9.8 Virtual Strain Energy Caused by Axial Load, Shear, Torsion, and Temperature

Although deflections of beams and frames are caused primarily by bending strain energy, in some structures the additional strain energy of axial load, shear, torsion, and perhaps temperature may become important. Each of these effects will now be considered.

Axial Load. Frame members can be subjected to axial loads, and the virtual strain energy caused by these loadings has been established in Sec. 9–4. For members having a constant cross-sectional area, we have

$$U_n = \frac{nNL}{AE} \tag{9–24}$$

where

n = internal virtual axial load caused by the external virtual unit load.

N = internal axial force in the member caused by the real loads.

E = modulus of elasticity for the material.

A = cross-sectional area of the member.

L = member's length.

Shear. In order to determine the virtual strain energy in a beam due to shear, we will consider the beam element dx shown in Fig. 9–22. The shearing distortion dy of the element as caused by the *real loads* is $dy = \gamma\, dx$. If the shearing strain γ is caused by *linear elastic material response,* then Hooke's law applies, $\gamma = \tau/G$. Therefore, $dy = (\tau/G)\, dx$. We can express the shear stress as $\tau = K(V/A)$, where K is a *form factor* that depends upon the shape of the beam's cross-sectional area A. Hence, we can write $dy = K(V/GA)\, dx$. The internal virtual work done by a virtual shear force v, acting on the element *while* it is deformed dy, is therefore $dU_s = v\, dy = v(KV/GA)\, dx$. For the entire beam, the virtual strain energy is determined by integration.

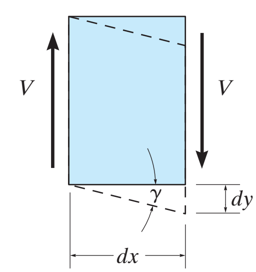

Fig. 9–22

$$U_s = \int_0^L K\left(\frac{vV}{GA}\right) dx \tag{9–25}$$

where

v = internal virtual shear in the member, expressed as a function of x and caused by the external virtual unit load.

V = internal shear in the member, expressed as a function of x and caused by the real loads.

A = cross-sectional area of the member.

K = form factor for the cross-sectional area:
- $K = 1.2$ for rectangular cross sections.
- $K = 10/9$ for circular cross sections.
- $K \approx 1$ for wide-flange and I-beams, where A is the area of the web.

G = shear modulus of elasticity for the material.

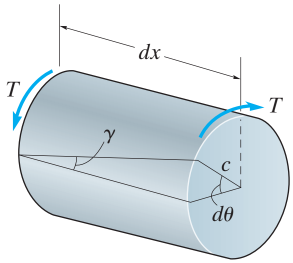

Fig. 9–23

Torsion. Often three-dimensional frameworks are subjected to torsional loadings. If the member has a *circular* cross-sectional area, no warping of its cross section will occur when it is loaded. As a result, the virtual strain energy in the member can easily be derived. To do so consider an element dx of the member that is subjected to an applied torque $\mathbf{T}$, Fig. 9–23. This torque causes a shear strain of $\gamma = (cd\theta)/dx$. Provided *linear elastic material response* occurs, then $\gamma = \tau/G$, where $\tau = Tc/J$. Thus, the angle of twist $d\theta = (\gamma\, dx)/c = (\tau/Gc)\, dx = (T/GJ)\, dx$. If a virtual unit load is applied to the structure that causes an internal virtual torque $\mathbf{t}$ in the member, then after applying the real loads, the virtual strain energy in the member of length dx will be $dU_t = t\, d\theta = tT\, dx/GJ$. Integrating over the length L of the member yields

$$U_t = \frac{tTL}{GJ} \tag{9–26}$$

where

t = internal virtual torque caused by the external virtual unit load.

T = internal torque in the member caused by the real loads.

G = shear modulus of elasticity for the material.

J = polar moment of inertia for the cross section, $J = \pi c^4/2$, where c is the radius of the cross-sectional area.

L = member's length.

The virtual strain energy due to torsion for members having noncircular cross-sectional areas is determined using a more rigorous analysis than that presented here.

Temperature. In Sec. 9–4 we considered the effect of a *uniform temperature change* ΔT on a truss member and indicated that the member will elongate or shorten by an amount $\Delta L = \alpha\, \Delta T L$. In some cases, however, a structural member can be subjected to a *temperature difference across its depth,* as in the case of the beam shown in Fig. 9–24*a*. If this occurs, it is possible to determine the displacement of points along the elastic curve of the beam by using the principle of virtual work. To do so we must first compute the amount of *rotation* of a differential element dx of the beam as caused by the thermal gradient that acts over the beam's cross section. For the sake of discussion we will choose the most common case of a beam having a neutral axis located at the mid-depth (c) of the beam. If we plot the temperature profile, Fig. 9–24*b*, it will be noted that the mean temperature is $T_m = (T_1 + T_2)/2$. If $T_1 > T_2$, the temperature difference at the top of the element causes strain elongation, while that at the bottom causes strain contraction. In both cases the difference in temperature is $\Delta T_m = T_1 - T_m = T_m - T_2$.

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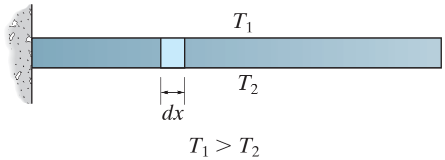
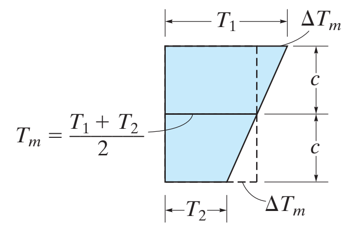


(a)



temperature profile

(b)

Since the thermal change of length at the top and bottom is $\delta x = \alpha\, \Delta T_m\, dx$, Fig. 9–24*c*, then the rotation of the element is

$$d\theta = \frac{\alpha\, \Delta T_m\, dx}{c}$$

If we apply a virtual unit load at a point on the beam where a displacement is to be determined, or apply a virtual unit couple moment at a point where a rotational displacement of the tangent is to be determined, then this loading creates a virtual moment **m** in the beam at the point where the element dx is located. When the temperature gradient is imposed, the virtual strain energy in the beam is then

$$U_{\text{temp}} = \int_0^L \frac{m\alpha\, \Delta T_m\, dx}{c} \tag{9–27}$$

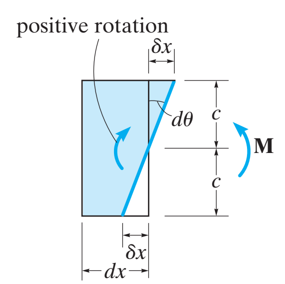
where

m = internal virtual moment in the beam expressed as a function of x and caused by the external virtual unit load or unit couple moment.

α = coefficient of thermal expansion.

ΔT_m = temperature difference between the mean temperature and the temperature at the top or bottom of the beam.

c = mid-depth of the beam.



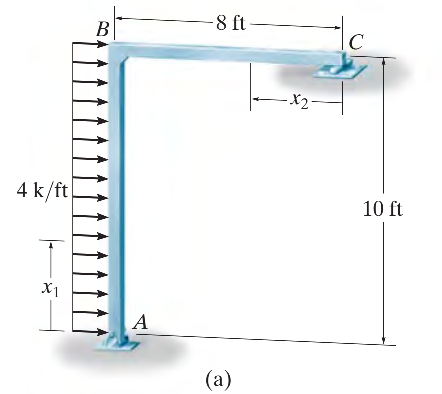
(c)

Fig. 9–24

Unless otherwise stated, *this text will consider only beam and frame deflections due to bending*. In general, though, beam and frame members may be subjected to several of the other loadings discussed in this section. However, as previously mentioned, the additional deflections caused by shear and axial force alter the deflection of beams by only a few percent and are therefore generally ignored for even "small" two- or three-member frame analysis of one-story height. If these and the other effects of torsion and temperature are to be considered for the analysis, then one simply adds their virtual strain energy as defined by Eqs. 9–24 through 9–27 to the equation of virtual work defined by Eq. 9–22 or Eq. 9–23. The following examples illustrate application of these equations.

EXAMPLE 9.12

Determine the horizontal displacement of point C on the frame shown in Fig. 9–25a. Take $E = 29(10^3)$ ksi, $G = 12(10^3)$ ksi, $I = 600$ in^4, and $A = 80$ in^2 for both members. The cross-sectional area is rectangular. Include the internal strain energy due to axial load and shear.

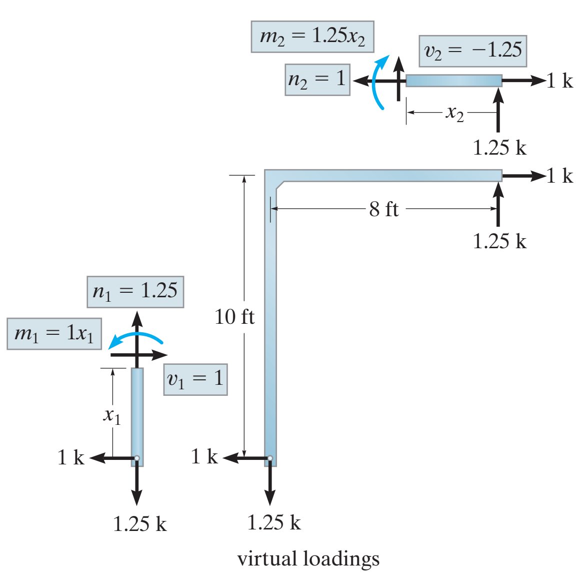


(a)

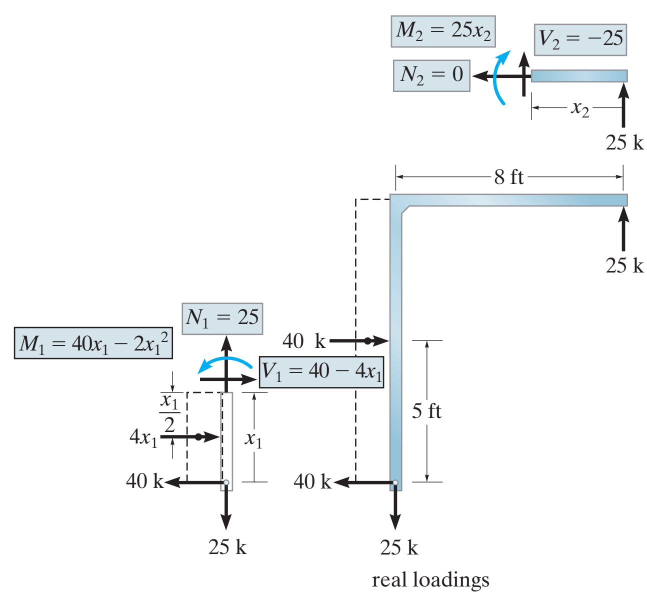
Fig. 9–25

SOLUTION

Here we must apply a horizontal unit load at C. The necessary free-body diagrams for the real and virtual loadings are shown in Figs. 9–25b and 9–25c.



(b)



(c)

Bending. The virtual strain energy due to bending has been determined in Example 9–10. There it was shown that

$$U_b = \int_0^L \frac{mM\,dx}{EI} = \frac{13\,666.7\ \text{k}^2\cdot\text{ft}^3}{EI} = \frac{13\,666.7\ \text{k}^2\cdot\text{ft}^3\ (12^3\ \text{in}^3/1\ \text{ft}^3)}{[29(10^3)\ \text{k/in}^2](600\ \text{in}^4)} = 1.357\ \text{in.}\cdot\text{k}$$

Axial load. From the data in Fig. 9–25*b* and 9–25*c*, we have

$$U_a = \sum \frac{nNL}{AE}$$

$$= \frac{1.25\ \text{k}(25\ \text{k})(120\ \text{in.})}{80\ \text{in}^2[29(10^3)\ \text{k/in}^2]} + \frac{1\ \text{k}(0)(96\ \text{in.})}{80\ \text{in}^2[29(10^3)\ \text{k/in}^2]}$$

$$= 0.001616\ \text{in.}\cdot\text{k}$$

Shear. Applying Eq. 9–25 with $K = 1.2$ for rectangular cross sections, and using the shear functions shown in Fig. 9–25*b* and 9–25*c*, we have

$$U_s = \int_0^L K\left(\frac{vV}{GA}\right) dx$$

$$= \int_0^{10} \frac{1.2(1)(40 - 4x_1)\,dx_1}{GA} + \int_0^{8} \frac{1.2(-1.25)(-25)\,dx_2}{GA}$$

$$= \frac{540\ \text{k}^2\cdot\text{ft}(12\ \text{in./ft})}{[12(10^3)\ \text{k/in}^2](80\ \text{in}^2)} = 0.00675\ \text{in.}\cdot\text{k}$$

Applying the equation of virtual work, we have

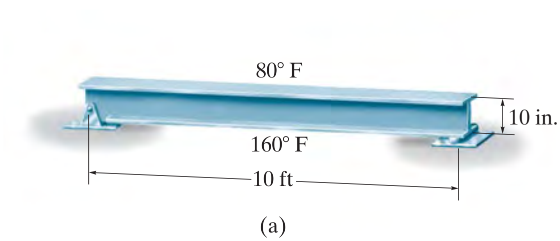
$$1\ \text{k}\cdot\Delta_{C_h} = 1.357\ \text{in.}\cdot\text{k} + 0.001616\ \text{in.}\cdot\text{k} + 0.00675\ \text{in.}\cdot\text{k}$$

$$\Delta_{C_h} = 1.37\ \text{in.}$$ *Ans.*

Including the effects of shear and axial load contributed only a 0.6% increase in the answer to that determined only from bending.

EXAMPLE 9.13

The beam shown in Fig. 9–26*a* is used in a building subjected to two different thermal environments. If the temperature at the top surface of the beam is 80°F and at the bottom surface is 160°F, determine the vertical deflection of the beam at its midpoint due to the temperature gradient. Take $\alpha = 6.5(10^{-6})/°\text{F}$.



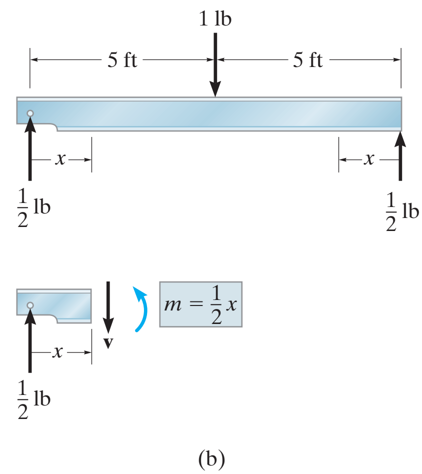


Fig. 9–26

SOLUTION

Since the deflection at the center of the beam is to be determined, a virtual unit load is placed there and the internal virtual moment in the beam is calculated, Fig. 9–26*b*.

The mean temperature at the center of the beam is $(160° + 80°)/2 = 120°\text{F}$, so that for application of Eq. 9–27, $\Delta T_m = 120°\text{F} - 80°\text{F} = 40°\text{F}$. Also, $c = 10\text{ in.}/2 = 5\text{ in.}$ Applying the principle of virtual work, we have

$$1\text{ lb}\cdot\Delta_{C_v} = \int_0^L \frac{m\alpha\,\Delta T_m\,dx}{c}$$

$$= 2\int_0^{60\text{ in.}} \frac{\left(\frac{1}{2}x\right)6.5(10^{-6})/°\text{F}(40°\text{F})}{5\text{ in.}}\,dx$$

$$\Delta_{C_v} = 0.0936\text{ in.} \qquad \textit{Ans.}$$

The result indicates a very negligible deflection.

9

9.9 Castigliano's Theorem for Beams and Frames

The internal bending strain energy for a beam or frame is given by Eq. 9–11 ($U_i = \int M^2\, dx/2EI$). Substituting this equation into Eq. 9–20 ($\Delta_i = \partial U_i/\partial P_i$) and omitting the subscript i, we have

$$\Delta = \frac{\partial}{\partial P}\int_0^L \frac{M^2\, dx}{2EI}$$

Rather than squaring the expression for internal moment M, integrating, and then taking the partial derivative, it is generally easier to differentiate prior to integration. Provided E and I are constant, we have

$$\boxed{\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}} \qquad (9\text{–}28)$$

where

Δ = external displacement of the point caused by the real loads acting on the beam or frame.

P = external force applied to the beam or frame in the direction of Δ.

M = internal moment in the beam or frame, expressed as a function of x and caused by both the force P and the real loads on the beam.

E = modulus of elasticity of beam material.

I = moment of inertia of cross-sectional area computed about the neutral axis.

If the slope θ at a point is to be determined, we must find the partial derivative of the internal moment M with respect to an *external couple moment* M' acting at the point, i.e.,

$$\boxed{\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}} \qquad (9\text{–}29)$$

The above equations are similar to those used for the method of virtual work, Eqs. 9–22 and 9–23, except $\partial M/\partial P$ and $\partial M/\partial M'$ replace m and m_θ, respectively. As in the case for trusses, slightly more calculation is generally required to determine the partial derivatives and apply Castigliano's theorem rather than use the method of virtual work. Also, recall that this theorem applies only to material having a linear elastic response. If a more complete accountability of strain energy in the structure is desired, the strain energy due to shear, axial force, and torsion must be included. The derivations for shear and torsion follow the same development as Eqs. 9–25 and 9–26. The strain energies and their derivatives are, respectively,

$$U_s = K\int_0^L \frac{V^2\,dx}{2AG} \qquad \frac{\partial U_s}{\partial P} = \int_0^L \frac{V}{AG}\left(\frac{\partial V}{\partial P}\right)dx$$

$$U_t = \int_0^L \frac{T^2\,dx}{2JG} \qquad \frac{\partial U_t}{\partial P} = \int_0^L \frac{T}{JG}\left(\frac{\partial T}{\partial P}\right)dx$$

These effects, however, will not be included in the analysis of the problems in this text since beam and frame deflections are caused mainly by bending strain energy. Larger frames, or those with unusual geometry, can be analyzed by computer, where these effects can readily be incorporated into the analysis.

Procedure for Analysis

The following procedure provides a method that may be used to determine the deflection and/or slope at a point in a beam or frame using Castigliano's theorem.

External Force P or Couple Moment M'

- Place a force **P** on the beam or frame at the point and in the direction of the desired displacement.
- If the slope is to be determined, place a couple moment $\mathbf{M}'$ at the point.
- It is assumed that both P and M' have a *variable magnitude* in order to obtain the changes $\partial M/\partial P$ or $\partial M/\partial M'$.

Internal Moments M

- Establish appropriate x coordinates that are valid within regions of the beam or frame where there is no discontinuity of force, distributed load, or couple moment.
- Calculate the internal moment M as a function of P or M' and each x coordinate. Also, compute the partial derivative $\partial M/\partial P$ or $\partial M/\partial M'$ for each coordinate x.
- After M and $\partial M/\partial P$ or $\partial M/\partial M'$ have been determined, assign P or M' its numerical value if it has replaced a real force or couple moment. Otherwise, set P or M' equal to zero.

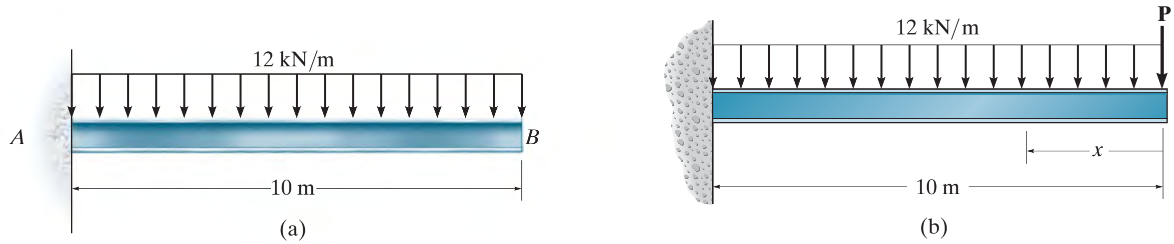
Castigliano's Theorem

- Apply Eq. 9–28 or 9–29 to determine the desired displacement Δ or slope θ. It is important to retain the algebraic signs for corresponding values of M and $\partial M/\partial P$ or $\partial M/\partial M'$.
- If the resultant sum of all the definite integrals is positive, Δ or θ is in the same direction as **P** or $\mathbf{M}'$.

9

EXAMPLE 9.14

Determine the displacement of point B of the beam shown in Fig. 9–27a. Take $E = 200$ GPa, $I = 500(10^6)$ mm^4.

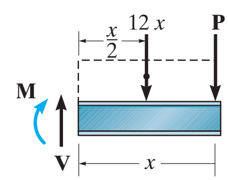


(a) (b)

SOLUTION

External Force P. A vertical force **P** is placed on the beam at B as shown in Fig. 9–27b.

Internal Moments M. A single x coordinate is needed for the solution, since there are no discontinuities of loading between A and B. Using the method of sections, Fig. 9–27c, we have



(c)

Fig. 9–27

$$\circlearrowleft + \Sigma M = 0; \qquad -M - (12x)\left(\frac{x}{2}\right) - Px = 0$$

$$M = -6x^2 - Px \qquad \frac{\partial M}{\partial P} = -x$$

Setting $P = 0$, its actual value, yields

$$M = -6x^2 \qquad \frac{\partial M}{\partial P} = -x$$

Castigliano's Theorem. Applying Eq. 9–28, we have

$$\Delta_B = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} = \int_0^{10} \frac{(-6x^2)(-x)\,dx}{EI} = \frac{15(10^3)\text{ kN}\cdot\text{m}^3}{EI}$$

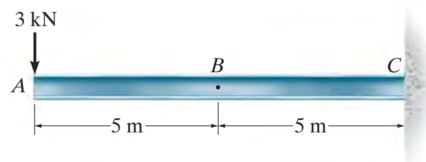
or

$$\Delta_B = \frac{15(10^3)\text{ kN}\cdot\text{m}^3}{200(10^6)\text{ kN/m}^2[500(10^6)\text{ mm}^4](10^{-12}\text{ m}^4/\text{mm}^4)}$$

$$= 0.150\text{ m} = 150\text{ mm} \qquad \textit{Ans.}$$

The similarity between this solution and that of the virtual-work method, Example 9–7, should be noted.

EXAMPLE 9.15

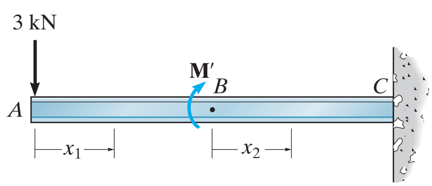


(a)

Determine the slope at point B of the beam shown in Fig. 9–28a. Take $E = 200$ GPa, $I = 60(10^6)$ mm^4.

SOLUTION

External Couple Moment M'. Since the slope at point B is to be determined, an external couple $\mathbf{M}'$ is placed on the beam at this point, Fig. 9–28b.



(b)

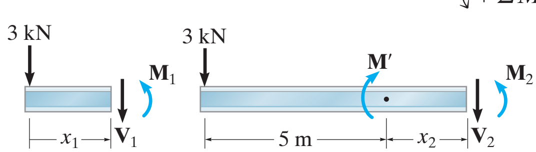
Internal Moments M. Two coordinates, x_1 and x_2, must be used to determine the internal moments within the beam since there is a discontinuity, M', at B. As shown in Fig. 9–28b, x_1 ranges from A to B and x_2 ranges from B to C. Using the method of sections, Fig. 9–28c, the internal moments and the partial derivatives are computed as follows:

For x_1:

$$\circlearrowright + \Sigma M = 0; \qquad M_1 + 3x_1 = 0$$
$$M_1 = -3x_1$$
$$\frac{\partial M_1}{\partial M'} = 0$$

For x_2:

$$\circlearrowright + \Sigma M = 0; \qquad M_2 - M' + 3(5 + x_2) = 0$$
$$M_2 = M' - 3(5 + x_2)$$
$$\frac{\partial M_2}{\partial M'} = 1$$



(c)

Fig. 9–28

Castigliano's Theorem. Setting $M' = 0$, its actual value, and applying Eq. 9–29, we have

$$\theta_B = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

$$= \int_0^5 \frac{(-3x_1)(0)\,dx_1}{EI} + \int_0^5 \frac{-3(5 + x_2)(1)\,dx_2}{EI} = -\frac{112.5 \text{ kN}\cdot\text{m}^2}{EI}$$

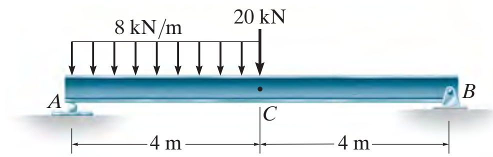
or

$$\theta_B = \frac{-112.5 \text{ kN}\cdot\text{m}^2}{200(10^6)\text{ kN/m}^2[60(10^6)\text{ mm}^4](10^{-12}\text{ m}^4/\text{mm}^4)}$$
$$= -0.00938 \text{ rad} \qquad \textit{Ans.}$$

The negative sign indicates that θ_B is opposite to the direction of the couple moment $\mathbf{M}'$. Note the similarity between this solution and that of Example 9–8.

EXAMPLE 9.16

Determine the vertical displacement of point C of the beam shown in Fig. 9–29*a*. Take $E = 200$ GPa, $I = 150(10^6)$ mm^4.

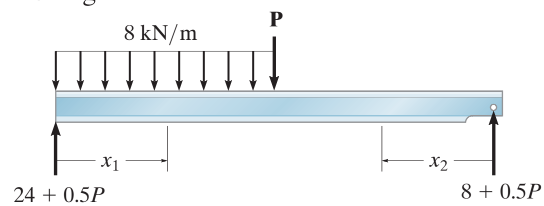


(a)

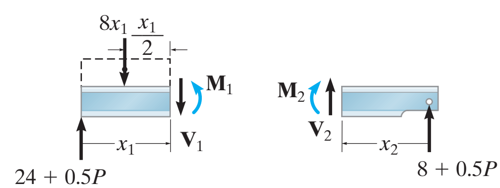
SOLUTION

External Force *P*. A vertical force **P** is applied at point C, Fig. 9–29*b*. Later this force will be set equal to a fixed value of 20 kN.

Internal Moments *M*. In this case two x coordinates are needed for the integration, Fig. 9–29*b*, since the load is discontinuous at C. Using the method of sections, Fig. 9–29*c*, we have



(b)



(c)

Fig. 9–29

For x_1:

$$\circlearrowleft + \Sigma M = 0; \quad -(24 + 0.5P)x_1 + 8x_1\left(\frac{x_1}{2}\right) + M_1 = 0$$

$$M_1 = (24 + 0.5P)x_1 - 4x_1^2$$

$$\frac{\partial M_1}{\partial P} = 0.5x_1$$

For x_2:

$$\circlearrowleft + \Sigma M = 0; \quad -M_2 + (8 + 0.5P)x_2 = 0$$

$$M_2 = (8 + 0.5P)x_2$$

$$\frac{\partial M_2}{\partial P} = 0.5x_2$$

Castigliano's Theorem. Setting $P = 20$ kN, its actual value, and applying Eq. 9–28 yields

$$\Delta_{C_v} = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}$$

$$= \int_0^4 \frac{(34x_1 - 4x_1^2)(0.5x_1)\,dx_1}{EI} + \int_0^4 \frac{(18x_2)(0.5x_2)\,dx_2}{EI}$$

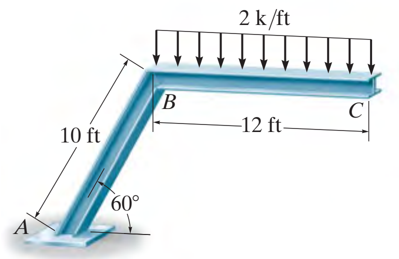
$$= \frac{234.7\text{ kN}\cdot\text{m}^3}{EI} + \frac{192\text{ kN}\cdot\text{m}^3}{EI} = \frac{426.7\text{ kN}\cdot\text{m}^3}{EI}$$

or

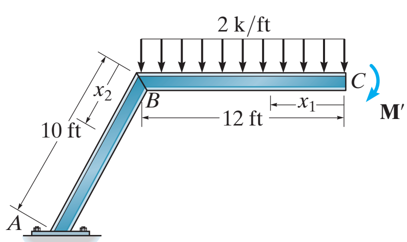
$$\Delta_{C_v} = \frac{426.7\text{ kN}\cdot\text{m}^3}{200(10^6)\text{ kN/m}^2[150(10^6)\text{ mm}^4](10^{-12}\text{ m}^4/\text{mm}^4)}$$

$$= 0.0142\text{ m} = 14.2\text{ mm} \qquad \textit{Ans.}$$

EXAMPLE 9.17



(a)



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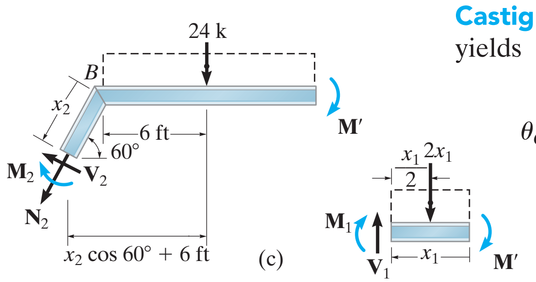


(c)

Fig. 9–30

Determine the slope at point C of the two-member frame shown in Fig. 9–30*a*. The support at A is fixed. Take $E = 29(10^3)$ ksi, $I = 600 \text{ in}^4$.

SOLUTION

External Couple Moment M'. A variable moment $\mathbf{M}'$ is applied to the frame at point C, since the slope at this point is to be determined, Fig. 9–30*b*. Later this moment will be set equal to zero.

Internal Moments M. Due to the discontinuity of internal loading at B, two coordinates, x_1 and x_2, are chosen as shown in Fig. 9–30*b*. Using the method of sections, Fig. 9–30*c*, we have

For x_1:

$$\circlearrowleft + \Sigma M = 0; \qquad -M_1 - 2x_1\left(\frac{x_1}{2}\right) - M' = 0$$

$$M_1 = -\left(x_1^2 + M'\right)$$

$$\frac{\partial M_1}{\partial M'} = -1$$

For x_2:

$$\circlearrowleft + \Sigma M = 0; \qquad -M_2 - 24(x_2 \cos 60° + 6) - M' = 0$$

$$M_2 = -24(x_2 \cos 60° + 6) - M'$$

$$\frac{\partial M_2}{\partial M'} = -1$$

Castigliano's Theorem. Setting $M' = 0$ and applying Eq. 9–29 yields

$$\theta_C = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

$$= \int_0^{12} \frac{\left(-x_1^2\right)(-1)\,dx_1}{EI} + \int_0^{10} \frac{-24(x_2 \cos 60° + 6)(-1)\,dx_2}{EI}$$

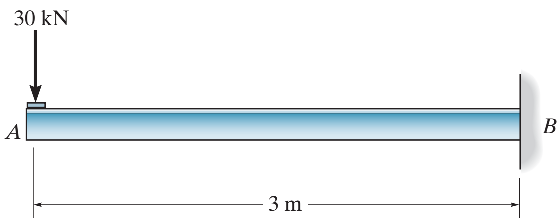
$$= \frac{576 \text{ k}\cdot\text{ft}^2}{EI} + \frac{2040 \text{ k}\cdot\text{ft}^2}{EI} = \frac{2616 \text{ k}\cdot\text{ft}^2}{EI}$$

$$\theta_C = \frac{2616 \text{ k}\cdot\text{ft}^2(144 \text{ in}^2/\text{ft}^2)}{29(10^3)\text{ k/in}^2(600 \text{ in}^4)} = 0.0216 \text{ rad} \qquad \textit{Ans.}$$

FUNDAMENTAL PROBLEMS

F9–13. Determine the slope and displacement at point A. EI is constant. Use the principle of virtual work.

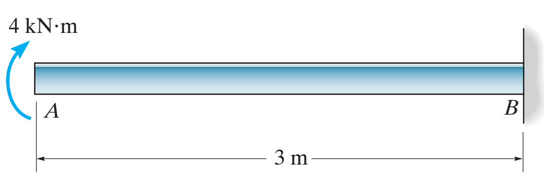
F9–14. Solve Prob. F9–13 using Castigliano's theorem.



F9–13/9–14

F9–15. Determine the slope and displacement at point A. EI is constant. Use the principle of virtual work.

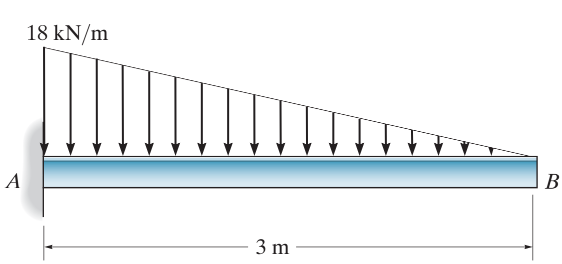
F9–16. Solve Prob. F9–15 using Castigliano's theorem.



F9–15/9–16

F9–17. Determine the slope and displacement at point B. EI is constant. Use the principle of virtual work.

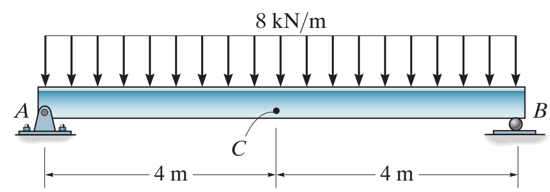
F9–18. Solve Prob. F9–17 using Castigliano's theorem.



F9–17/9–18

F9–19. Determine the slope at A and displacement at point C. EI is constant. Use the principle of virtual work.

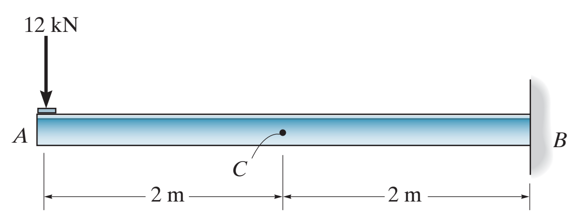
F9–20. Solve Prob. F9–19 using Castigliano's theorem.



F9–19/9–20

F9–21. Determine the slope and displacement at point C. EI is constant. Use the principle of virtual work.

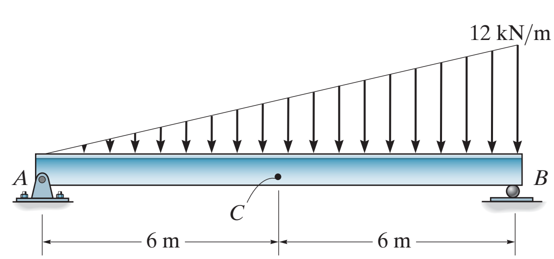
F9–22. Solve Prob. F9–21 using Castigliano's theorem.



F9–21/9–22

F9–23. Determine the displacement at point C. EI is constant. Use the principle of virtual work.

F9–24. Solve Prob. F9–23 using Castigliano's theorem.

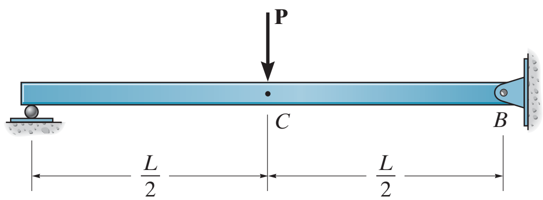


F9–23/9–24

PROBLEMS

9–21. Determine the displacement of point C and the slope at point B. EI is constant. Use the principle of virtual work.

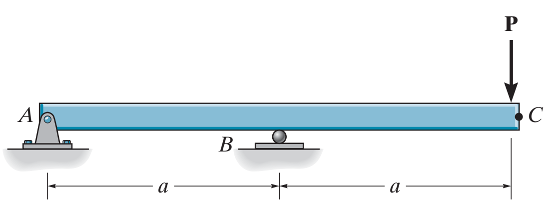
9–22. Solve Prob. 9–21 using Castigliano's theorem.



Probs. 9–21/9–22

9–23. Determine the displacement at point C. EI is constant. Use the method of virtual work.

***9–24.** Solve Prob. 9–23 using Castigliano's theorem.



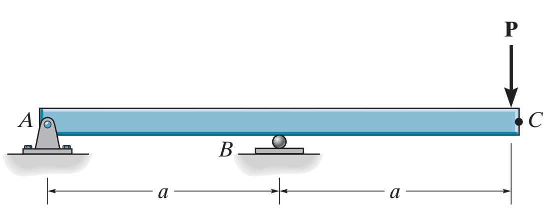
Probs. 9–23/9–24

9–25. Determine the slope at point C. EI is constant. Use the method of virtual work.

9–26. Solve Prob. 9–25 using Castigliano's theorem.

9–27. Determine the slope at point A. EI is constant. Use the method of virtual work.

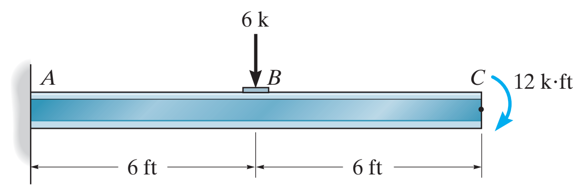
***9–28.** Solve Prob. 9–27 using Castigliano's theorem.



Probs. 9–25/9–26/9–27/9–28

9–29. Determine the slope and displacement at point C. Use the method of virtual work. $E = 29(10^3)$ ksi, $I = 800\text{ in}^4$.

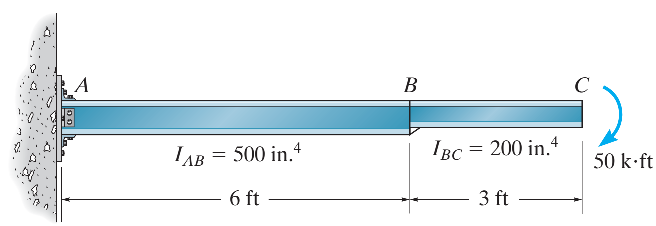
9–30. Solve Prob. 9–29 using Castigliano's theorem.



Probs. 9–29/9–30

9–31. Determine the displacement and slope at point C of the cantilever beam. The moment of inertia of each segment is indicated in the figure. Take $E = 29(10^3)$ ksi. Use the principle of virtual work.

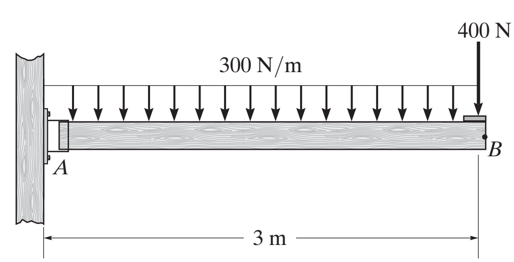
***9–32.** Solve Prob. 9–31 using Castigliano's theorem.



Probs. 9–31/9–32

9–33. Determine the slope and displacement at point B. EI is constant. Use the method of virtual work.

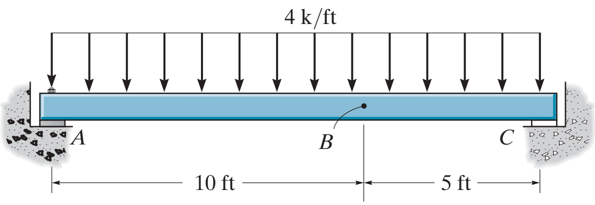
9–34. Solve Prob. 9–33 using Castigliano's theorem.



