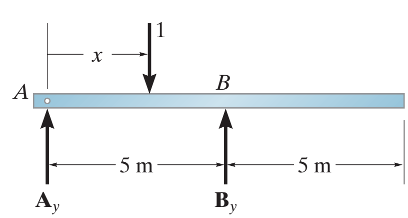
(c)

Influence-Line Equation. Applying the moment equation about A, in Fig. 6–2*d*,

$$\circlearrowleft+\Sigma M_A = 0; \qquad B_y(5) - 1(x) = 0$$

$$B_y = \tfrac{1}{5}x$$

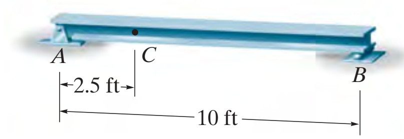
This is plotted in Fig. 6–2*c*.



(d)

EXAMPLE 6.3

Construct the influence line for the shear at point C of the beam in Fig. 6–3a.

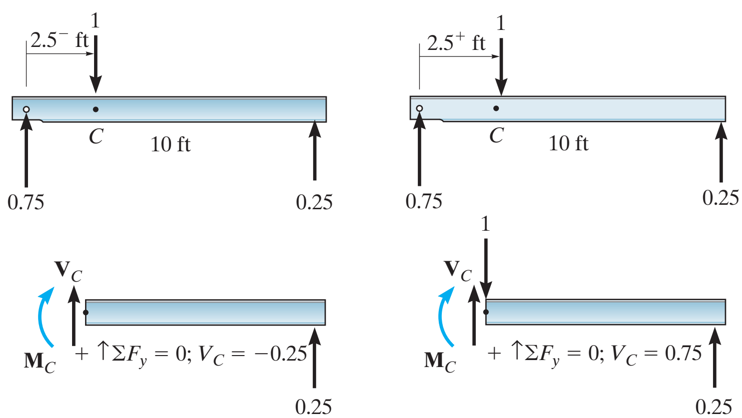


(a)

Fig. 6–3

SOLUTION

Tabulate Values. At each selected position x of the unit load, the method of sections is used to calculate the value of V_C. Note in particular that the unit load must be placed just to the left ($x = 2.5^-$) and just to the right ($x = 2.5^+$) of point C since the shear is discontinuous at C, Figs. 6–3b and 6–3c. A plot of the values in Fig. 6–3d yields the influence line for the shear at C, Fig. 6–3e.

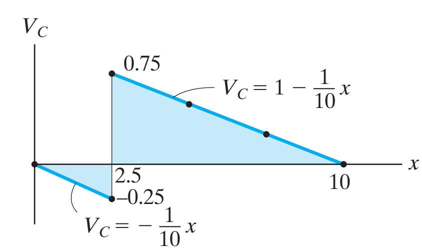


(b)

(c)

x	V_C
0	0
2.5^-	-0.25
2.5^+	0.75
5	0.5
7.5	0.25
10	0

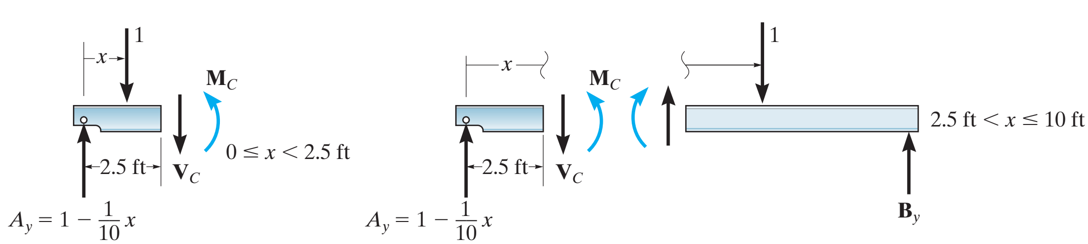
(d)



influence line for V_C

(e)

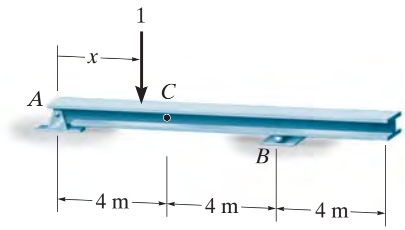
Influence-Line Equations. Here two equations have to be determined since there are two segments for the influence line due to the discontinuity of shear at C, Fig. 6–3f. These equations are plotted in Fig. 6–3e.



(f)

6

EXAMPLE 6.4



(a)

Fig. 6–4

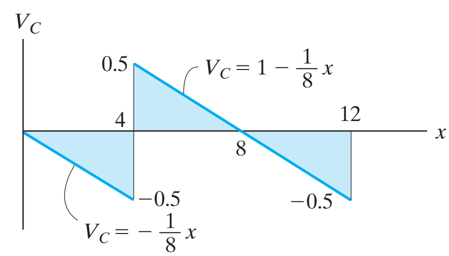
Construct the influence line for the shear at point C of the beam in Fig. 6–4*a*.

SOLUTION

Tabulate Values. Using statics and the method of sections, verify that the values of the shear V_C at point C in Fig. 6–4*b* correspond to each position x of the unit load on the beam. A plot of the values in Fig. 6–4*b* yields the influence line in Fig. 6–4*c*.

x	V_C
0	0
4^-	−0.5
4^+	0.5
8	0
12	−0.5

(b)



influence line for V_C

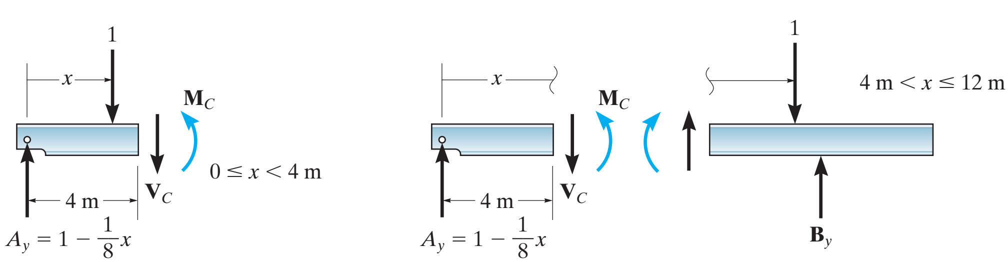
(c)

Influence-Line Equations. From Fig. 6–4*d*, verify that

$$V_C = -\tfrac{1}{8}x \qquad 0 \le x < 4 \text{ m}$$

$$V_C = 1 - \tfrac{1}{8}x \qquad 4 \text{ m} < x \le 12 \text{ m}$$

These equations are plotted in Fig. 6–4*c*.

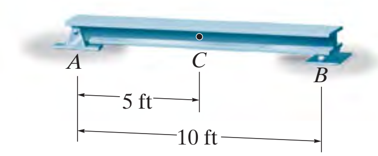


(d)

6

EXAMPLE 6.5

Construct the influence line for the moment at point C of the beam in Fig. 6–5*a*.

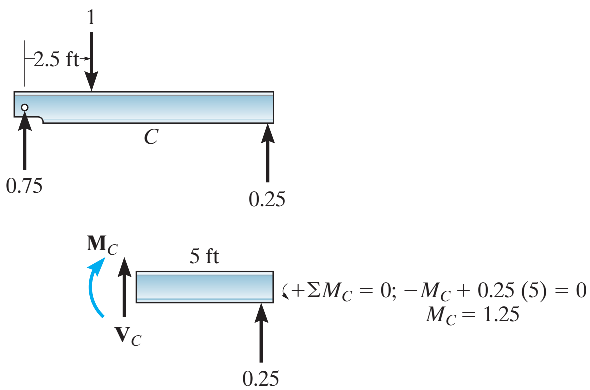


(a)

Fig. 6–5

SOLUTION

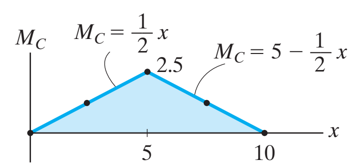
Tabulate Values. At each selected position of the unit load, the value of M_C is calculated using the method of sections. For example, see Fig. 6–5*b* for $x = 2.5$ ft. A plot of the values in Fig. 6–5*c* yields the influence line for the moment at C, Fig. 6–5*d*.



(b)

x	M_C
0	0
2.5	1.25
5	2.5
7.5	1.25
10	0

(c)



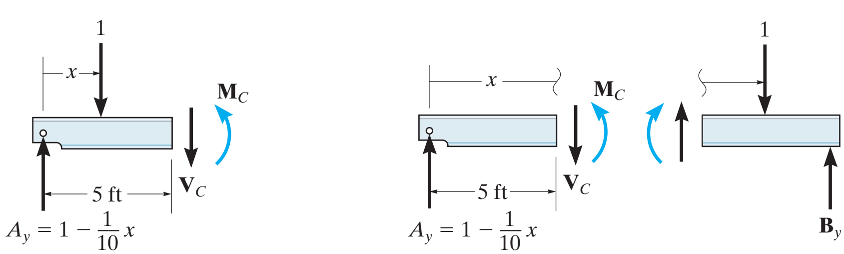
influence line for M_C

(d)

Influence-Line Equations. The two line segments for the influence line can be determined using $\Sigma M_C = 0$ along with the method of sections shown in Fig. 6–5*e*. These equations when plotted yield the influence line shown in Fig. 6–5*d*.

$$\circlearrowleft + \Sigma M_C = 0; \qquad M_C + 1(5 - x) - \left(1 - \tfrac{1}{10}x\right)5 = 0$$
$$M_C = \tfrac{1}{2}x \qquad 0 \le x < 5 \text{ ft}$$

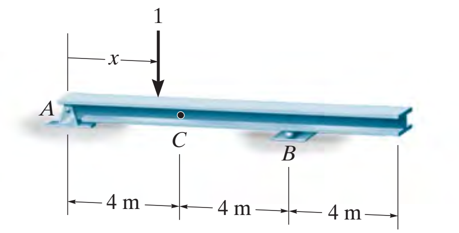
$$\circlearrowleft + \Sigma M_C = 0; \qquad M_C - \left(1 - \tfrac{1}{10}x\right)5 = 0$$
$$M_C = 5 - \tfrac{1}{2}x \qquad 5 \text{ ft} < x \le 10 \text{ ft}$$



(e)

EXAMPLE 6.6

Construct the influence line for the moment at point C of the beam in Fig. 6–6*a*.



(a)

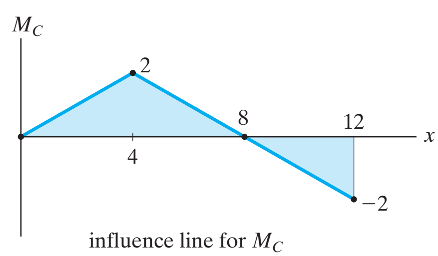
Fig. 6–6

SOLUTION

Tabulate Values. Using statics and the method of sections, verify that the values of the moment M_C at point C in Fig. 6–6*b* correspond to each position x of the unit load. A plot of the values in Fig. 6–6*b* yields the influence line in Fig. 6–6*c*.

x	M_C
0	0
4	2
8	0
12	−2

(b)

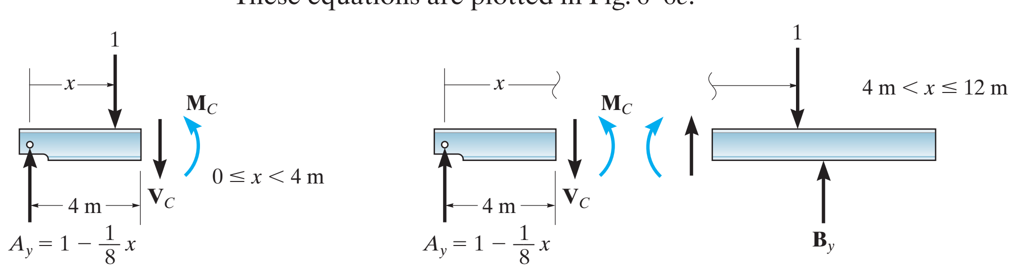


(c)

Influence-Line Equations. From Fig. 6–6*d* verify that

$$M_C = \tfrac{1}{2}x \qquad 0 \le x < 4\text{ m}$$
$$M_C = 4 - \tfrac{1}{2}x \qquad 4\text{ m} < x \le 12\text{ m}$$

These equations are plotted in Fig. 6–6*c*.



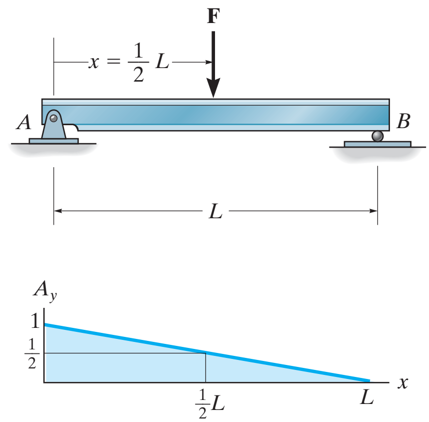
(d)

6.2 Influence Lines for Beams

Since beams (or girders) often form the main load-carrying elements of a floor system or bridge deck, it is important to be able to construct the influence lines for the reactions, shear, or moment at any specified point in a beam.

Loadings. Once the influence line for a function (reaction, shear, or moment) has been constructed, it will then be possible to position the live loads on the beam which will produce the maximum value of the function. Two types of loadings will now be considered.

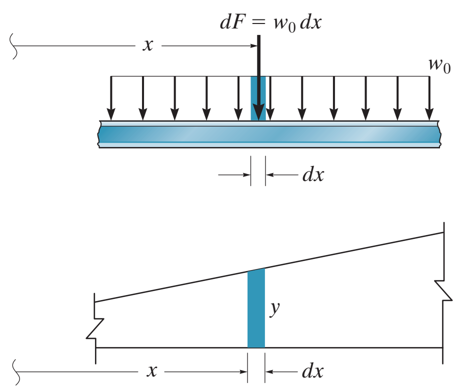
Concentrated Force. Since the numerical values of a function for an influence line are determined using a dimensionless unit load, then for any concentrated force **F** acting on the beam at any position x, *the value of the function can be found by multiplying the ordinate of the influence line at the position x by the magnitude of* **F**. For example, consider the influence line for the reaction at A for the beam AB, Fig. 6–7. If the *unit load* is at $x = \frac{1}{2}L$, the reaction at A is $A_y = \frac{1}{2}$ as indicated from the influence line. Hence, if the force F lb is at this same point, the reaction is $A_y = \left(\frac{1}{2}\right)(F)$ lb. Of course, this same value can also be determined by statics. Obviously, the *maximum influence* caused by **F** occurs when it is placed on the beam at the same location as the *peak* of the influence line—in this case at $x = 0$, where the reaction would be $A_y = (1)(F)$ lb.



influence line for A_y

Fig. 6–7

Uniform Load. Consider a portion of a beam subjected to a uniform load w_0, Fig. 6–8. As shown, each dx segment of this load creates a concentrated force of $dF = w_0\,dx$ on the beam. If $d\mathbf{F}$ is located at x, where the beam's influence-line ordinate for some function (reaction, shear, moment) is y, then the value of the function is $(dF)(y) = (w_0\,dx)y$. The effect of all the concentrated forces $d\mathbf{F}$ is determined by integrating over the entire length of the beam, that is, $\int w_0 y\,dx = w_0 \int y\,dx$. Also, since $\int y\,dx$ is equivalent to the *area* under the influence line, then, in general, *the value of a function caused by a uniform distributed load is simply the area under the influence line for the function multiplied by the intensity of the uniform load*. For example, in the case of a uniformly loaded beam shown in Fig. 6–9, the reaction $\mathbf{A}_y$ can be determined from the influence line as $A_y = (\text{area})(w_0) = \left[\frac{1}{2}(1)(L)\right]w_0 = \frac{1}{2}w_0 L$. This value can of course also be determined from statics.



influence line for function

Fig. 6–8

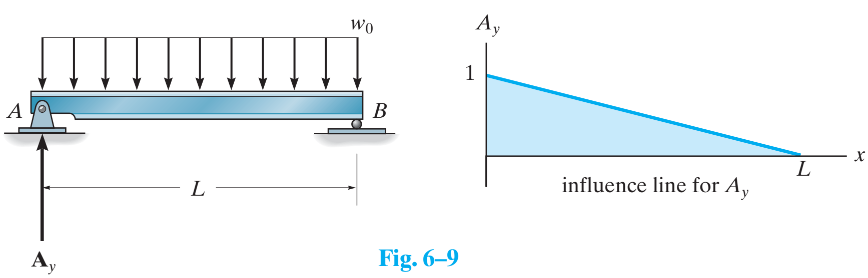
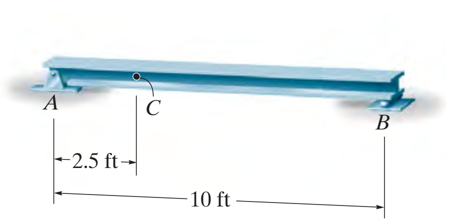


Fig. 6–9

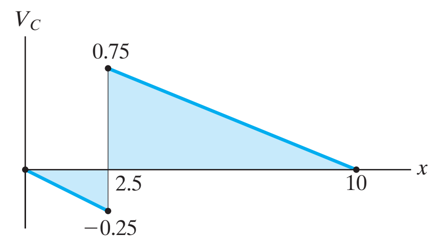
EXAMPLE 6.7

Determine the maximum *positive* shear that can be developed at point C in the beam shown in Fig. 6–10*a* due to a concentrated moving load of 4000 lb and a uniform moving load of 2000 lb/ft.



(a)

Fig. 6–10



influence line for V_C

(b)

SOLUTION

The influence line for the shear at C has been established in Example 6–3 and is shown in Fig. 6–10*b*.

Concentrated Force. The maximum positive shear at C will occur when the 4000-lb force is located at $x = 2.5^+$ ft, since this is the positive peak of the influence line. The ordinate of this peak is +0.75; so that

$$V_C = 0.75(4000 \text{ lb}) = 3000 \text{ lb}$$

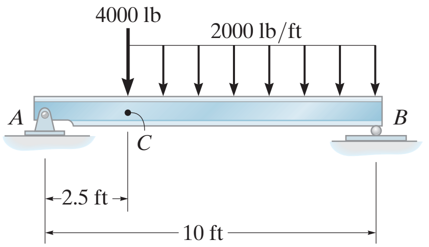
Uniform Load. The uniform moving load creates the maximum positive influence for V_C when the load acts on the beam between $x = 2.5^+$ ft and $x = 10$ ft, since within this region the influence line has a positive area. The magnitude of $\mathbf{V}_C$ due to this loading is

$$V_C = \left[\tfrac{1}{2}(10 \text{ ft} - 2.5 \text{ ft})(0.75)\right]2000 \text{ lb/ft} = 5625 \text{ lb}$$

Total Maximum Shear at C.

$$(V_C)_{\max} = 3000 \text{ lb} + 5625 \text{ lb} = 8625 \text{ lb} \qquad \textit{Ans.}$$

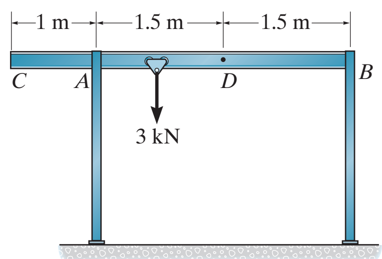
Notice that once the *positions* of the loads have been established using the influence line, Fig. 6–10*c*, this value of $(V_C)_{\max}$ can *also* be determined using statics and the method of sections. Show that this is the case.



(c)

EXAMPLE 6.8

The frame structure shown in Fig. 6–11*a* is used to support a hoist for transferring loads for storage at points underneath it. It is anticipated that the load on the dolly is 3 kN and the beam *CB* has a mass of 24 kg/m. Assume the dolly has negligible size and can travel the entire length of the beam. Also, assume *A* is a pin and *B* is a roller. Determine the maximum vertical support reactions at *A* and *B* and the maximum moment in the beam at *D*.



(a)

SOLUTION

Maximum Reaction at *A*. We first draw the influence line for A_y, Fig. 6–11*b*. Specifically, when a unit load is at *A* the reaction at *A* is 1 as shown. The ordinate at *C*, is 1.33. Here the maximum value for A_y occurs when the dolly is at *C*. Since the dead load (beam weight) must be placed over the entire length of the beam, we have,

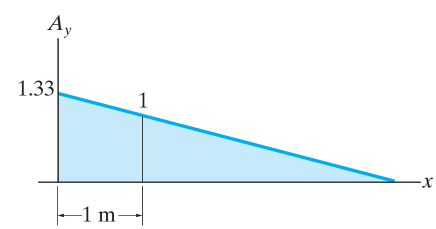
$$(A_y)_{\max} = 3000(1.33) + 24(9.81)\left[\tfrac{1}{2}(4)(1.33)\right]$$
$$= 4.63\text{ kN} \qquad \textit{Ans.}$$

Maximum Reaction at *B*. The influence line (or beam) takes the shape shown in Fig. 6–11*c*. The values at *C* and *B* are determined by statics. Here the dolly must be at *B*. Thus,

$$(B_y)_{\max} = 3000(1) + 24(9.81)\left[\tfrac{1}{2}(3)(1)\right] + 24(9.81)\left[\tfrac{1}{2}(1)(-0.333)\right]$$
$$= 3.31\text{ kN} \qquad \textit{Ans.}$$

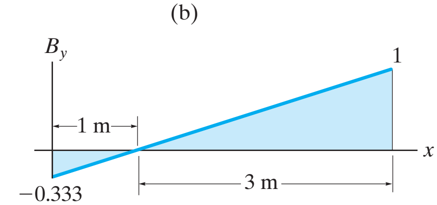
Maximum Moment at *D*. The influence line has the shape shown in Fig. 6–11*d*. The values at *C* and *D* are determined from statics. Here,

$$(M_D)_{\max} = 3000(0.75) + 24(9.81)\left[\tfrac{1}{2}(1)(-0.5)\right] + 24(9.81)\left[\tfrac{1}{2}(3)(0.75)\right]$$
$$= 2.46\text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$



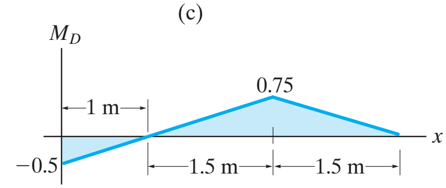
influence line for A_y

(b)



influence line for B_y

(c)



influence line for M_D

(d)

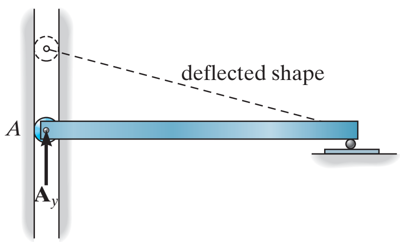
Fig. 6–11

6.3 Qualitative Influence Lines

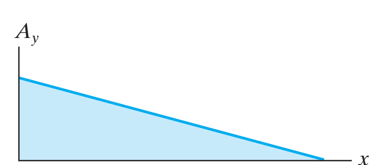
In 1886, Heinrich Müller-Breslau developed a technique for rapidly constructing the shape of an influence line. Referred to as the *Müller-Breslau principle, it states that the influence line for a function (reaction, shear, or moment) is to the same scale as the deflected shape of the beam when the beam is acted upon by the function.* In order to draw the deflected shape properly, the capacity of the beam to resist the applied function must be *removed* so the beam can deflect when the function is applied. For example, consider the beam in Fig. 6–12*a*. If the shape of the influence line for the *vertical reaction* at A is to be determined, the pin is first replaced by a *roller guide* as shown in Fig. 6–12*b*. A roller guide is necessary since the beam must still resist a horizontal force at A but *no vertical force*. When the positive (upward) force $\mathbf{A}_y$ is then applied at A, the beam deflects to the dashed position,* which represents the general shape of the influence line for A_y, Fig. 6–12*c*. (Numerical values for this specific case have been calculated in Example 6–1.) If the shape of the influence line for the *shear* at C is to be determined, Fig. 6–13*a*, the connection at C may be symbolized by a *roller guide* as shown in Fig. 6–13*b*. This device will resist a moment and axial force but no *shear*.† Applying a positive shear force $\mathbf{V}_C$ to the beam at C and allowing the beam to deflect to the dashed position, we find the influence-line shape as shown in Fig. 6–13*c*. Finally, if the shape of the influence line for the *moment* at C, Fig. 6–14*a*, is to be determined, an internal *hinge* or *pin* is placed at C, since this connection resists axial and shear forces but *cannot resist a moment,* Fig. 6–14*b*. Applying positive moments $\mathbf{M}_C$ to the beam, the beam then deflects to the dashed position, which is the shape of the influence line, Fig. 6–14*c*.

The proof of the Müller-Breslau principle can be established using the principle of virtual work. Recall that *work* is the product of either *a linear*

(a)

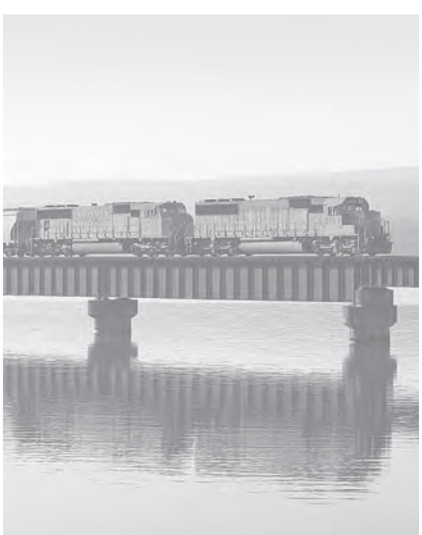


(b)



(c)

Fig. 6–12



Design of this bridge girder is based on influence lines that must be constructed for this train loading.

*Throughout the discussion all deflected positions are drawn to an exaggerated scale to illustrate the concept.

†Here the rollers *symbolize* supports that carry loads both in tension or compression. See Table 2–1, support (2).

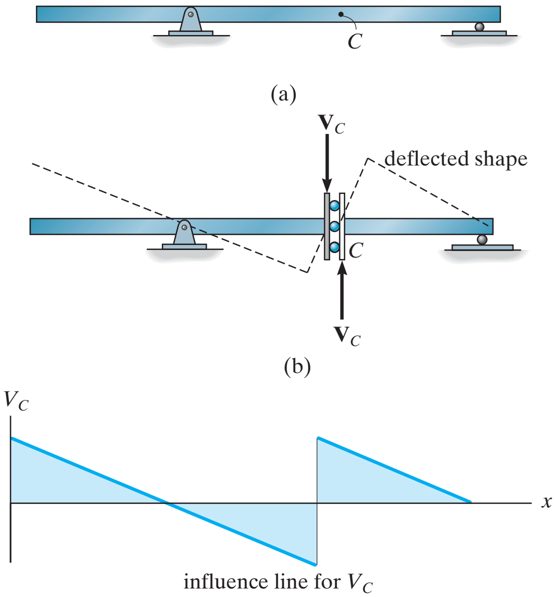


Fig. 6–13

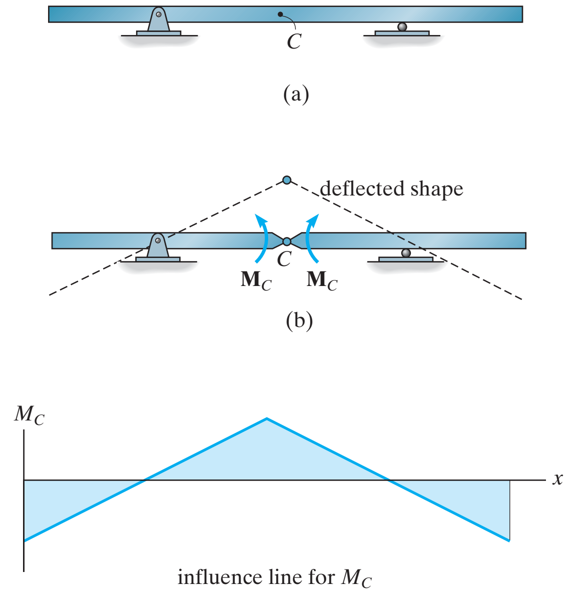


Fig. 6–14

displacement and force in the direction of the displacement or *a rotational displacement and moment in the direction of the displacement*. If a rigid body (beam) is in equilibrium, the sum of all the forces and moments on it must be equal to zero. Consequently, if the body is given an *imaginary* or *virtual displacement,* the work done by *all* these forces and couple moments must also be equal to zero. Consider, for example, the simply supported beam shown in Fig. 6–15*a*, which is subjected to a unit load placed at an arbitrary point along its length. If the beam is given a virtual (or imaginary) displacement δy at the support A, Fig. 6–15*b*, then only the support reaction $\mathbf{A}_y$ and the unit load do virtual work. Specifically, A_y does positive work $A_y\,\delta y$ and the unit load does negative work, $-1\delta y'$. (The support at B does not move and therefore the force at B does no work.) Since the beam is in equilibrium and therefore does not actually move, the virtual work sums to zero, i.e.,

$$A_y\,\delta y - 1\,\delta y' = 0$$

If δy is set equal to 1, then

$$A_y = \delta y'$$

In other words, the value of A_y represents the ordinate of the influence line at the position of the unit load. Since this value is equivalent to the displacement $\delta y'$ at the position of the unit load, it shows that the *shape* of the influence line for the reaction at A has been established. This proves the Müller-Breslau principle for reactions.

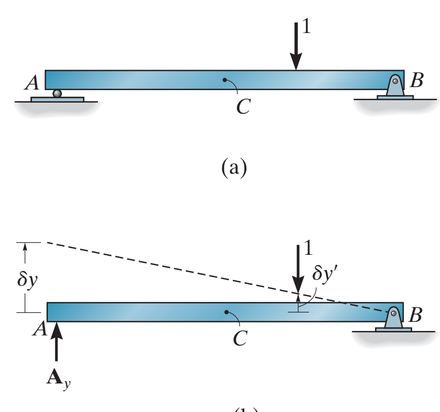
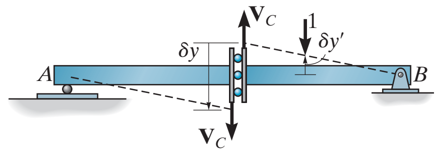


Fig. 6–15



(c)

Fig. 6–15

In the same manner, if the beam is sectioned at C, and the beam undergoes a virtual displacement δy at this point, Fig. 6–15*c*, then only the internal shear at C and the unit load do work. Thus, the virtual work equation is

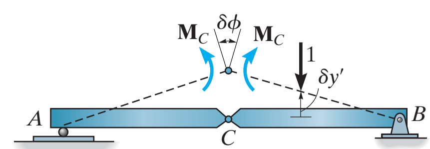
$$V_C\,\delta y - 1\,\delta y' = 0$$

Again, if $\delta y = 1$, then

$$V_C = \delta y'$$

and the *shape* of the influence line for the shear at C has been established.

6



(d)

Lastly, assume a hinge or pin is introduced into the beam at point C, Fig. 6–15*d*. If a virtual rotation $\delta\phi$ is introduced at the pin, virtual work will be done only by the internal moment and the unit load. So

$$M_C\,\delta\phi - 1\,\delta y' = 0$$

Setting $\delta\phi = 1$, it is seen that

$$M_C = \delta y'$$

which indicates that the deflected beam has the same *shape* as the influence line for the internal moment at point C (see Fig. 6–14).

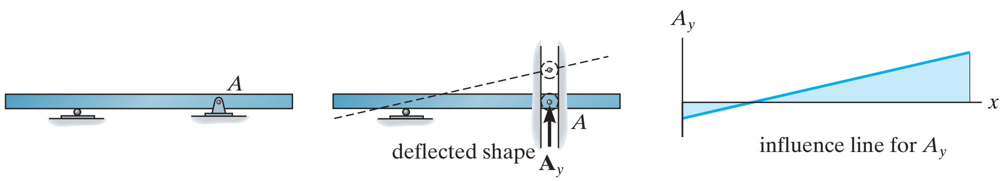
Obviously, the Müller-Breslau principle provides a quick method for establishing the *shape* of the influence line. Once this is known, the ordinates at the peaks can be determined by using the basic method discussed in Sec. 6–1. Also, by simply knowing the general shape of the influence line, it is possible to *locate* the live load on the beam and then determine the maximum value of the function by *using statics*. Example 6–12 illustrates this technique.

EXAMPLE 6.9

For each beam in Fig. 6–16*a* through 6–16*c*, sketch the influence line for the vertical reaction at A.

SOLUTION

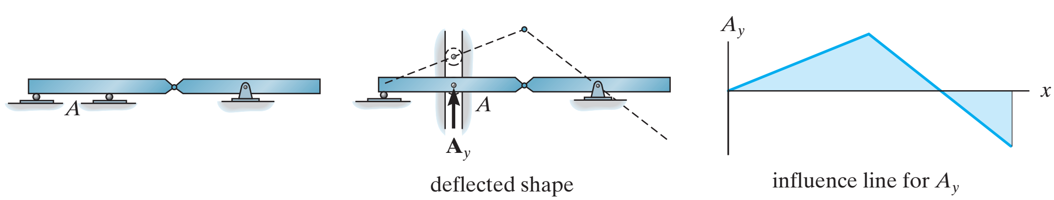
The support is replaced by a roller guide at A since it will resist $\mathbf{A}_x$, but not $\mathbf{A}_y$. The force $\mathbf{A}_y$ is then applied.



(a)

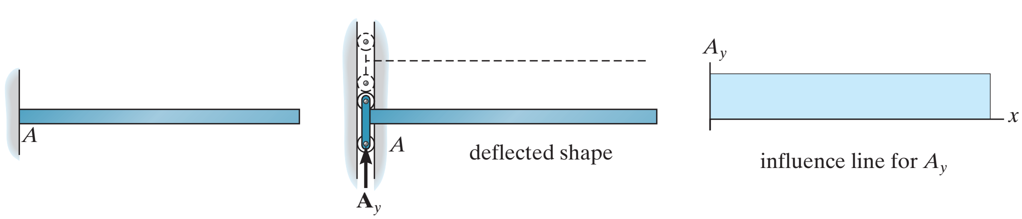
Fig. 6–16

Again, a roller guide is placed at A and the force $\mathbf{A}_y$ is applied.



(b)

A *double-roller guide* must be used at A in this case, since this type of support will resist both a moment $\mathbf{M}_A$ at the fixed support and axial load $\mathbf{A}_x$, but will not resist $\mathbf{A}_y$.



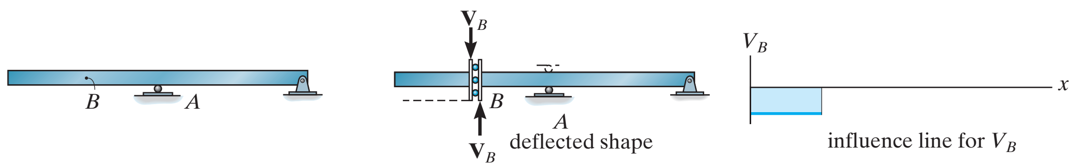
(c)

EXAMPLE 6.10

For each beam in Figs. 6–17*a* through 6–17*c*, sketch the influence line for the shear at *B*.

SOLUTION

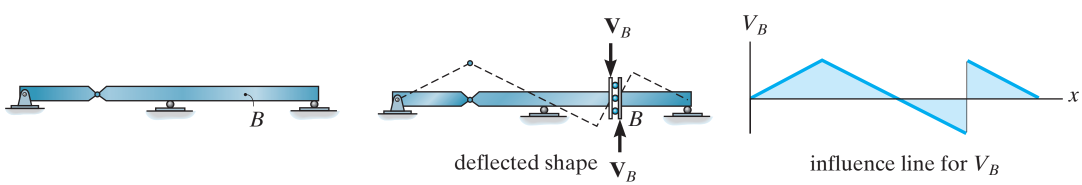
The roller guide is introduced at *B* and the positive shear $\mathbf{V}_B$ is applied. Notice that the right segment of the beam will *not deflect* since the roller at *A* actually constrains the beam from moving vertically, either up or down. [See support (2) in Table 2–1.]



(a)

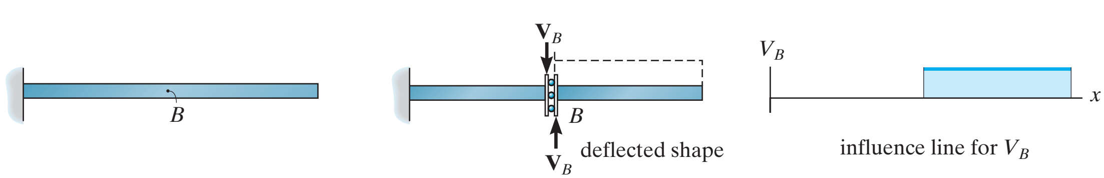
Fig. 6–17

Placing the roller guide at *B* and applying the positive shear at *B* yields the deflected shape and corresponding influence line.



(b)

Again, the roller guide is placed at *B*, the positive shear is applied, and the deflected shape and corresponding influence line are shown. Note that the left segment of the beam does not deflect, due to the fixed support.



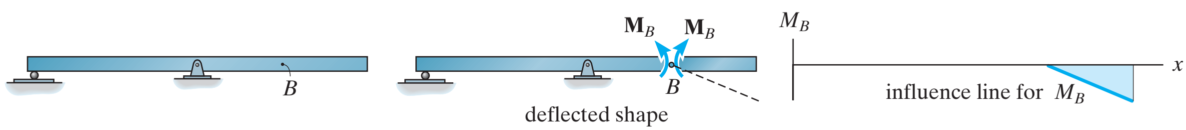
(c)

EXAMPLE 6.11

For each beam in Figs. 6–18*a* through 6–18*c*, sketch the influence line for the moment at *B*.

SOLUTION

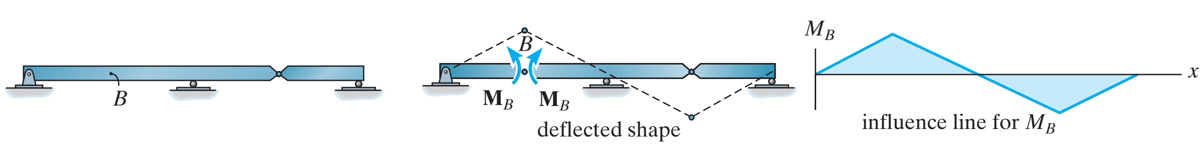
A hinge is introduced at *B* and positive moments $\mathbf{M}_B$ are applied to the beam. The deflected shape and corresponding influence line are shown.



(a)

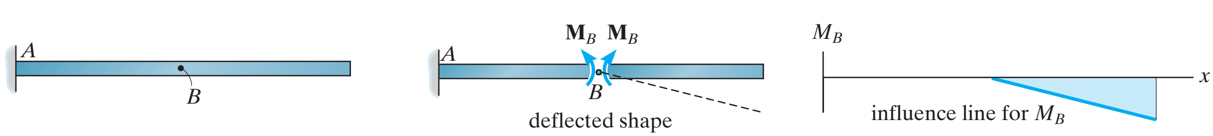
Fig. 6–18

Placing a hinge at *B* and applying positive moments $\mathbf{M}_B$ to the beam yields the deflected shape and influence line.



(b)

With the hinge and positive moment at *B* the deflected shape and influence line are shown. The left segment of the beam is constrained from moving due to the fixed wall at *A*.

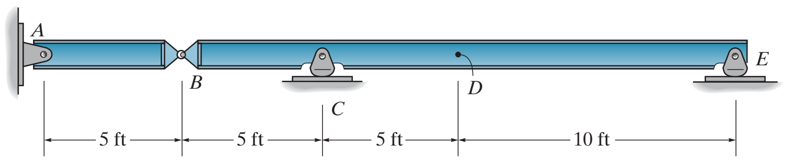


(c)

6

EXAMPLE 6.12

Determine the maximum positive moment that can be developed at point D in the beam shown in Fig. 6–19*a* due to a concentrated moving load of 4000 lb, a uniform moving load of 300 lb/ft, and a beam weight of 200 lb/ft.



(a)

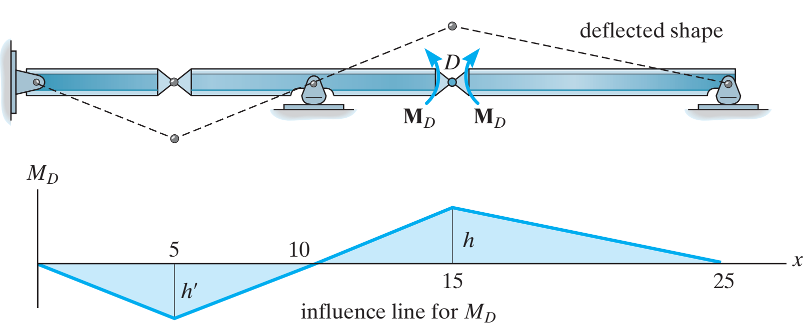
Fig. 6–19

SOLUTION

A hinge is placed at D and positive moments $\mathbf{M}_D$ are applied to the beam. The deflected shape and corresponding influence line are shown in Fig. 6–19*b*. Immediately one recognizes that the concentrated moving load of 4000 lb creates a maximum *positive* moment at D when it is placed at D, i.e., the peak of the influence line. Also, the uniform moving load of 300 lb/ft must extend from C to E in order to cover the region where the area of the influence line is positive. Finally, the uniform weight of 200 lb/ft acts over the *entire length* of the beam. The loading is shown on the beam in Fig. 6–19*c*. Knowing the position of the loads, we can now determine the maximum moment at D using statics. In Fig. 6–19*d* the reactions on BE have been computed. Sectioning the beam at D and using segment DE, Fig. 6–19*e*, we have

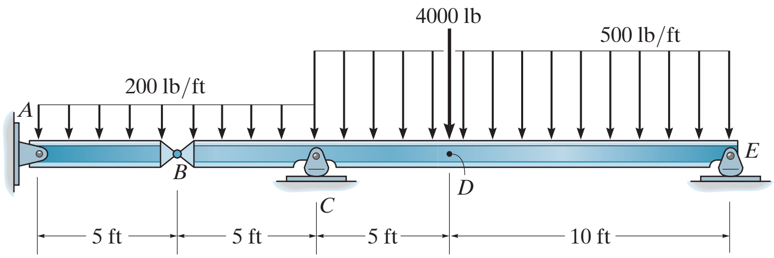
$$\circlearrowleft + \Sigma M_D = 0; \qquad -M_D - 5000(5) + 4750(10) = 0$$

$$M_D = 22\,500 \text{ lb}\cdot\text{ft} = 22.5 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

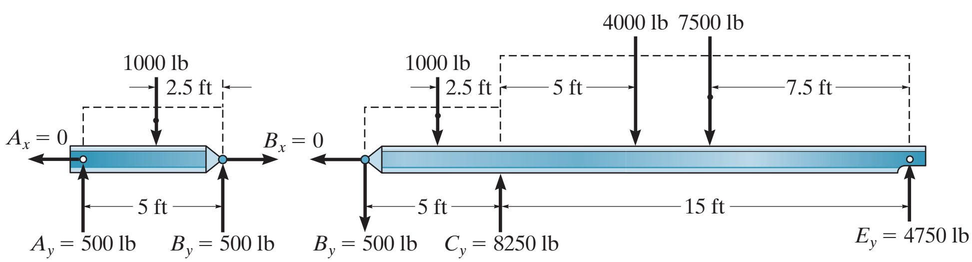


(b)

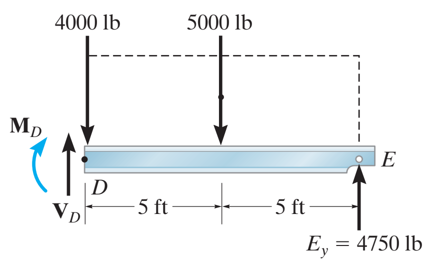
6



(c)



(d)



(e)

This problem can also be worked by using *numerical values* for the influence line as in Sec. 6–1. Actually, by inspection of Fig. 6–19*b*, only the peak value h at D must be determined. This requires placing a unit load on the beam at D in Fig. 6–19*a* and then solving for the internal moment in the beam at D. Show that the value obtained is $h = 3.33$. By proportional triangles, $h'/(10 - 5) = 3.33/(15 - 10)$ or $h' = 3.33$. Hence, with the loading on the beam as in Fig. 6–19*c*, using the areas and peak values of the influence line, Fig. 6–19*b*, we have

$$M_D = 500\left[\tfrac{1}{2}(25 - 10)(3.33)\right] + 4000(3.33) - 200\left[\tfrac{1}{2}(10)(3.33)\right]$$
$$= 22\,500 \text{ lb}\cdot\text{ft} = 22.5 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

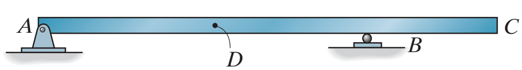
6

FUNDAMENTAL PROBLEMS

F6–1. Use the Müller-Breslau principle to sketch the influence lines for the vertical reaction at *A*, the shear at *C*, and the moment at *C*.

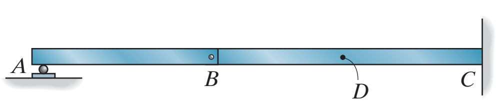
F6–1

F6–2. Use the Müller-Breslau principle to sketch the influence lines for the vertical reaction at *A*, the shear at *D*, and the moment at *B*.



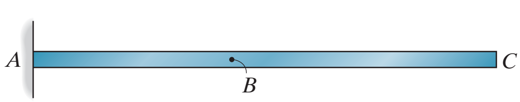
F6–2

F6–3. Use the Müller-Breslau principle to sketch the influence lines for the vertical reaction at *A*, the shear at *D*, and the moment at *D*.



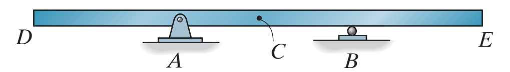
F6–3

F6–4. Use the Müller-Breslau principle to sketch the influence lines for the vertical reaction at *A*, the shear at *B*, and the moment at *B*.



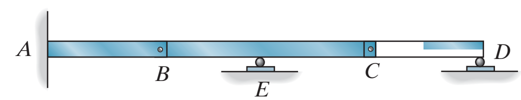
F6–4

F6–5. Use the Müller-Breslau principle to sketch the influence lines for the vertical reaction at *A*, the shear at *C*, and the moment at *C*.



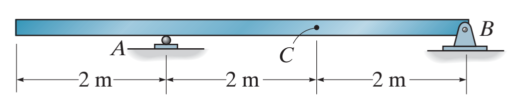
F 6–5

F6–6. Use the Müller-Breslau principle to sketch the influence lines for the vertical reaction at *A*, the shear just to the left of the roller support at *E*, and the moment at *A*.



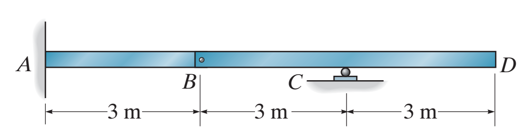
F6–6

F6–7. The beam supports a distributed live load of 1.5 kN/m and single concentrated load of 8 kN. The dead load is 2 kN/m. Determine (a) the maximum positive moment at *C*, (b) the maximum positive shear at *C*.



F6–7

F6–8. The beam supports a distributed live load of 2 kN/m and single concentrated load of 6 kN. The dead load is 4 kN/m. Determine (a) the maximum vertical positive reaction at *C*, (b) the maximum negative moment at *A*.

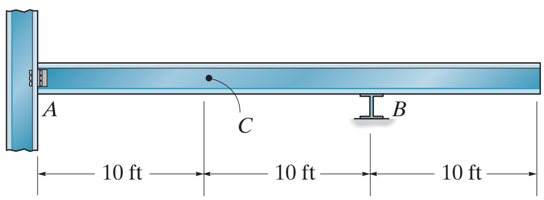


F6–8

PROBLEMS

6–1. Draw the influence lines for (a) the moment at *C*, (b) the reaction at *B*, and (c) the shear at *C*. Assume *A* is pinned and *B* is a roller. Solve this problem using the basic method of Sec. 6–1.

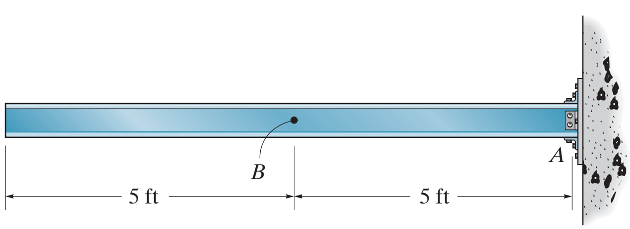
6–2. Solve Prob. 6–1 using the Müller-Breslau principle.



Probs. 6–1/6–2

6–3. Draw the influence lines for (a) the vertical reaction at *A*, (b) the moment at *A*, and (c) the shear at *B*. Assume the support at *A* is fixed. Solve this problem using the basic method of Sec. 6–1.

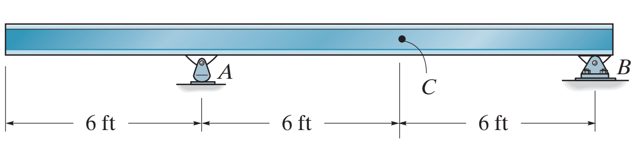
***6–4.** Solve Prob. 6–3 using the Müller-Breslau principle.



Probs. 6–3/6–4

6–5. Draw the influence lines for (a) the vertical reaction at *B*, (b) the shear just to the right of the rocker at *A*, and (c) the moment at *C*. Solve this problem using the basic method of Sec. 6–1.

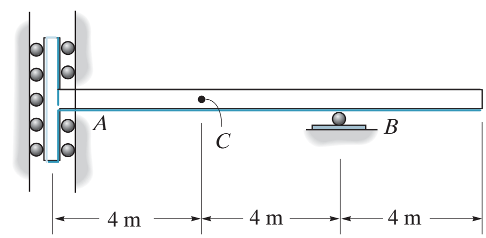
6–6. Solve Prob. 6–5 using Müller-Breslau's principle.



Probs. 6–5/6–6

6–7. Draw the influence line for (a) the moment at *B*, (b) the shear at *C*, and (c) the vertical reaction at *B*. Solve this problem using the basic method of Sec. 6–1. *Hint*: The support at *A* resists only a horizontal force and a bending moment.

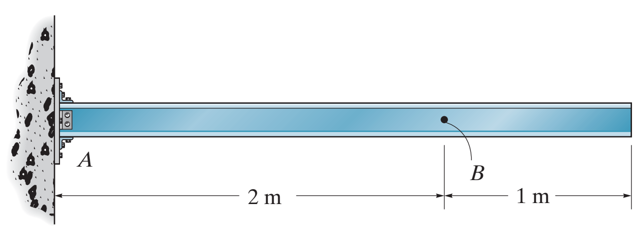
***6–8.** Solve Prob. 6–7 using the Müller-Breslau principle.



Probs. 6–7/6–8

6–9. Draw the influence line for (a) the vertical reaction at *A*, (b) the shear at *B*, and (c) the moment at *B*. Assume *A* is fixed. Solve this problem using the basic method of Sec. 6–1.

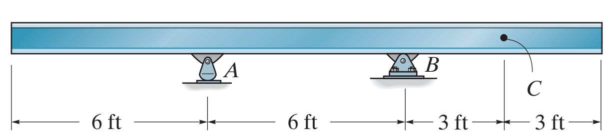
6–10. Solve Prob. 6–9 using the Müller-Breslau principle.



Probs. 6–9/6–10

6–11. Draw the influence lines for (a) the vertical reaction at *A*, (b) the shear at *C*, and (c) the moment at *C*. Solve this problem using the basic method of Sec. 6–1.

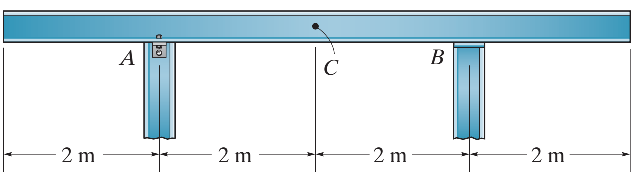
***6–12.** Solve Prob. 6–11 using Müller-Breslau's principle.



Probs. 6–11/6–12

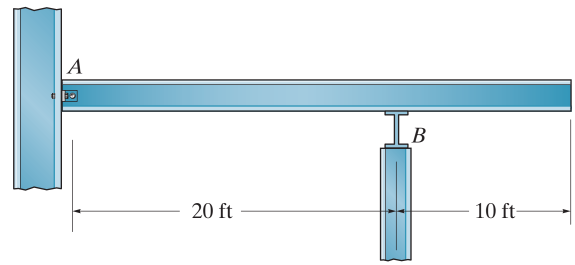
6–13. Draw the influence lines for (a) the vertical reaction at A, (b) the vertical reaction at B, (c) the shear just to the right of the support at A, and (d) the moment at C. Assume the support at A is a pin and B is a roller. Solve this problem using the basic method of Sec. 6–1.

6–14. Solve Prob. 6–13 using the Müller-Breslau principle.



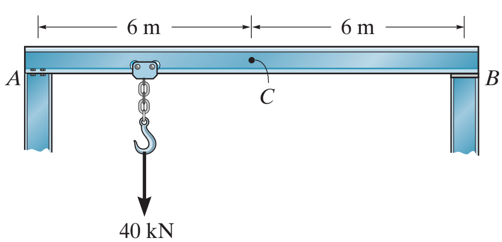
Probs. 6–13/6–14

6–17. A uniform live load of 300 lb/ft and a single live concentrated force of 1500 lb are to be placed on the beam. The beam has a weight of 150 lb/ft. Determine (a) the maximum vertical reaction at support B, and (b) the maximum negative moment at point B. Assume the support at A is a pin and B is a roller.



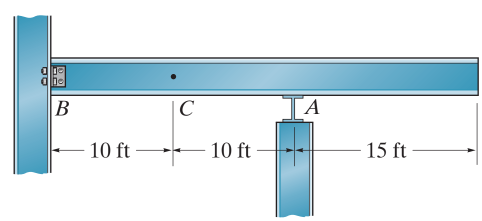
Prob. 6–17

6–15. The beam is subjected to a uniform dead load of 1.2 kN/m and a single live load of 40 kN. Determine (a) the maximum moment created by these loads at C, and (b) the maximum positive shear at C. Assume A is a pin. and B is a roller.



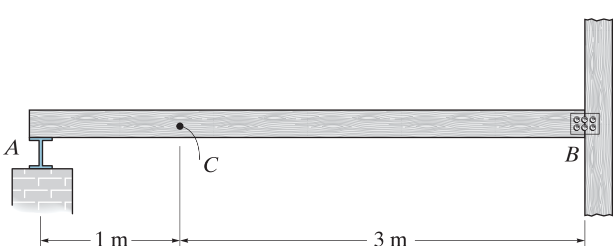
Prob. 6–15

6–18. The beam supports a uniform dead load of 0.4 k/ft, a live load of 1.5 k/ft, and a single live concentrated force of 8 k. Determine (a) the maximum positive moment at C, and (b) the maximum positive vertical reaction at B. Assume A is a roller and B is a pin.



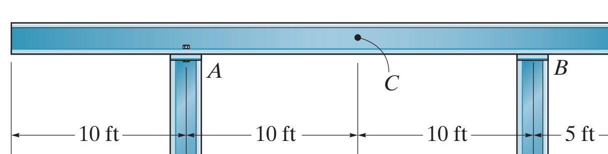
Prob. 6–18

***6–16.** The beam supports a uniform dead load of 500 N/m and a single live concentrated force of 3000 N. Determine (a) the maximum positive moment at C, and (b) the maximum positive shear at C. Assume the support at A is a roller and B is a pin.



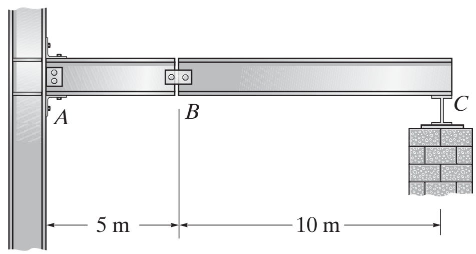
Prob. 6–16

6–19. The beam is used to support a dead load of 0.6 k/ft, a live load of 2 k/ft and a concentrated live load of 8 k. Determine (a) the maximum positive (upward) reaction at A, (b) the maximum positive moment at C, and (c) the maximum positive shear just to the right of the support at A. Assume the support at A is a pin and B is a roller.



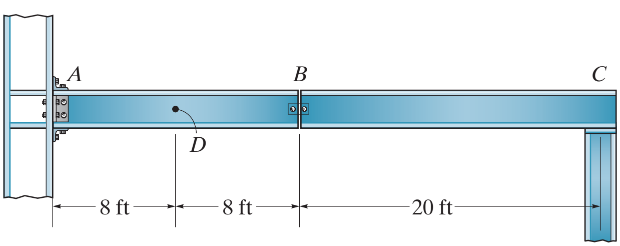
Prob. 6–19

***6–20.** The compound beam is subjected to a uniform dead load of 1.5 kN/m and a single live load of 10 kN. Determine (a) the maximum negative moment created by these loads at *A*, and (b) the maximum positive shear at *B*. Assume *A* is a fixed support, *B* is a pin, and *C* is a roller.



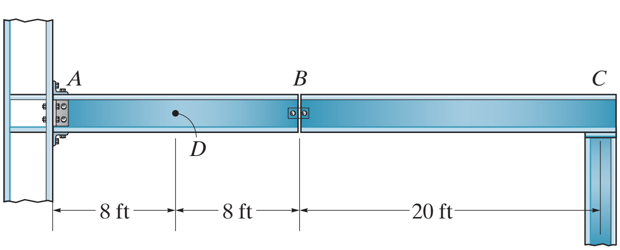
Prob. 6–20

6–21. Where should a single 500-lb live load be placed on the beam so it causes the largest moment at *D*? What is this moment? Assume the support at *A* is fixed, *B* is pinned, and *C* is a roller.



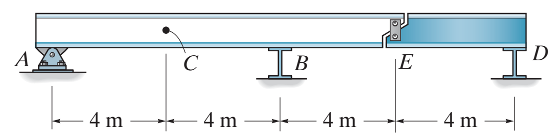
Prob. 6–21

6–22. Where should the beam *ABC* be loaded with a 300 lb/ft uniform distributed live load so it causes (a) the largest moment at point *A* and (b) the largest shear at *D*? Calculate the values of the moment and shear. Assume the support at *A* is fixed, *B* is pinned and *C* is a roller.



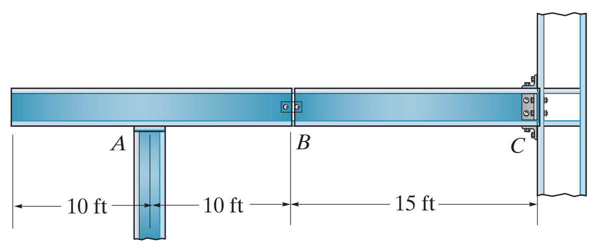
Prob. 6–22

6–23. The beam is used to support a dead load of 800 N/m, a live load of 4 kN/m, and a concentrated live load of 20 kN. Determine (a) the maximum positive (upward) reaction at *B*, (b) the maximum positive moment at *C*, and (c) the maximum negative shear at *C*. Assume *B* and *D* are pins.



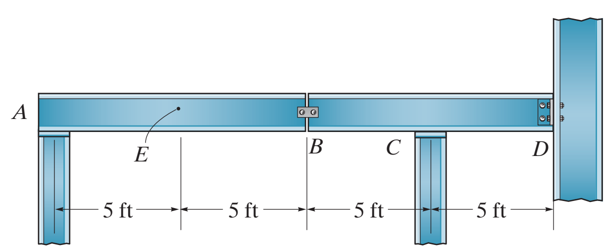
Prob. 6–23

***6–24.** The beam is used to support a dead load of 400 lb/ft, a live load of 2 k/ft, and a concentrated live load of 8 k. Determine (a) the maximum positive vertical reaction at *A*, (b) the maximum positive shear just to the right of the support at *A*, and (c) the maximum negative moment at *C*. Assume *A* is a roller, *C* is fixed, and *B* is pinned.



Prob. 6–24

6–25. The beam is used to support a dead load of 500 lb/ft, a live load of 2 k/ft, and a concentrated live load of 8 k. Determine (a) the maximum positive (upward) reaction at *A*, (b) the maximum positive moment at *E*, and (c) the maximum positive shear just to the right of the support at *C*. Assume *A* and *C* are rollers and *D* is a pin.



Prob. 6–25

6.4 Influence Lines for Floor Girders

Occasionally, floor systems are constructed as shown in Fig. 6–20*a*, where it can be seen that floor loads are transmitted from *slabs* to *floor beams,* then to *side girders,* and finally supporting *columns*. An idealized model of this system is shown in plane view, Fig. 6–20*b*. Here the slab is assumed to be a one-way slab and is segmented into simply supported spans resting on the floor beams. Furthermore, the girder is simply supported on the columns. Since the girders are main load-carrying members in this system, it is sometimes necessary to construct their shear and moment influence lines. This is especially true for industrial buildings subjected to heavy concentrated loads. In this regard, notice that a unit load on the floor slab is transferred to the girder only at points where it is in contact with the floor beams, i.e., points A, B, C, and D. These points are called *panel points,* and the region between these points is called a *panel,* such as BC in Fig. 6–20*b*.