Probs. 9–33/9–34

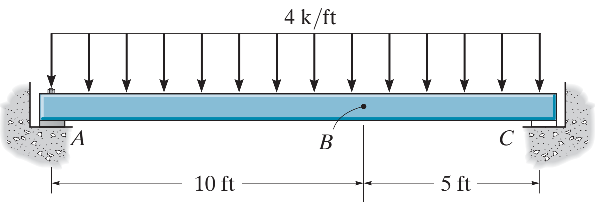
9–35. Determine the slope and displacement at point B. Assume the support at A is a pin and C is a roller. Take $E = 29(10^3)$ ksi, $I = 300$ in^4. Use the method of virtual work.

***9–36.** Solve Prob. 9–35 using Castigliano's theorem.



Probs. 9–35/9–36

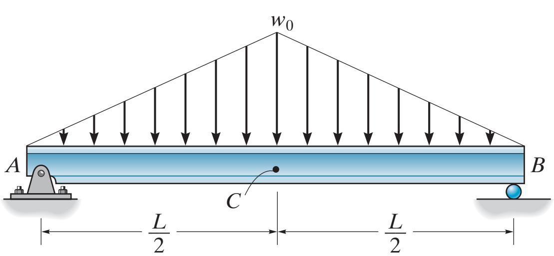
9–37. Determine the slope and displacement at point B. Assume the support at A is a pin and C is a roller. Account for the additional strain energy due to shear. Take $E = 29(10^3)$ ksi, $I = 300$ in^4, $G = 12(10^3)$ ksi, and assume AB has a cross-sectional area of $A = 7.50$ in^2. Use the method of virtual work.



Prob. 9–37

9–38. Determine the displacement of point C. Use the method of virtual work. EI is constant.

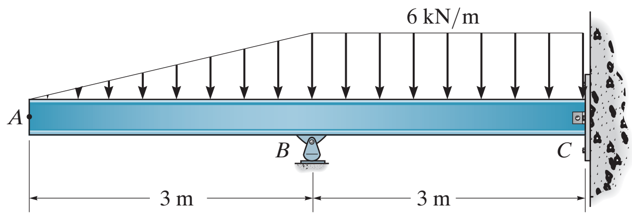
9–39. Solve Prob. 9–38 using Castigliano's theorem.



Probs. 9–38/9–39

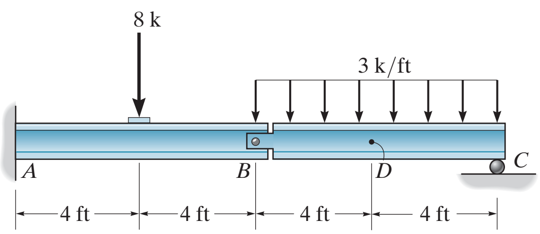
***9–40.** Determine the slope and displacement at point A. Assume C is pinned. Use the principle of virtual work. EI is constant.

9–41. Solve Prob. 9–40 using Castigliano's theorem.



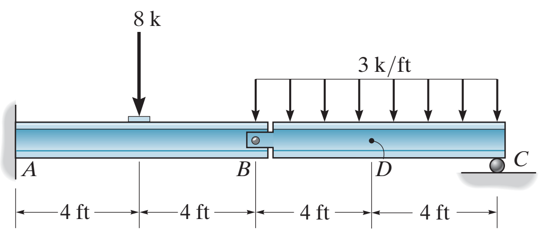
Probs. 9–40/9–41

9–42. Determine the displacement at point D. Use the principle of virtual work. EI is constant.



Prob. 9–42

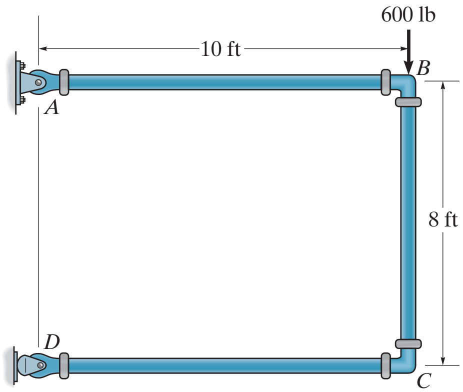
9–43. Determine the displacement at point D. Use Castigliano's theorem. EI is constant.



Prob. 9–43

***9–44.** Use the method of virtual work and determine the vertical deflection at the rocker support D. EI is constant.

9–45. Solve Prob. 9–44 using Castigliano's theorem.

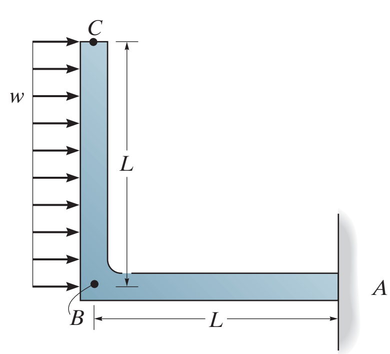


Probs. 9–44/9–45

9–46. The L-shaped frame is made from two segments, each of length L and flexural stiffness EI. If it is subjected to the uniform distributed load, determine the horizontal displacement of the end C. Use the method of virtual work.

9–47. The L-shaped frame is made from two segments, each of length L and flexural stiffness EI. If it is subjected to the uniform distributed load, determine the vertical displacement of point B. Use the method of virtual work.

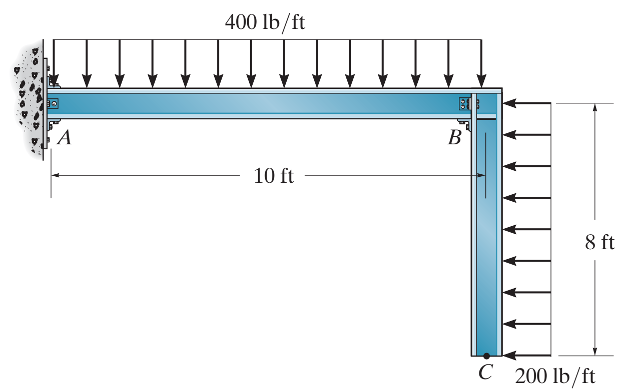
***9–48.** Solve Prob. 9–47 using Castigliano's theorem.



Probs. 9–46/9–47/9–48

9–49. Determine the horizontal displacement of point C. EI is constant. Use the method of virtual work.

9–50. Solve Prob. 9–49 using Castigliano's theorem.

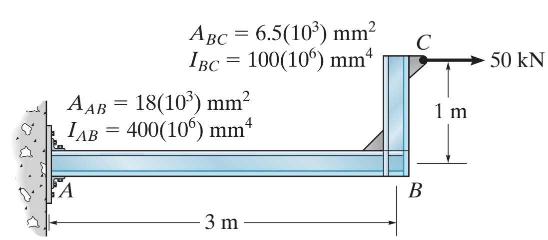


Probs. 9–49/9–50

9–51. Determine the vertical deflection at C. The cross-sectional area and moment of inertia of each segment is shown in the figure. Take $E = 200$ GPa. Assume A is a fixed support. Use the method of virtual work.

***9–52.** Solve Prob. 9–51, including the effect of shear and axial strain energy.

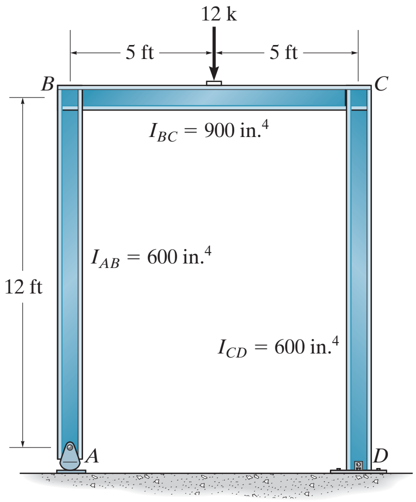
9–53. Solve Prob. 9–51 using Castigliano's theorem.



Probs. 9–51/9–52/9–53

9–54. Determine the slope at A. Take $E = 29(10^3)$ ksi. The moment of inertia of each segment of the frame is indicated in the figure. Assume D is a pin support. Use the method of virtual work.

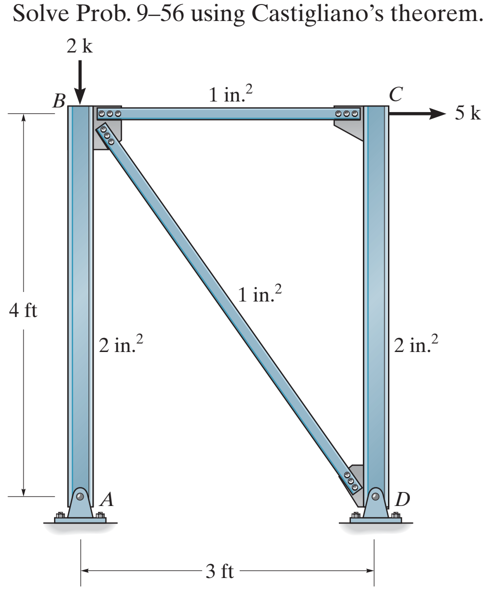
9–55. Solve Prob. 9–54 using Castigliano's theorem.



Probs. 9–54/9–55

***9–56.** Use the method of virtual work and determine the horizontal deflection at C. The cross-sectional area of each member is indicated in the figure. Assume the members are pin connected at their end points. $E = 29(10^3)$ ksi.

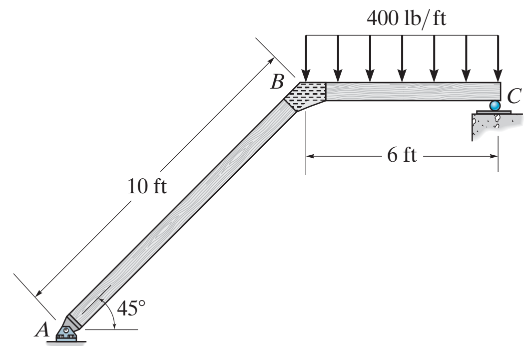
9–57. Solve Prob. 9–56 using Castigliano's theorem.



Probs. 9–56/9–57

9–58. Use the method of virtual work and determine the horizontal deflection at C. EI is constant. There is a pin at A. Assume C is a roller and B is a fixed joint.

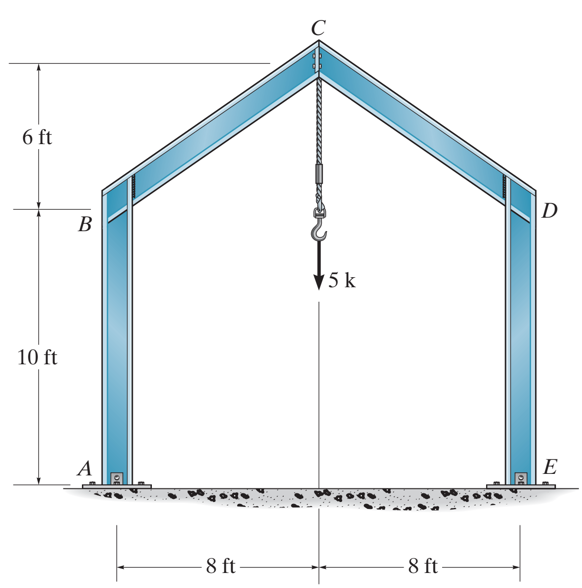
9–59. Solve Prob. 9–58 using Castigliano's theorem.



Probs. 9–58/9–59

***9–60.** The frame is subjected to the load of 5 k. Determine the vertical displacement at C. Assume that the members are pin connected at A, C, and E, and fixed connected at the knee joints B and D. EI is constant. Use the method of virtual work.

9–61. Solve Prob. 9–60 using Castigliano's theorem.



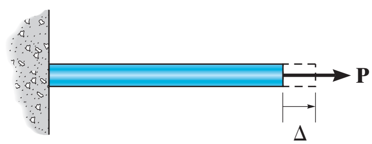
Probs. 9–60/9–61

CHAPTER REVIEW

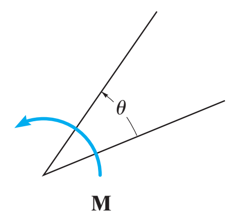
All energy methods are based on the conservation of energy principle, which states that the work done by all external forces acting on the structure, U_e, is transformed into internal work or strain energy, U_i, developed in the members when the structure deforms.

$$\boxed{U_e = U_i}$$

A force (moment) does work U when it undergoes a displacement (rotation) in the direction of the force (moment).



$U = P\Delta$



$U = M\theta$

The principle of virtual work is based upon the work done by a "virtual" or imaginary unit force. If the deflection (rotation) at a point on the structure is to be obtained, a unit virtual force (couple moment) is applied to the structure at the point. This causes internal virtual loadings in the structure. The virtual work is then developed when the real loads are placed on the structure causing it to deform.

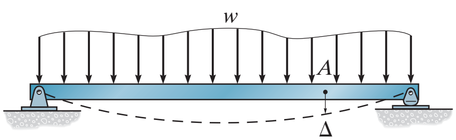
Truss displacements are found using

$$\boxed{1 \cdot \Delta = \sum \frac{nNL}{AE}}$$

If the displacement is caused by temperature, or fabrication errors, then

$$\boxed{1 \cdot \Delta = \Sigma n\alpha\,\Delta T\,L} \qquad \boxed{1 \cdot \Delta = \Sigma n\,\Delta L}$$

For beams and frames, the displacement (rotation) is defined from



$$1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$$

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx$$

Castigliano's second theorem, called the method of least work, can be used to determine the deflections in structures that respond elastically. It states that the displacement (rotation) at a point on a structure is equal to the first partial derivative of the strain energy in the structure with respect to a force P (couple moment M') acting at the point and in the direction of the displacement (rotation). For a truss

$$\Delta = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE}$$

For beams and frames

$$\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}$$

$$\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

The fixed-connected joints of this concrete framework make this a statically indeterminate structure.

Analysis of Statically Indeterminate Structures by the Force Method

10

In this chapter we will apply the *force* or *flexibility* method to analyze statically indeterminate trusses, beams, and frames. At the end of the chapter we will present a method for drawing the influence line for a statically indeterminate beam or frame.

10.1 Statically Indeterminate Structures

Recall from Sec. 2–4 that a structure of any type is classified as *statically indeterminate* when the number of unknown reactions or internal forces exceeds the number of equilibrium equations available for its analysis. In this section we will discuss the merits of using indeterminate structures and two fundamental ways in which they may be analyzed. Realize that most of the structures designed today are statically indeterminate. This indeterminacy may arise as a result of added supports or members, or by the general form of the structure. For example, reinforced concrete buildings are almost always statically indeterminate since the columns and beams are poured as continuous members through the joints and over supports.

Advantages and Disadvantages. Although the analysis of a statically indeterminate structure is more involved than that of a statically determinate one, there are usually several very important reasons for choosing this type of structure for design. Most important, for a given loading the maximum stress and deflection of an indeterminate structure are generally *smaller* than those of its statically determinate counterpart. For example, the statically indeterminate, fixed-supported beam in Fig. 10–1*a* will be subjected to a maximum moment of $M_{\max} = PL/8$, whereas the same beam, when simply supported, Fig. 10–1*b*, will be subjected to twice the moment, that is, $M_{\max} = PL/4$. As a result, the fixed-supported beam has one fourth the deflection and one half the stress at its center of the one that is simply supported.

Another important reason for selecting a statically indeterminate structure is because it has a tendency to redistribute its load to its redundant supports in cases where faulty design or overloading occurs. In these cases, the structure maintains its stability and collapse is prevented. This is particularly important when *sudden* lateral loads, such as wind or earthquake, are imposed on the structure. To illustrate, consider again the fixed-end beam in Fig. 10–1*a*. As **P** is increased, the beam's material at the walls and at the center of the beam begins to *yield* and forms localized "plastic hinges," which causes the beam to deflect as if it were hinged or pin connected at these points. Although the deflection becomes large, the walls will develop horizontal force and moment reactions that will hold the beam and thus prevent it from totally collapsing. In the case of the simply supported beam, Fig. 10–1*b*, an excessive load **P** will cause the "plastic hinge" to form only at the center of the beam, and due to the large vertical deflection, the supports will not develop the horizontal force and moment reactions that may be necessary to prevent total collapse.

Although statically indeterminate structures can support a loading with thinner members and with increased stability compared to their statically determinate counterparts, there are cases when these advantages may instead become disadvantages. The cost savings in material must be compared with the added cost necessary to fabricate the structure, since oftentimes it becomes more costly to construct the supports and joints of an indeterminate structure compared to one that is determinate. More important, though, because statically indeterminate structures have redundant support reactions, one has to be very careful to prevent differential displacement of the supports, since this effect will introduce internal stress in the structure. For example, if the wall at one end of the fixed-end beam in Fig. 10–1*a* were to settle, stress would be developed in the beam because of this "forced" deformation. On the other hand, if the beam were simply supported or statically determinate, Fig. 10–1*b*, then any settlement of its end would not cause the beam to deform, and therefore no stress would be developed in the beam. In general, then, any deformation, such as that caused by relative support displacement, or changes in member lengths caused by temperature or fabrication errors, will introduce additional stresses in the structure, which must be considered when designing indeterminate structures.

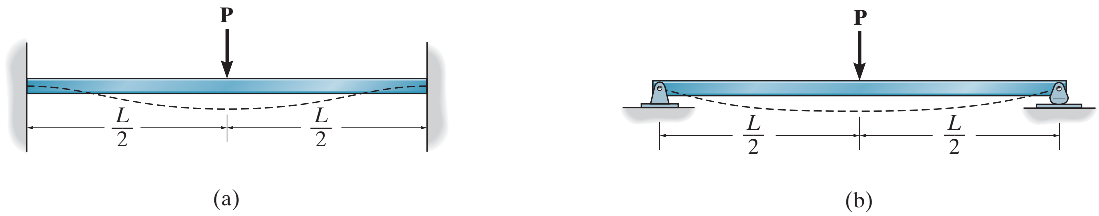


Fig. 10–1

Methods of Analysis. When analyzing any indeterminate structure, it is necessary to satisfy *equilibrium*, *compatibility*, and *force-displacement* requirements for the structure. *Equilibrium* is satisfied when the reactive forces hold the structure at rest, and *compatibility* is satisfied when the various segments of the structure fit together without intentional breaks or overlaps. The *force-displacement* requirements depend upon the way the material responds; in this text we have assumed linear elastic response. In general there are two different ways to satisfy these requirements when analyzing a statically indeterminate structure: the *force* or *flexibility method*, and the *displacement* or *stiffness method*.

Force Method. The force method was originally developed by James Clerk Maxwell in 1864 and later refined by Otto Mohr and Heinrich Müller-Breslau. This method was one of the first available for the analysis of statically indeterminate structures. Since compatibility forms the basis for this method, it has sometimes been referred to as the *compatibility method* or the *method of consistent displacements*. This method consists of writing equations that satisfy the *compatibility* and *force-displacement requirements* for the structure in order to determine the redundant *forces*. Once these forces have been determined, the remaining reactive forces on the structure are determined by satisfying the equilibrium requirements. The fundamental principles involved in applying this method are easy to understand and develop, and they will be discussed in this chapter.

Displacement Method. The displacement method of analysis is based on first writing force-displacement relations for the members and then satisfying the *equilibrium requirements* for the structure. In this case the *unknowns* in the equations are *displacements*. Once the displacements are obtained, the forces are determined from the compatibility and force-displacement equations. We will study some of the classical techniques used to apply the displacement method in Chapters 11 and 12. Since almost all present day computer software for structural analysis is developed using this method we will present a matrix formulation of the displacement method in Chapters 14, 15, and 16.

Each of these two methods of analysis, which are outlined in Fig. 10–2, has particular advantages and disadvantages, depending upon the geometry of the structure and its degree of indeterminacy. A discussion of the usefulness of each method will be given after each has been presented.

	Unknowns	Equations Used for Solution	Coefficients of the Unknowns
Force Method	Forces	Compatibility and Force Displacement	Flexibility Coefficients
Displacement Method	Displacements	Equilibrium and Force Displacement	Stiffness Coefficients

Fig. 10–2

10.2 Force Method of Analysis: General Procedure

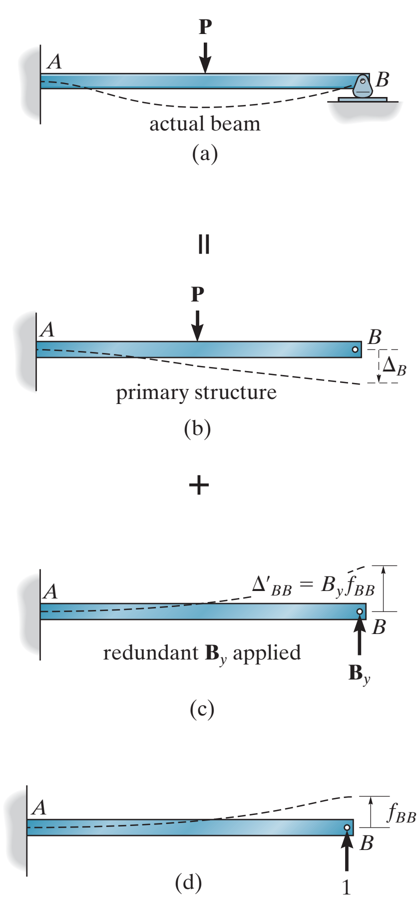


Fig. 10–3

Perhaps the best way to illustrate the principles involved in the force method of analysis is to consider the beam shown in Fig. 10–3*a*. If its free-body diagram were drawn, there would be four unknown support reactions; and since three equilibrium equations are available for solution, the beam is indeterminate to the first degree. Consequently, one additional equation is necessary for solution. To obtain this equation, we will use the principle of superposition and consider the *compatibility of displacement* at one of the supports. This is done by choosing one of the support reactions as "redundant" and temporarily removing its effect on the beam so that the beam then becomes statically determinate and stable. This beam is referred to as the *primary structure.* Here we will remove the restraining action of the rocker at B. As a result, the load $\mathbf{P}$ will cause B to be displaced downward by an amount Δ_B as shown in Fig. 10–3*b*. By superposition, however, the unknown reaction at B, i.e., $\mathbf{B}_y$, causes the beam at B to be displaced Δ'_{BB} upward, Fig. 10–3*c*. Here the first letter in this double-subscript notation refers to the point (B) where the deflection is specified, and the second letter refers to the point (B) where the unknown reaction acts. Assuming positive displacements act upward, then from Figs. 10–3*a* through 10–3*c* we can write the necessary compatibility equation at the rocker as

$$(+\uparrow) \qquad 0 = -\Delta_B + \Delta'_{BB}$$

Let us now denote the displacement at B caused by a *unit load* acting in the direction of $\mathbf{B}_y$ as the *linear flexibility coefficient* f_{BB}, Fig. 10–3*d*. Using the same scheme for this double-subscript notation as above, f_{BB} is the deflection at B caused by a unit load at B. Since the material behaves in a linear-elastic manner, a force of $\mathbf{B}_y$ acting at B, instead of the unit load, will cause a proportionate increase in f_{BB}. Thus we can write

$$\Delta'_{BB} = B_y f_{BB}$$

When written in this format, it can be seen that the linear flexibility coefficient f_{BB} is a *measure of the deflection per unit force,* and so its units are m/N, ft/lb, etc. The compatibility equation above can therefore be written in terms of the unknown B_y as

$$0 = -\Delta_B + B_y f_{BB}$$

Using the methods of Chapter 8 or 9, or the deflection table on the inside front cover of the book, the appropriate load-displacement relations for the deflection Δ_B, Fig. 10–3*b*, and the flexibility coefficient f_{BB}, Fig. 10–3*d*, can be obtained and the solution for B_y determined, that is, $B_y = \Delta_B/f_{BB}$. Once this is accomplished, the three reactions at the wall A can then be found from the equations of equilibrium.

As stated previously, the choice of the redundant is *arbitrary*. For example, the moment at A, Fig. 10–4*a*, can be determined *directly* by removing the capacity of the beam to support a moment at A, that is, by replacing the fixed support by a pin. As shown in Fig. 10–4*b*, the rotation at A caused by the load **P** is θ_A, and the rotation at A caused by the redundant $\mathbf{M_A}$ at A is θ'_{AA}, Fig. 10–4*c*. If we denote an *angular flexibility coefficient* α_{AA} as the angular displacement at A caused by a unit couple moment applied to A, Fig. 10–4*d*, then

$$\theta'_{AA} = M_A\alpha_{AA}$$

Thus, the angular flexibility coefficient measures the angular displacement per unit couple moment, and therefore it has units of rad/N·m or rad/lb·ft, etc. The compatibility equation for rotation at A therefore requires

$$(\curvearrowleft +) \qquad 0 = \theta_A + M_A\alpha_{AA}$$

In this case, $M_A = -\theta_A/\alpha_{AA}$, a negative value, which simply means that $\mathbf{M_A}$ acts in the opposite direction to the unit couple moment.

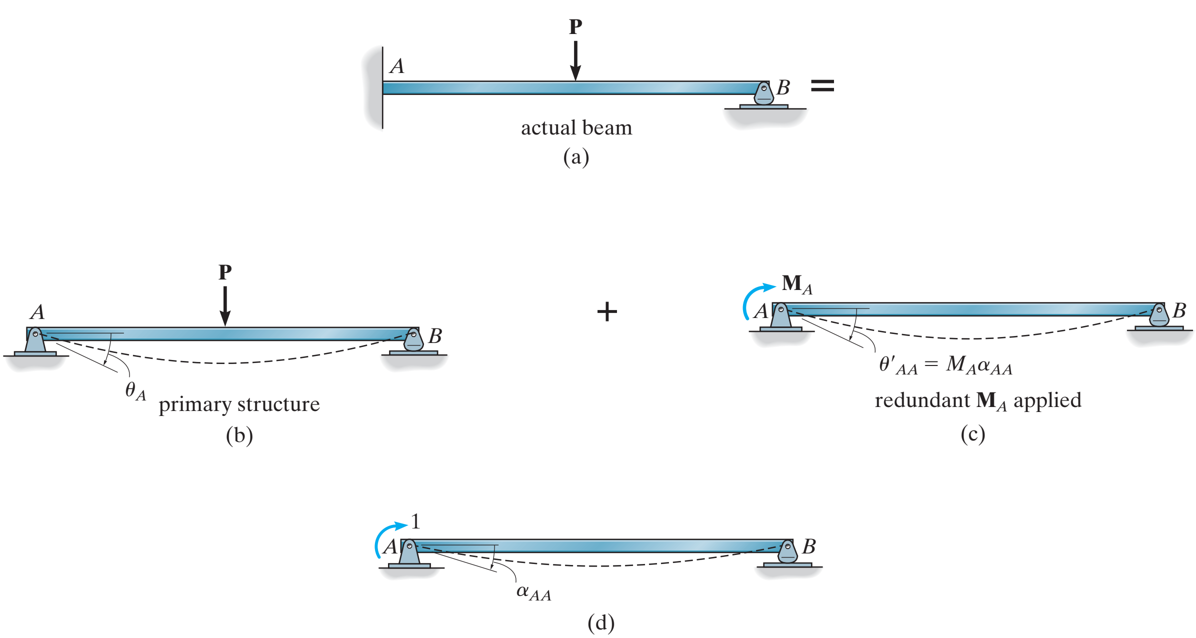


Fig. 10–4

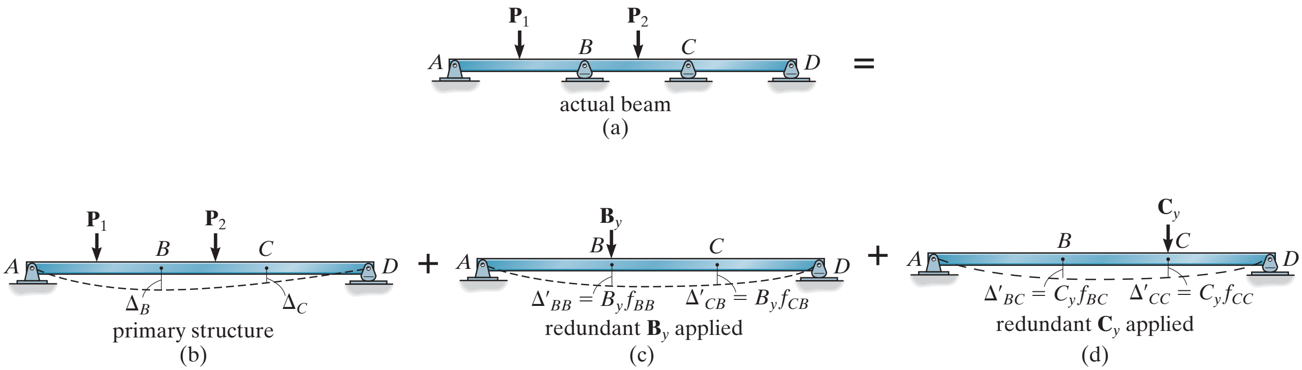
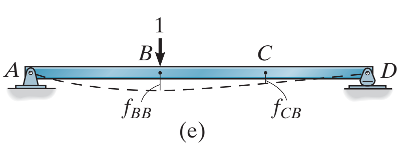
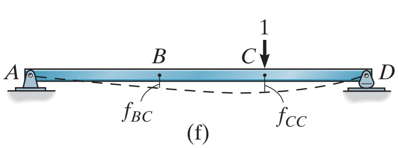


Fig. 10–5





A third example that illustrates application of the force method is given in Fig. 10–5*a*. Here the beam is indeterminate to the second degree and therefore two compatibility equations will be necessary for the solution. We will choose the vertical forces at the roller supports, B and C, as redundants. The resultant statically determinate beam deflects as shown in Fig. 10–5*b* when the redundants are removed. Each redundant force, which is *assumed* to act downward, deflects this beam as shown in Fig. 10–5*c* and 10–5*d*, respectively. Here the flexibility coefficients* f_{BB} and f_{CB} are found from a unit load acting at B, Fig. 10–5*e*; and f_{CC} and f_{BC} are found from a unit load acting at C, Fig. 10–5*f*. By superposition, the compatibility equations for the deflection at B and C, respectively, are

$$(+\downarrow) \qquad 0 = \Delta_B + B_y f_{BB} + C_y f_{BC}$$
$$(+\downarrow) \qquad 0 = \Delta_C + B_y f_{CB} + C_y f_{CC} \qquad (10\text{–}1)$$

Once the load-displacement relations are established using the methods of Chapter 8 or 9, these equations may be solved simultaneously for the two unknown forces B_y and C_y.

Having illustrated the application of the force method of analysis by example, we will now discuss its application in general terms and then we will use it as a basis for solving problems involving trusses, beams, and frames. For all these cases, however, realize that since the method depends on superposition of displacements, it is necessary that *the material remain linear elastic when loaded.* Also, recognize that *any* external reaction or internal loading at a point in the structure can be directly determined by first releasing the capacity of the structure to support the loading and then writing a compatibility equation at the point. See Example 10–4.

*f_{BB} is the deflection at B caused by a unit load at B; f_{CB} the deflection at C caused by a unit load at B.

Procedure for Analysis

The following procedure provides a general method for determining the reactions or internal loadings of statically indeterminate structures using the force or flexibility method of analysis.

Principle of Superposition

Determine the number of degrees n to which the structure is indeterminate. Then specify the n unknown redundant forces or moments that must be removed from the structure in order to make it statically determinate and stable. Using the principle of superposition, draw the statically indeterminate structure and show it to be equal to a series of corresponding statically *determinate* structures. The primary structure supports the same external loads as the statically indeterminate structure, and each of the other structures added to the primary structure shows the structure loaded with a separate redundant force or moment. Also, sketch the elastic curve on each structure and indicate symbolically the displacement or rotation at the point of each redundant force or moment.

Compatibility Equations

Write a compatibility equation for the displacement or rotation at each point where there is a redundant force or moment. These equations should be expressed in terms of the unknown redundants and their corresponding flexibility coefficients obtained from unit loads or unit couple moments that are collinear with the redundant forces or moments.

Determine all the deflections and flexibility coefficients using the table on the inside front cover or the methods of Chapter 8 or 9.* Substitute these load-displacement relations into the compatibility equations and solve for the unknown redundants. In particular, if a numerical value for a redundant is negative, it indicates the redundant acts opposite to its corresponding unit force or unit couple moment.

Equilibrium Equations

Draw a free-body diagram of the structure. Since the redundant forces and/or moments have been calculated, the remaining unknown reactions can be determined from the equations of equilibrium.

It should be realized that once all the support reactions have been obtained, the shear and moment diagrams can then be drawn, and the deflection at any point on the structure can be determined using the same methods outlined previously for statically determinate structures.

*It is suggested that if the M/EI diagram for a beam consists of simple segments, the moment-area theorems or the conjugate-beam method be used. Beams with complicated M/EI diagrams, that is, those with many curved segments (parabolic, cubic, etc.), can be readily analyzed using the method of virtual work or by Castigliano's second theorem.

10.3 Maxwell's Theorem of Reciprocal Displacements; Betti's Law

When Maxwell developed the force method of analysis, he also published a theorem that relates the flexibility coefficients of any two points on an elastic structure—be it a truss, a beam, or a frame. This theorem is referred to as the theorem of reciprocal displacements and may be stated as follows: *The displacement of a point B on a structure due to a unit load acting at point A is equal to the displacement of point A when the unit load is acting at point B, that is,* $f_{BA} = f_{AB}$.

Proof of this theorem is easily demonstrated using the principle of virtual work. For example, consider the beam in Fig. 10–6. When a real unit load acts at A, assume that the internal moments in the beam are represented by m_A. To determine the flexibility coefficient at B, that is, f_{BA}, a virtual unit load is placed at B, Fig. 10–7, and the internal moments m_B are computed. Then applying Eq. 9–18 yields

$$f_{BA} = \int \frac{m_B m_A}{EI}\,dx$$

Likewise, if the flexibility coefficient f_{AB} is to be determined when a real unit load acts at B, Fig. 10–7, then m_B represents the internal moments in the beam due to a real unit load. Furthermore, m_A represents the internal moments due to a virtual unit load at A, Fig. 10–6. Hence,

$$f_{AB} = \int \frac{m_A m_B}{EI}\,dx$$

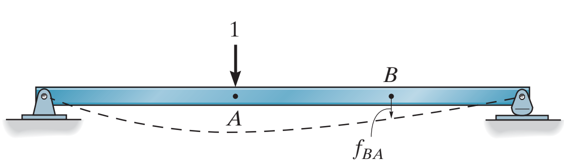


Fig. 10–6

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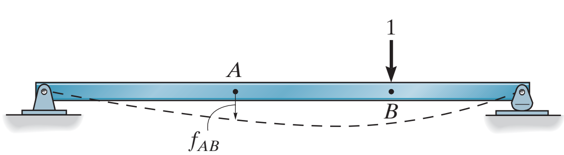


Fig. 10–7

Both integrals obviously give the same result, which proves the theorem. The theorem also applies for reciprocal rotations, and may be stated as follows: *The rotation at point B on a structure due to a unit couple moment acting at point A is equal to the rotation at point A when the unit couple moment is acting at point B.* Furthermore, using a unit force and unit couple moment, applied at separate points on the structure, we may also state: *The rotation in radians at point B on a structure due to a unit load acting at point A is equal to the displacement at point A when a unit couple moment is acting at point B.*

As a consequence of this theorem, some work can be saved when applying the force method to problems that are statically indeterminate to the second degree or higher. For example, only one of the two flexibility coefficients f_{BC} or f_{CB} has to be calculated in Eqs. 10–1, since $f_{BC} = f_{CB}$. Furthermore, the theorem of reciprocal displacements has applications in structural model analysis and for constructing influence lines using the Müller-Breslau principle (see Sec. 10–10).

When the theorem of reciprocal displacements is formalized in a more general sense, it is referred to as *Betti's law.* Briefly stated: The virtual work δU_{AB} done by a system of forces $\Sigma \mathbf{P}_B$ that undergo a displacement caused by a system of forces $\Sigma \mathbf{P}_A$ is equal to the virtual work δU_{BA} caused by the forces $\Sigma \mathbf{P}_A$ when the structure deforms due to the system of forces $\Sigma \mathbf{P}_B$ In other words, $\delta U_{AB} = \delta U_{BA}$. The proof of this statement is similar to that given above for the reciprocal-displacement theorem.

10.4 Force Method of Analysis: Beams

The force method applied to beams was outlined in Sec. 10–2. Using the "procedure for analysis" also given in Sec. 10–2, we will now present several examples that illustrate the application of this technique.

These bridge girders are statically indeterminate since they are continuous over their piers.

10

EXAMPLE 10.1

Determine the reaction at the roller support B of the beam shown in Fig. 10–8*a*. EI is constant.

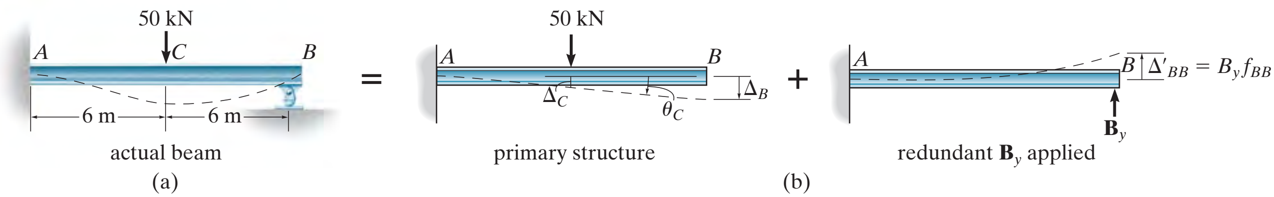


Fig. 10–8

SOLUTION

Principle of Superposition. By inspection, the beam is statically indeterminate to the first degree. The redundant will be taken as $\mathbf{B}_y$ so that this force can be determined *directly*. Figure 10–8*b* shows application of the principle of superposition. Notice that removal of the redundant requires that the roller support or the constraining action of the beam in the direction of $\mathbf{B}_y$ be removed. Here we have assumed that $\mathbf{B}_y$ acts upward on the beam.

Compatibility Equation. Taking positive displacement as upward, Fig. 10–8*b*, we have

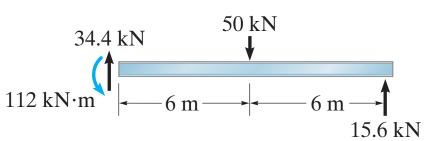
$$(+\uparrow) \qquad 0 = -\Delta_B + B_y f_{BB} \qquad (1)$$

The terms Δ_B and f_{BB} are easily obtained using the table on the inside front cover. In particular, note that $\Delta_B = \Delta_C + \theta_C(6\text{ m})$. Thus,

$$\Delta_B = \frac{P(L/2)^3}{3EI} + \frac{P(L/2)^2}{2EI}\left(\frac{L}{2}\right)$$

$$= \frac{(50\text{ kN})(6\text{ m})^3}{3EI} + \frac{(50\text{ kN})(6\text{ m})^2}{2EI}(6\text{ m}) = \frac{9000\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

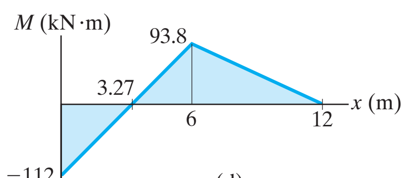
$$f_{BB} = \frac{PL^3}{3EI} = \frac{1(12\text{ m})^3}{3EI} = \frac{576\text{ m}^3}{EI}\uparrow$$



(c)

Substituting these results into Eq. (1) yields

$$(+\uparrow) \qquad 0 = -\frac{9000}{EI} + B_y\left(\frac{576}{EI}\right) \qquad B_y = 15.6\text{ kN} \qquad \textit{Ans.}$$



(d)

If this reaction is placed on the free-body diagram of the beam, the reactions at A can be obtained from the three equations of equilibrium, Fig. 10–8*c*.

Having determined all the reactions, the moment diagram can be constructed as shown in Fig. 10–8*d*.

EXAMPLE 10.2

Draw the shear and moment diagrams for the beam shown in Fig. 10–9*a*. The support at B settles 1.5 in. Take $E = 29(10^3)$ ksi, $I = 750$ in^4.

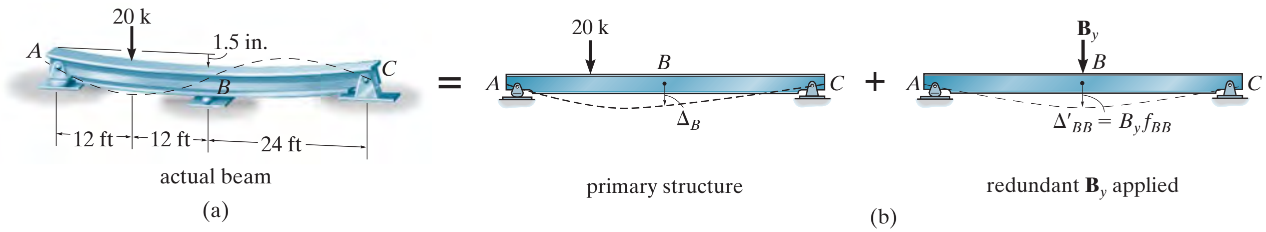


Fig. 10–9

SOLUTION

Principle of Superposition. By inspection, the beam is indeterminate to the first degree. The center support B will be chosen as the redundant, so that the roller at B is removed, Fig. 10–9*b*. Here $\mathbf{B}_y$ is assumed to act downward on the beam.

Compatibility Equation. With reference to point B in Fig. 10–9*b*, using units of inches, we require

$$(+\downarrow) \qquad 1.5 \text{ in.} = \Delta_B + B_y f_{BB} \qquad (1)$$

We will use the table on the inside front cover. Note that for Δ_B the equation for the deflection curve requires $0 < x < a$. Since $x = 24$ ft, then $a = 36$ ft. Thus,

$$\Delta_B = \frac{Pbx}{6LEI}(L^2 - b^2 - x^2) = \frac{20(12)(24)}{6(48)EI}[(48)^2 - (12)^2 - (24)^2]$$

$$= \frac{31{,}680 \text{ k}\cdot\text{ft}^3}{EI}$$

$$f_{BB} = \frac{PL^3}{48EI} = \frac{1(48)^3}{48\,EI} = \frac{2304 \text{ k}\cdot\text{ft}^3}{EI}$$

Substituting these values into Eq. (1), we get

$$1.5 \text{ in.}\,(29(10^3)\text{ k/in}^2)(750 \text{ in}^4)$$

$$= 31{,}680 \text{ k}\cdot\text{ft}^3(12 \text{ in./ft})^3 + B_y(2304 \text{ k}\cdot\text{ft}^3)(12 \text{ in./ft})^3$$

$$B_y = -5.56 \text{ k}$$

The negative sign indicates that $\mathbf{B}_y$ acts *upward* on the beam.

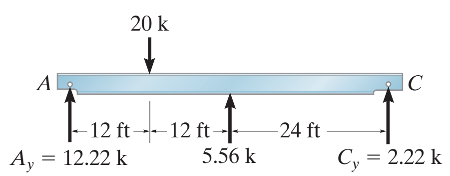
10

EXAMPLE 10.2 (Continued)

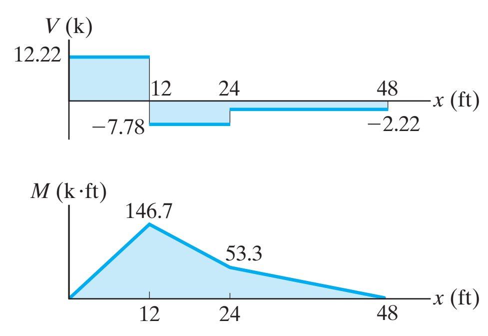
Equilibrium Equations. From the free-body diagram shown in Fig. 10–9*c* we have

$$\circlearrowleft + \Sigma M_A = 0; \qquad -20(12) + 5.56(24) + C_y(48) = 0$$
$$C_y = 2.22 \text{ k}$$
$$+\uparrow \Sigma F_y = 0; \qquad A_y - 20 + 5.56 + 2.22 = 0$$
$$A_y = 12.22 \text{ k}$$

Using these results, verify the shear and moment diagrams shown in Fig. 10–9*d*.



(c)



(d)

EXAMPLE 10.3

Draw the shear and moment diagrams for the beam shown in Figure 10–10*a*. *EI* is constant. Neglect the effects of axial load.

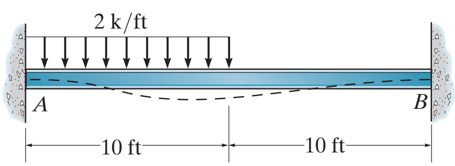
SOLUTION

Principle of Superposition. Since axial load is neglected, the beam is indeterminate to the second degree. The two end moments at A and B will be considered as the redundants. The beam's capacity to resist these moments is removed by placing a pin at A and a rocker at B. The principle of superposition applied to the beam is shown in Fig. 10–10*b*.

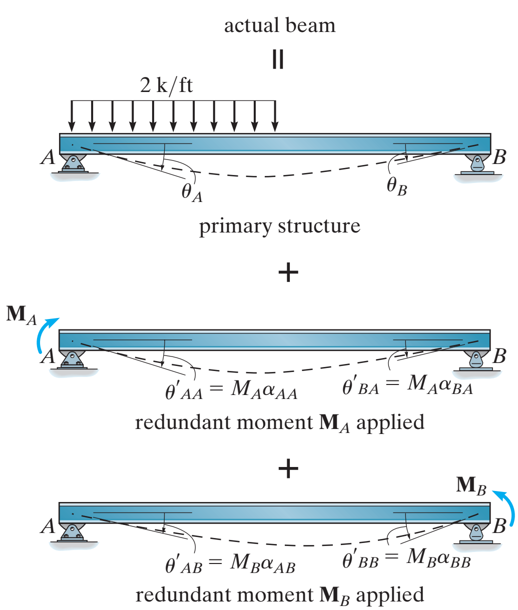
Compatibility Equations. Reference to points A and B, Fig. 10–10*b*, requires

$$(\curvearrowleft +) \qquad 0 = \theta_A + M_A\alpha_{AA} + M_B\alpha_{AB} \qquad (1)$$

$$(\curvearrowright +) \qquad 0 = \theta_B + M_A\alpha_{BA} + M_B\alpha_{BB} \qquad (2)$$



(a)



(b)

Fig. 10–10

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EXAMPLE 10.3 (Continued)

The required slopes and angular flexibility coefficients can be determined using the table on the inside front cover. We have

$$\theta_A = \frac{3wL^3}{128EI} = \frac{3(2)(20)^3}{128EI} = \frac{375}{EI}$$

$$\theta_B = \frac{7wL^3}{384EI} = \frac{7(2)(20)^3}{384EI} = \frac{291.7}{EI}$$

$$\alpha_{AA} = \frac{ML}{3EI} = \frac{1(20)}{3EI} = \frac{6.67}{EI}$$

$$\alpha_{BB} = \frac{ML}{3EI} = \frac{1(20)}{3EI} = \frac{6.67}{EI}$$

$$\alpha_{AB} = \frac{ML}{6EI} = \frac{1(20)}{6EI} = \frac{3.33}{EI}$$

Note that $\alpha_{BA} = \alpha_{AB}$, a consequence of Maxwell's theorem of reciprocal displacements.

Substituting the data into Eqs. (1) and (2) yields

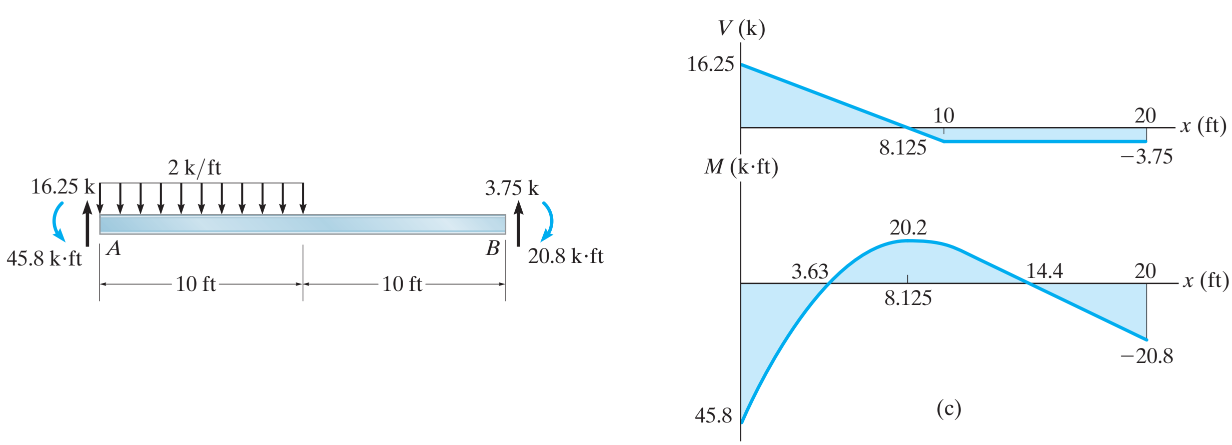
$$0 = \frac{375}{EI} + M_A\left(\frac{6.67}{EI}\right) + M_B\left(\frac{3.33}{EI}\right)$$

$$0 = \frac{291.7}{EI} + M_A\left(\frac{3.33}{EI}\right) + M_B\left(\frac{6.67}{EI}\right)$$

Canceling EI and solving these equations simultaneously, we have

$$M_A = -45.8 \text{ k} \cdot \text{ft} \qquad M_B = -20.8 \text{ k} \cdot \text{ft}$$

Using these results, the end shears are calculated, Fig. 10–10c, and the shear and moment diagrams plotted.



(c)

EXAMPLE 10.4

Determine the reactions at the supports for the beam shown in Fig. 10–11*a*. *EI* is constant.

SOLUTION

Principle of Superposition. By inspection, the beam is indeterminate to the first degree. Here, for the sake of illustration, we will choose the internal moment at support *B* as the redundant. Consequently, the beam is cut open and end pins or an internal hinge are placed at *B* in order to release *only* the capacity of the beam to resist moment at this point, Fig. 10–11*b*. The internal moment at *B* is applied to the beam in Fig. 10–11*c*.

Compatibility Equations. From Fig. 10–11*a* we require the relative rotation of one end of one beam with respect to the end of the other beam to be zero, that is,

$$(\curvearrowleft +) \qquad \theta_B + M_B \alpha_{BB} = 0$$

where

$$\theta_B = \theta'_B + \theta''_B$$

and

$$\alpha_{BB} = \alpha'_{BB} + \alpha''_{BB}$$

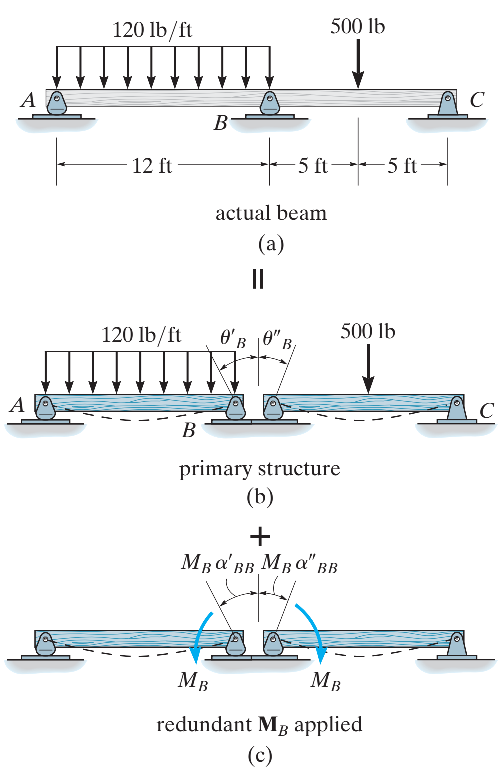


Fig. 10–11

EXAMPLE 10.4 (Continued)

The slopes and angular flexibility coefficients can be determined from the table on the inside front cover, that is,

$$\theta'_B = \frac{wL^3}{24EI} = \frac{120(12)^3}{24EI} = \frac{8640\ \text{lb}\cdot\text{ft}^2}{EI}$$

$$\theta''_B = \frac{PL^2}{16EI} = \frac{500(10)^2}{16EI} = \frac{3125\ \text{lb}\cdot\text{ft}^2}{EI}$$

$$\alpha'_{BB} = \frac{ML}{3EI} = \frac{1(12)}{3EI} = \frac{4\ \text{ft}}{EI}$$

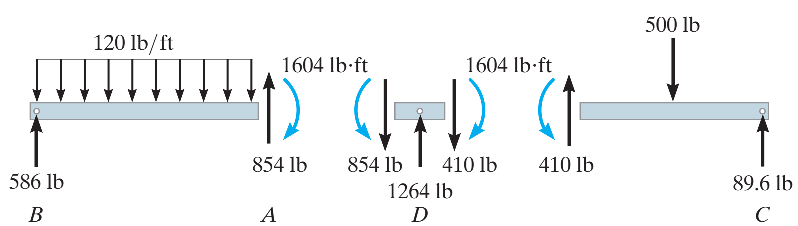
$$\alpha''_{BB} = \frac{ML}{3EI} = \frac{1(10)}{3EI} = \frac{3.33\ \text{ft}}{EI}$$

Thus

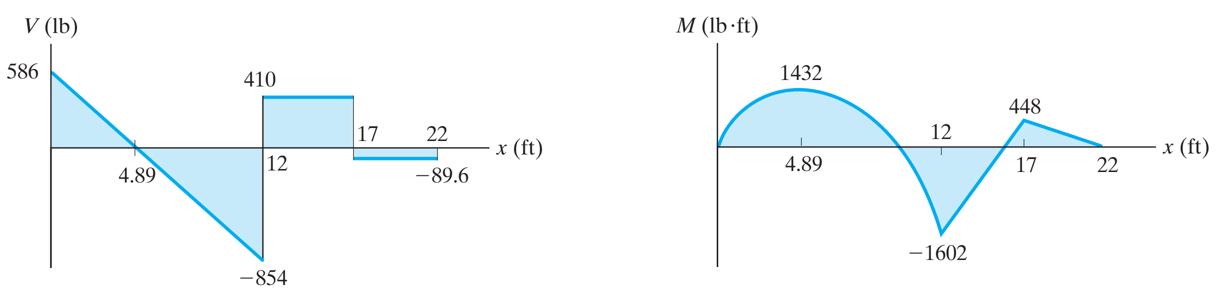
$$\frac{8640\ \text{lb}\cdot\text{ft}^2}{EI} + \frac{3125\ \text{lb}\cdot\text{ft}^2}{EI} + M_B\left(\frac{4\ \text{ft}}{EI} + \frac{3.33\ \text{ft}}{EI}\right) = 0$$

$$M_B = -1604\ \text{lb}\cdot\text{ft}$$

The negative sign indicates M_B acts in the opposite direction to that shown in Fig. 10–11*c*. Using this result, the reactions at the supports are calculated as shown in Fig. 10–11*d*. Furthermore, the shear and moment diagrams are shown in Fig. 10–11*e*.



(d)



(e)

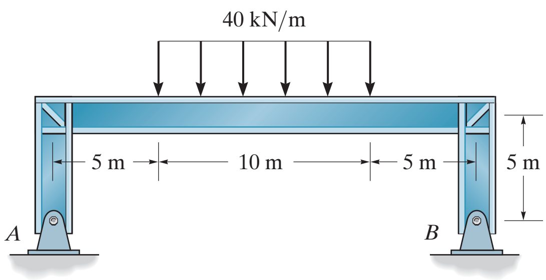
10.5 Force Method of Analysis: Frames

The force method is very useful for solving problems involving statically indeterminate frames that have a single story and unusual geometry, such as gabled frames. Problems involving multistory frames, or those with a high degree of indeterminacy, are best solved using the slope-deflection, moment-distribution, or the stiffness method discussed in later chapters.

The following examples illustrate the application of the force method using the procedure for analysis outlined in Sec. 10–2.

EXAMPLE 10.5

The frame, or bent, shown in the photo is used to support the bridge deck. Assuming EI is constant, a drawing of it along with the dimensions and loading is shown in Fig. 10–12*a*. Determine the support reactions.



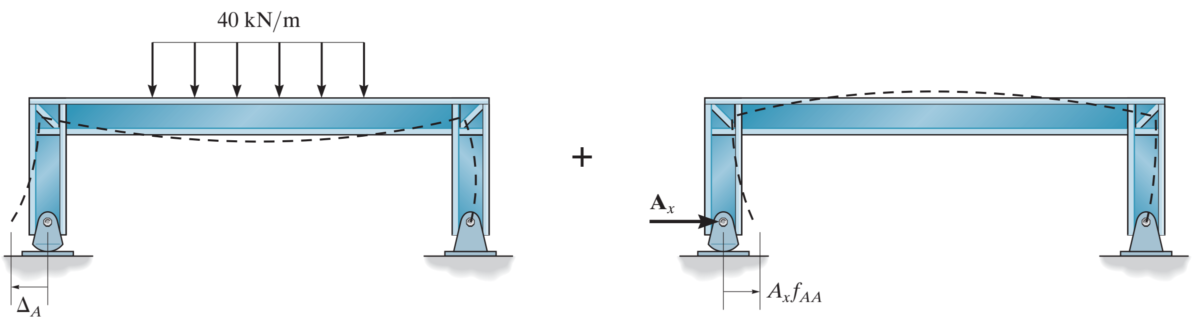
(a)

Fig. 10–12

SOLUTION

Principle of Superposition. By inspection the frame is statically indeterminate to the first degree. We will choose the horizontal reaction at A to be the redundant. Consequently, the pin at A is replaced by a rocker, since a rocker will not constrain A in the horizontal direction. The principle of superposition applied to the idealized model of the frame is shown in Fig. 10–12*b*. Notice how the frame deflects in each case.

EXAMPLE 10.5 (Continued)



Primary structure

Redundant force $\mathbf{A}_x$ applied

(b)

Compatibility Equation. Reference to point A in Fig. 10–12b requires

$$(\xrightarrow{+}) \qquad 0 = \Delta_A + A_x f_{AA} \qquad (1)$$

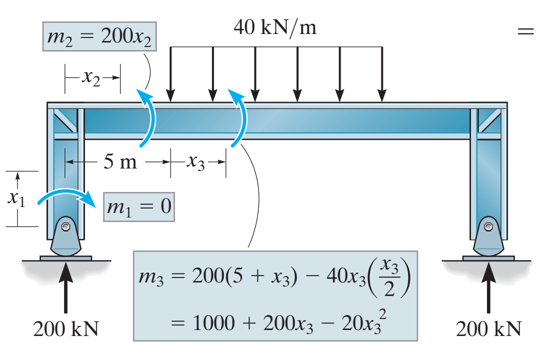
The terms Δ_A and f_{AA} will be determined using the method of virtual work. Because of symmetry of geometry *and* loading we need only three x coordinates. These and the internal moments are shown in Figs. 10–12c and 10–12d. It is important that each x coordinate be the *same* for both the real and virtual loadings. Also, the positive directions for **M** and **m** must be the *same.*

For Δ_A we require application of real loads, Fig. 10–12c, and a virtual unit load at A, Fig. 10–12d. Thus,

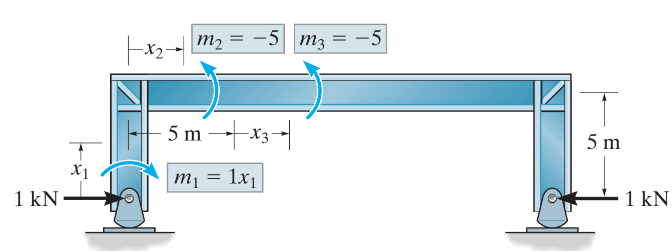
$$\Delta_A = \int_0^L \frac{Mm}{EI}dx = 2\int_0^5 \frac{(0)(1x_1)dx_1}{EI} + 2\int_0^5 \frac{(200x_2)(-5)dx_2}{EI}$$

$$+ 2\int_0^5 \frac{(1000 + 200x_3 - 20x_3^2)(-5)dx_3}{EI}$$

$$= 0 - \frac{25\,000}{EI} - \frac{66\,666.7}{EI} = -\frac{91\,666.7}{EI}$$



(c)



(d)

For f_{AA} we require application of a real unit load and a virtual unit load acting at A, Fig. 10–12d. Thus,

$$f_{AA} = \int_0^L \frac{mm}{EI}dx = 2\int_0^5 \frac{(1x_1)^2 dx_1}{EI} + 2\int_0^5 (5)^2 dx_2 + 2\int_0^5 (5)^2 dx_3$$

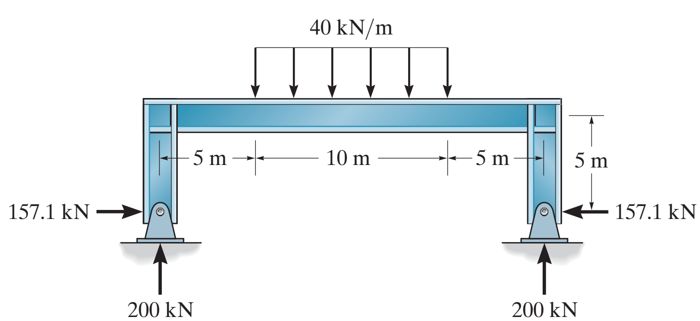
$$= \frac{583.33}{EI}$$

Substituting the results into Eq. (1) and solving yields

$$0 = \frac{-91\,666.7}{EI} + A_x\left(\frac{583.33}{EI}\right)$$

$$A_x = 157 \text{ kN} \qquad \textit{Ans.}$$

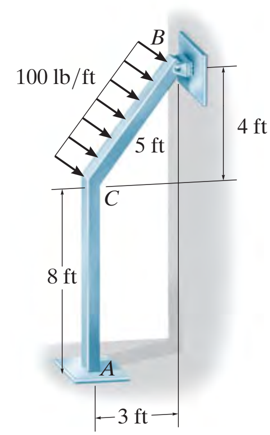
Equilibrium Equations. Using this result, the reactions on the idealized model of the frame are shown in Fig. 10–12e.



(e)

EXAMPLE 10.6

Determine the moment at the fixed support A for the frame shown in Fig. 10–13a. EI is constant.



(a)

Fig. 10–13

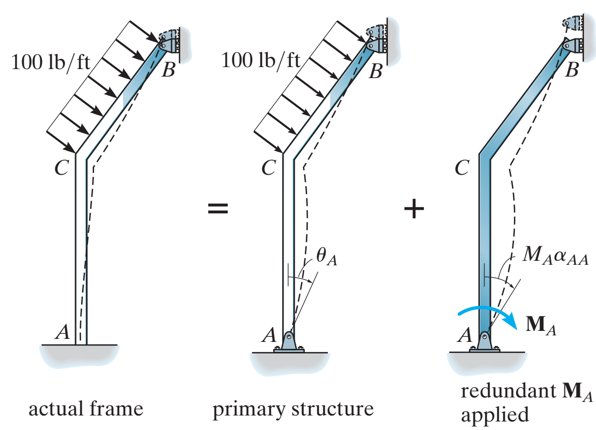
SOLUTION

Principle of Superposition. The frame is indeterminate to the first degree. A direct solution for $\mathbf{M}_A$ can be obtained by choosing this as the redundant. Thus the capacity of the frame to support a moment at A is removed and therefore a pin is used at A for support. The principle of superposition applied to the frame is shown in Fig. 10–13b.

Compatibility Equation. Reference to point A in Fig. 10–13b requires

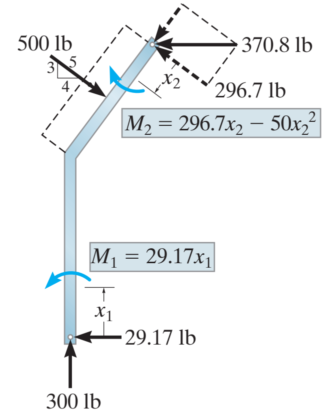
$$(\curvearrowleft +) \qquad 0 = \theta_A + M_A\alpha_{AA} \qquad (1)$$

As in the preceding example, θ_A and α_{AA} will be computed using the method of virtual work. The frame's x coordinates and internal moments are shown in Figs. 10–13c and 10–13d.



(b)

For θ_A we require application of the real loads, Fig. 10–13*c*, and a virtual unit couple moment at A, Fig. 10–13*d*. Thus,



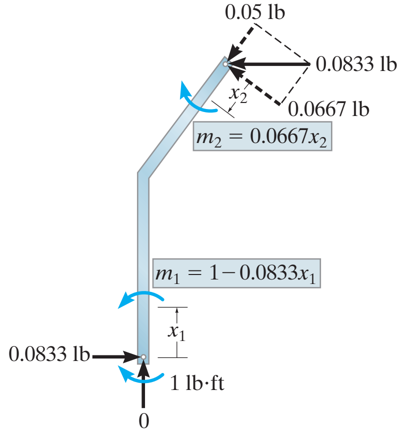
(c)

$$\theta_A = \sum \int_0^L \frac{M m_\theta \, dx}{EI}$$

$$= \int_0^8 \frac{(29.17x_1)(1 - 0.0833x_1)\, dx_1}{EI}$$

$$+ \int_0^5 \frac{\left(296.7x_2 - 50x_2^2\right)(0.0667x_2)\, dx_2}{EI}$$

$$= \frac{518.5}{EI} + \frac{303.2}{EI} = \frac{821.8}{EI}$$



(d)

For α_{AA} we require application of a real unit couple moment and a virtual unit couple moment acting at A, Fig. 10–13*d*. Thus,

$$\alpha_{AA} = \sum \int_0^L \frac{m_\theta m_\theta}{EI} dx$$

$$= \int_0^8 \frac{(1 - 0.0833x_1)^2\, dx_1}{EI} + \int_0^5 \frac{(0.0667x_2)^2\, dx_2}{EI}$$

$$= \frac{3.85}{EI} + \frac{0.185}{EI} = \frac{4.04}{EI}$$

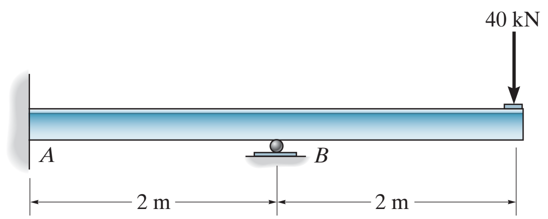
Substituting these results into Eq. (1) and solving yields

$$0 = \frac{821.8}{EI} + M_A\left(\frac{4.04}{EI}\right) \qquad M_A = -204 \text{ lb} \cdot \text{ft} \qquad \textit{Ans.}$$

The negative sign indicates $\mathbf{M}_A$ acts in the opposite direction to that shown in Fig. 10–13*b*.

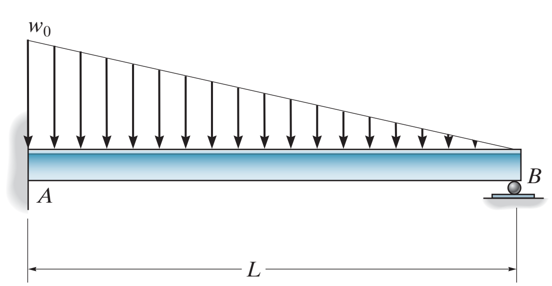
FUNDAMENTAL PROBLEMS

F10–1. Determine the reactions at the fixed support at A and the roller at B. EI is constant.



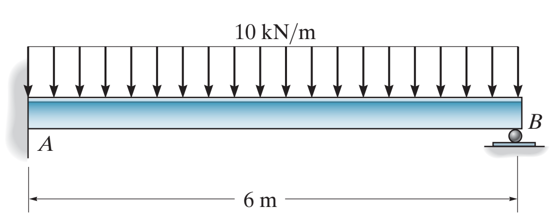
F10–1

F10–2. Determine the reactions at the fixed supports at A and the roller at B. EI is constant.



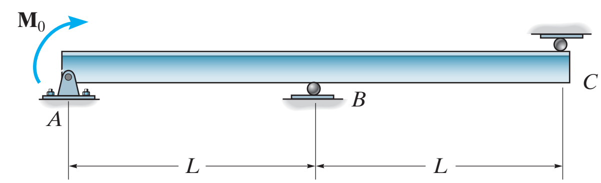
F10–2

F10–3. Determine the reactions at the fixed support at A and the roller at B. Support B settles 5 mm. Take $E = 200$ GPa and $I = 300(10^6)$ mm^4.



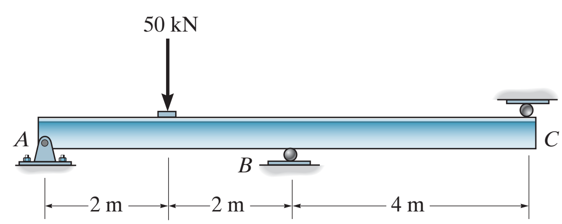
F10–3

F10–4. Determine the reactions at the pin at A and the rollers at B and C.



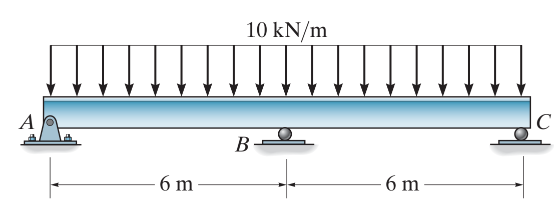
F10–4

F10–5. Determine the reactions at the pin A and the rollers at B and C on the beam. EI is constant.



F10–5

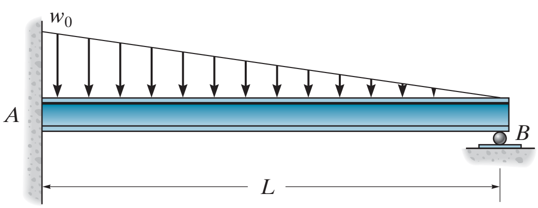
F10–6. Determine the reactions at the pin at A and the rollers at B and C on the beam. Support B settles 5 mm. Take $E = 200$ GPa, $I = 300(10^6)$ mm^4.



F10–6

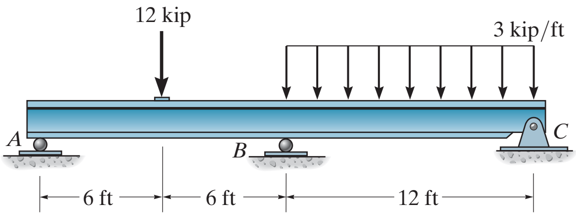
PROBLEMS

10–1. Determine the reactions at the supports A and B. EI is constant.



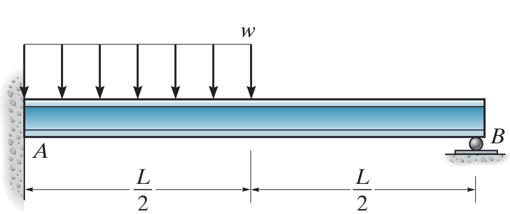
Prob. 10–1

10–2. Determine the reactions at the supports A, B, and C, then draw the shear and moment diagrams. EI is constant.



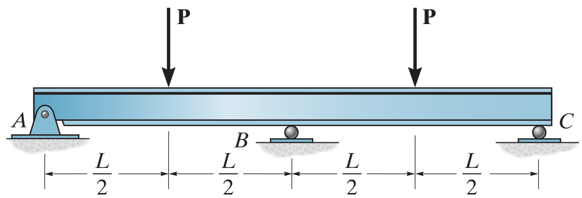
Prob. 10–2

10–3. Determine the reactions at the supports A and B. EI is constant.



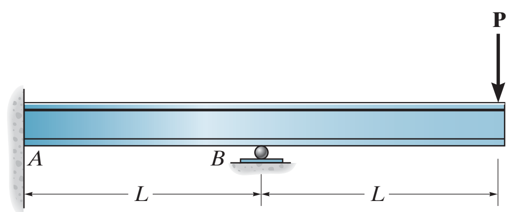
Prob. 10–3

***10–4.** Determine the reactions at the supports A, B, and C; then draw the shear and moment diagram. EI is constant.



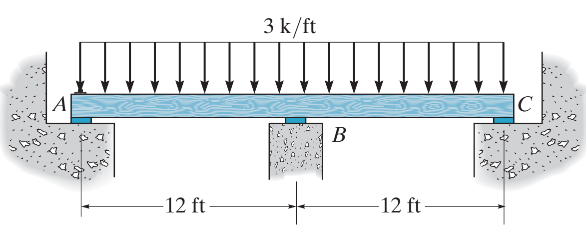
Prob. 10–4

10–5. Determine the reactions at the supports, then draw the shear and moment diagram. EI is constant.



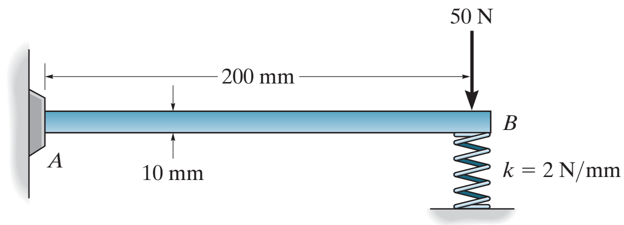
Prob. 10–5

10–6. Determine the reactions at the supports, then draw the moment diagram. Assume B and C are rollers and A is pinned. The support at B settles downward 0.25 ft. Take $E = 29(10^3)$ ksi, $I = 500\text{ in}^4$.



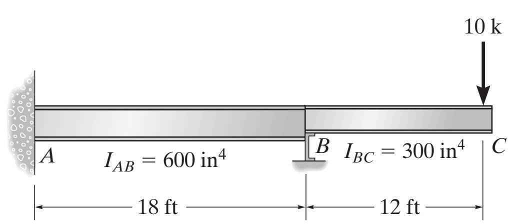
Prob. 10–6

10–7. Determine the deflection at the end B of the clamped A-36 steel strip. The spring has a stiffness of $k = 2$ N/mm. The strip is 5 mm wide and 10 mm high. Also, draw the shear and moment diagrams for the strip.



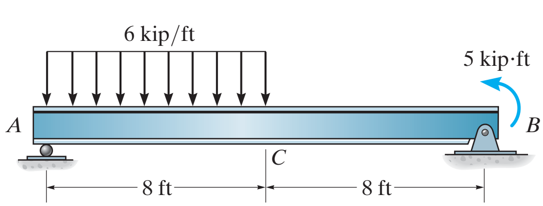
Prob. 10–7

***10–8.** Determine the reactions at the supports. The moment of inertia for each segment is shown in the figure. Assume the support at B is a roller. Take $E = 29(10^3)$ ksi.



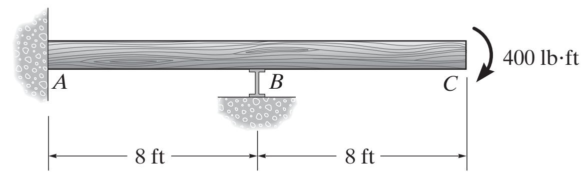
Prob. 10–8

10–9. The simply supported beam is subjected to the loading shown. Determine the deflection at its center C. EI is constant.



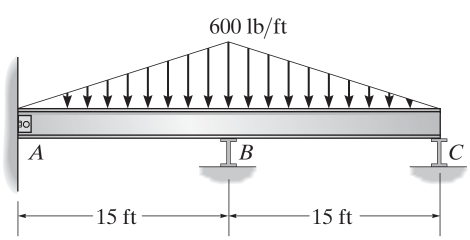
Prob. 10–9

10–10. Determine the reactions at the supports, then draw the moment diagram. Assume the support at B is a roller. EI is constant.



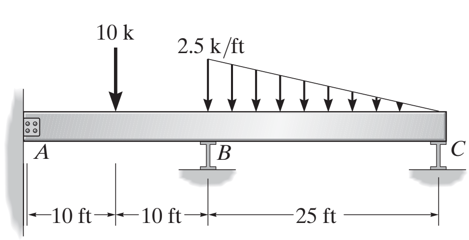
Prob. 10–10

10–11. Determine the reactions at the supports, then draw the moment diagram. Assume A is a pin and B and C are rollers. EI is constant.



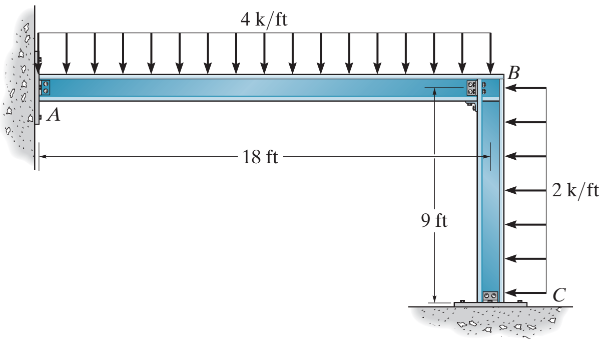
Prob. 10–11

***10–12.** Determine the reactions at the supports, then draw the moment diagram. Assume the support at A is a pin and B and C are rollers. EI is constant.



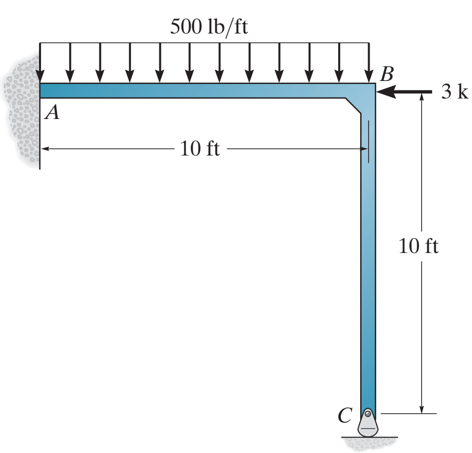
Prob. 10–12

10–13. Determine the reactions at the supports. Assume A and C are pins and the joint at B is fixed connected. EI is constant.



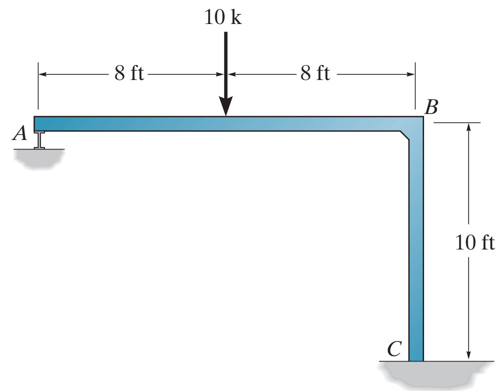
Prob. 10–13

10–14. Determine the reactions at the supports. EI is constant.



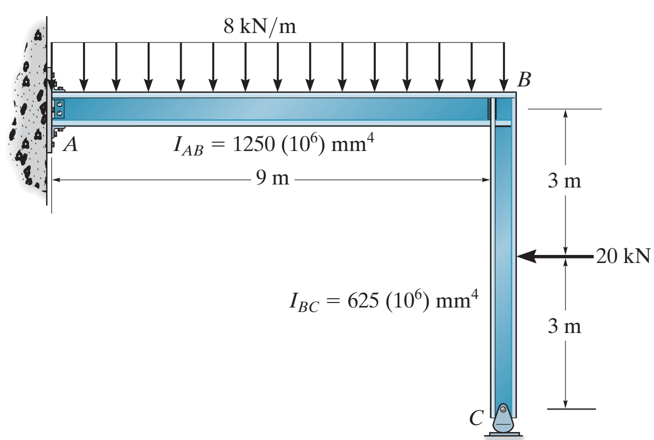
Prob. 10–14

10–15. Determine the reactions at the supports, then draw the moment diagram for each member. EI is constant.