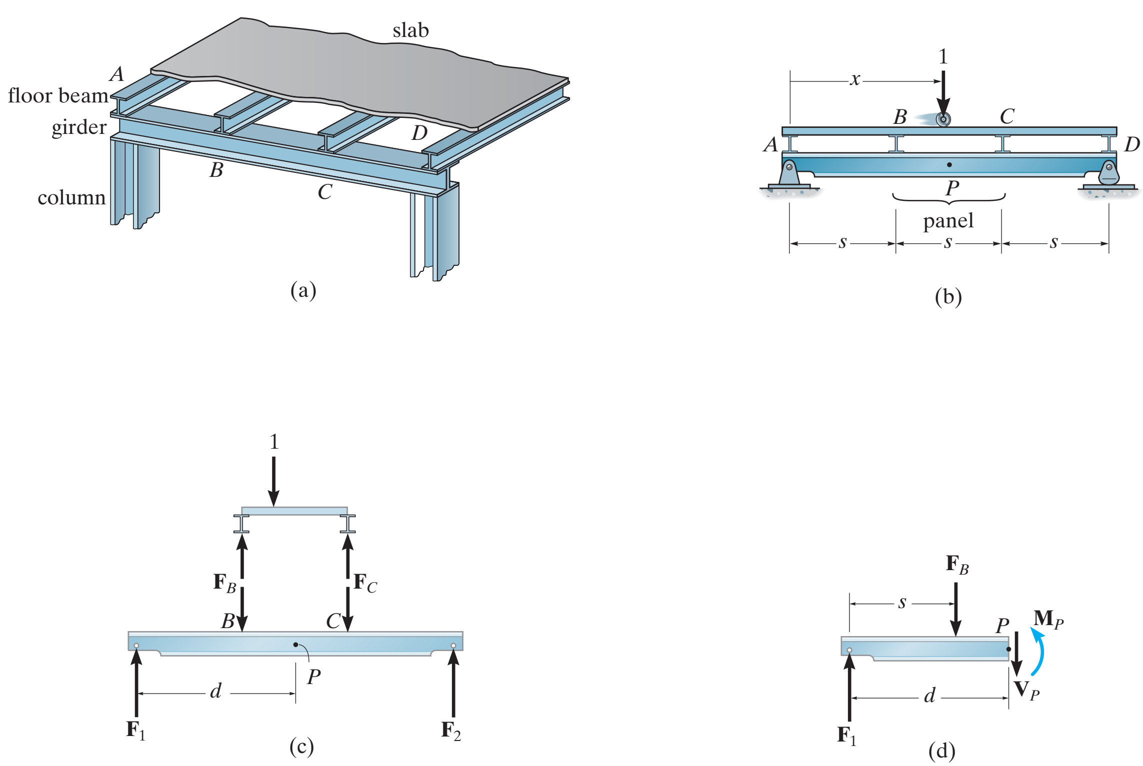


Fig. 6–20

The influence line for a specified point on the girder can be determined using the same statics procedure as in Sec. 6–1; i.e., place the unit load at various points x on the floor slab and always compute the function (shear or moment) at the specified point P in the girder, Fig. 6–20*b*. Plotting these values versus x yields the influence line for the function at P. In particular, the value for the internal moment in a girder panel will depend upon where point P is chosen for the influence line, since the magnitude of $\mathbf{M}_P$ depends upon the point's location from the end of the girder. For example, if the unit load acts on the floor slab as shown in Fig. 6–20*c*, one first finds the reactions $\mathbf{F}_B$ and $\mathbf{F}_C$ on the slab, then calculates the support reactions $\mathbf{F}_1$ and $\mathbf{F}_2$ on the girder. The internal moment at P is then determined by the method of sections, Fig. 6–20*d*. This gives $M_P = F_1 d - F_B(d - s)$. Using a similar analysis, the internal shear $\mathbf{V}_P$ can be determined. In this case, however, $\mathbf{V}_P$ will be *constant* throughout the panel $BC(V_P = F_1 - F_B)$ and so it does not depend upon the exact location d of P within the panel. For this reason, influence lines for shear in floor girders are specified for *panels* in the girder and not specific points along the girder. The shear is then referred to as *panel shear*. It should also be noted that since the girder is affected only by the loadings transmitted by the floor beams, the unit load is generally placed at each floor-beam location to establish the necessary data used to draw the influence line.

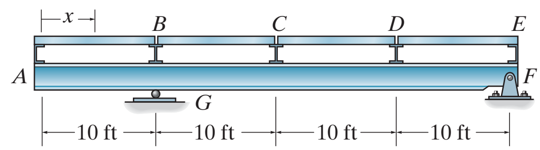
The following numerical examples should clarify the force analysis.

6

The design of the floor system of this warehouse building must account for critical locations of storage materials on the floor. Influence lines must be used for this purpose. (*Photo courtesy of Portland Cement Association*.)

EXAMPLE 6.13

Draw the influence line for the shear in panel CD of the floor girder in Fig. 6–21a.



(a)

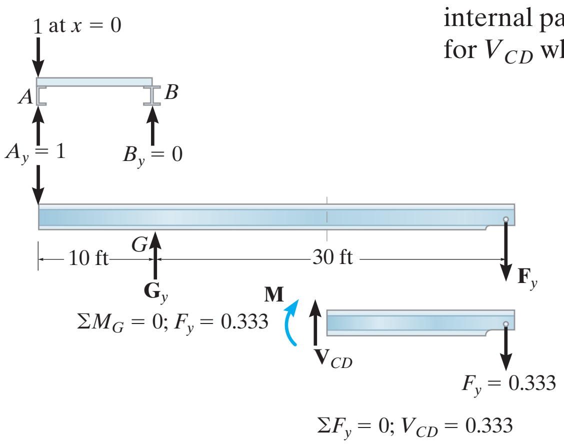
Fig. 6–21

SOLUTION

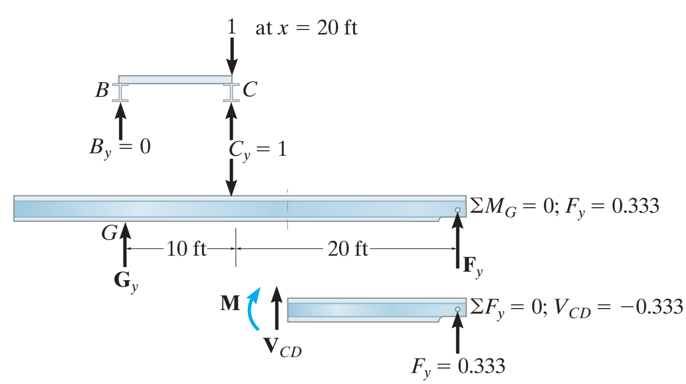
x	V_{CD}
0	0.333
10	0
20	−0.333
30	0.333
40	0

(b)

Tabulate Values. The unit load is placed at each floor beam location and the shear in panel CD is calculated. A table of the results is shown in Fig. 6–21b. The details for the calculations when $x = 0$ and $x = 20$ ft are given in Figs. 6–21c and 6–21d, respectively. Notice how in each case the reactions of the floor beams on the girder are calculated first, followed by a determination of the girder support reaction at F ($\mathbf{G}_y$ is not needed), and finally, a segment of the girder is considered and the internal panel shear V_{CD} is calculated. As an exercise, verify the values for V_{CD} when $x = 10$ ft, 30 ft, and 40 ft.

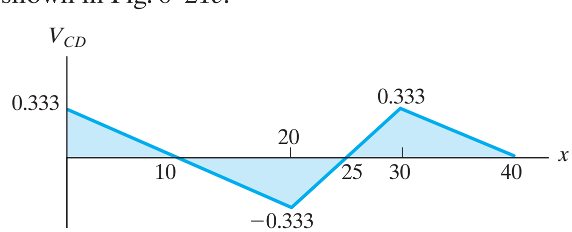


(c)



(d)

Influence Line. When the tabular values are plotted and the points connected with straight line segments, the resulting influence line for V_{CD} is as shown in Fig. 6–21e.

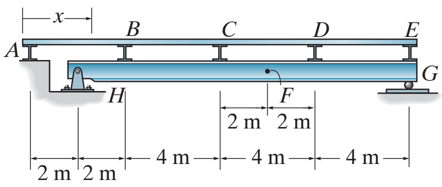


influence line for V_{CD}

(e)

EXAMPLE 6.14

Draw the influence line for the moment at point F for the floor girder in Fig. 6–22a.



(a)

x	M_F
0	0
2	0.429
4	0.857
8	2.571
10	2.429
12	2.286
16	0

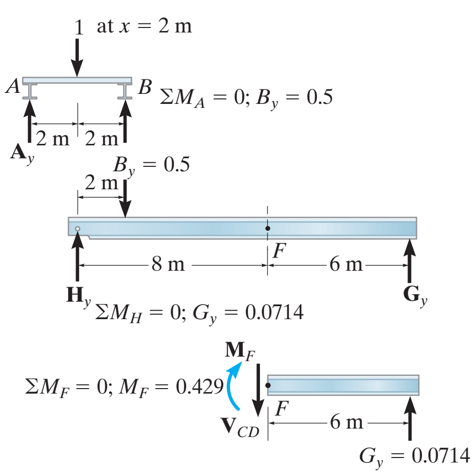
(b)

Fig. 6–22

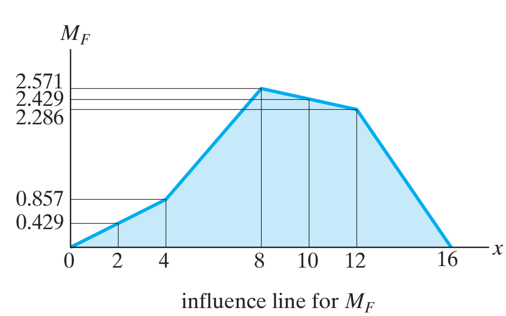
SOLUTION

Tabulate Values. The unit load is placed at $x = 0$ and each panel point thereafter. The corresponding values for M_F are calculated and shown in the table, Fig. 6–22b. Details of the calculations for $x = 2$ m are shown in Fig. 6–22c. As in the previous example, it is first necessary to determine the reactions of the floor beams on the girder, followed by a determination of the girder support reaction $\mathbf{G}_y$ ($\mathbf{H}_y$ is not needed), and finally segment GF of the girder is considered and the internal moment $\mathbf{M}_F$ is calculated. As an exercise, determine the other values of M_F listed in Fig. 6–22b.

Influence Line. A plot of the tabular values yields the influence line for M_F, Fig. 6–22d.



(c)



(d)

6.5 Influence Lines for Trusses

The members of this truss bridge were designed using influence lines in accordance with the AASHTO specifications.

Trusses are often used as primary load-carrying elements for bridges. Hence, for design it is important to be able to construct the influence lines for each of its members. As shown in Fig. 6–23, the loading on the bridge deck is transmitted to stringers, which in turn transmit the loading to floor beams and then to the *joints* along the bottom cord of the truss. Since the truss members are affected only by the joint loading, we can therefore obtain the ordinate values of the influence line for a member by loading each joint along the deck with a unit load and then use the method of joints or the method of sections to calculate the force in the member. The data can be arranged in tabular form, listing "unit load at joint" versus "force in member." As a convention, if the member force is *tensile* it is considered a *positive* value; if it is *compressive* it is *negative*. The influence line for the member is constructed by plotting the data and drawing straight lines between the points.

The following examples illustrate the method of construction.

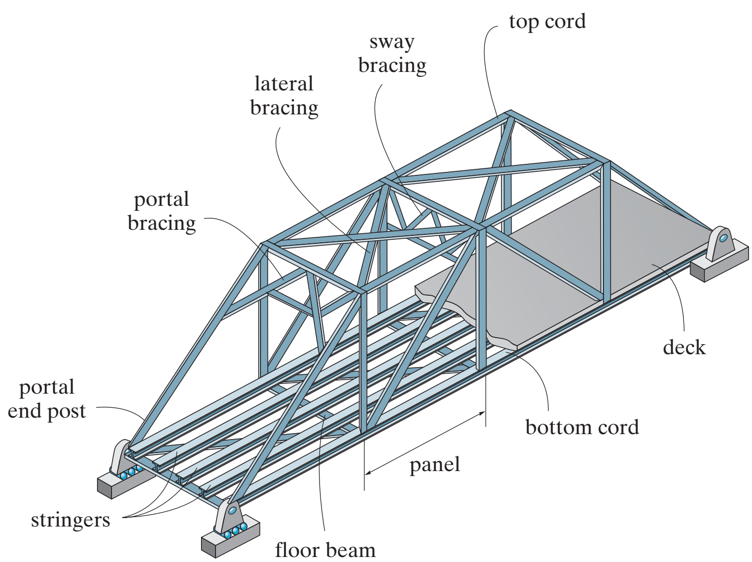
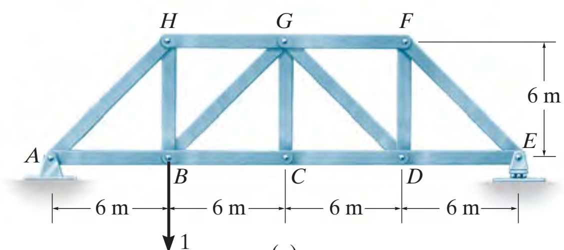


Fig. 6–23

EXAMPLE 6.15

Draw the influence line for the force in member *GB* of the bridge truss shown in Fig. 6–24*a*.



(a)

Fig. 6–24

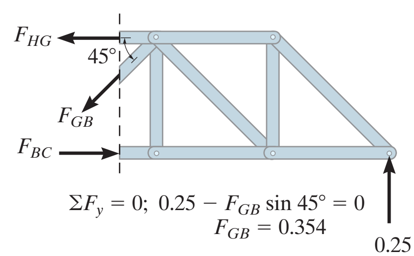
SOLUTION

Tabulate Values. Here each successive joint at the bottom cord is loaded with a unit load and the force in member *GB* is calculated using the method of sections, Fig. 6–24*b*. For example, placing the unit load at $x = 6$ m (joint *B*), the support reaction at *E* is calculated first, Fig. 6–24*a*, then passing a section through *HG*, *GB*, *BC* and isolating the right segment, the force in *GB* is determined, Fig. 6–24*c*. In the same manner, determine the other values listed in the table.

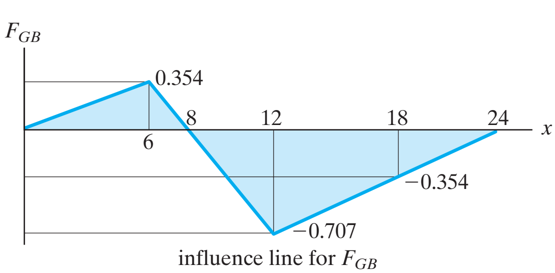
x	F_{GB}
0	0
6	0.354
12	−0.707
18	−0.354
24	0

(b)

Influence Line. Plotting the tabular data and connecting the points yields the influence line for member *GB*, Fig. 6–24*d*. Since the influence line extends over the entire span of the truss, member *GB* is referred to as a *primary member*. This means *GB* is subjected to a force regardless of where the bridge deck (roadway) is loaded, except, of course, at $x = 8$ m. The point of zero force, $x = 8$ m, is determined by similar triangles between $x = 6$ m and $x = 12$ m, that is, $(0.354 + 0.707)/(12 - 6) = 0.354/x'$, $x' = 2$ m, so $x = 6 + 2 = 8$ m.



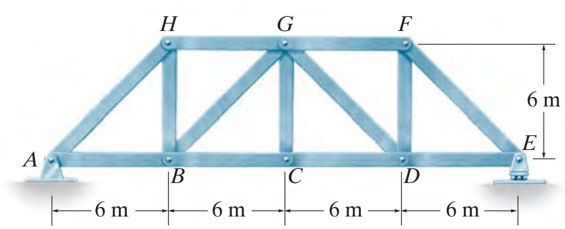
(c)



(d)

EXAMPLE 6.16

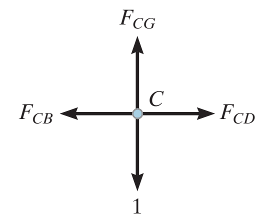
Draw the influence line for the force in member CG of the bridge truss shown in Fig. 6–25*a*.



(a)

x	F_{GC}
0	0
6	0
12	1
18	0
24	0

(b)



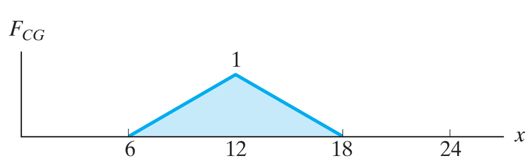
(c)

Fig. 6–25

SOLUTION

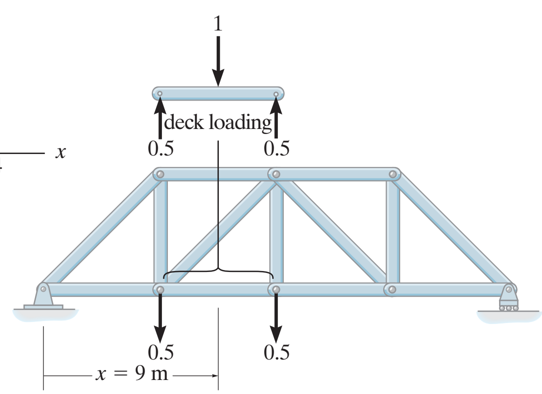
Tabulate Values. A table of unit-load position at the joints of the bottom cord versus the force in member CG is shown in Fig. 6–25*b*. These values are easily obtained by isolating joint C, Fig. 6–25*c*. Here it is seen that CG is a zero-force member unless the unit load is applied at joint C, in which case $F_{CG} = 1$ (T).

Influence Line. Plotting the tabular data and connecting the points yields the influence line for member CG as shown in Fig. 6–25*d*. In particular, notice that when the unit load is at $x = 9$ m, the force in member CG is $F_{CG} = 0.5$. This situation requires the unit load to be placed on the bridge deck *between* the joints. The transference of this load from the deck to the truss is shown in Fig. 6–25*e*. From this one can see that indeed $F_{CG} = 0.5$ by analyzing the equilibrium of joint C, Fig. 6–25*f*. Since the influence line for CG does *not* extend over the entire span of the truss, Fig. 6–25*d*, member CG is referred to as a *secondary member*.



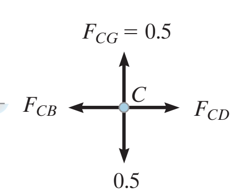
influence line for F_{CG}

(d)



truss loading

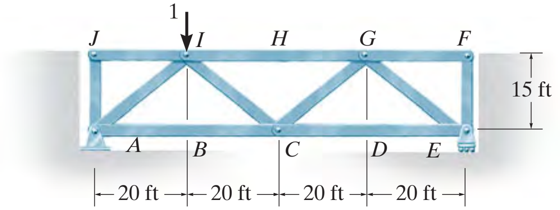
(e)



(f)

EXAMPLE 6.17

In order to determine the maximum force in each member of the Warren truss, shown in the photo, we must first draw the influence lines for each of its members. If we consider a similar truss as shown in Fig. 6–26*a*, determine the largest force that can be developed in member *BC* due to a moving force of 25 k and a moving distributed load of 0.6 k/ft. The loading is applied at the top cord.



(a)

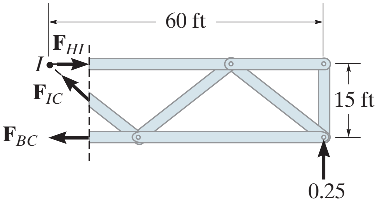
x	F_{BC}
0	0
20	1
40	0.667
60	0.333
80	0

(b)

Fig. 6–26

SOLUTION

Tabulate Values. A table of unit-load position x at the joints along the top cord versus the force in member *BC* is shown in Fig. 6–26*b*. The method of sections can be used for the calculations. For example, when the unit load is at joint I $(x = 20 \text{ ft})$, Fig. 6–26*a*, the reaction $\mathbf{E}_y$ is determined first $(E_y = 0.25)$. Then the truss is sectioned through *BC*, *IC*, and *HI*, and the right segment is isolated, Fig. 6–26*c*. One obtains $\mathbf{F}_{BC}$ by summing moments about point I, to eliminate $\mathbf{F}_{HI}$ and $\mathbf{F}_{IC}$. In a similar manner determine the other values in Fig. 6–26*b*.



$\circlearrowleft + \Sigma M_I = 0;\ -F_{BC}(15) + 0.25(60) = 0$

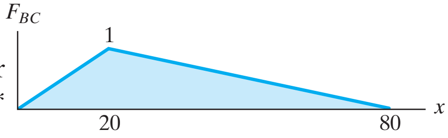
$F_{BC} = 1.00 \text{ (T)}$

(c)

Influence Line. A plot of the tabular values yields the influence line, Fig. 6–26*d*. By inspection, *BC* is a primary member. Why?

Concentrated Live Force. The largest force in member *BC* occurs when the moving force of 25 k is placed at $x = 20$ ft. Thus,

$$F_{BC} = (1.00)(25) = 25.0 \text{ k}$$



influence line for F_{BC}

(d)

Distributed Live Load. The uniform live load must be placed over the entire deck of the truss to create the largest tensile force in *BC*.* Thus,

$$F_{BC} = \left[\tfrac{1}{2}(80)(1.00)\right]0.6 = 24.0 \text{ k}$$

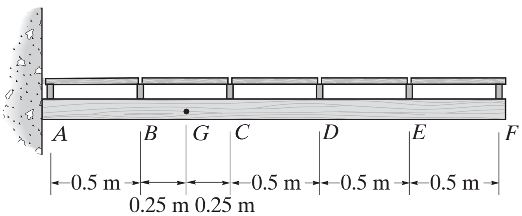
Total Maximum Force.

$$(F_{BC})_{max} = 25.0 \text{ k} + 24.0 \text{ k} = 49.0 \text{ k} \qquad \textit{Ans.}$$

*The largest *tensile* force in member *GB* of Example 6–15 is created when the distributed load acts on the deck of the truss from $x = 0$ to $x = 8$ m, Fig. 6–24*d*.

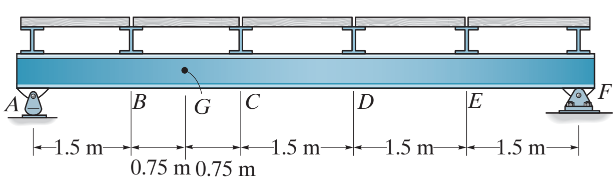
PROBLEMS

6–26. A uniform live load of 1.8 kN/m and a single concentrated live force of 4 kN are placed on the floor beams. Determine (a) the maximum positive shear in panel *BC* of the girder and (b) the maximum moment in the girder at *G*.



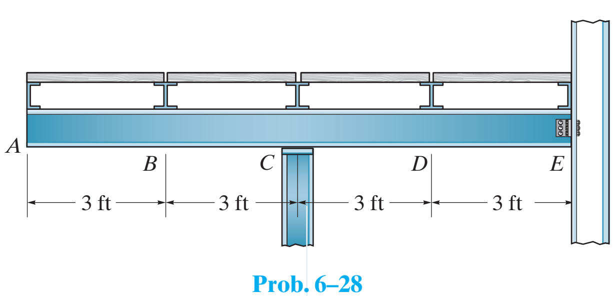
Prob. 6–26

6–27. A uniform live load of 2.8 kN/m and a single concentrated live force of 20 kN are placed on the floor beams. If the beams also support a uniform dead load of 700 N/m, determine (a) the maximum positive shear in panel *BC* of the girder and (b) the maximum positive moment in the girder at *G*.



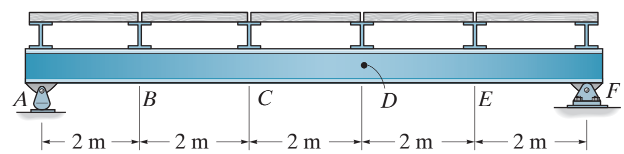
Prob. 6–27

***6–28.** A uniform live load of 2 k/ft and a single concentrated live force of 6 k are placed on the floor beams. If the beams also support a uniform dead load of 350 lb/ft, determine (a) the maximum positive shear in panel *CD* of the girder and (b) the maximum negative moment in the girder at *D*. Assume the support at *C* is a roller and *E* is a pin.



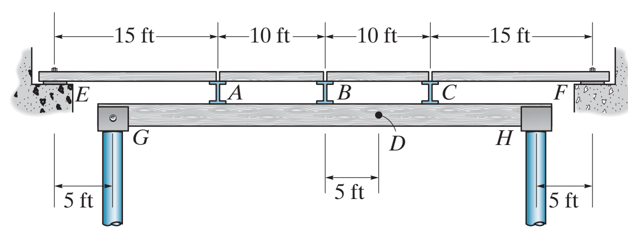
Prob. 6–28

6–29. Draw the influence line for (a) the shear in panel *BC* of the girder, and (b) the moment at *D*.



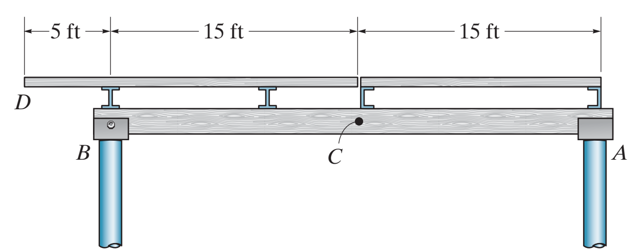
Prob. 6–29

6–30. A uniform live load of 250 lb/ft and a single concentrated live force of 1.5 k are to be placed on the floor beams. Determine (a) the maximum positive shear in panel *AB*, and (b) the maximum moment at *D*. Assume only vertical reaction occur at the supports.



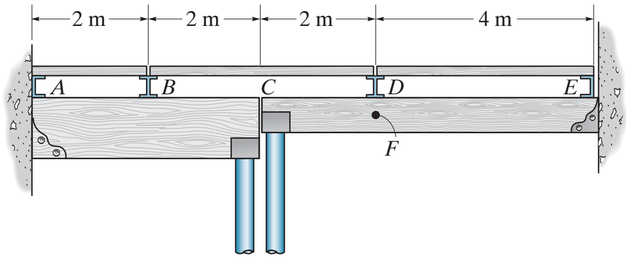
Prob. 6–30

6–31. A uniform live load of 0.6 k/ft and a single concentrated live force of 5 k are to be placed on the top beams. Determine (a) the maximum positive shear in panel *BC* of the girder, and (b) the maximum positive moment at *C*. Assume the support at *B* is a roller and at *D* a pin.



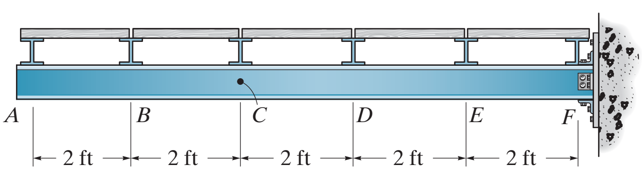
Prob. 6–31

***6–32.** Draw the influence line for the moment at F in the girder. Determine the maximum positive live moment in the girder at F if a single concentrated live force of 8 kN moves across the top floor beams. Assume the supports for all members can only exert either upward or downward forces on the members.



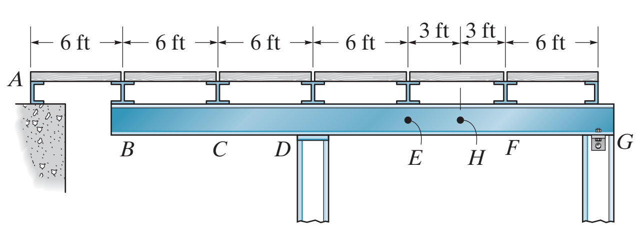
Prob. 6–32

6–33. A uniform live load of 4 k/ft and a single concentrated live force of 20 k are placed on the floor beams. If the beams also support a uniform dead load of 700 lb/ft, determine (a) the maximum negative shear in panel DE of the girder and (b) the maximum negative moment in the girder at C.



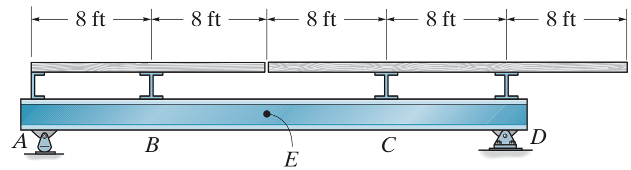
Prob. 6–33

6–34. A uniform live load of 0.2 k/ft and a single concentrated live force of 4 k are placed on the floor beams. Determine (a) the maximum positive shear in panel DE of the girder, and (b) the maximum positive moment at H.



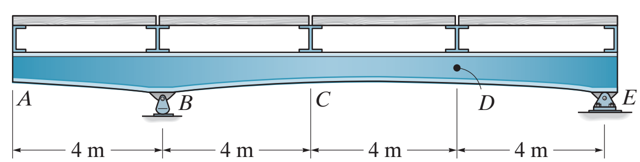
Prob. 6–34

6–35. Draw the influence line for the shear in panel CD of the girder. Determine the maximum negative live shear in panel CD due to a uniform live load of 500 lb/ft acting on the top beams.



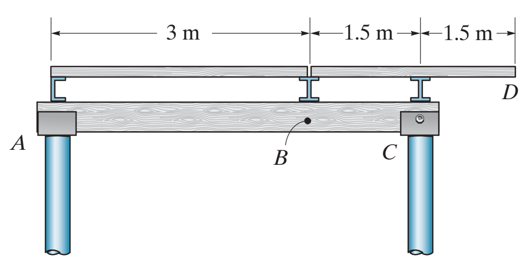
Prob. 6–35

***6–36.** A uniform live load of 6.5 kN/m and a single concentrated live force of 15 kN are placed on the floor beams. If the beams also support a uniform dead load of 600 N/m, determine (a) the maximum positive shear in panel CD of the girder and (b) the maximum positive moment in the girder at D.



Prob. 6–36

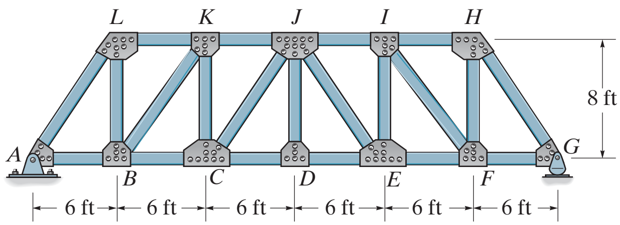
6–37. A uniform live load of 1.75 kN/m and a single concentrated live force of 8 kN are placed on the floor beams. If the beams also support a uniform dead load of 250 N/m, determine (a) the maximum negative shear in panel BC of the girder and (b) the maximum positive moment at B.



Prob. 6–37

6–38. Draw the influence line for the force in (a) member *KJ* and (b) member *CJ*.

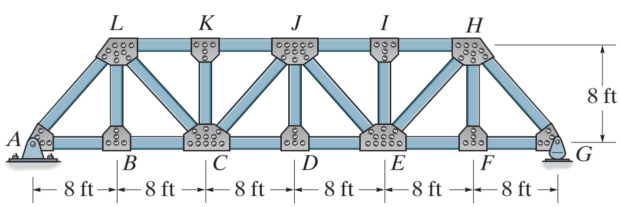
6–39. Draw the influence line for the force in (a) member *JI*, (b) member *IE*, and (c) member *EF*.



Probs. 6–38/6–39

***6–40.** Draw the influence line for the force in member *KJ*.

6–41. Draw the influence line for the force in member *JE*.

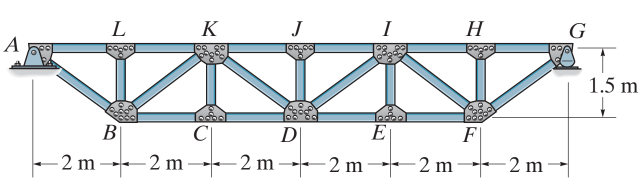


Probs. 6–40/6–41

6–42. Draw the influence line for the force in member *CD*.

6–43. Draw the influence line for the force in member *JK*.

***6–44.** Draw the influence line for the force in member *DK*.

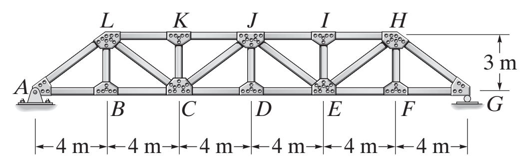


Probs. 6–42/6–43/6–44

6–45. Draw the influence line for the force in (a) member *EH* and (b) member *JE*.

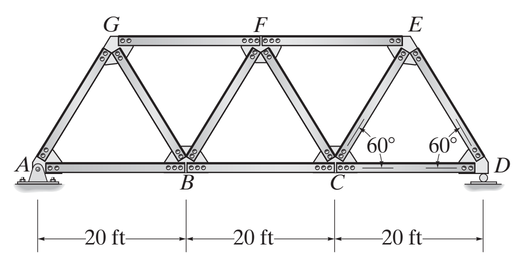
6–46. Draw the influence line for the force in member *JI*.

6–47. Draw the influence line for the force in member *AL*.



Probs. 6–45/6–46/6–47

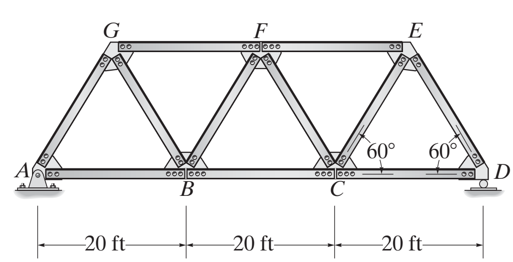
***6–48.** Draw the influence line for the force in member *BC* of the Warren truss. Indicate numerical values for the peaks. All members have the same length.



Prob. 6–48

6–49. Draw the influence line for the force in member *BF* of the Warren truss. Indicate numerical values for the peaks. All members have the same length.

6–50. Draw the influence line for the force in member *FE* of the Warren truss. Indicate numerical values for the peaks. All members have the same length.

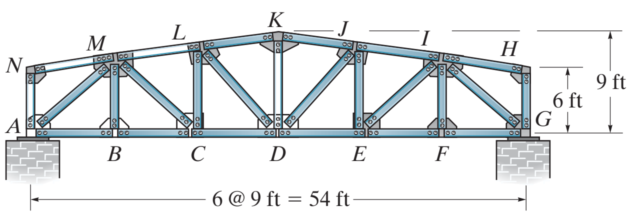


Probs. 6–49/6–50

6–51. Draw the influence line for the force in member CL.

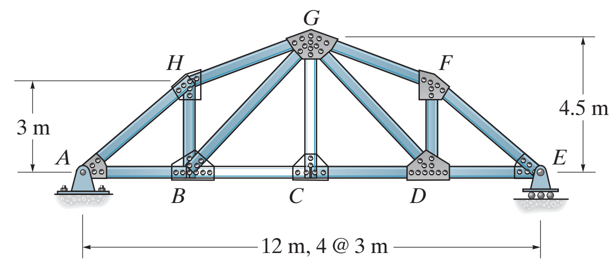
***6–52.** Draw the influence line for the force in member DL.

6–53. Draw the influence line for the force in member CD.



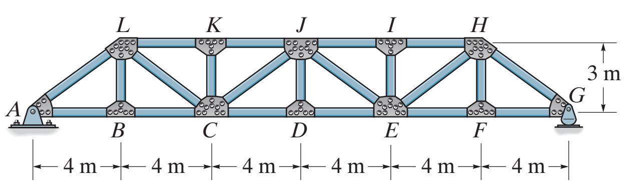
Probs. 6–51/6–52/6–53

***6–56.** Draw the influence line for the force in member GD, then determine the maximum force (tension or compression) that can be developed in this member due to a uniform live load of 3 kN/m that acts on the bridge deck along the bottom cord of the truss.



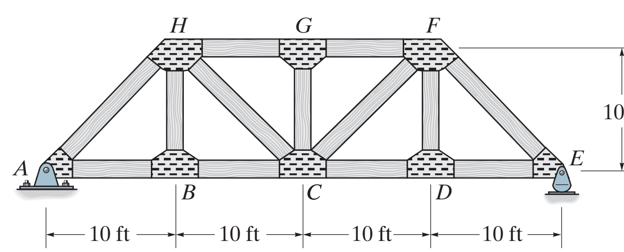
Prob. 6–56

6–54. Draw the influence line for the force in member CD.



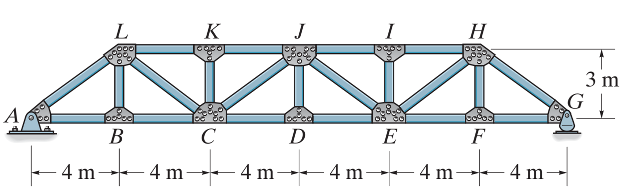
Prob. 6–54

6–57. Draw the influence line for the force in member CD, and then determine the maximum force (tension or compression) that can be developed in this member due to a uniform live load of 800 lb/ft which acts along the bottom cord of the truss.



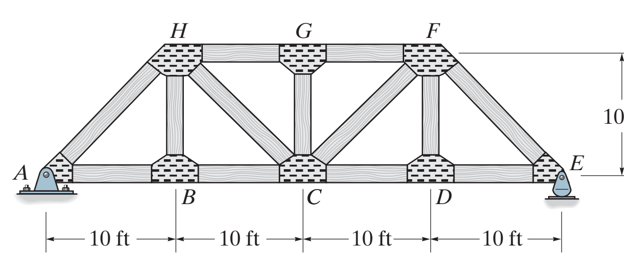
Prob. 6–57

6–55. Draw the influence line for the force in member KJ.



Prob. 6–55

6–58. Draw the influence line for the force in member CF, and then determine the maximum force (tension or compression) that can be developed in this member due to a uniform live load of 800 lb/ft which is transmitted to the truss along the bottom cord.



Prob. 6–58

6

6.6 Maximum Influence at a Point due to a Series of Concentrated Loads

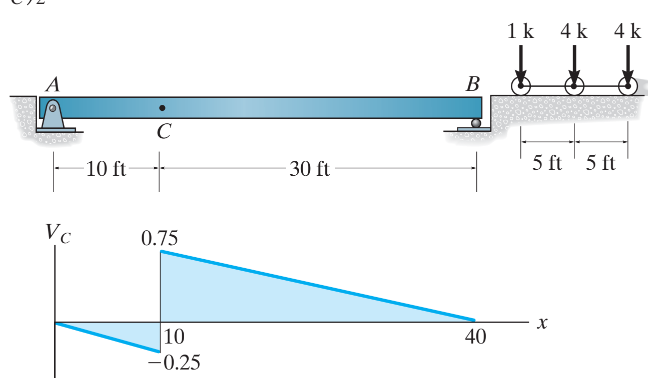
As the train passes over this girder bridge the engine and its cars will exert vertical reactions on the girder. These along with the dead load of the bridge must be considered for design.

Once the influence line of a function has been established for a point in a structure, the maximum effect caused by a live concentrated force is determined by multiplying the peak ordinate of the influence line by the magnitude of the force. In some cases, however, *several* concentrated forces must be placed on the structure. An example would be the wheel loadings of a truck or train. In order to determine the maximum effect in this case, either a trial-and-error procedure can be used or a method that is based on the change in the function that takes place as the load is moved. Each of these methods will now be explained specifically as it applies to shear and moment.

Shear. Consider the simply supported beam with the associated influence line for the shear at point C in Fig. 6–27*a*. The maximum *positive shear* at point C is to be determined due to the series of concentrated (wheel) loads which move from right to left over the beam. The critical loading will occur when one of the loads is placed *just to the right* of point C, which is coincident with the positive peak of the influence line. By trial and error each of three possible cases can therefore be investigated, Fig. 6–27*b*. We have

$$\text{Case 1:}\quad (V_C)_1 = 1(0.75) + 4(0.625) + 4(0.5) = 5.25\text{ k}$$
$$\text{Case 2:}\quad (V_C)_2 = 1(-0.125) + 4(0.75) + 4(0.625) = 5.375\text{ k}$$
$$\text{Case 3:}\quad (V_C)_3 = 1(0) + 4(-0.125) + 4(0.75) = 2.5\text{ k}$$

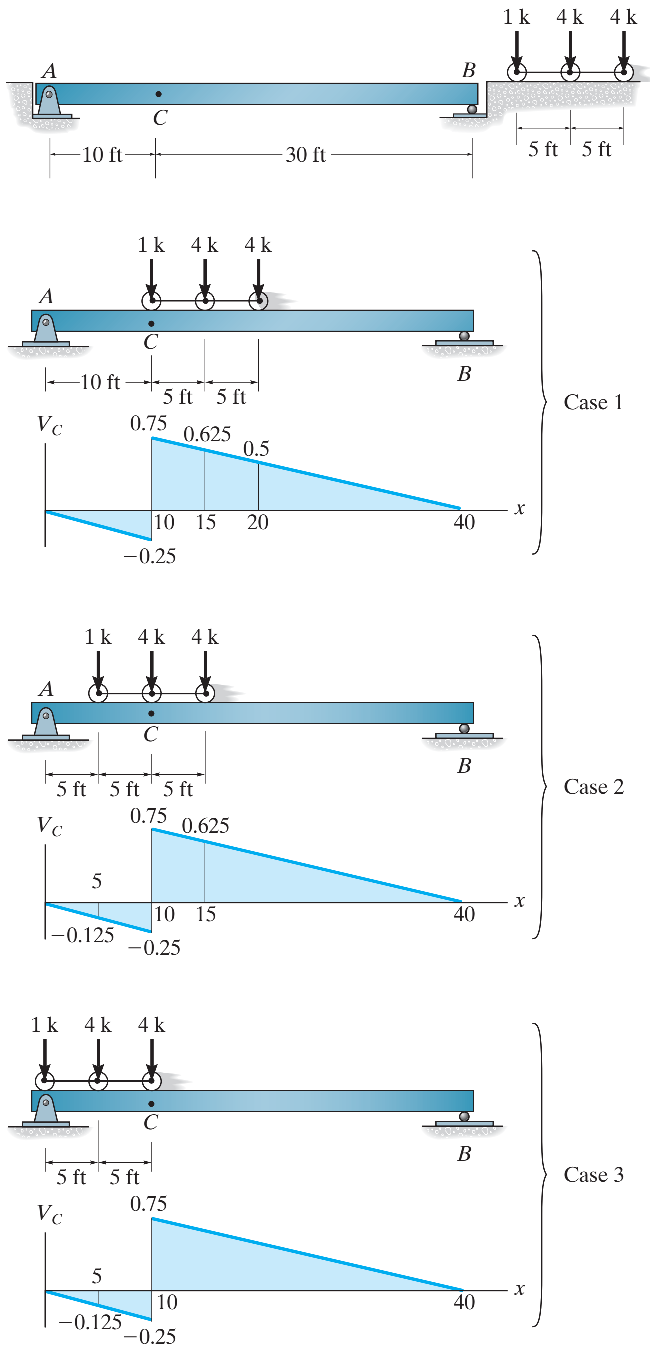
Case 2, with the 1-k force located 5^+ ft from the left support, yields the largest value for V_C and therefore represents the critical loading. Actually investigation of Case 3 is unnecessary, since by inspection such an arrangement of loads would yield a value of $(V_C)_3$ that would be less than $(V_C)_2$.



influence line for V_C

(a)

Fig. 6–27



1 k
4 k
4 k
A
B
C
10 ft
30 ft
5 ft
5 ft
1 k
4 k
4 k
A
C
B
10 ft
5 ft
5 ft
V_C
0.75
0.625
0.5
10
15
20
40
x
−0.25
Case 1
1 k
4 k
4 k
A
C
B
5 ft
5 ft
5 ft
V_C
0.75
0.625
5
10
15
40
x
−0.125
−0.25
Case 2
1 k
4 k
4 k
A
C
B
5 ft
5 ft
V_C
0.75
5
10
40
x
−0.125
−0.25
Case 3

(b)

Fig. 6–27

When many concentrated loads act on the span, as in the case of the E-72 load of Fig. 1–11, the trial-and-error computations used above can be tedious. Instead, the critical position of the loads can be determined in a more direct manner by finding the change in shear, ΔV, which occurs when the loads are moved from Case 1 to Case 2, then from Case 2 to Case 3, and so on. As long as each computed ΔV is *positive,* the new position will yield a larger shear in the beam at C than the previous position. Each movement is investigated until a negative change in shear is computed. When this occurs, the previous position of the loads will give the critical value. The change in shear ΔV for a load P that moves from position x_1 to x_2 over a beam can be determined by multiplying P by the change in the ordinate of the influence line, that is, $(y_2 - y_1)$. If the slope of the influence line is s, then $(y_2 - y_1) = s(x_2 - x_1)$, and therefore

$$\boxed{\begin{gathered}\Delta V = Ps(x_2 - x_1)\\ \text{Sloping Line}\end{gathered}} \qquad (6\text{–}1)$$

If the load moves past a point where there is a discontinuity or "jump" in the influence line, as point C in Fig. 6–27*a*, then the change in shear is simply

$$\boxed{\begin{gathered}\Delta V = P(y_2 - y_1)\\ \text{Jump}\end{gathered}} \qquad (6\text{–}2)$$

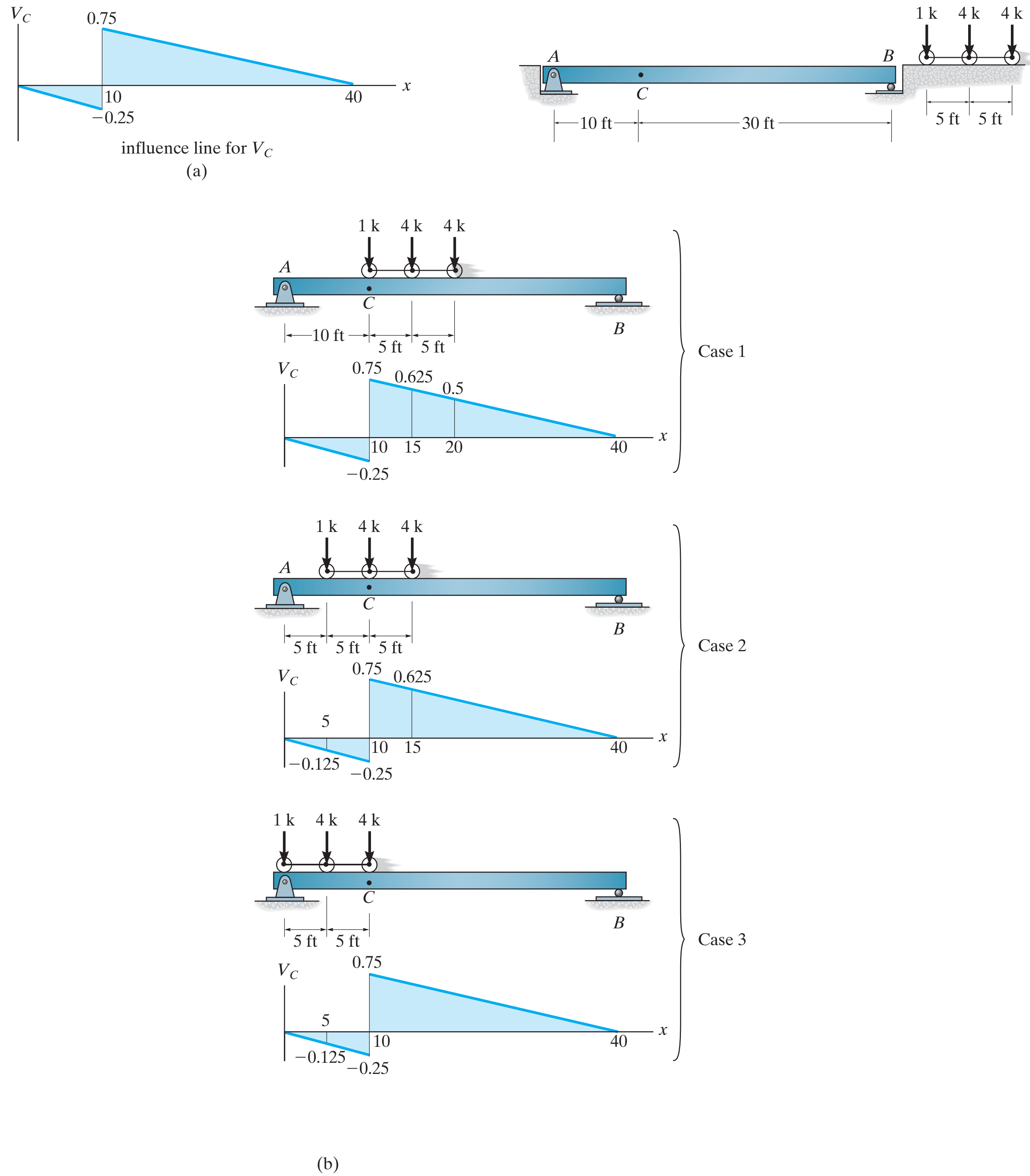
Use of the above equations will be illustrated with reference to the beam, loading, and influence line for V_C, shown in Fig. 6–28*a*. Notice that the magnitude of the slope of the influence line is $s = 0.75/(40 - 10) = 0.25/10 = 0.025$, and the jump at C has a magnitude of $0.75 + 0.25 = 1$. Consider the loads of Case 1 moving 5 ft to Case 2, Fig. 6–28*b*. When this occurs, the 1-k load jumps *down* (-1) and *all* the loads move *up* the slope of the influence line. This causes a change of shear,

$$\Delta V_{1-2} = 1(-1) + [1 + 4 + 4](0.025)(5) = +0.125 \text{ k}$$

Since ΔV_{1-2} is positive, Case 2 will yield a larger value for V_C than Case 1. [Compare the answers for $(V_C)_1$ and $(V_C)_2$ previously computed, where indeed $(V_C)_2 = (V_C)_1 + 0.125$.] Investigating ΔV_{2-3}, which occurs when Case 2 moves to Case 3, Fig. 6–28*b*, we must account for the downward (negative) jump of the 4-k load and the 5-ft horizontal movement of all the loads *up* the slope of the influence line. We have

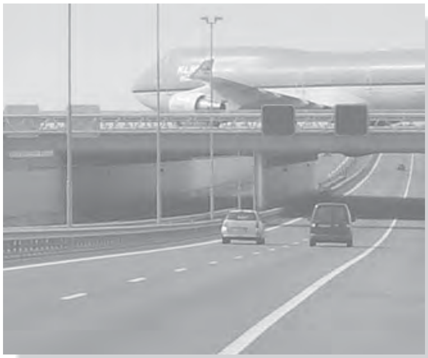
$$\Delta V_{2-3} = 4(-1) + (1 + 4 + 4)(0.025)(5) = -2.875 \text{ k}$$

Since ΔV_{2-3} is negative, Case 2 is the position of the critical loading, as determined previously.



V_C
0.75
10
40
x
−0.25
influence line for V_C
(a)
1 k
4 k
4 k
A
B
C
10 ft
30 ft
5 ft
5 ft
Case 1
0.625
0.5
15
20
Case 2
5
−0.125
Case 3
(b)

Fig. 6–28



The girders of this bridge must resist the maximum moment caused by the weight of this jet plane as it passes over it.

Moment. We can also use the foregoing methods to determine the critical position of a series of concentrated forces so that they create the largest internal moment at a specific point in a structure. Of course, it is first necessary to draw the influence line for the moment at the point and determine the slopes s of its line segments. For a horizontal movement $(x_2 - x_1)$ of a concentrated force P, the change in moment, ΔM, is equivalent to the magnitude of the force times the change in the influence-line ordinate under the load, that is,

$$\boxed{\begin{array}{c}\Delta M = Ps(x_2 - x_1)\\ \text{Sloping Line}\end{array}} \tag{6–3}$$

As an example, consider the beam, loading, and influence line for the moment at point C in Fig. 6–29*a*. If each of the three concentrated forces is placed on the beam, coincident with the peak of the influence line, we will obtain the greatest influence from each force. The three cases of loading are shown in Fig. 6–29*b*. When the loads of Case 1 are moved 4 ft to the left to Case 2, it is observed that the 2-k load *decreases* ΔM_{1-2}, since the *slope* (7.5/10) is *downward,* Fig. 6–29*a*. Likewise, the 4-k and 3-k forces cause an *increase* of ΔM_{1-2}, since the *slope* [7.5/(40 − 10)] is *upward*. We have

$$\Delta M_{1-2} = -2\left(\frac{7.5}{10}\right)(4) + (4 + 3)\left(\frac{7.5}{40 - 10}\right)(4) = 1.0\ \text{k}\cdot\text{ft}$$

Since ΔM_{1-2} is positive, we must further investigate moving the loads 6 ft from Case 2 to Case 3.

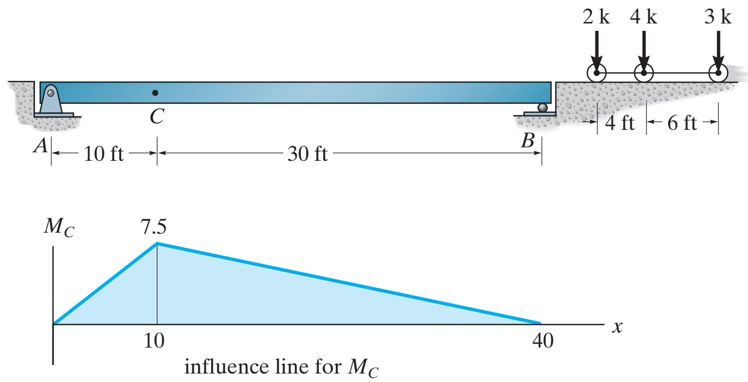
$$\Delta M_{2-3} = -(2 + 4)\left(\frac{7.5}{10}\right)(6) + 3\left(\frac{7.5}{40 - 10}\right)(6) = -22.5\ \text{k}\cdot\text{ft}$$

Here the change is negative, so the greatest moment at C will occur when the beam is loaded as shown in Case 2, Fig. 6–29*c*. The maximum moment at C is therefore

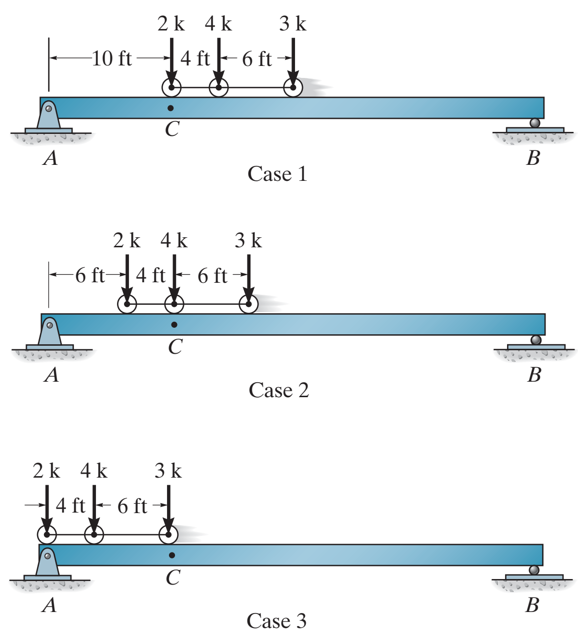
$$(M_C)_{\max} = 2(4.5) + 4(7.5) + 3(6.0) = 57.0\ \text{k}\cdot\text{ft}$$

The following examples further illustrate this method.

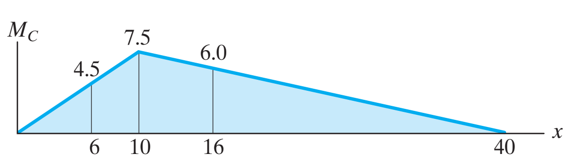
6


2 k
4 k
3 k
C
A
B
10 ft
30 ft
4 ft
6 ft
M_C
7.5
10
40
x
influence line for M_C

(a)


2 k
4 k
3 k
10 ft
4 ft
6 ft
C
A
B
Case 1
2 k
4 k
3 k
6 ft
4 ft
6 ft
C
A
B
Case 2
2 k
4 k
3 k
4 ft
6 ft
C
A
B
Case 3

(b)

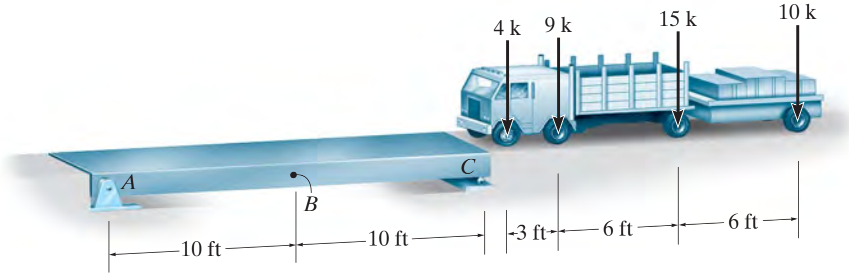

M_C
4.5
7.5
6.0
6
10
16
40
x

(c)

Fig. 6–29

EXAMPLE 6.18

Determine the maximum positive shear created at point B in the beam shown in Fig. 6–30a due to the wheel loads of the moving truck.

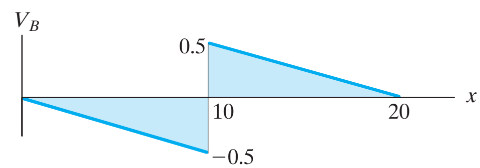


(a)

Fig. 6–30

SOLUTION

The influence line for the shear at B is shown in Fig. 6–30b.



influence line for V_B
(b)

3-ft Movement of 4-k Load. Imagine that the 4-k load acts just to the right of point B so that we obtain its maximum positive influence. Since the beam segment BC is 10 ft long, the 10-k load is not as yet on the beam. When the truck moves 3 ft to the left, the 4-k load jumps *downward* on the influence line 1 unit and the 4-k, 9-k, and 15-k loads create a positive increase in ΔV_B, since the slope is upward to the left. Although the 10-k load also moves forward 3 ft, it is still not on the beam. Thus,

$$\Delta V_B = 4(-1) + (4 + 9 + 15)\left(\frac{0.5}{10}\right)3 = +0.2 \text{ k}$$

6-ft Movement of 9-k Load. When the 9-k load acts just to the right of B, and then the truck moves 6 ft to the left, we have

$$\Delta V_B = 9(-1) + (4 + 9 + 15)\left(\frac{0.5}{10}\right)(6) + 10\left(\frac{0.5}{10}\right)(4) = +1.4 \text{ k}$$

Note in the calculation that the 10-k load only moves 4 ft on the beam.

6

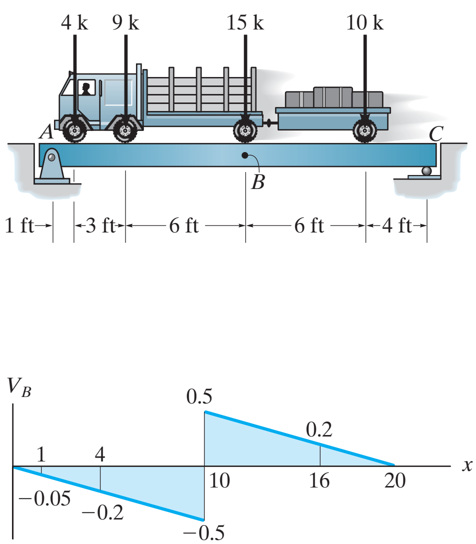
6-ft Movement of 15-k Load. If the 15-k load is positioned just to the right of B and then the truck moves 6 ft to the left, the 4-k load moves only 1 ft until it is off the beam, and likewise the 9-k load moves only 4 ft until it is off the beam. Hence,

$$\Delta V_B = 15(-1) + 4\left(\frac{0.5}{10}\right)(1) + 9\left(\frac{0.5}{10}\right)(4) + (15 + 10)\left(\frac{0.5}{10}\right)(6)$$
$$= -5.5 \text{ k}$$

Since ΔV_B is now negative, the correct position of the loads occurs when the 15-k load is just to the right of point B, Fig. 6–30*c*. Consequently,

$$(V_B)_{\max} = 4(-0.05) + 9(-0.2) + 15(0.5) + 10(0.2)$$
$$= 7.5 \text{ k} \qquad \textit{Ans.}$$

In practice one also has to consider motion of the truck from left to right and then choose the maximum value between these two situations.



(c)

EXAMPLE 6.19

Determine the maximum positive moment created at point B in the beam shown in Fig. 6–31a due to the wheel loads of the crane.