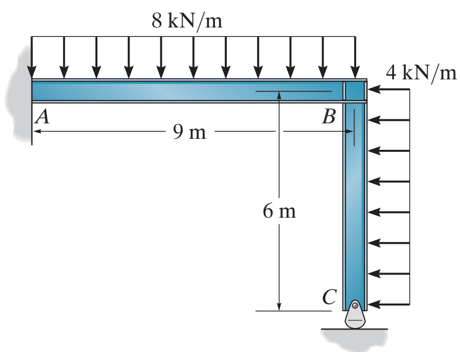
Prob. 10–15

***10–16.** Determine the reactions at the supports. Assume A is fixed connected. E is constant.



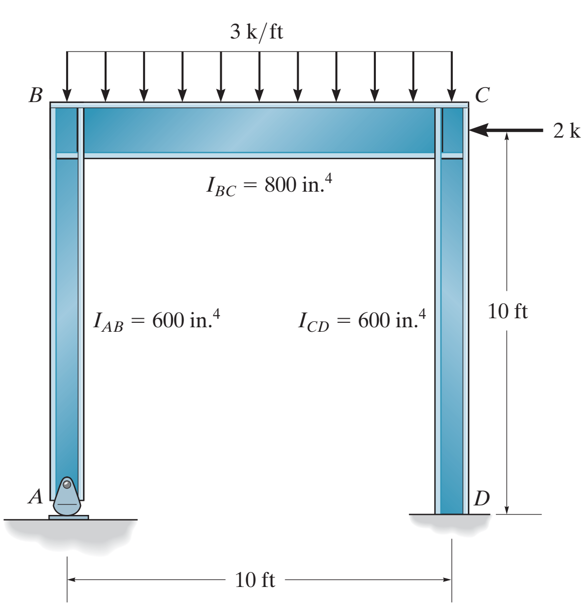
Prob. 10–16

10–17. Determine the reactions at the supports. EI is constant.



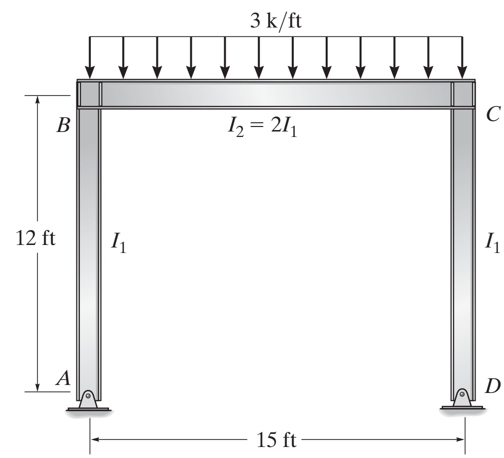
Prob. 10–17

10–18. Determine the reactions at the supports A and D. The moment of inertia of each segment of the frame is listed in the figure. Take $E = 29(10^3)$ ksi.



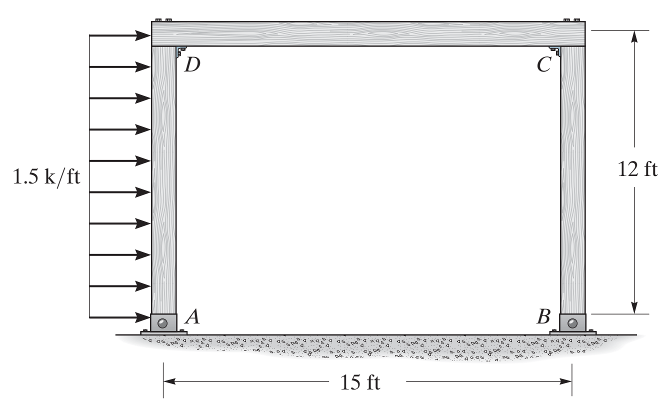
Prob. 10–18

10–19. The steel frame supports the loading shown. Determine the horizontal and vertical components of reaction at the supports A and D. Draw the moment diagram for the frame members. E is constant.



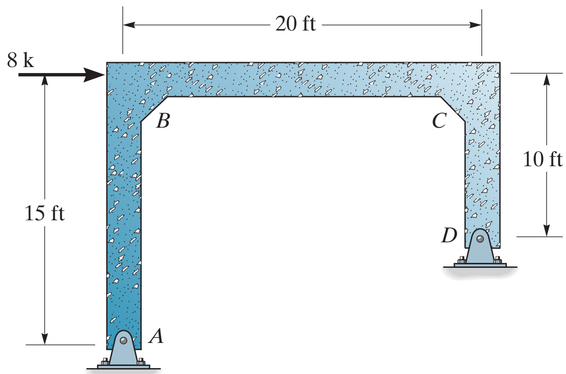
Prob. 10–19

***10–20.** Determine the reactions at the supports. Assume A and B are pins and the joints at C and D are fixed connections. EI is constant.



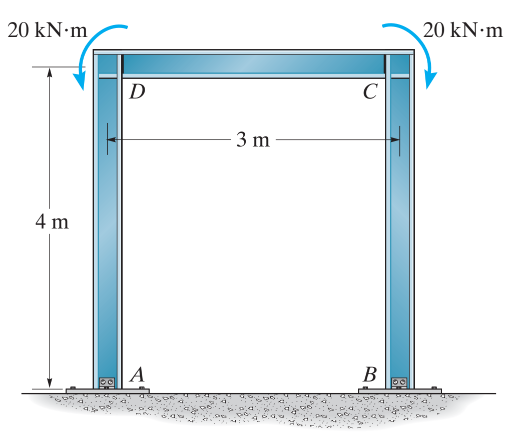
Prob. 10–20

10–21. Determine the reactions at the supports. Assume *A* and *D* are pins. *EI* is constant.



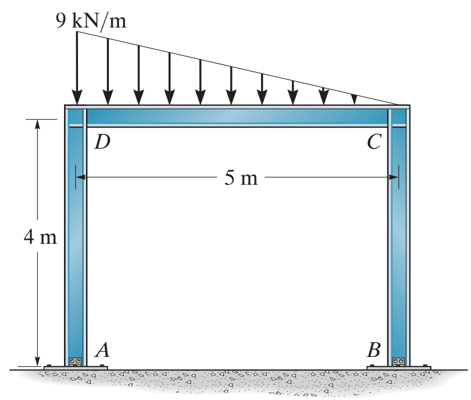
Prob. 10–21

10–22. Determine the reactions at the supports. Assume *A* and *B* are pins. *EI* is constant.



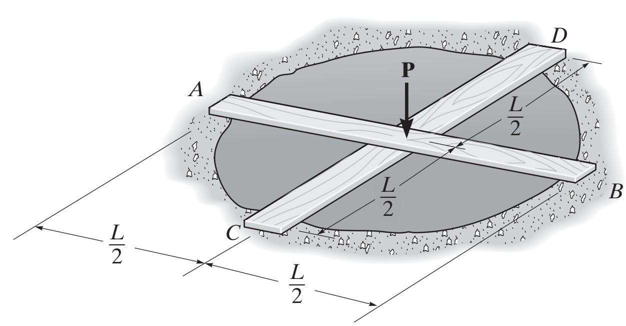
Prob. 10–22

10–23. Determine the reactions at the supports. Assume *A* and *B* are pins. *EI* is constant.



Prob. 10–23

***10–24.** Two boards each having the same *EI* and length *L* are crossed perpendicular to each other as shown. Determine the vertical reactions at the supports. Assume the boards just touch each other before the load **P** is applied.



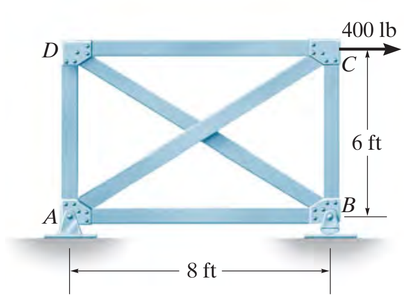
Prob. 10–24

10.6 Force Method of Analysis: Trusses

The degree of indeterminacy of a truss can usually be determined by inspection; however, if this becomes difficult, use Eq. 3–1, $b + r > 2j$. Here the unknowns are represented by the number of bar forces (b) plus the support reactions (r), and the number of available equilibrium equations is $2j$ since two equations can be written for each of the (j) joints.

The force method is quite suitable for analyzing trusses that are statically indeterminate to the first or second degree. The following examples illustrate application of this method using the procedure for analysis outlined in Sec. 10–2.

EXAMPLE 10.7



(a)

Fig. 10–14

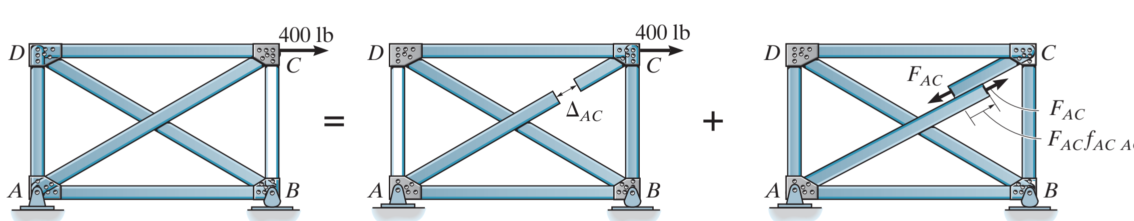
Determine the force in member AC of the truss shown in Fig. 10–14a. AE is the same for all the members.

SOLUTION

Principle of Superposition. By inspection the truss is indeterminate to the first degree.* Since the force in member AC is to be determined, member AC will be chosen as the redundant. This requires "cutting" this member so that it cannot sustain a force, thereby making the truss statically determinate and stable. The principle of superposition applied to the truss is shown in Fig. 10–14b.

Compatibility Equation. With reference to member AC in Fig. 10–14b, we require the relative displacement Δ_{AC}, which occurs at the ends of the cut member AC due to the 400-lb load, plus the relative displacement $F_{AC}f_{AC\,AC}$ caused by the redundant force acting alone, to be equal to zero, that is,

$$0 = \Delta_{AC} + F_{AC}f_{AC\,AC} \tag{1}$$



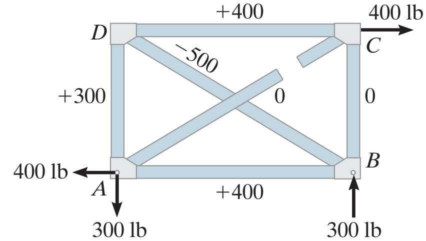
actual truss   primary structure   redundant $\mathbf{F}_{AC}$ applied

(b)

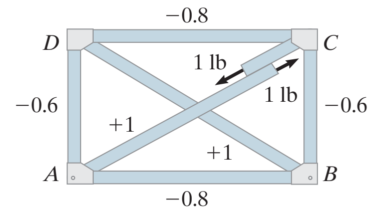
*Applying Eq. 3–1, $b + r > 2j$ or $6 + 3 > 2(4)$, $9 > 8$, $9 - 8 = $ 1st degree.

Here the flexibility coefficient $f_{AC\,AC}$ represents the relative displacement of the cut ends of member AC caused by a "real" unit load acting at the cut ends of member AC. This term, $f_{AC\,AC}$, and Δ_{AC} will be computed using the method of virtual work. The force analysis, using the method of joints, is summarized in Fig. 10–14*c* and 10–14*d*.

For Δ_{AC} we require application of the real load of 400 lb, Fig. 10–14*c*, and a virtual unit force acting at the cut ends of member AC, Fig. 10–14*d*. Thus,



(c)



(d)

$$\Delta_{AC} = \sum \frac{nNL}{AE}$$

$$= 2\left[\frac{(-0.8)(400)(8)}{AE}\right] + \frac{(-0.6)(0)(6)}{AE} + \frac{(-0.6)(300)(6)}{AE}$$

$$+ \frac{(1)(-500)(10)}{AE} + \frac{(1)(0)(10)}{AE}$$

$$= -\frac{11\,200}{AE}$$

For $f_{AC\,AC}$ we require application of real unit forces and virtual unit forces acting on the cut ends of member AC, Fig. 10–14*d*. Thus,

$$f_{AC\,AC} = \sum \frac{n^2 L}{AE}$$

$$= 2\left[\frac{(-0.8)^2(8)}{AE}\right] + 2\left[\frac{(-0.6)^2(6)}{AE}\right] + 2\left[\frac{(1)^2 10}{AE}\right]$$

$$= \frac{34.56}{AE}$$

Substituting the data into Eq. (1) and solving yields

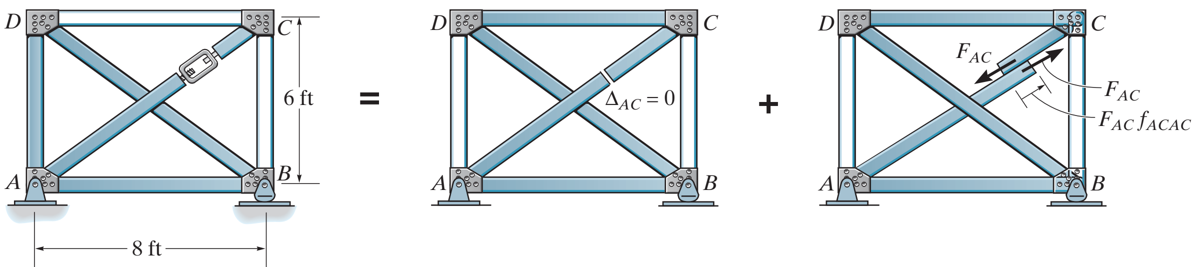
$$0 = -\frac{11\,200}{AE} + \frac{34.56}{AE} F_{AC}$$

$$F_{AC} = 324 \text{ lb (T)} \qquad \textit{Ans.}$$

Since the numerical result is positive, AC is subjected to tension as assumed, Fig. 10–14b. Using this result, the forces in the other members can be found by equilibrium, using the method of joints.

EXAMPLE 10.8

Determine the force in each member of the truss shown in Fig. 10–15*a* if the turnbuckle on member AC is used to shorten the member by 0.5 in. Each bar has a cross-sectional area of 0.2 in^2, and $E = 29(10^6)$ psi.



actual truss

(a)

primary structure

redundant $\mathbf{F}_{AC}$ applied

(b)

Fig. 10–15

SOLUTION

Principle of Superposition. This truss has the same geometry as that in Example 10–7. Since AC has been shortened, we will choose it as the redundant, Fig. 10–15*b*.

Compatibility Equation. Since no external loads act on the primary structure (truss), there will be no relative displacement between the ends of the sectioned member caused by load; that is, $\Delta_{AC} = 0$. The flexibility coefficient $f_{AC\,AC}$ has been determined in Example 10–7, so

$$f_{AC\,AC} = \frac{34.56}{AE}$$

Assuming the amount by which the bar is shortened is positive, the compatibility equation for the bar is therefore

$$0.5 \text{ in.} = 0 + \frac{34.56}{AE} F_{AC}$$

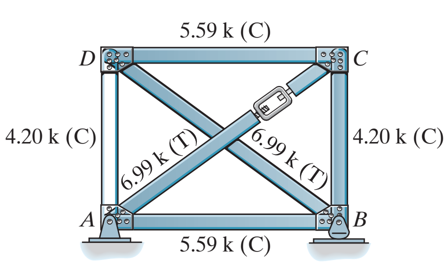
Realizing that $f_{AC\,AC}$ is a measure of displacement per unit force, we have

$$0.5 \text{ in.} = 0 + \frac{34.56 \text{ ft}(12 \text{ in./ft})}{(0.2 \text{ in}^2)[29(10^6) \text{ lb/in}^2]} F_{AC}$$

Thus,

$$F_{AC} = 6993 \text{ lb} = 6.99 \text{ k (T)} \qquad \textit{Ans.}$$

Since no external forces act on the truss, the external reactions are zero. Therefore, using F_{AC} and analyzing the truss by the method of joints yields the results shown in Fig. 10–15*c*.



(c)

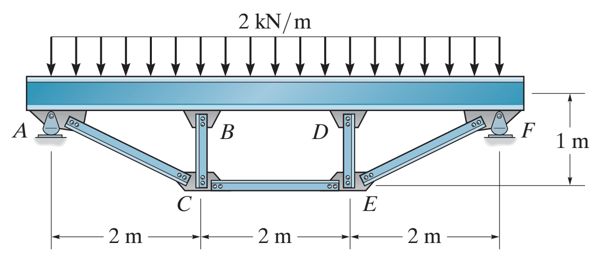
10

10.7 Composite Structures

Composite structures are composed of some members subjected only to axial force, while other members are subjected to bending. If the structure is statically indeterminate, the force method can conveniently be used for its analysis. The following example illustrates the procedure.

EXAMPLE 10.9

The simply supported queen-post trussed beam shown in the photo is to be designed to support a uniform load of 2 kN/m. The dimensions of the structure are shown in Fig. 10–16*a*. Determine the force developed in member *CE*. Neglect the thickness of the beam and assume the truss members are pin connected to the beam. Also, neglect the effect of axial compression and shear in the beam. The cross-sectional area of each strut is 400 mm^2, and for the beam $I = 20(10^6)\text{ mm}^4$. Take $E = 200\text{ GPa}$.

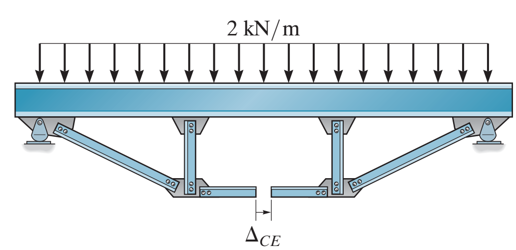


Actual structure

(a)

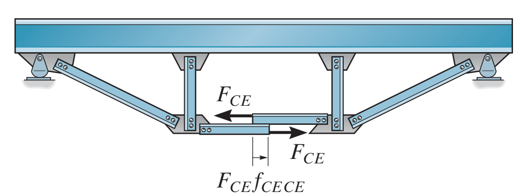
Fig. 10–16

=



Primary structure

+



Redundant $\mathbf{F}_{CE}$ applied

(b)

SOLUTION

Principle of Superposition. If the force in one of the truss members is known, then the force in all the other members, as well as in the beam, can be determined by statics. Hence, the structure is indeterminate to the first degree. For solution the force in member *CE* is chosen as the redundant. This member is therefore sectioned to eliminate its capacity to sustain a force. The principle of superposition applied to the structure is shown in Fig. 10–16*b*.

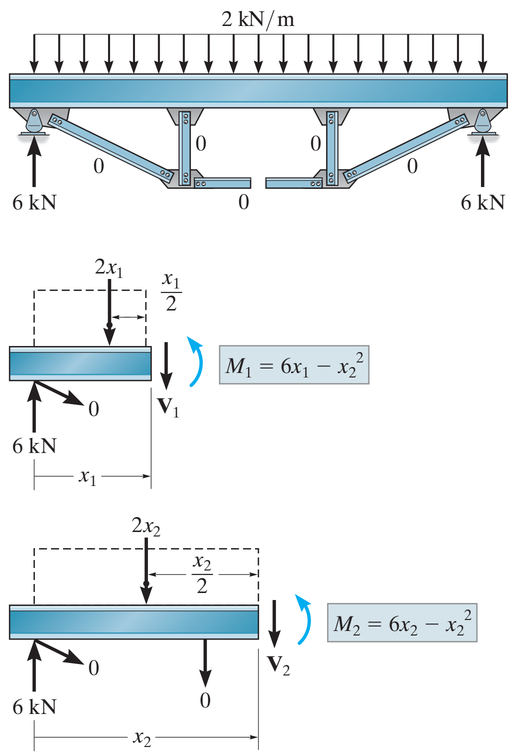
Compatibility Equation. With reference to the relative displacement of the cut ends of member *CE*, Fig. 10–16*b*, we require

$$0 = \Delta_{CE} + F_{CE} f_{CE\,CE} \tag{1}$$

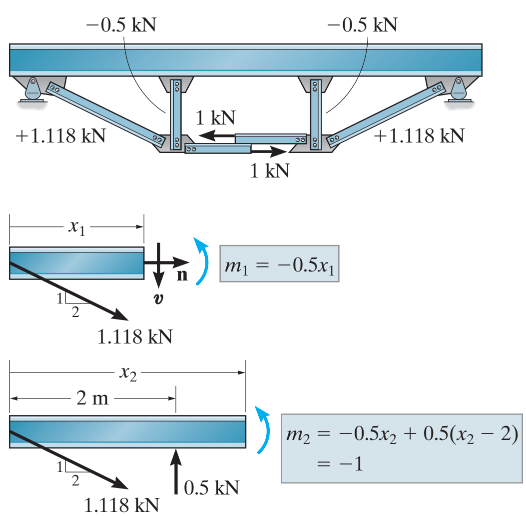
10

EXAMPLE 10.9 (Continued)

The method of virtual work will be used to find Δ_{CE} and $f_{CE\,CE}$. The necessary force analysis is shown in Figs. 10–16*c* and 10–16*d*.



(c)



(d)

For Δ_{CE} we require application of the real loads, Fig. 10–16*c*, and a virtual unit load applied to the cut ends of member *CE*, Fig. 10–16*d*. Here we will use symmetry of *both* loading and geometry, and only consider the bending strain energy in the beam and, of course, the axial strain energy in the truss members. Thus,

$$\Delta_{CE} = \int_0^L \frac{Mm}{EI}dx + \sum \frac{nNL}{AE} = 2\int_0^2 \frac{(6x_1 - x_1^2)(-0.5x_1)dx_1}{EI}$$

$$+ 2\int_2^3 \frac{(6x_2 - x_2^2)(-1)dx_2}{EI} + 2\left(\frac{(1.118)(0)(\sqrt{5})}{AE}\right)$$

$$+ 2\left(\frac{(-0.5)(0)(1)}{AE}\right) + \left(\frac{1(0)2}{AE}\right)$$

$$= -\frac{12}{EI} - \frac{17.33}{EI} + 0 + 0 + 0$$

$$= \frac{-29.33(10^3)}{200(10^9)(20)(10^{-6})} = -7.333(10^{-3})\text{ m}$$

For $f_{CE\,CE}$ we require application of a real unit load and a virtual unit load at the cut ends of member *CE*, Fig. 10–16*d*. Thus,

$$f_{CE\,CE} = \int_0^L \frac{m^2dx}{EI} + \sum \frac{n^2L}{AE} = 2\int_0^2 \frac{(-0.5x_1)^2dx_1}{EI} + 2\int_2^3 \frac{(-1)^2dx_2}{EI}$$

$$+ 2\left(\frac{(1.118)^2(\sqrt{5})}{AE}\right) + 2\left(\frac{(-0.5)^2(1)}{AE}\right) + \left(\frac{(1)^2(2)}{AE}\right)$$

$$= \frac{1.3333}{EI} + \frac{2}{EI} + \frac{5.590}{AE} + \frac{0.5}{AE} + \frac{2}{AE}$$

$$= \frac{3.333(10^3)}{200(10^9)(20)(10^{-6})} + \frac{8.090(10^3)}{400(10^{-6})(200(10^9))}$$

$$= 0.9345(10^{-3})\text{ m/kN}$$

Substituting the data into Eq. (1) yields

$$0 = -7.333(10^{-3})\text{ m} + F_{CE}(0.9345(10^{-3})\text{ m/kN})$$

$$F_{CE} = 7.85\text{ kN}$$ *Ans.*

10

10.8 Additional Remarks on the Force Method of Analysis

Now that the basic ideas regarding the force method have been developed, we will proceed to generalize its application and discuss its usefulness.

When computing the flexibility coefficients, f_{ij} (or α_{ij}), for the structure, it will be noticed that they depend only on the material and geometrical properties of the members and *not* on the loading of the primary structure. Hence these values, once determined, can be used to compute the reactions for any loading.

For a structure having n redundant reactions, $\mathbf{R}_n$, we can write n compatibility equations, namely:

$$\begin{aligned}
\Delta_1 + f_{11}R_1 + f_{12}R_2 + \cdots + f_{1n}R_n &= 0\\
\Delta_2 + f_{21}R_1 + f_{22}R_2 + \cdots + f_{2n}R_n &= 0\\
&\vdots\\
\Delta_n + f_{n1}R_1 + f_{n2}R_2 + \cdots + f_{nn}R_n &= 0
\end{aligned}$$

Here the displacements, $\Delta_1, \ldots, \Delta_n$, are caused by *both* the *real loads* on the primary structure and by *support settlement* or *dimensional changes* due to temperature differences or fabrication errors in the members. To simplify computation for structures having a large degree of indeterminacy, the above equations can be recast into a matrix form,

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n}\\ f_{21} & f_{22} & \cdots & f_{2n}\\ & & \vdots & \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} R_1\\ R_2\\ \\ R_n \end{bmatrix} = -\begin{bmatrix} \Delta_1\\ \Delta_2\\ \vdots\\ \Delta_n \end{bmatrix} \qquad (10\text{–}2)$$

or simply

$$\mathbf{fR} = -\boldsymbol{\Delta}$$

In particular, note that $f_{ij} = f_{ji}(f_{12} = f_{21}$, etc.), a consequence of Maxwell's theorem of reciprocal displacements (or Betti's law). Hence the *flexibility matrix* will be *symmetric,* and this feature is beneficial when solving large sets of linear equations, as in the case of a highly indeterminate structure.

Throughout this chapter we have determined the flexibility coefficients using the method of virtual work as it applies to the *entire structure*. It is possible, however, to obtain these coefficients for *each member* of the structure, and then, using transformation equations, to obtain their values for the entire structure. This approach is covered in books devoted to matrix analysis of structures, and will not be covered in this text.*

*See, for example, H. C. Martin, *Introduction to Matrix Methods of Structural Analysis,* McGraw-Hill, New York.

Although the details for applying the force method of analysis using computer methods will also be omitted here, we can make some general observations and comments that apply when using this method to solve problems that are highly indeterminate and thus involve large sets of equations. In this regard, numerical accuracy for the solution is improved if the flexibility coefficients located near the main diagonal of the **f** matrix are larger than those located off the diagonal. To achieve this, some thought should be given to selection of the primary structure. To facilitate computations of $\mathbf{f}_{ij}$, it is also desirable to choose the primary structure so that it is somewhat symmetric. This will tend to yield some flexibility coefficients that are similar or may be zero. Lastly, the deflected shape of the primary structure should be *similar* to that of the actual structure. If this occurs, then the redundants will induce only *small* corrections to the primary structure, which results in a more accurate solution of Eq. 10–2.

10.9 Symmetric Structures

A structural analysis of any highly indeterminate structure, or for that matter, even a statically determinate structure, can be simplified provided the designer or analyst can recognize those structures that are symmetric and support either symmetric or antisymmetric loadings. In a general sense, a structure can be classified as being *symmetric* provided half of it develops the same internal loadings and deflections as its mirror image reflected about its central axis. Normally symmetry requires the material composition, geometry, supports, and loading to be the same on each side of the structure. However, this does not always have to be the case. Notice that for horizontal stability a pin is required to support the beam and truss in Figs. 10–17*a* and 10–17*b*. Here the horizontal reaction at the pin is zero, and so both of these structures will deflect and produce the same internal loading as their reflected counterpart. As a result, they can be classified as being symmetric. Realize that this would not be the case for the frame, Figs. 10–17*c*, if the fixed support at *A* was replaced by a pin, since then the deflected shape and internal loadings would not be the same on its left and right sides.

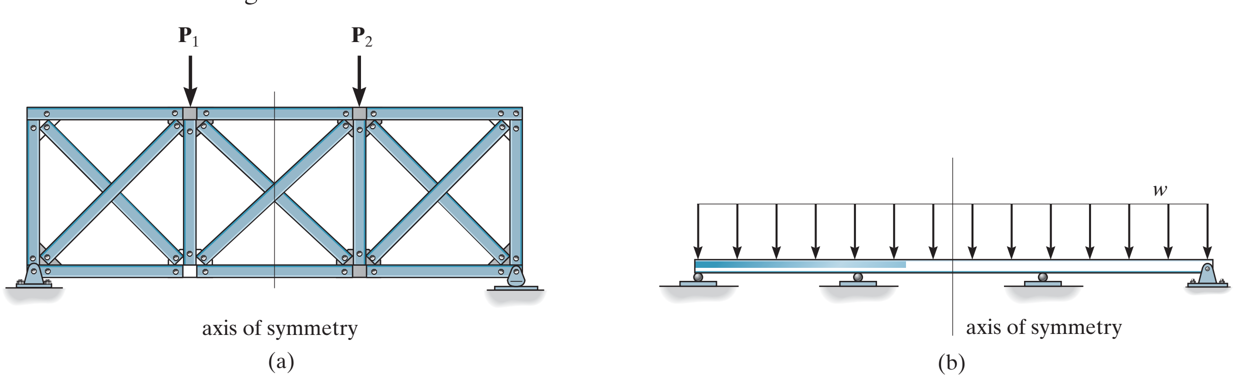
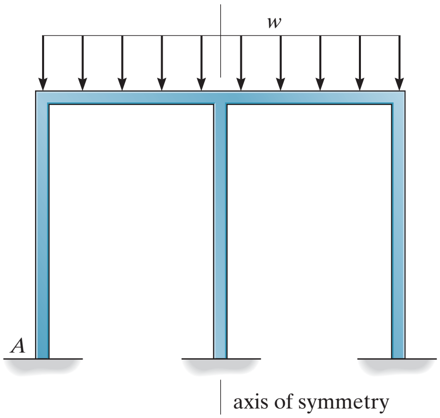


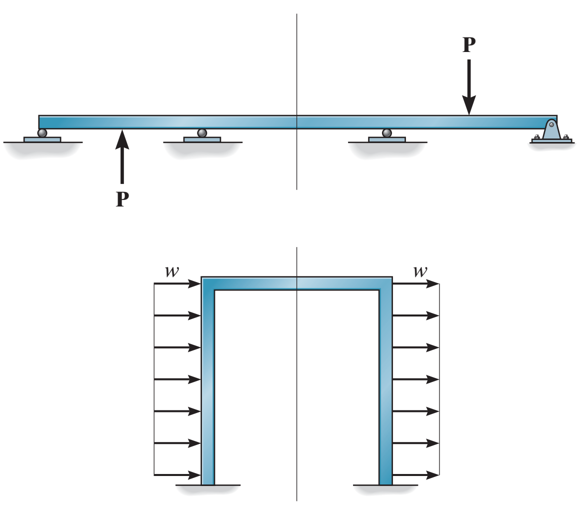
Fig. 10–17



(c)

Fig. 10–17

Sometimes a symmetric structure supports an antisymmetric loading, that is, the loading on its reflected side has the opposite direction, such as shown by the two examples in Fig. 10–18. Provided the structure is symmetric and its loading is either symmetric or antisymmetric, then a structural analysis will only have to be performed on half the members of the structure since the same (symmetric) or opposite (antisymmetric) results will be produced on the other half. If a structure is symmetric and its applied loading is unsymmetrical, then it is possible to transform this loading into symmetric and antisymmetric components. To do this, *the loading is first divided in half, then it is reflected to the other side of the structure and both symmetric and antisymmetric components are produced*. For example, the loading on the beam in Fig. 10–19*a* is divided by two and reflected about the beam's axis of symmetry. From this, the symmetric and antisymmetric components of the load are produced as shown in Fig. 10–19*b*. When added together these components produce the original loading. A separate structural analysis can now be performed using the symmetric and antisymmetric loading components and the results superimposed to obtain the actual behavior of the structure.



Antisymmetric loading

Fig. 10–18

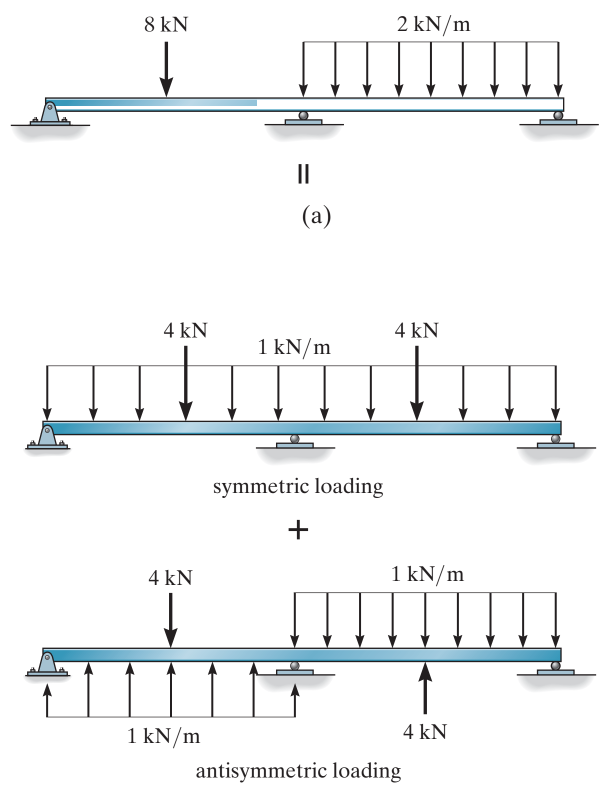
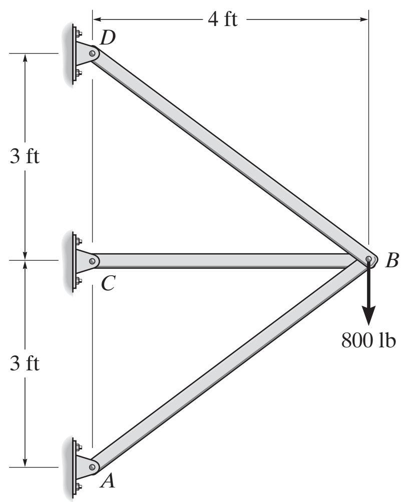


Fig. 10–19

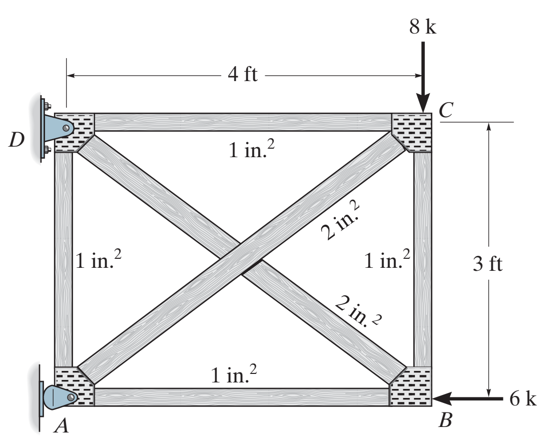
PROBLEMS

10–25. Determine the force in each member of the truss. AE is constant.



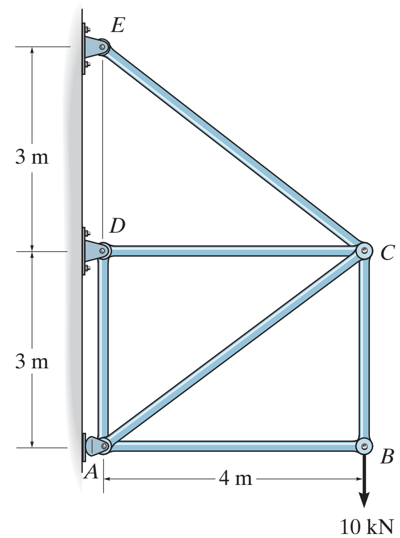
Prob. 10–25

10–26. Determine the force in each member of the truss. The cross-sectional area of each member is indicated in the figure. $E = 29(10^3)$ ksi. Assume the members are pin connected at their ends.



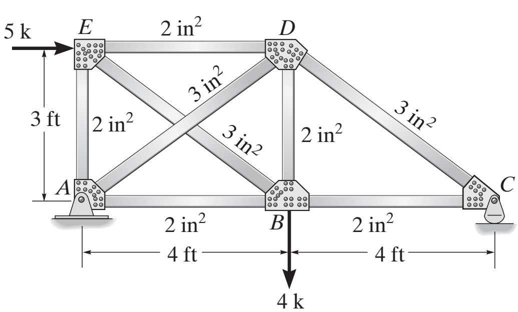
Prob. 10–26

10–27. Determine the force in member AC of the truss. AE is constant.



Prob. 10–27

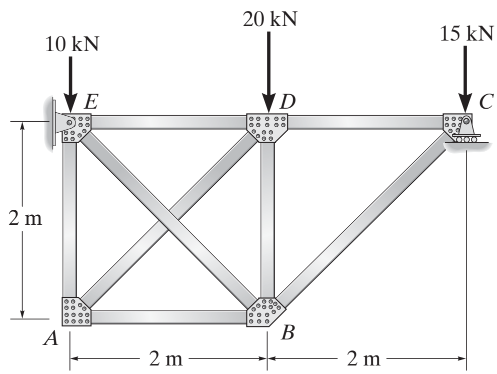
***10–28.** Determine the force in member AD of the truss. The cross-sectional area of each member is shown in the figure. Assume the members are pin connected at their ends. Take $E = 29(10^3)$ ksi.



Prob. 10–28

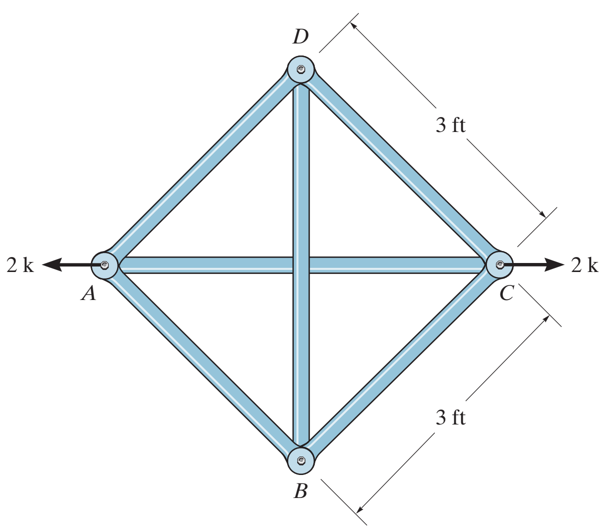
10

10–29. Determine the force in each member of the truss. Assume the members are pin connected at their ends. AE is constant.



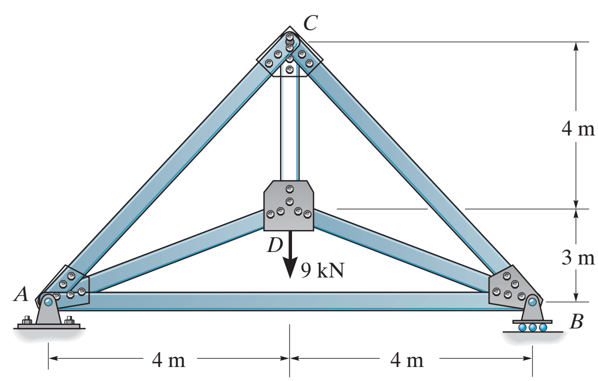
Prob. 10–29

10–30. Determine the force in each member of the pin-connected truss. AE is constant.



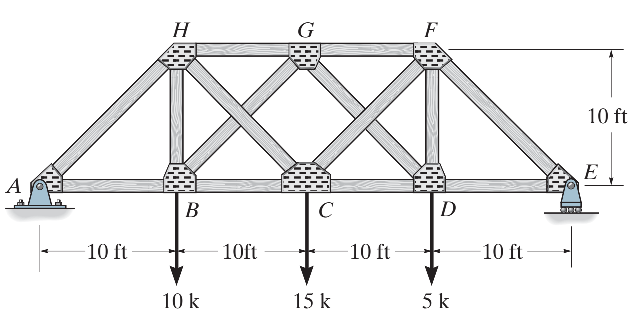
Prob. 10–30

10–31. Determine the force in member CD of the truss. AE is constant.



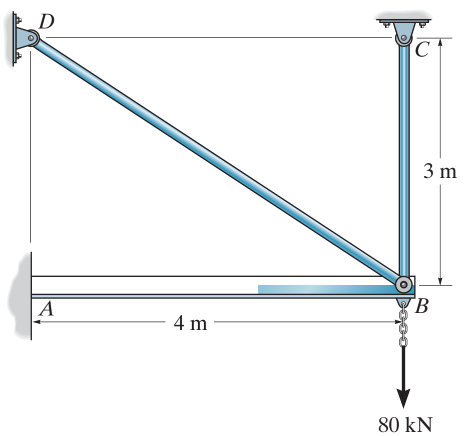
Prob. 10–31

***10–32.** Determine the force in member GB of the truss. AE is constant.



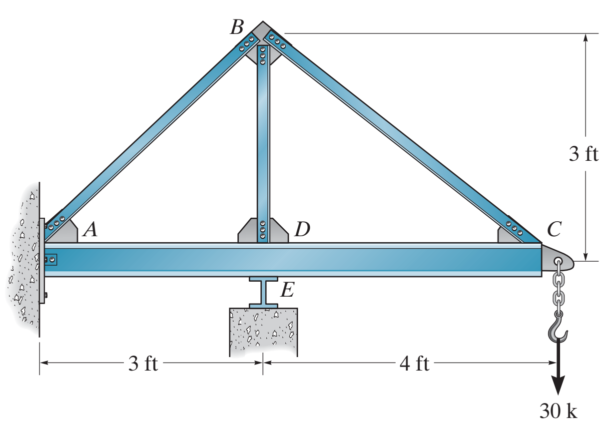
Prob. 10–32

10–33. The cantilevered beam AB is additionally supported using two tie rods. Determine the force in each of these rods. Neglect axial compression and shear in the beam. For the beam, $I_b = 200(10^6)\ \text{mm}^4$, and for each tie rod, $A = 100\ \text{mm}^2$. Take $E = 200$ GPa.



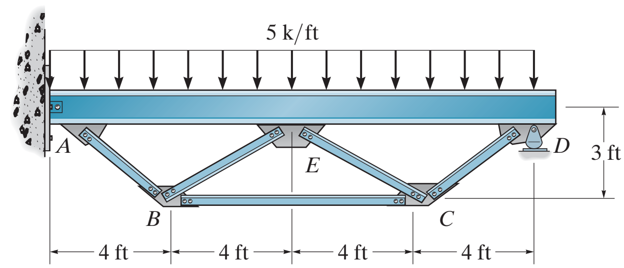
Prob. 10–33

10–34. Determine the force in member AB, BC and BD which is used in conjunction with the beam to carry the 30-k load. The beam has a moment of inertia of $I = 600\ \text{in}^4$, the members AB and BC each have a cross-sectional area of 2 in^2, and BD has a cross-sectional area of 4 in^2. Take $E = 29(10^3)$ ksi. Neglect the thickness of the beam and its axial compression, and assume all members are pin connected. Also assume the support at A is a pin and E is a roller.



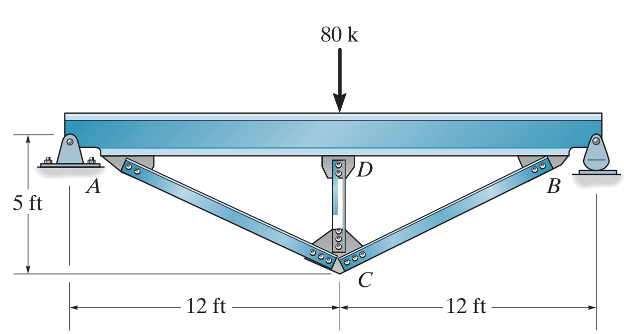
Prob. 10–34

10–35. The trussed beam supports the uniform distributed loading. If all the truss members have a cross-sectional area of 1.25 in^2, determine the force in member BC. Neglect both the depth and axial compression in the beam. Take $E = 29(10^3)$ ksi for all members. Also, for the beam $I_{AD} = 750\ \text{in}^4$. Assume A is a pin and D is a rocker.



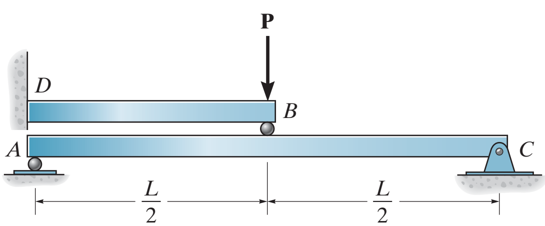
Prob. 10–35

***10–36.** The trussed beam supports a concentrated force of 80 k at its center. Determine the force in each of the three struts and draw the bending-moment diagram for the beam. The struts each have a cross-sectional area of 2 in^2. Assume they are pin connected at their end points. Neglect both the depth of the beam and the effect of axial compression in the beam. Take $E = 29(10^3)$ ksi for both the beam and struts. Also, for the beam $I = 400\ \text{in}^4$.



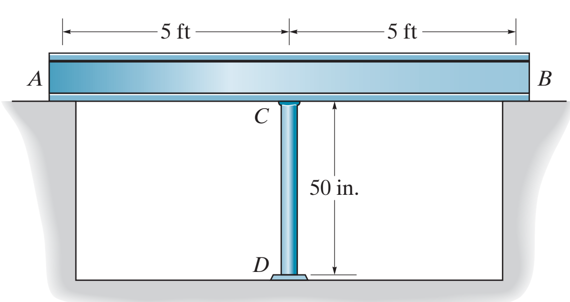
Prob. 10–36

10–37. Determine the reactions at support C. EI is constant for both beams.



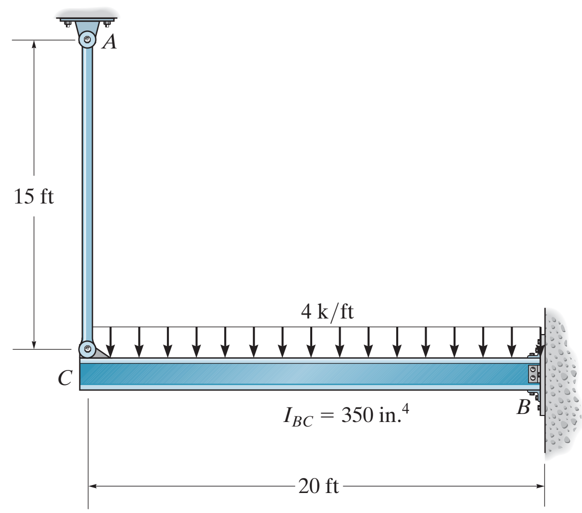
Prob. 10–37

10–38. The beam AB has a moment of inertia $I = 475\,\text{in}^4$ and rests on the smooth supports at its ends. A 0.75-in.-diameter rod CD is welded to the center of the beam and to the fixed support at D. If the temperature of the rod is decreased by 150°F, determine the force developed in the rod. The beam and rod are both made of steel for which $E = 200$ GPa and $\alpha = 6.5(10^{-6})/\text{F}°$.



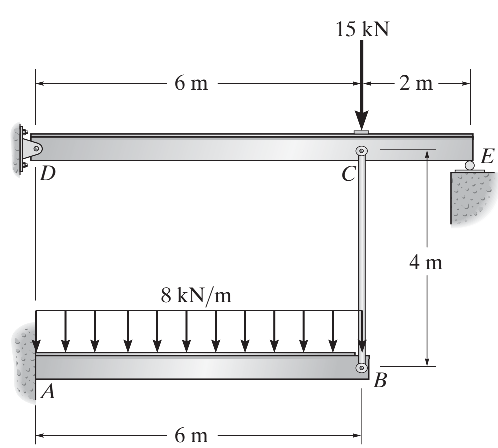
Prob. 10–38

10–39. The contilevered beam is supported at one end by a $\frac{1}{2}$-in.-diameter suspender rod AC and fixed at the other end B. Determine the force in the rod due to a uniform loading of 4 k/ft. $E = 29(10^3)$ ksi for both the beam and rod.



Prob. 10–39

***10–40.** The structural assembly supports the loading shown. Draw the moment diagrams for each of the beams. Take $I = 100(10^6)\,\text{mm}^4$ for the beams and $A = 200\,\text{mm}^2$ for the tie rod. All members are made of steel for which $E = 200$ GPa.



Prob. 10–40

10.10 Influence Lines for Statically Indeterminate Beams

In Sec. 6–3 we discussed the use of the Müller-Breslau principle for drawing the influence line for the reaction, shear, and moment at a point in a statically determinate beam. In this section we will extend this method and apply it to statically indeterminate beams.

Recall that, for a beam, *the Müller-Breslau principle states that the influence line for a function (reaction, shear, or moment) is to the same scale as the deflected shape of the beam when the beam is acted upon by the function.* To draw the deflected shape properly, the capacity of the beam to resist the applied function must be *removed* so the beam can deflect when the function is applied. For *statically determinate beams,* the deflected shapes (or the influence lines) will be a series of *straight line segments.* For *statically indeterminate beams, curves* will result. Construction of each of the three types of influence lines (reaction, shear, and moment) will now be discussed for a statically indeterminate beam. In each case we will illustrate the validity of the Müller-Breslau principle using Maxwell's theorem of reciprocal displacements.

Reaction at *A*.

To determine the influence line for the reaction at A in Fig. 10–20*a*, a unit load is placed on the beam at successive points, and at each point the reaction at A must be determined. A plot of these results yields the influence line. For example, when the load is at point D, Fig. 10–20*a*, the reaction at A, which represents the ordinate of the influence line at D, can be determined by the force method. To do this, the principle of superposition is applied, as shown in Figs. 10–20*a* through 10–20*c*. The compatibility equation for point A is thus $0 = f_{AD} + A_y f_{AA}$ or $A_y = -f_{AD}/f_{AA}$; however, by Maxwell's theorem of reciprocal displacements $f_{AD} = -f_{DA}$, Fig. 10–20*d*, so that we can also compute A_y (or the ordinate of the influence line at D) using the equation

$$A_y = \left(\frac{1}{f_{AA}}\right) f_{DA}$$

By comparison, the Müller-Breslau principle requires removal of the support at A and application of a vertical unit load. The resulting deflection curve, Fig. 10–20*d*, is to some scale the shape of the influence line for A_y. From the equation above, however, it is seen that the scale factor is $1/f_{AA}$.

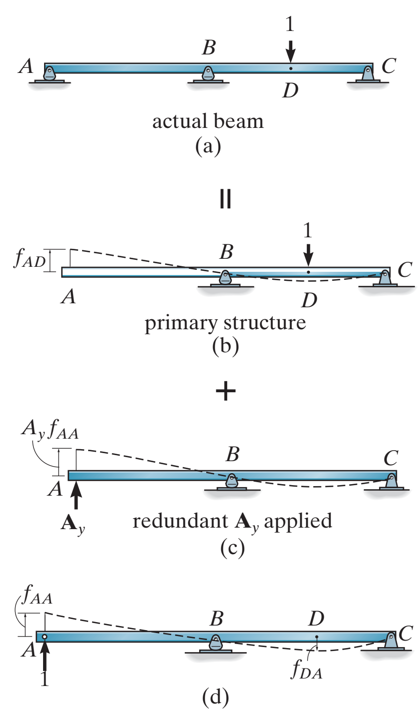


Fig. 10–20

Shear at *E*. If the influence line for the shear at point E of the beam in Fig. 10–21*a* is to be determined, then by the Müller-Breslau principle the beam is imagined cut open at this point and a *sliding device* is inserted at E, Fig. 10–21*b*. This device will transmit a moment and normal force but no shear. When the beam deflects due to positive unit shear loads acting at E, the slope on each side of the guide remains the same, and the deflection curve represents to some scale the influence line for the shear at E, Fig. 10–21*c*. Had the basic method for establishing the influence line for the shear at E been applied, it would then be necessary to apply a unit load at each point D and compute the internal shear at E, Fig. 10–21*a*. This value, V_E, would represent the ordinate of the influence line at D. Using the force method and Maxwell's theorem of reciprocal displacements, as in the previous case, it can be shown that

$$V_E = \left(\frac{1}{f_{EE}}\right) f_{DE}$$

This again establishes the validity of the Müller-Breslau principle, namely, a positive unit shear load applied to the beam at E, Fig. 10–21*c*, will cause the beam to deflect into the *shape* of the influence line for the shear at E. Here the scale factor is $(1/f_{EE})$.

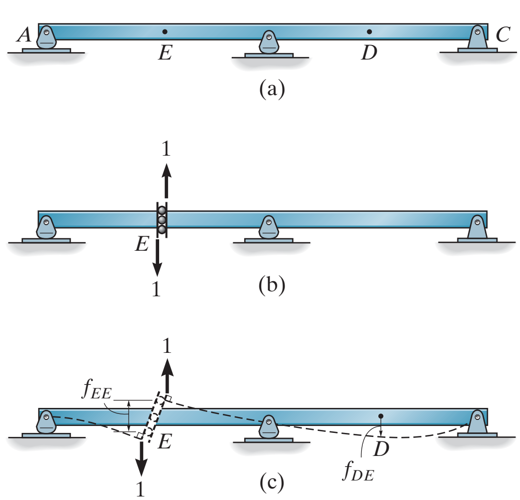


Fig. 10–21

Moment at *E*. The influence line for the moment at E in Fig. 10–22*a* can be determined by placing a *pin* or *hinge* at E, since this connection transmits normal and shear forces but cannot resist a moment, Fig. 10–22*b*. Applying a positive unit couple moment, the beam then deflects to the dashed position in Fig. 10–22*c*, which yields to some scale the influence line, again a consequence of the Müller-Breslau principle. Using the force method and Maxwell's reciprocal theorem, we can show that

$$M_E = \left(\frac{1}{\alpha_{EE}}\right) f_{DE}$$

The scale factor here is $(1/\alpha_{EE})$.

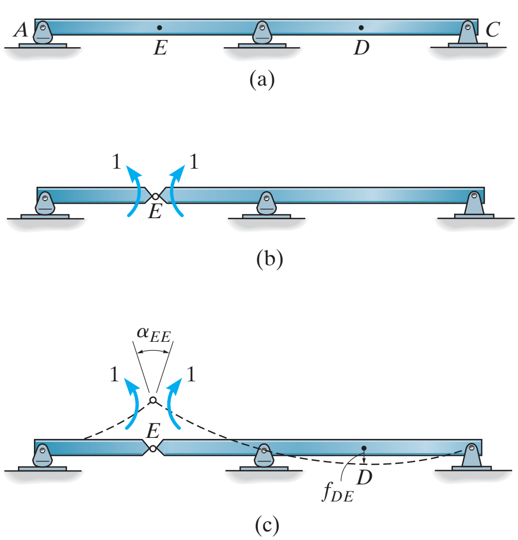


Fig. 10–22

10

Procedure for Analysis

The following procedure provides a method for establishing the influence line for the reaction, shear, or moment at a point on a beam using the Müller-Breslau technique.

Qualitative Influence Line

At the point on the beam for which the influence line is to be determined, place a connection that will remove the capacity of the beam to support the function of the influence line. If the function is a vertical *reaction,* use a vertical *roller guide;* if the function is *shear,* use a *sliding device;* or if the function is *moment,* use a *pin* or *hinge.* Place a unit load at the connection acting on the beam in the "positive direction" of the function. Draw the deflection curve for the beam. This curve represents to some scale the shape of the influence line for the beam.

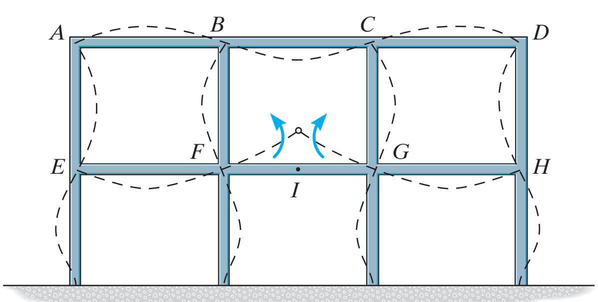
Quantitative Influence Line

If numerical values of the influence line are to be determined, compute the *displacement* of successive points along the beam when the beam is subjected to the unit load placed at the connection mentioned above. Divide each value of displacement by the displacement determined at the point where the unit load acts. By applying this scalar factor, the resulting values are the ordinates of the influence line.

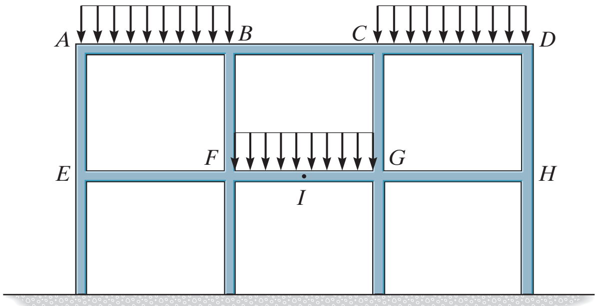
Influence lines for the continuous girder of this trestle were constructed in order to properly design the girder.

10.11 Qualitative Influence Lines for Frames

The Müller-Breslau principle provides a quick method and is of great value for establishing the general shape of the influence line for building frames. Once the influence-line *shape* is known, one can immediately specify the *location* of the live loads so as to create the greatest influence of the function (reaction, shear, or moment) in the frame. For example, the shape of the influence line for the *positive* moment at the center I of girder FG of the frame in Fig. 10–23a is shown by the dashed lines. Thus, uniform loads would be placed only on girders AB, CD, and FG in order to create the largest positive moment at I. With the frame loaded in this manner, Fig. 10–23b, an indeterminate analysis of the frame could then be performed to determine the critical moment at I.



(a)

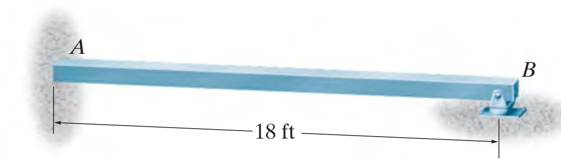


(b)

Fig. 10–23

EXAMPLE 10.10

Draw the influence line for the vertical reaction at A for the beam in Fig. 10–24*a*. *EI* is constant. Plot numerical values every 6 ft.



(a)

SOLUTION

The capacity of the beam to resist the reaction $\mathbf{A}_y$ is removed. This is done using a vertical roller device shown in Fig. 10–24*b*. Applying a vertical unit load at A yields the shape of the influence line shown in Fig. 10–24*c*.

In order to determine ordinates of the influence line we will use the conjugate-beam method. The reactions at A and B on the "real beam," when subjected to the unit load at A, are shown in Fig. 10–24*b*. The corresponding conjugate beam is shown in Fig. 10–24*d*. Notice that the support at A' remains the *same* as that for A in Fig. 10–24*b*. This is because a vertical roller device on the conjugate beam supports a moment but no shear, corresponding to a displacement but no slope at A on the real beam, Fig. 10–24*c*. The reactions at the supports of the conjugate beam have been computed and are shown in Fig. 10–24*d*. The displacements of points on the real beam, Fig. 10–24*b*, will now be computed.

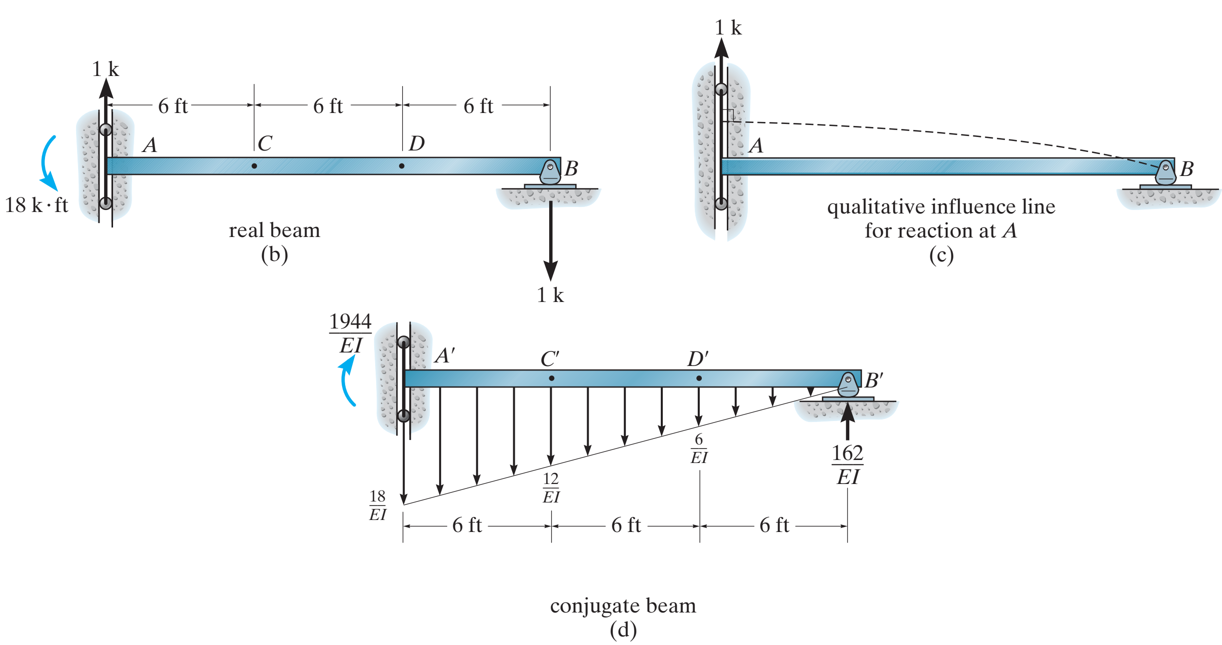
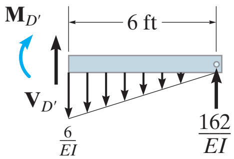


Fig. 10–24

For B', since no moment exists on the conjugate beam at B', Fig. 10–24*d*, then

$$\Delta_B = M_{B'} = 0$$



(e)

For D', Fig. 10–24*e*:

$$\Sigma M_{D'} = 0; \qquad \Delta_D = M_{D'} = \frac{162}{EI}(6) - \frac{1}{2}\left(\frac{6}{EI}\right)(6)(2) = \frac{936}{EI}$$

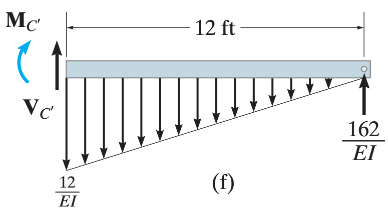
For C', Fig. 10–24*f*:

$$\Sigma M_{C'} = 0; \qquad \Delta_C = M_{C'} = \frac{162}{EI}(12) - \frac{1}{2}\left(\frac{12}{EI}\right)(12)(4) = \frac{1656}{EI}$$

For A', Fig. 10–24*d*:

$$\Delta_A = M_{A'} = \frac{1944}{EI}$$

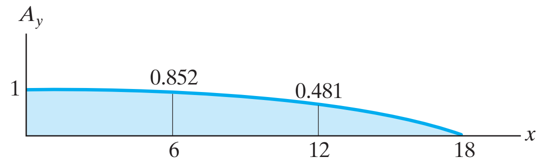
Since a vertical 1-k load acting at A on the beam in Fig. 10–24*a* will cause a vertical reaction at A of 1 k, the displacement at A, $\Delta_A = 1944/EI$, should correspond to a numerical value of 1 for the influence-line ordinate at A. Thus, dividing the other computed displacements by this factor, we obtain



(f)

x	A_y
A	1
C	0.852
D	0.481
B	0

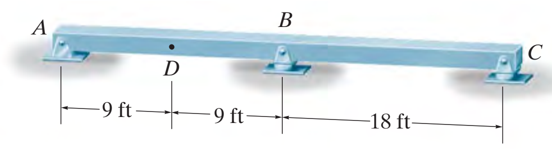
A plot of these values yields the influence line shown in Fig. 10–24*g*.



quantitative influence line
for reaction at A
(g)

EXAMPLE 10.11

Draw the influence line for the shear at D for the beam in Fig. 10–25*a*. EI is constant. Plot numerical values every 9 ft.



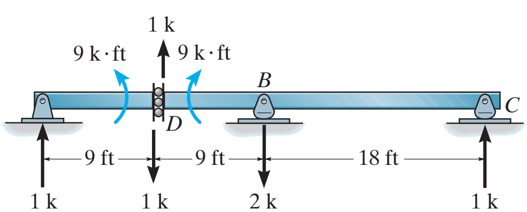
(a)

Fig. 10–25

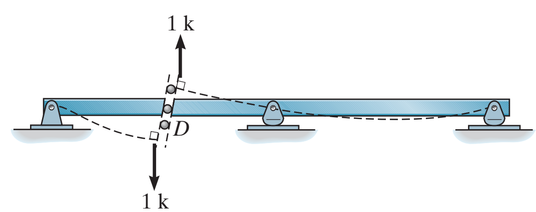
SOLUTION

The capacity of the beam to resist shear at D is removed. This is done using the roller device shown in Fig. 10–25*b*. Applying a positive unit shear at D yields the shape of the influence line shown in Fig. 10–25*c*.

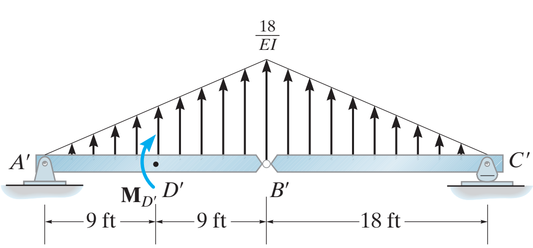
The support reactions at A, B, and C on the "real beam" when subjected to the unit shear at D are shown in Fig. 10–25*b*. The corresponding conjugate beam is shown in Fig. 10–25*d*. Here an external couple moment $\mathbf{M}_{D'}$ must be applied at D' in order to cause a different *internal moment* just to the left and just to the right of D'. These internal moments correspond to the displacements just to the left and just to the right of D on the real beam, Fig. 10–25*c*. The reactions at the supports A', B', C' and the external moment $\mathbf{M}_{D'}$ on the conjugate beam have been computed and are shown in Fig. 10–25*e*. As an exercise verify the calculations.



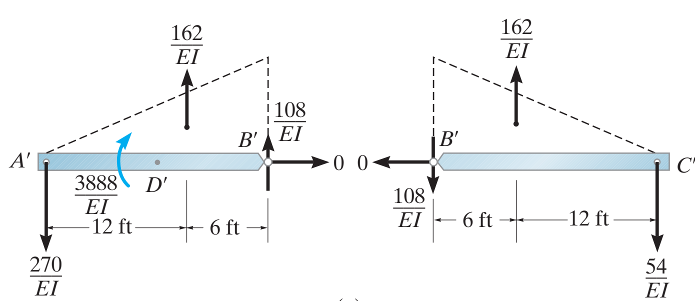
real beam
(b)



qualitative influence line
for shear at D
(c)



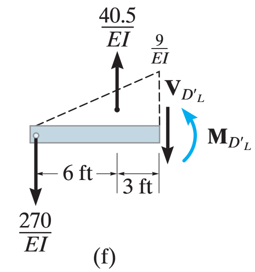
conjugate beam
(d)



(e)

Since there is a *discontinuity* of moment at D', the internal moment just to the left and right of D' will be computed. Just to the left of D', Fig. 10–25*f*, we have

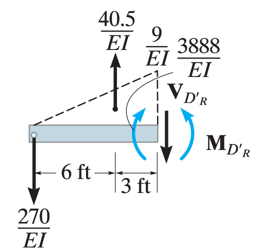
$$\Sigma M_{D'_L} = 0; \qquad \Delta_{D_L} = M_{D'_L} = \frac{40.5}{EI}(3) - \frac{270}{EI}(9) = -\frac{2308.5}{EI}$$



(f)

Just to the right of D', Fig. 10–25*g*, we have

$$\Sigma M_{D'_R} = 0; \qquad \Delta_{D_R} = M_{D'_R} = \frac{40.5}{EI}(3) - \frac{270}{EI}(9) + \frac{3888}{EI} = \frac{1579.5}{EI}$$



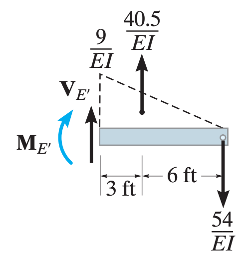
(g)

From Fig. 10–25*e*,

$$\Delta_A = M_{A'} = 0 \qquad \Delta_B = M_{B'} = 0 \qquad \Delta_C = M_{C'} = 0$$

For point E, Fig. 10–25*b*, using the method of sections at the corresponding point E' on the conjugate beam, Fig. 10–25*h*, we have

$$\Sigma M_{E'} = 0; \qquad \Delta_E = M_{E'} = \frac{40.5}{EI}(3) - \frac{54}{EI}(9) = -\frac{364.5}{EI}$$

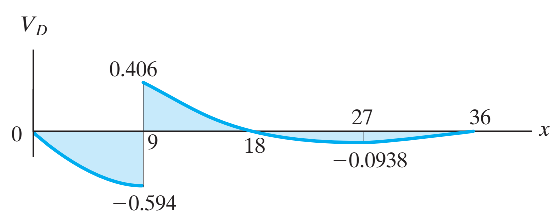


(h)

The ordinates of the influence line are obtained by dividing each of the above values by the scale factor $M_{D'} = 3888/EI$. We have

x	V_D
A	0
D_L	-0.594
D_R	0.406
B	0
E	-0.0938
C	0

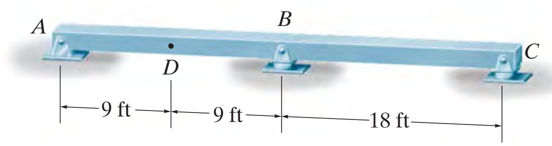
A plot of these values yields the influence line shown in Fig. 10–25*i*.



quantitative influence line for shear at D
(i)

EXAMPLE 10.12

Draw the influence line for the moment at D for the beam in Fig. 10–26*a*. EI is constant. Plot numerical values every 9 ft.



(a)

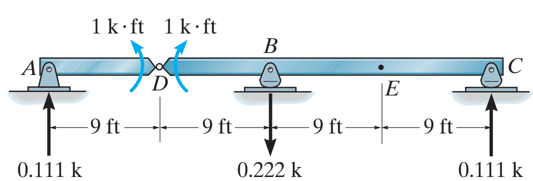
Fig. 10–26

SOLUTION

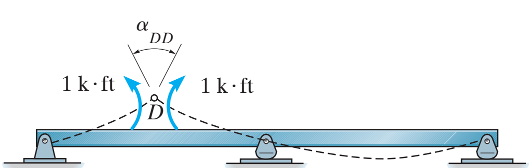
A hinge is inserted at D in order to remove the capacity of the beam to resist moment at this point, Fig. 10–26*b*. Applying positive unit couple moments at D yields the influence line shown in Fig. 10–26*c*.

The reactions at A, B, and C on the "real beam" when subjected to the unit couple moments at D are shown in Fig. 10–26*b*. The corresponding conjugate beam and its reactions are shown in Fig. 10–26*d*. It is suggested that the reactions be verified in both cases. From Fig. 10–26*d*, note that

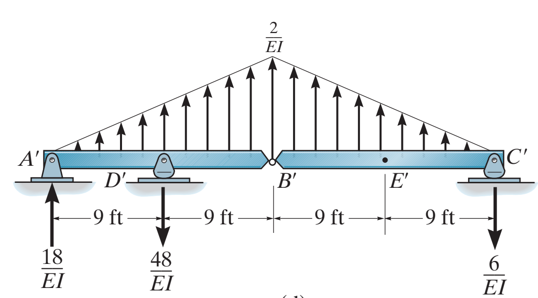
$$\Delta_A = M_{A'} = 0 \qquad \Delta_B = M_{B'} = 0 \qquad \Delta_C = M_{C'} = 0$$



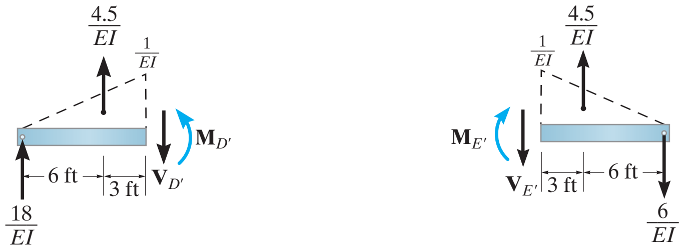
real beam
(b)



qualitative influence line for moment at D
(c)



(d)



(e) (f)

For point D', Fig. 10–26*e*:

$$\Sigma M_{D'} = 0; \qquad \Delta_D = M_{D'} = \frac{4.5}{EI}(3) + \frac{18}{EI}(9) = \frac{175.5}{EI}$$

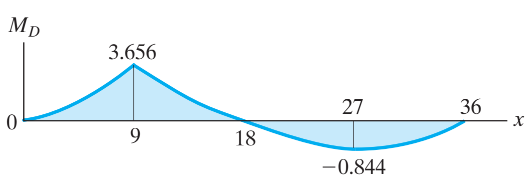
For point E', Fig. 10–26*f*:

$$\Sigma M_{E'} = 0; \qquad \Delta_E = M_{E'} = \frac{4.5}{EI}(3) - \frac{6}{EI}(9) = -\frac{40.5}{EI}$$

The angular displacement α_{DD} at D of the "real beam" in Fig. 10–26*c* is defined by the reaction at D' on the conjugate beam. This factor, $D'_y = 48/EI$, is divided into the above values to give the ordinates of the influence line, that is,

x	M_D
A	0
D	3.656
B	0
E	−0.844
C	0

A plot of these values yields the influence line shown in Fig. 10–26*g*.

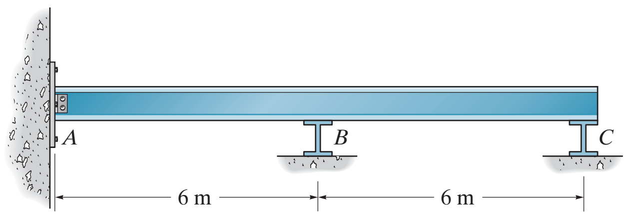


quantitative influence line
for moment at D
(g)

10

PROBLEMS

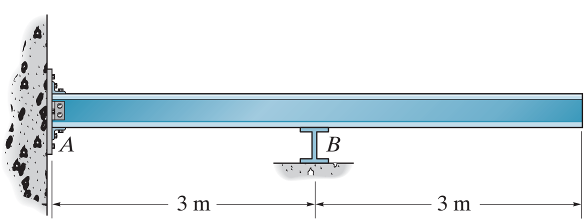
10–41. Draw the influence line for the reaction at C. Plot numerical values at the peaks. Assume A is a pin and B and C are rollers. EI is constant.



Prob. 10–41

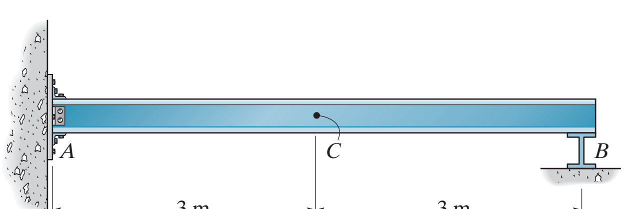
10–42. Draw the influence line for the moment at A. Plot numerical values at the peaks. Assume A is fixed and the support at B is a roller. EI is constant.

10–43. Draw the influence line for the vertical reaction at B. Plot numerical values at the peaks. Assume A is fixed and the support at B is a roller. EI is constant.



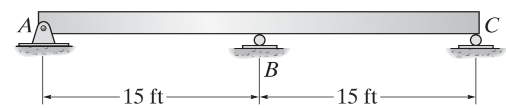
Probs. 10–42/10–43

***10–44.** Draw the influence line for the shear at C. Plot numerical values every 1.5 m. Assume A is fixed and the support at B is a roller. EI is constant.



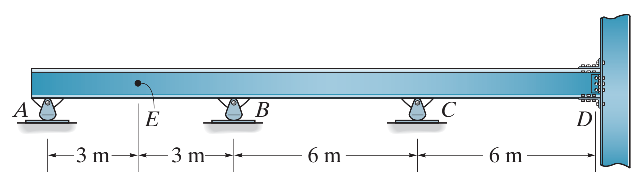
Prob. 10–44

10–45. Draw the influence line for the reaction at C. Plot the numerical values every 5 ft. EI is constant.



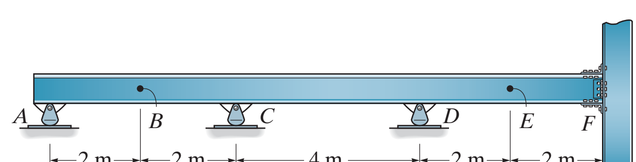
Prob. 10–45

10–46. Sketch the influence line for (a) the moment at E, (b) the reaction at C, and (c) the shear at E. In each case, indicate on a sketch of the beam where a uniform distributed live load should be placed so as to cause a maximum positive value of these functions. Assume the beam is fixed at D.



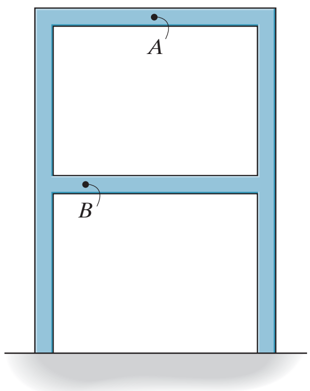
Prob. 10–46

10–47. Sketch the influence line for (a) the vertical reaction at C, (b) the moment at B, and (c) the shear at E. In each case, indicate on a sketch of the beam where a uniform distributed live load should be placed so as to cause a maximum positive value of these functions. Assume the beam is fixed at F.



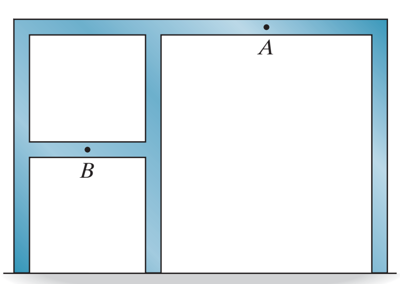
Prob. 10–47

***10–48.** Use the Müller-Breslau principle to sketch the general shape of the influence line for (a) the moment at A and (b) the shear at B.



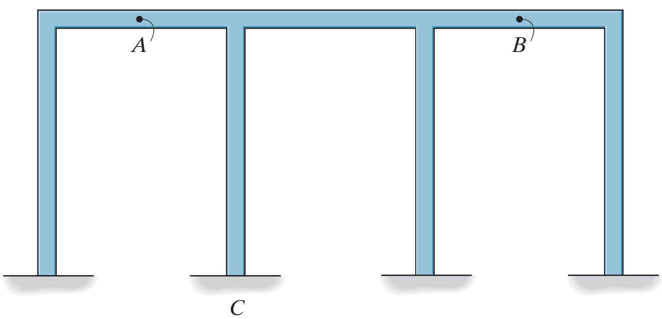
Prob. 10–48

10–49. Use the Müller-Breslau principle to sketch the general shape of the influence line for (a) the moment at A and (b) the shear at B.