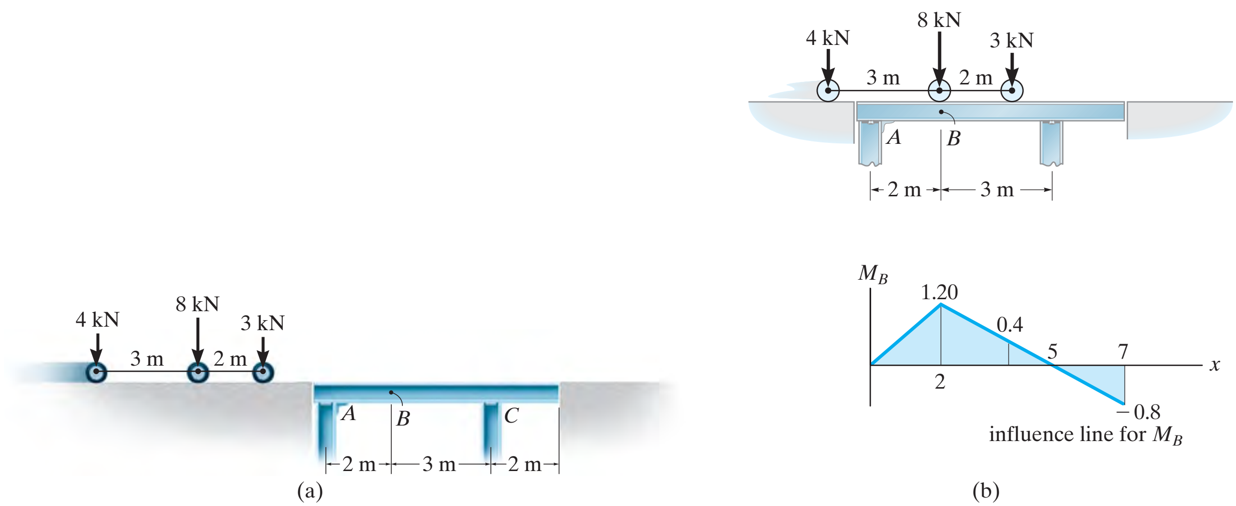


Fig. 6–31

SOLUTION

The influence line for the moment at B is shown in Fig. 6–31b.

2-m Movement of 3-kN Load. If the 3-kN load is assumed to act at B and then moves 2 m to the right, Fig. 6–31b, the change in moment is

$$\Delta M_B = -3\left(\frac{1.20}{3}\right)(2) + 8\left(\frac{1.20}{3}\right)(2) = 7.20\ \text{kN}\cdot\text{m}$$

Why is the 4-kN load not included in the calculations?

3-m Movement of 8-kN Load. If the 8-kN load is assumed to act at B and then moves 3 m to the right, the change in moment is

$$\begin{aligned}\Delta M_B &= -3\left(\frac{1.20}{3}\right)(3) - 8\left(\frac{1.20}{3}\right)(3) + 4\left(\frac{1.20}{2}\right)(2)\\ &= -8.40\ \text{kN}\cdot\text{m}\end{aligned}$$

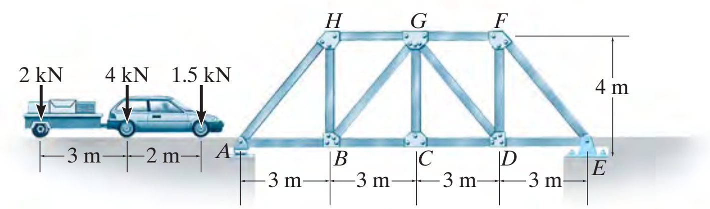
Notice here that the 4-kN load was initially 1 m off the beam, and therefore moves only 2 m on the beam.

Since there is a sign change in ΔM_B, the correct position of the loads for maximum positive moment at B occurs when the 8-kN force is at B, Fig. 6–31b. Therefore,

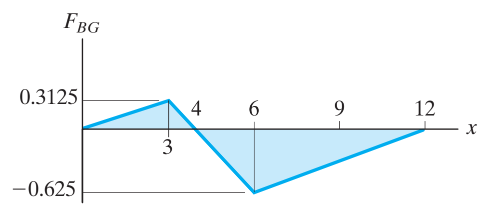
$$(M_B)_{\max} = 8(1.20) + 3(0.4) = 10.8\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

EXAMPLE 6.20

Determine the maximum compressive force developed in member *BG* of the side truss in Fig. 6–32*a* due to the right side wheel loads of the car and trailer. Assume the loads are applied directly to the truss and move only to the right.



(a)



influence line for F_{BG}

(b)

Fig. 6–32

SOLUTION

The influence line for the force in member *BG* is shown in Fig. 6–32*b*. Here a trial-and-error approach for the solution will be used. Since we want the greatest negative (compressive) force in *BG*, we begin as follows:

1.5-kN Load at Point C. In this case

$$F_{BG} = 1.5\text{ kN}(-0.625) + 4(0) + 2\text{ kN}\left(\frac{0.3125}{3\text{ m}}\right)(1\text{ m})$$
$$= -0.729\text{ kN}$$

4-kN Load at Point C. By inspection this would seem a more reasonable case than the previous one.

$$F_{BG} = 4\text{ kN}(-0.625) + 1.5\text{ kN}\left(\frac{-0.625}{6\text{ m}}\right)(4\text{ m}) + 2\text{ kN}(0.3125)$$
$$= -2.50\text{ kN}$$

2-kN Load at Point C. In this case all loads will create a compressive force in *BC*.

$$F_{BG} = 2\text{ kN}(-0.625) + 4\text{ kN}\left(\frac{-0.625}{6\text{ m}}\right)(3\text{ m}) + 1.5\text{ kN}\left(\frac{-0.625}{6\text{ m}}\right)(1\text{ m})$$
$$= -2.66\text{ kN} \qquad \textit{Ans.}$$

Since this final case results in the largest answer, the critical loading occurs when the 2-kN load is at *C*.

6

6.7 Absolute Maximum Shear and Moment

In Sec. 6–6 we developed the methods for computing the maximum shear and moment at a *specified point* in a beam due to a series of concentrated moving loads. A more general problem involves the determination of both the *location of the point* in the beam *and the position of the loading* on the beam so that one can obtain the *absolute maximum* shear and moment caused by the loads. If the beam is cantilevered or simply supported, this problem can be readily solved.

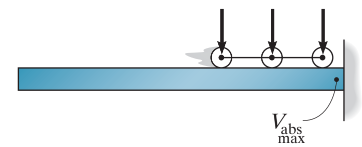


Fig. 6–33

Shear. For a *cantilevered beam* the absolute maximum shear will occur at a point located just next to the fixed support. The maximum shear is found by the method of sections, with the loads positioned anywhere on the span, Fig. 6–33.

For *simply supported beams* the absolute maximum shear will occur just next to one of the supports. For example, if the loads are equivalent, they are positioned so that the first one in sequence is placed close to the support, as in Fig. 6–34.

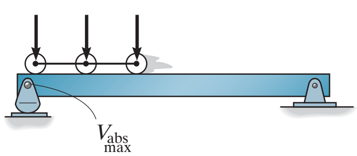


Fig. 6–34

Moment. The absolute maximum moment for a *cantilevered beam* occurs at the same point where absolute maximum shear occurs, although in this case the concentrated loads should be positioned at the *far end* of the beam, as in Fig. 6–35.

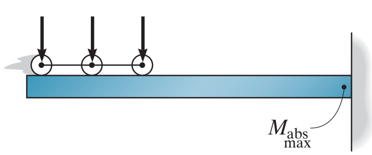


Fig. 6–35

For a *simply supported beam* the critical position of the loads and the associated absolute maximum moment cannot, in general, be determined by inspection. We can, however, determine the position analytically. For purposes of discussion, consider a beam subjected to the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ shown in Fig. 6–36*a*. Since the moment diagram for a series of concentrated forces consists of straight line segments having peaks at each force, the absolute maximum moment will occur under one of the forces. Assume this maximum moment occurs under $\mathbf{F}_2$. The position of the loads $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ on the beam will be specified by the distance x, measured from $\mathbf{F}_2$ to the beam's centerline as shown. To determine a specific value of x, we first obtain the resultant force of the system, $\mathbf{F}_R$, and its distance

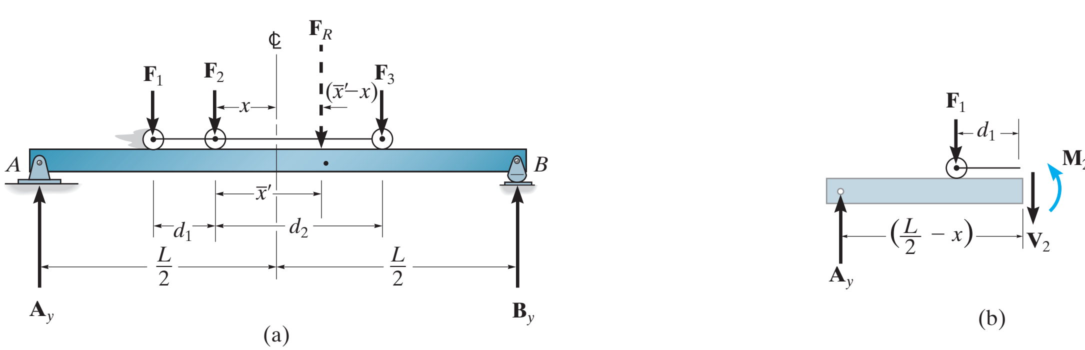


Fig. 6–36

$\bar{x}'$ measured from $\mathbf{F}_2$. Once this is done, moments are summed about B, which yields the beam's left reaction, $\mathbf{A}_y$, that is,

$$\Sigma M_B = 0; \qquad A_y = \frac{1}{L}(F_R)\left[\frac{L}{2} - (\bar{x}' - x)\right]$$

If the beam is sectioned just to the left of $\mathbf{F}_2$, the resulting free-body diagram is shown in Fig. 6–36*b*. The moment $\mathbf{M}_2$ under $\mathbf{F}_2$ is therefore

$$\Sigma M = 0; \qquad M_2 = A_y\left(\frac{L}{2} - x\right) - F_1 d_1$$

$$= \frac{1}{L}(F_R)\left[\frac{L}{2} - (\bar{x}' - x)\right]\left(\frac{L}{2} - x\right) - F_1 d_1$$

$$= \frac{F_R L}{4} - \frac{F_R \bar{x}'}{2} - \frac{F_R x^2}{L} + \frac{F_R x \bar{x}'}{L} - F_1 d_1$$

For maximum M_2 we require

$$\frac{dM_2}{dx} = \frac{-2F_R x}{L} + \frac{F_R \bar{x}'}{L} = 0$$

or

$$x = \frac{\bar{x}'}{2}$$

Hence, we may conclude that the *absolute maximum moment in a simply supported beam occurs under one of the concentrated forces, such that this force is positioned on the beam so that it and the resultant force of the system are equidistant from the beam's centerline*. Since there are a series of loads on the span (for example, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ in Fig. 6–36*a*), this principle will have to be applied to each load in the series and the corresponding maximum moment computed. By comparison, the largest moment is the absolute maximum. As a general rule, though, the absolute maximum moment often occurs under the largest force lying nearest the resultant force of the system.

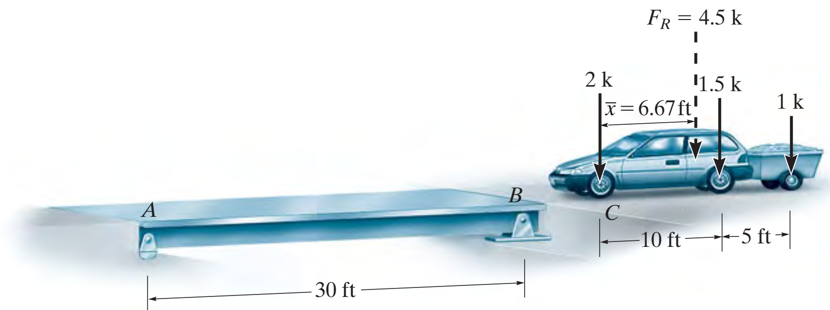
Envelope of Maximum Influence-Line Values. Rules or formulations for determining the absolute maximum shear or moment are difficult to establish for beams supported in ways other than the cantilever or simple support discussed here. An elementary way to proceed to solve this problem, however, requires constructing influence lines for the shear or moment at selected points along the entire length of the beam and then computing the maximum shear or moment in the beam for each point using the methods of Sec. 6–6. These values when plotted yield an "envelope of maximums," from which both the absolute maximum value of shear or moment and its location can be found. Obviously, a computer solution for this problem is desirable for complicated situations, since the work can be rather tedious if carried out by hand calculations.

The absolute maximum moment in this girder bridge is the result of the moving concentrated loads caused by the wheels of these train cars. The cars must be in the critical position, and the location of the point in the girder where the absolute maximum moment occurs must be identified.

6

EXAMPLE 6.21

Determine the absolute maximum moment in the simply supported bridge deck shown in Fig. 6–37*a*.



(a)

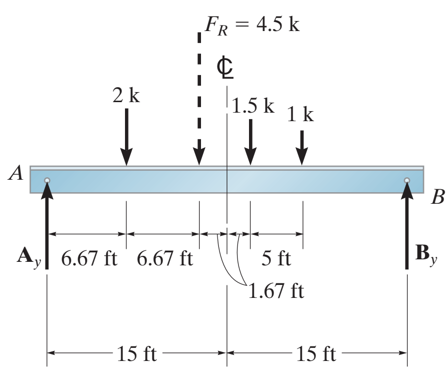
SOLUTION

The magnitude and position of the resultant force of the system are determined first, Fig. 6–37*a*. We have

$$+\downarrow F_R = \Sigma F; \qquad F_R = 2 + 1.5 + 1 = 4.5 \text{ k}$$

$$\circlearrowleft + M_{R_C} = \Sigma M_C; \qquad 4.5\bar{x} = 1.5(10) + 1(15)$$

$$\bar{x} = 6.67 \text{ ft}$$



(b)

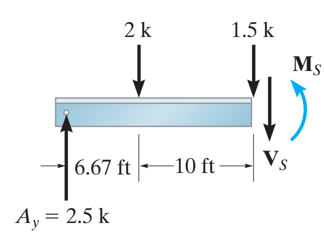
Let us first assume the absolute maximum moment occurs under the 1.5-k load. The load and the resultant force are positioned equidistant from the beam's centerline, Fig. 6–37*b*. Calculating A_y first, Fig. 6–37*b*, we have

$$\circlearrowright + \Sigma M_B = 0; \qquad -A_y(30) + 4.5(16.67) = 0 \qquad A_y = 2.50 \text{ k}$$

Now using the left section of the beam, Fig. 6–37*c*, yields

$$\circlearrowright + \Sigma M_S = 0; \qquad -2.50(16.67) + 2(10) + M_S = 0$$

$$M_S = 21.7 \text{ k}\cdot\text{ft}$$



(c)

Fig. 6–37

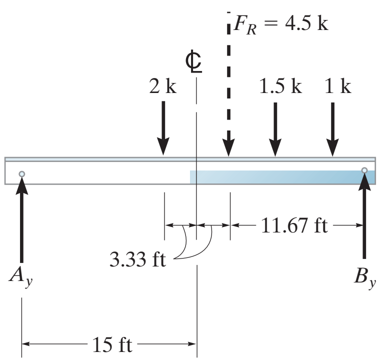
There is a possibility that the absolute maximum moment may occur under the 2-k load, since 2 k > 1.5 k and $\mathbf{F}_R$ is between both 2 k and 1.5 k. To investigate this case, the 2-k load and $\mathbf{F}_R$ are positioned equidistant from the beam's centerline, Fig. 6–37*d*. Show that $A_y = 1.75$ k as indicated in Fig. 6–37*e* and that

$$M_S = 20.4 \text{ k} \cdot \text{ft}$$

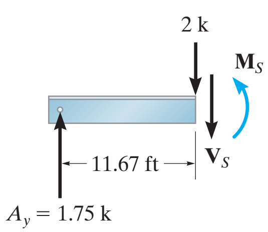
By comparison, the absolute maximum moment is

$$M_S = 21.7 \text{ k} \cdot \text{ft} \qquad \textit{Ans.}$$

which occurs under the 1.5-k load, when the loads are positioned on the beam as shown in Fig. 6–37*b*.



(d)



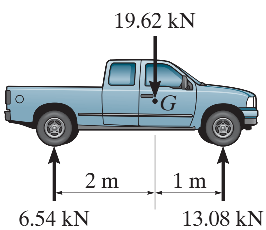
(e)

EXAMPLE 6.22

The truck has a mass of 2 Mg and a center of gravity at G as shown in Fig. 6–38*a*. Determine the absolute maximum moment developed in the simply supported bridge deck due to the truck's weight. The bridge has a length of 10 m.

SOLUTION

As noted in Fig. 6–38*a*, the weight of the truck, $2(10^3)\,\text{kg}(9.81\ \text{m/s}^2) = 19.62\,\text{kN}$, and the wheel reactions have been calculated by statics. Since the largest reaction occurs at the front wheel, we will select this wheel along with the resultant force and position them *equidistant* from the centerline of the bridge, Fig. 6–38*b*. Using the resultant force rather than the wheel loads, the vertical reaction at B is then



(a)

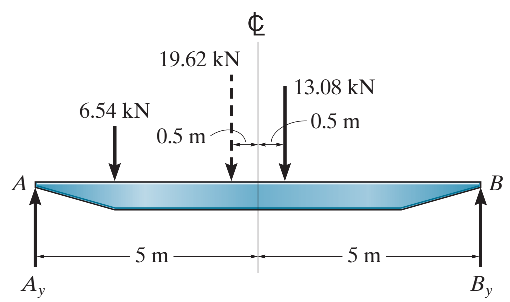
$$\circlearrowleft + \Sigma M_A = 0; \qquad B_y(10) - 19.62(4.5) = 0$$

$$B_y = 8.829\ \text{kN}$$

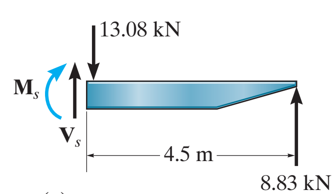
The maximum moment occurs under the front wheel loading. Using the right section of the bridge deck, Fig. 6–38*c*, we have

$$\circlearrowleft + \Sigma M_s = 0; \qquad 8.829(4.5) - M_s = 0$$

$$M_s = 39.7\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$



(b)

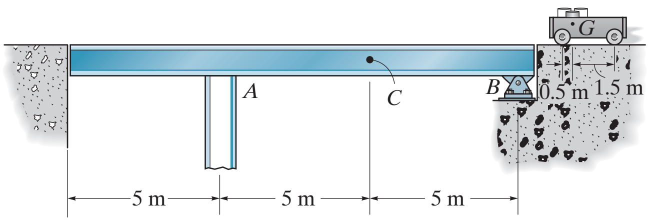


(c)

Fig. 6–38

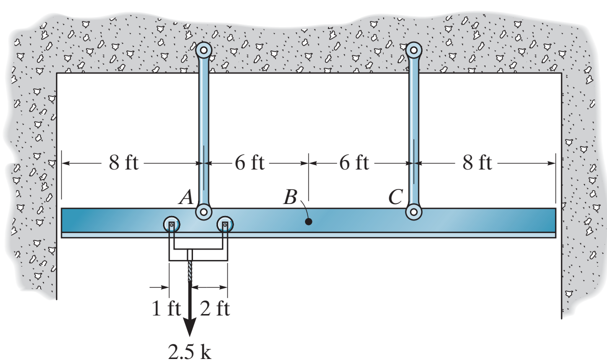
PROBLEMS

6–59. Determine the maximum moment at point C on the single girder caused by the moving dolly that has a mass of 2 Mg and a mass center at G. Assume A is a roller.



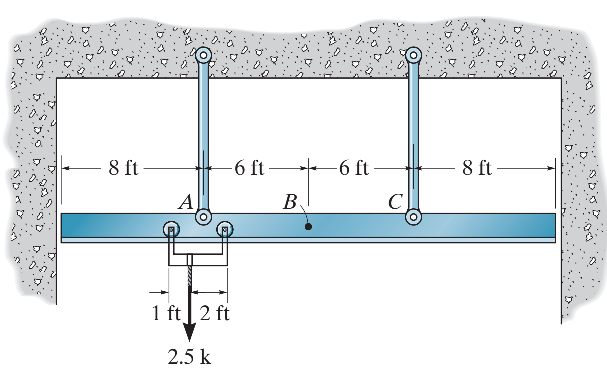
Prob. 6–59

***6–60.** Determine the maximum moment in the suspended rail at point B if the rail supports the load of 2.5 k on the trolley.



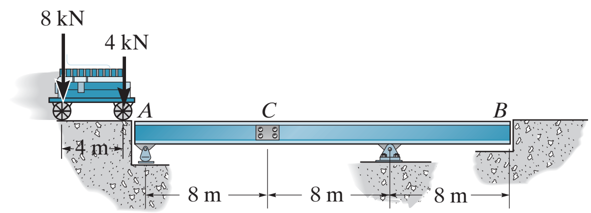
Prob. 6–60

6–61. Determine the maximum positive shear at point B if the rail supports the load of 2.5 k on the trolley.



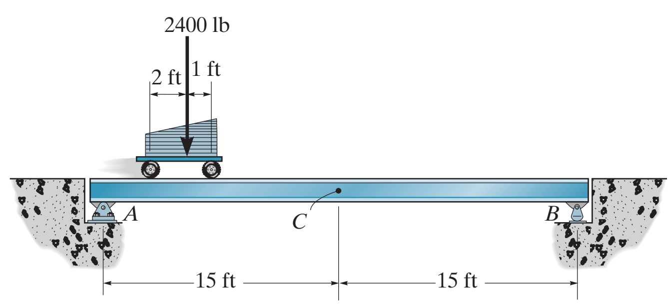
Prob. 6–61

6–62. Determine the maximum positive moment at the splice C on the side girder caused by the moving load which travels along the center of the bridge.



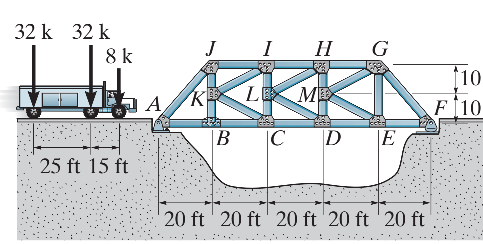
Prob. 6–62

6–63. Determine the maximum moment at C caused by the moving load.



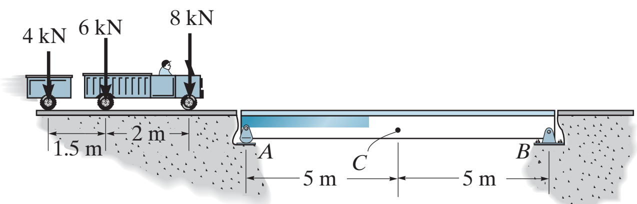
Prob. 6–63

***6–64.** Draw the influence line for the force in member IH of the bridge truss. Determine the maximum force (tension or compression) that can be developed in this member due to a 72-k truck having the wheel loads shown. Assume the truck can travel in *either direction* along the *center* of the deck, so that half its load is transferred to each of the two side trusses. Also assume the members are pin connected at the gusset plates.



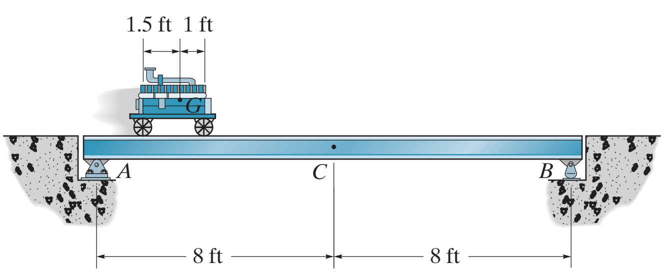
Prob. 6–64

6–65. Determine the maximum positive moment at point C on the single girder caused by the moving load.



Prob. 6–65

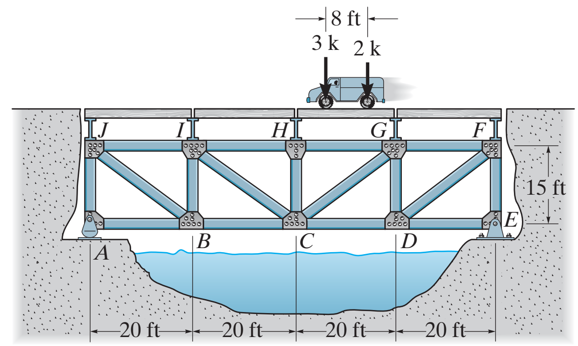
6–66. The cart has a weight of 2500 lb and a center of gravity at G. Determine the maximum positive moment created in the side girder at C as it crosses the bridge. Assume the car can travel in either direction along the *center* of the deck, so that *half* its load is transferred to each of the two side girders.



Prob. 6–66

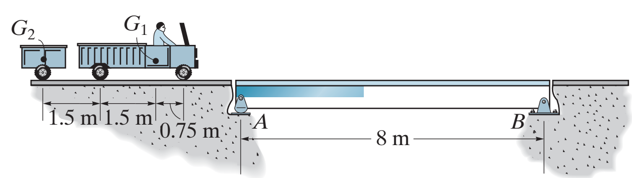
6–67. Draw the influence line for the force in member BC of the bridge truss. Determine the maximum force (tension or compression) that can be developed in the member due to a 5-k truck having the wheel loads shown. Assume the truck can travel in *either direction* along the *center* of the deck, so that *half* the load shown is transferred to each of the two side trusses. Also assume the members are pin connected at the gusset plates.

***6–68.** Draw the influence line for the force in member IC of the bridge truss. Determine the maximum force (tension or compression) that can be developed in the member due to a 5-k truck having the wheel loads shown. Assume the truck can travel in *either direction* along the *center* of the deck, so that half the load shown is transferred to each of the two side trusses. Also assume the members are pin connected at the gusset plates.



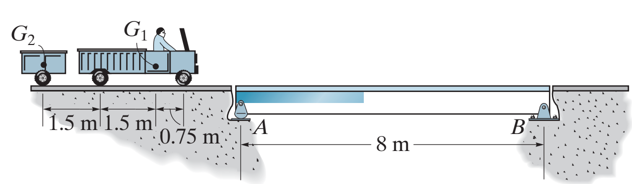
Probs. 6–67/6–68

6–69. The truck has a mass of 4 Mg and mass center at G_1, and the trailer has a mass of 1 Mg and mass center at G_2. Determine the absolute maximum live moment developed in the bridge.



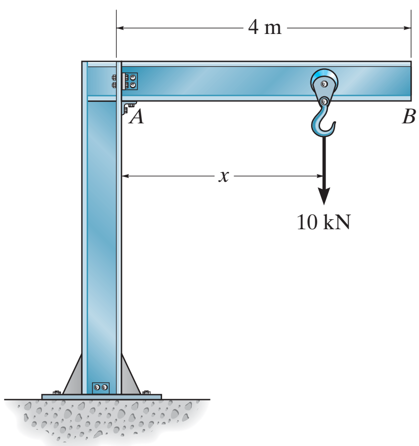
Prob. 6–69

6–70. Determine the absolute maximum live moment in the bridge in Problem 6–69 if the trailer is removed.



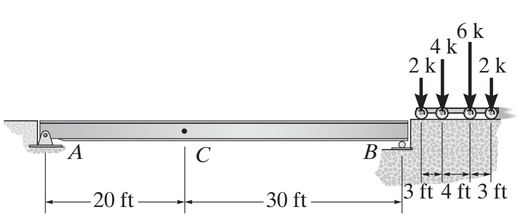
Prob. 6–70

6–71. Determine the absolute maximum live shear and absolute maximum moment in the jib beam AB due to the 10-kN loading. The end constraints require $0.1 \text{ m} \leq x \leq 3.9 \text{ m}$.



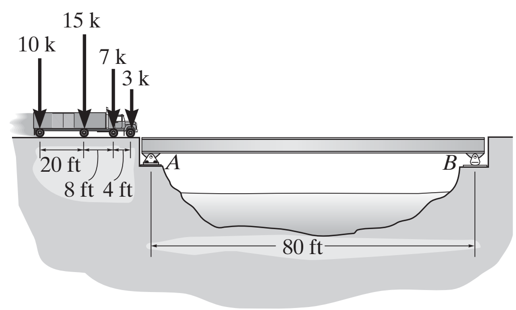
Prob. 6–71

***6–72.** Determine the maximum moment at C caused by the moving loads.



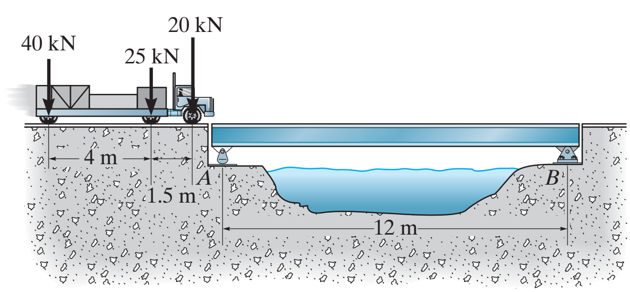
Prob. 6–72

6–73. Determine the absolute maximum moment in the girder bridge due to the truck loading shown. The load is applied directly to the girder.



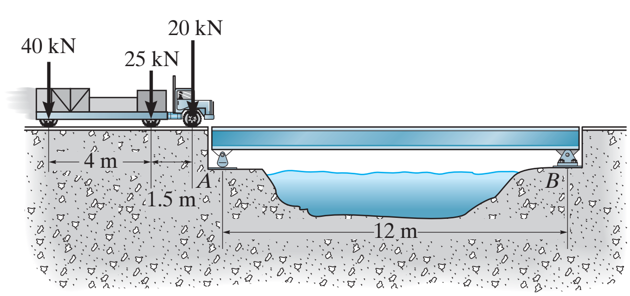
Prob. 6–73

6–74. Determine the absolute maximum shear in the beam due to the loading shown.



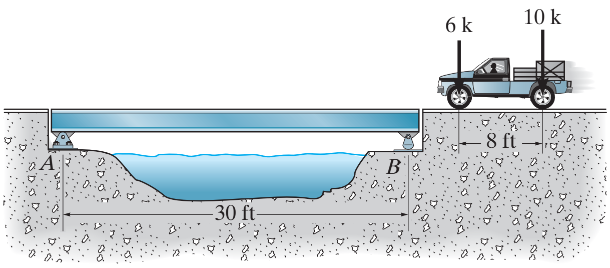
Prob. 6–74

6–75. Determine the absolute maximum moment in the beam due to the loading shown.



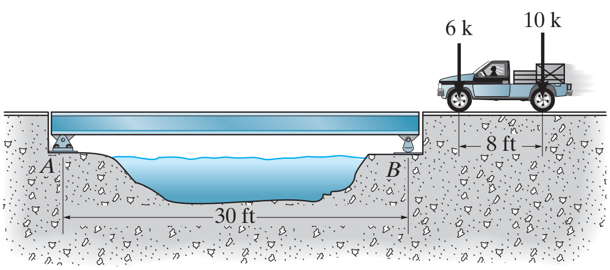
Prob. 6–75

***6–76.** Determine the absolute maximum shear in the bridge girder due to the loading shown.



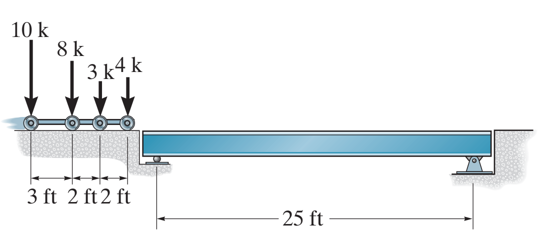
Prob. 6–76

6–77. Determine the absolute maximum moment in the bridge girder due to the loading shown.



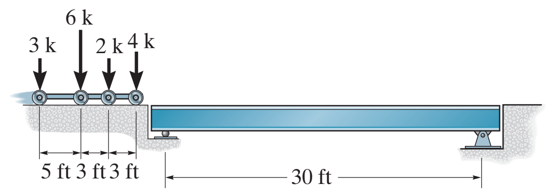
Prob. 6–77

6–78. Determine the absolute maximum moment in the girder due to the loading shown.



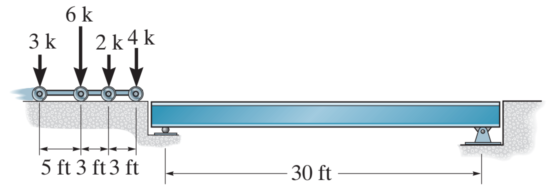
Prob. 6–78

6–79. Determine the absolute maximum shear in the beam due to the loading shown.



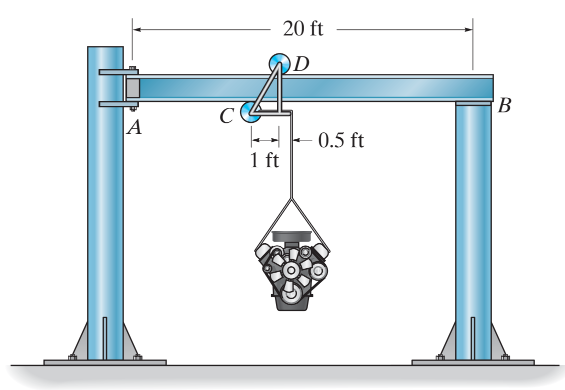
Prob. 6–79

***6–80.** Determine the absolute maximum moment in the bridge due to the loading shown.



Prob. 6–80

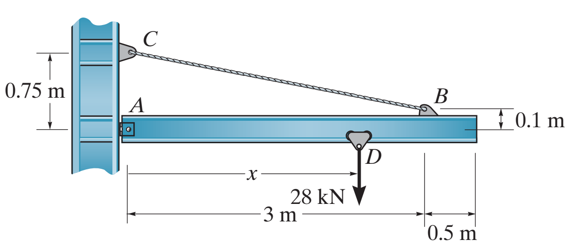
6–81. The trolley rolls at C and D along the bottom and top flange of beam AB. Determine the absolute maximum moment developed in the beam if the load supported by the trolley is 2 k. Assume the support at A is a pin and at B a roller.



Prob. 6–81

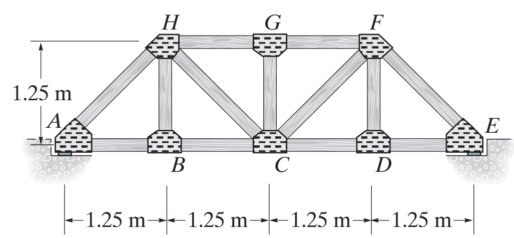
PROJECT PROBLEMS

6–1P. The chain hoist on the wall crane can be placed anywhere along the boom ($0.1 \text{ m} < x < 3.4 \text{ m}$) and has a rated capacity of 28 kN. Use an impact factor of 0.3 and determine the absolute maximum bending moment in the boom and the maximum force developed in the tie rod *BC*. The boom is pinned to the wall column at its left end *A*. Neglect the size of the trolley at *D*.

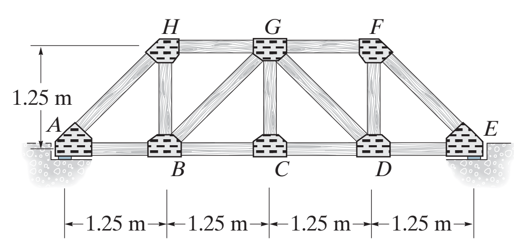


Prob. 6–1P

6–2P. A simply supported pedestrian bridge is to be constructed in a city park and two designs have been proposed as shown in case *a* and case *b*. The truss members are to be made from timber. The deck consists of 1.5-m-long planks that have a mass of 20 kg/m^2. A local code states the live load on the deck is required to be 5 kPa with an impact factor of 0.2. Consider the deck to be simply supported on stringers. Floor beams then transmit the load to the bottom joints of the truss. (See Fig. 6–23.) In each case find the member subjected to the largest tension and largest compression load and suggest why you would choose one design over the other. Neglect the weights of the truss members.



case *a*



case *b*

Prob. 6–2P

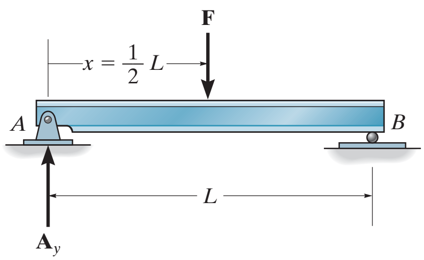
6

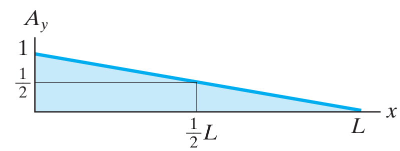
CHAPTER REVIEW

An influence line indicates the value of a reaction, shear, or moment at a specific point on a member, as a unit load moves over the member.

Once the influence line for a reaction, shear, or moment (function) is constructed, then it will be possible to locate the live load on the member to produce the maximum positive or negative value of the function.

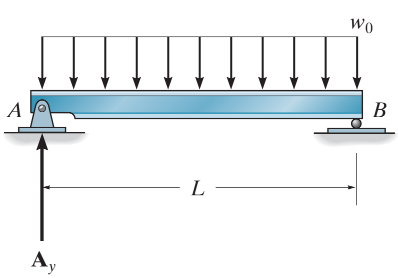
A concentrated live force is applied at the positive (negative) peaks of the influence line. The value of the function is then equal to the product of the influence line ordinate and the magnitude of the force.

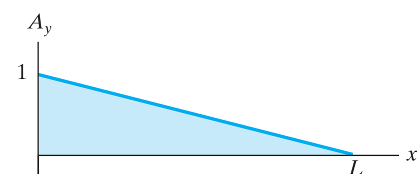




$$A_y = \left(\tfrac{1}{2}\right)F$$

A uniform distributed load extends over a positive (negative) region of the influence line. The value of the function is then equal to the product of the area under the influence line for the region and the magnitude of the uniform load.

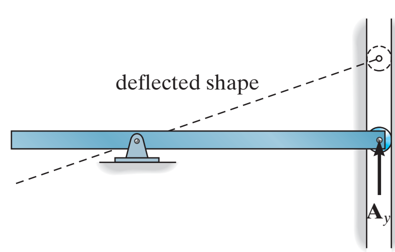




$$A_y = \tfrac{1}{2}(1)(L)(w_0)$$

The general shape of the influence line can be established using the Müller-Breslau principle, which states that the influence line for a reaction, shear, or moment is to the same scale as the deflected shape of the member when it is acted upon by the reaction, shear, or moment.

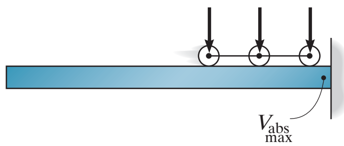
(a)



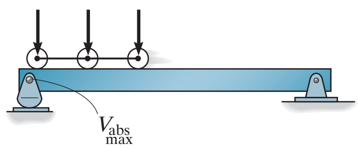
(b)

Influence lines for floor girders and trusses can be established by placing the unit load at each panel point or joint, and calculating the value of the required reaction, shear, or moment.

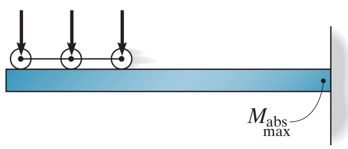
When a series of concentrated loads pass over the member, then the various positions of the load on the member have to be considered to determine the largest shear or moment in the member. In general, place the loadings so that each contributes its maximum influence, as determined by multiplying each load by the ordinate of the influence line. This process of finding the actual position can be done using a trial-and-error procedure, or by finding the change in either the shear or moment when the loads are moved from one position to another. Each moment is investigated until a negative value of shear or moment occurs. Once this happens the previous position will define the critical loading.



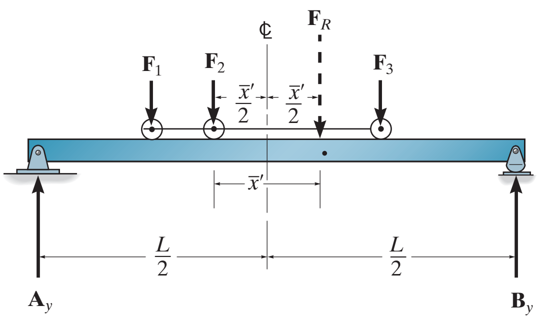
Absolute maximum *shear* in a cantilever or simply supported beam will occur at a support, when one of the loads is placed next to the support.



Absolute maximum *moment* in a cantilevered beam occurs when the series of concentrated loads are placed at the farthest point away from the fixed support.



To determine the absolute maximum moment in a simply supported beam, the resultant of the force system is first determined. Then it, along with one of the concentrated forces in the system is positioned so that these two forces are equidistant from the centerline of the beam. The maximum moment then occurs under the selected force. Each force in the system is selected in this manner, and by comparison the largest for all these cases is the absolute maximum moment.



The portal to this bridge must resist loteral loads due to wind and traffic. An approximate analysis can be made of the forces produced for a preliminary design of the members, before a more exact structural analysis is done.

Approximate Analysis of Statically Indeterminate Structures

7

In this chapter we will present some of the approximate methods used to analyze statically indeterminate trusses and frames. These methods were developed on the basis of structural behavior, and their accuracy in most cases compares favorably with more exact methods of analysis. Although not all types of structural forms will be discussed here, it is hoped that enough insight is gained from the study of these methods so that one can judge what would be the best approximations to make when performing an approximate force analysis of a statically indeterminate structure.

7.1 Use of Approximate Methods

When a *model* is used to represent any structure, the analysis of it must satisfy *both* the conditions of equilibrium and compatibility of displacement at the joints. As will be shown in later chapters of this text, the compatibility conditions for a *statically indeterminate* structure can be related to the loads provided we know the material's modulus of elasticity and the size and shape of the members. For an initial design, however, we will *not* know a member's size, and so a statically indeterminate analysis cannot be considered. For analysis a simpler model of the structure must be developed, one that is statically determinate. Once this model is specified, the analysis of it is called an *approximate analysis*. By performing an approximate analysis, a preliminary design of the members of a structure can be made, and when this is complete, the more exact indeterminate analysis can then be performed and the design refined. An approximate analysis also provides insight as to a structure's behavior under load and is beneficial when checking a more exact analysis or when time, money, or capability are not available for performing the more exact analysis.

Realize that, in a general sense, all methods of structural analysis are approximate, simply because the actual conditions of loading, geometry, material behavior, and joint resistance at the supports are never known in an *exact sense*. In this text, however, the statically indeterminate analysis of a structure will be referred to as an *exact analysis*, and the simpler statically determinate analysis will be referred to as the *approximate analysis*.

7.2 Trusses

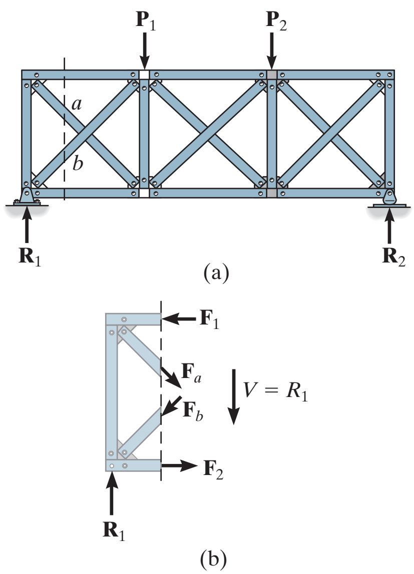


Fig. 7–1

A common type of truss often used for lateral bracing of a building or for the top and bottom cords of a bridge is shown in Fig. 7–1*a*. (Also see Fig. 3–4.) When used for this purpose, this truss is not considered a primary element for the support of the structure, and as a result it is often analyzed by approximate methods. In the case shown, it will be noticed that if a diagonal is removed from each of the three panels, it will render the truss statically determinate. Hence, the truss is statically indeterminate to the third degree (using Eq. 3–1, $b + r > 2j$, or $16 + 3 > 8(2)$) and therefore we must make three assumptions regarding the bar forces in order to reduce the truss to one that is statically determinate. These assumptions can be made with regard to the cross-diagonals, realizing that when one diagonal in a panel is in tension the corresponding cross-diagonal will be in compression. This is evident from Fig. 7–1*b*, where the "panel shear" **V** is carried by the *vertical component* of tensile force in member *a* and the *vertical component* of compressive force in member *b*. Two methods of analysis are generally acceptable.

Method 1: If the diagonals are intentionally designed to be *long and slender*, it is reasonable to assume that they *cannot* support a compressive force; otherwise, they may easily buckle. Hence the panel shear is resisted entirely by the *tension diagonal*, whereas the *compressive diagonal is assumed to be a zero-force member*.

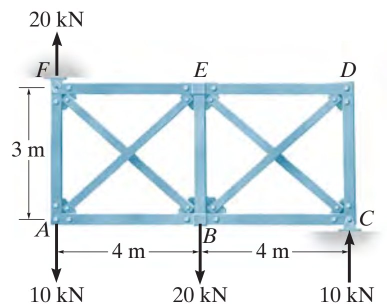
Method 2: If the diagonal members are intended to be constructed from large rolled sections such as angles or channels, they may be equally capable of supporting a tensile and compressive force. Here we will assume that the tension and compression diagonals each carry *half* the panel shear.

Both of these methods of approximate analysis are illustrated numerically in the following examples.

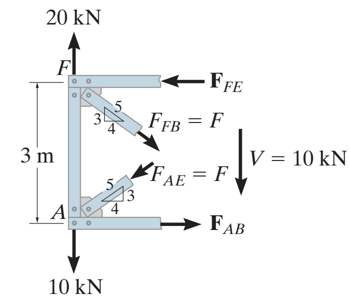
An approximate method can be used to determine the forces in the cross bracing in each panel of this bascule railroad bridge. Here the cross members are thin and so we can assume they carry no compressive force.

EXAMPLE 7.1

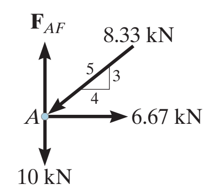
Determine (approximately) the forces in the members of the truss shown in Fig. 7–2*a*. The diagonals are to be designed to support both tensile and compressive forces, and therefore each is assumed to carry half the panel shear. The support reactions have been computed.



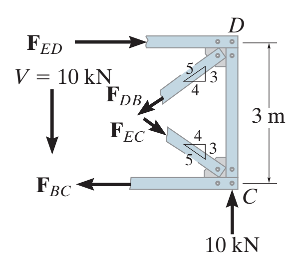
(a)



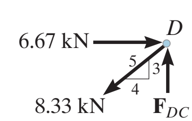
(b)



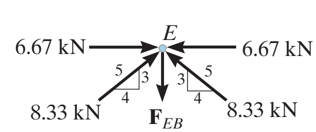
(c)



(d)



(e)



(f)

Fig. 7–2

SOLUTION

By inspection the truss is statically indeterminate to the second degree. The two assumptions require the tensile and compressive diagonals to carry equal forces, that is, $F_{FB} = F_{AE} = F$. For a vertical section through the left panel, Fig. 7–2*b*, we have

$$+\uparrow\Sigma F_y = 0; \qquad 20 - 10 - 2\left(\tfrac{3}{5}\right)F = 0 \qquad F = 8.33 \text{ kN} \qquad \textit{Ans.}$$

so that

$$F_{FB} = 8.33 \text{ kN (T)} \qquad \textit{Ans.}$$

$$F_{AE} = 8.33 \text{ kN (C)} \qquad \textit{Ans.}$$

$$\circlearrowright{+}\Sigma M_A = 0; \quad -8.33\left(\tfrac{4}{5}\right)(3) + F_{FE}(3) = 0 \quad F_{FE} = 6.67 \text{ kN (C)} \quad \textit{Ans.}$$

$$\circlearrowright{+}\Sigma M_F = 0; \quad -8.33\left(\tfrac{4}{5}\right)(3) + F_{AB}(3) = 0 \quad F_{AB} = 6.67 \text{ kN (T)} \quad \textit{Ans.}$$

From joint *A*, Fig. 7–2*c*,

$$+\uparrow\Sigma F_y = 0; \quad F_{AF} - 8.33\left(\tfrac{3}{5}\right) - 10 = 0 \qquad F_{AF} = 15 \text{ kN (T)} \quad \textit{Ans.}$$

A vertical section through the right panel is shown in Fig. 7–2*d*. Show that

$$F_{DB} = 8.33 \text{ kN (T)}, \quad F_{ED} = 6.67 \text{ kN (C)} \qquad \textit{Ans.}$$

$$F_{EC} = 8.33 \text{ kN (C)}, \quad F_{BC} = 6.67 \text{ kN (T)} \qquad \textit{Ans.}$$

Furthermore, using the free-body diagrams of joints *D* and *E*, Figs. 7–2*e* and 7–2*f*, show that

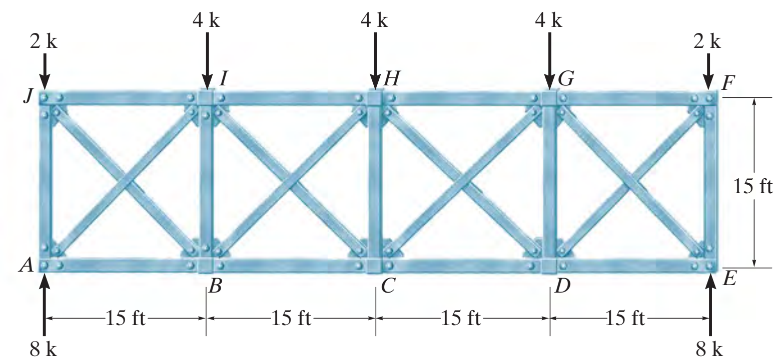
$$F_{DC} = 5 \text{ kN (C)} \qquad \textit{Ans.}$$

$$F_{EB} = 10 \text{ kN (T)} \qquad \textit{Ans.}$$

7

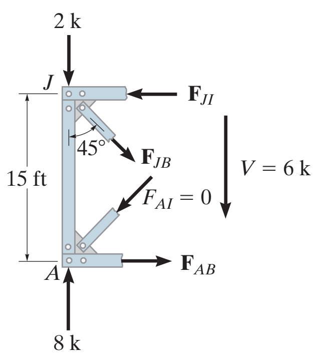
EXAMPLE 7.2

Cross bracing is used to provide lateral support for this bridge deck due to the wind and unbalanced traffic loads. Determine (approximately) the forces in the members of this truss. Assume the diagonals are slender and therefore will not support a compressive force. The loads and support reactions are shown in Fig. 7–3*a*.



(a)

Fig. 7–3



(b)

SOLUTION

By inspection the truss is statically indeterminate to the fourth degree. Thus the four assumptions to be used require that each compression diagonal sustain zero force. Hence, from a vertical section through the left panel, Fig. 7–3*b*, we have

$$F_{AI} = 0 \qquad \textit{Ans.}$$

$$+\uparrow \Sigma F_y = 0; \qquad 8 - 2 - F_{JB} \cos 45^\circ = 0$$

$$F_{JB} = 8.49 \text{ k (T)} \qquad \textit{Ans.}$$

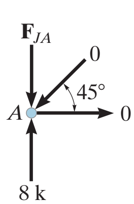
$$\circlearrowleft + \Sigma M_A = 0; \qquad -8.49 \sin 45^\circ(15) + F_{JI}(15) = 0$$

$$F_{JI} = 6 \text{ k (C)} \qquad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_J = 0; \qquad -F_{AB}(15) = 0$$

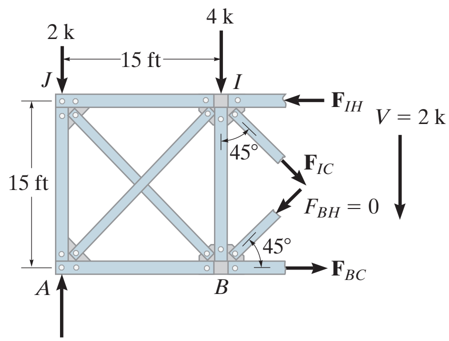
$$F_{AB} = 0 \qquad \textit{Ans.}$$

From joint *A*, Fig. 7–3*c*,



(c)

$$F_{JA} = 8 \text{ k (C)} \qquad \textit{Ans.}$$



(d)

A vertical section of the truss through members IH, IC, BH, and BC is shown in Fig. 7–3d. The panel shear is $V = \Sigma F_y = 8 - 2 - 4 = 2$ k. We require

$$F_{BH} = 0 \qquad \textit{Ans.}$$

$$+\uparrow\Sigma F_y = 0; \qquad 8 - 2 - 4 - F_{IC}\cos 45^\circ = 0$$

$$F_{IC} = 2.83 \text{ k (T)} \qquad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_B = 0; \quad -8(15) + 2(15) - 2.83 \sin 45^\circ(15) + F_{IH}(15) = 0$$

$$F_{IH} = 8 \text{ k (C)} \qquad \textit{Ans.}$$

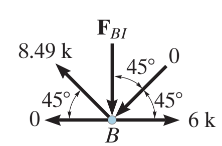
$$\circlearrowleft + \Sigma M_I = 0; \qquad -8(15) + 2(15) + F_{BC}(15) = 0$$

$$F_{BC} = 6 \text{ k (T)} \qquad \textit{Ans.}$$

From joint B, Fig. 7–3e,

$$+\uparrow\Sigma \mathrm{F}_y = 0; \qquad 8.49 \sin 45^\circ - F_{BI} = 0$$

$$F_{BI} = 6 \text{ k (C)} \qquad \textit{Ans.}$$

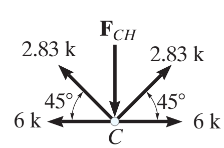


(e)

The forces in the other members can be determined by symmetry, except F_{CH}; however, from joint C, Fig. 7–3f, we have

$$+\uparrow\Sigma \mathrm{F}_y = 0; \qquad 2(2.83 \sin 45^\circ) - F_{CH} = 0$$

$$F_{CH} = 4 \text{ k (C)} \qquad \textit{Ans.}$$

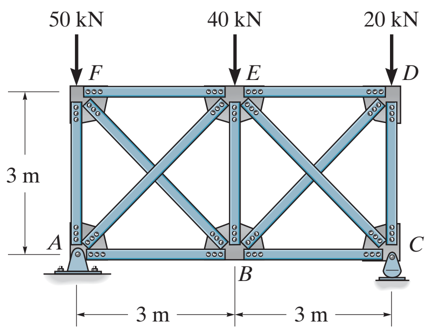


(f)

PROBLEMS

7–1. Determine (approximately) the force in each member of the truss. Assume the diagonals can support either a tensile or a compressive force.

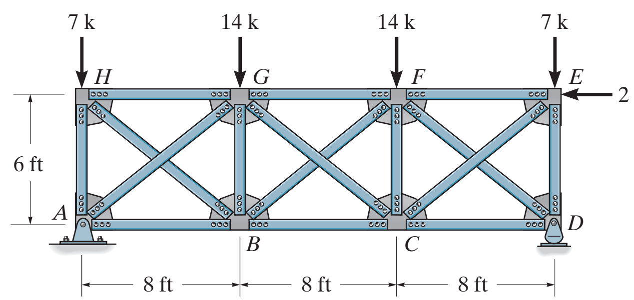
7–2. Solve Prob. 7–1 assuming that the diagonals cannot support a compressive force.



Probs. 7–1/7–2

7–5. Determine (approximately) the force in each member of the truss. Assume the diagonals can support either a tensile or a compressive force.

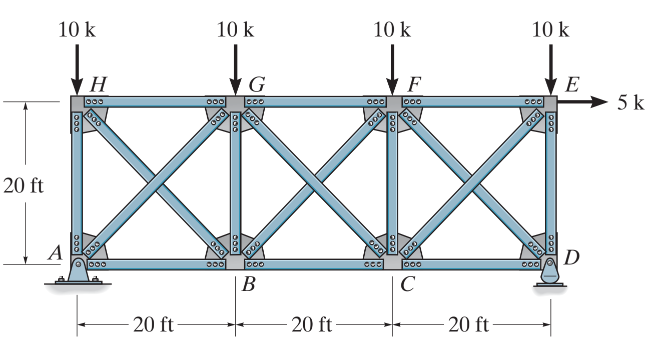
7–6. Solve Prob. 7–5 assuming that the diagonals cannot support a compressive force.



Probs. 7–5/7–6

7–3. Determine (approximately) the force in each member of the truss. Assume the diagonals can support either a tensile or a compressive force.

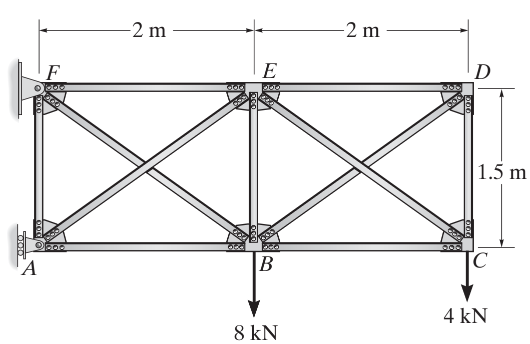
***7–4.** Solve Prob. 7–3 assuming that the diagonals cannot support a compressive force.



Probs. 7–3/7–4

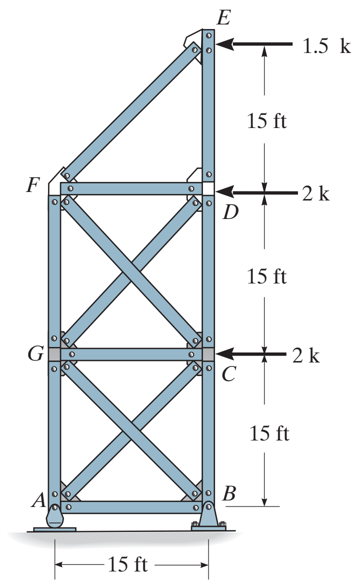
7–7. Determine (approximately) the force in each member of the truss. Assume the diagonals can support either a tensile or compressive force.

***7–8.** Solve Prob. 7–7 assuming that the diagonals cannot support a compressive force.



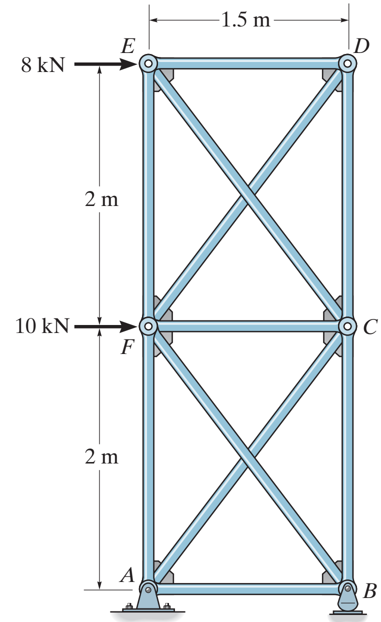
Probs. 7–7/7–8

7–9. Determine (approximately) the force in each member of the truss. Assume the diagonals can support both tensile and compressive forces.



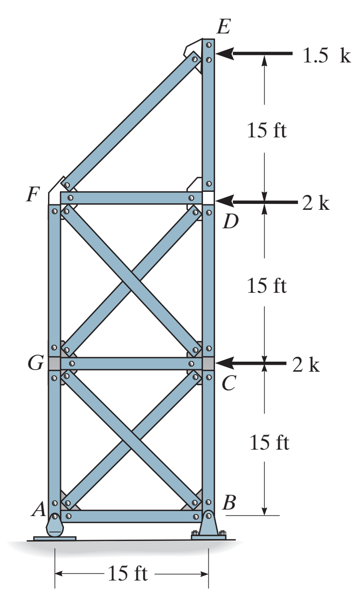
Prob. 7–9

7–11. Determine (approximately) the force in each member of the truss. Assume the diagonals can support either a tensile or compressive force.



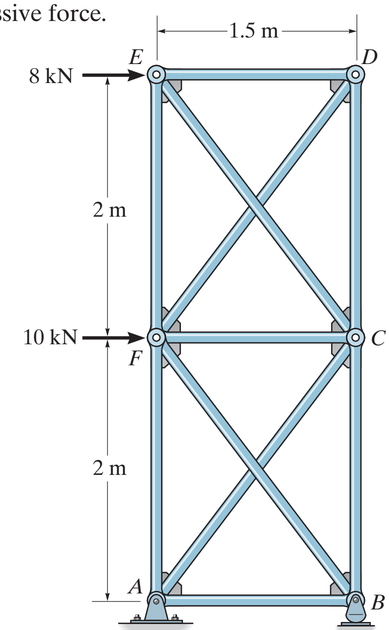
Prob. 7–11

7–10. Determine (approximately) the force in each member of the truss. Assume the diagonals *DG* and *AC* cannot support a compressive force.



Prob. 7–10

***7–12.** Determine (approximately) the force in each member of the truss. Assume the diagonals cannot support a compressive force.



Prob. 7–12

7.3 Vertical Loads on Building Frames

Building frames often consist of girders that are *rigidly connected* to columns so that the entire structure is better able to resist the effects of lateral forces due to wind and earthquake. An example of such a rigid framework, often called a building bent, is shown in Fig. 7–4.