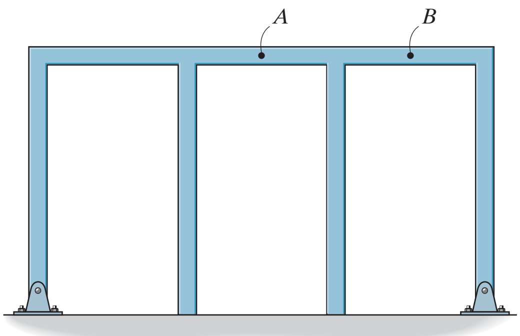
Prob. 10–49

10–50. Use the Müller-Breslau principle to sketch the general shape of the influence line for (a) the moment at A and (b) the shear at B.



Prob. 10–50

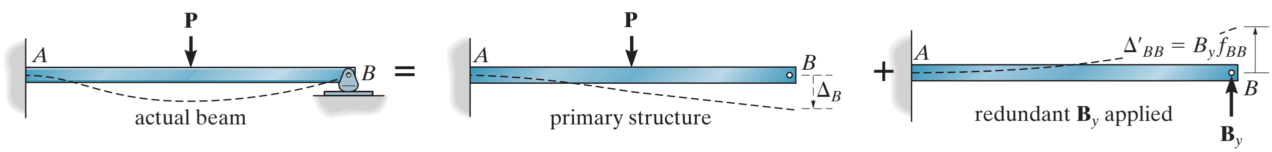
10–51. Use the Müller-Breslau principle to sketch the general shape of the influence line for (a) the moment at A and (b) the shear at B.



Prob. 10–51

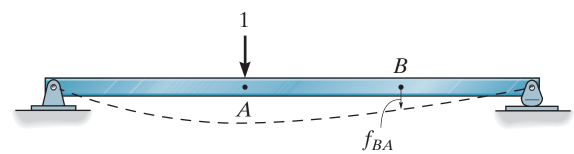
CHAPTER REVIEW

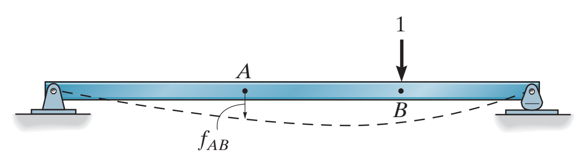
The analysis of a statically indeterminate structure requires satisfying equilibrium, compatibility, and the force-displacement relationships for the structure. A force method of analysis consists of writing equations that satisfy compatibility and the force-displacement requirements, which then gives a direct solution for the redundant reactions. Once obtained, the remaining reactions are found from the equilibrium equations.



$$+\downarrow \; 0 = \Delta_B - B_y f_{BB}$$

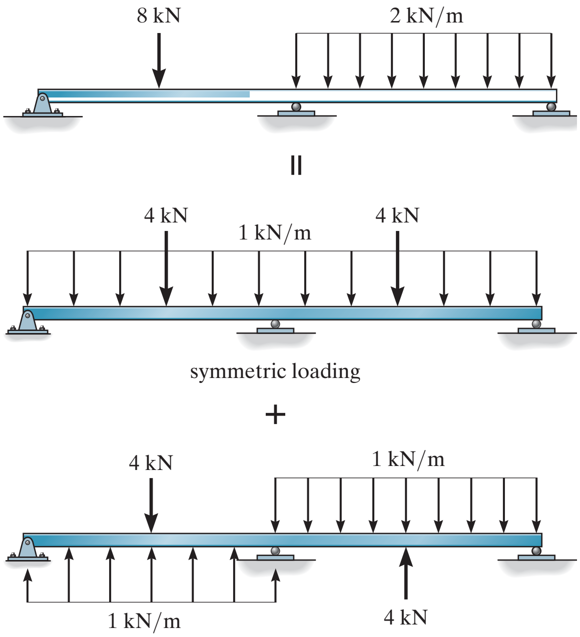
Simplification of the force method is possible, using Maxwell's theorem of reciprocal displacements, which states that the displacement of a point B on a structure due to a unit load acting at point A, f_{BA}, is equal to the displacement of point A when the load acts at B, f_{AB}.



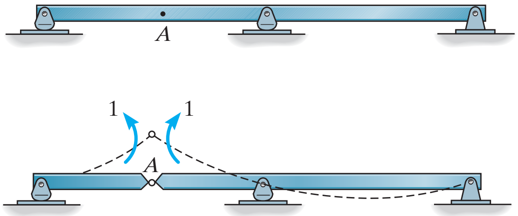


$$f_{BA} = f_{AB}$$

The analysis of a statically indeterminate structure can be simplified if the structure has symmetry of material, geometry, and loading about its central axis. In particular, structures having an asymmetric loading can be replaced with a superposition of a symmetric and antisymmetric load.



Influence lines for statically indeterminate structures will consist of *curved lines*. They can be sketched using the Müller-Breslau principle, which states that the influence line shape for either the reaction, shear, or moment is to the same scale as the deflected shape of the structure when it is acted upon by the reaction, shear, or moment, respectively. By using Maxwell's theorem of reciprocal deflections, it is possible to obtain specific values of the ordinates of an influence line.



influence line shape for moment at *A*

The members of this building frame are all fixed connected, so the framework
is statically indeterminate.

Displacement Method of Analysis: Slope-Deflection Equations

11

In this chapter we will briefly outline the basic ideas for analyzing structures using the displacement method of analysis. Once these concepts have been presented, we will develop the general equations of slope deflection and then use them to analyze statically indeterminate beams and frames.

11.1 Displacement Method of Analysis: General Procedures

All structures must satisfy equilibrium, load-displacement, and compatibility of displacements requirements in order to ensure their safety. It was stated in Sec. 10–1 that there are two different ways to satisfy these requirements when analyzing a statically indeterminate structure. The force method of analysis, discussed in the previous chapter, is based on identifying the unknown redundant forces and then satisfying the structure's compatibility equations. This is done by expressing the displacements in terms of the loads by using the load-displacement relations. The solution of the resultant equations yields the redundant reactions, and then the equilibrium equations are used to determine the remaining reactions on the structure.

The *displacement method* works the opposite way. It first requires satisfying equilibrium equations for the structure. To do this the unknown displacements are written in terms of the loads by using the load-displacement relations, then these equations are solved for the displacements. Once the displacements are obtained, the unknown loads are determined from the compatibility equations using the load-displacement relations. Every displacement method follows this general procedure. In this chapter, the procedure will be generalized to produce the slope-deflection equations. In Chapter 12, the moment-distribution method will be developed. This method sidesteps the calculation of the displacements and instead makes it possible to apply a series of converging corrections that allow direct calculation of the end moments. Finally, in Chapters 14, 15, and 16, we will illustrate how to apply this method using matrix analysis, making it suitable for use on a computer.

In the discussion that follows we will show how to identify the unknown displacements in a structure and we will develop some of the important load-displacement relations for beam and frame members. The results will be used in the next section and in later chapters as the basis for applying the displacement method of analysis.

Degrees of Freedom. When a structure is loaded, specified points on it, called *nodes*, will undergo unknown displacements. These displacements are referred to as the *degrees of freedom* for the structure, and in the displacement method of analysis it is important to specify these degrees of freedom since they become the unknowns when the method is applied. The number of these unknowns is referred to as the degree in which the structure is kinematically indeterminate.

To determine the kinematic indeterminacy we can imagine the structure to consist of a series of members connected to nodes, which are usually located at *joints, supports,* at the *ends* of a member, or where the members have a sudden *change in cross section.* In three dimensions, each node on a frame or beam can have at most three linear displacements and three rotational displacements; and in two dimensions, each node can have at most two linear displacements and one rotational displacement. Furthermore, nodal displacements may be restricted by the supports, or due to assumptions based on the behavior of the structure. For example, if the structure is a beam and only deformation due to bending is considered, then there can be no linear displacement along the axis of the beam since this displacement is caused by axial-force deformation.

To clarify these concepts we will consider some examples, beginning with the beam in Fig. 11–1*a*. Here any load **P** applied to the beam will cause node A only to rotate (neglecting axial deformation), while node B is completely restricted from moving. Hence the beam has only one unknown degree of freedom, θ_A, and is therefore kinematically indeterminate to the first degree. The beam in Fig. 11–1*b* has nodes at A, B, and C, and so has four degrees of freedom, designated by the rotational displacements θ_A, θ_B, θ_C, and the vertical displacement Δ_C. It is kinematically indeterminate to the fourth degree. Consider now the frame in Fig. 11–1*c*. Again, if we neglect axial deformation of the members, an arbitrary loading **P** applied to the frame can cause nodes B and C to rotate, and these nodes can be displaced horizontally by an *equal* amount. The frame therefore has three degrees of freedom, θ_B, θ_C, Δ_B, and thus it is kinematically indeterminate to the third degree.

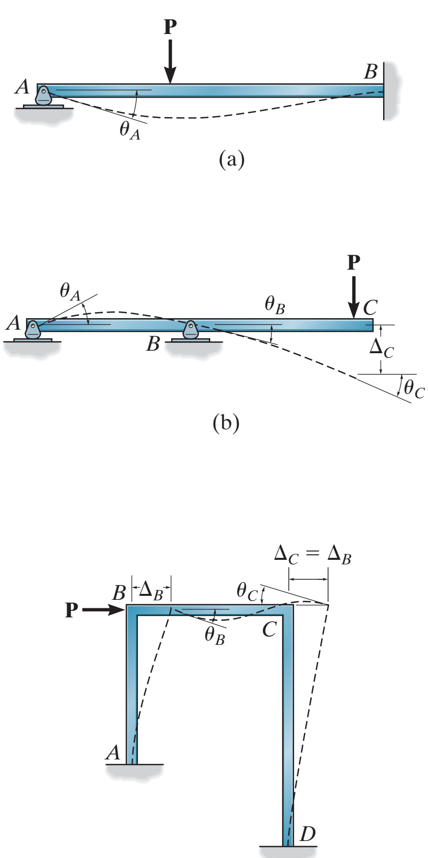


Fig. 11–1

In summary, specifying the kinematic indeterminacy or the number of unconstrained degrees of freedom for the structure is a necessary first step when applying a displacement method of analysis. It identifies the number of unknowns in the problem, based on the assumptions made regarding the deformation behavior of the structure. Furthermore, once these nodal displacements are known, the deformation of the structural members can be completely specified, and the loadings within the members obtained.

11.2 Slope-Deflection Equations

As indicated previously, the method of consistent displacements studied in Chapter 10 is called a force method of analysis, because it requires writing equations that relate the unknown forces or moments in a structure. Unfortunately, its use is limited to structures which are *not* highly indeterminate. This is because much work is required to set up the compatibility equations, and furthermore each equation written involves *all the unknowns,* making it difficult to solve the resulting set of equations unless a computer is available. By comparison, the slope-deflection method is not as involved. As we shall see, it requires less work both to write the necessary equations for the solution of a problem and to solve these equations for the unknown displacements and associated internal loads. Also, the method can be easily programmed on a computer and used to analyze a wide range of indeterminate structures.

The slope-deflection method was originally developed by Heinrich Manderla and Otto Mohr for the purpose of studying secondary stresses in trusses. Later, in 1915, G. A. Maney developed a refined version of this technique and applied it to the analysis of indeterminate beams and framed structures.

General Case. The slope-deflection method is so named since it relates the unknown slopes and deflections to the applied load on a structure. In order to develop the general form of the slope-deflection equations, we will consider the typical span AB of a continuous beam as shown in Fig. 11–2, which is subjected to the arbitrary loading and has a constant EI. We wish to relate the beam's internal end moments M_{AB} and M_{BA} in terms of its three degrees of freedom, namely, its angular displacements θ_A and θ_B, and linear displacement Δ which could be caused by a relative settlement between the supports. Since we will be developing a formula, *moments* and *angular displacements* will be considered *positive* when they act *clockwise on the span,* as shown in Fig. 11–2. Furthermore, the *linear displacement* Δ is considered *positive* as shown, since this displacement causes the cord of the span and the span's cord angle ψ to rotate *clockwise*.

The slope-deflection equations can be obtained by using the principle of superposition by considering *separately* the moments developed at each support due to each of the displacements, θ_A, θ_B, and Δ, and then the loads.

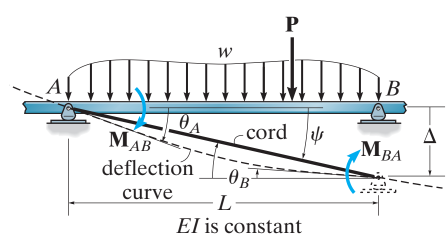


Fig. 11–2

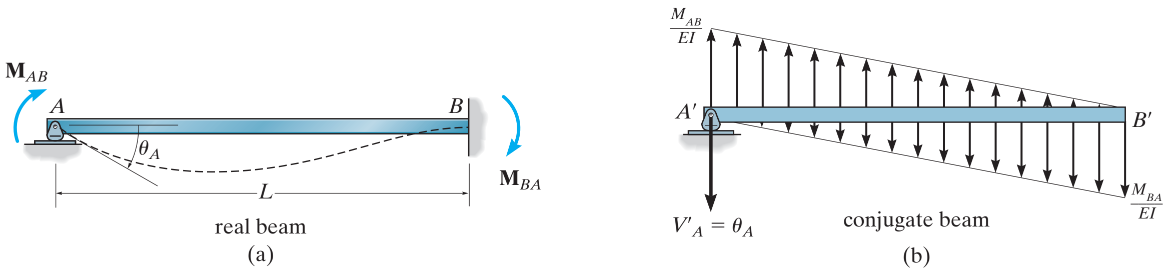


Fig. 11–3

Angular Displacement at A, θ_A. Consider node A of the member shown in Fig. 11–3a to rotate θ_A while its far-end node B is *held fixed*. To determine the moment M_{AB} needed to cause this displacement, we will use the conjugate-beam method. For this case the conjugate beam is shown in Fig. 11–3b. Notice that the end shear at A' acts downward on the beam, since θ_A is clockwise. The deflection of the "real beam" in Fig. 11–3a is to be zero at A and B, and therefore the corresponding sum of the *moments* at each end A' and B' of the conjugate beam must also be zero. This yields

$$\circlearrowleft + \Sigma M_{A'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3} = 0$$

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3} + \theta_A L = 0$$

from which we obtain the following load-displacement relationships.

$$\boxed{M_{AB} = \frac{4EI}{L}\theta_A} \qquad (11\text{–}1)$$

$$\boxed{M_{BA} = \frac{2EI}{L}\theta_A} \qquad (11\text{–}2)$$

Angular Displacement at B, θ_B. In a similar manner, if end B of the beam rotates to its final position θ_B, while end A is *held fixed*, Fig. 11–4, we can relate the applied moment M_{BA} to the angular displacement θ_B and the reaction moment M_{AB} at the wall. The results are

$$\boxed{M_{BA} = \frac{4EI}{L}\theta_B} \qquad (11\text{–}3)$$

$$\boxed{M_{AB} = \frac{2EI}{L}\theta_B} \qquad (11\text{–}4)$$

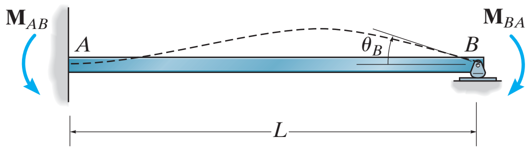


Fig. 11–4

Relative Linear Displacement, Δ. If the far node B of the member is displaced relative to A, so that the cord of the member rotates clockwise (positive displacement) and yet both ends do not rotate, then equal but opposite moment and shear reactions are developed in the member, Fig. 11–5*a*. As before, the moment $\mathbf{M}$ can be related to the displacement Δ using the conjugate-beam method. In this case, the conjugate beam, Fig. 11–5*b*, is free at both ends, since the real beam (member) is fixed supported. However, due to the *displacement* of the real beam at B, the *moment* at the end B' of the conjugate beam must have a magnitude of Δ as indicated.* Summing moments about B', we have

$$\circlearrowright + \Sigma M_{B'} = 0; \qquad \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{2}{3}L\right)\right] - \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{1}{3}L\right)\right] - \Delta = 0$$

$$\boxed{M_{AB} = M_{BA} = M = \frac{-6EI}{L^2}\Delta} \qquad (11\text{–}5)$$

By our sign convention, this induced moment is negative since for equilibrium it acts counterclockwise on the member.

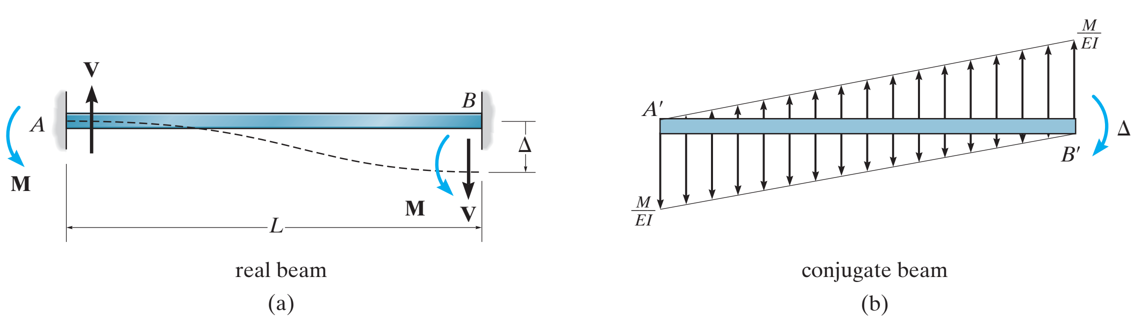


Fig. 11–5

*The moment diagrams shown on the conjugate beam were determined by the method of superposition for a simply supported beam, as explained in Sec. 4–5.

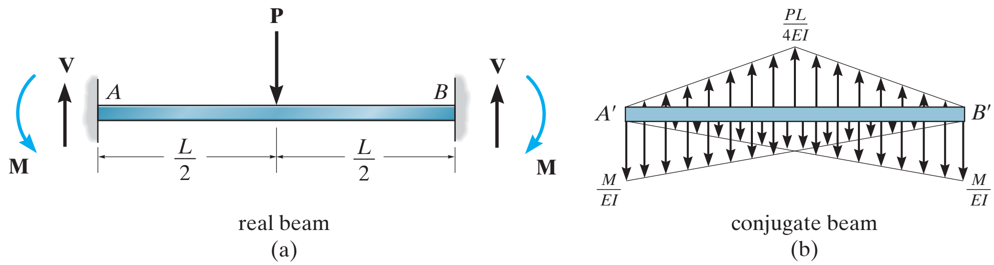


Fig. 11–6

Fixed-End Moments. In the previous cases we have considered relationships between the displacements and the necessary moments M_{AB} and M_{BA} acting at nodes A and B, respectively. In general, however, the linear or angular displacements of the nodes are caused by loadings acting on the *span* of the member, not by moments acting at its nodes. In order to develop the slope-deflection equations, we must transform these *span loadings* into equivalent moments acting at the nodes and then use the load-displacement relationships just derived. This is done simply by finding the reaction moment that each load develops at the nodes. For example, consider the fixed-supported member shown in Fig. 11–6*a*, which is subjected to a concentrated load **P** at its center. The conjugate beam for this case is shown in Fig. 11–6*b*. Since we require the slope at each end to be zero,

$$+\uparrow \Sigma F_y = 0; \qquad \left[\frac{1}{2}\left(\frac{PL}{4EI}\right)L\right] - 2\left[\frac{1}{2}\left(\frac{M}{EI}\right)L\right] = 0$$

$$M = \frac{PL}{8}$$

This moment is called a *fixed-end moment* (FEM). Note that according to our sign convention, it is negative at node A (counterclockwise) and positive at node B (clockwise). For convenience in solving problems, fixed-end moments have been calculated for other loadings and are tabulated on the inside back cover of the book. Assuming these FEMs have been determined for a specific problem (Fig. 11–7), we have

$$M_{AB} = (\text{FEM})_{AB} \qquad M_{BA} = (\text{FEM})_{BA} \tag{11–6}$$

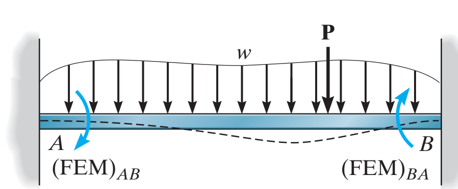


Fig. 11–7

Slope-Deflection Equation. If the end moments due to each displacement (Eqs. 11–1 through 11–5) and the loading (Eq. 11–6) are added together, the resultant moments at the ends can be written as

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{AB}$$
$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{BA} \qquad (11\text{–}7)$$

This pedestrian bridge has a reinforced concrete deck. Since it extends over all its supports, it is indeterminate to the second degree. The slope deflection equations provide a convenient method for finding the internal moments in each span.

Since these two equations are similar, the result can be expressed as a single equation. Referring to one end of the span as the near end (N) and the other end as the far end (F), and letting the *member stiffness* be represented as $k = I/L$, and the *span's cord rotation* as ψ (psi) $= \Delta/L$, we can write

$$\boxed{\begin{array}{c} M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N \\ \text{For Internal Span or End Span with Far End Fixed} \end{array}} \qquad (11\text{–}8)$$

where

M_N = internal moment in the near end of the span; this moment is *positive clockwise* when acting on the span.

E, k = modulus of elasticity of material and span stiffness $k = I/L$.

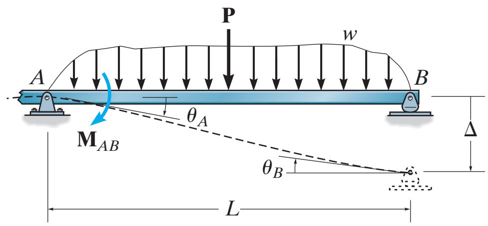
θ_N, θ_F = near- and far-end slopes or angular displacements of the span at the supports; the angles are measured in *radians* and are *positive clockwise.*

ψ = span rotation of its cord due to a linear displacement, that is, $\psi = \Delta/L$; this angle is measured in *radians* and is *positive clockwise.*

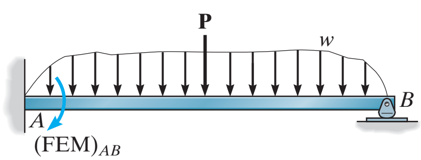
$(\text{FEM})_N$ = fixed-end moment at the near-end support; the moment is *positive clockwise* when acting on the span; refer to the table on the inside back cover for various loading conditions.

From the derivation Eq. 11–8 is both a compatibility and load-displacement relationship found by considering only the effects of bending and neglecting axial and shear deformations. It is referred to as the general *slope-deflection equation*. When used for the solution of problems, this equation is applied *twice* for each member span (AB); that is, application is from A to B and from B to A for span AB in Fig. 11–2.

11



(a)



(b)

Fig. 11–8

Pin-Supported End Span. Occasionally an end span of a beam or frame is supported by a pin or roller at its *far end,* Fig. 11–8*a*. When this occurs, the moment at the roller or pin must be zero; and provided the angular displacement θ_B at this support does not have to be determined, we can modify the general slope-deflection equation so that it has to be applied *only once* to the span rather than twice. To do this we will apply Eq. 11–8 or Eqs. 11–7 to each end of the beam in Fig. 11–8. This results in the following two equations:

$$\begin{aligned} M_N &= 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N \\ 0 &= 2Ek(2\theta_F + \theta_N - 3\psi) + 0 \end{aligned} \qquad (11\text{–}9)$$

Here the $(\text{FEM})_F$ is equal to zero since the far end is pinned, Fig. 11–8*b*. Furthermore, the $(\text{FEM})_N$ can be obtained, for example, using the table in the right-hand column on the inside back cover of this book. Multiplying the first equation by 2 and subtracting the second equation from it *eliminates* the unknown θ_F and yields

$$\boxed{\begin{gathered} M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N \\ \text{Only for End Span with Far End Pinned or Roller Supported} \end{gathered}} \qquad (11\text{–}10)$$

Since the moment at the far end is zero, only *one* application of this equation is necessary for the end span. This simplifies the analysis since the general equation, Eq. 11–8, would require *two* applications for this span and therefore involve the (extra) unknown angular displacement θ_B (or θ_F) at the end support.

To summarize application of the slope-deflection equations, consider the continuous beam shown in Fig. 11–9 which has four degrees of freedom. Here Eq. 11–8 can be applied twice to each of the three spans, i.e., from A to B, B to A, B to C, C to B, C to D, and D to C. These equations would involve the four unknown rotations, θ_A, θ_B, θ_C, θ_D. Since the end moments at A and D are zero, however, it is not necessary to determine θ_A and θ_D. A shorter solution occurs if we apply Eq. 11–10 from B to A and C to D and then apply Eq. 11–8 from B to C and C to B. These four equations will involve only the unknown rotations θ_B and θ_C.

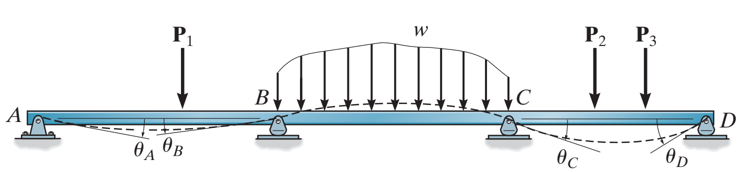


Fig. 11–9

11.3 Analysis of Beams

Procedure for Analysis

Degrees of Freedom

Label all the supports and joints (nodes) in order to identify the spans of the beam or frame between the nodes. By drawing the deflected shape of the structure, it will be possible to identify the number of degrees of freedom. Here each node can possibly have an angular displacement and a linear displacement. *Compatibility* at the nodes is maintained provided the members that are fixed connected to a node undergo the same displacements as the node. If these displacements are unknown, and in general they will be, then for convenience *assume* they act in the *positive direction* so as to cause *clockwise* rotation of a member or joint, Fig. 11–2.

Slope-Deflection Equations

The slope-deflection equations relate the unknown moments applied to the nodes to the displacements of the nodes for any span of the structure. If a load exists on the span, compute the FEMs using the table given on the inside back cover. Also, if a node has a linear displacement, Δ, compute $\psi = \Delta/L$ for the adjacent spans. Apply Eq. 11–8 to each end of the span, thereby generating *two* slope-deflection equations for each span. However, if a span at the *end* of a continuous beam or frame is pin supported, apply Eq. 11–10 only to the restrained end, thereby generating *one* slope-deflection equation for the span.

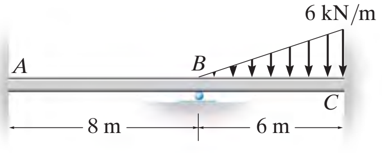
Equilibrium Equations

Write an equilibrium equation for each unknown degree of freedom for the structure. Each of these equations should be expressed in terms of unknown internal moments as specified by the slope-deflection equations. For beams and frames write the moment equation of equilibrium at each support, and for frames also write joint moment equations of equilibrium. If the frame sidesways or deflects horizontally, column shears should be related to the moments at the ends of the column. This is discussed in Sec. 11.5.

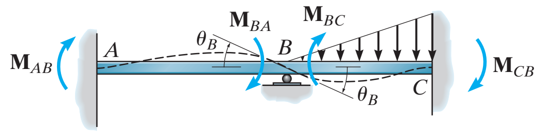
Substitute the slope-deflection equations into the equilibrium equations and solve for the unknown joint displacements. These results are then substituted into the slope-deflection equations to determine the internal moments at the ends of each member. If any of the results are *negative,* they indicate *counterclockwise* rotation; whereas *positive* moments and displacements are applied *clockwise.*

EXAMPLE 11.1

Draw the shear and moment diagrams for the beam shown in Fig. 11–10*a*. *EI* is constant.



(a)



(b)

Fig. 11–10

SOLUTION

Slope-Deflection Equations. Two spans must be considered in this problem. Since there is *no* span having the far end pinned or roller supported, Eq. 11–8 applies to the solution. Using the formulas for the FEMs tabulated for the triangular loading given on the inside back cover, we have

$$(\text{FEM})_{BC} = -\frac{wL^2}{30} = -\frac{6(6)^2}{30} = -7.2 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{20} = \frac{6(6)^2}{20} = 10.8 \text{ kN}\cdot\text{m}$$

Note that $(\text{FEM})_{BC}$ is negative since it acts counterclockwise *on the beam* at B. Also, $(\text{FEM})_{AB} = (\text{FEM})_{BA} = 0$ since there is no load on span AB.

In order to identify the unknowns, the elastic curve for the beam is shown in Fig. 11–10*b*. As indicated, there are four unknown internal moments. Only the slope at B, θ_B, is unknown. Since A and C are fixed supports, $\theta_A = \theta_C = 0$. Also, since the supports do not settle, nor are they displaced up or down, $\psi_{AB} = \psi_{BC} = 0$. For span AB, considering A to be the near end and B to be the far end, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{8}\right)[2(0) + \theta_B - 3(0)] + 0 = \frac{EI}{4}\theta_B \qquad (1)$$

Now, considering B to be the near end and A to be the far end, we have

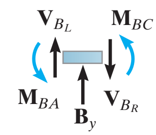
$$M_{BA} = 2E\left(\frac{I}{8}\right)[2\theta_B + 0 - 3(0)] + 0 = \frac{EI}{2}\theta_B \qquad (2)$$

In a similar manner, for span BC we have

$$M_{BC} = 2E\left(\frac{I}{6}\right)[2\theta_B + 0 - 3(0)] - 7.2 = \frac{2EI}{3}\theta_B - 7.2 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_B - 3(0)] + 10.8 = \frac{EI}{3}\theta_B + 10.8 \qquad (4)$$

Equilibrium Equations. The above four equations contain five unknowns. The necessary fifth equation comes from the condition of moment equilibrium at support B. The free-body diagram of a segment of the beam at B is shown in Fig. 11–10c. Here $\mathbf{M}_{BA}$ and $\mathbf{M}_{BC}$ are assumed to act in the positive direction to be consistent with the slope-deflection equations.* The beam shears contribute negligible moment about B since the segment is of differential length. Thus,



(c)

$$\circlearrowright + \Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad (5)$$

To solve, substitute Eqs. (2) and (3) into Eq. (5), which yields

$$\theta_B = \frac{6.17}{EI}$$

Resubstituting this value into Eqs. (1)–(4) yields

$$M_{AB} = 1.54 \text{ kN}\cdot\text{m}$$

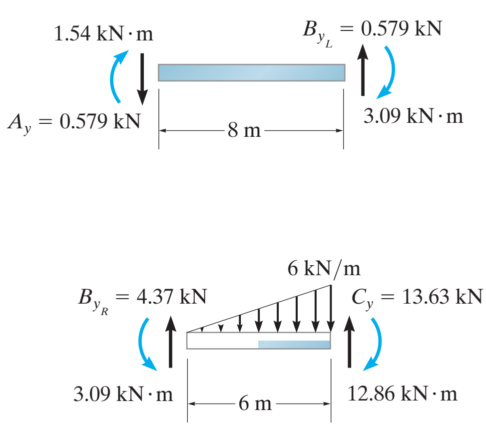
$$M_{BA} = 3.09 \text{ kN}\cdot\text{m}$$

$$M_{BC} = -3.09 \text{ kN}\cdot\text{m}$$

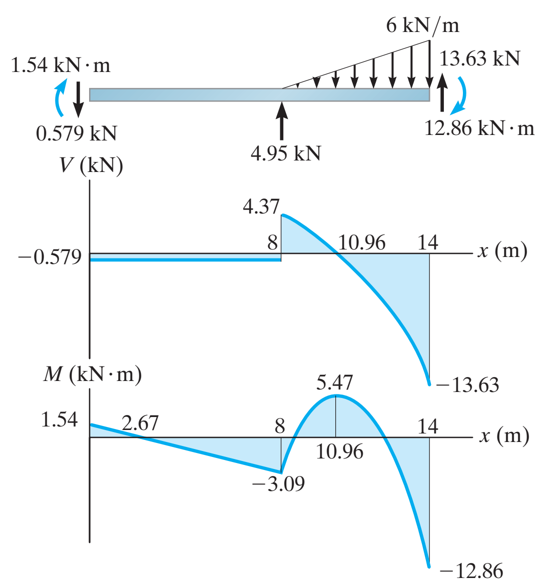
$$M_{CB} = 12.86 \text{ kN}\cdot\text{m}$$

The negative value for M_{BC} indicates that this moment acts counterclockwise on the beam, not clockwise as shown in Fig. 11–10b.

Using these results, the shears at the end spans are determined from the equilibrium equations, Fig. 11–10d. The free-body diagram of the entire beam and the shear and moment diagrams are shown in Fig. 11–10e.



(d)

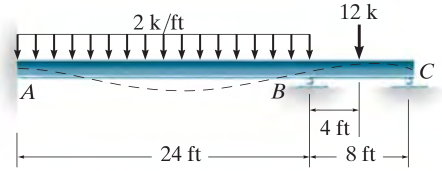


(e)

*Clockwise on the beam segment, but—by the principle of action, equal but opposite reaction—counterclockwise on the support.

EXAMPLE 11.2

Draw the shear and moment diagrams for the beam shown in Fig. 11–11*a*. *EI* is constant.



(a)

Fig. 11–11

SOLUTION

Slope-Deflection Equations. Two spans must be considered in this problem. Equation 11–8 applies to span AB. We can use Eq. 11–10 for span BC since the *end* C *is on a roller*. Using the formulas for the FEMs tabulated on the inside back cover, we have

$$(\text{FEM})_{AB} = -\frac{wL^2}{12} = -\frac{1}{12}(2)(24)^2 = -96\ \text{k}\cdot\text{ft}$$

$$(\text{FEM})_{BA} = \frac{wL^2}{12} = \frac{1}{12}(2)(24)^2 = 96\ \text{k}\cdot\text{ft}$$

$$(\text{FEM})_{BC} = -\frac{3PL}{16} = -\frac{3(12)(8)}{16} = -18\ \text{k}\cdot\text{ft}$$

Note that $(\text{FEM})_{AB}$ and $(\text{FEM})_{BC}$ are negative since they act counterclockwise on the beam at A and B, respectively. Also, since the supports do not settle, $\psi_{AB} = \psi_{BC} = 0$. Applying Eq. 11–8 for span AB and realizing that $\theta_A = 0$, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{24}\right)[2(0) + \theta_B - 3(0)] - 96$$

$$M_{AB} = 0.08333EI\theta_B - 96 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{24}\right)[2\theta_B + 0 - 3(0)] + 96$$

$$M_{BA} = 0.1667EI\theta_B + 96 \qquad (2)$$

Applying Eq. 11–10 with B as the near end and C as the far end, we have

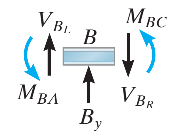
$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{BC} = 3E\left(\frac{I}{8}\right)(\theta_B - 0) - 18$$

$$M_{BC} = 0.375EI\theta_B - 18 \qquad (3)$$

Remember that Eq. 11–10 is *not* applied from C (near end) to B (far end).

Equilibrium Equations. The above three equations contain four unknowns. The necessary fourth equation comes from the conditions of equilibrium at the support B. The free-body diagram is shown in Fig. 11–11b. We have



(b)

$$\circlearrowright + \Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad (4)$$

To solve, substitute Eqs. (2) and (3) into Eq. (4), which yields

$$\theta_B = -\frac{144.0}{EI}$$

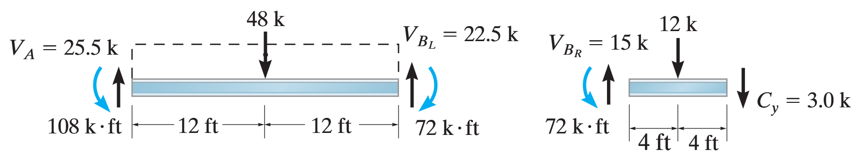
Since θ_B is negative (counterclockwise) the elastic curve for the beam has been correctly drawn in Fig. 11–11a. Substituting θ_B into Eqs. (1)–(3), we get

$$M_{AB} = -108.0 \text{ k} \cdot \text{ft}$$

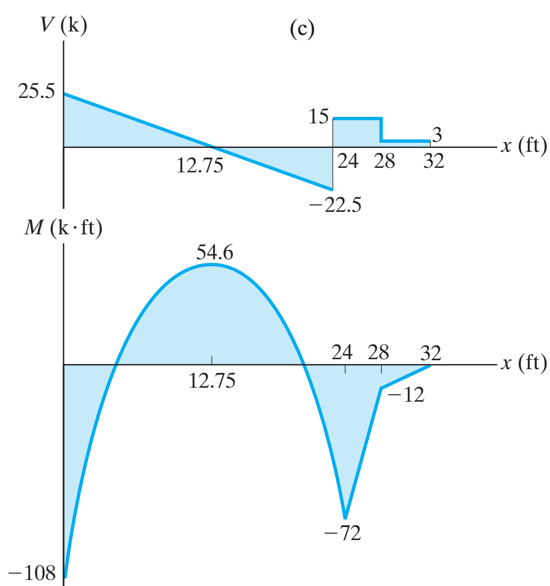
$$M_{BA} = 72.0 \text{ k} \cdot \text{ft}$$

$$M_{BC} = -72.0 \text{ k} \cdot \text{ft}$$

Using these data for the moments, the shear reactions at the ends of the beam spans have been determined in Fig. 11–11c. The shear and moment diagrams are plotted in Fig. 11–11d.



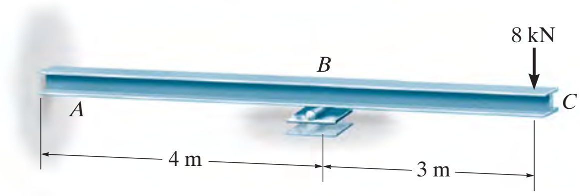
(c)



(d)

EXAMPLE 11.3

Determine the moment at A and B for the beam shown in Fig. 11–12a. The support at B is displaced (settles) 80 mm. Take $E = 200$ GPa, $I = 5(10^6)$ mm^4.

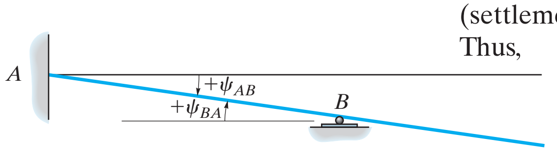


(a)

Fig. 11–12

SOLUTION

Slope-Deflection Equations. Only one span (AB) must be considered in this problem since the moment $\mathbf{M}_{BC}$ due to the overhang can be calculated from statics. Since there is no loading on span AB, the FEMs are zero. As shown in Fig. 11–12b, the downward displacement (settlement) of B causes the cord for span AB to rotate clockwise. Thus,



(b)

$$\psi_{AB} = \psi_{BA} = \frac{0.08 \text{ m}}{4} = 0.02 \text{ rad}$$

The stiffness for AB is

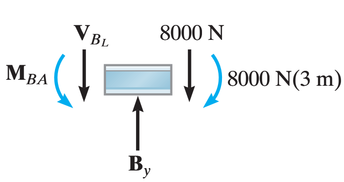
$$k = \frac{I}{L} = \frac{5(10^6)\text{ mm}^4(10^{-12})\text{ m}^4/\text{mm}^4}{4\text{ m}} = 1.25(10^{-6})\text{ m}^3$$

Applying the slope-deflection equation, Eq. 11–8, to span AB, with $\theta_A = 0$, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2(200(10^9)\text{ N/m}^2)[1.25(10^{-6})\text{ m}^3][2(0) + \theta_B - 3(0.02)] + 0 \quad (1)$$

$$M_{BA} = 2(200(10^9)\text{ N/m}^2)[1.25(10^{-6})\text{ m}^3][2\theta_B + 0 - 3(0.02)] + 0 \quad (2)$$



(c)

Equilibrium Equations. The free-body diagram of the beam at support B is shown in Fig. 11–12c. Moment equilibrium requires

$$\circlearrowleft + \Sigma M_B = 0; \qquad M_{BA} - 8000\text{ N}(3\text{ m}) = 0$$

Substituting Eq. (2) into this equation yields

$$1(10^6)\theta_B - 30(10^3) = 24(10^3)$$

$$\theta_B = 0.054 \text{ rad}$$

Thus, from Eqs. (1) and (2),

$$M_{AB} = -3.00 \text{ kN}\cdot\text{m}$$

$$M_{BA} = 24.0 \text{ kN}\cdot\text{m}$$

EXAMPLE 11.4

Determine the internal moments at the supports of the beam shown in Fig. 11–13*a*. The roller support at C is pushed downward 0.1 ft by the force $\mathbf{P}$. Take $E = 29(10^3)$ ksi, $I = 1500\ \text{in}^4$.

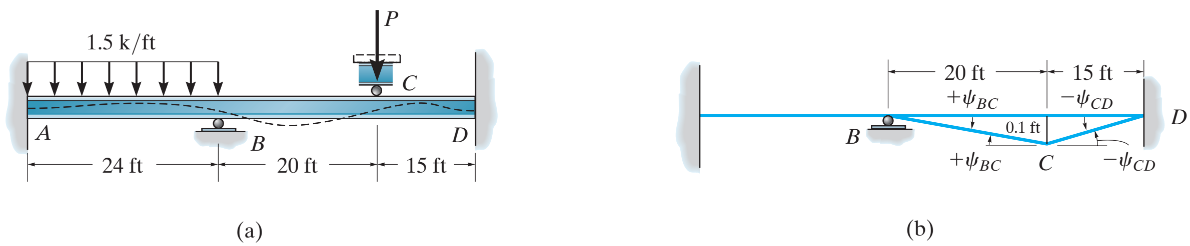


Fig. 11–13

SOLUTION

Slope-Deflection Equations. Three spans must be considered in this problem. Equation 11–8 applies since the end supports A and D are fixed. Also, only span AB has FEMs.

$$(\text{FEM})_{AB} = -\frac{wL^2}{12} = -\frac{1}{12}(1.5)(24)^2 = -72.0\ \text{k}\cdot\text{ft}$$

$$(\text{FEM})_{BA} = \frac{wL^2}{12} = \frac{1}{12}(1.5)(24)^2 = 72.0\ \text{k}\cdot\text{ft}$$

As shown in Fig. 11–13*b*, the displacement (or settlement) of the support C causes ψ_{BC} to be positive, since the cord for span BC rotates clockwise, and ψ_{CD} to be negative, since the cord for span CD rotates counterclockwise. Hence,

$$\psi_{BC} = \frac{0.1\ \text{ft}}{20\ \text{ft}} = 0.005\ \text{rad} \qquad \psi_{CD} = -\frac{0.1\ \text{ft}}{15\ \text{ft}} = -0.00667\ \text{rad}$$

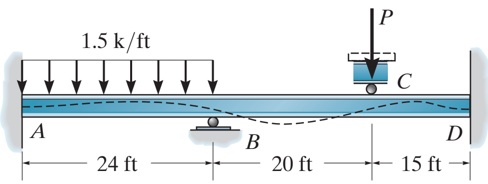
Also, expressing the units for the stiffness in feet, we have

$$k_{AB} = \frac{1500}{24(12)^4} = 0.003014\ \text{ft}^3 \qquad k_{BC} = \frac{1500}{20(12)^4} = 0.003617\ \text{ft}^3$$

$$k_{CD} = \frac{1500}{15(12)^4} = 0.004823\ \text{ft}^3$$

Noting that $\theta_A = \theta_D = 0$ since A and D are fixed supports, and applying the slope-deflection Eq. 11–8 twice to each span, we have

EXAMPLE 11.4 (Continued)



(a)

For span AB:

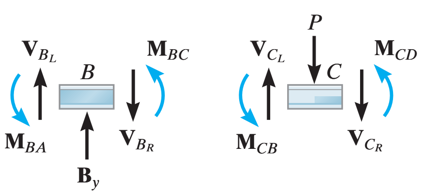
$$M_{AB} = 2[29(10^3)(12)^2](0.003014)[2(0) + \theta_B - 3(0)] - 72$$
$$M_{AB} = 25\,173.6\theta_B - 72 \tag{1}$$
$$M_{BA} = 2[29(10^3)(12)^2](0.003014)[2\theta_B + 0 - 3(0)] + 72$$
$$M_{BA} = 50\,347.2\theta_B + 72 \tag{2}$$

For span BC:

$$M_{BC} = 2[29(10^3)(12)^2](0.003617)[2\theta_B + \theta_C - 3(0.005)] + 0$$
$$M_{BC} = 60\,416.7\theta_B + 30\,208.3\theta_C - 453.1 \tag{3}$$
$$M_{CB} = 2[29(10^3)(12)^2](0.003617)[2\theta_C + \theta_B - 3(0.005)] + 0$$
$$M_{CB} = 60\,416.7\theta_C + 30\,208.3\theta_B - 453.1 \tag{4}$$

For span CD:

$$M_{CD} = 2[29(10^3)(12)^2](0.004823)[2\theta_C + 0 - 3(-0.00667)] + 0$$
$$M_{CD} = 80\,555.6\theta_C + 0 + 805.6 \tag{5}$$
$$M_{DC} = 2[29(10^3)(12)^2](0.004823)[2(0) + \theta_C - 3(-0.00667)] + 0$$
$$M_{DC} = 40\,277.8\theta_C + 805.6 \tag{6}$$



(c)

Equilibrium Equations. These six equations contain eight unknowns. Writing the moment equilibrium equations for the supports at B and C, Fig. 10–13c, we have

$$\circlearrowright+\Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0 \tag{7}$$
$$\circlearrowright+\Sigma M_C = 0; \qquad M_{CB} + M_{CD} = 0 \tag{8}$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), and Eqs. (4) and (5) into Eq. (8). This yields

$$\theta_C + 3.667\theta_B = 0.01262$$
$$-\theta_C - 0.214\theta_B = 0.00250$$

Thus,

$$\theta_B = 0.00438 \text{ rad} \qquad \theta_C = -0.00344 \text{ rad}$$

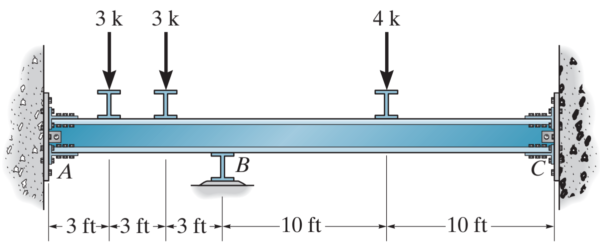
The negative value for θ_C indicates counterclockwise rotation of the tangent at C, Fig. 11–13a. Substituting these values into Eqs. (1)–(6) yields

$$M_{AB} = 38.2 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$
$$M_{BA} = 292 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$
$$M_{BC} = -292 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$
$$M_{CB} = -529 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$
$$M_{CD} = 529 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$
$$M_{DC} = 667 \text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

Apply these end moments to spans BC and CD and show that $V_{C_L} = 41.05$ k, $V_{C_R} = -79.73$ k and the force on the roller is $P = 121$ k.

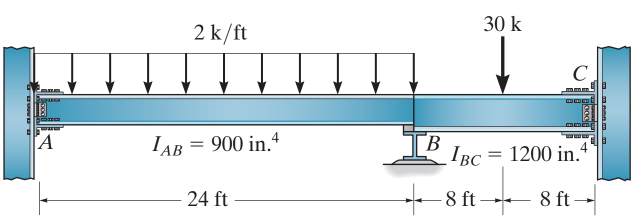
PROBLEMS

11–1. Determine the moments at A, B, and C and then draw the moment diagram. EI is constant. Assume the support at B is a roller and A and C are fixed.



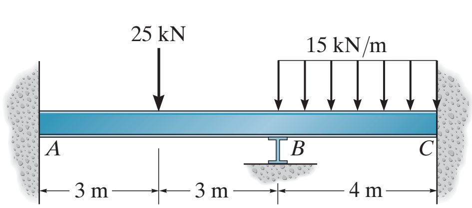
Prob. 11–1

11–2. Determine the moments at A, B, and C, then draw the moment diagram for the beam. The moment of inertia of each span is indicated in the figure. Assume the support at B is a roller and A and C are fixed. $E = 29(10^3)$ ksi.



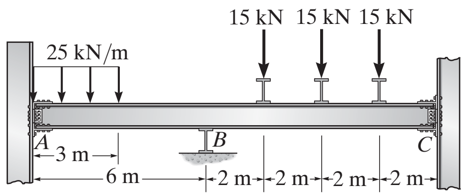
Prob. 11–2

11–3. Determine the moments at the supports A and C, then draw the moment diagram. Assume joint B is a roller. EI is constant.



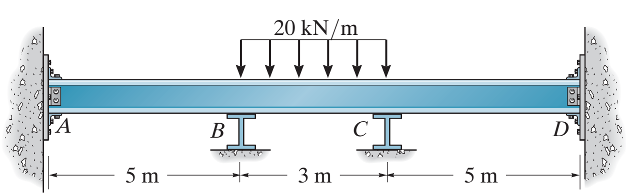
Prob. 11–3

***11–4.** Determine the moments at the supports, then draw the moment diagram. Assume B is a roller and A and C are fixed. EI is constant.



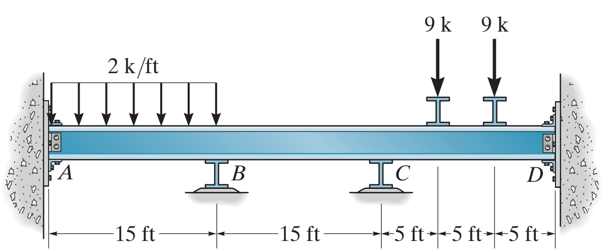
Prob. 11–4

11–5. Determine the moment at A, B, C and D, then draw the moment diagram for the beam. Assume the supports at A and D are fixed and B and C are rollers. EI is constant.



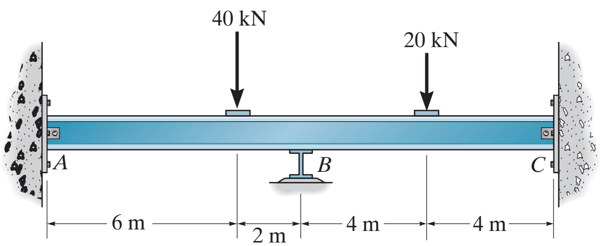
Prob. 11–5

11–6. Determine the moments at A, B, C and D, then draw the moment diagram for the beam. Assume the supports at A and D are fixed and B and C are rollers. EI is constant.



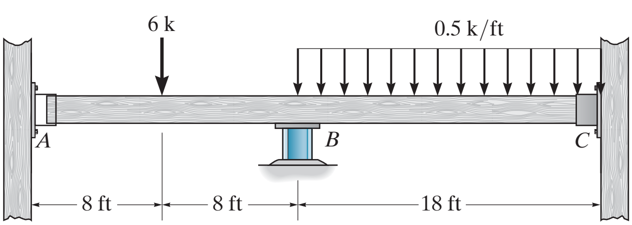
Prob. 11–6

11–7. Determine the moment at B, then draw the moment diagram for the beam. Assume the supports at A and C are pins and B is a roller. EI is constant.



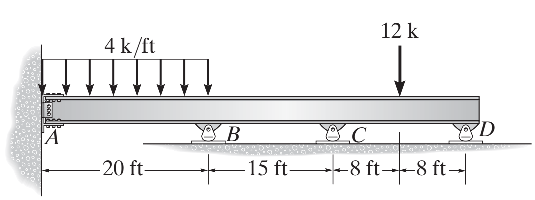
Prob. 11–7

***11–8.** Determine the moments at A, B, and C, then draw the moment diagram. EI is constant. Assume the support at B is a roller and A and C are fixed.



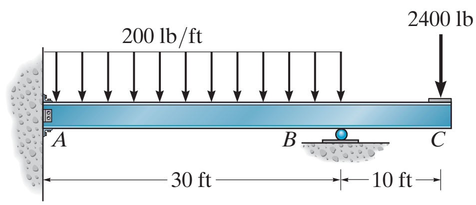
Prob. 11–8

11–9. Determine the moments at each support, then draw the moment diagram. Assume A is fixed. EI is constant.



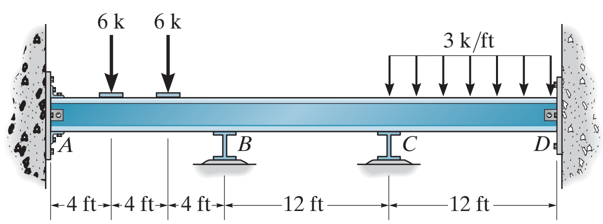
Prob. 11–9

11–10. Determine the moments at A and B, then draw the moment diagram for the beam. EI is constant.



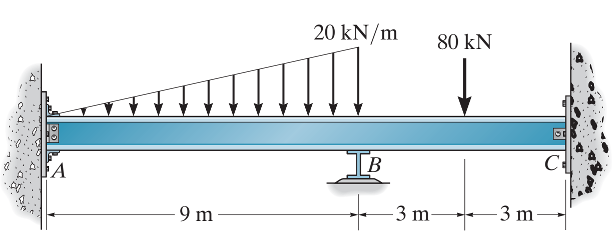
Prob. 11–10

11–11. Determine the moments at A, B, and C, then draw the moment diagram for the beam. Assume the support at A is fixed, B and C are rollers, and D is a pin. EI is constant.



Prob. 11–11

***11–12.** Determine the moments acting at A and B. Assume A is fixed supported, B is a roller, and C is a pin. EI is constant.



Prob. 11–12

11.4 Analysis of Frames: No Sidesway

A frame will not sidesway, or be displaced to the left or right, provided it is properly restrained. Examples are shown in Fig. 11–14. Also, no sidesway will occur in an unrestrained frame provided it is symmetric with respect to both loading and geometry, as shown in Fig. 11–15. For both cases the term ψ in the slope-deflection equations is equal to zero, since bending does not cause the joints to have a linear displacement.

The following examples illustrate application of the slope-deflection equations using the procedure for analysis outlined in Sec. 11–3 for these types of frames.

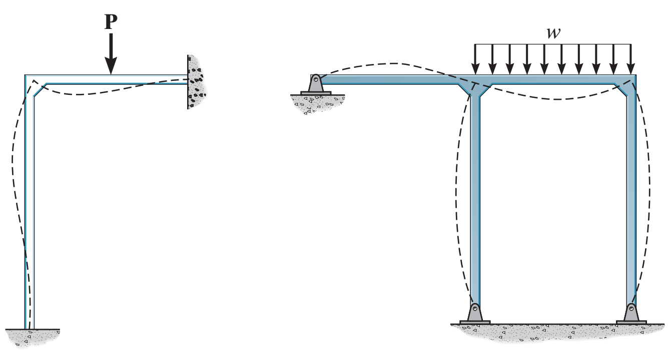


Fig. 11–14

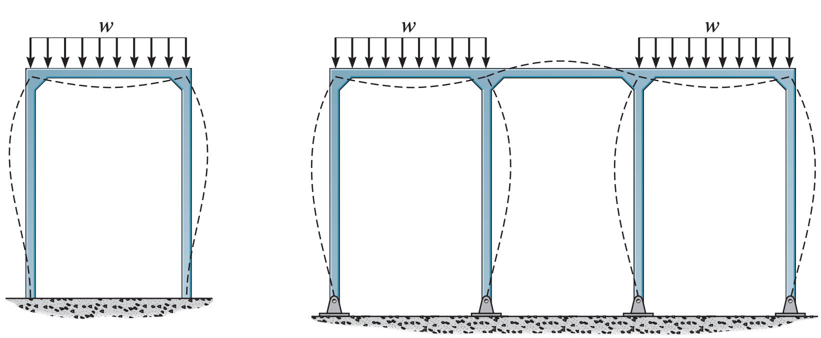
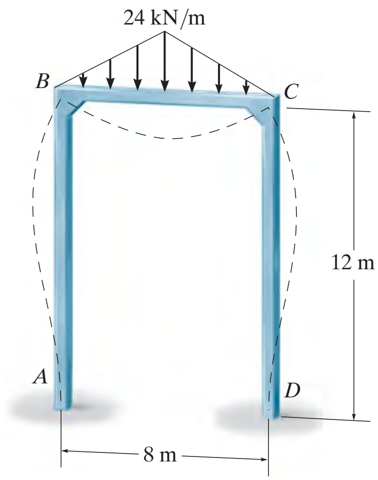


Fig. 11–15

EXAMPLE 11.5



(a)

Fig. 11–16

Determine the moments at each joint of the frame shown in Fig. 11–16*a*. *EI* is constant.

SOLUTION

Slope-Deflection Equations. Three spans must be considered in this problem: AB, BC, and CD. Since the spans are fixed supported at A and D, Eq. 11–8 applies for the solution.

From the table on the inside back cover, the FEMs for BC are

$$(\text{FEM})_{BC} = -\frac{5wL^2}{96} = -\frac{5(24)(8)^2}{96} = -80 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{5wL^2}{96} = \frac{5(24)(8)^2}{96} = 80 \text{ kN}\cdot\text{m}$$

Note that $\theta_A = \theta_D = 0$ and $\psi_{AB} = \psi_{BC} = \psi_{CD} = 0$, since no sidesway will occur.

Applying Eq. 11–8, we have

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{12}\right)[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 0.1667EI\theta_B \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{12}\right)[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 0.333EI\theta_B \qquad (2)$$

$$M_{BC} = 2E\left(\frac{I}{8}\right)[2\theta_B + \theta_C - 3(0)] - 80$$

$$M_{BC} = 0.5EI\theta_B + 0.25EI\theta_C - 80 \qquad (3)$$

$$M_{CB} = 2E\left(\frac{I}{8}\right)[2\theta_C + \theta_B - 3(0)] + 80$$

$$M_{CB} = 0.5EI\theta_C + 0.25EI\theta_B + 80 \qquad (4)$$

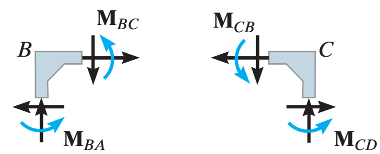
$$M_{CD} = 2E\left(\frac{I}{12}\right)[2\theta_C + 0 - 3(0)] + 0$$

$$M_{CD} = 0.333EI\theta_C \qquad (5)$$

$$M_{DC} = 2E\left(\frac{I}{12}\right)[2(0) + \theta_C - 3(0)] + 0$$

$$M_{DC} = 0.1667EI\theta_C \qquad (6)$$

Equilibrium Equations. The preceding six equations contain eight unknowns. The remaining two equilibrium equations come from moment equilibrium at joints B and C, Fig. 11–16b. We have



(b)

$$M_{BA} + M_{BC} = 0 \qquad (7)$$
$$M_{CB} + M_{CD} = 0 \qquad (8)$$

To solve these eight equations, substitute Eqs. (2) and (3) into Eq. (7) and substitute Eqs. (4) and (5) into Eq. (8). We get

$$0.833EI\theta_B + 0.25EI\theta_C = 80$$
$$0.833EI\theta_C + 0.25EI\theta_B = -80$$

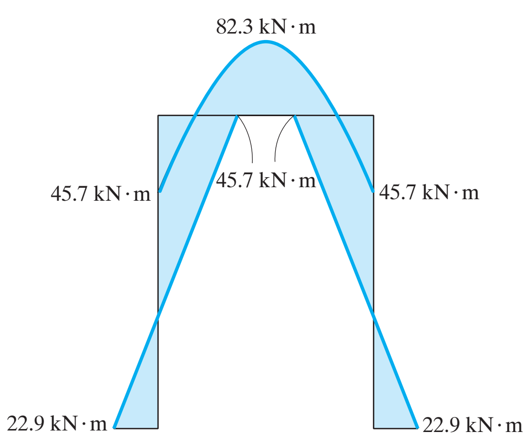
Solving simultaneously yields

$$\theta_B = -\theta_C = \frac{137.1}{EI}$$

which conforms with the way the frame deflects as shown in Fig. 11–16a. Substituting into Eqs. (1)–(6), we get

$$M_{AB} = 22.9 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{BA} = 45.7 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{BC} = -45.7 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{CB} = 45.7 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{CD} = -45.7 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$
$$M_{DC} = -22.9 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

Using these results, the reactions at the ends of each member can be determined from the equations of equilibrium, and the moment diagram for the frame can be drawn, Fig. 11–16c.

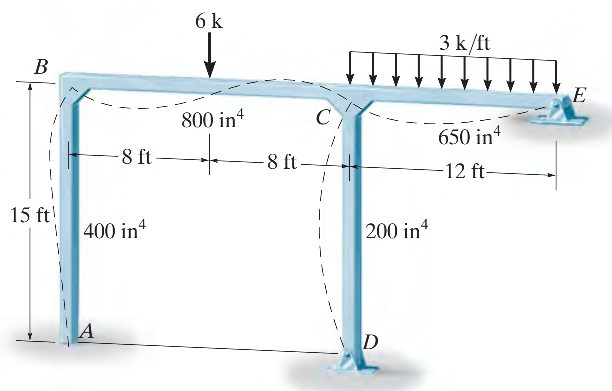


(c)

11

EXAMPLE 11.6

Determine the internal moments at each joint of the frame shown in Fig. 11–17*a*. The moment of inertia for each member is given in the figure. Take $E = 29(10^3)$ ksi.



(a)

Fig. 11–17

SOLUTION

Slope-Deflection Equations. Four spans must be considered in this problem. Equation 11–8 applies to spans *AB* and *BC*, and Eq. 11–10 will be applied to *CD* and *CE*, because the ends at *D* and *E* are pinned.

Computing the member stiffnesses, we have

$$k_{AB} = \frac{400}{15(12)^4} = 0.001286 \text{ ft}^3 \qquad k_{CD} = \frac{200}{15(12)^4} = 0.000643 \text{ ft}^3$$

$$k_{BC} = \frac{800}{16(12)^4} = 0.002411 \text{ ft}^3 \qquad k_{CE} = \frac{650}{12(12)^4} = 0.002612 \text{ ft}^3$$

The FEMs due to the loadings are

$$(\text{FEM})_{BC} = -\frac{PL}{8} = -\frac{6(16)}{8} = -12 \text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{CB} = \frac{PL}{8} = \frac{6(16)}{8} = 12 \text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{CE} = -\frac{wL^2}{8} = -\frac{3(12)^2}{8} = -54 \text{ k}\cdot\text{ft}$$

Applying Eqs. 11–8 and 11–10 to the frame and noting that $\theta_A = 0$, $\psi_{AB} = \psi_{BC} = \psi_{CD} = \psi_{CE} = 0$ since no sidesway occurs, we have

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2[29(10^3)(12)^2](0.001286)[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 10740.7\theta_B \qquad (1)$$

$$M_{BA} = 2[29(10^3)(12)^2](0.001286)[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 21\,481.5\theta_B \qquad (2)$$

$$M_{BC} = 2[29(10^3)(12)^2](0.002411)[2\theta_B + \theta_C - 3(0)] - 12$$

$$M_{BC} = 40\,277.8\theta_B + 20\,138.9\theta_C - 12 \qquad (3)$$

$$M_{CB} = 2[29(10^3)(12)^2](0.002411)[2\theta_C + \theta_B - 3(0)] + 12$$

$$M_{CB} = 20\,138.9\theta_B + 40\,277.8\theta_C + 12 \qquad (4)$$

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

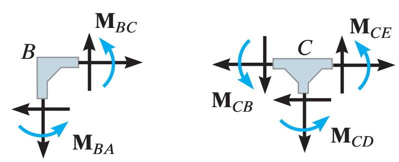
$$M_{CD} = 3[29(10^3)(12)^2](0.000643)[\theta_C - 0] + 0 \qquad (5)$$

$$M_{CD} = 8055.6\theta_C$$

$$M_{CE} = 3[29(10^3)(12)^2](0.002612)[\theta_C - 0] - 54$$

$$M_{CE} = 32\,725.7\theta_C - 54 \qquad (6)$$

Equations of Equilibrium. These six equations contain eight unknowns. Two moment equilibrium equations can be written for joints B and C, Fig. 11–17b. We have



(b)

$$M_{BA} + M_{BC} = 0 \qquad (7)$$

$$M_{CB} + M_{CD} + M_{CE} = 0 \qquad (8)$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), and Eqs. (4)–(6) into Eq. (8). This gives

$$61\,759.3\theta_B + 20\,138.9\theta_C = 12$$

$$20\,138.9\theta_B + 81\,059.0\theta_C = 42$$

Solving these equations simultaneously yields

$$\theta_B = 2.758(10^{-5})\text{ rad} \qquad \theta_C = 5.113(10^{-4})\text{ rad}$$

These values, being clockwise, tend to distort the frame as shown in Fig. 11–17a. Substituting these values into Eqs. (1)–(6) and solving, we get

$$M_{AB} = 0.296\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BA} = 0.592\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BC} = -0.592\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CB} = 33.1\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CD} = 4.12\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CE} = -37.3\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

11.5 Analysis of Frames: Sidesway

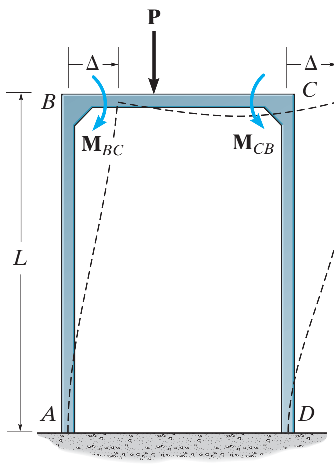
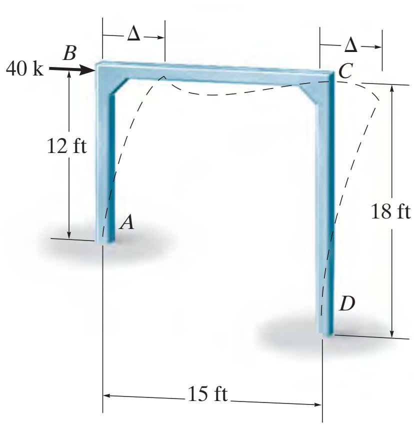


Fig. 11–18

A frame will sidesway, or be displaced to the side, when it or the loading acting on it is nonsymmetric. To illustrate this effect, consider the frame shown in Fig. 11–18. Here the loading **P** causes *unequal* moments $\mathbf{M}_{BC}$ and $\mathbf{M}_{CB}$ at the joints B and C, respectively. $\mathbf{M}_{BC}$ tends to displace joint B to the right, whereas $\mathbf{M}_{CB}$ tends to displace joint C to the left. Since $\mathbf{M}_{BC}$ is larger than $\mathbf{M}_{CB}$, the net result is a sidesway Δ of both joints B and C to the right, as shown in the figure.* When applying the slope-deflection equation to each column of this frame, we must therefore consider the column rotation ψ (since $\psi = \Delta/L$) as unknown in the equation. As a result an extra equilibrium equation must be included for the solution. In the previous sections it was shown that unknown *angular displacements* θ were related by joint *moment equilibrium equations*. In a similar manner, when unknown joint *linear displacements* Δ (or span rotations ψ) occur, we must write *force equilibrium equations* in order to obtain the complete solution. The unknowns in these equations, however, must only involve the internal *moments* acting at the ends of the columns, since the slope-deflection equations involve these moments. The technique for solving problems for frames with sidesway is best illustrated by examples.

EXAMPLE 11.7



(a)

Fig. 11–19

Determine the moments at each joint of the frame shown in Fig. 11–19*a*. EI is constant.

SOLUTION

Slope-Deflection Equations. Since the ends A and D are fixed, Eq. 11–8 applies for all three spans of the frame. Sidesway occurs here since both the applied loading and the geometry of the frame are nonsymmetric. Here the load is applied directly to joint B and therefore no FEMs act at the joints. As shown in Fig. 11–19*a*, both joints B and C are assumed to be displaced an *equal amount* Δ. Consequently, $\psi_{AB} = \Delta/12$ and $\psi_{DC} = \Delta/18$. Both terms are positive since the cords of members AB and CD "rotate" clockwise. Relating ψ_{AB} to ψ_{DC}, we have $\psi_{AB} = (18/12)\psi_{DC}$. Applying Eq. 11–8 to the frame, we have

$$M_{AB} = 2E\left(\frac{I}{12}\right)\left[2(0) + \theta_B - 3\left(\frac{18}{12}\psi_{DC}\right)\right] + 0 = EI(0.1667\theta_B - 0.75\psi_{DC}) \quad (1)$$

$$M_{BA} = 2E\left(\frac{I}{12}\right)\left[2\theta_B + 0 - 3\left(\frac{18}{12}\psi_{DC}\right)\right] + 0 = EI(0.333\theta_B - 0.75\psi_{DC}) \quad (2)$$

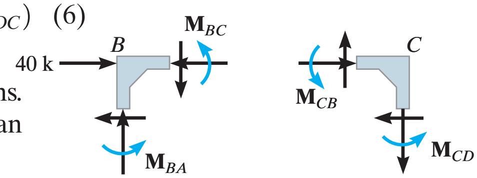
$$M_{BC} = 2E\left(\frac{I}{15}\right)[2\theta_B + \theta_C - 3(0)] + 0 = EI(0.267\theta_B + 0.133\theta_C) \quad (3)$$

*Recall that the deformation of all three members due to shear and axial force is neglected.

$$M_{CB} = 2E\left(\frac{I}{15}\right)[2\theta_C + \theta_B - 3(0)] + 0 = EI(0.267\theta_C + 0.133\theta_B) \quad (4)$$

$$M_{CD} = 2E\left(\frac{I}{18}\right)[2\theta_C + 0 - 3\psi_{DC}] + 0 = EI(0.222\theta_C - 0.333\psi_{DC}) \quad (5)$$

$$M_{DC} = 2E\left(\frac{I}{18}\right)[2(0) + \theta_C - 3\psi_{DC}] + 0 = EI(0.111\theta_C - 0.333\psi_{DC}) \quad (6)$$



(b)

Equations of Equilibrium. The six equations contain nine unknowns. Two moment equilibrium equations for joints B and C, Fig. 11–19b, can be written, namely,

$$M_{BA} + M_{BC} = 0 \quad (7)$$

$$M_{CB} + M_{CD} = 0 \quad (8)$$

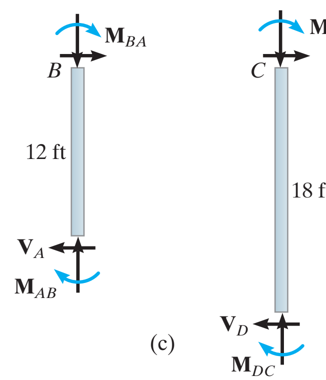
Since a horizontal displacement Δ occurs, we will consider summing forces on the *entire frame* in the x direction. This yields

$$\xrightarrow{+} \Sigma F_x = 0; \qquad 40 - V_A - V_D = 0$$

The horizontal reactions or column shears V_A and V_D can be related to the internal moments by considering the free-body diagram of each column separately, Fig. 11–19c. We have

$$\Sigma M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{12}$$

$$\Sigma M_C = 0; \qquad V_D = -\frac{M_{DC} + M_{CD}}{18}$$



(c)

Thus,

$$40 + \frac{M_{AB} + M_{BA}}{12} + \frac{M_{DC} + M_{CD}}{18} = 0 \quad (9)$$

In order to solve, substitute Eqs. (2) and (3) into Eq. (7), Eqs. (4) and (5) into Eq. (8), and Eqs. (1), (2), (5), (6) into Eq. (9). This yields

$$0.6\theta_B + 0.133\theta_C - 0.75\psi_{DC} = 0$$

$$0.133\theta_B + 0.489\theta_C - 0.333\psi_{DC} = 0$$

$$0.5\theta_B + 0.222\theta_C - 1.944\psi_{DC} = -\frac{480}{EI}$$

Solving simultaneously, we have

$$EI\theta_B = 438.81 \qquad EI\theta_C = 136.18 \qquad EI\psi_{DC} = 375.26$$

Finally, using these results and solving Eqs. (1)–(6) yields

$M_{AB} = -208\ \text{k}\cdot\text{ft}$ *Ans.*

$M_{BA} = -135\ \text{k}\cdot\text{ft}$ *Ans.*

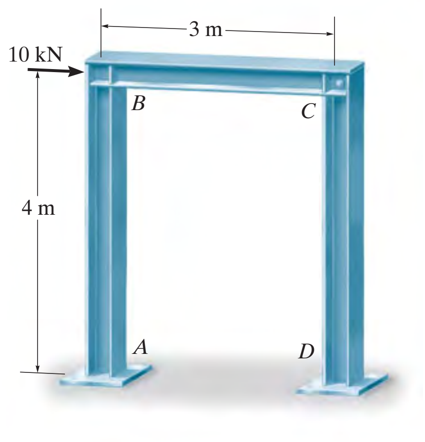
$M_{BC} = 135\ \text{k}\cdot\text{ft}$ *Ans.*

$M_{CB} = 94.8\ \text{k}\cdot\text{ft}$ *Ans.*

$M_{CD} = -94.8\ \text{k}\cdot\text{ft}$ *Ans.*

$M_{DC} = -110\ \text{k}\cdot\text{ft}$ *Ans.*

EXAMPLE 11.8



(a)

Fig. 11–20

Determine the moments at each joint of the frame shown in Fig. 11–20*a*. The supports at A and D are fixed and joint C is assumed pin connected. EI is constant for each member.

SOLUTION

Slope-Deflection Equations. We will apply Eq. 11–8 to member AB since it is fixed connected at both ends. Equation 11–10 can be applied from B to C and from D to C since the pin at C supports zero moment. As shown by the deflection diagram, Fig. 11–20*b*, there is an unknown linear displacement Δ of the frame and unknown angular displacement θ_B at joint B.* Due to Δ, the cord members AB and CD rotate clockwise, $\psi = \psi_{AB} = \psi_{DC} = \Delta/4$. Realizing that $\theta_A = \theta_D = 0$ and that there are no FEMs for the members, we have

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

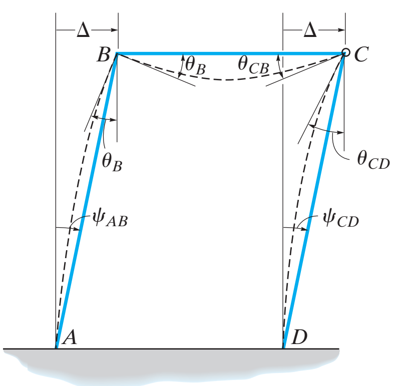
$$M_{AB} = 2E\left(\frac{I}{4}\right)[2(0) + \theta_B - 3\psi] + 0 \qquad (1)$$

$$M_{BA} = 2E\left(\frac{I}{4}\right)(2\theta_B + 0 - 3\psi) + 0 \qquad (2)$$

$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{BC} = 3E\left(\frac{I}{3}\right)(\theta_B - 0) + 0 \qquad (3)$$

$$M_{DC} = 3E\left(\frac{I}{4}\right)(0 - \psi) + 0 \qquad (4)$$



(b)

Equilibrium Equations. Moment equilibrium of joint B, Fig. 11–20*c*, requires

$$M_{BA} + M_{BC} = 0 \qquad (5)$$

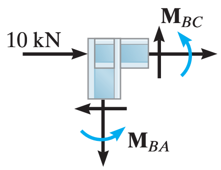
If forces are summed for the *entire frame* in the horizontal direction, we have

$$\xrightarrow{+} \Sigma F_x = 0; \qquad 10 - V_A - V_D = 0 \qquad (6)$$

As shown on the free-body diagram of each column, Fig. 11–20*d*, we have

$$\Sigma M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{4}$$

$$\Sigma M_C = 0; \qquad V_D = -\frac{M_{DC}}{4}$$



(c)

*The angular displacements θ_{CB} and θ_{CD} at joint C (pin) are not included in the analysis since Eq. 11–10 is to be used.

Thus, from Eq. (6),

$$10 + \frac{M_{AB} + M_{BA}}{4} + \frac{M_{DC}}{4} = 0 \qquad (7)$$

Substituting the slope-deflection equations into Eqs. (5) and (7) and simplifying yields

$$\theta_B = \frac{3}{4}\psi$$

$$10 + \frac{EI}{4}\left(\frac{3}{2}\theta_B - \frac{15}{4}\psi\right) = 0$$

Thus,

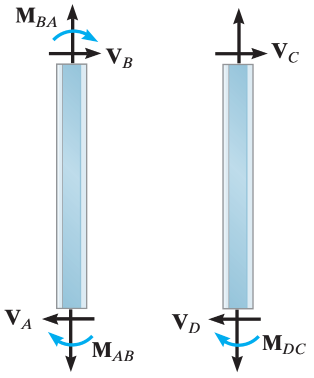
$$\theta_B = \frac{240}{21EI} \qquad \psi = \frac{320}{21EI}$$

Substituting these values into Eqs. (1)–(4), we have

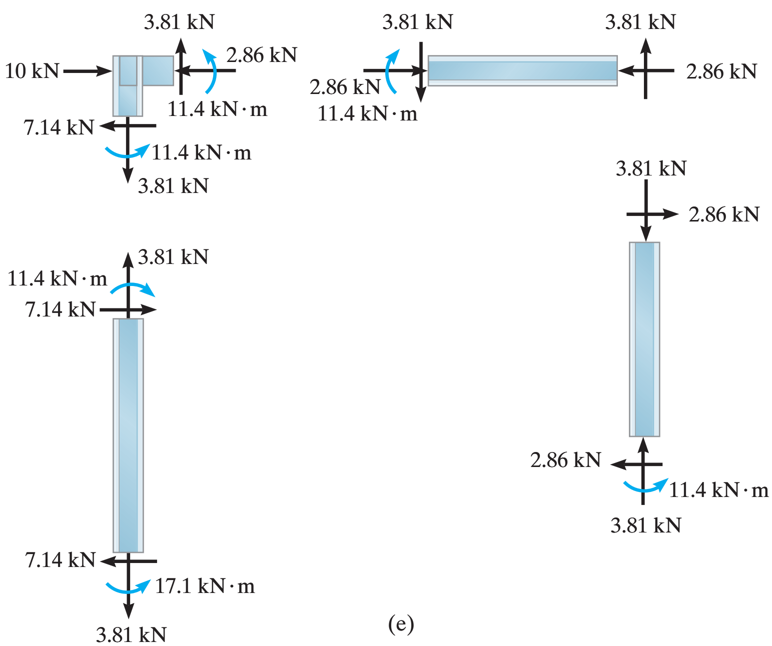
$M_{AB} = -17.1\ \text{kN}\cdot\text{m}$, $M_{BA} = -11.4\ \text{kN}\cdot\text{m}$ *Ans.*

$M_{BC} = 11.4\ \text{kN}\cdot\text{m}$, $M_{DC} = -11.4\ \text{kN}\cdot\text{m}$ *Ans.*

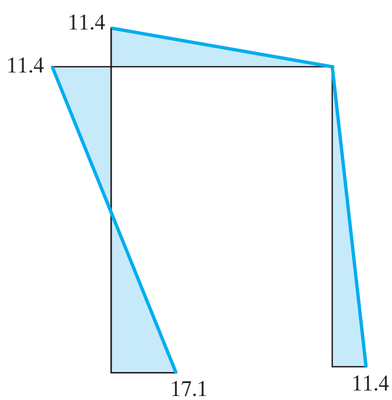
Using these results, the end reactions on each member can be determined from the equations of equilibrium, Fig. 11–20*e*. The moment diagram for the frame is shown in Fig. 11–20*f*.