In practice, a structural engineer can use several techniques for performing an approximate analysis of a building bent. Each is based upon knowing how the structure *will deform under load*. One technique would be to consider only the members within a localized region of the structure. This is possible provided the deflections of the members within the region cause little disturbance to the members outside the region. Most often, however, the deflection curve of the entire structure is considered. From this, the approximate location of points of inflection, that is, the points where the member changes its curvature, can be specified. These points can be considered as *pins* since there is zero moment within the member at the points of inflection. We will use this idea in this section to analyze the forces on building frames due to vertical loads, and in Secs. 7–5 and 7–6 an approximate analysis for frames subjected to lateral loads will be presented. Since the frame can be subjected to both of these loadings simultaneously, then, provided the material remains elastic, the resultant loading is determined by superposition.

Assumptions for Approximate Analysis. Consider a typical girder located within a building bent and subjected to a uniform vertical load, as shown in Fig. 7–5*a*. The column supports at A and B will each exert three reactions on the girder, and therefore the girder will be statically indeterminate to the third degree (6 reactions – 3 equations of equilibrium). To make the girder statically determinate, an approximate analysis will therefore require three assumptions. If the columns are extremely stiff, no rotation at A and B will occur, and the deflection

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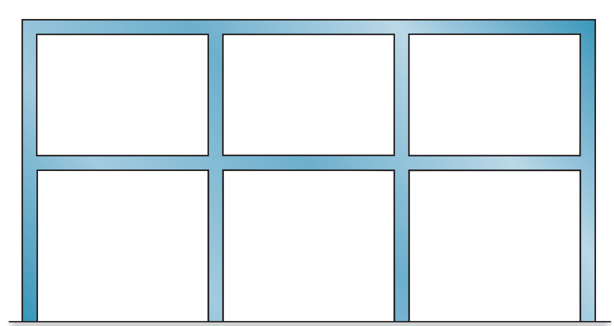


typical building frame

Fig. 7–4

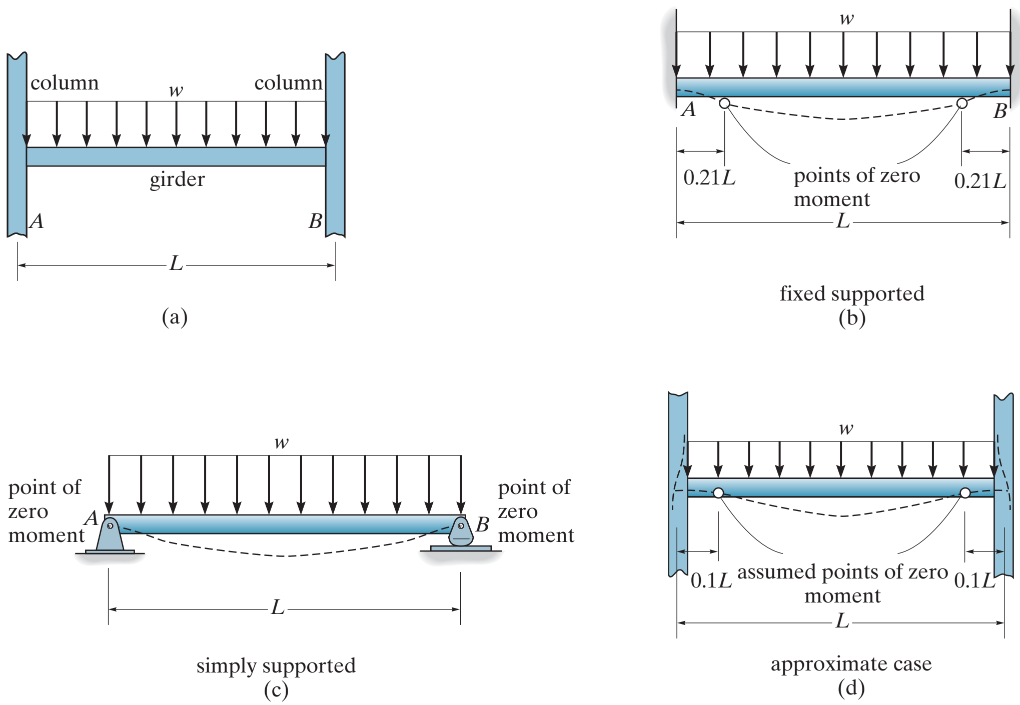


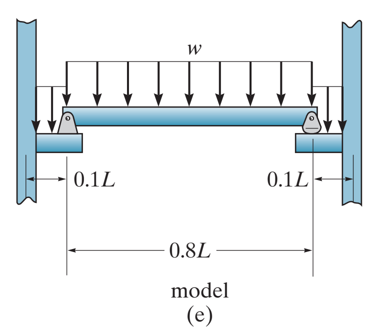
Fig. 7–5

curve for the girder will look like that shown in Fig. 7–5*b*. Using one of the methods presented in Chapters 9 through 11, an exact analysis reveals that for this case inflection points, or points of zero moment, occur at $0.21L$ from each support. If, however, the column connections at A and B are very flexible, then like a simply supported beam, zero moment will occur at the supports, Fig. 7–5*c*. In reality, however, the columns will provide some flexibility at the supports, and therefore we will assume that zero moment occurs at the *average point* between the two extremes, i.e., at $(0.21L + 0)/2 \approx 0.1L$ from each support, Fig. 7–5*d*. Furthermore, an exact analysis of frames supporting vertical loads indicates that the axial forces in the girder are negligible.

In summary then, each girder of length L may be modeled by a simply supported span of length $0.8L$ resting on two cantilevered ends, each having a length of $0.1L$, Fig. 7–5*e*. The following three assumptions are incorporated in this model:

1. There is zero moment in the girder, $0.1L$ from the left support.
2. There is zero moment in the girder, $0.1L$ from the right support.
3. The girder does not support an axial force.

By using statics, the internal loadings in the girders can now be obtained and a preliminary design of their cross sections can be made. The following example illustrates this numerically.



7

EXAMPLE 7.3

Determine (approximately) the moment at the joints E and C caused by members EF and CD of the building bent in Fig. 7–6a.

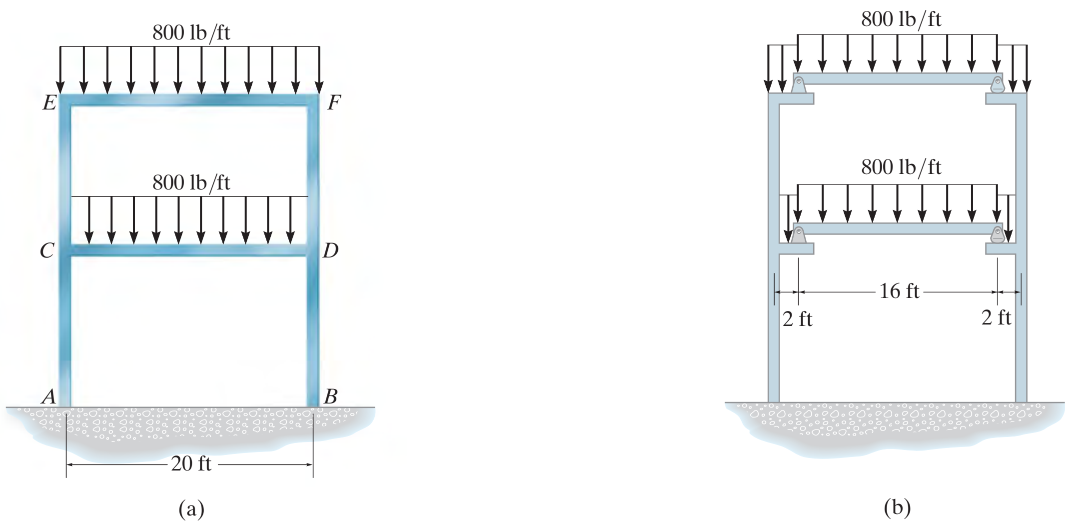


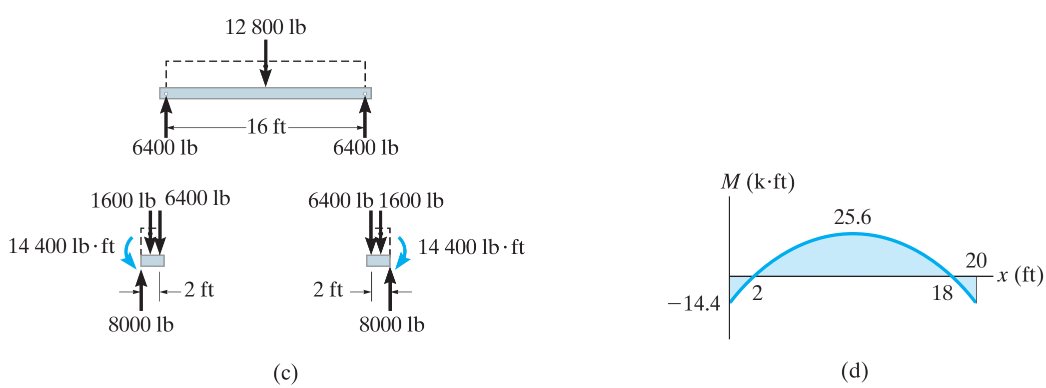
Fig. 7–6

SOLUTION

For an approximate analysis the frame is modeled as shown in Fig. 7–6b. Note that the cantilevered spans supporting the center portion of the girder have a length of $0.1L = 0.1(20) = 2$ ft. Equilibrium requires the end reactions for the center portion of the girder to be 6400 lb, Fig. 7–6c. The cantilevered spans are then subjected to a reaction moment of

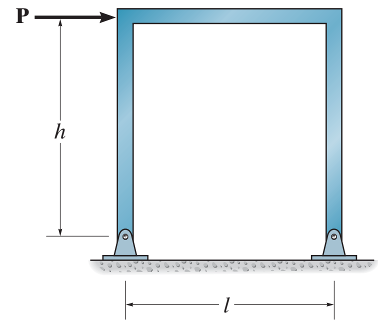
$$M = 1600(1) + 6400(2) = 14\,400 \text{ lb}\cdot\text{ft} = 14.4 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

This approximate moment, with opposite direction, acts on the joints at E and C, Fig. 7–6a. Using the results, the approximate moment diagram for one of the girders is shown in Fig. 7–6d.



7.4 Portal Frames and Trusses

Frames. Portal frames are frequently used over the entrance of a bridge* and as a main stiffening element in building design in order to transfer horizontal forces applied at the top of the frame to the foundation. On bridges, these frames resist the forces caused by wind, earthquake, and unbalanced traffic loading on the bridge deck. Portals can be pin supported, fixed supported, or supported by partial fixity. The approximate analysis of each case will now be discussed for a simple three-member portal.

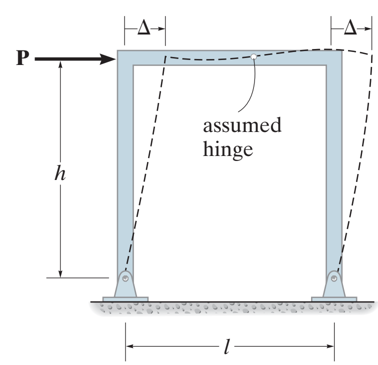


(a)

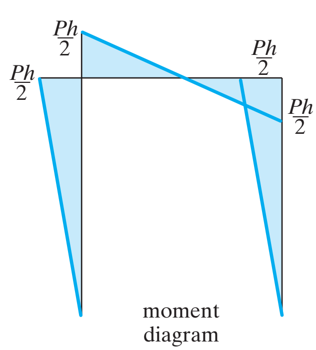
Fig. 7–7

Pin Supported. A typical pin-supported portal frame is shown in Fig. 7–7*a*. Since four unknowns exist at the supports but only three equilibrium equations are available for solution, this structure is statically indeterminate to the first degree. Consequently, only one assumption must be made to reduce the frame to one that is statically determinate.

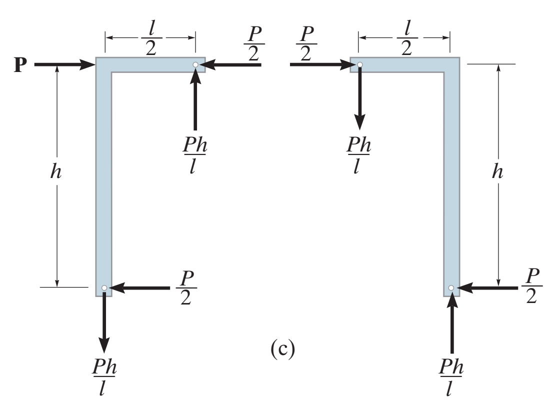
The elastic deflection of the portal is shown in Fig. 7–7*b*. This diagram indicates that a point of inflection, that is, where the moment changes from positive bending to negative bending, is located *approximately* at the girder's midpoint. Since the moment in the girder is zero at this point, we can *assume* a hinge exists there and then proceed to determine the reactions at the supports using statics. If this is done, it is found that the horizontal reactions (shear) at the base of each column are *equal* and the other reactions are those indicated in Fig. 7–7*c*. Furthermore, the moment diagrams for this frame are indicated in Fig. 7–7*d*.



(b)

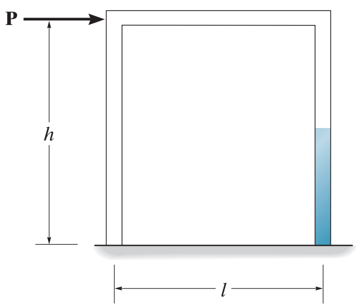


(d)



(c)

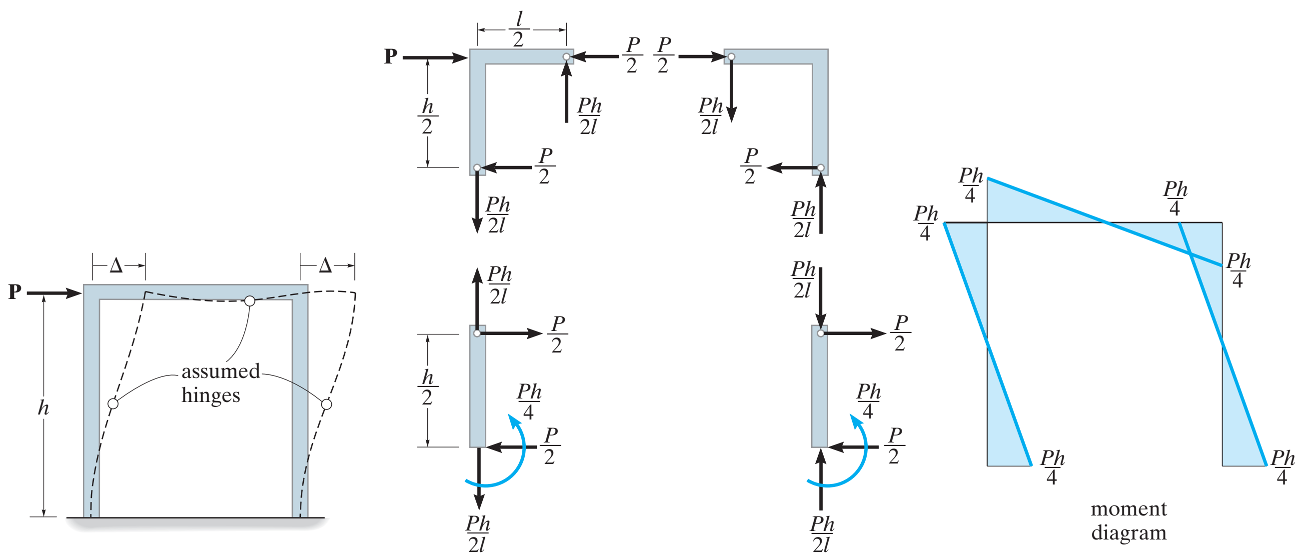
*See Fig. 3–4.



(a)

Fig. 7–8

Fixed Supported. Portals with two fixed supports, Fig. 7–8*a*, are statically indeterminate to the third degree since there are a total of six unknowns at the supports. If the vertical members have equal lengths and cross-sectional areas, the frame will deflect as shown in Fig. 7–8*b*. For this case we will *assume* points of inflection occur at the midpoints of all three members, and therefore hinges are placed at these points. The reactions and moment diagrams for each member can therefore be determined by dismembering the frame at the hinges and applying the equations of equilibrium to each of the four parts. The results are shown in Fig. 7–8*c*. Note that, as in the case of the pin-connected portal, the horizontal reactions (shear) at the base of each column are *equal*. The moment diagram for this frame is indicated in Fig. 7–8*d*.



(b) (c) (d)

Partial Fixity. Since it is both difficult and costly to construct a perfectly fixed support or foundation for a portal frame, it is conservative and somewhat realistic to assume a slight rotation occurs at the supports, Fig. 7–9*a*. As a result, the points of inflection on the columns lie somewhere between the case of having a pin-supported portal, Fig. 7–7*a*, where the "inflection points" are at the supports (base of columns), and a fixed-supported portal, Fig. 7–8*a*, where the inflection points are at the center of the columns. Many engineers arbitrarily define the location at $h/3$, Fig. 7–9*b*, and therefore place hinges at these points, and also at the center of the girder.

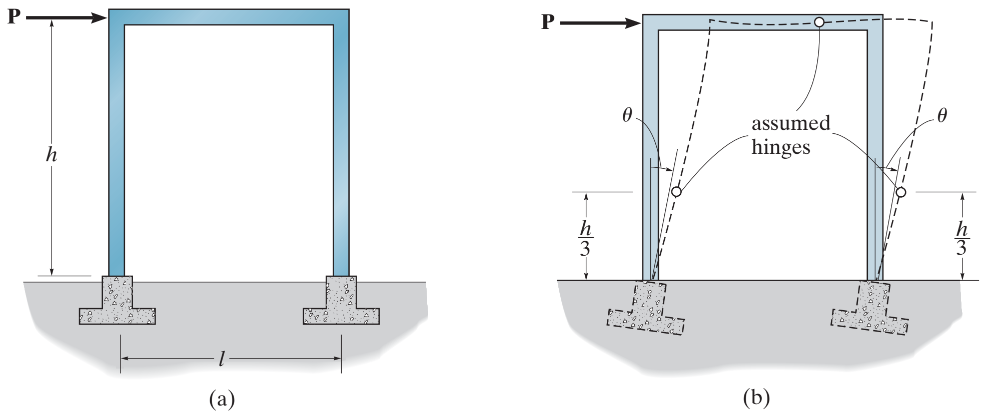


Fig. 7–9

Trusses. When a portal is used to span large distances, a truss may be used in place of the horizontal girder. Such a structure is used on large bridges and as transverse bents for large auditoriums and mill buildings. A typical example is shown in Fig. 7–10*a*. In all cases, the suspended truss is assumed to be pin connected at its points of attachment to the columns. Furthermore, the truss keeps the columns straight within the region of attachment when the portal is subjected to the sidesway Δ, Fig. 7–10*b*. Consequently, we can analyze trussed portals using the same assumptions as those used for simple portal frames. For pin-supported columns, assume the horizontal reactions (shear) are equal, as in Fig. 7–7*c*. For fixed-supported columns, assume the horizontal reactions are equal and an inflection point (or hinge) occurs on each column, measured midway between the base of the column and the *lowest point* of truss member connection to the column. See Fig. 7–8*c* and Fig. 7–10*b*.

The following example illustrates how to determine the forces in the members of a trussed portal using the approximate method of analysis described above.

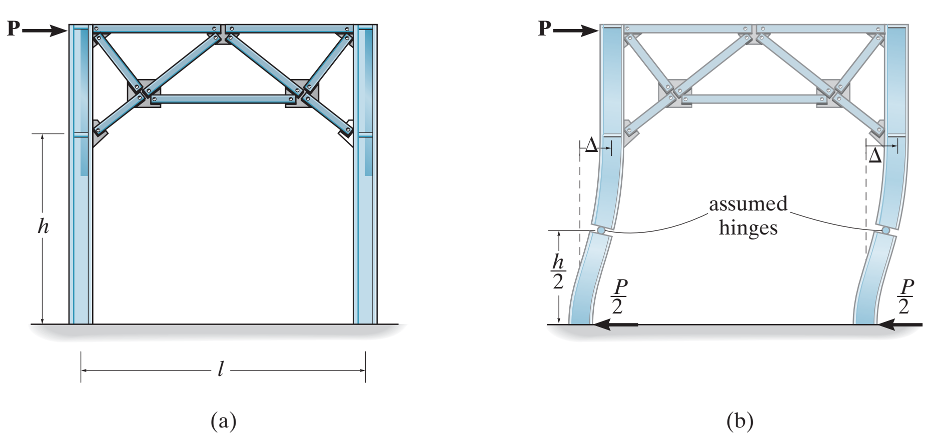


Fig. 7–10

EXAMPLE 7.4

Determine by approximate methods the forces acting in the members of the Warren portal shown in Fig. 7–11*a*.

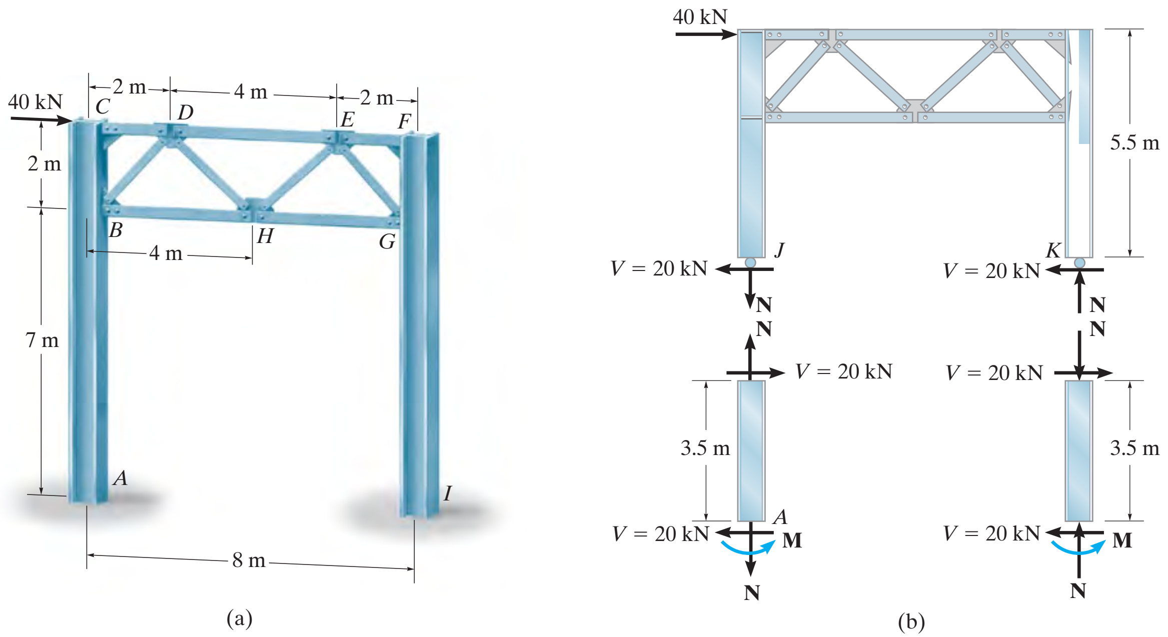


Fig. 7–11

SOLUTION

The truss portion B, C, F, G acts as a rigid unit. Since the supports are fixed, a point of inflection is assumed to exist $7\text{ m}/2 = 3.5\text{ m}$ above A and I, and equal horizontal reactions or shear act at the base of the columns, i.e., $\Sigma F_x = 0$; $V = 40\text{ kN}/2 = 20\text{ kN}$. With these assumptions, we can separate the structure at the hinges J and K, Fig. 7–11*b*, and determine the reactions on the columns as follows:

Lower Half of Column

$$\circlearrowleft + \Sigma M_A = 0; \qquad M - 3.5(20) = 0 \qquad M = 70\text{ kN}\cdot\text{m}$$

Upper Portion of Column

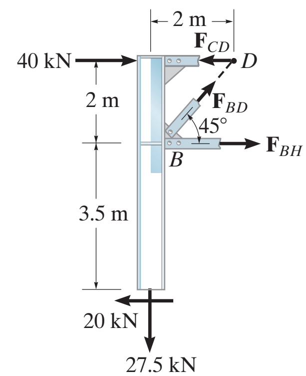
$$\circlearrowleft + \Sigma M_J = 0; \qquad -40(5.5) + N(8) = 0 \qquad N = 27.5\text{ kN}$$

Using the method of sections, Fig. 7–11c, we can now proceed to obtain the forces in members CD, BD, and BH.

$$+\uparrow \Sigma F_y = 0; \quad -27.5 + F_{BD} \sin 45^\circ = 0 \qquad F_{BD} = 38.9 \text{ kN (T)} \quad \textit{Ans.}$$

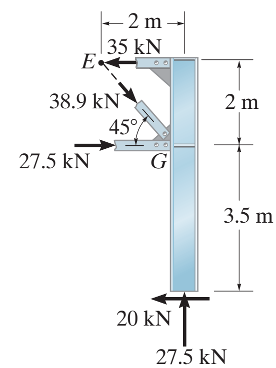
$$\circlearrowleft + \Sigma M_B = 0; \quad -20(3.5) - 40(2) + F_{CD}(2) = 0 \quad F_{CD} = 75 \text{ kN (C)} \quad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_D = 0; \quad F_{BH}(2) - 20(5.5) + 27.5(2) = 0 \quad F_{BH} = 27.5 \text{ kN (T)} \quad \textit{Ans.}$$



(c)

In a similar manner, show that one obtains the results on the free-body diagram of column FGI in Fig. 7–11d. Using these results, we can now find the force in each of the other truss members of the portal using the method of joints.



(d)

Joint *D*, Fig. 7–11*e*

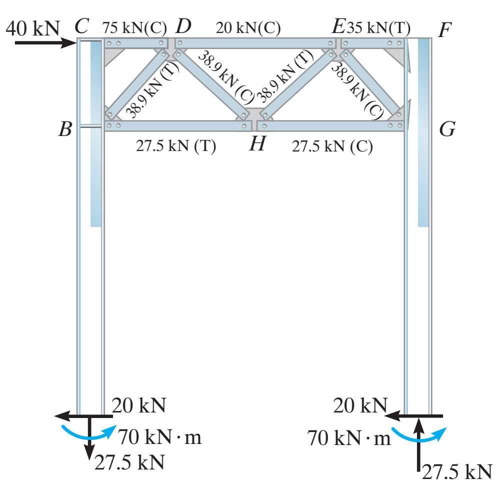
$$+\uparrow \Sigma F_y = 0; \quad F_{DH} \sin 45^\circ - 38.9 \sin 45^\circ = 0 \quad F_{DH} = 38.9 \text{ kN (C)} \quad \textit{Ans.}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \quad 75 - 2(38.9 \cos 45^\circ) - F_{DE} = 0 \quad F_{DE} = 20 \text{ kN (C)} \quad \textit{Ans.}$$

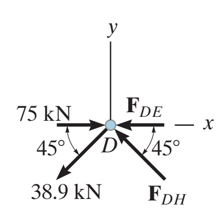
Joint *H*, Fig. 7–11*f*

$$+\uparrow \Sigma F_y = 0; \quad F_{HE} \sin 45^\circ - 38.9 \sin 45^\circ = 0 \quad F_{HE} = 38.9 \text{ kN (T)} \quad \textit{Ans.}$$

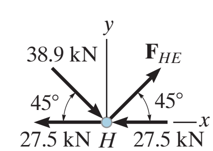
These results are summarized in Fig. 7–11g.



(g)



(e)

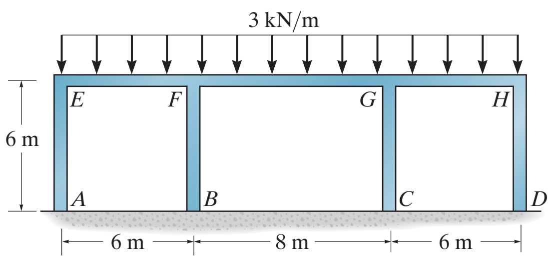


(f)

7

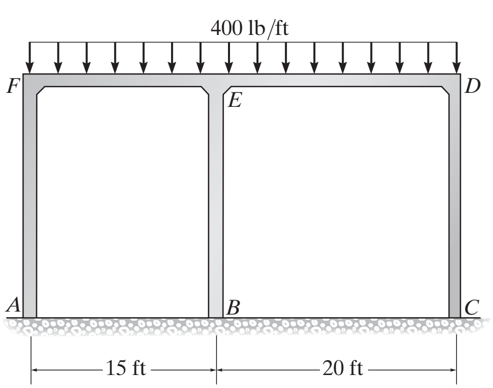
PROBLEMS

7–13. Determine (approximately) the internal moments at joints A and B of the frame.



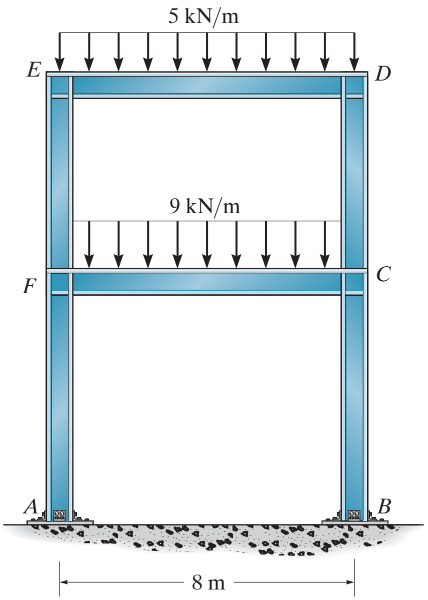
Prob. 7–13

7–14. Determine (approximately) the internal moments at joints F and D of the frame.



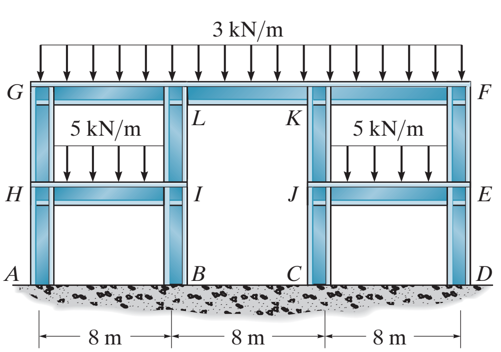
Prob. 7–14

7–15. Determine (approximately) the internal moment at A caused by the vertical loading.



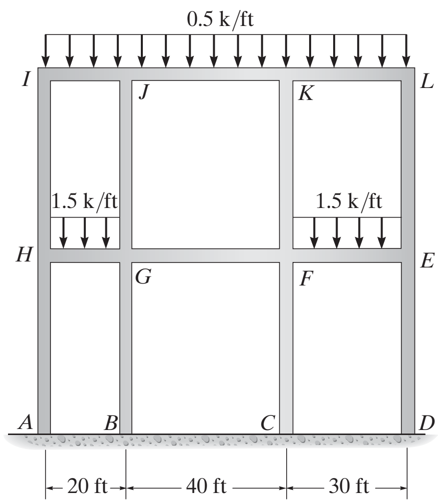
Prob. 7–15

***7–16.** Determine (approximately) the internal moments at A and B caused by the vertical loading.



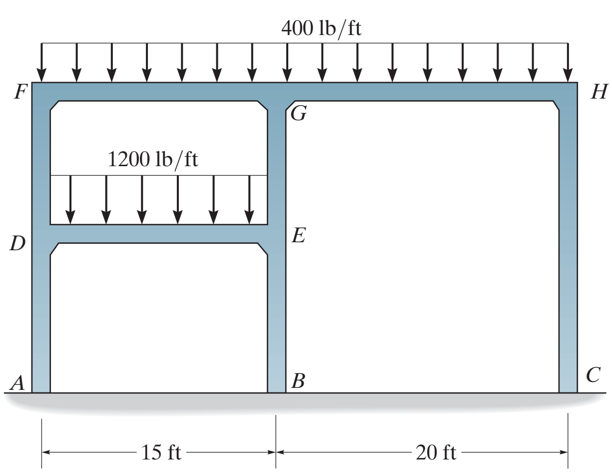
Prob. 7–16

7–17. Determine (approximately) the internal moments at joints I and L. Also, what is the internal moment at joint H caused by member HG?



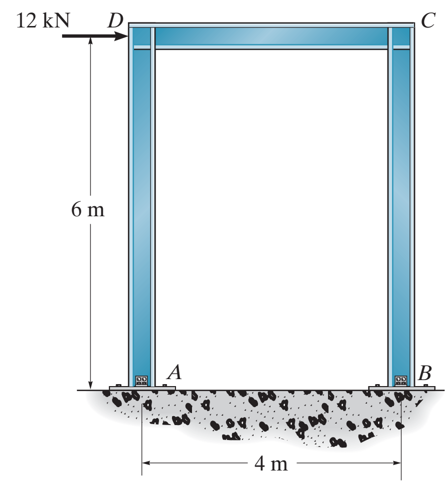
Prob. 7–17

7–18. Determine (approximately) the support actions at A, B, and C of the frame.



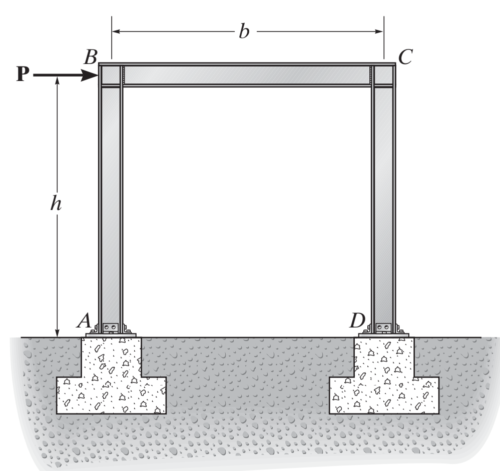
Prob. 7–18

7–19. Determine (approximately) the support reactions at A and B of the portal frame. Assume the supports are (a) pinned, and (b) fixed.



Prob. 7–19

***7–20.** Determine (approximately) the internal moment and shear at the ends of each member of the portal frame. Assume the supports at A and D are partially fixed, such that an inflection point is located at $h/3$ from the bottom of each column.

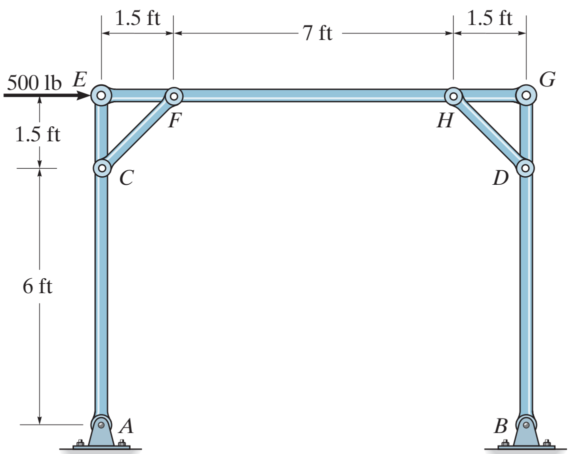


Prob. 7–20

7

7–21. Draw (approximately) the moment diagram for member ACE of the portal constructed with a *rigid* member EG and knee braces CF and DH. Assume that all points of connection are pins. Also determine the force in the knee brace CF.

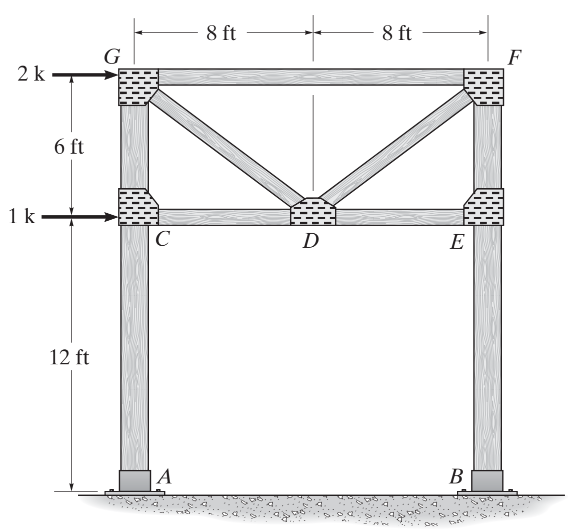
7–22. Solve Prob. 7–21 if the supports at A and B are fixed instead of pinned.



Probs. 7–21/7–22

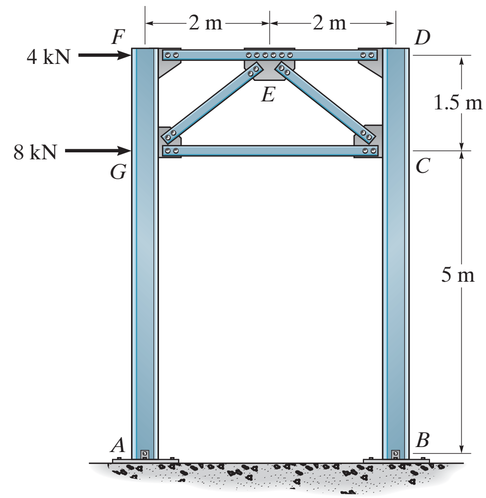
7–23. Determine (approximately) the force in each truss member of the portal frame. Also find the reactions at the fixed column supports A and B. Assume all members of the truss to be pin connected at their ends.

***7–24.** Solve Prob. 7–23 if the supports at A and B are pinned instead of fixed.



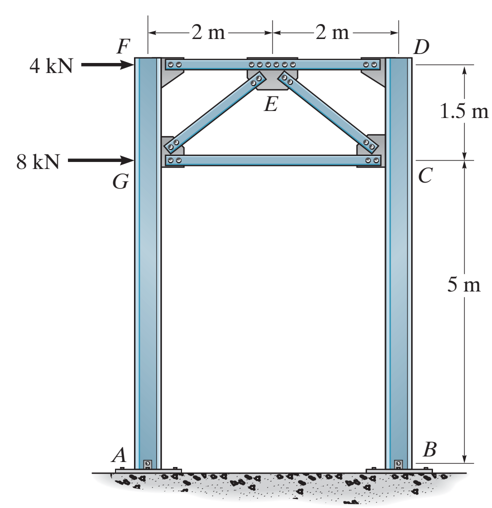
Probs. 7–23/7–24

7–25. Draw (approximately) the moment diagram for column AGF of the portal. Assume all truss members and the columns to be pin connected at their ends. Also determine the force in all the truss members.



Prob. 7–25

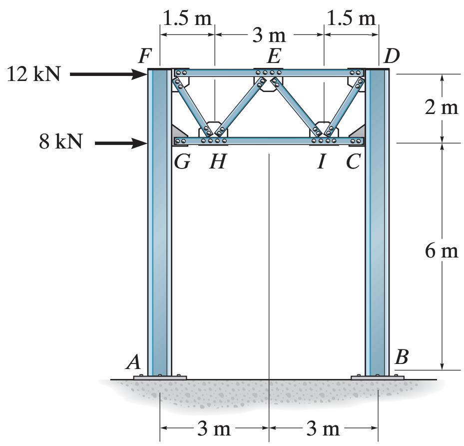
7–26. Draw (approximately) the moment diagram for column AGF of the portal. Assume all the members of the truss to be pin connected at their ends. The columns are fixed at A and B. Also determine the force in all the truss members.



Prob. 7–26

7–27. Determine (approximately) the force in each truss member of the portal frame. Also find the reactions at the fixed column supports A and B. Assume all members of the truss to be pin connected at their ends.

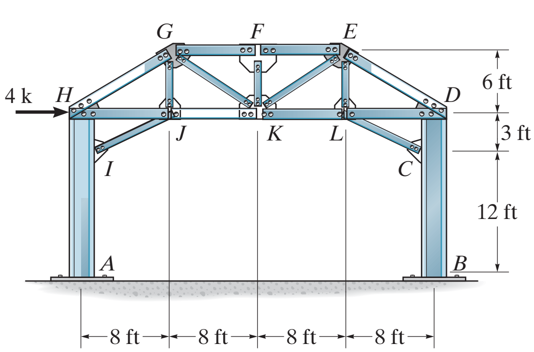
***7–28.** Solve Prob. 7–27 if the supports at A and B are pinned instead of fixed.



Probs. 7–27/7–28

7–29. Determine (approximately) the force in members GF, GK, and JK of the portal frame. Also find the reactions at the fixed column supports A and B. Assume all members of the truss to be connected at their ends.

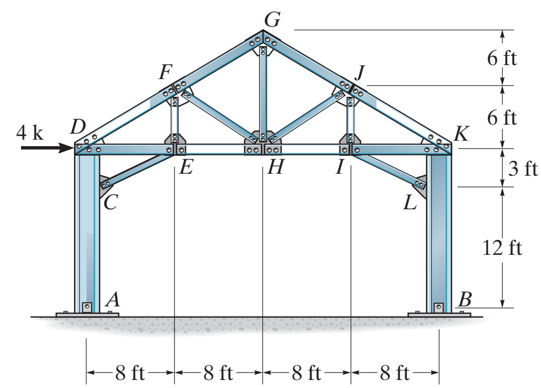
7–30. Solve Prob. 7–29 if the supports at A and B are pin connected instead of fixed.



Probs. 7–29/7–30

7–31. Draw (approximately) the moment diagram for column ACD of the portal. Assume all truss members and the columns to be pin connected at their ends. Also determine the force in members FG, FH, and EH.

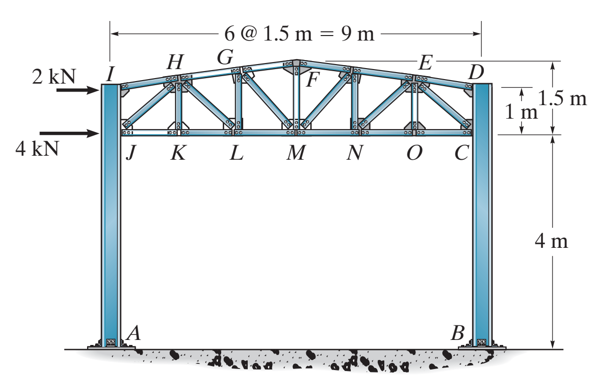
***7–32.** Solve Prob. 7–31 if the supports at A and B are fixed instead of pinned.



Probs. 7–31/7–32

7–33. Draw (approximately) the moment diagram for column AJI of the portal. Assume all truss members and the columns to be pin connected at their ends. Also determine the force in members HG, HL, and KL.

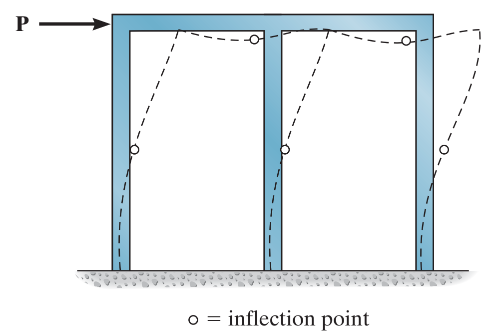
7–34. Solve Prob. 7–33 if the supports at A and B are fixed instead of pinned.



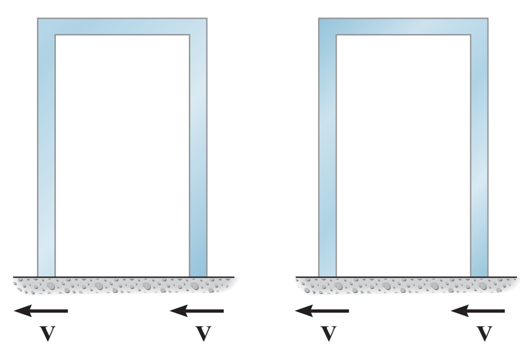
Probs. 7–33/7–34

7.5 Lateral Loads on Building Frames: Portal Method

In Sec. 7–4 we discussed the action of lateral loads on portal frames and found that for a frame fixed supported at its base, points of inflection occur at approximately the center of each girder and column and the columns carry equal shear loads, Fig. 7–8. A building bent deflects in the same way as a portal frame, Fig. 7–12*a*, and therefore it would be appropriate to assume inflection points occur at the center of the columns and girders. If we consider each bent of the frame to be composed of a series of portals, Fig. 7–12*b*, then as a further assumption, the *interior columns* would represent the effect of *two portal columns* and would therefore carry twice the shear V as the two exterior columns.



(a)



(b)

Fig. 7–12

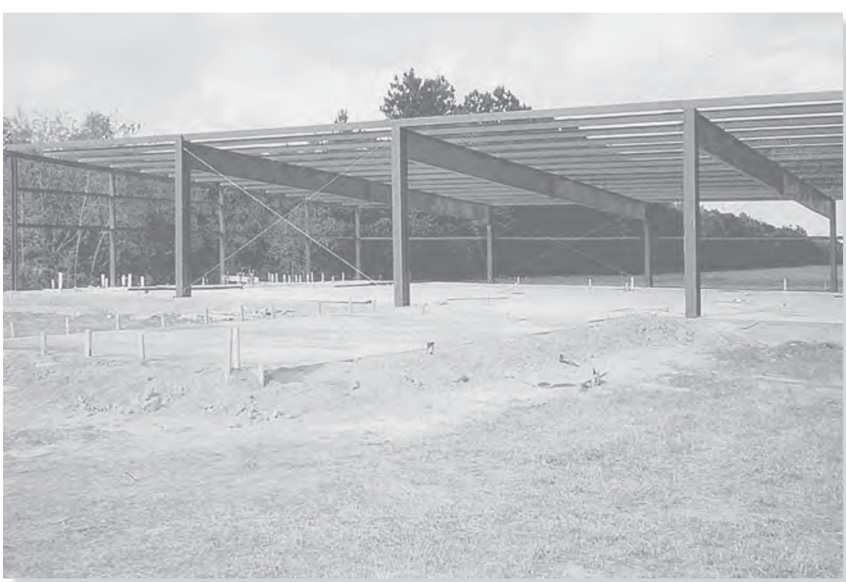
In summary, the portal method for analyzing fixed-supported building frames requires the following assumptions:

1. A hinge is placed at the center of each girder, since this is assumed to be a point of zero moment.
2. A hinge is placed at the center of each column, since this is assumed to be a point of zero moment.
3. At a given floor level the shear at the interior column hinges is twice that at the exterior column hinges, since the frame is considered to be a superposition of portals.

These assumptions provide an adequate reduction of the frame to one that is statically determinate yet stable under loading.

By comparison with the more exact statically indeterminate analysis, *the portal method is most suitable for buildings having low elevation* and *uniform framing*. The reason for this has to do with the structure's action under load. In this regard, *consider the frame as acting like a cantilevered beam* that is fixed to the ground. Recall from mechanics of materials that *shear resistance* becomes more important in the design of *short* beams, whereas *bending* is more important if the beam is *long*. (See Sec. 7–6.) The portal method is based on the assumption related to shear as stated in item 3 above.

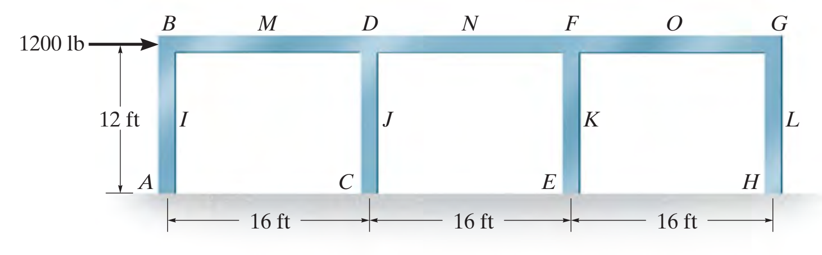
The following examples illustrate how to apply the portal method to analyze a building bent.



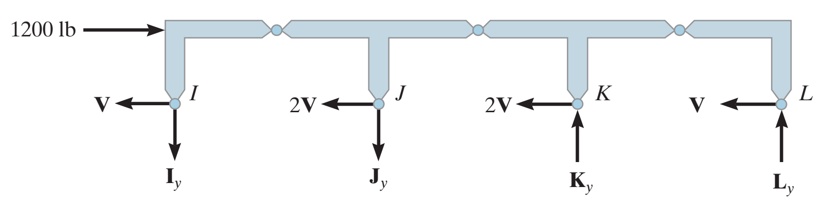
The portal method of analysis can be used to (approximately) perform a lateral-load analysis of this single-story frame.

EXAMPLE 7.5

Determine (approximately) the reactions at the base of the columns of the frame shown in Fig. 7–13*a*. Use the portal method of analysis.



(a)



(b)

Fig. 7–13

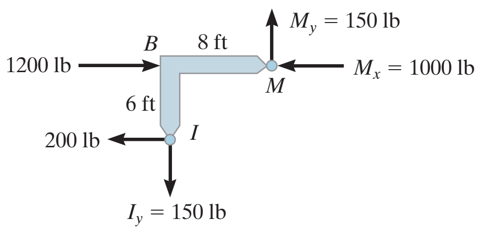
SOLUTION

Applying the first two assumptions of the portal method, we place hinges at the centers of the girders and columns of the frame, Fig. 7–13*a*. A section through the column hinges at I, J, K, L yields the free-body diagram shown in Fig. 7–13*b*. Here the third assumption regarding the column shears applies. We require

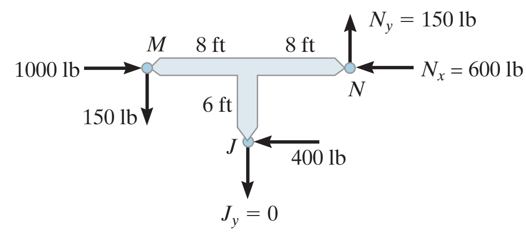
$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 1200 - 6V = 0 \qquad V = 200 \text{ lb}$$

Using this result, we can now proceed to dismember the frame at the hinges and determine their reactions. *As a general rule, always start this analysis at the corner or joint where the horizontal load is applied*. Hence, the free-body diagram of segment IBM is shown in Fig. 7–13*c*. The three reaction components at the hinges I_y, M_x, and M_y are determined by applying $\Sigma M_M = 0$, $\Sigma F_x = 0$, $\Sigma F_y = 0$, respectively. The adjacent segment MJN is analyzed next, Fig. 7–13*d*, followed by segment NKO, Fig. 7–13*e*, and finally segment OGL, Fig. 7–13*f*. Using these results, the free-body diagrams of the columns with their support reactions are shown in Fig. 7–13*g*.

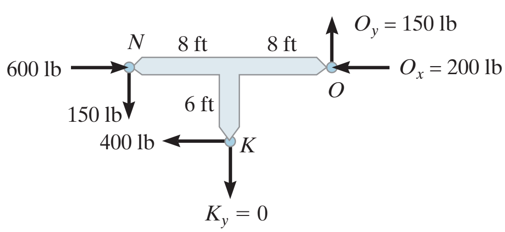
If the horizontal segments of the girders in Figs. 7–13*c*, *d*, *e* and *f* are considered, show that the moment diagram for the girder looks like that shown in Fig. 7–13*h*.



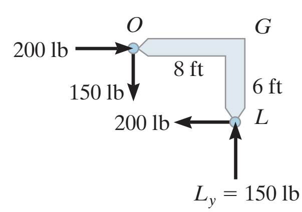
(c)



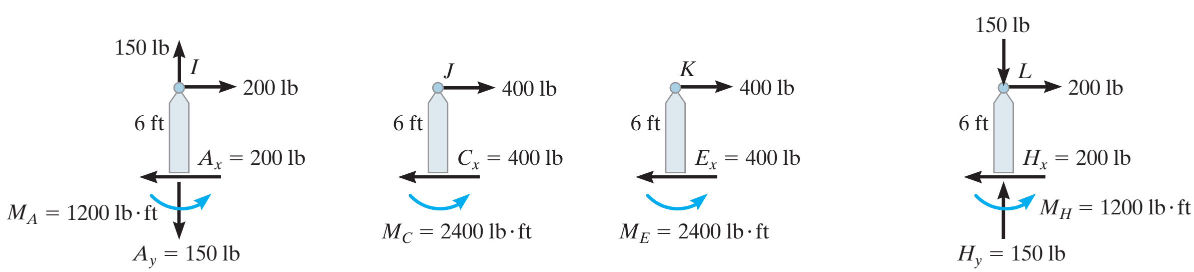
(d)



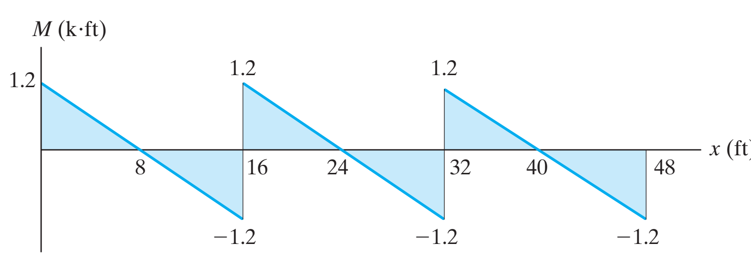
(e)



(f)



(g)

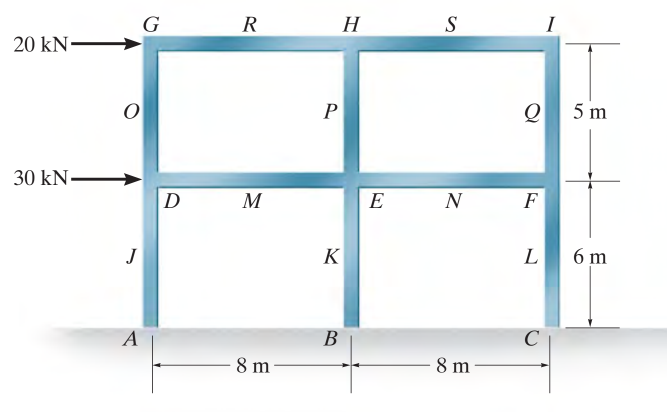


(h)

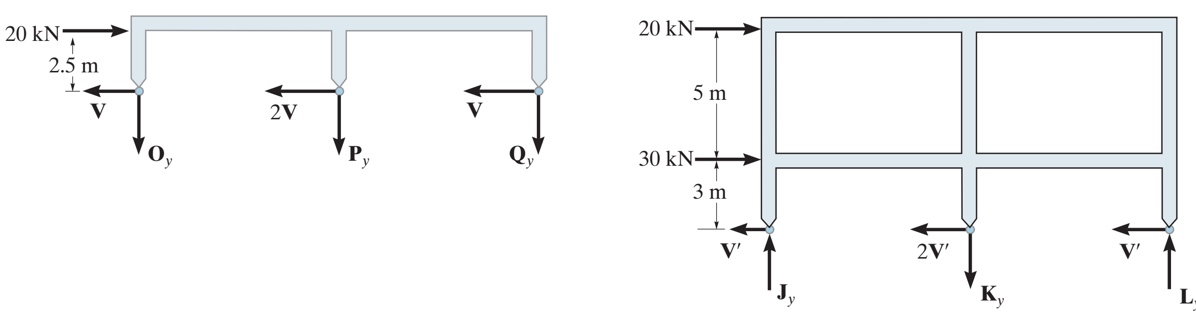
7

EXAMPLE 7.6

Determine (approximately) the reactions at the base of the columns of the frame shown in Fig. 7–14*a*. Use the portal method of analysis.



(a)



(b)

Fig. 7–14

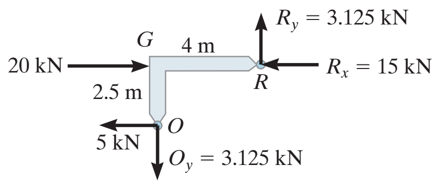
SOLUTION

First hinges are placed at the *centers* of the girders and columns of the frame, Fig. 7–14*a*. A section through the hinges at O, P, Q and J, K, L yields the free-body diagrams shown in Fig. 7–14*b*. The column shears are calculated as follows:

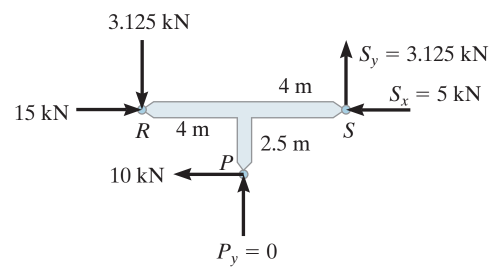
$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 20 - 4V = 0 \qquad V = 5 \text{ kN}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 20 + 30 - 4V' = 0 \qquad V' = 12.5 \text{ kN}$$

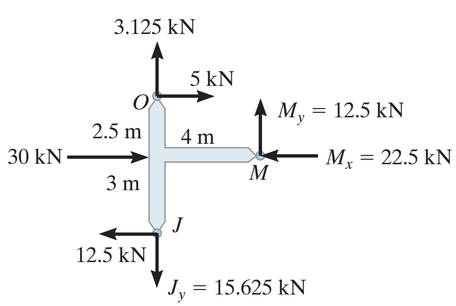
Using these results, we can now proceed to analyze each part of the frame. The analysis starts with the *corner* segment OGR, Fig. 7–14*c*. The three unknowns O_y, R_x, and R_y have been calculated using the equations of equilibrium. With these results segment OJM is analyzed next, Fig. 7–14*d*; then segment JA, Fig. 7–14*e*; RPS, Fig. 7–14*f*; $PMKN$, Fig. 7–14*g*; and KB, Fig. 7–14*h*. Complete this example and analyze segments SIQ, then QNL, and finally LC, and show that $C_x = 12.5$ kN, $C_y = 15.625$ kN, and $M_C = 37.5$ kN·m. Also, use the results and show that the moment diagram for $DMENF$ is given in Fig. 7–14*i*.



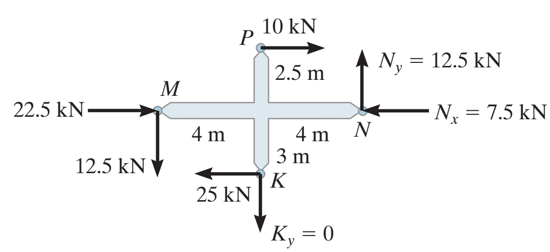
(c)



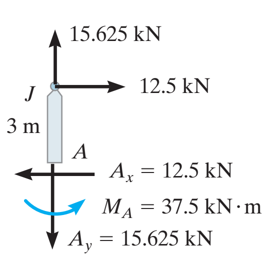
(f)



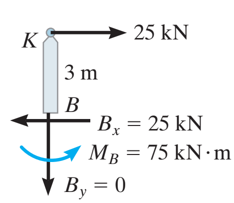
(d)



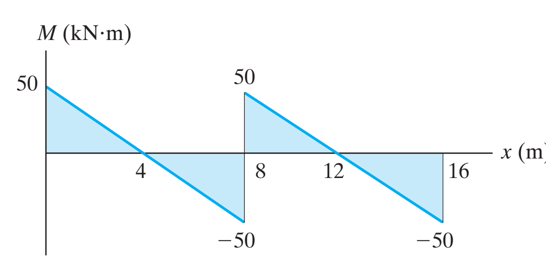
(g)



(e)



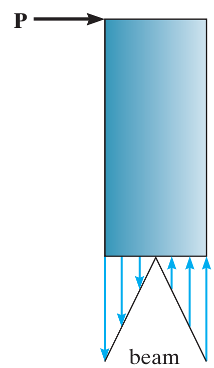
(h)



(i)

7.6 Lateral Loads on Building Frames: Cantilever Method

The cantilever method is based on the same action as a long cantilevered beam subjected to a transverse load. It may be recalled from mechanics of materials that such a loading causes a bending stress in the beam that varies linearly from the beam's neutral axis, Fig. 7–15*a*. In a similar manner, the lateral loads on a frame tend to tip the frame over, or cause a rotation of the frame about a "neutral axis" lying in a horizontal plane that passes through the columns between each floor. To counteract this tipping, the axial forces (or stress) in the columns will be tensile on one side of the neutral axis and compressive on the other side, Fig. 7–15*b*. Like the cantilevered beam, it therefore seems reasonable to assume this axial stress has a linear variation from the centroid of the column areas or neutral axis. *The cantilever method is therefore appropriate if the frame is tall and slender, or has columns with different cross-sectional areas*.



beam
(a)

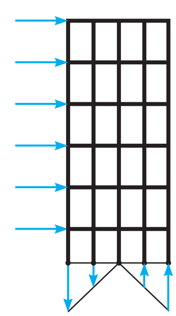

building frame
(b)

Fig. 7–15

In summary, using the cantilever method, the following assumptions apply to a fixed-supported frame.

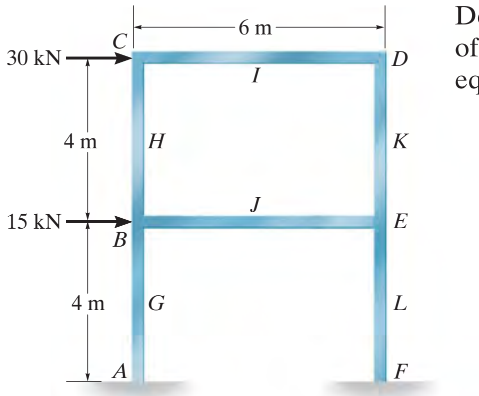
1. A hinge is placed at the center of each girder, since this is assumed to be a point of zero moment.
2. A hinge is placed at the center of each column, since this is assumed to be a point of zero moment.
3. The axial *stress* in a column is proportional to its distance from the centroid of the cross-sectional areas of the columns at a given floor level. Since stress equals force per area, then in the special case of the *columns having equal cross-sectional areas,* the *force* in a column is also proportional to its distance from the centroid of the column areas.

These three assumptions reduce the frame to one that is both stable and statically determinate.

The following examples illustrate how to apply the cantilever method to analyze a building bent.

The building framework has rigid connections. A lateral-load analysis can be performed (approximately) by using the cantilever method of analysis.

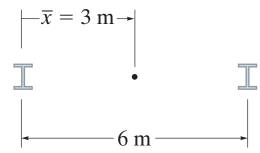
EXAMPLE 7.7



(a)

Fig. 7–16

Determine (approximately) the reactions at the base of the columns of the frame shown in Fig. 7–16*a*. The columns are assumed to have equal cross-sectional areas. Use the cantilever method of analysis.

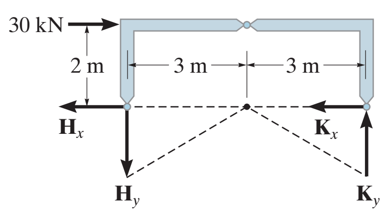


(b)

SOLUTION

First hinges are placed at the midpoints of the columns and girders. The locations of these points are indicated by the letters G through L in Fig. 7–16*a*. The centroid of the columns' cross-sectional areas can be determined by inspection, Fig. 7–16*b*, or analytically as follows:

$$\bar{x} = \frac{\Sigma \tilde{x} A}{\Sigma A} = \frac{0(A) + 6(A)}{A + A} = 3 \text{ m}$$



(c)

The axial *stress* in each column is thus proportional to its distance from this point. Here the columns have the same cross-sectional area, and so the force in each column is also proportional to its distance from the centroid. Hence, a section through the hinges H and K at the top story yields the free-body diagram shown in Fig. 7–16*c*. Note how the column to the left of the centroid must be subjected to tension and the one on the right is subjected to compression. This is necessary in order to counteract the tipping caused by the 30-kN force. Summing moments about the neutral axis, we have

$$\circlearrowright + \Sigma M = 0; \qquad -30(2) + 3H_y + 3K_y = 0$$

The unknowns can be related by proportional triangles, Fig. 7–16*c*, that is,

$$\frac{H_y}{3} = \frac{K_y}{3} \qquad \text{or} \qquad H_y = K_y$$

Thus,

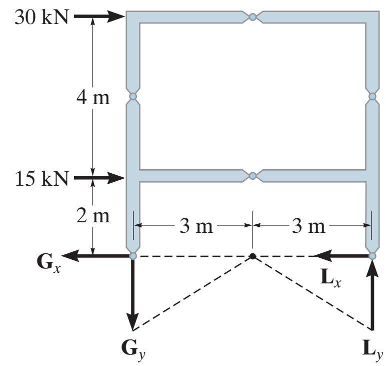
$$H_y = K_y = 10 \text{ kN}$$

In a similar manner, using a section of the frame through the hinges at G and L, Fig. 7–16*d*, we have

$$\circlearrowright + \Sigma M = 0; \qquad -30(6) - 15(2) + 3G_y + 3L_y = 0$$

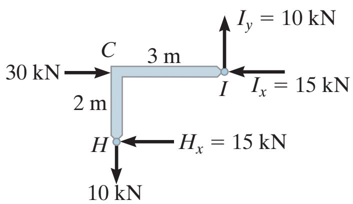
Since $G_y/3 = L_y/3$ or $G_y = L_y$, then

$$G_y = L_y = 35 \text{ kN}$$

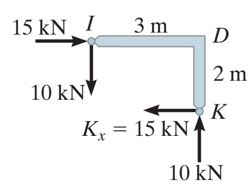


(d)

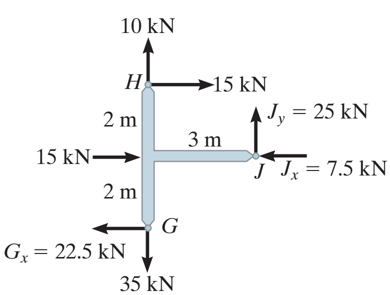
Each part of the frame can now be analyzed using the above results. As in Examples 7–5 and 7–6, we begin at the upper corner where the applied loading occurs, i.e., segment *HCI*, Fig. 7–16*a*. Applying the three equations of equilibrium, $\Sigma M_I = 0$, $\Sigma F_x = 0$, $\Sigma F_y = 0$, yields the results for H_x, I_x, and I_y, respectively, shown on the free-body diagram in Fig. 7–16*e*. Using these results, segment *IDK* is analyzed next, Fig. 7–16*f*; followed by *HJG*, Fig. 7–16*g*; then *KJL*, Fig. 7–16*h*; and finally the bottom portions of the columns, Fig. 7–16*i* and Fig. 7–16*j*. The moment diagrams for each girder are shown in Fig. 7–16*k*.



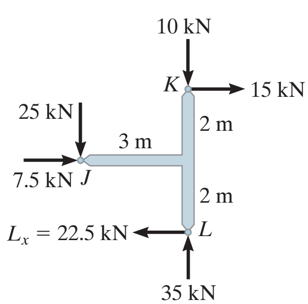
(e)



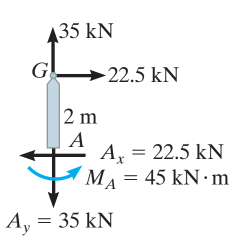
(f)



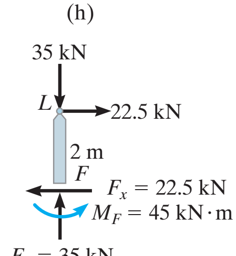
(g)



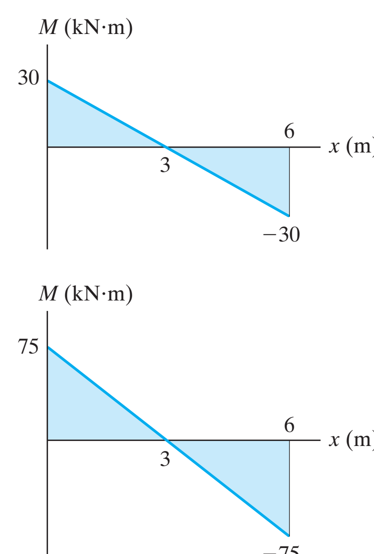
(h)



(i)



(j)



(k)

EXAMPLE 7.8

Show how to determine (approximately) the reactions at the base of the columns of the frame shown in Fig. 7–17*a*. The columns have the cross-sectional areas shown in Fig. 7–17*b*. Use the cantilever method of analysis.

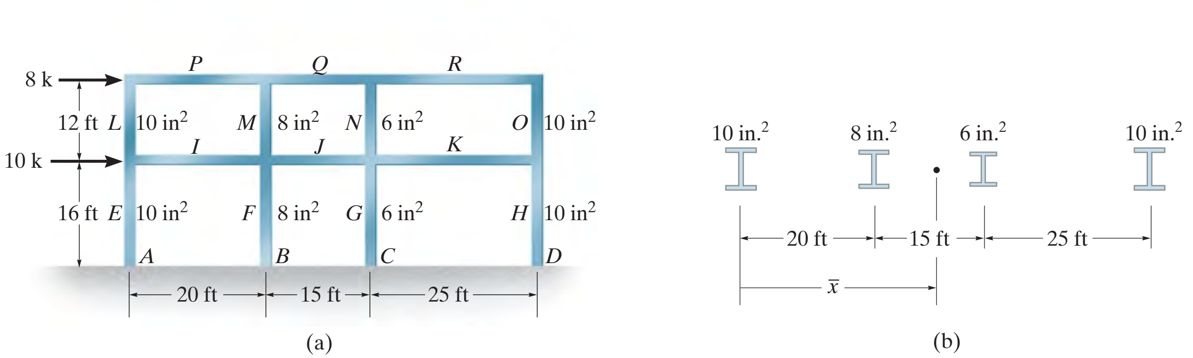
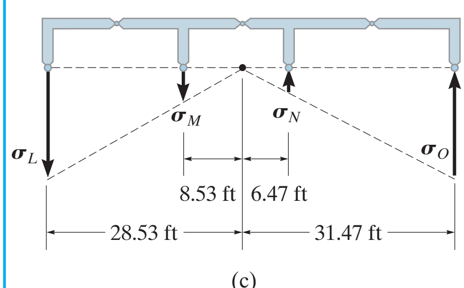


Fig. 7–17

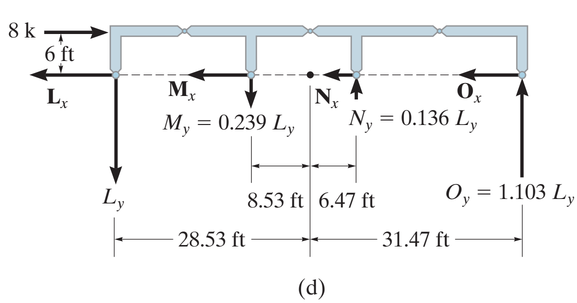


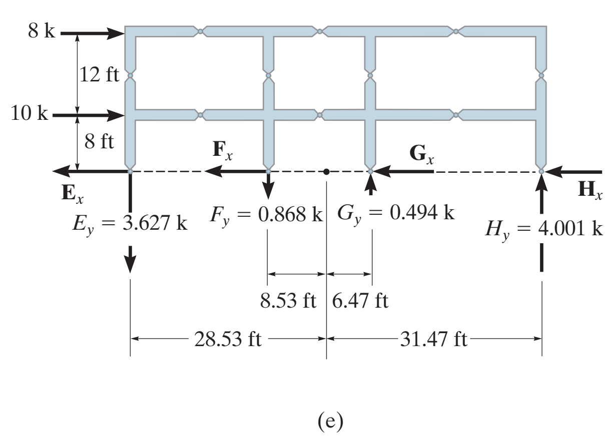
SOLUTION

First, hinges are assumed to exist at the centers of the girders and columns of the frame, Fig. 7–17*d* and Fig. 7–17*e*. The centroid of the columns' cross-sectional areas is determined from Fig. 7–17*b* as follows:

$$\bar{x} = \frac{\Sigma\tilde{x}A}{\Sigma A} = \frac{0(10) + 20(8) + 35(6) + 60(10)}{10 + 8 + 6 + 10} = 28.53 \text{ ft}$$

First we will consider the section through hinges at L, M, N and O.





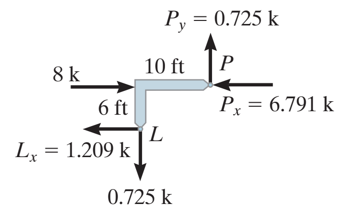
In this problem the columns have *different* cross-sectional areas, so we must consider the *axial stress* in each column to be proportional to its distance from the neutral axis, located at $\bar{x} = 28.53$ ft.

We can relate the column stresses by proportional triangles, Fig. 7–17*c*. Expressing the relations in terms of the force in each column, since $\sigma = F/A$, we have

$$\sigma_M = \frac{8.53\text{ ft}}{28.53\text{ ft}}\sigma_L; \qquad \frac{M_y}{8\text{ in}^2} = \frac{8.53}{28.53}\left(\frac{L_y}{10\text{ in}^2}\right) \qquad M_y = 0.239L_y$$

$$\sigma_N = \frac{6.47\text{ ft}}{28.53\text{ ft}}\sigma_L; \qquad \frac{N_y}{6\text{ in}^2} = \frac{6.47}{28.53}\left(\frac{L_y}{10\text{ in}^2}\right) \qquad N_y = 0.136L_y$$

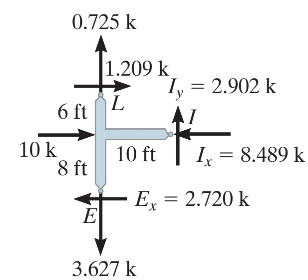
$$\sigma_O = \frac{31.47\text{ ft}}{28.53\text{ ft}}\sigma_L; \qquad \frac{O_y}{10\text{ in}^2} = \frac{31.47}{28.53}\left(\frac{L_y}{10\text{ in}^2}\right) \qquad O_y = 1.103L_y$$



(f)

Now that each force is related to L_y, the free-body diagram is shown in Fig. 7–17*d*.

Note how the columns to the left of the centroid are subjected to tension and those on the right are subjected to compression. Why? Summing moments about the neutral axis, we have



(g)

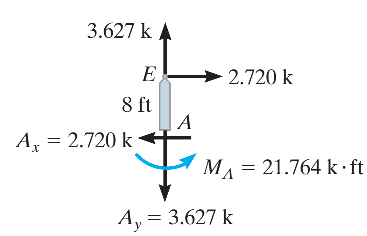
$$\circlearrowleft + \Sigma M = 0; \quad -8\text{ k }(6\text{ ft}) + L_y(28.53\text{ ft}) + (0.239L_y)(8.53\text{ ft})$$
$$+ (0.136L_y)(6.47\text{ ft}) + (1.103L_y)(31.47\text{ ft}) = 0$$

Solving,

$$L_y = 0.725\text{ k} \qquad M_y = 0.174\text{ k} \qquad N_y = 0.0987\text{ k} \qquad O_y = 0.800\text{ k}$$

Using this same method, show that one obtains the results in Fig. 7–17*e* for the columns at *E*, *F*, *G*, and *H*.

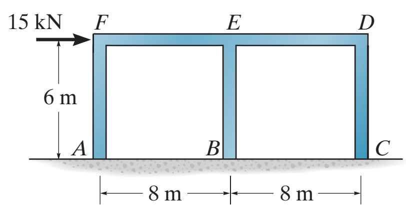
We can now proceed to analyze each part of the frame. As in the previous examples, we begin with the upper corner segment *LP*, Fig. 7–17*f*. Using the calculated results, segment *LEI* is analyzed next, Fig. 7–17*g*, followed by segment *EA*, Fig. 7–17*h*. One can continue to analyze the other segments in sequence, i.e., *PQM*, then *MJFI*, then *FB*, and so on.



(h)

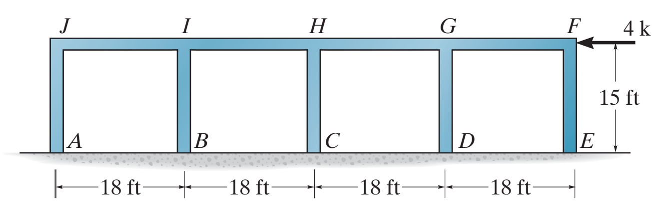
PROBLEMS

7–35. Use the portal method of analysis and draw the moment diagram for girder *FED*.



Prob. 7–35

***7–36.** Use the portal method of analysis and draw the moment diagram for girder *JIHGF*.



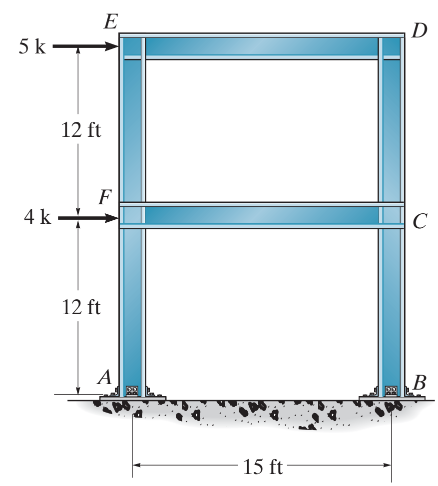
Prob. 7–36

7–37. Use the portal method and determine (approximately) the reactions at supports *A*, *B*, *C*, and *D*.

7–38. Use the cantilever method and determine (approximately) the reactions at supports *A*, *B*, *C*, and *D*. All columns have the same cross-sectional area.

7–39. Use the portal method of analysis and draw the moment diagram for column *AFE*.

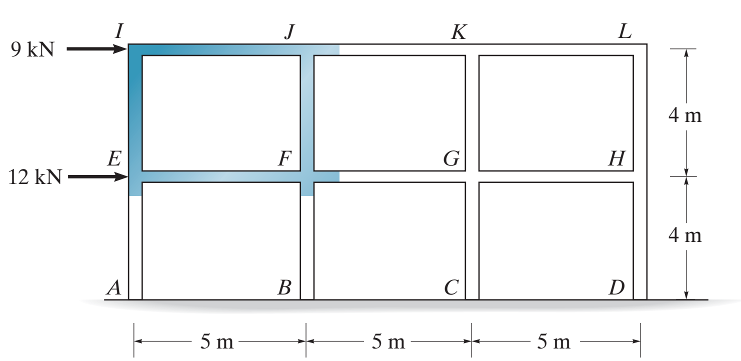
***7–40.** Solve Prob. 7–39 using the cantilever method of analysis. All the columns have the same cross-sectional area.



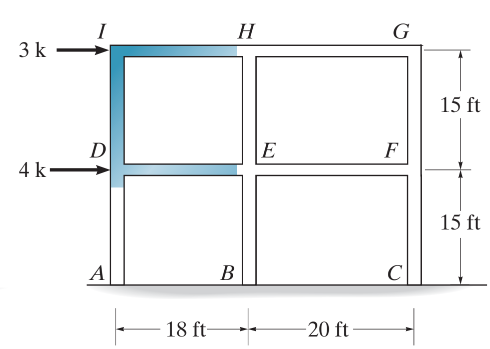
Probs. 7–39/7–40

7–41. Use the portal method and determine (approximately) the reactions at *A*.

7–42. Use the cantilever method and determine (approximately) the reactions at *A*. All of the columns have the same cross-sectional area.



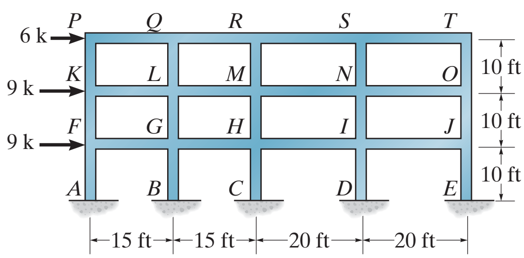
Probs. 7–37/7–38



Probs. 7–41/7–42

7–43. Draw (approximately) the moment diagram for girder *PQRST* and column *BGLQ* of the building frame. Use the portal method.

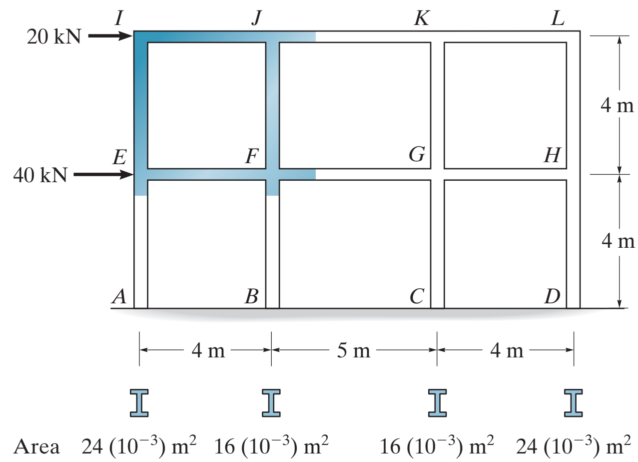
***7–44.** Draw (approximately) the moment diagram for girder *PQRST* and column *BGLQ* of the building frame. All columns have the same cross-sectional area. Use the cantilever method.



Probs. 7–43/7–44

7–45. Draw the moment diagram for girder *IJKL* of the building frame. Use the portal method of analysis.

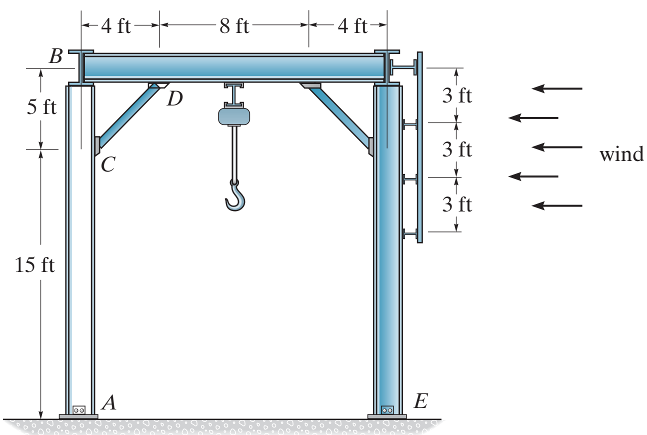
7–46. Solve Prob. 7–45 using the cantilever method of analysis. Each column has the cross-sectional area indicated.



Probs. 7–45/7–46

PROJECT PROBLEMS

7–1P. The storage building bents shown in the photo are spaced 10 ft apart and can be assumed pin connected at all points of support. Use the idealized model shown and determine the anticipated wind loading on the bent. Note that the wind loading is transmitted from the wall to the four purlins, then to the columns on the right side. Do an approximate analysis and determine the maximum axial load and maximum moment in column *AB*. Assume the columns and knee braces are pinned at their ends. The building is located on flat terrain in New Orleans, Louisiana, where $V = 125$ mi/h.

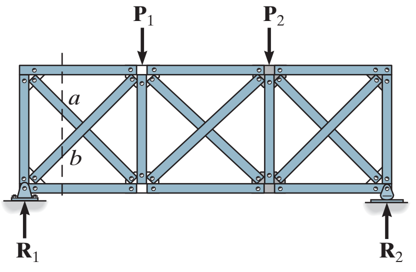


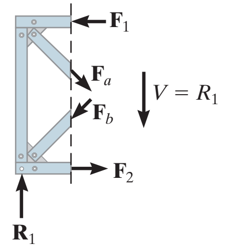
Prob. 7–1P

CHAPTER REVIEW

An approximate structural analysis is used to reduce a statically indeterminate structure to one that is statically determinate. By doing so a preliminary design of the members can be made, and once complete, the more exact indeterminate analysis can then be performed and the design refined.

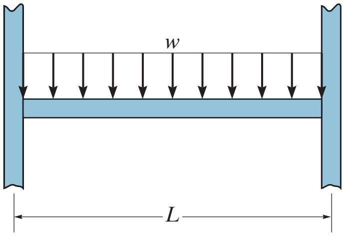
Trusses having cross-diagonal bracing within their panels can be analyzed by assuming the tension diagonal supports the panel shear and the compressive diagonal is a zero-force member. This is reasonable if the members are long and slender. For larger cross sections, it is reasonable to assume each diagonal carries one-half the panel shear.

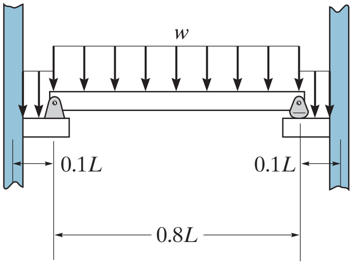




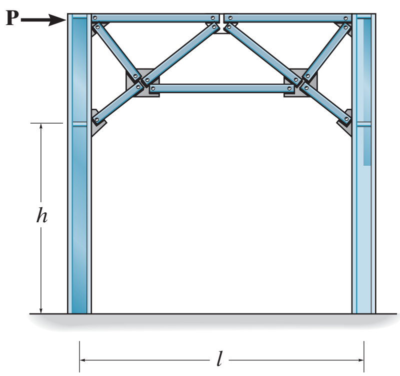
7

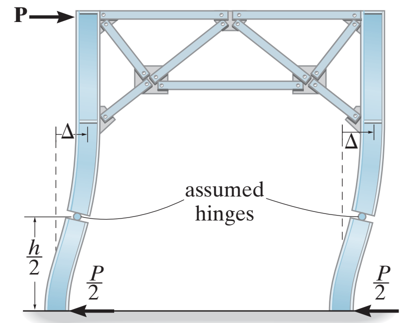
The approximate analysis of a vertical uniform load acting on a girder of length L of a fixed-connected building frame can be approximated by assuming that the girder does not support an axial load, and there are inflection points (hinges) located $0.1L$ from the supports.



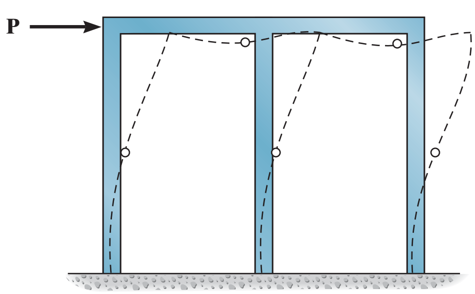


Portal frames having fixed supports are approximately analyzed by assuming there are hinges at the midpoint of each column height, measured to the bottom of the truss bracing. Also, for these, and pin-supported frames, each column is assumed to support half the shear load on the frame.

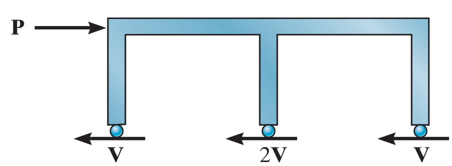




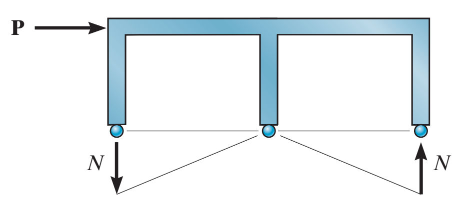
For fixed-connected building frames subjected to lateral loads, we can assume there are hinges at the centers of the columns and girders. If the frame has a low elevation, shear resistance is important and so we can use the portal method, where the interior columns at any floor level carry twice the shear as that of the exterior columns. For tall slender frames, the cantilever method can be used, where the axial stress in a column is proportional to its distance from the centroid of the cross-sectional area of all the columns at a given floor level.



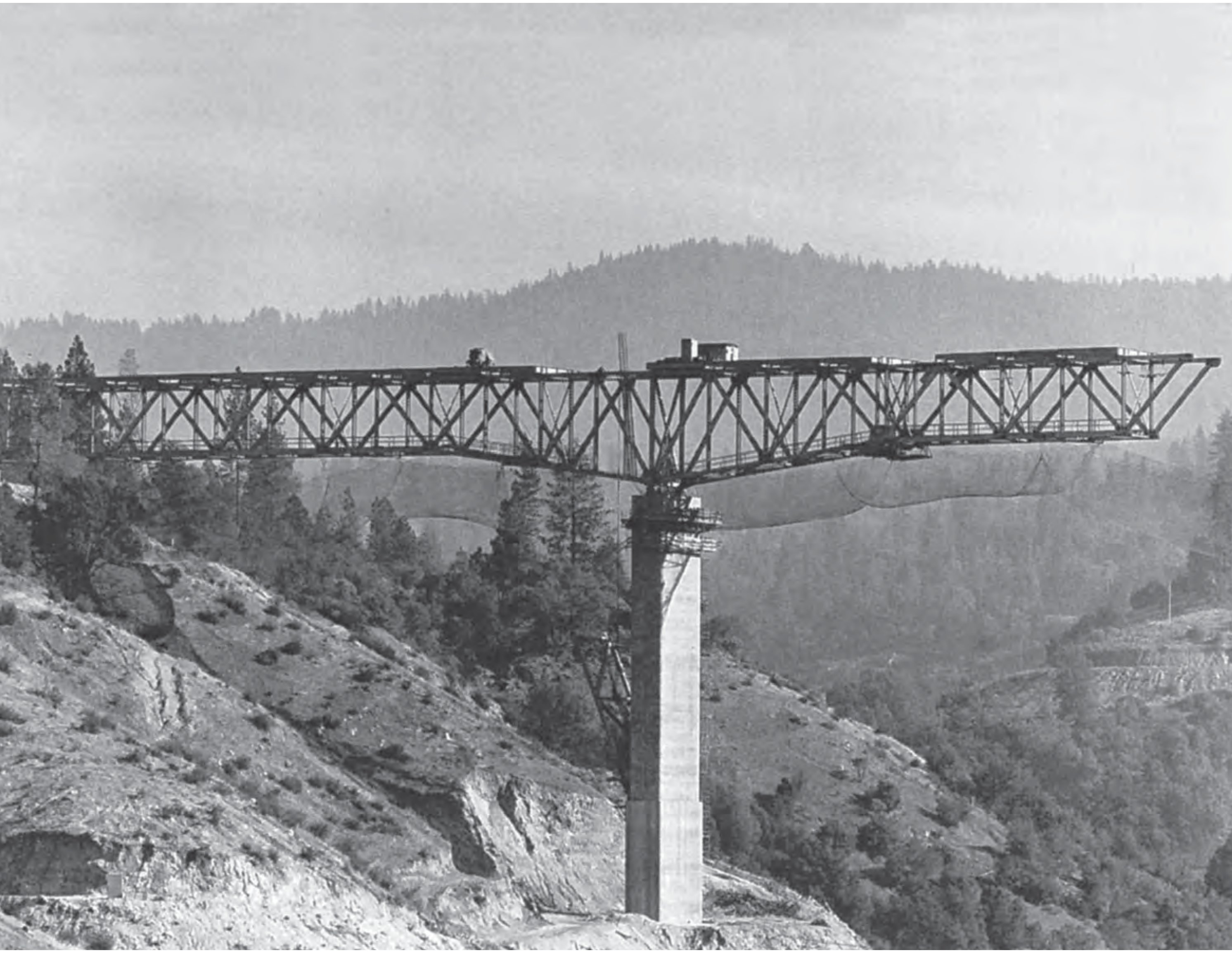
○ = inflection point



Portal method



Cantilever method



The deflection of this arch bridge must be carefully monitored while it is under construction.

Deflections

8

In this chapter we will show how to determine the elastic deflections of a beam using the method of double integration and two important geometrical methods, namely, the moment-area theorems and the conjugate-beam method. Double integration is used to obtain equations which define the slope and the elastic curve. The geometric methods provide a way to obtain the slope and deflection at specific points on the beam. Each of these methods has particular advantages or disadvantages, which will be discussed when each method is presented.

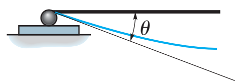
8.1 Deflection Diagrams and the Elastic Curve

Deflections of structures can occur from various sources, such as loads, temperature, fabrication errors, or settlement. In design, deflections must be limited in order to provide integrity and stability of roofs, and prevent cracking of attached brittle materials such as concrete, plaster or glass. Furthermore, a structure must not vibrate or deflect severely in order to "appear" safe for its occupants. More important, though, deflections at specified points in a structure must be determined if one is to analyze statically indeterminate structures.

The deflections to be considered throughout this text apply only to structures having *linear elastic material response*. Under this condition, a structure subjected to a load will return to its original undeformed position after the load is removed. The deflection of a structure is caused

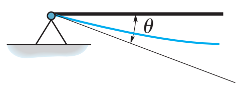
TABLE 8–1

(1)



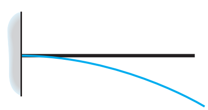
$\Delta = 0$
roller or rocker

(2)



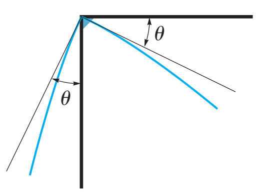
$\Delta = 0$
pin

(3)



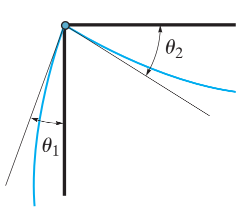
$\Delta = 0$
$\theta = 0$
fixed support

(4)



fixed-connected joint

(5)



pin-connected joint

by its internal loadings such as normal force, shear force, or bending moment. For *beams* and *frames,* however, the greatest deflections are most often caused by *internal bending*, whereas *internal axial forces* cause the deflections of a *truss*.

Before the slope or displacement of a point on a beam or frame is determined, it is often helpful to sketch the deflected shape of the structure when it is loaded in order to partially check the results. This *deflection diagram* represents the *elastic curve* or locus of points which defines the displaced position of the centroid of the cross section along the members. For most problems the elastic curve can be sketched without much difficulty. When doing so, however, it is necessary to know the restrictions as to slope or displacement that often occur at a support or a connection. With reference to Table 8–1, supports that *resist a force,* such as a pin, *restrict displacement;* and those that *resist moment,* such as a fixed wall, *restrict rotation*. Note also that deflection of frame members that are fixed connected (4) causes the joint to rotate the connected members by the same amount θ. On the other hand, if a pin connection is used at the joint, the members will each have a *different slope* or rotation at the pin, since the pin cannot support a moment (5).

The two-member frames support both the dead load of the roof and a live snow loading. The frame can be considered pinned at the wall, fixed at the ground, and having a fixed-connected joint.

If the elastic curve seems difficult to establish, it is suggested that the moment diagram for the beam or frame be drawn first. By our sign convention for moments established in Chapter 4, a *positive moment* tends to bend a beam or horizontal member *concave upward,* Fig. 8–1. Likewise, a *negative moment* tends to bend the beam or member *concave downward,* Fig. 8–2. Therefore, *if the shape of the moment diagram is known, it will be easy to construct the elastic curve and vice versa*. For example, consider the beam in Fig. 8–3 with its associated moment diagram. Due to the pin-and-roller support, the displacement at A and D must be zero. Within the region of negative moment, the elastic curve is concave downward; and within the region of positive moment, the elastic curve is concave upward. In particular, there must be an *inflection point* at the point where the curve changes from concave down to concave up, since this is a point of zero moment. Using these same principles, note how the elastic curve for the beam in Fig. 8–4 was drawn based on its moment diagram. In particular, realize that the positive moment reaction from the wall keeps the initial slope of the beam horizontal.

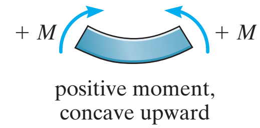


Fig. 8–1

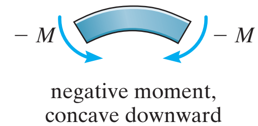


Fig. 8–2

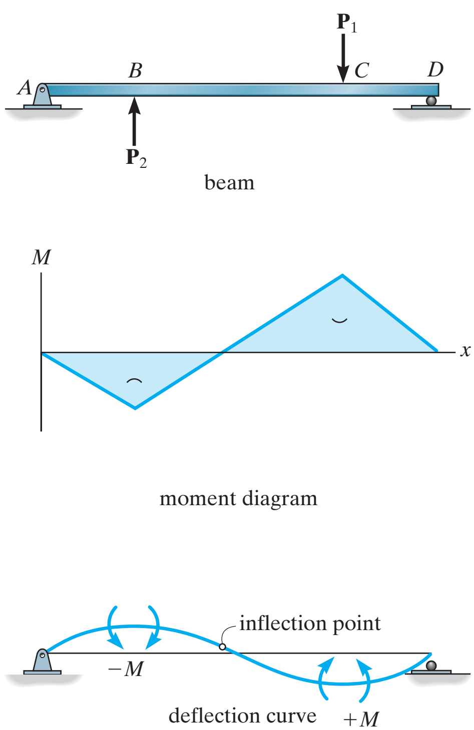


Fig. 8–3

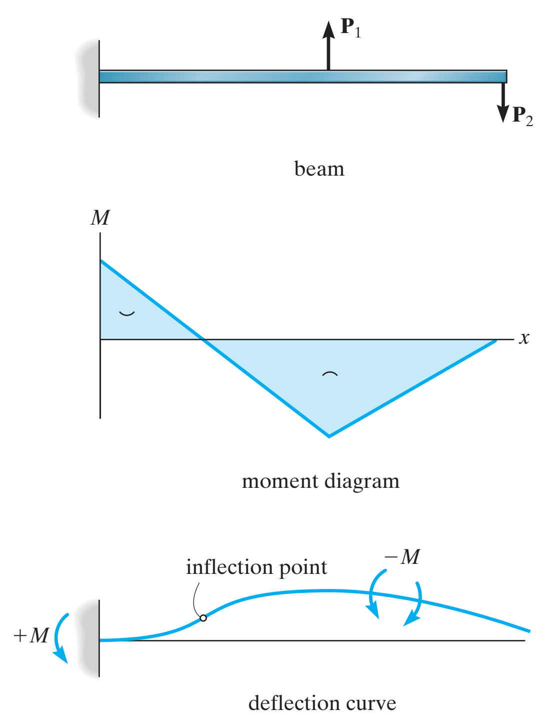


Fig. 8–4

EXAMPLE 8.1

Draw the deflected shape of each of the beams shown in Fig. 8–5.

SOLUTION

In Fig. 8–5*a* the roller at *A* allows free rotation with no deflection while the fixed wall at *B* prevents both rotation and deflection. The deflected shape is shown by the bold line. In Fig. 8–5*b*, no rotation or deflection can occur at *A* and *B*. In Fig. 8–5*c*, the couple moment will rotate end *A*. This will cause deflections at both ends of the beam since no deflection is possible at *B* and *C*. Notice that segment *CD* remains undeformed (a straight line) since no internal load acts within it. In Fig. 8–5*d*, the pin (internal hinge) at *B* allows free rotation, and so the slope of the deflection curve will suddenly change at this point while the beam is constrained by its supports. In Fig. 8–5*e*, the compound beam deflects as shown. The slope abruptly changes on each side of the pin at *B*. Finally, in Fig. 8–5*f*, span *BC* will deflect concave upwards due to the load. Since the beam is continuous, the end spans will deflect concave downwards.

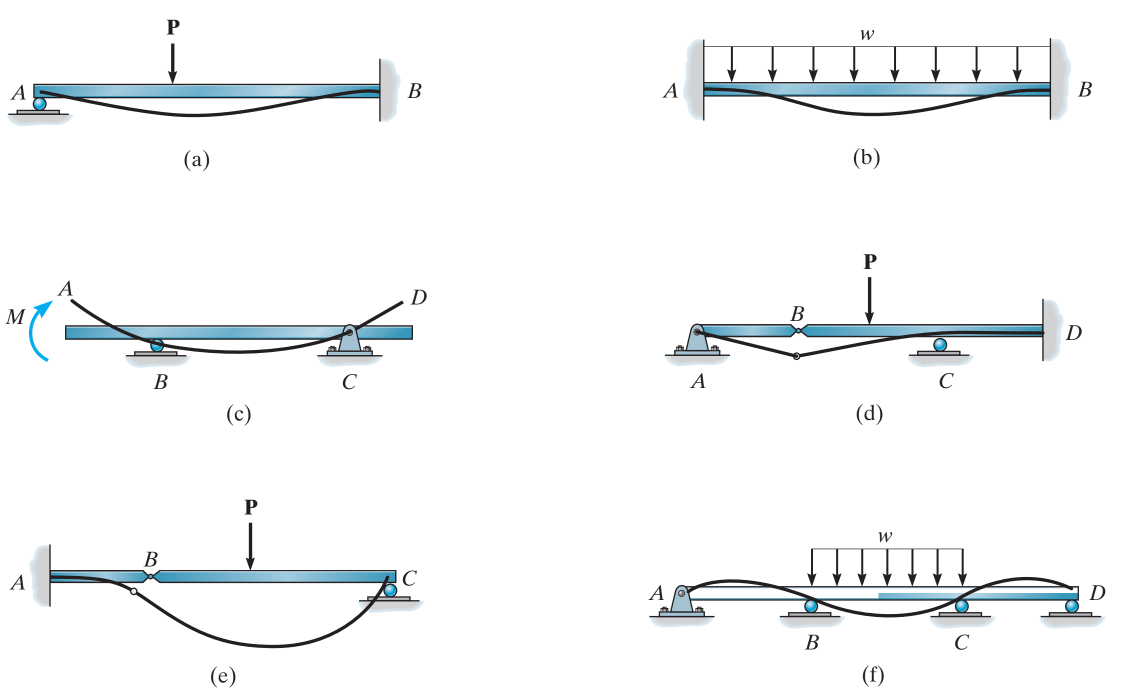
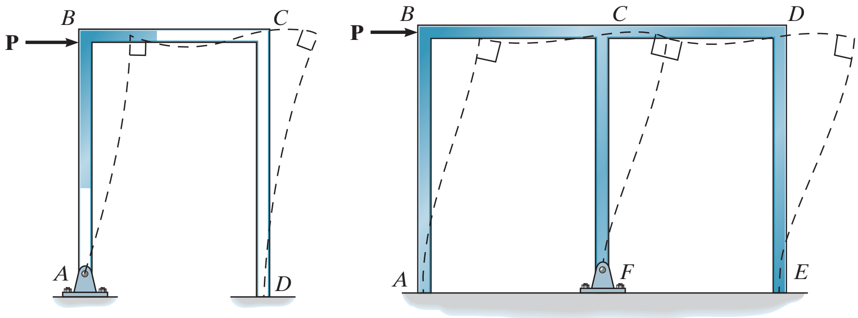


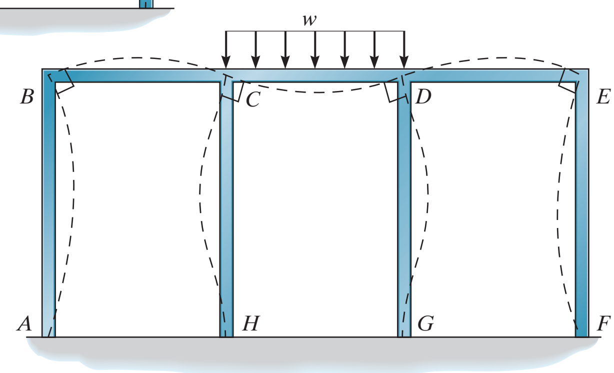
Fig. 8–5

EXAMPLE 8.2

Draw the deflected shapes of each of the frames shown in Fig. 8–6.



(a) (b)



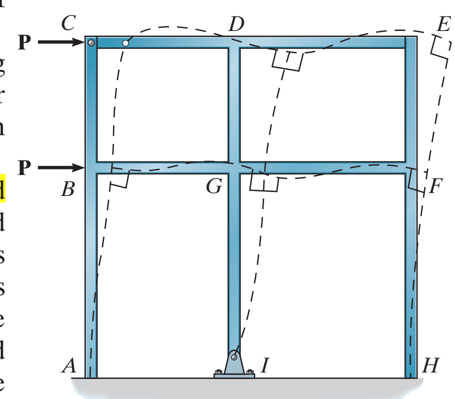
(c)

SOLUTION

In Fig. 8–6*a*, when the load **P** pushes joints *B* and *C* to the right, it will cause clockwise rotation of each column as shown. As a result, joints *B* and *C* must rotate clockwise. Since the 90° angle between the connected members must be maintained at these joints, the beam *BC* will deform so that its curvature is reversed from concave up on the left to concave down on the right. Note that this produces a point of inflection within the beam.

In Fig. 8–6*b*, **P** displaces joints *B*, *C*, and *D* to the right, causing each column to bend as shown. The fixed joints must maintain their 90° angles, and so *BC* and *CD* must have a reversed curvature with an inflection point near their midpoint.

In Fig. 8–6*c*, the vertical loading on this symmetric frame will bend beam *CD* concave upwards, causing clockwise rotation of joint *C* and counterclockwise rotation of joint *D*. Since the 90° angle at the joints must be maintained, the columns bend as shown. This causes spans *BC* and *DE* to be concave downwards, resulting in counterclockwise rotation at *B* and clockwise rotation at *E*. The columns therefore bend as shown. Finally, in Fig. 8–6*d*, the loads push joints *B* and *C* to the right, which bends the columns as shown. The fixed joint *B* maintains its 90° angle; however, no restriction on the relative rotation between the members at *C* is possible since the joint is a pin. Consequently, only beam *CD* does not have a reverse curvature.



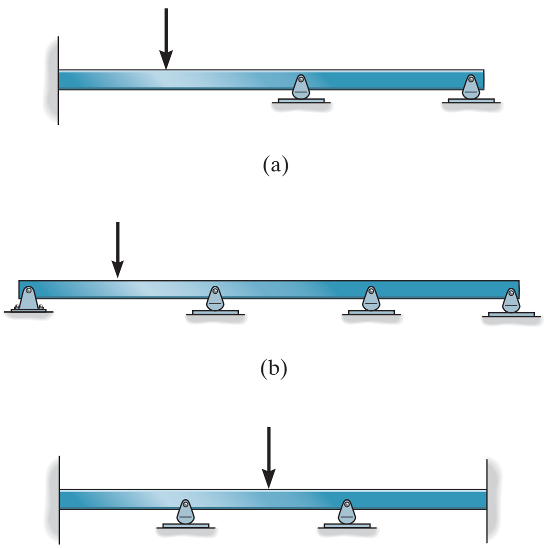
(d)

Fig. 8–6

8

FUNDAMENTAL PROBLEMS

F8–1. Draw the deflected shape of each beam. Indicate the inflection points.



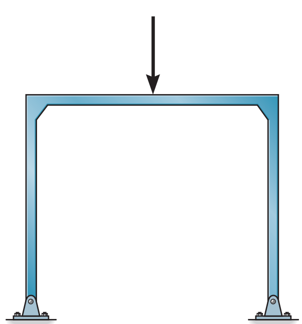
(a)

(b)

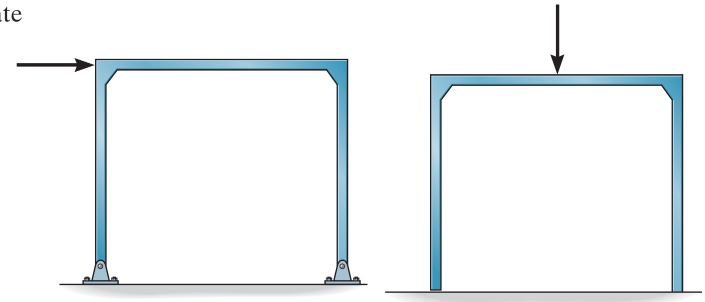
(c)

F8–1

F8–2. Draw the deflected shape of each frame. Indicate the inflection points.



(a)

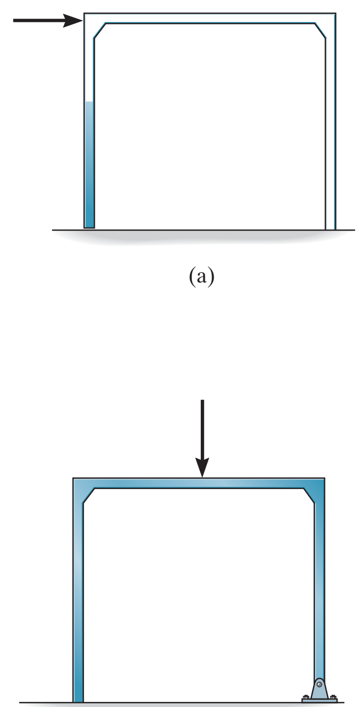


(b)

(c)

F8–2

F8–3. Draw the deflected shape of each frame. Indicate the inflection points.



(a)

(b)

F8–3

8.2 Elastic-Beam Theory

In this section we will develop two important differential equations that relate the internal moment in a beam to the displacement and slope of its elastic curve. These equations form the basis for the deflection methods presented in this chapter, and for this reason the assumptions and limitations used in their development should be fully understood.

To derive these relationships, we will limit the analysis to the most common case of an initially straight beam that is elastically deformed by loads applied perpendicular to the beam's x axis and lying in the x–v plane of symmetry for the beam's cross-sectional area, Fig. 8–7*a*. Due to the loading, the deformation of the beam is caused by both the internal shear force and bending moment. If the beam has a length that is much greater than its depth, the greatest deformation will be caused by bending, and therefore we will direct our attention to its effects. Deflections caused by shear will be discussed later in the chapter.

When the internal moment M deforms the element of the beam, each cross section remains plane and the angle between them becomes $d\theta$, Fig. 8–7*b*. The arc dx that represents a portion of the elastic curve intersects the neutral axis for each cross section. The *radius of curvature* for this arc is defined as the distance ρ, which is measured from the *center of curvature* O' to dx. Any arc on the element other than dx is subjected to a normal strain. For example, the strain in arc ds, located at a position y from the neutral axis, is $\epsilon = (ds' - ds)/ds$. However, $ds = dx = \rho\, d\theta$ and $ds' = (\rho - y)\, d\theta$, and so

$$\epsilon = \frac{(\rho - y)\, d\theta - \rho\, d\theta}{\rho\, d\theta} \quad \text{or} \quad \frac{1}{\rho} = -\frac{\epsilon}{y}$$

If the material is homogeneous and behaves in a linear elastic manner, then Hooke's law applies, $\epsilon = \sigma/E$. Also, since the flexure formula applies, $\sigma = -My/I$. Combining these equations and substituting into the above equation, we have

$$\frac{1}{\rho} = \frac{M}{EI} \tag{8–1}$$

Here

ρ = the radius of curvature at a specific point on the elastic curve ($1/\rho$ is referred to as the *curvature*)

M = the internal moment in the beam at the point where ρ is to be determined

E = the material's modulus of elasticity

I = the beam's moment of inertia computed about the neutral axis

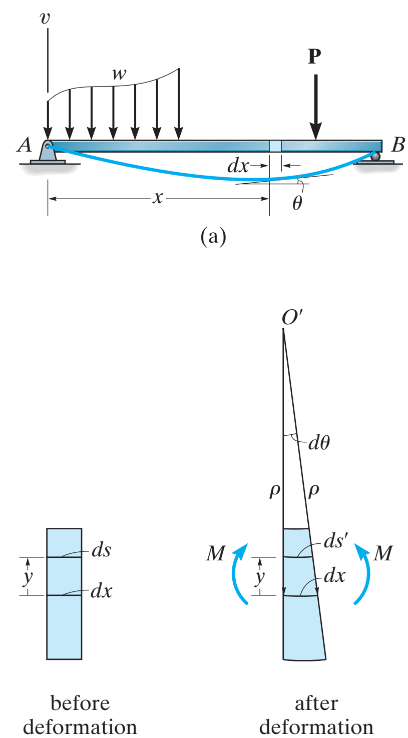


Fig. 8–7

8

The product EI in this equation is referred to as the *flexural rigidity*, and it is always a positive quantity. Since $dx = \rho\, d\theta$, then from Eq. 8–1,

$$d\theta = \frac{M}{EI}\,dx \tag{8–2}$$

If we choose the v axis positive upward, Fig. 8–7a, and if we can express the curvature $(1/\rho)$ in terms of x and v, we can then determine the elastic curve for the beam. In most calculus books it is shown that this curvature relationship is

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

Therefore,

$$\frac{M}{EI} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}} \tag{8–3}$$

This equation represents a nonlinear second-order differential equation. Its solution, $v = f(x)$, gives the exact shape of the elastic curve—assuming, of course, that beam deflections occur only due to bending. In order to facilitate the solution of a greater number of problems, Eq. 8–3 will be modified by making an important simplification. Since the slope of the elastic curve for most structures is very small, we will use small deflection theory and assume $dv/dx \approx 0$. Consequently its square will be negligible compared to unity and therefore Eq. 8–3 reduces to

$$\boxed{\frac{d^2v}{dx^2} = \frac{M}{EI}} \tag{8–4}$$

It should also be pointed out that by assuming $dv/dx \approx 0$, the original length of the beam's axis x and the *arc* of its elastic curve will be approximately the same. In other words, ds in Fig. 8–7b is approximately equal to dx, since

$$ds = \sqrt{dx^2 + dv^2} = \sqrt{1 + (dv/dx)^2}\,dx \approx dx$$

This result implies that points on the elastic curve will only be displaced vertically and not horizontally.

Tabulated Results. In the next section we will show how to apply Eq. 8–4 to find the slope of a beam and the equation of its elastic curve. The results from such an analysis for some common beam loadings often encountered in structural analysis are given in the table on the inside front cover of this book. Also listed are the slope and displacement at critical points on the beam. Obviously, no single table can account for the many different cases of loading and geometry that are encountered in practice. When a table is not available or is incomplete, the displacement or slope of a specific point on a beam or frame can be determined by using the double integration method or one of the other methods discussed in this and the next chapter.

8.3 The Double Integration Method

Once M is expressed as a function of position x, then successive integrations of Eq. 8–4 will yield the beam's slope, $\theta \approx \tan\theta = dv/dx = \int(M/EI)\,dx$ (Eq. 8–2), and the equation of the elastic curve, $v = f(x) = \int\int(M/EI)\,dx$, respectively. For each integration it is necessary to introduce a "constant of integration" and then solve for the constants to obtain a unique solution for a particular problem. Recall from Sec. 4–2 that if the loading on a beam is discontinuous—that is, it consists of a series of several distributed and concentrated loads—then several functions must be written for the internal moment, each valid within the region between the discontinuities. For example, consider the beam shown in Fig. 8–8. The internal moment in regions AB, BC, and CD must be written in terms of the x_1, x_2, and x_3 coordinates. Once these functions are integrated through the application of Eq. 8–4 and the constants of integration determined, the functions will give the slope and deflection (elastic curve) for each region of the beam for which they are valid.

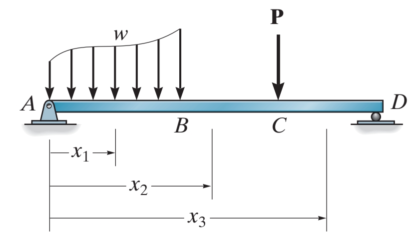


Fig. 8–8

Sign Convention. When applying Eq. 8–4, it is important to use the proper sign for M as established by the sign convention that was used in the derivation of this equation, Fig. 8–9a. Furthermore, recall that positive deflection, v, is upward, and as a result, the positive slope angle θ will be measured counterclockwise from the x axis. The reason for this is shown in Fig. 8–9b. Here, positive increases dx and dv in x and v create an increase $d\theta$ that is counterclockwise. Also, since the slope angle θ will be very small, its value in radians can be determined directly from $\theta \approx \tan\theta = dv/dx$.

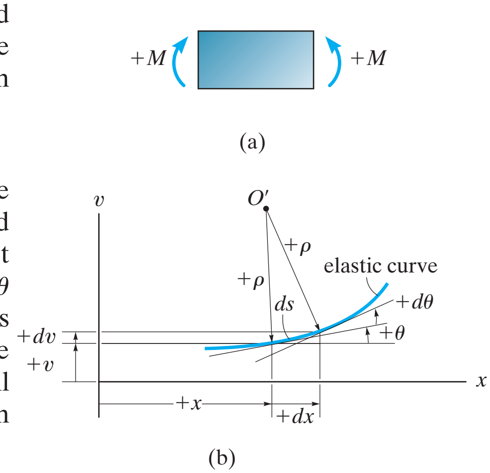


Fig. 8–9

Boundary and Continuity Conditions. The constants of integration are determined by evaluating the functions for slope or displacement at a particular point on the beam where the value of the function is known. These values are called *boundary conditions*. For example, if the beam is supported by a roller or pin, then it is required that the displacement be zero at these points. Also, at a fixed support the slope and displacement are both zero.

If a single x coordinate cannot be used to express the equation for the beam's slope or the elastic curve, then continuity conditions must be used to evaluate some of the integration constants. Consider the beam in Fig. 8–10. Here the x_1 and x_2 coordinates are valid only within the regions AB and BC, respectively. Once the functions for the slope and deflection are obtained, they must give the same values for the slope and deflection at point B, $x_1 = x_2 = a$, so that the elastic curve is physically continuous. Expressed mathematically, this requires $\theta_1(a) = \theta_2(a)$ and $v_1(a) = v_2(a)$. These equations can be used to determine two constants of integration.

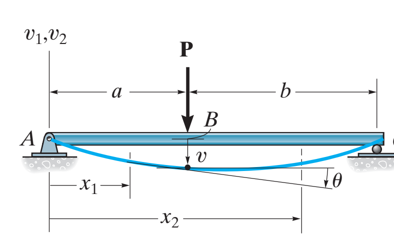


Fig. 8–10

Procedure for Analysis

The following procedure provides a method for determining the slope and deflection of a beam (or shaft) using the method of double integration. It should be realized that this method is suitable only for *elastic deflections* for which the beam's slope is very small. Furthermore, the method considers *only deflections due to bending*. Additional deflection due to shear generally represents only a few percent of the bending deflection, and so it is usually neglected in engineering practice.

Elastic Curve

- Draw an exaggerated view of the beam's elastic curve. Recall that points of zero slope and zero displacement occur at a fixed support, and zero displacement occurs at pin and roller supports.
- Establish the x and v coordinate axes. The x axis must be parallel to the undeflected beam and its origin at the left side of the beam, with a positive direction to the right.
- If several discontinuous loads are present, establish x coordinates that are valid for each region of the beam between the discontinuities.
- In all cases, the associated positive v axis should be directed upward.

Load or Moment Function

- For each region in which there is an x coordinate, express the internal moment M as a function of x.
- *Always* assume that M acts in the positive direction when applying the equation of moment equilibrium to determine $M = f(x)$.

Slope and Elastic Curve

- Provided EI is constant, apply the moment equation $EI\, d^2v/dx^2 = M(x)$, which requires two integrations. For each integration it is important to include a constant of integration. The constants are determined using the boundary conditions for the supports and the continuity conditions that apply to slope and displacement at points where two functions meet.
- Once the integration constants are determined and substituted back into the slope and deflection equations, the slope and displacement at *specific points* on the elastic curve can be determined. The numerical values obtained can be checked graphically by comparing them with the sketch of the elastic curve.
- *Positive* values for *slope* are *counterclockwise* and *positive displacement* is *upward*.

EXAMPLE 8.3

Each simply supported floor joist shown in the photo is subjected to a uniform design loading of 4 kN/m, Fig. 8–11*a*. Determine the maximum deflection of the joist. *EI* is constant.

Elastic Curve. Due to symmetry, the joist's maximum deflection will occur at its center. Only a single *x* coordinate is needed to determine the internal moment.

Moment Function. From the free-body diagram, Fig. 8–11*b*, we have

$$M = 20x - 4x\left(\frac{x}{2}\right) = 20x - 2x^2$$

Slope and Elastic Curve. Applying Eq. 8–4 and integrating twice gives

$$EI\frac{d^2v}{dx^2} = 20x - 2x^2$$

$$EI\frac{dv}{dx} = 10x^2 - 0.6667x^3 + C_1$$

$$EI\,v = 3.333x^3 - 0.1667x^4 + C_1x + C_2$$

Here $v = 0$ at $x = 0$ so that $C_2 = 0$, and $v = 0$ at $x = 10$, so that $C_1 = -166.7$. The equation of the elastic curve is therefore

$$EI\,v = 3.333x^3 - 0.1667x^4 - 166.7x$$

At $x = 5$ m, note that $dv/dx = 0$. The maximum deflection is therefore

$$v_{\max} = -\frac{521}{EI} \qquad \textit{Ans.}$$

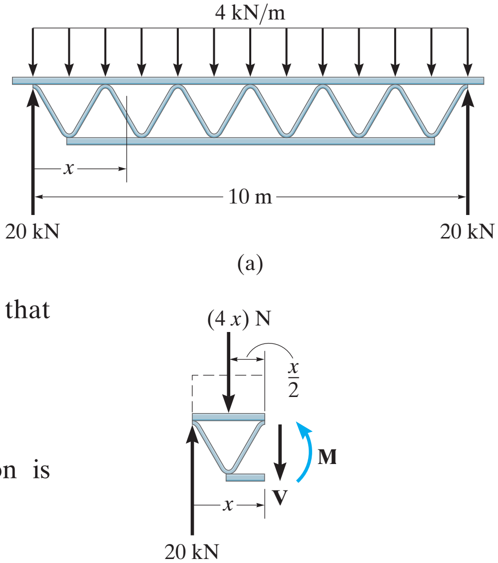


Fig. 8–11

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EXAMPLE 8.4

The cantilevered beam shown in Fig. 8–12*a* is subjected to a couple moment $\mathbf{M}_0$ at its end. Determine the equation of the elastic curve. *EI* is constant.

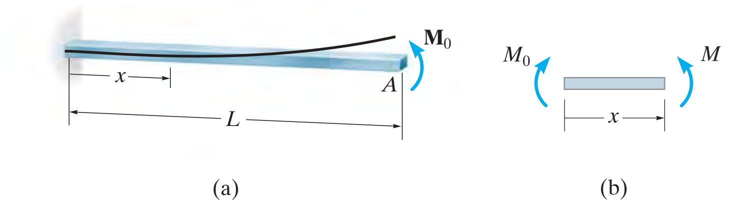


Fig. 8–12

SOLUTION

Elastic Curve. The load tends to deflect the beam as shown in Fig. 8–9*a*. By inspection, the internal moment can be represented throughout the beam using a single *x* coordinate.

Moment Function. From the free-body diagram, with **M** acting in the *positive direction*, Fig. 8–12*b*, we have

$$M = M_0$$

Slope and Elastic Curve. Applying Eq. 8–4 and integrating twice yields

$$EI\frac{d^2v}{dx^2} = M_0 \tag{1}$$

$$EI\frac{dv}{dx} = M_0x + C_1 \tag{2}$$

$$EIv = \frac{M_0x^2}{2} + C_1x + C_2 \tag{3}$$

Using the boundary conditions $dv/dx = 0$ at $x = 0$ and $v = 0$ at $x = 0$, then $C_1 = C_2 = 0$. Substituting these results into Eqs. (2) and (3) with $\theta = dv/dx$, we get

$$\theta = \frac{M_0x}{EI}$$

$$v = \frac{M_0x^2}{2EI} \qquad \textit{Ans.}$$

Maximum slope and displacement occur at A $(x = L)$, for which

$$\theta_A = \frac{M_0 L}{EI} \tag{4}$$

$$v_A = \frac{M_0 L^2}{2EI} \tag{5}$$

The *positive* result for θ_A indicates *counterclockwise* rotation and the *positive* result for v_A indicates that v_A is *upward*. This agrees with the results sketched in Fig. 8–12*a*.

In order to obtain some idea as to the actual *magnitude* of the slope and displacement at the end A, consider the beam in Fig. 8–12*a* to have a length of 12 ft, support a couple moment of 15 k · ft, and be made of steel having $E_{st} = 29(10^3)$ ksi. If this beam were designed *without* a factor of safety by assuming the allowable normal stress is equal to the yield stress $\sigma_{allow} = 36$ ksi, then a $W6 \times 9$ would be found to be adequate ($I = 16.4$ in.^{4}). From Eqs. (4) and (5) we get

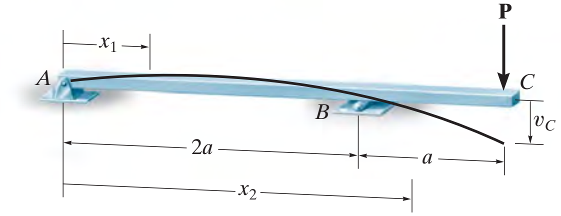
$$\theta_A = \frac{15\text{ k}\cdot\text{ft}(12\text{ in./ft})(12\text{ ft})(12\text{ in./ft})}{29(10^3)\text{ k/in}^2(16.4\text{ in}^4)} = 0.0545\text{ rad}$$

$$v_A = \frac{15\text{ k}\cdot\text{ft}(12\text{ in./ft})(12\text{ ft})^2(12\text{ in./1 ft})^2}{2(29(10^3)\text{ k/in}^2)(16.4\text{ in}^4)} = 3.92\text{ in.}$$

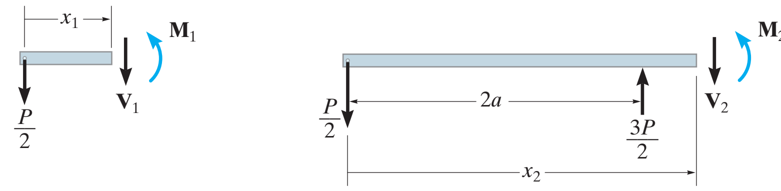
Since $\theta_A^2 = 0.00297\text{ rad}^2 \ll 1$, this justifies the use of Eq. 8–4, rather than applying the more exact Eq. 8–3, for computing the deflection of beams. Also, since this numerical application is for a *cantilevered beam,* we have obtained *larger values* for maximum θ and v than would have been obtained if the beam were supported using pins, rollers, or other supports.

EXAMPLE 8.5

The beam in Fig. 8–13*a* is subjected to a load **P** at its end. Determine the displacement at *C*. *EI* is constant.



(a)



(b)

Fig. 8–13

SOLUTION

Elastic Curve. The beam deflects into the shape shown in Fig. 8–13*a*. Due to the loading, two *x* coordinates must be considered.

Moment Functions. Using the free-body diagrams shown in Fig. 8–13*b*, we have

$$M_1 = -\frac{P}{2}x_1 \qquad 0 \le x_1 \le 2a$$

$$M_2 = -\frac{P}{2}x_2 + \frac{3P}{2}(x_2 - 2a)$$

$$= Px_2 - 3Pa \qquad 2a \le x_2 \le 3a$$

Slope and Elastic Curve. Applying Eq. 8–4,

for x_1, $$EI\frac{d^2v_1}{dx_1^2} = -\frac{P}{2}x_1$$

$$EI\frac{dv_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

8

For x_2, $$EI\frac{d^2v_2}{dx_2^2} = Px_2 - 3Pa$$

$$EI\frac{dv_2}{dx_2} = \frac{P}{2}x_2^2 - 3Pax_2 + C_3 \tag{3}$$

$$EIv_2 = \frac{P}{6}x_2^3 - \frac{3}{2}Pax_2^2 + C_3x_2 + C_4 \tag{4}$$

The *four* constants of integration are determined using *three* boundary conditions, namely, $v_1 = 0$ at $x_1 = 0$, $v_1 = 0$ at $x_1 = 2a$, and $v_2 = 0$ at $x_2 = 2a$, and *one* continuity equation. Here the continuity of slope at the roller requires $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$. (Note that continuity of displacement at B has been indirectly considered in the boundary conditions, since $v_1 = v_2 = 0$ at $x_1 = x_2 = 2a$.) Applying these four conditions yields

$v_1 = 0$ at $x_1 = 0$; $$0 = 0 + 0 + C_2$$

$v_1 = 0$ at $x_1 = 2a$; $$0 = -\frac{P}{12}(2a)^3 + C_1(2a) + C_2$$

$v_2 = 0$ at $x_2 = 2a$; $$0 = \frac{P}{6}(2a)^3 - \frac{3}{2}Pa(2a)^2 + C_3(2a) + C_4$$

$$\frac{dv_1(2a)}{dx_1} = \frac{dv_2(2a)}{dx_2}; \qquad -\frac{P}{4}(2a)^2 + C_1 = \frac{P}{2}(2a)^2 - 3Pa(2a) + C_3$$

Solving, we obtain

$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{10}{3}Pa^2 \qquad C_4 = -2Pa^3$$

Substituting C_3 and C_4 into Eq. (4) gives

$$v_2 = \frac{P}{6EI}x_2^3 - \frac{3Pa}{2EI}x_2^2 + \frac{10Pa^2}{3EI}x_2 - \frac{2Pa^3}{EI}$$

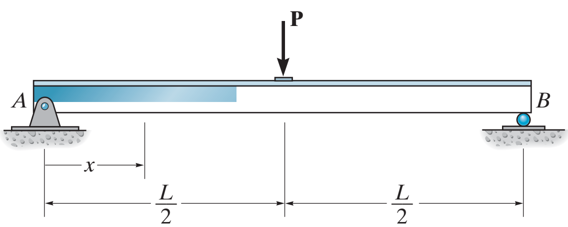
The displacement at C is determined by setting $x_2 = 3a$. We get

$$v_C = -\frac{Pa^3}{EI} \qquad \textit{Ans.}$$

8

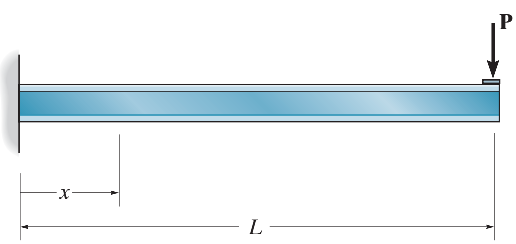
FUNDAMENTAL PROBLEMS

F8–4. Determine the equation of the elastic curve for the beam using the x coordinate that is valid for $0 < x < L$. EI is constant.



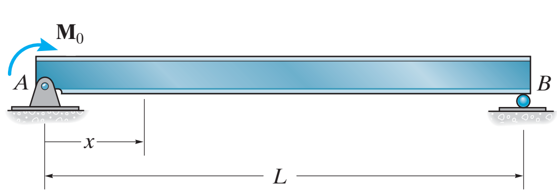
F8–4

F8–5. Determine the equation of the elastic curve for the beam using the x coordinate that is valid for $0 < x < L$. EI is constant.



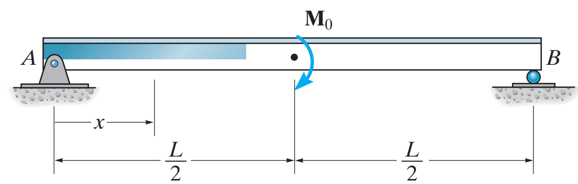
F8–5

F8–6. Determine the equation of the elastic curve for the beam using the x coordinate that is valid for $0 < x < L$. EI is constant.



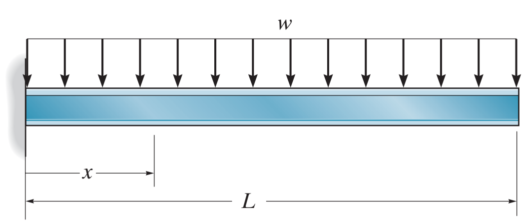
F8–6

F8–7. Determine the equation of the elastic curve for the beam using the x coordinate that is valid for $0 < x < L$. EI is constant.



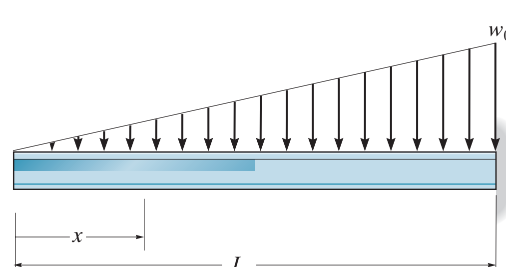
F8–7

F8–8. Determine the equation of the elastic curve for the beam using the x coordinate that is valid for $0 < x < L$. EI is constant.



F8–8

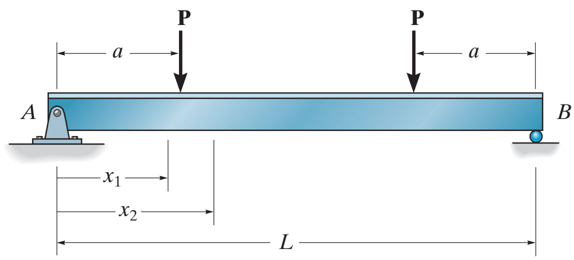
F8–9. Determine the equation of the elastic curve for the beam using the x coordinate that is valid for $0 < x < L$. EI is constant.



F8–9

PROBLEMS

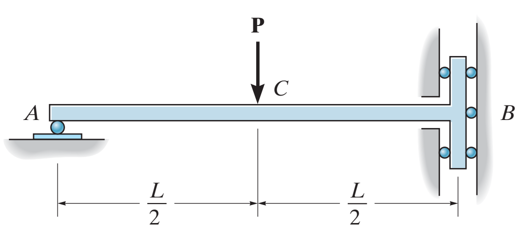
8–1. Determine the equations of the elastic curve for the beam using the x_1 and x_2 coordinates. Specify the slope at A and the maximum deflection. EI is constant.



Prob. 8–1

8–2. The bar is supported by a roller constraint at B, which allows vertical displacement but resists axial load and moment. If the bar is subjected to the loading shown, determine the slope at A and the deflection at C. EI is constant.

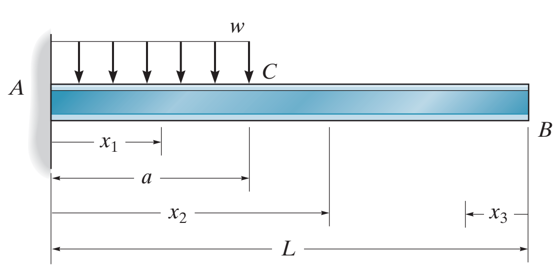
8–3. Determine the deflection at B of the bar in Prob. 8–2.



Probs. 8–2/8–3

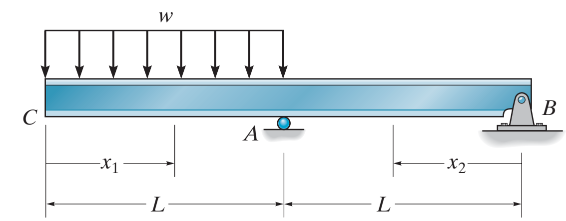
***8–4.** Determine the equations of the elastic curve using the coordinates x_1 and x_2, specify the slope and deflection at B. EI is constant.

8–5. Determine the equations of the elastic curve using the coordinates x_1 and x_3, and specify the slope and deflection at point B. EI is constant.



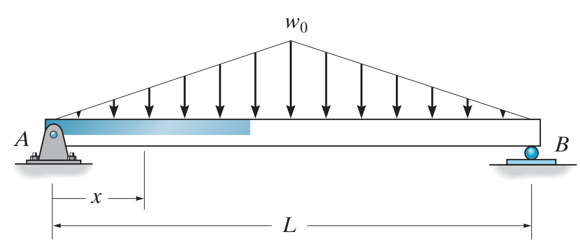
Probs. 8–4/8–5

8–6. Determine the maximum deflection between the supports A and B. EI is constant. Use the method of integration.



Prob. 8–6

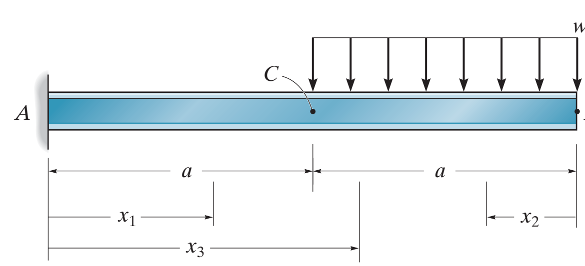
8–7. Determine the elastic curve for the simply supported beam using the x coordinate $0 \leq x \leq L/2$. Also, determine the slope at A and the maximum deflection of the beam. EI is constant.



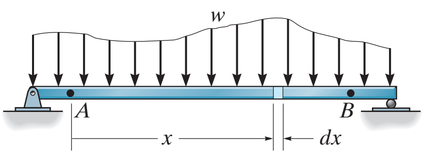
Prob. 8–7

***8–8.** Determine the equations of the elastic curve using the coordinates x_1 and x_2, and specify the slope at C and displacement at B. EI is constant.

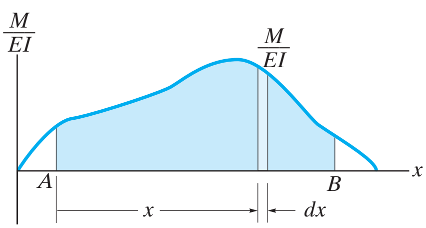
8–9. Determine the equations of the elastic curve using the coordinates x_1 and x_3, and specify the slope at B and deflection at C. EI is constant.



Probs. 8–8/8–9



(a)



(b)

8.4 Moment-Area Theorems

The initial ideas for the two moment-area theorems were developed by Otto Mohr and later stated formally by Charles E. Greene in 1873. These theorems provide a semigraphical technique for determining the slope of the elastic curve and its deflection due to bending. They are particularly advantageous when used to solve problems involving beams, especially those subjected to a series of concentrated loadings or having segments with different moments of inertia.

To develop the theorems, reference is made to the beam in Fig. 8–14*a*. If we draw the moment diagram for the beam and then divide it by the flexural rigidity, *EI*, the "*M/EI* diagram" shown in Fig. 8–14*b* results. By Eq. 8–2,

$$d\theta = \left(\frac{M}{EI}\right) dx$$

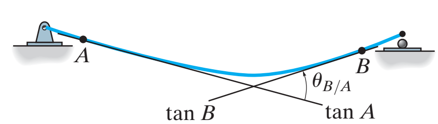
Thus it can be seen that the change $d\theta$ in the slope of the tangents on either side of the element dx is equal to the lighter-shaded *area* under the *M/EI* diagram. Integrating from point A on the elastic curve to point B, Fig. 8–14*c*, we have

$$\theta_{B/A} = \int_A^B \frac{M}{EI}\, dx \tag{8–5}$$

This equation forms the basis for the first moment-area theorem.

Theorem 1: The change in slope between any two points on the elastic curve equals the area of the *M/EI* diagram between these two points.

The notation $\theta_{B/A}$ is referred to as the angle of the tangent at B measured with respect to the tangent at A. From the proof it should be evident that this angle is measured *counterclockwise* from tangent A to tangent B if the area of the *M/EI* diagram is *positive,* Fig. 8–14*c*. Conversely, if this area is *negative,* or below the x axis, the angle $\theta_{B/A}$ is measured *clockwise* from tangent A to tangent B. Furthermore, from the dimensions of Eq. 8–5, $\theta_{B/A}$ is measured in radians.



elastic curve
(c)

Fig. 8–14

8

The second moment-area theorem is based on the relative deviation of *tangents* to the elastic curve. Shown in Fig. 8–15*c* is a greatly exaggerated view of the *vertical deviation dt* of the tangents on each side of the differential element dx. This deviation is measured along a vertical line passing through point A. Since the slope of the elastic curve and its deflection are assumed to be very small, it is satisfactory to approximate the length of each tangent line by x and the arc ds' by dt. Using the circular-arc formula $s = \theta r$, where r is of length x, we can write $dt = x\,d\theta$. Using Eq. 8–2, $d\theta = (M/EI)\,dx$, the vertical deviation of the tangent at *A with respect to the tangent* at B can be found by integration, in which case

$$t_{A/B} = \int_A^B x \frac{M}{EI}\,dx \qquad (8\text{–}6)$$

Recall from statics that the centroid of an area is determined from $\bar{x}\int dA = \int x\,dA$. Since $\int M/EI\,dx$ represents an area of the M/EI diagram, we can also write

$$t_{A/B} = \bar{x}\int_A^B \frac{M}{EI}\,dx \qquad (8\text{–}7)$$

Here $\bar{x}$ is the distance from the vertical axis through A to the *centroid* of the area between A and B, Fig. 8–15*b*.

The second moment-area theorem can now be stated as follows:

Theorem 2: The vertical deviation of the tangent at a point (A) on the elastic curve with respect to the tangent extended from another point (B) equals the "moment" of the area under the M/EI diagram between the two points (A and B). This moment is computed about point A (the point on the elastic curve), where the deviation $t_{A/B}$ is to be determined.

Provided the moment of a *positive M/EI* area from A to B is computed, as in Fig. 8–15*b*, it indicates that the tangent at point A is *above* the tangent to the curve extended from point B, Fig. 8–15*c*. Similarly, *negative M/EI* areas indicate that the tangent at A is *below* the tangent extended from B. Note that in general $t_{A/B}$ is not equal to $t_{B/A}$, which is shown in Fig. 8–15*d*. Specifically, the moment of the area under the M/EI diagram between A and B is computed about point A to determine $t_{A/B}$, Fig. 8–15*b*, and it is computed about point B to determine $t_{B/A}$.

It is important to realize that the moment-area theorems can only be used to determine the angles or deviations between two tangents on the beam's elastic curve. In general, they *do not* give a direct solution for the slope or displacement at a point on the beam. These unknowns must first be related to the angles or vertical deviations of tangents at points on the elastic curve. Usually the tangents at the supports are drawn in this regard since these points do not undergo displacement and/or have zero slope. Specific cases for establishing these geometric relationships are given in the example problems.

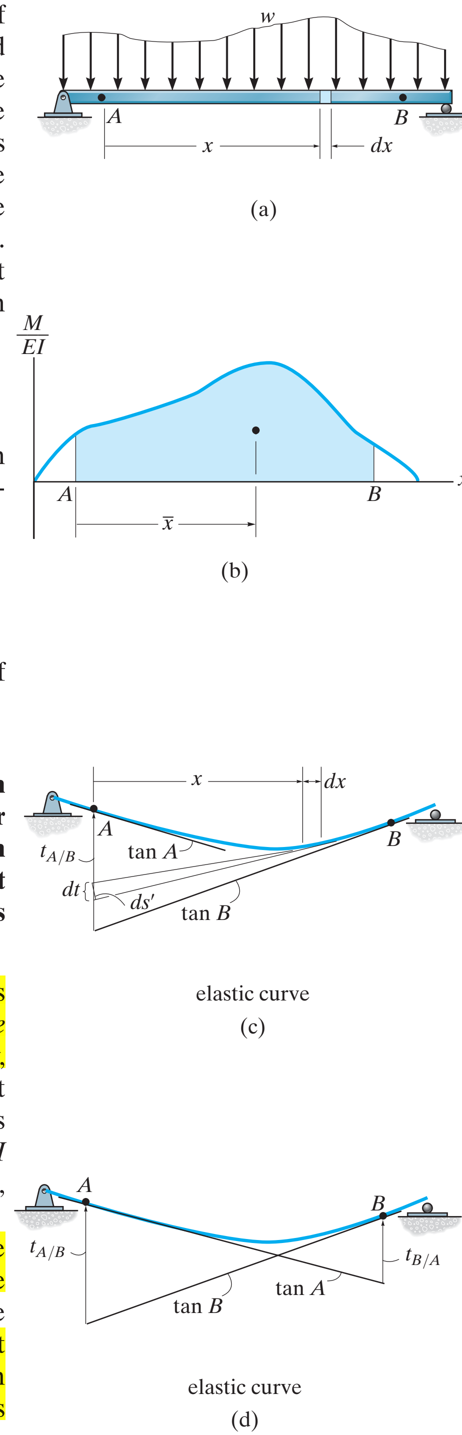


Fig. 8–15

8

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the moment-area theorems.

M/EI Diagram

- Determine the support reactions and draw the beam's *M/EI* diagram.
- If the beam is loaded with concentrated forces, the *M/EI* diagram will consist of a series of straight line segments, and the areas and their moments required for the moment-area theorems will be relatively easy to compute.
- If the loading consists of a *series* of concentrated forces and distributed loads, it may be simpler to compute the required *M/EI* areas and their moments by drawing the *M/EI* diagram in parts, using the method of superposition as discussed in Sec. 4–5. In any case, the *M/EI* diagram will consist of parabolic or perhaps higher-order curves, and it is suggested that the table on the inside back cover be used to locate the area and centroid under each curve.

Elastic Curve

- Draw an exaggerated view of the beam's elastic curve. Recall that points of zero slope occur at fixed supports and zero displacement occurs at all fixed, pin, and roller supports.
- If it becomes difficult to draw the general shape of the elastic curve, use the moment (or *M/EI*) diagram. Realize that when the beam is subjected to a *positive moment* the beam bends *concave up,* whereas *negative moment* bends the beam *concave down.* Furthermore, an inflection point or change in curvature occurs where the moment in the beam (or *M/EI*) is zero.
- The displacement and slope to be determined should be indicated on the curve. Since the moment-area theorems apply only between two tangents, attention should be given as to which tangents should be constructed so that the angles or deviations between them will lead to the solution of the problem. In this regard, *the tangents at the points of unknown slope and displacement and at the supports should be considered,* since the beam usually has zero displacement and/or zero slope at the supports.

Moment-Area Theorems

- Apply Theorem 1 to determine the angle between two tangents, and Theorem 2 to determine vertical deviations between these tangents.
- Realize that Theorem 2 in general *will not* yield the displacement of a point on the elastic curve. When applied properly, it will only give the vertical distance or deviation of a tangent at point *A* on the elastic curve from the tangent at *B*.
- After applying either Theorem 1 or Theorem 2, the algebraic sign of the answer can be verified from the angle or deviation as indicated on the elastic curve.

EXAMPLE 8.6

Determine the slope at points B and C of the beam shown in Fig. 8–16*a*. Take $E = 29(10^3)$ ksi and $I = 600\,\text{in}^4$.

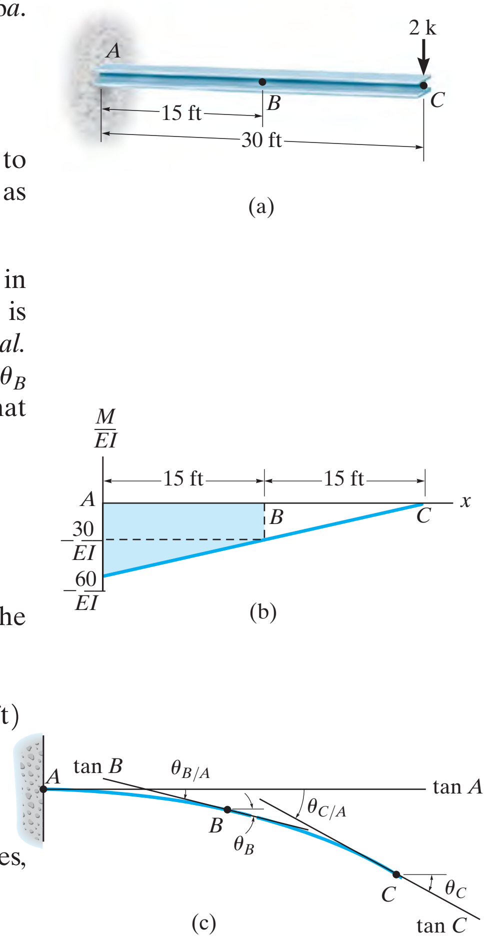


Fig. 8–16

SOLUTION

M/EI Diagram. This diagram is shown in Fig. 8–16*b*. It is easier to solve the problem in terms of EI and substitute the numerical data as a last step.

Elastic Curve. The 2-k load causes the beam to deflect as shown in Fig. 8–16*c*. (The beam is deflected concave down, since M/EI is negative.) Here the tangent at A (the support) is *always horizontal.* The tangents at B and C are also indicated. We are required to find θ_B and θ_C. By the construction, the angle between tan A and tan B, that is, $\theta_{B/A}$, is equivalent to θ_B.

$$\theta_B = \theta_{B/A}$$

Also,

$$\theta_C = \theta_{C/A}$$

Moment-Area Theorem. Applying Theorem 1, $\theta_{B/A}$ is equal to the area under the M/EI diagram between points A and B; that is,

$$\theta_B = \theta_{B/A} = -\left(\frac{30\text{ k}\cdot\text{ft}}{EI}\right)(15\text{ ft}) - \frac{1}{2}\left(\frac{60\text{ k}\cdot\text{ft}}{EI} - \frac{30\text{ k}\cdot\text{ft}}{EI}\right)(15\text{ ft})$$

$$= -\frac{675\text{ k}\cdot\text{ft}^2}{EI}$$

Substituting numerical data for E and I, and converting feet to inches, we have

$$\theta_B = \frac{-675\text{ k}\cdot\text{ft}^2(144\text{ in}^2/1\text{ ft}^2)}{29(10^3)\text{ k/in}^2(600\text{ in}^4)}$$

$$= -0.00559\text{ rad} \qquad \textit{Ans.}$$

The *negative sign* indicates that the angle is measured clockwise from A, Fig. 8–16*c*.

In a similar manner, the area under the M/EI diagram between points A and C equals $\theta_{C/A}$. We have

$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{60\text{ k}\cdot\text{ft}}{EI}\right)(30\text{ ft}) = -\frac{900\text{ k}\cdot\text{ft}^2}{EI}$$

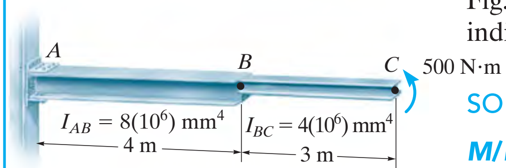
Substituting numerical values for EI, we have

$$\theta_C = \frac{-900\text{ k}\cdot\text{ft}^2(144\text{ in}^2/\text{ft}^2)}{29(10^3)\text{ k/in}^2(600\text{ in}^4)}$$

$$= -0.00745\text{ rad} \qquad \textit{Ans.}$$

8

EXAMPLE 8.7



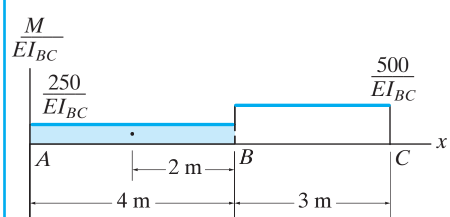
(a)

Determine the deflection at points B and C of the beam shown in Fig. 8–17a. Values for the moment of inertia of each segment are indicated in the figure. Take $E = 200$ GPa.

SOLUTION

***M/EI* Diagram.** By inspection, the moment diagram for the beam is a rectangle. Here we will construct the M/EI diagram relative to I_{BC}, realizing that $I_{AB} = 2I_{BC}$. Fig. 8–17b. Numerical data for EI_{BC} will be substituted as a last step.

Elastic Curve. The couple moment at C causes the beam to deflect as shown in Fig. 8–17c. The tangents at A (the support), B, and C are indicated. We are required to find Δ_B and Δ_C. These displacements can be related *directly* to the deviations between the tangents, so that from the construction Δ_B is equal to the deviation of tan B relative to tan A; that is,



(b)

$$\Delta_B = t_{B/A}$$

Also,

$$\Delta_C = t_{C/A}$$

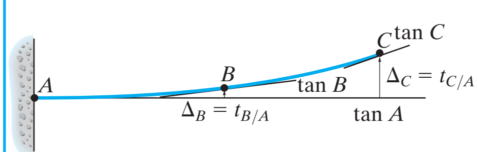
Moment-Area Theorem. Applying Theorem 2, $t_{B/A}$ is equal to the moment of the area under the M/EI_{BC} diagram between A and B computed about point B, since this is the point where the tangential deviation is to be determined. Hence, from Fig. 8–17b,

$$\Delta_B = t_{B/A} = \left[\frac{250\ \text{N}\cdot\text{m}}{EI_{BC}}(4\ \text{m})\right](2\ \text{m}) = \frac{2000\ \text{N}\cdot\text{m}^3}{EI_{BC}}$$

8

Substituting the numerical data yields

$$\Delta_B = \frac{2000\ \text{N}\cdot\text{m}^3}{[200(10^9)\ \text{N/m}^2][4(10^6)\ \text{mm}^4(1\ \text{m}^4/(10^3)^4\ \text{mm}^4)]}$$
$$= 0.0025\ \text{m} = 2.5\ \text{mm}. \qquad \textit{Ans.}$$



(c)

Fig. 8–17

Likewise, for $t_{C/A}$ we must compute the moment of the entire M/EI_{BC} diagram from A to C about point C. We have

$$\Delta_C = t_{C/A} = \left[\frac{250\ \text{N}\cdot\text{m}}{EI_{BC}}(4\ \text{m})\right](5\ \text{m}) + \left[\frac{500\ \text{N}\cdot\text{m}}{EI_{BC}}(3\ \text{m})\right](1.5\ \text{m})$$
$$= \frac{7250\ \text{N}\cdot\text{m}^3}{EI_{BC}} = \frac{7250\ \text{N}\cdot\text{m}^3}{[200(10^9)\ \text{N/m}^2][4(10^6)(10^{-12})\ \text{m}^4]}$$
$$= 0.00906\ \text{m} = 9.06\ \text{mm} \qquad \textit{Ans.}$$

Since both answers are *positive,* they indicate that points B and C lie *above* the tangent at A.

EXAMPLE 8.8

Determine the slope at point C of the beam in Fig. 8–18*a*. $E = 200$ GPa, $I = 6(10^6)$ mm^4.

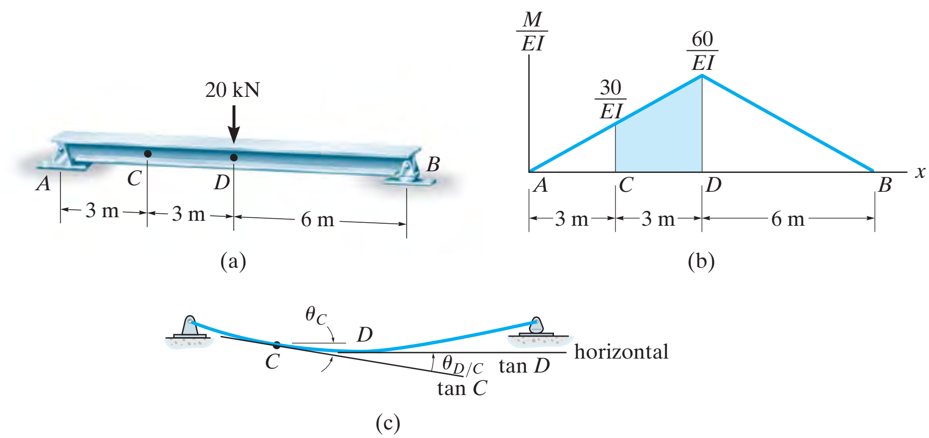


Fig. 8–18

SOLUTION

***M/EI* Diagram.** Fig. 8–18*b*.

Elastic Curve. Since the loading is applied symmetrically to the beam, the elastic curve is symmetric, as shown in Fig. 8–18*c*. We are required to find θ_C. This can easily be done, realizing that the tangent at D is *horizontal*, and therefore, by the construction, the angle $\theta_{D/C}$ between tan C and tan D is equal to θ_C; that is,

$$\theta_C = \theta_{D/C}$$

Moment-Area Theorem. Using Theorem 1, $\theta_{D/C}$ is equal to the shaded area under the M/EI diagram between points C and D. We have

$$\theta_C = \theta_{D/C} = 3\text{ m}\left(\frac{30\text{ kN}\cdot\text{m}}{EI}\right) + \frac{1}{2}(3\text{ m})\left(\frac{60\text{ kN}\cdot\text{m}}{EI} - \frac{30\text{ kN}\cdot\text{m}}{EI}\right)$$

$$= \frac{135\text{ kN}\cdot\text{m}^2}{EI}$$

Thus,

$$\theta_C = \frac{135\text{ kN}\cdot\text{m}^2}{[200(10^6)\text{ kN/m}^2][6(10^6)(10^{-12})\text{ m}^4]} = 0.112\text{ rad} \qquad \textit{Ans.}$$

EXAMPLE 8.9

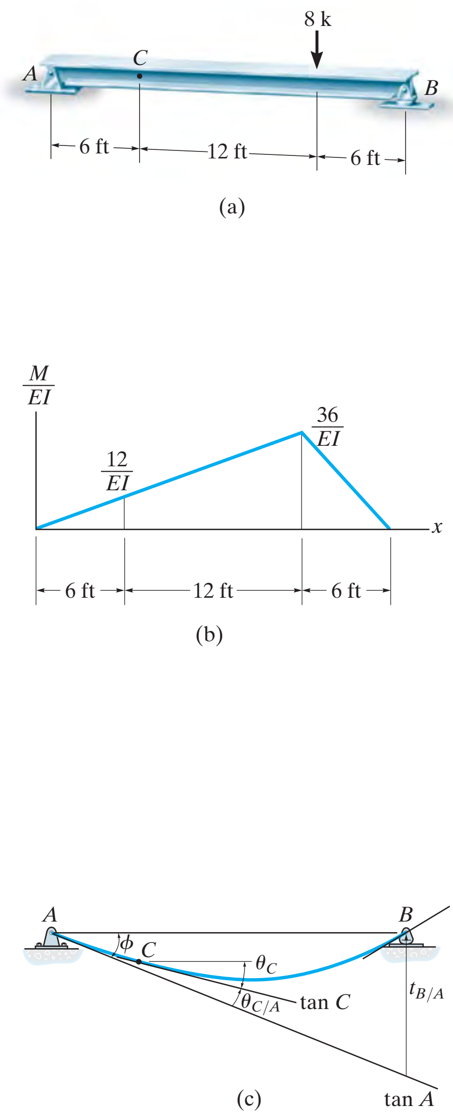


Fig. 8–19

Determine the slope at point C of the beam in Fig. 8–19*a*. $E = 29(10^3)$ ksi, $I = 600$ in^4.

SOLUTION

M/EI Diagram. Fig. 8–19*b*.

Elastic Curve. The elastic curve is shown in Fig. 8–19*c*. We are required to find θ_C. To do this, establish tangents at A, B (the supports), and C and note that $\theta_{C/A}$ is the angle between the tangents at A and C. Also, the angle ϕ in Fig. 8–19*c* can be found using $\phi = t_{B/A}/L_{AB}$. This equation is valid since $t_{B/A}$ is actually very small, so that $t_{B/A}$ can be approximated by the length of a circular arc defined by a radius of $L_{AB} = 24$ ft and sweep of ϕ. (Recall that $s = \theta r$.) From the geometry of Fig. 8–19*c*, we have

$$\theta_C = \phi - \theta_{C/A} = \frac{t_{B/A}}{24} - \theta_{C/A} \qquad (1)$$

Moment-Area Theorems. Using Theorem 1, $\theta_{C/A}$ is equivalent to the area under the M/EI diagram between points A and C; that is,

$$\theta_{C/A} = \frac{1}{2}(6\text{ ft})\left(\frac{12\text{ k}\cdot\text{ft}}{EI}\right) = \frac{36\text{ k}\cdot\text{ft}^2}{EI}$$

Applying Theorem 2, $t_{B/A}$ is equivalent to the moment of the area under the M/EI diagram between B and A about point B, since this is the point where the tangential deviation is to be determined. We have

$$\begin{aligned} t_{B/A} &= \left[6\text{ ft} + \frac{1}{3}(18\text{ ft})\right]\left[\frac{1}{2}(18\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right] \\ &\quad + \frac{2}{3}(6\text{ ft})\left[\frac{1}{2}(6\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right] \\ &= \frac{4320\text{ k}\cdot\text{ft}^3}{EI} \end{aligned}$$

Substituting these results into Eq. 1, we have

$$\theta_C = \frac{4320\text{ k}\cdot\text{ft}^3}{(24\text{ ft})\,EI} - \frac{36\text{ k}\cdot\text{ft}^2}{EI} = \frac{144\text{ k}\cdot\text{ft}^2}{EI}$$

so that

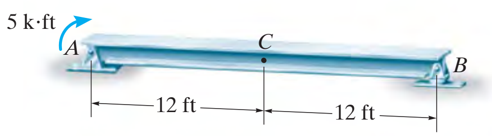
$$\begin{aligned} \theta_C &= \frac{144\text{ k}\cdot\text{ft}^2}{29(10^3)\text{ k/in}^2(144\text{ in}^2/\text{ft}^2)\,600\text{ in}^4(1\text{ ft}^4/(12)^4\text{ in}^4)} \\ &= 0.00119\text{ rad} \end{aligned}$$

Ans.

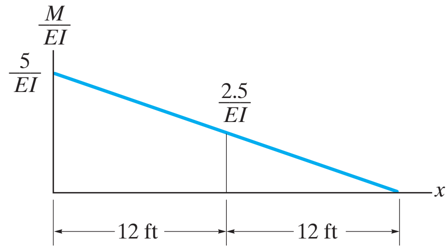
8

EXAMPLE 8.10

Determine the deflection at C of the beam shown in Fig. 8–20*a*. Take $E = 29(10^3)$ ksi, $I = 21$ in^4.



(a)



(b)

SOLUTION

***M/EI* Diagram.** Fig. 8–20*b*.

Elastic Curve. Here we are required to find Δ_C, Fig. 8–20*c*. This is not necessarily the maximum deflection of the beam, since the loading and hence the elastic curve are *not symmetric*. Also indicated in Fig. 8–20*c* are the tangents at A, B (the supports), and C. If $t_{A/B}$ is determined, then Δ' can be found from proportional triangles, that is, $\Delta'/12 = t_{A/B}/24$ or $\Delta' = t_{A/B}/2$. From the construction in Fig. 8–20*c*, we have

$$\Delta_C = \frac{t_{A/B}}{2} - t_{C/B} \qquad (1)$$

Moment-Area Theorem. We will apply Theorem 2 to determine $t_{A/B}$ and $t_{C/B}$. Here $t_{A/B}$ is the moment of the M/EI diagram between A and B about point A,

$$t_{A/B} = \left[\frac{1}{3}(24\text{ ft})\right]\left[\frac{1}{2}(24\text{ ft})\left(\frac{5\text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{480\text{ k}\cdot\text{ft}^3}{EI}$$

and $t_{C/B}$ is the moment of the M/EI diagram between C and B about C.

$$t_{C/B} = \left[\frac{1}{3}(12\text{ ft})\right]\left[\frac{1}{2}(12\text{ ft})\left(\frac{2.5\text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{60\text{ k}\cdot\text{ft}^3}{EI}$$

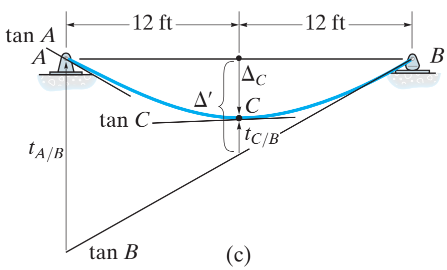
Substituting these results into Eq. (1) yields

$$\Delta_C = \frac{1}{2}\left(\frac{480\text{ k}\cdot\text{ft}^3}{EI}\right) - \frac{60\text{ k}\cdot\text{ft}^3}{EI} = \frac{180\text{ k}\cdot\text{ft}^3}{EI}$$

Working in units of kips and inches, we have

$$\Delta_C = \frac{180\text{ k}\cdot\text{ft}^3(1728\text{ in}^3/\text{ft}^3)}{29(10^3)\text{ k/in}^2(21\text{ in}^4)}$$

$$= 0.511\text{ in.} \qquad \textit{Ans.}$$



(c)

Fig. 8–20

EXAMPLE 8.11

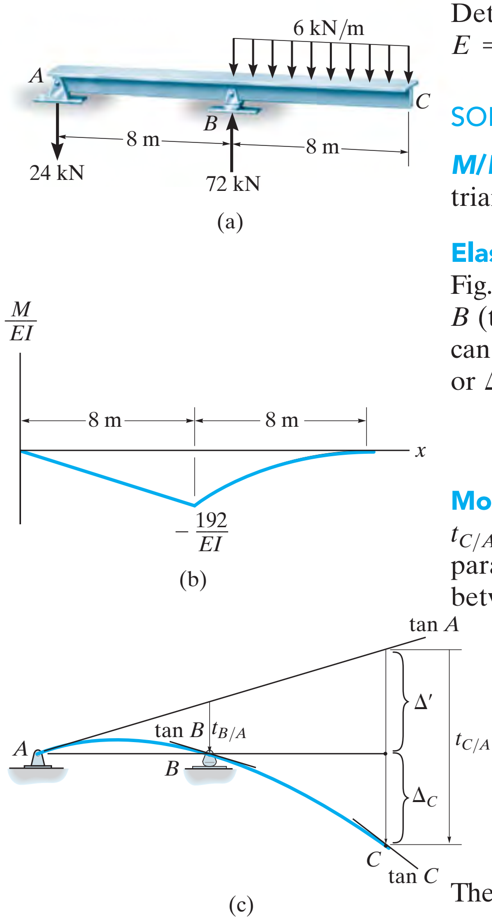


Fig. 8–21

Determine the deflection at point C of the beam shown in Fig. 8–21a. $E = 200$ GPa, $I = 250(10^6)$ mm^4.

SOLUTION

M/EI Diagram. As shown in Fig. 8–21b, this diagram consists of a triangular and a parabolic segment.

Elastic Curve. The loading causes the beam to deform as shown in Fig. 8–21c. We are required to find Δ_C. By constructing tangents at A, B (the supports), and C, it is seen that $\Delta_C = t_{C/A} - \Delta'$. However, Δ' can be related to $t_{B/A}$ by proportional triangles, that is, $\Delta'/16 = t_{B/A}/8$ or $\Delta' = 2t_{B/A}$. Hence

$$\Delta_C = t_{C/A} - 2t_{B/A} \qquad (1)$$

Moment-Area Theorem. We will apply Theorem 2 to determine $t_{C/A}$ and $t_{B/A}$. Using the table on the inside back cover for the parabolic segment and considering the moment of the M/EI diagram between A and C about point C, we have

$$\begin{aligned} t_{C/A} &= \left[\frac{3}{4}(8\text{ m})\right]\left[\frac{1}{3}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right] \\ &\quad + \left[\frac{1}{3}(8\text{ m}) + 8\text{ m}\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right] \\ &= -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI} \end{aligned}$$

The moment of the M/EI diagram between A and B about point B gives

$$t_{B/A} = \left[\frac{1}{3}(8\text{ m})\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right] = -\frac{2048\text{ kN}\cdot\text{m}^3}{EI}$$

Why are these terms negative? Substituting the results into Eq. (1) yields

$$\begin{aligned} \Delta_C &= -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI} - 2\left(-\frac{2048\text{ kN}\cdot\text{m}^3}{EI}\right) \\ &= -\frac{7168\text{ kN}\cdot\text{m}^3}{EI} \end{aligned}$$

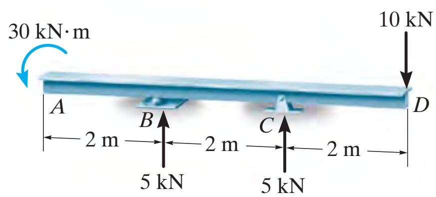
Thus,

$$\begin{aligned} \Delta_C &= \frac{-7168\text{ kN}\cdot\text{m}^3}{[200(10^6)\text{ kN/m}^2][250(10^6)(10^{-12})\text{ m}^4]} \\ &= -0.143\text{ m} \end{aligned}$$ *Ans.*

8

EXAMPLE 8.12

Determine the slope at the roller B of the double overhang beam shown in Fig. 8–22a. Take $E = 200$ GPa, $I = 18(10^6)$ mm^4.



(a)

SOLUTION

***M/EI* Diagram.** The M/EI diagram can be simplified by drawing it in parts and considering the M/EI diagrams for the three loadings each acting on a cantilever beam fixed at D, Fig. 8–22b. (The 10-kN load is not considered since it produces no moment about D.)

Elastic Curve. If tangents are drawn at B and C, Fig. 8–22c, the slope B can be determined by finding $t_{C/B}$, and for small angles,

$$\theta_B = \frac{t_{C/B}}{2\text{ m}} \tag{1}$$

Moment Area Theorem. To determine $t_{C/B}$ we apply the moment area theorem by finding the moment of the M/EI diagram between BC about point C. This only involves the shaded area under two of the diagrams in Fig. 8–22b. Thus,

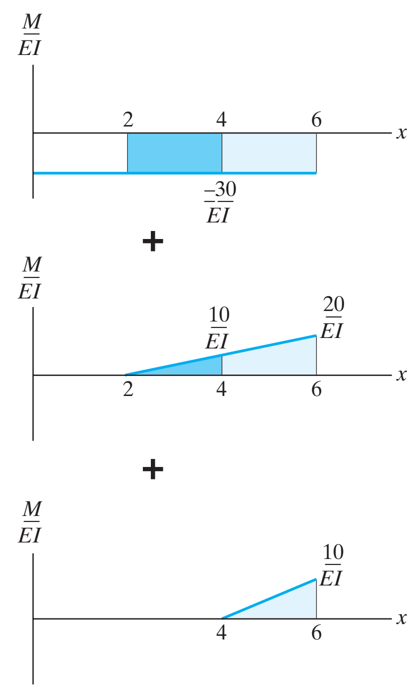
$$t_{C/B} = (1\text{ m})\left[(2\text{ m})\left(\frac{-30\text{ kN}\cdot\text{m}}{EI}\right)\right] + \left(\frac{2\text{ m}}{3}\right)\left[\frac{1}{2}(2\text{ m})\left(\frac{10\text{ kN}\cdot\text{m}}{EI}\right)\right]$$

$$= \frac{53.33\text{ kN}\cdot\text{m}^3}{EI}$$

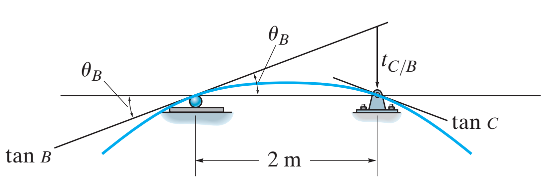
Substituting into Eq. (1),

$$\theta_B = \frac{53.33\text{ kN}\cdot\text{m}^3}{(2\text{ m})[200(10^6)\text{ kN/m}^3][18(10^6)(10^{-12})\text{ m}^4]}$$

$$= 0.00741\text{ rad}$$ *Ans.*



(b)



(c)

Fig. 8–22

8.5 Conjugate-Beam Method

The conjugate-beam method was developed by H. Müller-Breslau in 1865. Essentially, it requires the same amount of computation as the moment-area theorems to determine a beam's slope or deflection; however, this method relies only on the principles of statics, and hence its application will be more familiar.

The basis for the method comes from the *similarity* of Eq. 4–1 and Eq. 4–2 to Eq. 8–2 and Eq. 8–4. To show this similarity, we can write these equations as follows:

$$\frac{dV}{dx} = w \qquad \Bigg| \qquad \frac{d^2M}{dx^2} = w$$

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \Bigg| \qquad \frac{d^2v}{dx^2} = \frac{M}{EI}$$

Or integrating,

$$V = \int w\,dx \qquad \Bigg| \qquad M = \int\left[\int w\,dx\right]dx$$

$$\updownarrow \qquad \updownarrow \qquad \Bigg| \qquad \updownarrow \qquad \updownarrow$$

$$\theta = \int\left(\frac{M}{EI}\right)dx \qquad \Bigg| \qquad v = \int\left[\int\left(\frac{M}{EI}\right)dx\right]dx$$

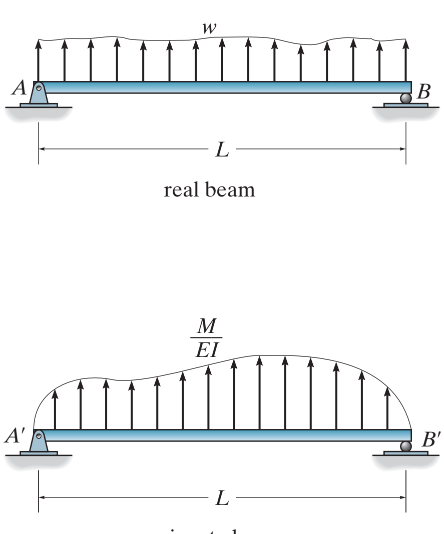


Fig. 8–23

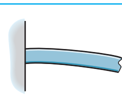
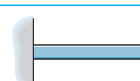
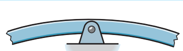
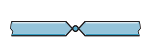
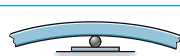
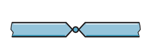
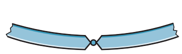
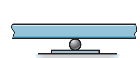
Here the *shear V* compares with the *slope θ*, the *moment M* compares with the *displacement v*, and the *external load w* compares with the *M/EI* diagram. To make use of this comparison we will now consider a beam having the same length as the real beam, but referred to here as the "conjugate beam," Fig. 8–23. The conjugate beam is "loaded" with the *M/EI* diagram derived from the load *w* on the real beam. From the above comparisons, we can state two theorems related to the conjugate beam, namely,

Theorem 1: The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam.

Theorem 2: The displacement of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam.

Conjugate-Beam Supports. When drawing the conjugate beam it is important that the shear and moment developed at the supports of the conjugate beam account for the corresponding slope and displacement of the real beam at its supports, a consequence of Theorems 1 and 2. For example, as shown in Table 8–2, a pin or roller support at the end of the real beam provides *zero displacement,* but the beam has a nonzero slope. Consequently, from Theorems 1 and 2, the conjugate beam must be supported by a pin or roller, since this support has *zero moment* but has a shear or end reaction. When the real beam is fixed supported (3), both the slope and displacement at the support are zero. Here the conjugate beam has a free end, since at this end there is zero shear and zero moment. Corresponding real and conjugate-beam supports for other cases are listed in the table. Examples of real and conjugate beams are shown in Fig. 8–24. Note that, as a rule, neglecting axial force, statically determinate real beams have statically determinate conjugate beams; and statically indeterminate real beams, as in the last case in Fig. 8–24, become unstable conjugate beams. Although this occurs, the *M/EI* loading will provide the necessary "equilibrium" to hold the conjugate beam stable.

TABLE 8–2

	Real Beam		Conjugate Beam	
1)	θ $\Delta = 0$	 pin	V $M = 0$	 pin
2)	θ $\Delta = 0$	 roller	V $M = 0$	 roller
3)	$\theta = 0$ $\Delta = 0$	 fixed	$V = 0$ $M = 0$	 free
4)	θ Δ	 free	V M	 fixed
5)	θ $\Delta = 0$	 internal pin	V $M = 0$	 hinge
6)	θ $\Delta = 0$	 internal roller	V $M = 0$	 hinge
7)	θ Δ	 hinge	V M	 internal roller

8

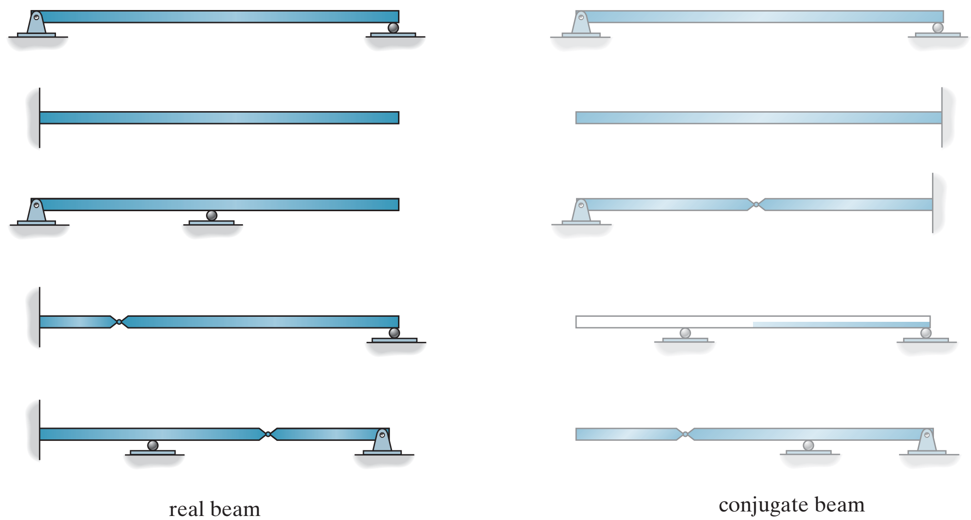


Fig. 8–24

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement and slope at a point on the elastic curve of a beam using the conjugate-beam method.

Conjugate Beam

- Draw the conjugate beam for the real beam. This beam has the same length as the real beam and has corresponding supports as listed in Table 8–2.
- In general, if the real support allows a *slope,* the conjugate support must develop a *shear*; and if the real support allows a *displacement,* the conjugate support must develop a *moment*.
- The conjugate beam is loaded with the real beam's M/EI diagram. This loading is assumed to be *distributed* over the conjugate beam and is directed *upward* when M/EI is *positive* and *downward* when M/EI is *negative*. In other words, the loading always acts *away* from the beam.

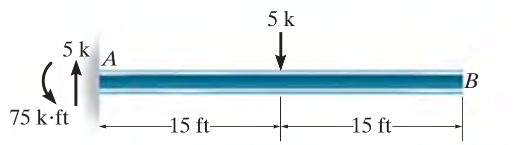
Equilibrium

- Using the equations of equilibrium, determine the reactions at the conjugate beam's supports.
- Section the conjugate beam at the point where the slope θ and displacement Δ of the real beam are to be determined. At the section show the unknown shear V' and moment M' acting in their positive sense.
- Determine the shear and moment using the equations of equilibrium. V' and M' equal θ and Δ, respectively, for the real beam. In particular, if these values are *positive,* the *slope* is *counterclockwise* and the *displacement* is *upward*.

8

EXAMPLE 8.13

Determine the slope and deflection at point B of the steel beam shown in Fig. 8–25*a*. The reactions have been computed. $E = 29(10^3)$ ksi, $I = 800\ \text{in}^4$.

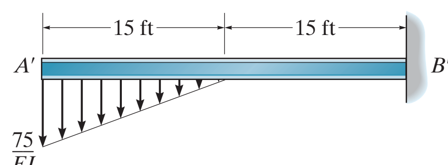


real beam
(a)

Fig. 8–25

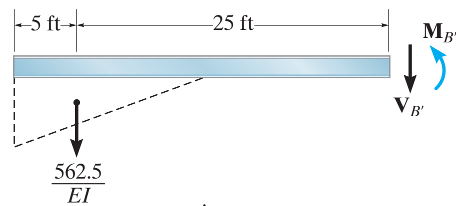
SOLUTION

Conjugate Beam. The conjugate beam is shown in Fig. 8–25*b*. The supports at A' and B' correspond to supports A and B on the real beam, Table 8–2. It is important to understand why this is so. The M/EI diagram is *negative*, so the distributed load acts *downward*, i.e., away from the beam.



conjugate beam
(b)

Equilibrium. Since θ_B and Δ_B are to be determined, we must compute $V_{B'}$ and $M_{B'}$ in the conjugate beam, Fig. 8–25*c*.



reactions
(c)

$$+\uparrow \Sigma F_y = 0; \qquad -\frac{562.5\ \text{k}\cdot\text{ft}^2}{EI} - V_{B'} = 0$$

$$\theta_B = V_{B'} = -\frac{562.5\ \text{k}\cdot\text{ft}^2}{EI}$$

$$= \frac{-562.5\ \text{k}\cdot\text{ft}^2}{29(10^3)\ \text{k/in}^2(144\ \text{in}^2/\text{ft}^2)800\ \text{in}^4(1\ \text{ft}^4/(12)^4\ \text{in}^4)}$$

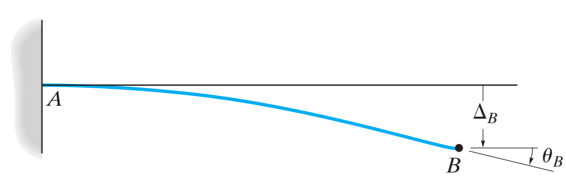
$$= -0.00349\ \text{rad} \qquad \textit{Ans.}$$

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad \frac{562.5\ \text{k}\cdot\text{ft}^2}{EI}(25\ \text{ft}) + M_{B'} = 0$$

$$\Delta_B = M_{B'} = -\frac{14\,062.5\ \text{k}\cdot\text{ft}^3}{EI}$$

$$= \frac{-14\,062.5\ \text{k}\cdot\text{ft}^3}{29(10^3)(144)\ \text{k/ft}^2[800/(12)^4]\ \text{ft}^4}$$

$$= -0.0873\ \text{ft} = -1.05\ \text{in.} \qquad \textit{Ans.}$$



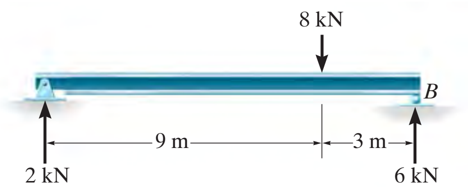
(d)

The negative signs indicate the slope of the beam is measured clockwise and the displacement is downward, Fig. 8–25*d*.

8

EXAMPLE 8.14

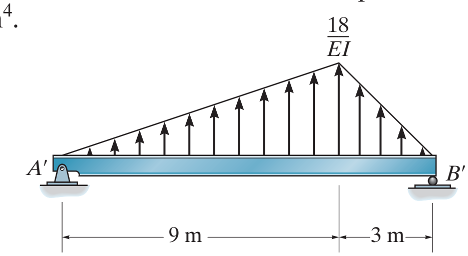
Determine the maximum deflection of the steel beam shown in Fig. 8–26*a*. The reactions have been computed. $E = 200$ GPa, $I = 60(10^6)\ \text{mm}^4$.



real beam

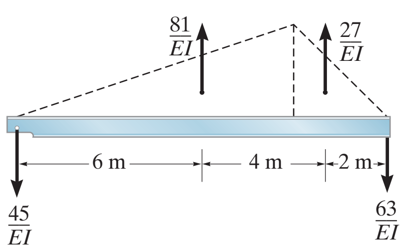
(a)

Fig. 8–26



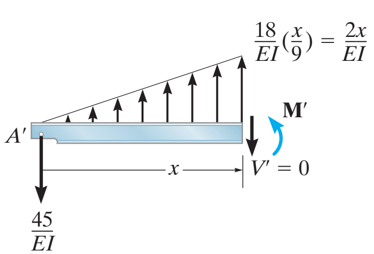
conjugate beam

(b)



external reactions

(c)



internal reactions

(d)

SOLUTION

Conjugate Beam. The conjugate beam loaded with the *M/EI* diagram is shown in Fig. 8–26*b*. Since the *M/EI* diagram is positive, the distributed load acts upward (away from the beam).

Equilibrium. The external reactions on the conjugate beam are determined first and are indicated on the free-body diagram in Fig. 8–26*c*. *Maximum deflection* of the real beam occurs at the point where the *slope* of the beam is *zero*. This corresponds to the same point in the conjugate beam where the *shear* is *zero*. Assuming this point acts within the region $0 \le x \le 9$ m from A', we can isolate the section shown in Fig. 8–26*d*. Note that the peak of the distributed loading was determined from proportional triangles, that is, $w/x = (18/EI)/9$. We require $V' = 0$ so that

$$+\uparrow \Sigma F_y = 0; \qquad -\frac{45}{EI} + \frac{1}{2}\left(\frac{2x}{EI}\right)x = 0$$

$$x = 6.71\ \text{m} \qquad (0 \le x \le 9\ \text{m})\ \text{OK}$$

Using this value for x, the maximum deflection in the real beam corresponds to the moment M'. Hence,

$$\circlearrowleft + \Sigma M = 0; \qquad \frac{45}{EI}(6.71) - \left[\frac{1}{2}\left(\frac{2(6.71)}{EI}\right)6.71\right]\frac{1}{3}(6.71) + M' = 0$$

$$\Delta_{\max} = M' = -\frac{201.2\ \text{kN}\cdot\text{m}^3}{EI}$$

$$= \frac{-201.2\ \text{kN}\cdot\text{m}^3}{[200(10^6)\ \text{kN/m}^2][60(10^6)\ \text{mm}^4(1\ \text{m}^4/(10^3)^4\ \text{mm}^4)]}$$

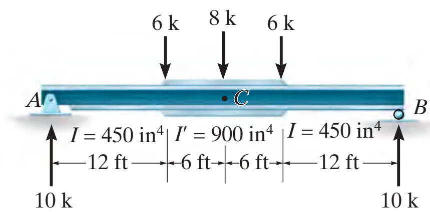
$$= -0.0168\ \text{m} = -16.8\ \text{mm} \qquad \textit{Ans.}$$

The negative sign indicates the deflection is downward.

8

EXAMPLE 8.15

The girder in Fig. 8–27*a* is made from a continuous beam and reinforced at its center with cover plates where its moment of inertia is larger. The 12-ft end segments have a moment of inertia of $I = 450\text{ in}^4$, and the center portion has a moment of inertia of $I' = 900\text{ in}^4$. Determine the deflection at the center C. Take $E = 29(10^3)$ ksi. The reactions have been calculated.



real beam

(a)

Fig. 8–27

SOLUTION

Conjugate Beam. The moment diagram for the beam is determined first, Fig. 8–27*b*. Since $I' = 2I$, for simplicity, we can express the load on the conjugate beam in terms of the constant EI, as shown in Fig. 8–27*c*.

Equilibrium. The reactions on the conjugate beam can be calculated by the symmetry of the loading or using the equations of equilibrium. The results are shown in Fig. 8–27*d*. Since the deflection at C is to be determined, we must compute the internal moment at C'. Using the method of sections, segment $A'C'$ is isolated and the resultants of the distributed loads and their locations are determined, Fig. 8–27*e*. Thus,

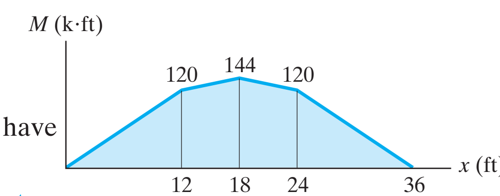
$$\circlearrowleft + \Sigma M_{C'} = 0; \quad \frac{1116}{EI}(18) - \frac{720}{EI}(10) - \frac{360}{EI}(3) - \frac{36}{EI}(2) + M_{C'} = 0$$

$$M_{C'} = -\frac{11\,736\text{ k}\cdot\text{ft}^3}{EI}$$

Substituting the numerical data for EI and converting units, we have

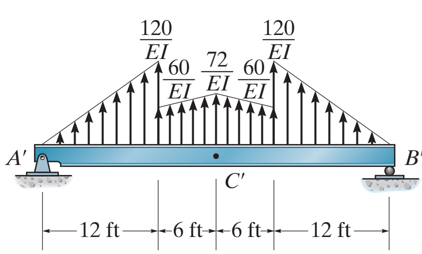
$$\Delta_C = M_{C'} = -\frac{11\,736\text{ k}\cdot\text{ft}^3(1728\text{ in}^3/\text{ft}^3)}{29(10^3)\text{ k/in}^2(450\text{ in}^4)} = -1.55\text{ in.} \qquad \textit{Ans.}$$

The negative sign indicates that the deflection is downward.



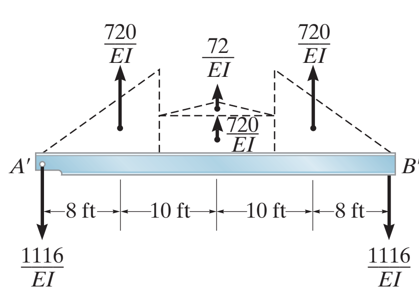
moment diagram

(b)



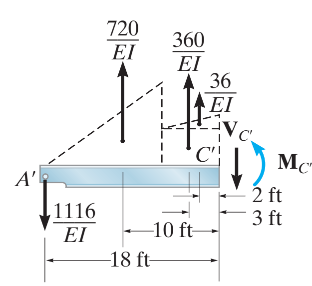
conjugate beam

(c)



external reactions

(d)



internal reactions

(e)

EXAMPLE 8.16

Determine the displacement of the pin at B and the slope of each beam segment connected to the pin for the compound beam shown in Fig. 8–28*a*. $E = 29(10^3)$ ksi, $I = 30$ in^4.

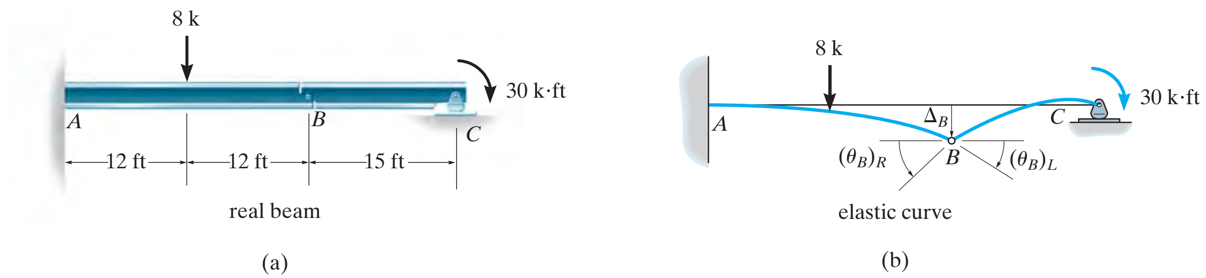
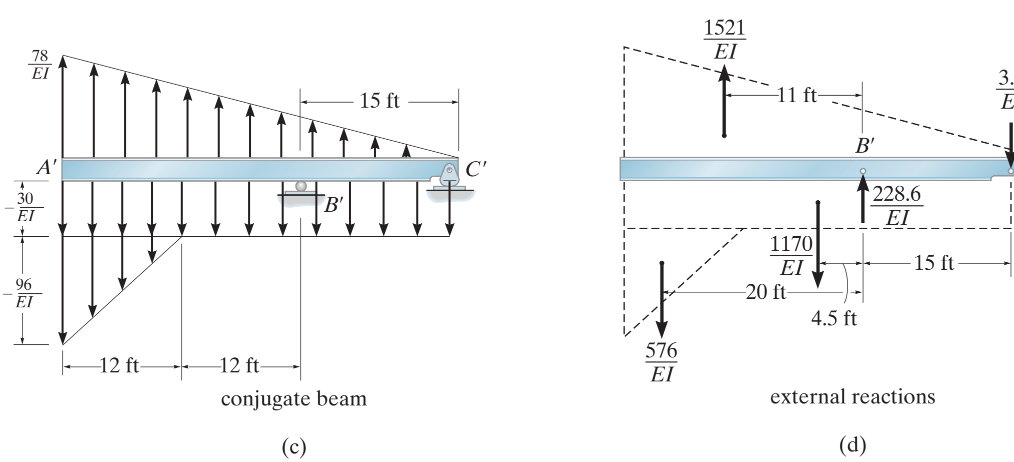


Fig. 8–28

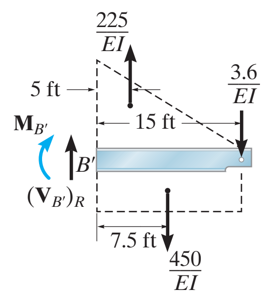
SOLUTION

Conjugate Beam. The elastic curve for the beam is shown in Fig. 8–28*b* in order to identify the unknown displacement Δ_B and the slopes $(\theta_B)_L$ and $(\theta_B)_R$ to the left and right of the pin. Using Table 8–2, the conjugate beam is shown in Fig. 8–28*c*. For simplicity in calculation, the M/EI diagram has been drawn in *parts* using the principle of superposition as described in Sec. 4–5. In this regard, the real beam is thought of as cantilevered from the left support, A. The moment diagrams for the 8-k load, the reactive force $C_y = 2$ k, and the 30-k · ft loading are given. Notice that negative regions of this diagram develop a downward distributed load and positive regions have a distributed load that acts upward.



8

Equilibrium. The external reactions at B' and C' are calculated first and the results are indicated in Fig. 8–28*d*. In order to determine $(\theta_B)_R$, the conjugate beam is sectioned just to the *right* of B' and the shear force $(V_{B'})_R$ is computed, Fig. 8–28*e*. Thus,



(e)

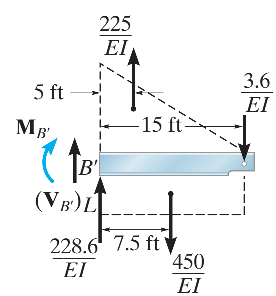
$$+\uparrow \Sigma F_y = 0; \qquad (V_{B'})_R + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

$$(\theta_B)_R = (V_{B'})_R = \frac{228.6 \text{ k}\cdot\text{ft}^2}{EI}$$

$$= \frac{228.6 \text{ k}\cdot\text{ft}^2}{[29(10^3)(144) \text{ k/ft}^2][30/(12)^4] \text{ ft}^4}$$

$$= 0.0378 \text{ rad} \qquad \textit{Ans.}$$

The internal moment at B' yields the displacement of the pin. Thus,



(f)

$$\circlearrowright + \Sigma M_{B'} = 0; \qquad -M_{B'} + \frac{225}{EI}(5) - \frac{450}{EI}(7.5) - \frac{3.6}{EI}(15) = 0$$

$$\Delta_B = M_{B'} = -\frac{2304 \text{ k}\cdot\text{ft}^3}{EI}$$

$$= \frac{-2304 \text{ k}\cdot\text{ft}^3}{[29(10^3)(144) \text{ k/ft}^2][30/(12)^4] \text{ ft}^4}$$

$$= -0.381 \text{ ft} = -4.58 \text{ in.} \qquad \textit{Ans.}$$

The slope $(\theta_B)_L$ can be found from a section of beam just to the *left* of B', Fig. 8–28*f*. Thus,

$$+\uparrow \Sigma F_y = 0; \qquad (V_{B'})_L + \frac{228.6}{EI} + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

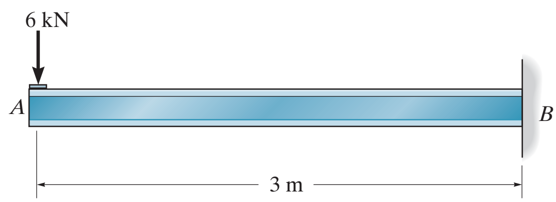
$$(\theta_B)_L = (V_{B'})_L = 0 \qquad \textit{Ans.}$$

Obviously, $\Delta_B = M_{B'}$ for this segment is the *same* as previously calculated, since the moment arms are only slightly different in Figs. 8–28*e* and 8–28*f*.

FUNDAMENTAL PROBLEMS

F8–10. Use the moment-area theorems and determine the slope at *A* and deflection at *A*. *EI* is constant.

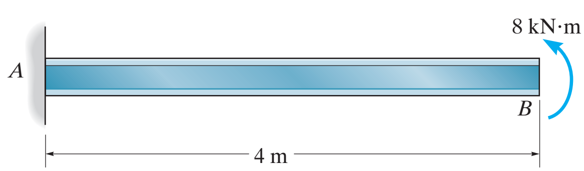
F8–11. Solve Prob. F8–10 using the conjugate beam method.



F8–10/8–11

F8–12. Use the moment-area theorems and determine the slope at *B* and deflection at *B*. *EI* is constant.

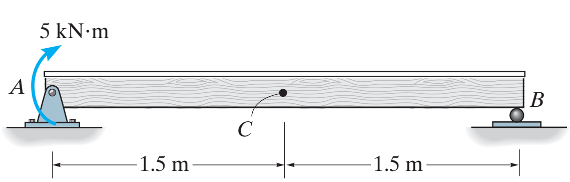
F8–13. Solve Prob. F8–12 using the conjugate beam method.



F8–12/8–13

F8–14. Use the moment-area theorems and determine the slope at *A* and displacement at *C*. *EI* is constant.

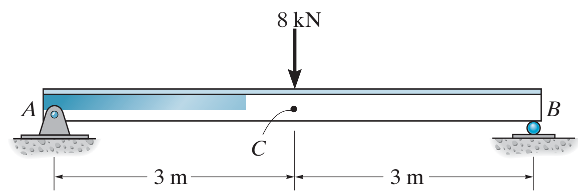
F8–15. Solve Prob. F8–14 using the conjugate beam method.



F8–14/8–15

F8–16. Use the moment-area theorems and determine the slope at *A* and displacement at *C*. *EI* is constant.

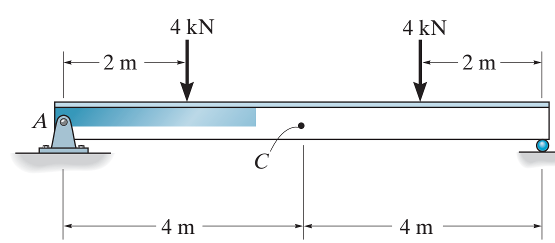
F8–17. Solve Prob. F8–16 using the conjugate beam method.



F8–16/8–17

F8–18. Use the moment-area theorems and determine the slope at *A* and displacement at *C*. *EI* is constant.

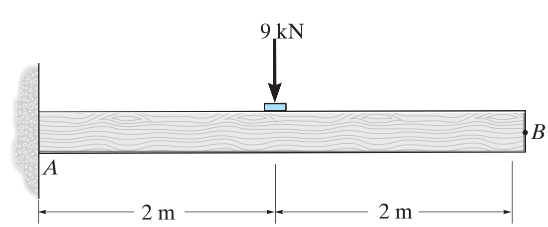
F8–19. Solve Prob. F8–18 using the conjugate beam method.



F8–18/8–19

F8–20. Use the moment-area theorems and determine the slope at *B* and displacement at *B*. *EI* is constant.

F8–21. Solve Prob. F8–20 using the conjugate beam method.

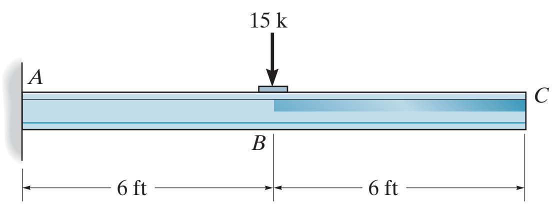


F8–20/8–21

PROBLEMS

8–10. Determine the slope at B and the maximum displacement of the beam. Use the moment-area theorems. Take $E = 29(10^3)$ ksi, $I = 500$ in^4.

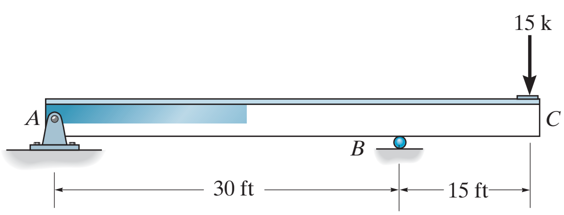
8–11. Solve Prob. 8–10 using the conjugate-beam method.



Probs. 8–10/8–11

***8–12.** Determine the slope and displacement at C. EI is constant. Use the moment-area theorems.

8–13. Solve Prob. 8–12 using the conjugate-beam method.



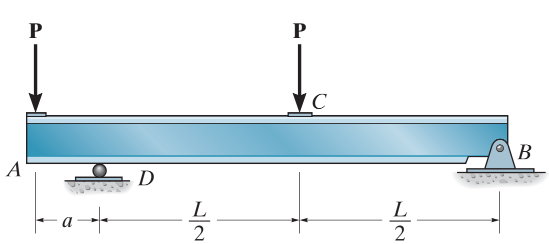
Probs. 8–12/8–13

8–14. Determine the value of a so that the slope at A is equal to zero. EI is constant. Use the moment-area theorems.

8–15. Solve Prob. 8–14 using the conjugate-beam method.

***8–16.** Determine the value of a so that the displacement at C is equal to zero. EI is constant. Use the moment-area theorems.

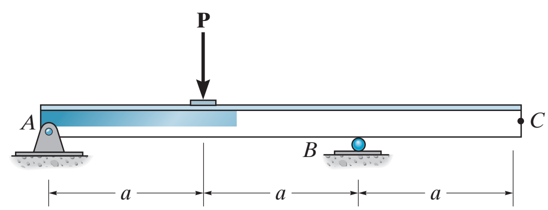
8–17. Solve Prob. 8–16 using the conjugate-beam method.



Probs. 8–14/8–15/8–16/8–17

8–18. Determine the slope and the displacement at C. EI is constant. Use the moment-area theorems.

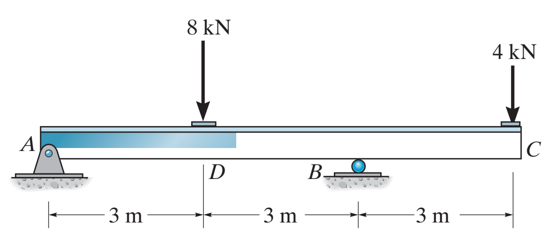
8–19. Solve Prob. 8–18 using the conjugate-beam method.



Probs. 8–18/8–19

***8–20.** Determine the slope and the displacement at the end C of the beam. $E = 200$ GPa, $I = 70(10^6)$ mm^4. Use the moment-area theorems.

8–21. Solve Prob. 8–20 using the conjugate-beam method.

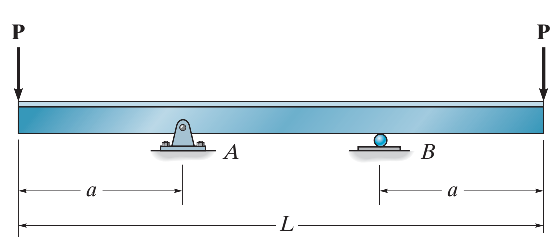


Probs. 8–20/8–21

8

8–22. At what distance a should the bearing supports at A and B be placed so that the displacement at the center of the shaft is equal to the deflection at its ends? The bearings exert only vertical reactions on the shaft. EI is constant. Use the moment-area theorems.

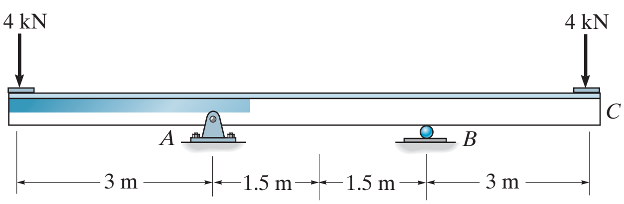
8–23. Solve Prob. 8–22 using the conjugate-beam method.



Probs. 8–22/8–23

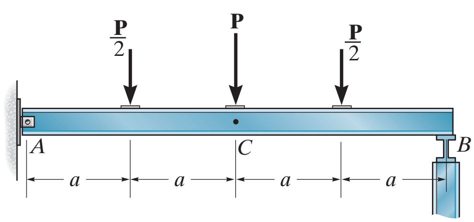
***8–24.** Determine the displacement at C and the slope at B. EI is constant. Use the moment-area theorems.

8–25. Solve Prob. 8–24 using the conjugate-beam method.



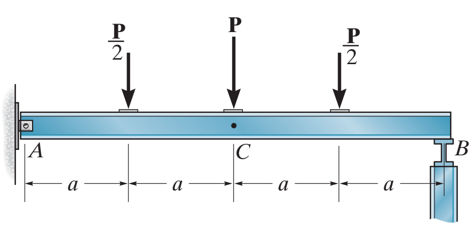
Probs. 8–24/8–25

8–26. Determine the displacement at C and the slope at B. EI is constant. Use the moment-area theorems.



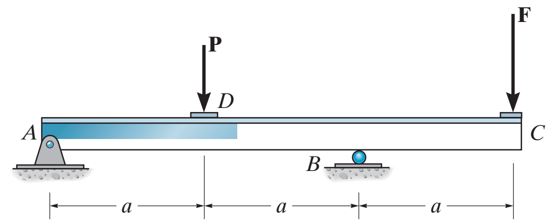
Prob. 8–26

8–27. Determine the displacement at C and the slope at B. EI is constant. Use the conjugate-beam method.



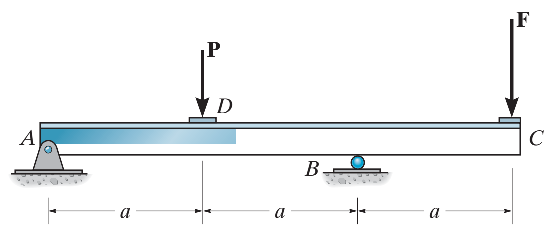
Prob. 8–27

***8–28.** Determine the force **F** at the end of the beam C so that the displacement at C is zero. EI is constant. Use the moment-area theorems.



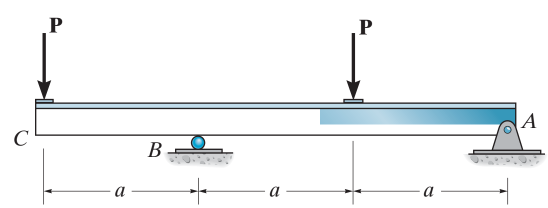
Prob. 8–28

8–29. Determine the force **F** at the end of the beam C so that the displacement at C is zero. EI is constant. Use the conjugate-beam method.



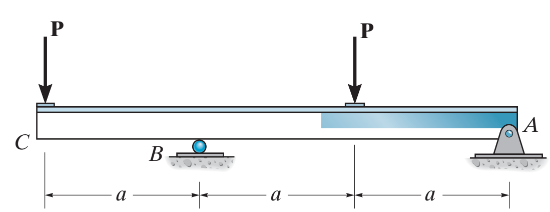
Prob. 8–29

8–30. Determine the slope at B and the displacement at C. EI is constant. Use the moment-area theorems.



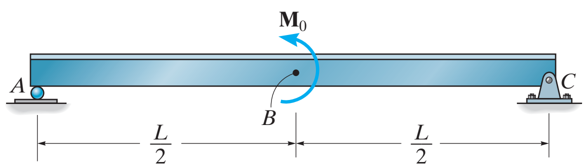
Prob. 8–30

8–31. Determine the slope at B and the displacement at C. EI is constant. Use the conjugate-beam method.



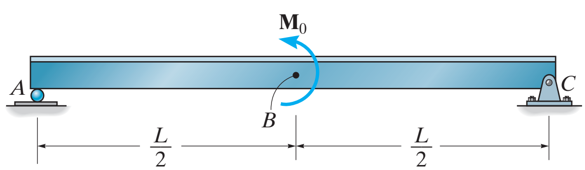
Prob. 8–31

***8–32.** Determine the maximum displacement and the slope at *A*. *EI* is constant. Use the moment-area theorems.



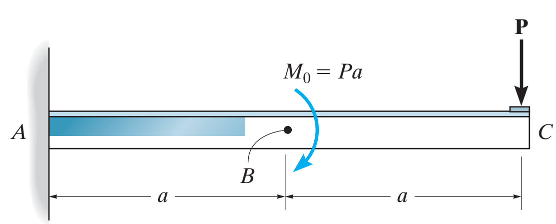
Prob. 8–32

8–33. Determine the maximum displacement at *B* and the slope at *A*. *EI* is constant. Use the conjugate-beam method.



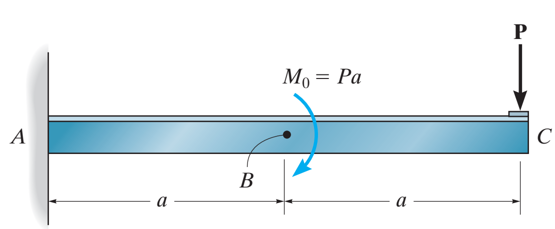
Prob. 8–33

8–34. Determine the slope and displacement at *C*. *EI* is constant. Use the moment-area theorems.



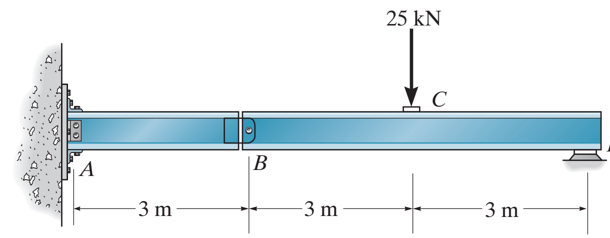
Prob. 8–34

8–35. Determine the slope and displacement at *C*. *EI* is constant. Use the conjugate-beam method.



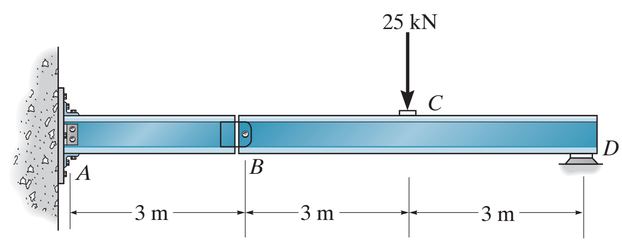
Prob. 8–35

***8–36.** Determine the displacement at *C*. Assume *A* is a fixed support, *B* is a pin, and *D* is a roller. *EI* is constant. Use the moment-area theorems.



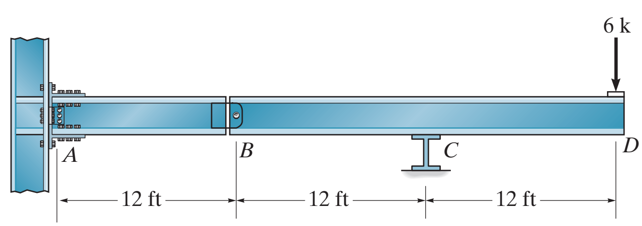
Prob. 8–36

8–37. Determine the displacement at *C*. Assume *A* is a fixed support, *B* is a pin, and *D* is a roller. *EI* is constant. Use the conjugate-beam method.



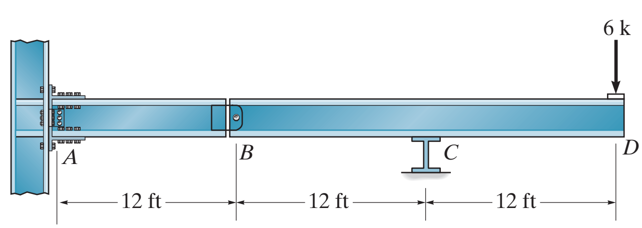
Prob. 8–37

8–38. Determine the displacement at *D* and the slope at *D*. Assume *A* is a fixed support, *B* is a pin, and *C* is a roller. Use the moment-area theorems.



Prob. 8–38

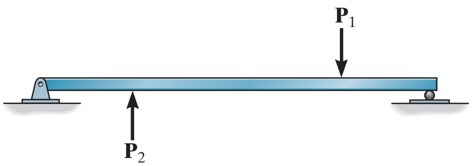
8–39. Determine the displacement at *D* and the slope at *D*. Assume *A* is a fixed support, *B* is a pin, and *C* is a roller. Use the conjugate-beam method.



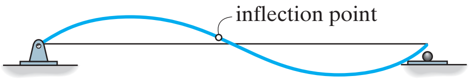
Prob. 8–39

CHAPTER REVIEW

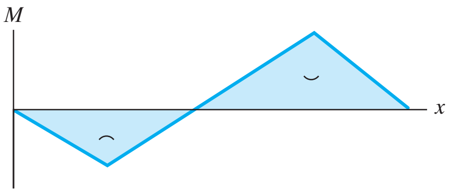
The deflection of a member (or structure) can always be established provided the moment diagram is known, because positive moment will tend to bend the member concave upwards, and negative moment will tend to bend the member concave downwards. Likewise, the general shape of the moment diagram can be determined if the deflection curve is known.



beam



deflection curve

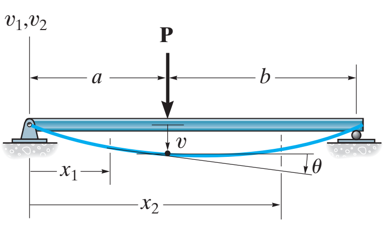


moment diagram

Deflection of a beam due to bending can be determined by using double integration of the equation.

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

Here the internal moment M must be expressed as a function of the x coordinates that extend across the beam. The constants of integration are obtained from the boundary conditions, such as zero deflection at a pin or roller support and zero deflection and slope at a fixed support. If several x coordinates are necessary, then the continuity of slope and deflection must be considered, where at $x_1 = x_2 = a$, $\theta_1(a) = \theta_2(a)$ and $v_1(a) = v_2(a)$.



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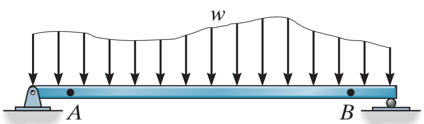
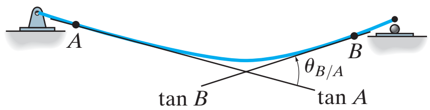
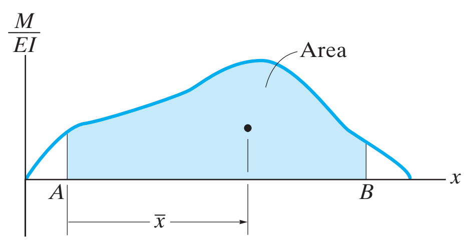
If the moment diagram has a simple shape, the moment-area theorems or the conjugate beam method can be used to determine the deflection and slope at a point on the beam.

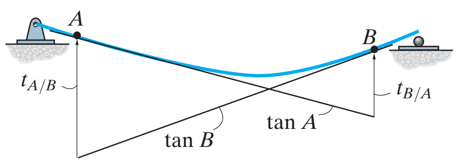
The moment-area theorems consider the angles and vertical deviation between the tangents at two points A and B on the elastic curve. The change in slope is found from the area under the M/EI diagram between the two points, and the deviation is determined from the moment of the M/EI diagram area about the point where the deviation occurs.





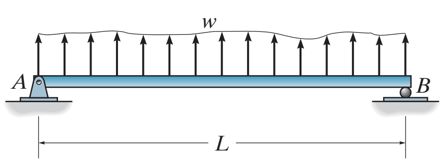
$\theta_{B/A}$ = Area of M/EI diagram



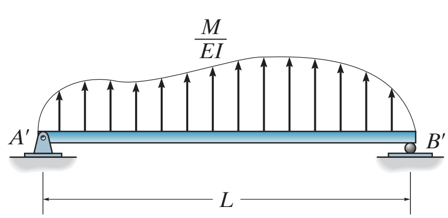


$t_{A/B} = \bar{x}$ (Area of M/EI diagram)

The conjugate beam method is very methodical and requires application of the principles of statics. Quite simply, one establishes the conjugate beam using Table 8–2, then considers the loading as the M/EI diagram. The slope (deflection) at a point on the real beam is then equal to the shear (moment) at the same point on the conjugate beam.



real beam



conjugate beam

8

The displacement at the ends of this bridge deck, as it is being constructed, can be determined using energy methods.

Deflections Using Energy Methods

9

In this chapter, we will show how to apply energy methods to solve problems involving slope and deflection. The chapter begins with a discussion of work and strain energy, followed by a development of the principle of work and energy. The method of virtual work and Castigliano's theorem are then developed, and these methods are used to determine the displacements at points on trusses, beams, and frames.

9.1 External Work and Strain Energy

The semigraphical methods presented in the previous chapters are very effective for finding the displacements and slopes at points in *beams* subjected to rather simple loadings. For more complicated loadings or for structures such as trusses and frames, it is suggested that energy methods be used for the computations. Most energy methods are based on the *conservation of energy principle,* which states that the work done by all the external forces acting on a structure, U_e, is transformed into internal work or strain energy, U_i, which is developed when the structure deforms. If the material's elastic limit is not exceeded, the *elastic strain energy* will return the structure to its undeformed state when the loads are removed. The conservation of energy principle can be stated mathematically as

$$\boxed{U_e = U_i} \tag{9–1}$$

Before developing any of the energy methods based on this principle, however, we will first determine the external work and strain energy caused by a force and a moment. The formulations to be presented will provide a basis for understanding the work and energy methods that follow.

External Work—Force. When a force **F** undergoes a displacement dx in the *same direction* as the force, the work done is $dU_e = F\,dx$. If the total displacement is x, the work becomes

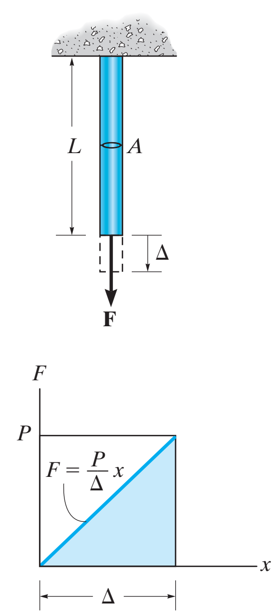
$$U_e = \int_0^x F\,dx \tag{9–2}$$

Consider now the effect caused by an axial force applied to the end of a bar as shown in Fig. 9–1a. As the magnitude of **F** is *gradually* increased from zero to some limiting value $F = P$, the final elongation of the bar becomes Δ. If the material has a linear elastic response, then $F = (P/\Delta)x$. Substituting into Eq. 9–2, and integrating from 0 to Δ, we get

$$U_e = \tfrac{1}{2}P\Delta \tag{9–3}$$

which represents the shaded *triangular area* in Fig. 9–1a.

We may also conclude from this that as a force is gradually applied to the bar, and its magnitude builds linearly from zero to some value P, the work done is equal to the *average force magnitude* $(P/2)$ times the displacement (Δ).



(a)

Fig. 9–1

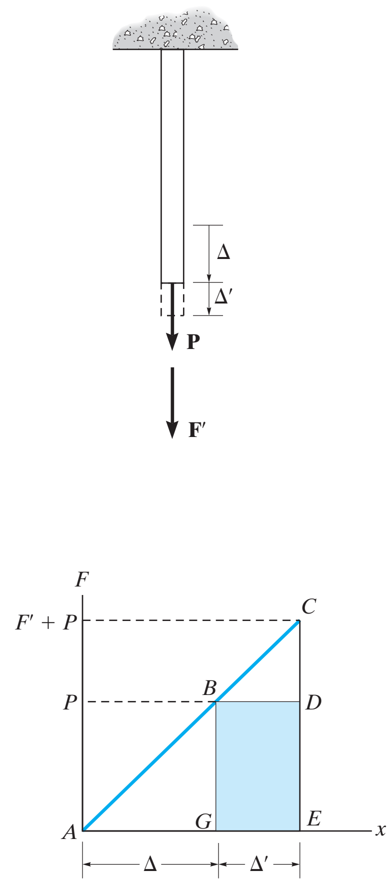
9

Suppose now that **P** is already applied to the bar and that *another force* $\mathbf{F}'$ is now applied, so the bar deflects further by an amount Δ', Fig. 9–1*b*. The work done by **P** (not $\mathbf{F}'$) when the bar undergoes the further deflection Δ' is then

$$U_e' = P\Delta' \tag{9–4}$$

Here the work represents the shaded *rectangular area* in Fig. 9–1*b*. In this case **P** does not change its magnitude since Δ' is caused only by $\mathbf{F}'$. Therefore, work is simply the force magnitude (P) times the displacement (Δ').

In summary, then, when a force **P** is applied to the bar, followed by application of the force $\mathbf{F}'$, the total work done by both forces is represented by the triangular area ACE in Fig. 9–1*b*. The triangular area ABG represents the work of **P** that is caused by its displacement Δ, the triangular area BCD represents the work of $\mathbf{F}'$ since this force causes a displacement Δ', and lastly, the shaded rectangular area $BDEG$ represents the additional work done by **P** when displaced Δ' as caused by $\mathbf{F}'$.



(b)

Fig. 9–1

External Work—Moment. The work of a moment is defined by the product of the magnitude of the moment **M** and the angle $d\theta$ through which it rotates, that is, $dU_e = M\,d\theta$, Fig. 9–2. If the total angle of rotation is θ radians, the work becomes

$$U_e = \int_0^\theta M\,d\theta \tag{9–5}$$

As in the case of force, if the moment is applied *gradually* to a structure having linear elastic response from zero to M, the work is then

$$U_e = \tfrac{1}{2}M\theta \tag{9–6}$$

However, if the moment is already applied to the structure and other loadings further distort the structure by an amount θ', then **M** rotates θ', and the work is

$$U_e' = M\theta' \tag{9–7}$$

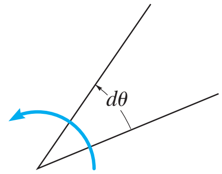


Fig. 9–2

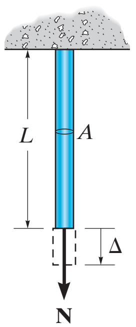


Fig. 9–3

Strain Energy—Axial Force. When an axial force **N** is applied gradually to the bar in Fig. 9–3, it will strain the material such that the *external work* done by **N** will be converted into *strain energy,* which is stored in the bar (Eq. 9–1). Provided the material is *linearly elastic,* Hooke's law is valid, $\sigma = E\epsilon$, and if the bar has a constant cross-sectional area A and length L, the normal stress is $\sigma = N/A$ and the final strain is $\epsilon = \Delta/L$. Consequently, $N/A = E(\Delta/L)$, and the final deflection is

$$\Delta = \frac{NL}{AE} \tag{9–8}$$

Substituting into Eq. 9–3, with $P = N$, the strain energy in the bar is therefore

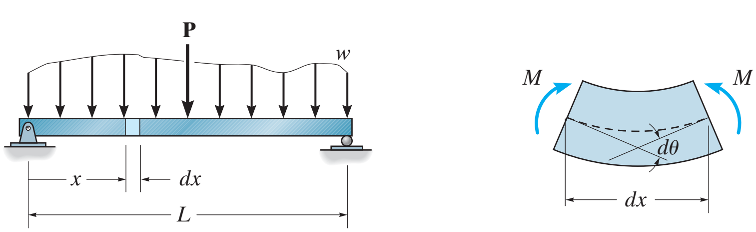
$$U_i = \frac{N^2L}{2AE} \tag{9–9}$$

Strain Energy—Bending. Consider the beam shown in Fig. 9–4*a*, which is distorted by the *gradually* applied loading **P** and w. These loads create an internal moment **M** in the beam at a section located a distance x from the left support. The resulting rotation of the differential element dx, Fig. 9–4*b*, can be found from Eq. 8–2, that is, $d\theta = (M/EI)\,dx$. Consequently, the strain energy, or work stored in the element, is determined from Eq. 9–6 since the internal moment is gradually developed. Hence,

$$dU_i = \frac{M^2\,dx}{2EI} \tag{9–10}$$

The strain energy for the beam is determined by integrating this result over the beam's entire length L. The result is

$$U_i = \int_0^L \frac{M^2\,dx}{2EI} \tag{9–11}$$



(a) (b)

Fig. 9–4

9.2 Principle of Work and Energy

Now that the work and strain energy for a force and a moment have been formulated, we will illustrate how the conservation of energy or the principle of work and energy can be applied to determine the displacement at a point on a structure. To do this, consider finding the displacement Δ at the point where the force $\mathbf{P}$ is applied to the cantilever beam in Fig. 9–5. From Eq. 9–3, the external work is $U_e = \frac{1}{2}P\Delta$. To obtain the resulting strain energy, we must first determine the internal moment as a function of position x in the beam and then apply Eq. 9–11. In this case $M = -Px$, so that

$$U_i = \int_0^L \frac{M^2\,dx}{2EI} = \int_0^L \frac{(-Px)^2\,dx}{2EI} = \frac{1}{6}\frac{P^2L^3}{EI}$$

Equating the external work to internal strain energy and solving for the unknown displacement Δ, we have

$$U_e = U_i$$

$$\frac{1}{2}P\Delta = \frac{1}{6}\frac{P^2L^3}{EI}$$

$$\Delta = \frac{PL^3}{3EI}$$

Although the solution here is quite direct, application of this method is limited to only a few select problems. It will be noted that only *one load* may be applied to the structure, since if more than one load were applied, there would be an unknown displacement under each load, and yet it is possible to write only *one* "work" equation for the beam. Furthermore, *only the displacement under the force can be obtained,* since the external work depends upon both the force and its corresponding displacement. One way to circumvent these limitations is to use the method of virtual work or Castigliano's theorem, both of which are explained in the following sections.

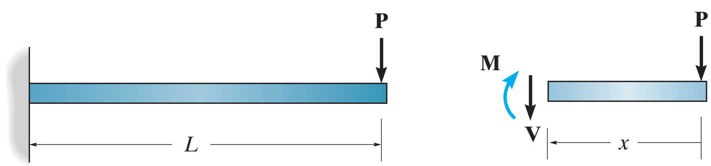


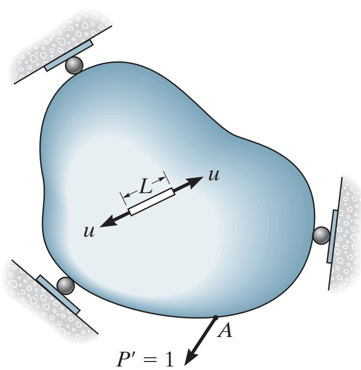
Fig. 9–5

9.3 Principle of Virtual Work

The principle of virtual work was developed by John Bernoulli in 1717 and is sometimes referred to as the unit-load method. It provides a general means of obtaining the displacement and slope at a specific point on a structure, be it a beam, frame, or truss.

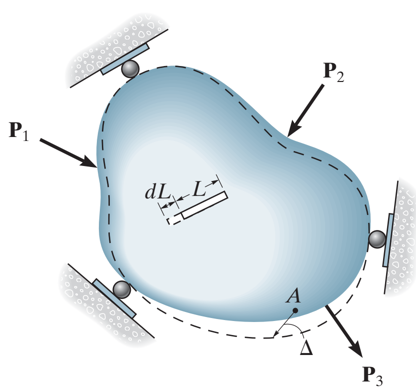
Before developing the principle of virtual work, it is necessary to make some general statements regarding the principle of work and energy, which was discussed in the previous section. If we take a deformable structure of any shape or size and apply a series of *external loads* **P** to it, it will cause *internal* loads **u** at points throughout the structure. *It is necessary that the external and internal loads be related by the equations of equilibrium*. As a consequence of these loadings, external displacements Δ will occur at the **P** loads and internal displacements δ will occur at each point of internal load **u**. In general, *these displacements do not have to be elastic,* and they may not be related to the loads; however, *the external and internal displacements must be related by the compatibility of the displacements*. In other words, if the external displacements are known, the corresponding internal displacements are uniquely defined. In general, then, the principle of work and energy states:

$$\underset{\substack{\text{Work of}\\\text{External Loads}}}{\Sigma P\Delta} = \underset{\substack{\text{Work of}\\\text{Internal Loads}}}{\Sigma u\delta} \qquad (9\text{–}12)$$



Apply virtual load $P' = 1$

(a)



Apply real loads $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$

(b)

Fig. 9–6

Based on this concept, the principle of virtual work will now be developed. To do this, we will consider the structure (or body) to be of arbitrary shape as shown in Fig. 9–6*b*.* Suppose it is necessary to determine the displacement Δ of point A on the body caused by the "real loads" $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. It is to be understood that these loads cause no movement of the supports; in general, however, they can strain the material *beyond the elastic limit*. Since no external load acts on the body at A and in the direction of Δ, the displacement Δ can be determined by *first* placing on the body a *"virtual" load* such that this force $\mathbf{P}'$ acts in the *same direction* as Δ, Fig. 9–6*a*. For convenience, which will be apparent later, we will choose $\mathbf{P}'$ to have a "unit" magnitude, that is, $P' = 1$. The term "virtual" is used to describe the load, since *it is imaginary and does not actually exist as part of the real loading*. The unit load ($\mathbf{P}'$) does, however, create an internal virtual load **u** in a representative element or fiber of the body, as shown in Fig. 9–6*a*. Here it is required that $\mathbf{P}'$ and **u** be related by the equations of equilibrium.†

*This arbitrary shape will later represent a specific truss, beam, or frame.

†Although these loads will cause virtual displacements, we will not be concerned with their magnitudes.

Once the virtual loadings are applied, *then* the body is subjected to the *real loads* $\mathbf{P_1}$, $\mathbf{P_2}$, and $\mathbf{P_3}$, Fig. 9–6*b*. Point A will be displaced an amount Δ, causing the element to deform an amount dL. As a result, the external virtual force $\mathbf{P}'$ and internal virtual load $\mathbf{u}$ “ride along” by Δ and dL, respectively, and therefore perform *external virtual work* of $1 \cdot \Delta$ on the body and internal virtual work of $u \cdot dL$ on the element. Realizing that the external virtual work is equal to the internal virtual work done on all the elements of the body, we can write the virtual-work equation as

virtual loadings

$$1 \cdot \Delta = \Sigma u \cdot dL \qquad (9\text{–}13)$$

real displacements

where

$P' = 1 =$ external virtual unit load acting in the direction of Δ.

$u =$ internal virtual load acting on the element in the direction of dL.

$\Delta =$ external displacement caused by the real loads.

$dL =$ internal deformation of the element caused by the real loads.

By choosing $P' = 1$, it can be seen that the solution for Δ follows directly, since $\Delta = \Sigma u\, dL$.

In a similar manner, if the rotational displacement or slope of the tangent at a point on a structure is to be determined, a virtual *couple moment* $\mathbf{M}'$ having a “unit” magnitude is applied at the point. As a consequence, this couple moment causes a virtual load $\mathbf{u}_{\boldsymbol{\theta}}$ in one of the elements of the body. Assuming that the real loads deform the element an amount dL, the rotation θ can be found from the virtual-work equation

virtual loadings

$$1 \cdot \theta = \Sigma u_\theta \cdot dL \qquad (9\text{–}14)$$

real displacements

where

$M' = 1 =$ external virtual unit couple moment acting in the direction of θ.

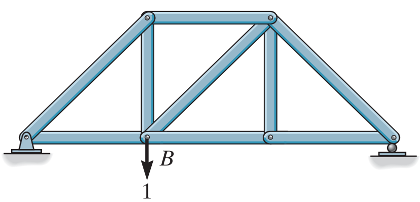
$u_\theta =$ internal virtual load acting on an element in the direction of dL.

$\theta =$ external rotational displacement or slope in radians caused by the real loads.

$dL =$ internal deformation of the element caused by the real loads.

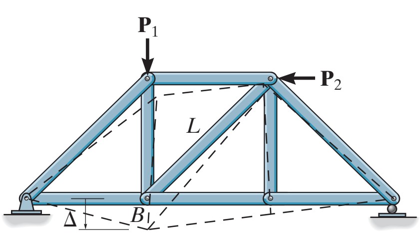
This method for applying the principle of virtual work is often referred to as the *method of virtual forces,* since a virtual force is applied resulting in the calculation of a *real displacement*. The equation of virtual work in this case represents a *compatibility requirement* for the structure. Although not important here, realize that we can also apply the principle of virtual work as a *method of virtual displacements*. In this case virtual displacements are imposed on the structure while the structure is subjected to *real loadings*. This method can be used to determine a force on or in a structure,* so that the equation of virtual work is then expressed as an *equilibrium requirement*.

9.4 Method of Virtual Work: Trusses



Apply virtual unit load to B

(a)



Apply real loads $\mathbf{P}_1$, $\mathbf{P}_2$

(b)

Fig. 9–7

We can use the method of virtual work to determine the displacement of a truss joint when the truss is subjected to an external loading, temperature change, or fabrication errors. Each of these situations will now be discussed.

External Loading. For the purpose of explanation let us consider the vertical displacement Δ of joint B of the truss in Fig. 9–7*a*. Here a typical element of the truss would be one of its *members* having a length L, Fig. 9–7*b*. If the applied loadings $\mathbf{P}_1$ and $\mathbf{P}_2$ cause a *linear elastic material response,* then this element deforms an amount $\Delta L = NL/AE$, where N is the normal or axial force in the member, caused by the loads. Applying Eq. 9–13, the virtual-work equation for the truss is therefore

$$\boxed{1 \cdot \Delta = \sum \frac{nNL}{AE}} \tag{9–15}$$

where

1 = external virtual unit load acting on the truss joint in the stated direction of Δ.

n = internal virtual normal force in a truss member caused by the external virtual unit load.

Δ = external joint displacement caused by the real loads on the truss.

N = internal normal force in a truss member caused by the real loads.

L = length of a member.

A = cross-sectional area of a member.

E = modulus of elasticity of a member.

The formulation of this equation follows naturally from the development in Sec. 9–3. Here the external virtual unit load creates internal virtual forces **n** in each of the truss members. The real loads then cause the truss joint to be displaced Δ in the same direction as the virtual unit load, and each member is displaced NL/AE in the same direction as its respective **n** force. Consequently, the external virtual work $1 \cdot \Delta$ equals the internal virtual work or the internal (virtual) strain energy stored in *all* the truss members, that is, $\Sigma nNL/AE$.

*It was used in this manner in Sec. 6–3 with reference to the Müller-Breslau principle.

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Temperature. In some cases, truss members may change their length due to temperature. If α is the coefficient of thermal expansion for a member and ΔT is the change in its temperature, the change in length of a member is $\Delta L = \alpha\,\Delta T\,L$. Hence, we can determine the displacement of a selected truss joint due to this temperature change from Eq. 9–13, written as

$$\boxed{1 \cdot \Delta = \Sigma n\alpha\,\Delta T\,L} \qquad (9\text{–}16)$$

where

1 = external virtual unit load acting on the truss joint in the stated direction of Δ.

n = internal virtual normal force in a truss member caused by the external virtual unit load.

Δ = external joint displacement caused by the temperature change.

α = coefficient of thermal expansion of member.

ΔT = change in temperature of member.

L = length of member.

Fabrication Errors and Camber. Occasionally, errors in fabricating the lengths of the members of a truss may occur. Also, in some cases truss members must be made slightly longer or shorter in order to give the truss a camber. Camber is often built into a bridge truss so that the bottom cord will curve upward by an amount equivalent to the downward deflection of the cord when subjected to the bridge's full dead weight. If a truss member is shorter or longer than intended, the displacement of a truss joint from its expected position can be determined from direct application of Eq. 9–13, written as

$$\boxed{1 \cdot \Delta = \Sigma n\,\Delta L} \qquad (9\text{–}17)$$

where

1 = external virtual unit load acting on the truss joint in the stated direction of Δ.

n = internal virtual normal force in a truss member caused by the external virtual unit load.

Δ = external joint displacement caused by the fabrication errors.

ΔL = difference in length of the member from its intended size as caused by a fabrication error.

A combination of the right sides of Eqs. 9–15 through 9–17 will be necessary if both external loads act on the truss and some of the members undergo a thermal change or have been fabricated with the wrong dimensions.

Procedure for Analysis

The following procedure may be used to determine a specific displacement of any joint on a truss using the method of virtual work.

Virtual Forces **n**

- Place the unit load on the truss at the joint where the desired displacement is to be determined. The load should be in the same direction as the specified displacement, e.g., horizontal or vertical.
- With the unit load so placed, and all the real loads *removed* from the truss, use the method of joints or the method of sections and calculate the internal **n** force in each truss member. Assume that tensile forces are positive and compressive forces are negative.

Real Forces **N**

- Use the method of sections or the method of joints to determine the **N** force in each member. These forces are caused only by the real loads acting on the truss. Again, assume tensile forces are positive and compressive forces are negative.

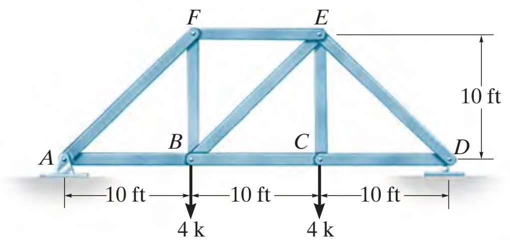
Virtual-Work Equation

- Apply the equation of virtual work, to determine the desired displacement. It is important to retain the algebraic sign for each of the corresponding **n** and **N** forces when substituting these terms into the equation.
- If the resultant sum $\Sigma nNL/AE$ is positive, the displacement Δ is in the same direction as the unit load. If a negative value results, Δ is opposite to the unit load.
- When applying $1 \cdot \Delta = \Sigma n\alpha\, \Delta T L$, realize that if any of the members undergoes an *increase in temperature,* ΔT will be *positive,* whereas a *decrease in temperature* results in a *negative* value for ΔT.
- For $1 \cdot \Delta = \Sigma n\, \Delta L$, when a fabrication error *increases the length* of a member, ΔL is *positive,* whereas a *decrease in length* is *negative*.
- When applying any formula, attention should be paid to the units of each numerical quantity. In particular, the virtual unit load can be assigned any arbitrary unit (lb, kip, N, etc.), since the **n** forces will have these *same units,* and as a result the units for both the virtual unit load and the **n** forces will cancel from both sides of the equation.

9

EXAMPLE 9.1

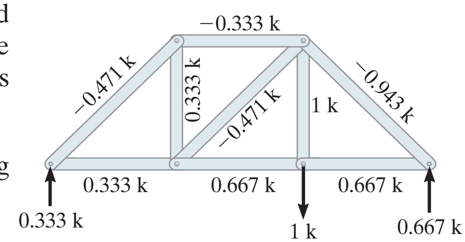
Determine the vertical displacement of joint C of the steel truss shown in Fig. 9–8*a*. The cross-sectional area of each member is $A = 0.5\ \text{in}^2$ and $E = 29(10^3)$ ksi.



(a)

SOLUTION

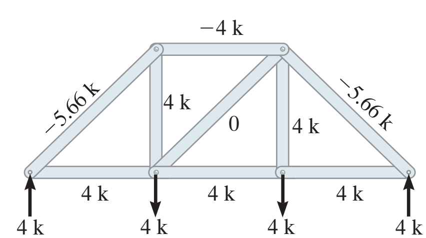
Virtual Forces n. Only a vertical 1-k load is placed at joint C, and the force in each member is calculated using the method of joints. The results are shown in Fig. 9–8*b*. Positive numbers indicate tensile forces and negative numbers indicate compressive forces.



virtual forces **n**

(b)

Real Forces N. The real forces in the members are calculated using the method of joints. The results are shown in Fig. 9–8*c*.



real forces **N**

(c)

Fig. 9–8

Virtual-Work Equation. Arranging the data in tabular form, we have

Member	n (k)	N (k)	L (ft)	nNL ($\text{k}^2 \cdot \text{ft}$)
AB	0.333	4	10	13.33
BC	0.667	4	10	26.67
CD	0.667	4	10	26.67
DE	−0.943	−5.66	14.14	75.42
FE	−0.333	−4	10	13.33
EB	−0.471	0	14.14	0
BF	0.333	4	10	13.33
AF	−0.471	−5.66	14.14	37.71
CE	1	4	10	40
				$\Sigma 246.47$

Thus

$$1\ \text{k} \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{246.47\ \text{k}^2 \cdot \text{ft}}{AE}$$

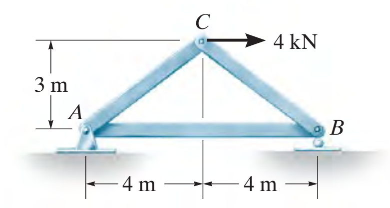
Converting the units of member length to inches and substituting the numerical values for A and E, we have

$$1\ \text{k} \cdot \Delta_{C_v} = \frac{(246.47\ \text{k}^2 \cdot \text{ft})(12\ \text{in./ft})}{(0.5\ \text{in}^2)(29(10^3)\ \text{k/in}^2)}$$

$$\Delta_{C_v} = 0.204\ \text{in.} \qquad \textit{Ans.}$$

EXAMPLE 9.2

The cross-sectional area of each member of the truss shown in Fig. 9–9*a* is $A = 400\ \text{mm}^2$ and $E = 200\ \text{GPa}$. (a) Determine the vertical displacement of joint C if a 4-kN force is applied to the truss at C. (b) If no loads act on the truss, what would be the vertical displacement of joint C if member AB were 5 mm too short?



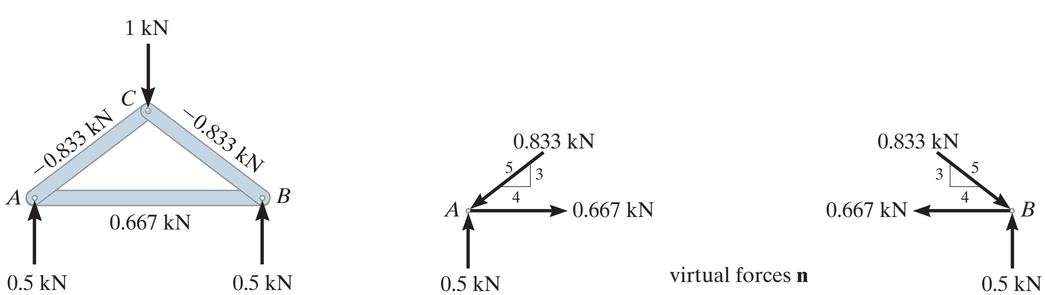
(a)

Fig. 9–9

SOLUTION

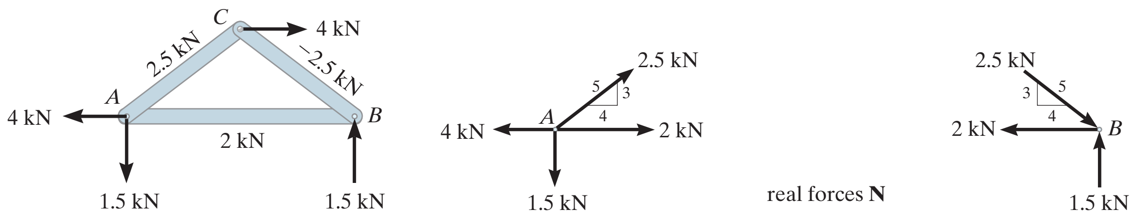
Part (a)

Virtual Forces n. Since the *vertical displacement* of joint C is to be determined, a virtual force of 1 kN is applied at C in the vertical direction. The units of this force are the *same* as those of the real loading. The support reactions at A and B are calculated and the **n** force in each member is determined by the method of joints as shown on the free-body diagrams of joints A and B, Fig. 9–9*b*.



(b)

Real Forces N. The joint analysis of A and B when the real load of 4 kN is applied to the truss is given in Fig. 9–9*c*.



(c)

Virtual-Work Equation. Since AE is constant, each of the terms nNL can be arranged in tabular form and computed. Here positive numbers indicate tensile forces and negative numbers indicate compressive forces.

Member	n (kN)	N (kN)	L (m)	$n\,NL$ (kN^2 • m)
AB	0.667	2	8	10.67
AC	−0.833	2.5	5	−10.41
CB	−0.833	−2.5	5	10.41
				$\Sigma 10.67$

Thus,

$$1\text{ kN}\cdot\Delta_{C_v} = \sum\frac{nNL}{AE} = \frac{10.67\text{ kN}^2\cdot\text{m}}{AE}$$

Substituting the values $A = 400\text{ mm}^2 = 400(10^{-6})\text{ m}^2$, $E = 200\text{ GPa} = 200(10^6)\text{ kN/m}^2$, we have

$$1\text{ kN}\cdot\Delta_{C_v} = \frac{10.67\text{ kN}^2\cdot\text{m}}{400(10^{-6})\text{ m}^2(200(10^6)\text{ kN/m}^2)}$$

$$\Delta_{C_v} = 0.000133\text{ m} = 0.133\text{ mm} \qquad \textit{Ans.}$$

Part (b). Here we must apply Eq. 9–17. Since the vertical displacement of C is to be determined, we can use the results of Fig. 9–7*b*. Only member AB undergoes a change in length, namely, of $\Delta L = -0.005$ m. Thus,

$$1\cdot\Delta = \Sigma n\,\Delta L$$

$$1\text{ kN}\cdot\Delta_{C_v} = (0.667\text{ kN})(-0.005\text{ m})$$

$$\Delta_{C_v} = -0.00333\text{ m} = -3.33\text{ mm} \qquad \textit{Ans.}$$

The negative sign indicates joint C is displaced *upward,* opposite to the 1-kN vertical load. Note that if the 4-kN load and fabrication error are both accounted for, the resultant displacement is then $\Delta_{C_v} = 0.133 - 3.33 = -3.20$ mm (upward).

9

EXAMPLE 9.3

Determine the vertical displacement of joint C of the steel truss shown in Fig. 9–10*a*. Due to radiant heating from the wall, member AD is subjected to an *increase* in temperature of $\Delta T = +120°\text{F}$. Take $\alpha = 0.6(10^{-5})/°\text{F}$ and $E = 29(10^3)$ ksi. The cross-sectional area of each member is indicated in the figure.

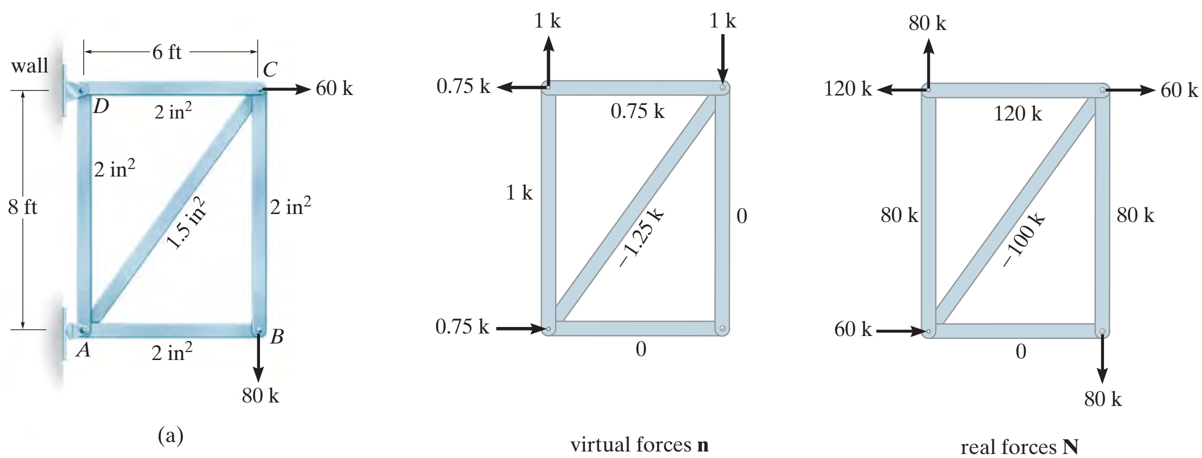


Fig. 9–10

(b)

(c)

SOLUTION

Virtual Forces n. A *vertical* 1-k load is applied to the truss at joint C, and the forces in the members are computed, Fig. 9–10*b*.

Real Forces N. Since the **n** forces in members AB and BC are *zero,* the **N** forces in these members do *not* have to be computed. Why? For completion, though, the entire real-force analysis is shown in Fig. 9–10*c*.

Virtual-Work Equation. Both loads and temperature affect the deformation; therefore, Eqs. 9–15 and 9–16 are combined. Working in units of kips and inches, we have

$$1 \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} + \Sigma n\alpha\,\Delta T\,L$$

$$= \frac{(0.75)(120)(6)(12)}{2[29(10^3)]} + \frac{(1)(80)(8)(12)}{2[29(10^3)]}$$

$$+ \frac{(-1.25)(-100)(10)(12)}{1.5[29(10^3)]} + (1)[0.6(10^{-5})](120)(8)(12)$$

$$\Delta_{C_v} = 0.658 \text{ in.} \qquad \textit{Ans.}$$

9

9.5 Castigliano's Theorem

In 1879 Alberto Castigliano, an Italian railroad engineer, published a book in which he outlined a method for determining the deflection or slope at a point in a structure, be it a truss, beam, or frame. This method, which is referred to as *Castigliano's second theorem,* or the *method of least work,* applies only to structures that have constant temperature, unyielding supports, and *linear elastic* material response. If the displacement of a point is to be determined, the theorem states that it is equal to the first partial derivative of the strain energy in the structure with respect to a force acting at the point and in the direction of displacement. In a similar manner, the slope at a point in a structure is equal to the first partial derivative of the strain energy in the structure with respect to a couple moment acting at the point and in the direction of rotation.

To derive Castigliano's second theorem, consider a body (structure) of any arbitrary shape which is subjected to a series of n forces $P_1, P_2, \ldots, P_n$. Since the external work done by these loads is equal to the internal strain energy stored in the body, we can write

$$U_i = U_e$$

The external work is a function of the external loads $\left(U_e = \Sigma \int P\, dx\right)$. Thus,

$$U_i = U_e = f(P_1, P_2, \ldots, P_n)$$

Now, if any one of the forces, say P_i, is increased by a differential amount dP_i, the internal work is also increased such that the new strain energy becomes

$$U_i + dU_i = U_i + \frac{\partial U_i}{\partial P_i} dP_i \tag{9–18}$$

This value, however, should not depend on the sequence in which the n forces are applied to the body. For example, if we apply dP_i to the body *first,* then this will cause the body to be displaced a differential amount $d\Delta_i$ in the direction of dP_i. By Eq. 9–3 $\left(U_e = \frac{1}{2}P\Delta\right)$, the increment of strain energy would be $\frac{1}{2}dP_i\, d\Delta_i$. This quantity, however, is a second-order differential and may be neglected. Further application of the loads $P_1, P_2, \ldots, P_n$, which displace the body $\Delta_1, \Delta_2, \ldots, \Delta_n$, yields the strain energy.

$$U_i + dU_i = U_i + dP_i\Delta_i \tag{9–19}$$

Here, as before, U_i is the internal strain energy in the body, caused by the loads $P_1, P_2, \ldots, P_n$, and $dU_i = dP_i\Delta_i$ is the *additional* strain energy caused by dP_i. (Eq. 9–4, $U_e = P\Delta'$.)

In summary, then, Eq. 9–18 represents the strain energy in the body determined by first applying the loads $P_1, P_2, \ldots, P_n$, *then* dP_i, and Eq. 9–19 represents the strain energy determined by first applying dP_i and *then* the loads $P_1, P_2, \ldots, P_n$. Since these two equations must be equal, we require

$$\Delta_i = \frac{\partial U_i}{\partial P_i} \tag{9–20}$$

which proves the theorem; i.e., the displacement Δ_i in the direction of P_i is equal to the first partial derivative of the strain energy with respect to P_i.*

It should be noted that Eq. 9–20 is a statement regarding the *structure's compatibility*. Also, the above derivation requires that *only conservative forces* be considered for the analysis. These forces do work that is independent of the path and therefore create no energy loss. Since forces causing a linear elastic response are conservative, the theorem is restricted to *linear elastic behavior* of the material. This is unlike the method of virtual force discussed in the previous section, which applied to *both* elastic and inelastic behavior.

9.6 Castigliano's Theorem for Trusses

The strain energy for a member of a truss is given by Eq. 9–9, $U_i = N^2L/2AE$. Substituting this equation into Eq. 9–20 and omitting the subscript i, we have

$$\Delta = \frac{\partial}{\partial P}\sum \frac{N^2L}{2AE}$$

It is generally easier to perform the differentiation prior to summation. In the general case L, A, and E are constant for a given member, and therefore we may write

$$\boxed{\Delta = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE}} \tag{9–21}$$

where

Δ = external joint displacement of the truss.

P = external force applied to the truss joint in the direction of Δ.

N = internal force in a member caused by *both* the force P and the loads on the truss.

L = length of a member.

A = cross-sectional area of a member.

E = modulus of elasticity of a member.

*Castigliano's first theorem is similar to his second theorem; however, it relates the load P_i to the partial derivative of the strain energy with respect to the corresponding displacement, that is, $P_i = \partial U_i/\partial \Delta_i$. The proof is similar to that given above and, like the method of virtual displacement, Castigliano's first theorem applies to both elastic and inelastic material behavior. This theorem is another way of expressing the *equilibrium requirements* for a structure, and since it has very limited use in structural analysis, it will not be discussed in this book.

This equation is similar to that used for the method of virtual work, Eq. 9–15 ($1 \cdot \Delta = \Sigma nNL/AE$), except n is replaced by $\partial N/\partial P$. Notice that in order to determine this partial derivative it will be necessary to treat P as a *variable* (not a specific numerical quantity), and furthermore, each member force N must be expressed as a function of P. As a result, computing $\partial N/\partial P$ generally requires slightly more calculation than that required to compute each n force directly. These terms will of course be the same, since n or $\partial N/\partial P$ is simply the change of the internal member force with respect to the load P, or the change in member force per unit load.

Procedure for Analysis

The following procedure provides a method that may be used to determine the displacement of any joint of a truss using Castigliano's theorem.

External Force *P*

- Place a force P on the truss at the joint where the desired displacement is to be determined. This force is assumed to have a *variable magnitude* in order to obtain the change $\partial N/\partial P$. Be sure $\mathbf{P}$ is directed along the line of action of the displacement.

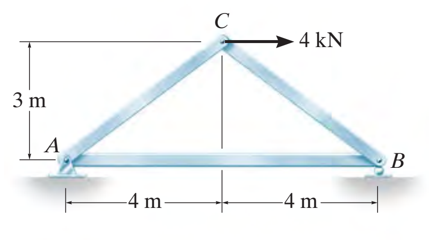
Internal Forces *N*

- Determine the force N in each member caused by both the real (numerical) loads and the variable force P. Assume tensile forces are positive and compressive forces are negative.
- Compute the respective partial derivative $\partial N/\partial P$ for each member.
- After N and $\partial N/\partial P$ have been determined, assign P its numerical value if it has replaced a real force on the truss. Otherwise, set P equal to zero.

Castigliano's Theorem

- Apply Castigliano's theorem to determine the desired displacement Δ. It is important to retain the algebraic signs for corresponding values of N and $\partial N/\partial P$ when substituting these terms into the equation.
- If the resultant sum $\Sigma N(\partial N/\partial P)L/AE$ is positive, Δ is in the same direction as P. If a negative value results, Δ is opposite to P.

EXAMPLE 9.4



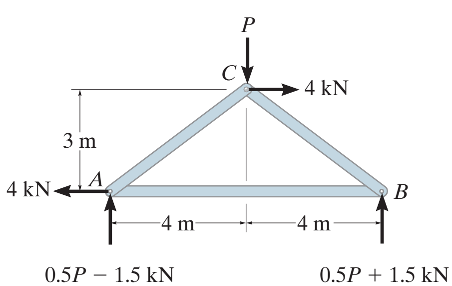
(a)

Determine the vertical displacement of joint C of the truss shown in Fig. 9–11a. The cross-sectional area of each member is $A = 400\ \text{mm}^2$ and $E = 200$ GPa.

SOLUTION

External Force *P*. A vertical force $\mathbf{P}$ is applied to the truss at joint C, since this is where the vertical displacement is to be determined, Fig. 9–11b.

Internal Forces *N*. The reactions at the truss supports at A and B are determined and the results are shown in Fig. 9–11b. Using the method of joints, the N forces in each member are determined, Fig. 9–11c.* For convenience, these results along with the partial derivatives $\partial N/\partial P$ are listed in tabular form as follows:



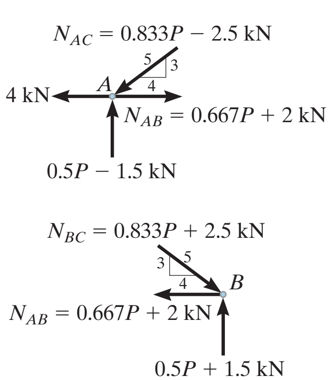
(b)

Member	N	$\dfrac{\partial N}{\partial P}$	$N\ (P = 0)$	L	$N\left(\dfrac{\partial N}{\partial P}\right)L$
AB	$0.667P + 2$	0.667	2	8	10.67
AC	$-(0.833P - 2.5)$	-0.833	2.5	5	-10.42
BC	$-(0.833P + 2.5)$	-0.833	-2.5	5	10.42
					$\Sigma = 10.67\ \text{kN}\cdot\text{m}$

Since P does not actually exist as a real load on the truss, we require $P = 0$ in the table above.

Castigliano's Theorem. Applying Eq. 9–21, we have

$$\Delta_{C_v} = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = \frac{10.67\ \text{kN}\cdot\text{m}}{AE}$$



(c)

Fig. 9–11

Substituting $A = 400\ \text{mm}^2 = 400(10^{-6})\ \text{m}^2$, $E = 200\ \text{GPa} = 200(10^9)$ Pa, and converting the units of N from kN to N, we have

$$\Delta_{C_v} = \frac{10.67(10^3)\ \text{N}\cdot\text{m}}{400(10^{-6})\ \text{m}^2(200(10^9)\ \text{N/m}^2)} = 0.000133\ \text{m} = 0.133\ \text{mm}$$

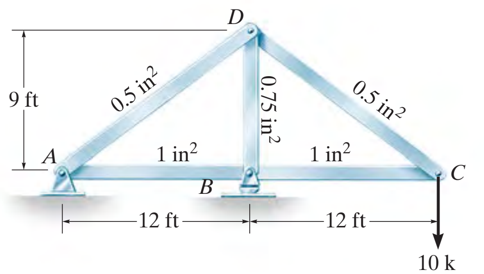
Ans.

This solution should be compared with the virtual-work method of Example 9–2.

*It may be more convenient to analyze the truss with just the 4-kN load on it, then analyze the truss with the P load on it. The results can then be added together to give the N forces.

EXAMPLE 9.5

Determine the horizontal displacement of joint D of the truss shown in Fig. 9–12a. Take $E = 29(10^3)$ ksi. The cross-sectional area of each member is indicated in the figure.



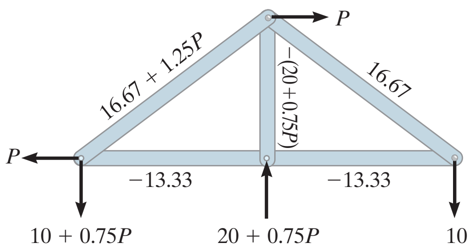


Fig. 9–12

SOLUTION

External Force *P*. Since the horizontal displacement of D is to be determined, a horizontal variable force P is applied to joint D, Fig. 9–12b.

Internal Forces *N*. Using the method of joints, the force N in each member is computed.* Again, when applying Eq. 9–21, we set $P = 0$ since this force does not actually exist on the truss. The results are shown in Fig. 9–12b. Arranging the data in tabular form, we have

Member	N	$\dfrac{\partial N}{\partial P}$	$N\,(P = 0)$	L	$N\left(\dfrac{\partial N}{\partial P}\right)L$
AB	-13.33	0	-13.33	12	0
BC	-13.33	0	-13.33	12	0
CD	16.67	0	16.67	15	0
DA	$16.67 + 1.25P$	1.25	16.67	15	312.50
BD	$-(20 + 0.75P)$	-0.75	-20	9	135.00

Castigliano's Theorem. Applying Eq. 9–21, we have

$$\Delta_{D_h} = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = 0 + 0 + 0 + \frac{312.50\ \text{k}\cdot\text{ft}(12\ \text{in./ft})}{(0.5\ \text{in}^2)[29(10^3)\ \text{k/in}^2]} + \frac{135.00\ \text{k}\cdot\text{ft}(12\ \text{in./ft})}{(0.75\ \text{in}^2)[29(10^3)\ \text{k/in}^2]}$$

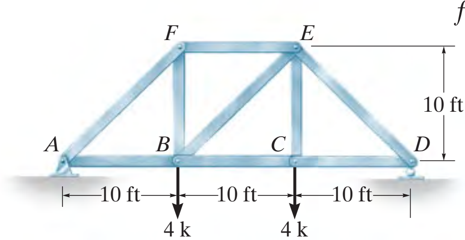
$$= 0.333\ \text{in.}$$ *Ans.*

*As in the preceding example, it may be preferable to perform a separate analysis of the truss loaded with 10 k and loaded with P and then superimpose the results.

9

EXAMPLE 9.6

Determine the vertical displacement of joint C of the truss shown in Fig. 9–13*a*. Assume that $A = 0.5\ \text{in}^2$ and $E = 29(10^3)$ ksi.



(a)

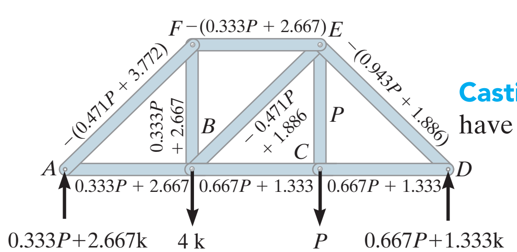
SOLUTION

External Force P. The 4-k force at C is replaced with a *variable force* P at joint C, Fig. 9–13*b*.

Internal Forces N. The method of joints is used to determine the force N in each member of the truss. The results are summarized in Fig. 9–13*b*. Here $P = 4$ k when we apply Eq. 9–21. The required data can be arranged in tabulated form as follows:

Member	N	$\dfrac{\partial N}{\partial P}$	$N\ (P = 4\ \text{k})$	L	$N\left(\dfrac{\partial N}{\partial P}\right)L$
AB	$0.333P + 2.667$	0.333	4	10	13.33
BC	$0.667P + 1.333$	0.667	4	10	26.67
CD	$0.667P + 1.333$	0.667	4	10	26.67
DE	$-(0.943P + 1.886)$	-0.943	-5.66	14.14	75.42
EF	$-(0.333P + 2.667)$	-0.333	-4	10	13.33
FA	$-(0.471P + 3.771)$	-0.471	-5.66	14.14	37.71
BF	$0.333P + 2.667$	0.333	4	10	13.33
BE	$-0.471P + 1.886$	-0.471	0	14.14	0
CE	P	1	4	10	40

$\Sigma = 246.47\ \text{k} \cdot \text{ft}$



(b)

Fig. 9–13

Castigliano's Theorem. Substituting the data into Eq. 9–21, we have

$$\Delta_{C_v} = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = \frac{246.47\ \text{k} \cdot \text{ft}}{AE}$$

Converting the units of member length to inches and substituting the numerical value for AE, we have

$$\Delta_{C_v} = \frac{(246.47\ \text{k} \cdot \text{ft})(12\ \text{in./ft})}{(0.5\ \text{in}^2)(29(10^3)\ \text{k/in}^2)} = 0.204\ \text{in.} \qquad \textit{Ans.}$$

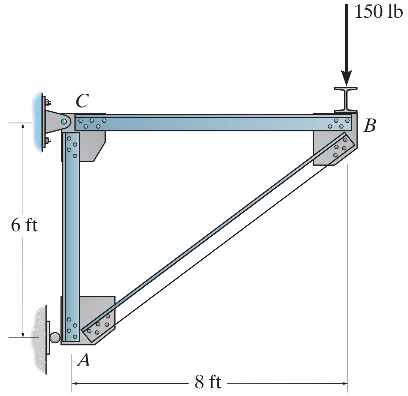
The similarity between this solution and that of the virtual-work method, Example 9–1, should be noted.

9

FUNDAMENTAL PROBLEMS

F9–1. Determine the vertical displacement of joint *B*. *AE* is constant. Use the principle of virtual work.

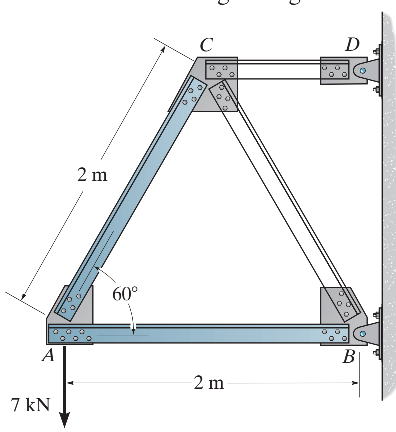
F9–2. Solve Prob. F9–2 using Castigliano's theorem.



F9–1/9–2

F9–3. Determine the horizontal displacement of joint *A*. *AE* is constant. Use the principle of virtual work.

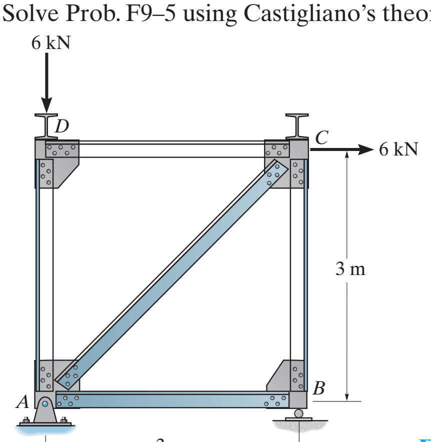
F9–4. Solve Prob. F9–3 using Castigliano's theorem.



F9–3/9–4

F9–5. Determine the horizontal displacement of joint *D*. *AE* is constant. Use the principle of virtual work.

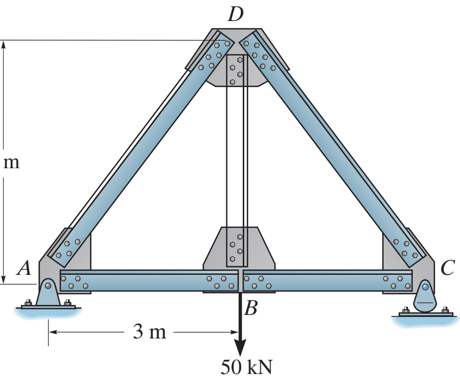
F9–6. Solve Prob. F9–5 using Castigliano's theorem.



F9–5/9–6

F9–7. Determine the vertical displacement of joint *D*. *AE* is constant. Use the principle of virtual work.

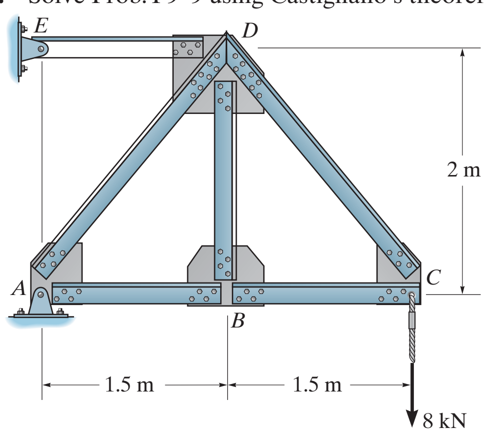
F9–8. Solve Prob. F9–7 using Castigliano's theorem.



F9–7/9–8

F9–9. Determine the vertical displacement of joint *B*. *AE* is constant. Use the principle of virtual work.

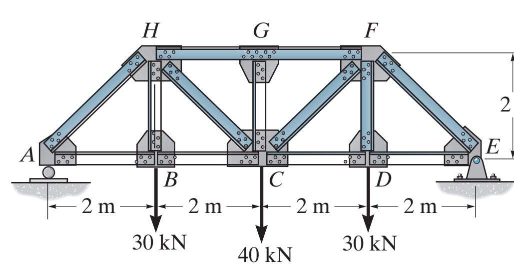
F9–10. Solve Prob. F9–9 using Castigliano's theorem.



F9–9/9–10

F9–11. Determine the vertical displacement of joint *C*. *AE* is constant. Use the principle of virtual work.

F9–12. Solve Prob. F9–11 using Castigliano's theorem.

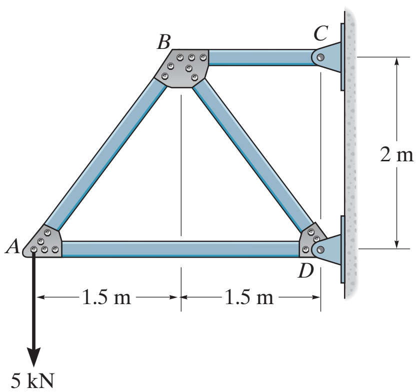


F9–11/9–12

PROBLEMS

9–1. Determine the vertical displacement of joint A. Each bar is made of steel and has a cross-sectional area of 600 mm^2. Take $E = 200$ GPa. Use the method of virtual work.

9–2. Solve Prob. 9–1 using Castigliano's theorem.



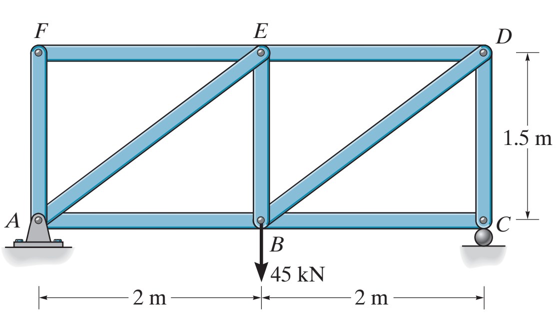
Probs. 9–1/9–2

9–3. Determine the vertical displacement of joint B. For each member $A = 400$ mm^2, $E = 200$ GPa. Use the method of virtual work.

***9–4.** Solve Prob. 9–3 using Castigliano's theorem.

9–5. Determine the vertical displacement of joint E. For each member $A = 400$ mm^2, $E = 200$ GPa. Use the method of virtual work.

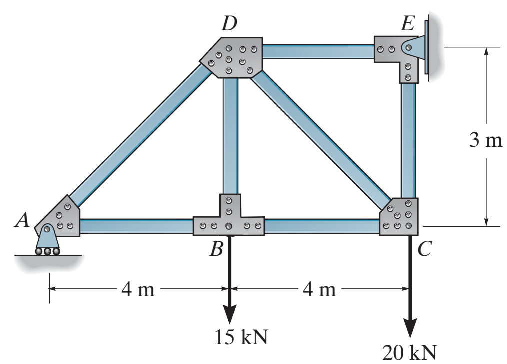
9–6. Solve Prob. 9–5 using Castigliano's theorem.



Probs. 9–3/9–4/9–5/9–6

9–7. Determine the vertical displacement of joint D. Use the method of virtual work. AE is constant. Assume the members are pin connected at their ends.

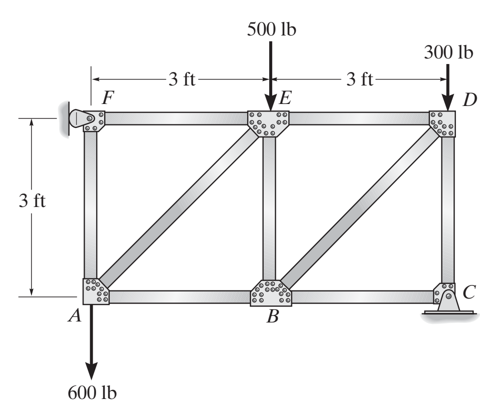
***9–8.** Solve Prob. 9–7 using Castigliano's theorem.



Probs. 9–7/9–8

9–9. Use the method of virtual work.

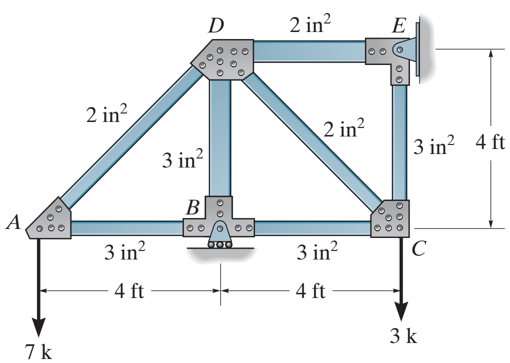
9–10. Solve Prob. 9–9 using Castigliano's theorem.



Probs. 9–9/9–10

9–11. Determine the vertical displacement of joint A. The cross-sectional area of each member is indicated in the figure. Assume the members are pin connected at their end points. $E = 29\ (10)^3$ ksi. Use the method of virtual work.

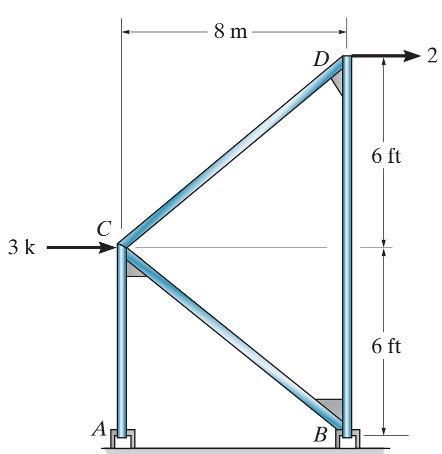
***9–12.** Solve Prob. 9–11 using Castigliano's theorem.



Probs. 9–11/9–12

9–13. Determine the horizontal displacement of joint D. Assume the members are pin connected at their end points. AE is constant. Use the method of virtual work.

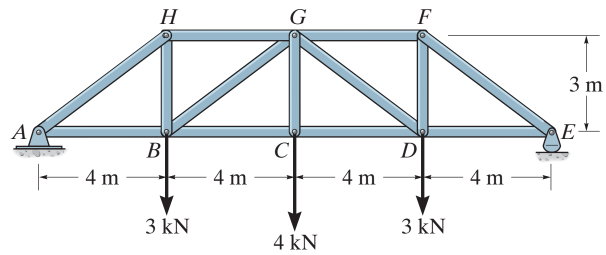
9–14. Solve Prob. 9–13 using Castigliano's theorem.



Probs. 9–13/9–14

9–15. Determine the vertical displacement of joint C of the truss. Each member has a cross-sectional area of $A = 300\ \text{mm}^2$. $E = 200$ GPa. Use the method of virtual work.

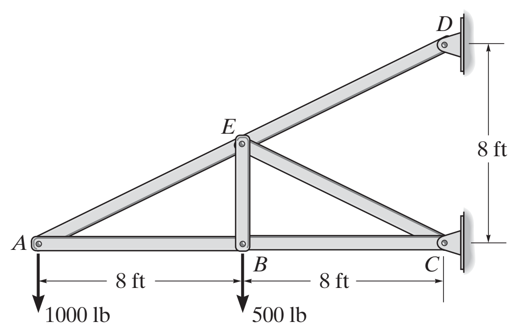
***9–16.** Solve Prob. 9–15 using Castigliano's theorem.



Probs. 9–15/9–16

9–17. Determine the vertical displacement of joint A. Assume the members are pin connected at their end points. Take $A = 2\ \text{in}^2$ and $E = 29\ (10^3)$ for each member. Use the method of virtual work.

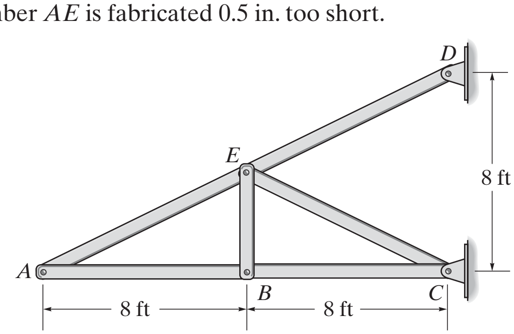
9–18. Solve Prob. 9–17 using Castigliano's theorem.



Probs. 9–17/9–18

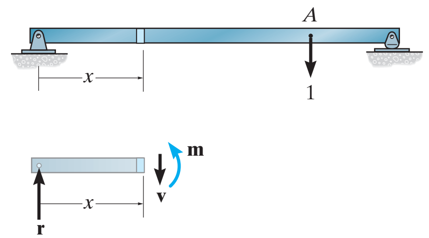
9–19. Determine the vertical displacement of joint A if members AB and BC experience a temperature increase of $\Delta T = 200°\text{F}$. Take $A = 2\ \text{in}^2$ and $E = 29(10^3)$ ksi. Also, $\alpha = 6.60\ (10^{-6})/°\text{F}$.

***9–20.** Determine the vertical displacement of joint A if member AE is fabricated 0.5 in. too short.



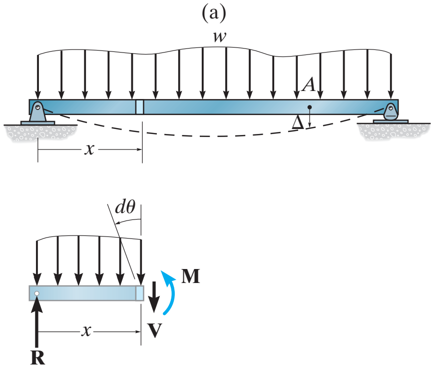
Probs. 9–19/9–20

9.7 Method of Virtual Work: Beams and Frames



Apply virtual unit load to point A

(a)



Apply real load w

(b)

Fig. 9–14

The method of virtual work can also be applied to deflection problems involving beams and frames. Since strains due to *bending* are the *primary cause* of beam or frame deflections, we will discuss their effects first. Deflections due to shear, axial and torsional loadings, and temperature will be considered in Sec. 9–8.

The principle of virtual work, or more exactly, the method of virtual force, may be formulated for beam and frame deflections by considering the beam shown in Fig. 9–14*b*. Here the displacement Δ of point A is to be determined. To compute Δ a virtual unit load acting in the direction of Δ is placed on the beam at A, and the *internal virtual moment* **m** is determined by the method of sections at an arbitrary location x from the left support, Fig. 9–14*a*. When the real loads act on the beam, Fig. 9–14*b*, point A is displaced Δ. Provided these loads cause *linear elastic material response,* then from Eq. 8–2, the element dx deforms or rotates $d\theta = (M/EI)\,dx$.* Here M is the internal moment at x caused by the real loads. Consequently, the *external virtual work* done by the unit load is $1 \cdot \Delta$, and the *internal virtual work* done by the moment **m** is $m\,d\theta = m(M/EI)\,dx$. Summing the effects on all the elements dx along the beam requires an integration, and therefore Eq. 9–13 becomes

$$\boxed{1 \cdot \Delta = \int_0^L \frac{mM}{EI}\,dx} \qquad (9\text{–}22)$$

where

1 = external virtual unit load acting on the beam or frame in the direction of Δ.

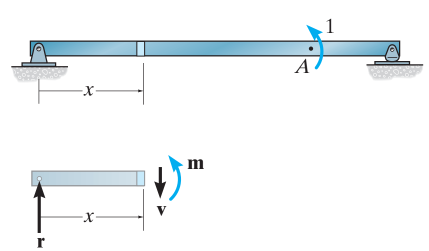
m = internal virtual moment in the beam or frame, expressed as a function of x and caused by the external virtual unit load.

Δ = external displacement of the point caused by the real loads acting on the beam or frame.

M = internal moment in the beam or frame, expressed as a function of x and caused by the real loads.

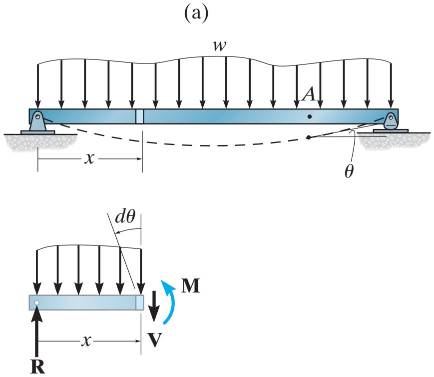
E = modulus of elasticity of the material.

I = moment of inertia of cross-sectional area, computed about the neutral axis.



Apply virtual unit couple moment to point A

(a)



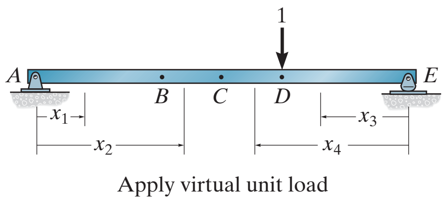
Apply real load w

Fig. 9–15

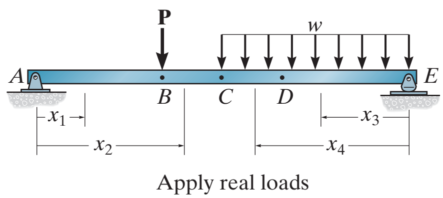
In a similar manner, if the tangent rotation or slope angle θ at a point A on the beam's elastic curve is to be determined, Fig. 9–15, a unit couple moment is first applied at the point, and the corresponding internal moments m_θ have to be determined. Since the work of the unit couple is $1 \cdot \theta$, then

$$\boxed{1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI}\,dx} \qquad (9\text{–}23)$$

*Recall that if the material is strained beyond its elastic limit, the principle of virtual work can still be applied, although in this case a nonlinear or plastic analysis must be used.



(a)



(b)

Fig. 9–16

When applying Eqs. 9–22 and 9–23, it is important to realize that the definite integrals on the right side actually represent the amount of virtual strain energy that is *stored* in the beam. If concentrated forces or couple moments act on the beam or the distributed load is discontinuous, a single integration cannot be performed across the beam's entire length. Instead, separate x coordinates will have to be chosen within regions that have no discontinuity of loading. Also, it is not necessary that each x have the same origin; however, the x selected for determining the real moment M in a particular region must be the *same* x as that selected for determining the virtual moment m or m_θ within the same region. For example, consider the beam shown in Fig. 9–16. In order to determine the displacement of D, four regions of the beam must be considered, and therefore four integrals having the form $\int (mM/EI)\,dx$ must be evaluated. We can use x_1 to determine the strain energy in region AB, x_2 for region BC, x_3 for region DE, and x_4 for region DC. In any case, each x coordinate should be selected so that both M and m (or m_θ) can be easily formulated.

Integration Using Tables. When the structure is subjected to a relatively simple loading, and yet the solution for a displacement requires several integrations, a *tabular method* may be used to perform these integrations. To do so the moment diagrams for each member are drawn first for both the real and virtual loadings. By matching these diagrams for m and M with those given in the table on the inside front cover, the integral $\int mM\,dx$ can be determined from the appropriate formula. Examples 9–8 and 9–10 illustrate the application of this method.

Procedure for Analysis

The following procedure may be used to determine the displacement and/or the slope at a point on the elastic curve of a beam or frame using the method of virtual work.

Virtual Moments m or m_θ

- Place a *unit load* on the beam or frame at the point and in the direction of the desired *displacement*.
- If the *slope* is to be determined, place a *unit couple moment* at the point.
- Establish appropriate x coordinates that are valid within regions of the beam or frame where there is no discontinuity of real or virtual load.
- With the virtual load in place, and all the real loads *removed* from the beam or frame, calculate the internal moment m or m_θ as a function of each x coordinate.
- Assume m or m_θ acts in the conventional positive direction as indicated in Fig. 4–1.

Real Moments

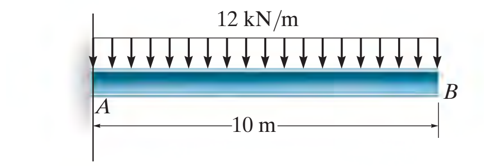
- Using the *same* x coordinates as those established for m or m_θ, determine the internal moments M caused only by the real loads.
- Since m or m_θ was assumed to act in the conventional positive direction, *it is important that positive M* acts in this *same* direction. This is necessary since positive or negative internal work depends upon the directional sense of load (defined by $\pm m$ or $\pm m_\theta$) and displacement (defined by $\pm M\,dx/EI$).

Virtual-Work Equation

- Apply the equation of virtual work to determine the desired displacement Δ or rotation θ. It is important to retain the algebraic sign of each integral calculated within its specified region.
- If the algebraic sum of all the integrals for the entire beam or frame is positive, Δ or θ is in the same direction as the virtual unit load or unit couple moment, respectively. If a negative value results, the direction of Δ or θ is opposite to that of the unit load or unit couple moment.

EXAMPLE 9.7

Determine the displacement of point B of the steel beam shown in Fig. 9–17a. Take $E = 200$ GPa, $I = 500(10^6)$ mm^4.



(a)

SOLUTION

Virtual Moment *m*. The vertical displacement of point B is obtained by placing a virtual unit load of 1 kN at B, Fig. 9–17b. By inspection there are no discontinuities of loading on the beam for *both* the real and virtual loads. Thus, a *single* x coordinate can be used to determine the virtual strain energy. This coordinate will be selected with its origin at B, since then the reactions at A do not have to be determined in order to find the internal moments m and M. Using the method of sections, the internal moment m is formulated as shown in Fig. 9–17b.

Real Moment *M*. Using the *same* x coordinate, the internal moment M is formulated as shown in Fig. 9–17c.

Virtual-Work Equation. The vertical displacement of B is thus

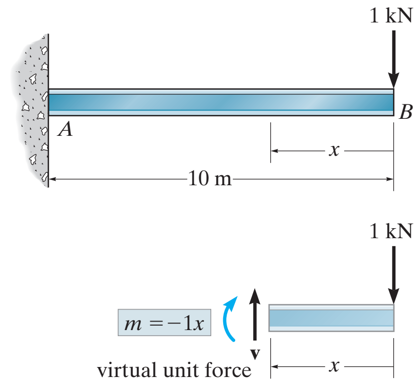
$$1\text{ kN}\cdot\Delta_B = \int_0^L \frac{mM}{EI}\,dx = \int_0^{10} \frac{(-1x)(-6x^2)\,dx}{EI}$$

$$1\text{ kN}\cdot\Delta_B = \frac{15(10^3)\text{ kN}^2\cdot\text{m}^3}{EI}$$

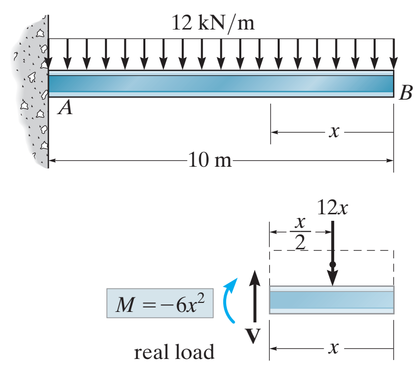
or

$$\Delta_B = \frac{15(10^3)\text{ kN}\cdot\text{m}^3}{200(10^6)\text{ kN/m}^2(500(10^6)\text{ mm}^4)(10^{-12}\text{ m}^4/\text{mm}^4)}$$

$$= 0.150\text{ m} = 150\text{ mm} \qquad \textit{Ans.}$$



(b)



(c)

Fig. 9–17

EXAMPLE 9.8

Determine the slope θ at point B of the steel beam shown in Fig. 9–18a. Take $E = 200$ GPa, $I = 60(10^6)$ mm^4.

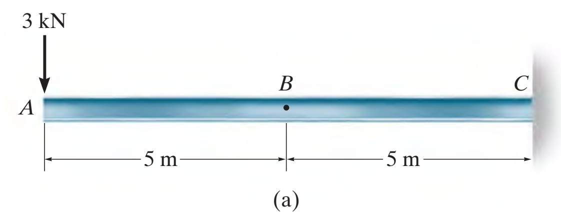
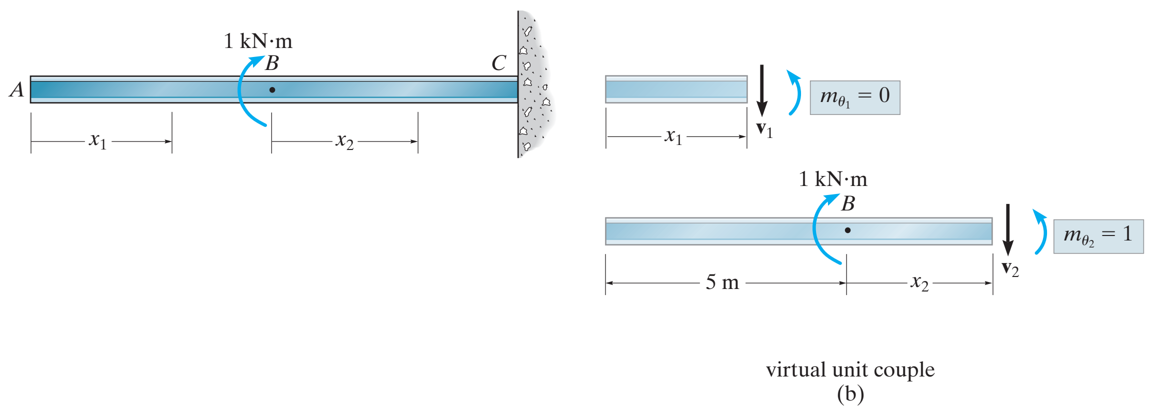
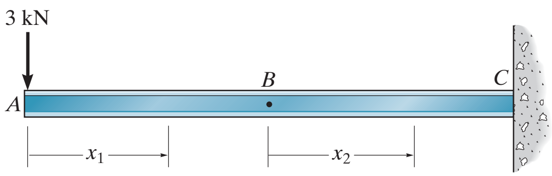


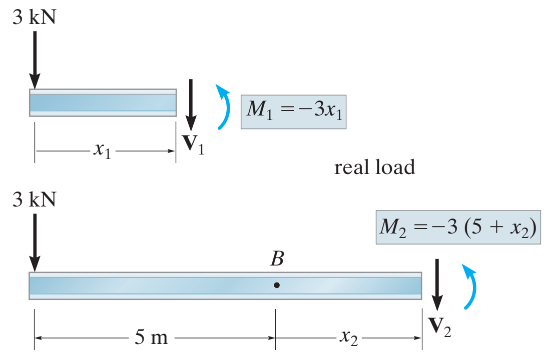
Fig. 9–18

SOLUTION

Virtual Moment m_θ. The slope at B is determined by placing a virtual unit couple moment of 1 kN · m at B, Fig. 9–18b. Here two x coordinates must be selected in order to determine the total virtual strain energy in the beam. Coordinate x_1 accounts for the strain energy within segment AB and coordinate x_2 accounts for that in segment BC. The internal moments m_θ within each of these segments are computed using the method of sections as shown in Fig. 9–18b.







(c)

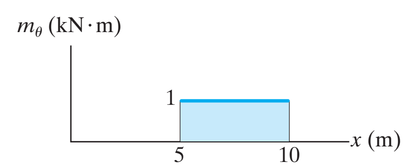
Real Moments M. Using the *same* coordinates x_1 and x_2, the internal moments M are computed as shown in Fig. 9–18*c*.

Virtual-Work Equation. The slope at B is thus given by

$$1 \cdot \theta_B = \int_0^L \frac{m_\theta M}{EI} dx$$

$$= \int_0^5 \frac{(0)(-3x_1)\,dx_1}{EI} + \int_0^5 \frac{(1)[-3(5 + x_2)]\,dx_2}{EI}$$

$$\theta_B = \frac{-112.5 \text{ kN}\cdot\text{m}^2}{EI} \qquad (1)$$



(d)

We can also evaluate the integrals $\int m_\theta M\,dx$ graphically, using the table given on the inside front cover of the book. To do so it is first necessary to draw the moment diagrams for the beams in Figs. 9–18*b* and 9–18*c*. These are shown in Figs. 9–18*d* and 9–18*e*, respectively. Since there is no moment m for $0 \le x < 5$ m, we use only the shaded rectangular and trapezoidal areas to evaluate the integral. Finding these shapes in the appropriate row and column of the table, we have

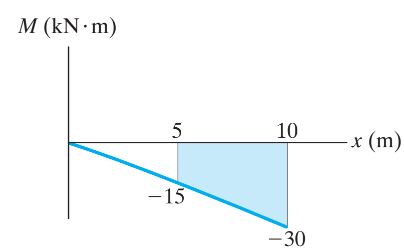
$$\int_5^{10} m_\theta M\,dx = \tfrac{1}{2}m_\theta(M_1 + M_2)L = \tfrac{1}{2}(1)(-15 - 30)5$$

$$= -112.5 \text{ kN}^2\cdot\text{m}^3$$

This is the same value as that determined in Eq. 1. Thus,

$$(1 \text{ kN}\cdot\text{m})\cdot\theta_B = \frac{-112.5 \text{ kN}^2\cdot\text{m}^3}{200(10^6)\text{ kN/m}^2[60(10^6)\text{ mm}^4](10^{-12}\text{ m}^4/\text{mm}^4)}$$

$$\theta_B = -0.00938 \text{ rad} \qquad \textit{Ans.}$$

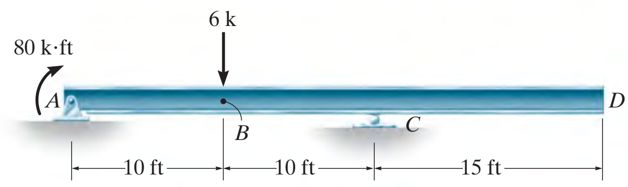


(e)

The *negative sign* indicates θ_B is *opposite* to the direction of the virtual couple moment shown in Fig. 9–18*b*.

EXAMPLE 9.9

Determine the displacement at D of the steel beam in Fig. 9–19a. Take $E = 29(10^3)$ ksi, $I = 800$ in^4.

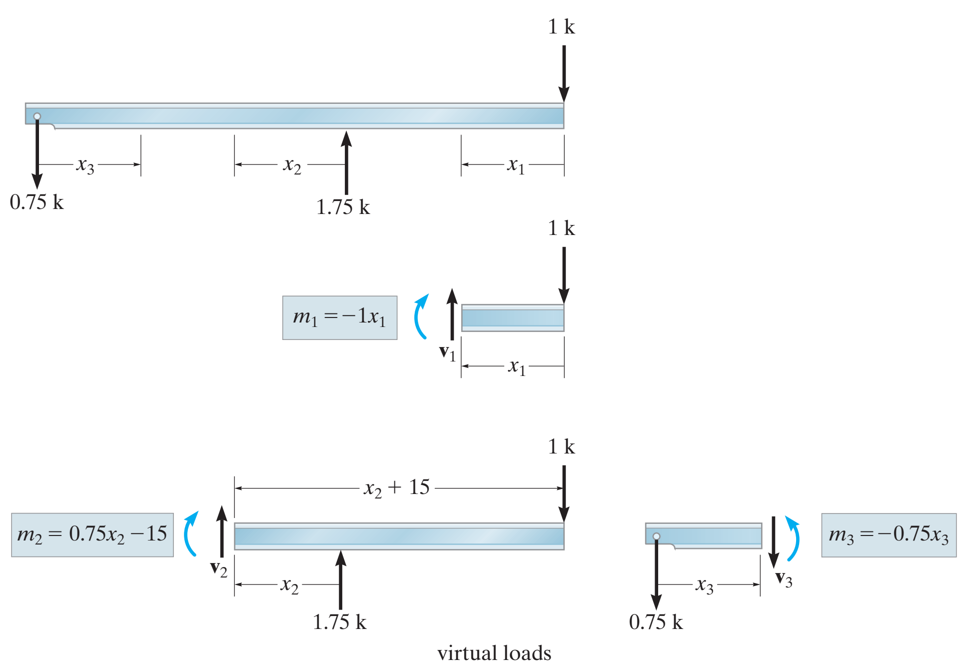


(a)

Fig. 9–19

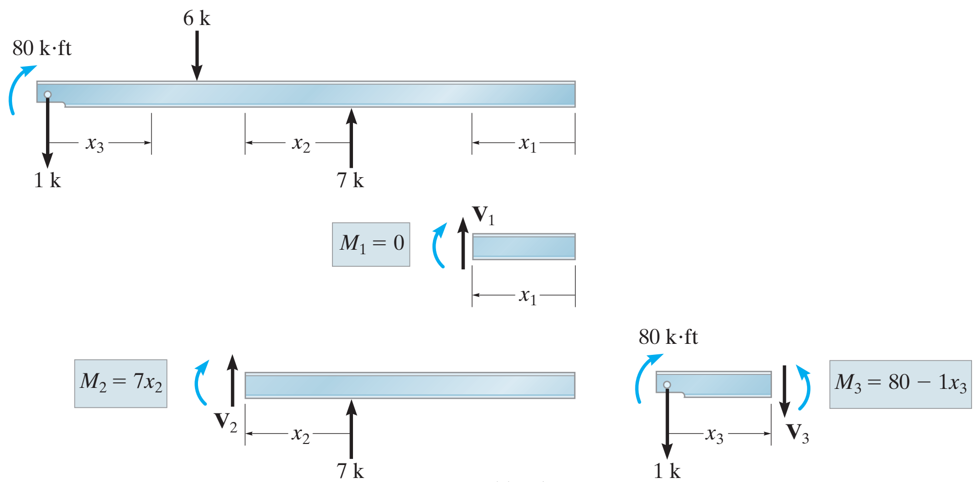
SOLUTION

Virtual Moments *m*. The beam is subjected to a virtual unit load at D as shown in Fig. 9–19b. By inspection, *three coordinates,* such as x_1, x_2, and x_3, must be used to cover all the regions of the beam. Notice that these coordinates cover regions where no discontinuities in either real or virtual load occur. The internal moments m have been computed in Fig. 9–19b using the method of sections.



(b)

9



real loads
(c)

Real Moments *M*. The reactions on the beam are computed first; then, using the *same* x coordinates as those used for m, the internal moments M are determined as shown in Fig. 9–19*c*.

Virtual-Work Equation. Applying the equation of virtual work to the beam using the data in Figs. 9–19*b* and 9–19*c*, we have

$$1 \cdot \Delta_D = \int_0^L \frac{mM}{EI} dx$$

$$= \int_0^{15} \frac{(-1x_1)(0)\,dx_1}{EI} + \int_0^{10} \frac{(0.75x_2 - 15)(7x_2)\,dx_2}{EI}$$

$$+ \int_0^{10} \frac{(-0.75x_3)(80 - 1x_3)\,dx_3}{EI}$$

$$\Delta_D = \frac{0}{EI} - \frac{3500}{EI} - \frac{2750}{EI} = -\frac{6250 \text{ k} \cdot \text{ft}^3}{EI}$$

or

$$\Delta_D = \frac{-6250 \text{ k} \cdot \text{ft}^3 (12)^3 \text{ in}^3/\text{ft}^3}{29(10^3) \text{ k/in}^2 (800 \text{ in}^4)}$$

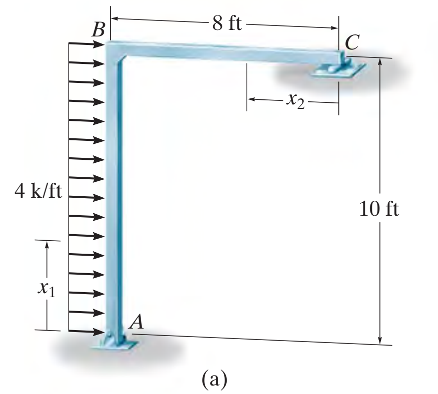
$$= -0.466 \text{ in.}$$ *Ans.*

The negative sign indicates the displacement is upward, opposite to the downward unit load, Fig. 9–19*b*. Also note that m_1 did not actually have to be calculated since $M_1 = 0$.

9

EXAMPLE 9.10

Determine the horizontal displacement of point C on the frame shown in Fig. 9–20*a*. Take $E = 29(10^3)$ ksi and $I = 600$ in^4 for both members.

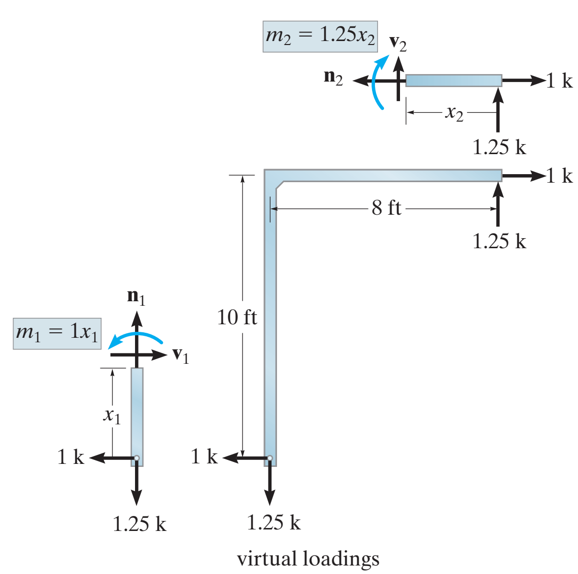


(a)

Fig. 9–20

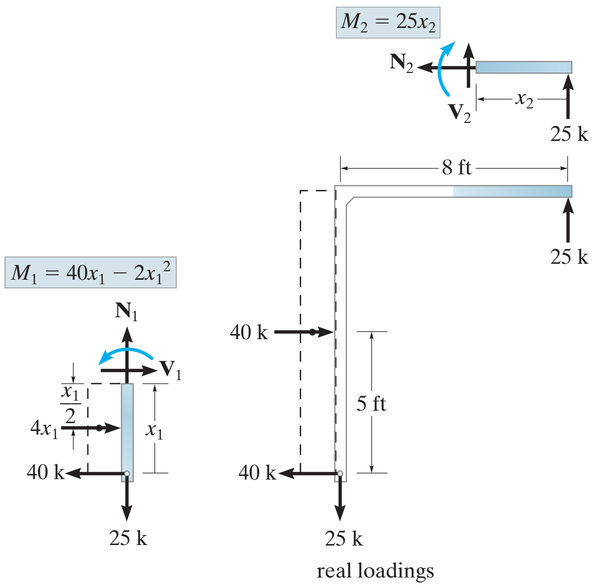
SOLUTION

Virtual Moments *m*. For convenience, the coordinates x_1 and x_2 in Fig. 9–20*a* will be used. A *horizontal* unit load is applied at C, Fig. 9–20*b*. Why? The support reactions and internal virtual moments are computed as shown.



(b)

9



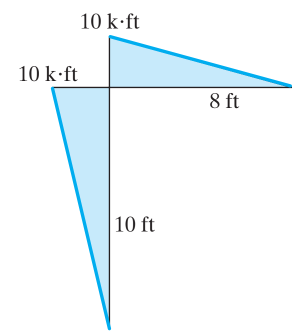
(c)

Real Moments *M*. In a similar manner the support reactions and real moments are computed as shown in Fig. 9–20*c*.

Virtual-Work Equation. Using the data in Figs. 9–20*b* and 9–20*c*, we have

$$1 \cdot \Delta_{C_h} = \int_0^L \frac{mM}{EI} dx = \int_0^{10} \frac{(1x_1)\left(40x_1 - 2x_1^2\right) dx_1}{EI} + \int_0^8 \frac{(1.25x_2)(25x_2)\, dx_2}{EI}$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI} = \frac{13\,666.7 \text{ k} \cdot \text{ft}^3}{EI} \qquad (1)$$



(d)

If desired, the integrals $\int mM/dx$ can also be evaluated graphically using the table on the inside front cover. The moment diagrams for the frame in Figs. 9–20*b* and 9–20*c* are shown in Figs. 9–20*d* and 9–20*e*, respectively. Thus, using the formulas for similar shapes in the table yields

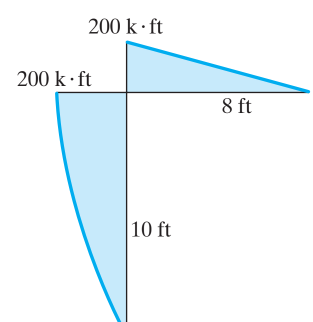
$$\int mM\, dx = \tfrac{5}{12}(10)(200)(10) + \tfrac{1}{3}(10)(200)(8)$$

$$= 8333.3 + 5333.3 = 13\,666.7 \text{ k}^2 \cdot \text{ft}^3$$

This is the same as that calculated in Eq. 1. Thus

$$\Delta_{C_h} = \frac{13\,666.7 \text{ k} \cdot \text{ft}^3}{[29(10^3) \text{ k/in}^2((12)^2 \text{ in}^2/\text{ft}^2)][600 \text{ in}^4(\text{ft}^4/(12)^4 \text{ in}^4)]}$$

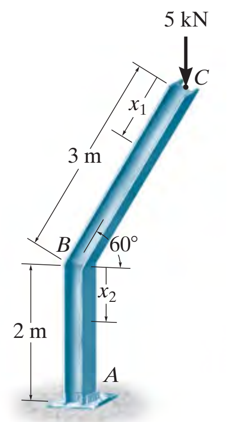
$$= 0.113 \text{ ft} = 1.36 \text{ in.} \qquad \textit{Ans.}$$



(e)

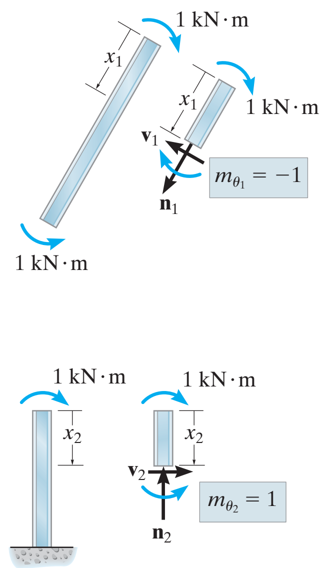
EXAMPLE 9.11

Determine the tangential rotation at point C of the frame shown in Fig. 9–21a. Take $E = 200$ GPa, $I = 15(10^6)$ mm^4.



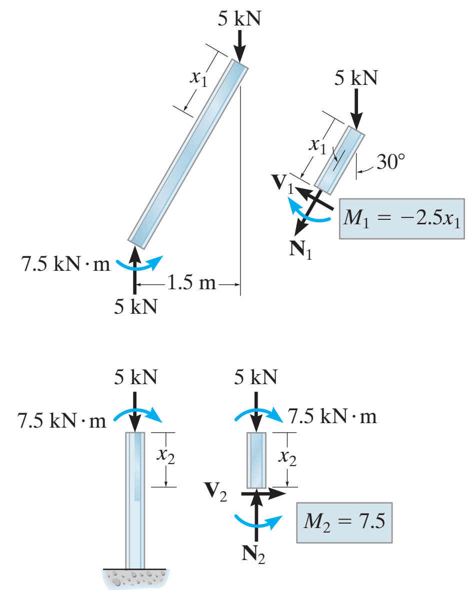
(a)

Fig. 9–21



virtual loads

(b)



real loads

(c)

SOLUTION

Virtual Moments m_θ. The coordinates x_1 and x_2 shown in Fig. 9–21a will be used. A unit couple moment is applied at C and the internal moments m_θ are calculated, Fig. 9–21b.

Real Moments M. In a similar manner, the real moments M are calculated as shown in Fig. 9–21c.

Virtual-Work Equation. Using the data in Figs. 9–21b and 9–21c, we have

$$1 \cdot \theta_C = \int_0^L \frac{m_\theta M}{EI}\,dx = \int_0^3 \frac{(-1)(-2.5x_1)\,dx_1}{EI} + \int_0^2 \frac{(1)(7.5)\,dx_2}{EI}$$

$$\theta_C = \frac{11.25}{EI} + \frac{15}{EI} = \frac{26.25\ \text{kN}\cdot\text{m}^2}{EI}$$

or

$$\theta_C = \frac{26.25\ \text{kN}\cdot\text{m}^2}{200(10^6)\ \text{kN/m}^2[15(10^6)\ \text{mm}^4](10^{-12}\ \text{m}^4/\text{mm}^4)}$$

$$= 0.00875\ \text{rad} \qquad \textit{Ans.}$$

9