(d)

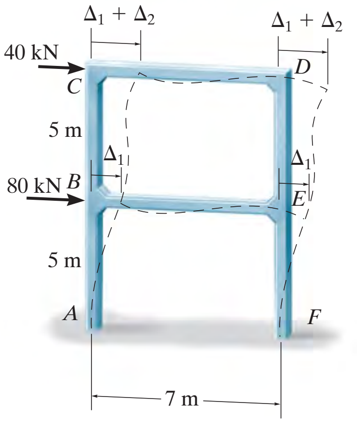


(e)



(f)

EXAMPLE 11.9



(a)

Fig. 11–21

Explain how the moments in each joint of the two-story frame shown in Fig. 11–21*a* are determined. *EI* is constant.

SOLUTION

Slope-Deflection Equation. Since the supports at A and F are fixed, Eq. 11–8 applies for all six spans of the frame. No FEMs have to be calculated, since the applied loading acts at the joints. Here the loading displaces joints B and E an amount Δ_1, and C and D an amount $\Delta_1 + \Delta_2$. As a result, members AB and FE undergo rotations of $\psi_1 = \Delta_1/5$ and BC and ED undergo rotations of $\psi_2 = \Delta_2/5$.

Applying Eq. 11–8 to the frame yields

$$M_{AB} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_B - 3\psi_1] + 0 \quad (1)$$

$$M_{BA} = 2E\left(\frac{I}{5}\right)[2\theta_B + 0 - 3\psi_1] + 0 \quad (2)$$

$$M_{BC} = 2E\left(\frac{I}{5}\right)[2\theta_B + \theta_C - 3\psi_2] + 0 \quad (3)$$

$$M_{CB} = 2E\left(\frac{I}{5}\right)[2\theta_C + \theta_B - 3\psi_2] + 0 \quad (4)$$

$$M_{CD} = 2E\left(\frac{I}{7}\right)[2\theta_C + \theta_D - 3(0)] + 0 \quad (5)$$

$$M_{DC} = 2E\left(\frac{I}{7}\right)[2\theta_D + \theta_C - 3(0)] + 0 \quad (6)$$

$$M_{BE} = 2E\left(\frac{I}{7}\right)[2\theta_B + \theta_E - 3(0)] + 0 \quad (7)$$

$$M_{EB} = 2E\left(\frac{I}{7}\right)[2\theta_E + \theta_B - 3(0)] + 0 \quad (8)$$

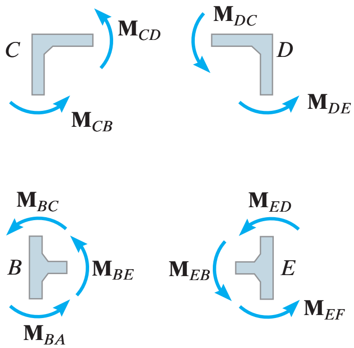
$$M_{ED} = 2E\left(\frac{I}{5}\right)[2\theta_E + \theta_D - 3\psi_2] + 0 \quad (9)$$

$$M_{DE} = 2E\left(\frac{I}{5}\right)[2\theta_D + \theta_E - 3\psi_2] + 0 \quad (10)$$

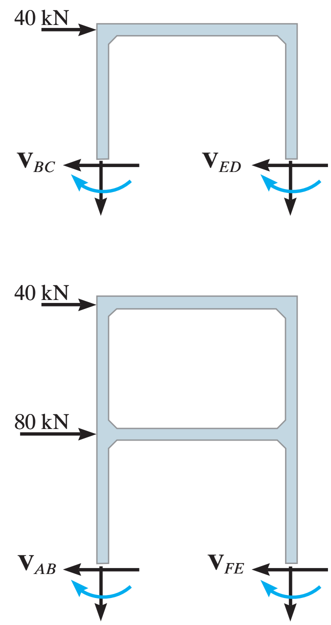
$$M_{FE} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_E - 3\psi_1] + 0 \quad (11)$$

$$M_{EF} = 2E\left(\frac{I}{5}\right)[2\theta_E + 0 - 3\psi_1] + 0 \quad (12)$$

These 12 equations contain 18 unknowns.



(b)



(c)

Equilibrium Equations. Moment equilibrium of joints B, C, D, and E, Fig. 11–21b, requires

$$M_{BA} + M_{BE} + M_{BC} = 0 \tag{13}$$

$$M_{CB} + M_{CD} = 0 \tag{14}$$

$$M_{DC} + M_{DE} = 0 \tag{15}$$

$$M_{EF} + M_{EB} + M_{ED} = 0 \tag{16}$$

As in the preceding examples, the shear at the base of all the columns for any story must balance the applied horizontal loads, Fig. 11–21c. This yields

$$\xrightarrow{+} \Sigma F_x = 0; \qquad 40 - V_{BC} - V_{ED} = 0$$

$$40 + \frac{M_{BC} + M_{CB}}{5} + \frac{M_{ED} + M_{DE}}{5} = 0 \tag{17}$$

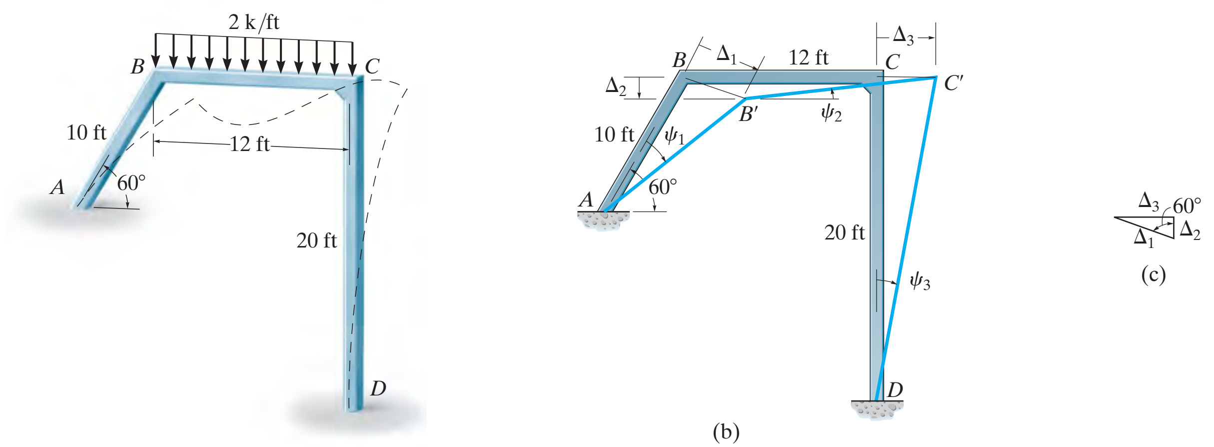
$$\xrightarrow{+} \Sigma F_x = 0; \qquad 40 + 80 - V_{AB} - V_{FE} = 0$$

$$120 + \frac{M_{AB} + M_{BA}}{5} + \frac{M_{EF} + M_{FE}}{5} = 0 \tag{18}$$

Solution requires substituting Eqs. (1)–(12) into Eqs. (13)–(18), which yields six equations having six unknowns, ψ_1, ψ_2, θ_B, θ_C, θ_D, and θ_E. These equations can then be solved simultaneously. The results are resubstituted into Eqs. (1)–(12), which yields the moments at the joints.

EXAMPLE 11.10

Determine the moments at each joint of the frame shown in Fig. 11–22*a*. *EI* is constant for each member.



(a) (b) (c)

Fig. 11–22

SOLUTION

Slope-Deflection Equations. Equation 11–8 applies to each of the three spans. The FEMs are

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{2(12)^2}{12} = -24\ \text{k}\cdot\text{ft}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{2(12)^2}{12} = 24\ \text{k}\cdot\text{ft}$$

The sloping member *AB* causes the frame to sidesway to the right as shown in Fig. 11–22*a*. As a result, joints *B* and *C* are subjected to both rotational *and* linear displacements. The linear displacements are shown in Fig. 11–22*b*, where *B* moves Δ_1 to B' and *C* moves Δ_3 to C'. These displacements cause the members' cords to rotate ψ_1, ψ_3 (clockwise) and $-\psi_2$ (counterclockwise) as shown.* Hence,

$$\psi_1 = \frac{\Delta_1}{10} \qquad \psi_2 = -\frac{\Delta_2}{12} \qquad \psi_3 = \frac{\Delta_3}{20}$$

As shown in Fig. 11–22*c*, the three displacements can be related. For example, $\Delta_2 = 0.5\Delta_1$ and $\Delta_3 = 0.866\Delta_1$. Thus, from the above equations we have

$$\psi_2 = -0.417\psi_1 \qquad \psi_3 = 0.433\psi_1$$

Using these results, the slope-deflection equations for the frame are

*Recall that distortions due to axial forces are neglected and the arc displacements BB' and CC' can be considered as straight lines, since ψ_1 and ψ_3 are actually very small.

$$M_{AB} = 2E\left(\frac{I}{10}\right)[2(0) + \theta_B - 3\psi_1] + 0 \quad (1)$$

$$M_{BA} = 2E\left(\frac{I}{10}\right)[2\theta_B + 0 - 3\psi_1] + 0 \quad (2)$$

$$M_{BC} = 2E\left(\frac{I}{12}\right)[2\theta_B + \theta_C - 3(-0.417\psi_1)] - 24 \quad (3)$$

$$M_{CB} = 2E\left(\frac{I}{12}\right)[2\theta_C + \theta_B - 3(-0.417\psi_1)] + 24 \quad (4)$$

$$M_{CD} = 2E\left(\frac{I}{20}\right)[2\theta_C + 0 - 3(0.433\psi_1)] + 0 \quad (5)$$

$$M_{DC} = 2E\left(\frac{I}{20}\right)[2(0) + \theta_C - 3(0.433\psi_1)] + 0 \quad (6)$$

These six equations contain nine unknowns.

Equations of Equilibrium. Moment equilibrium at joints B and C yields

$$M_{BA} + M_{BC} = 0 \quad (7)$$

$$M_{CD} + M_{CB} = 0 \quad (8)$$

The necessary third equilibrium equation can be obtained by summing moments about point O on the entire frame, Fig. 11–22d. This eliminates the unknown normal forces $\mathbf{N}_A$ and $\mathbf{N}_D$, and therefore

$\curvearrowleft + \Sigma M_O = 0;$

$$M_{AB} + M_{DC} - \left(\frac{M_{AB} + M_{BA}}{10}\right)(34) - \left(\frac{M_{DC} + M_{CD}}{20}\right)(40.78) - 24(6) = 0$$

$$-2.4M_{AB} - 3.4M_{BA} - 2.04M_{CD} - 1.04M_{DC} - 144 = 0 \quad (9)$$

Substituting Eqs. (2) and (3) into Eq. (7), Eqs. (4) and (5) into Eq. (8), and Eqs. (1), (2), (5), and (6) into Eq. (9) yields

$$0.733\theta_B + 0.167\theta_C - 0.392\psi_1 = \frac{24}{EI}$$

$$0.167\theta_B + 0.533\theta_C + 0.0784\psi_1 = -\frac{24}{EI}$$

$$-1.840\theta_B - 0.512\theta_C + 3.880\psi_1 = \frac{144}{EI}$$

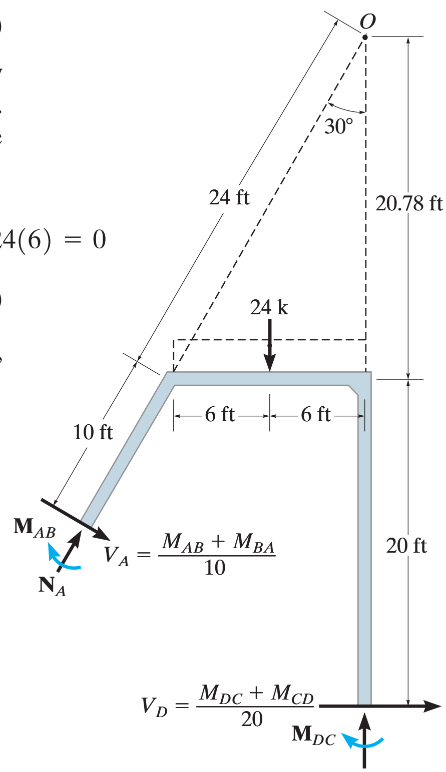
Solving these equations simultaneously yields

$$EI\theta_B = 87.67 \qquad EI\theta_C = -82.3 \qquad EI\psi_1 = 67.83$$

Substituting these values into Eqs. (1)–(6), we have

$M_{AB} = -23.2 \text{ k}\cdot\text{ft}$ $M_{BC} = 5.63 \text{ k}\cdot\text{ft}$ $M_{CD} = -25.3 \text{ k}\cdot\text{ft}$ *Ans.*

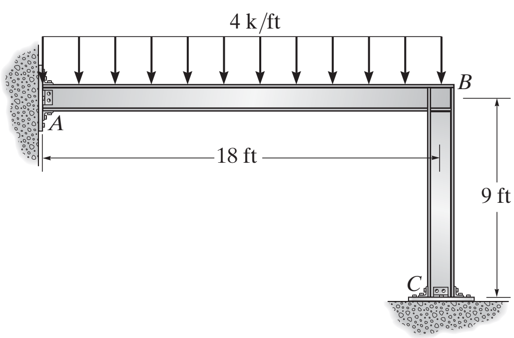
$M_{BA} = -5.63 \text{ k}\cdot\text{ft}$ $M_{CB} = 25.3 \text{ k}\cdot\text{ft}$ $M_{DC} = -17.0 \text{ k}\cdot\text{ft}$ *Ans.*



(d)

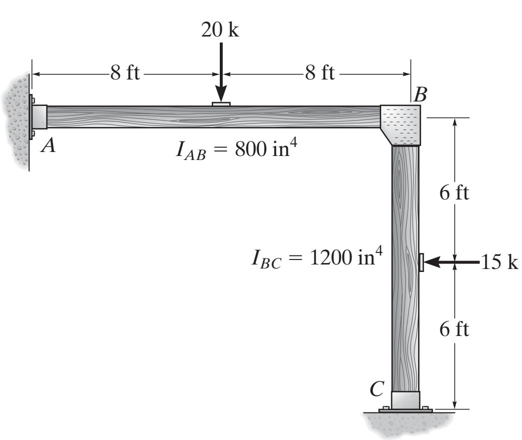
PROBLEMS

11–13. Determine the moments at A, B, and C, then draw the moment diagram for each member. Assume all joints are fixed connected. EI is constant.



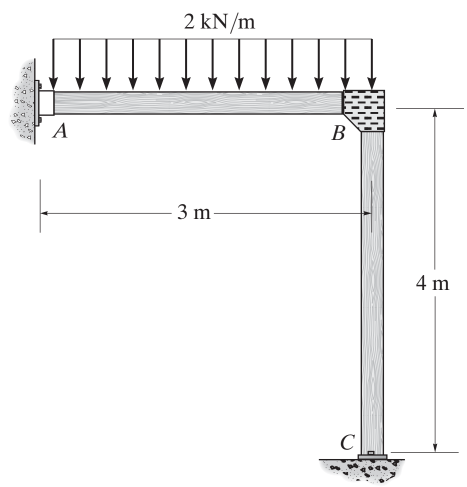
Prob. 11–13

11–14. Determine the moments at the supports, then draw the moment diagram. The members are fixed connected at the supports and at joint B. The moment of inertia of each member is given in the figure. Take $E = 29(10^3)$ ksi.



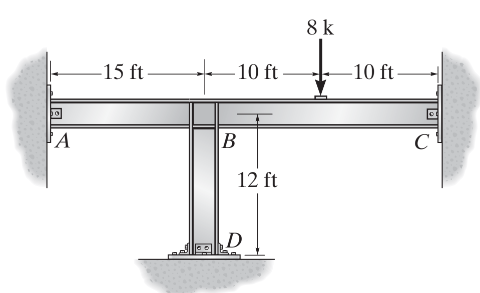
Prob. 11–14

11–15. Determine the moment at B, then draw the moment diagram for each member of the frame. Assume the support at A is fixed and C is pinned. EI is constant.



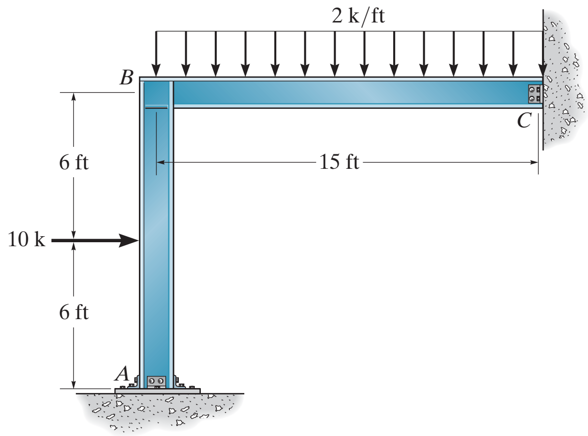
Prob. 11–15

***11–16.** Determine the moments at B and D, then draw the moment diagram. Assume A and C are pinned and B and D are fixed connected. EI is constant.



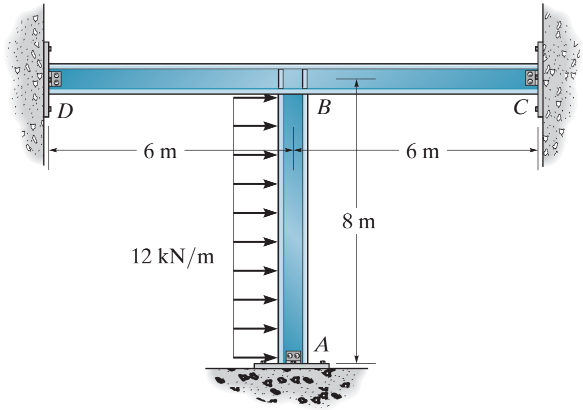
Prob. 11–16

11–17. Determine the moment that each member exerts on the joint at B, then draw the moment diagram for each member of the frame. Assume the support at A is fixed and C is a pin. EI is constant.



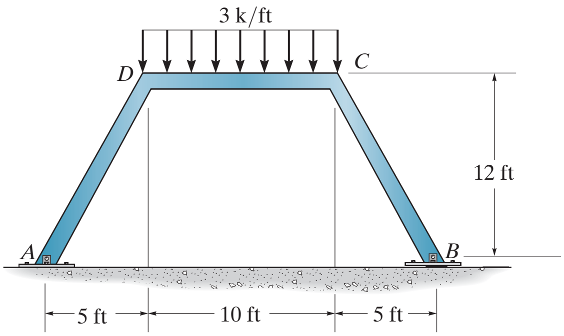
Prob. 11–17

11–18. Determine the moment that each member exerts on the joint at B, then draw the moment diagram for each member of the frame. Assume the supports at A, C, and D are pins. EI is constant.



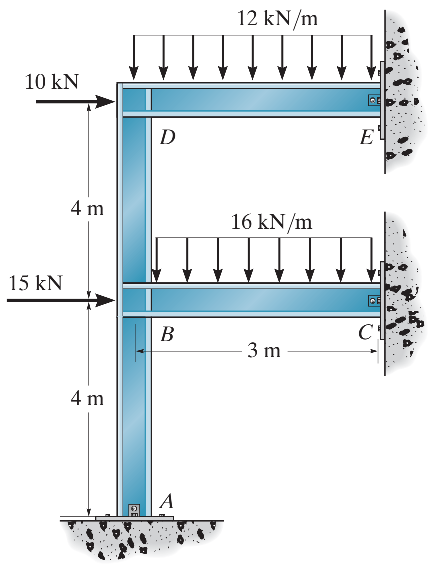
Prob. 11–18

11–19. Determine the moment at joints D and C, then draw the moment diagram for each member of the frame. Assume the supports at A and B are pins. EI is constant.



Prob. 11–19

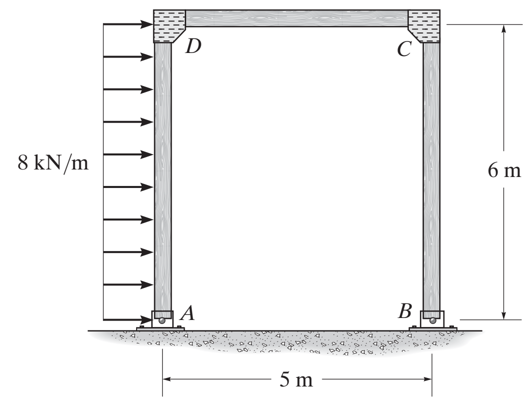
***11–20.** Determine the moment that each member exerts on the joints at B and D, then draw the moment diagram for each member of the frame. Assume the supports at A, C, and E are pins. EI is constant.



Prob. 11–20

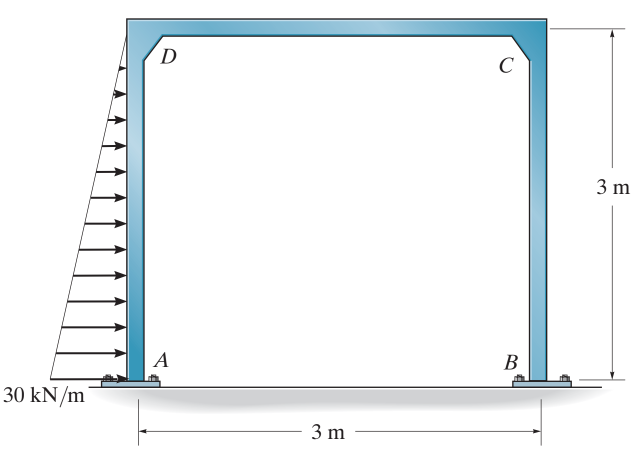
11

11–21. Determine the moment at joints C and D, then draw the moment diagram for each member of the frame. Assume the supports at A and B are pins. EI is constant.



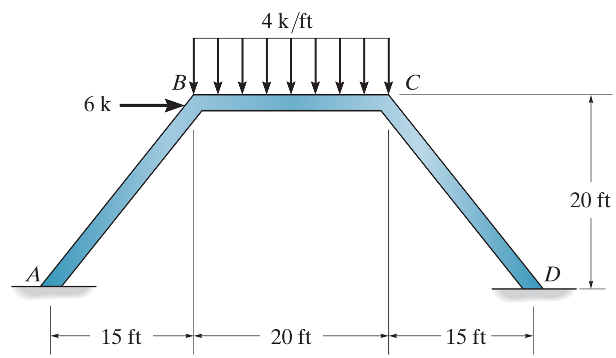
Prob. 11–21

11–22. Determine the moment at joints A, B, C, and D, then draw the moment diagram for each member of the frame. Assume the supports at A and B are fixed. EI is constant.



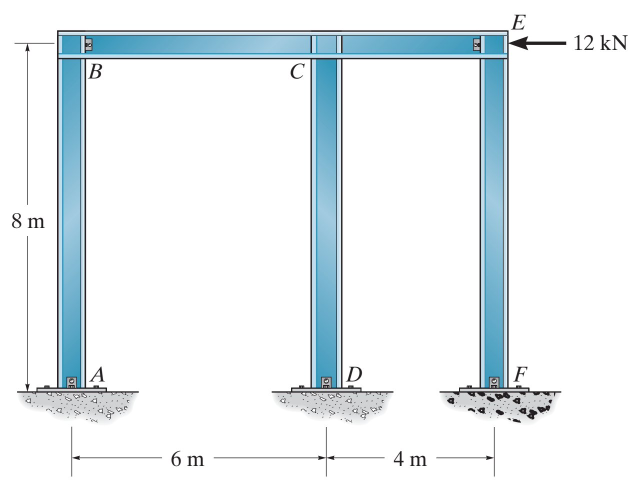
Prob. 11–22

11–23. Determine the moments acting at the supports A and D of the battered-column frame. Take $E = 29(10^3)$ ksi, $I = 600 \text{ in}^4$.



Prob. 11–23

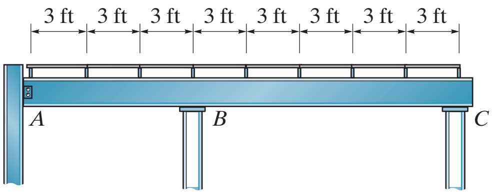
***11–24.** Wind loads are transmitted to the frame at joint E. If A, B, E, D, and F are all pin connected and C is fixed connected, determine the moments at joint C and draw the bending moment diagrams for the girder BCE. EI is constant.



Prob. 11–24

PROJECT PROBLEM

11–1P. The roof is supported by joists that rest on two girders. Each joist can be considered simply supported, and the front girder can be considered attached to the three columns by a pin at A and rollers at B and C. Assume the roof will be made from 3 in.-thick cinder concrete, and each joist has a weight of 550 lb. According to code the roof will be subjected to a snow loading of 25 psf. The joists have a length of 25 ft. Draw the shear and moment diagrams for the girder. Assume the supporting columns are rigid.



Project Prob. 11–1P

CHAPTER REVIEW

The unknown displacements of a structure are referred to as the degrees of freedom for the structure. They consist of either joint displacements or rotations.

The slope-deflection equations relate the unknown moments at each joint of a structural member to the unknown rotations that occur there. The following equation is applied twice to each member or span, considering each side as the "near" end and its counterpart as the far end.

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

For Internal Span or End Span with Far End Fixed

This equation is only applied once, where the "far" end is at the pin or roller support.

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

Only for End Span with Far End Pinned or Roller Supported

Once the slope-deflection equations are written, they are substituted into the equations of moment equilibrium at each joint and then solved for the unknown displacements. If the structure (frame) has sidesway, then an unknown horizontal displacement at each floor level will occur, and the unknown column shears must be related to the moments at the joints, using both the force and moment equilibrium equations. Once the unknown displacements are obtained, the unknown reactions are found from the load-displacement relations.

The girders of this concrete building are all fixed connected, so the statically indeterminate analysis of the framework can be done using the moment distribution method.

Displacement Method of Analysis: Moment Distribution

12

The moment-distribution method is a displacement method of analysis that is easy to apply once certain elastic constants have been determined. In this chapter we will first state the important definitions and concepts for moment distribution and then apply the method to solve problems involving statically indeterminate beams and frames. Application to multistory frames is discussed in the last part of the chapter.

12.1 General Principles and Definitions

The method of analyzing beams and frames using moment distribution was developed by Hardy Cross, in 1930. At the time this method was first published it attracted immediate attention, and it has been recognized as one of the most notable advances in structural analysis during the twentieth century.

As will be explained in detail later, moment distribution is a method of successive approximations that may be carried out to any desired degree of accuracy. Essentially, the method begins by assuming each joint of a structure is fixed. Then, by unlocking and locking each joint in succession, the internal moments at the joints are "distributed" and balanced until the joints have rotated to their final or nearly final positions. It will be found that this process of calculation is both repetitive and easy to apply. Before explaining the techniques of moment distribution, however, certain definitions and concepts must be presented.

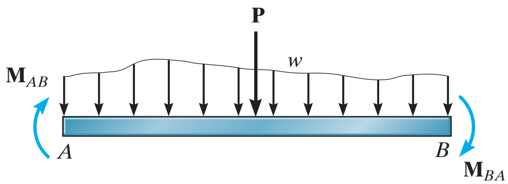


Fig. 12–1

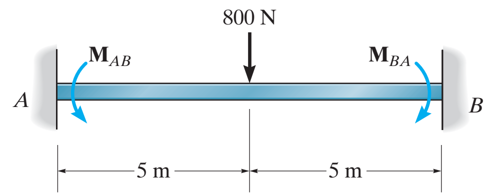


Fig. 12–2

Sign Convention.

We will establish the same sign convention as that established for the slope-deflection equations: *Clockwise moments* that act *on the member* are considered *positive*, whereas *counterclockwise moments* are *negative*, Fig. 12–1.

Fixed-End Moments (FEMs).

The moments at the "walls" or fixed joints of a loaded member are called *fixed-end moments*. These moments can be determined from the table given on the inside back cover, depending upon the type of loading on the member. For example, the beam loaded as shown in Fig. 12–2 has fixed-end moments of FEM $= PL/8 = 800(10)/8 = 1000\ \text{N}\cdot\text{m}$. Noting the action of these moments *on the beam* and applying our sign convention, it is seen that $M_{AB} = -1000\ \text{N}\cdot\text{m}$ and $M_{BA} = +1000\ \text{N}\cdot\text{m}$.

Member Stiffness Factor.

Consider the beam in Fig. 12–3, which is pinned at one end and fixed at the other. Application of the moment **M** causes the end A to rotate through an angle θ_A. In Chapter 11 we related M to θ_A using the conjugate-beam method. This resulted in Eq. 11–1, that is, $M = (4EI/L)\,\theta_A$. The term in parentheses

$$\boxed{K = \frac{4EI}{L} \quad \text{Far End Fixed}} \tag{12–1}$$

is referred to as the *stiffness factor* at A and can be defined as the amount of moment M required to rotate the end A of the beam $\theta_A = 1$ rad.

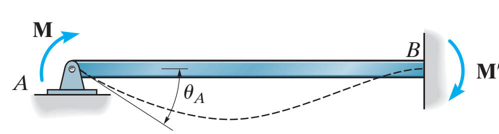


Fig. 12–3

Joint Stiffness Factor. If several members are fixed connected to a joint and each of their far ends is fixed, then by the principle of superposition, the *total stiffness factor* at the joint is the sum of the member stiffness factors at the joint, that is, $K_T = \Sigma K$. For example, consider the frame joint A in Fig. 12–4*a*. The numerical value of each member stiffness factor is determined from Eq. 12–1 and listed in the figure. Using these values, the total stiffness factor of joint A is $K_T = \Sigma K = 4000 + 5000 + 1000 = 10\,000$. This value represents the amount of moment needed to rotate the joint through an angle of 1 rad.

Distribution Factor (DF). If a moment **M** is applied to a fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint. That fraction of the total resisting moment supplied by the member is called the *distribution factor* (DF). To obtain its value, imagine the joint is fixed connected to n members. If an applied moment **M** causes the joint to rotate an amount θ, then each member i rotates by this same amount. If the stiffness factor of the ith member is K_i, then the moment contributed by the member is $M_i = K_i\theta$. Since equilibrium requires $M = M_1 + M_n = K_1\theta + K_n\theta = \theta\Sigma K_i$ then the distribution factor for the ith member is

$$\mathrm{DF}_i = \frac{M_i}{M} = \frac{K_i\theta}{\theta\Sigma K_i}$$

Canceling the common term θ, it is seen that the distribution factor for a member is equal to the stiffness factor of the member divided by the total stiffness factor for the joint; that is, in general,

$$\boxed{\mathrm{DF} = \frac{K}{\Sigma K}} \qquad (12\text{–}2)$$

For example, the distribution factors for members AB, AC, and AD at joint A in Fig. 12–4*a* are

$$\mathrm{DF}_{AB} = 4000/10\,000 = 0.4$$
$$\mathrm{DF}_{AC} = 5000/10\,000 = 0.5$$
$$\mathrm{DF}_{AD} = 1000/10\,000 = 0.1$$

As a result, if $M = 2000\ \mathrm{N \cdot m}$ acts at joint A, Fig. 12–4*b*, the equilibrium moments exerted by the members on the joint, Fig. 12–4*c*, are

$$M_{AB} = 0.4(2000) = 800\ \mathrm{N \cdot m}$$
$$M_{AC} = 0.5(2000) = 1000\ \mathrm{N \cdot m}$$
$$M_{AD} = 0.1(2000) = 200\ \mathrm{N \cdot m}$$

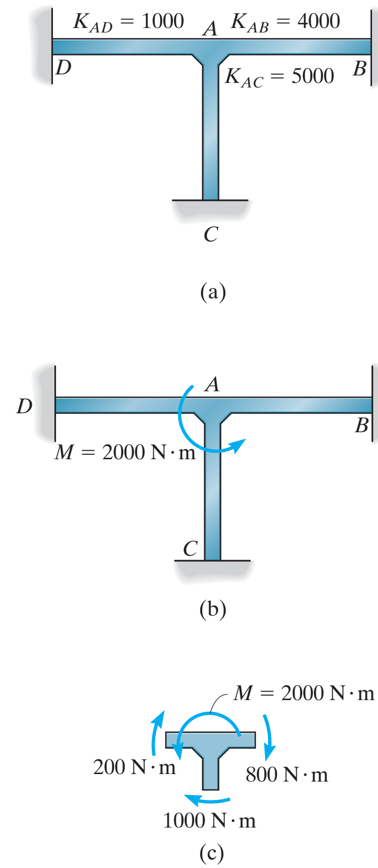


Fig. 12–4

The statically indeterminate loading in bridge girders that are continuous over their piers can be determined using the method of moment distribution.

Member Relative-Stiffness Factor. Quite often a continuous beam or a frame will be made from the same material so its modulus of elasticity E will be the *same* for all the members. If this is the case, the common factor $4E$ in Eq. 12–1 will *cancel* from the numerator and denominator of Eq. 12–2 when the distribution factor for a joint is determined. Hence, it is *easier* just to determine the member's *relative-stiffness factor*

$$K_R = \frac{I}{L} \quad \text{Far End Fixed} \tag{12–3}$$

and use this for the computations of the DF.

Carry-Over Factor. Consider again the beam in Fig. 12–3. It was shown in Chapter 11 that $M_{AB} = (4EI/L)\,\theta_A$ (Eq. 11–1) and $M_{BA} = (2EI/L)\,\theta_A$ (Eq. 11–2). Solving for θ_A and equating these equations we get $M_{BA} = M_{AB}/2$. In other words, the moment $\mathbf{M}$ at the pin induces a moment of $\mathbf{M}' = \frac{1}{2}\mathbf{M}$ at the wall. The carry-over factor represents the fraction of $\mathbf{M}$ that is "carried over" from the pin to the wall. Hence, in the case of a beam with *the far end fixed,* the carry-over factor is $+\frac{1}{2}$. The plus sign indicates both moments act in the same direction.

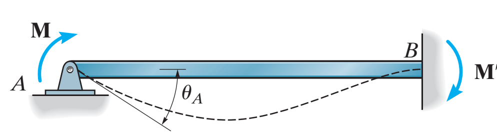


Fig. 12–3

12.2 Moment Distribution for Beams

Moment distribution is based on the principle of successively locking and unlocking the joints of a structure in order to allow the moments at the joints to be distributed and balanced. The best way to explain the method is by examples.

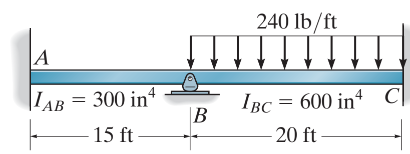
Consider the beam with a constant modulus of elasticity E and having the dimensions and loading shown in Fig. 12–5*a*. Before we begin, we must first determine the distribution factors at the two ends of each span. Using Eq. 12–1, $K = 4EI/L$, the stiffness factors on either side of B are

$$K_{BA} = \frac{4E(300)}{15} = 4E(20)\text{ in}^4/\text{ft} \qquad K_{BC} = \frac{4E(600)}{20} = 4E(30)\text{ in}^4/\text{ft}$$

Thus, using Eq. 12–2, $\text{DF} = K/\Sigma K$, for the ends connected to joint B, we have

$$\text{DF}_{BA} = \frac{4E(20)}{4E(20) + 4E(30)} = 0.4$$

$$\text{DF}_{BC} = \frac{4E(30)}{4E(20) + 4E(30)} = 0.6$$



(a)

Fig. 12–5

At the walls, joint A and joint C, the distribution factor depends on the member stiffness factor and the "stiffness factor" of the wall. Since in theory it would take an "infinite" size moment to rotate the wall one radian, the wall stiffness factor is infinite. Thus for joints A and C we have

$$\text{DF}_{AB} = \frac{4E(20)}{\infty + 4E(20)} = 0$$

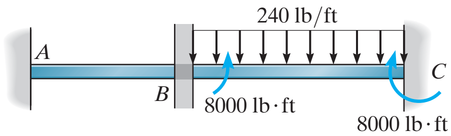
$$\text{DF}_{CB} = \frac{4E(30)}{\infty + 4E(30)} = 0$$

Note that the above results could also have been obtained if the relative stiffness factor $K_R = I/L$ (Eq. 12–3) had been used for the calculations. Furthermore, as long as a *consistent* set of units is used for the stiffness factor, the DF will always be dimensionless, and at a joint, except where it is located at a fixed wall, the sum of the DFs will always equal 1.

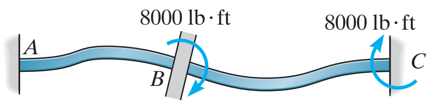
Having computed the DFs, we will now determine the FEMs. Only span BC is loaded, and using the table on the inside back cover for a uniform load, we have

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000\text{ lb}\cdot\text{ft}$$

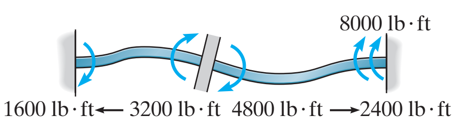
$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = 8000\text{ lb}\cdot\text{ft}$$



joint B held fixed
(b)



correction moment applied to joint B
(c)



moment at B distributed
(d)

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4	0.6	0
FEM Dist,CO	 1600 ←	 3200	−8000 4800 →	8000 2400
ΣM	1600	3200	−3200	10 400

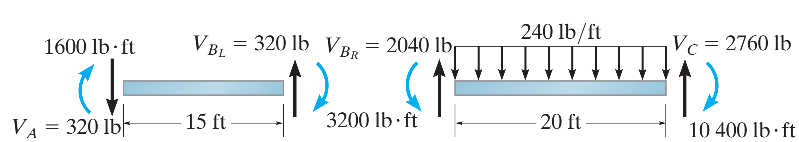
(e)

We begin by assuming joint B is fixed or locked. The fixed-end moment at B then holds span BC in this fixed or locked position as shown in Fig. 12–5*b*. This, of course, does not represent the actual equilibrium situation at B, since the moments on *each side* of this joint must be equal but opposite. To correct this, we will apply an equal, but opposite moment of 8000 lb · ft to the joint and allow the joint to rotate freely, Fig. 12–5*c*. As a result, portions of this moment are distributed in spans BC and BA in accordance with the DFs (or stiffness) of these spans at the joint. Specifically, the moment in BA is $0.4(8000) = 3200$ lb · ft and the moment in BC is $0.6(8000) = 4800$ lb · ft. Finally, due to the released rotation that takes place at B, these moments must be "carried over" since moments are developed at the far ends of the span. Using the carry-over factor of $+\frac{1}{2}$, the results are shown in Fig. 12–5*d*.

This example indicates the basic steps necessary when distributing moments at a joint: Determine the unbalanced moment acting at the initially "locked" joint, unlock the joint and apply an equal but opposite unbalanced moment to correct the equilibrium, distribute the moment among the connecting spans, and carry the moment in each span over to its other end. The steps are usually presented in tabular form as indicated in Fig. 12–5*e*. Here the notation Dist, CO indicates a line where moments are distributed, then carried over. In this particular case only *one cycle* of moment distribution is necessary, since the wall supports at A and C "absorb" the moments and no further joints have to be balanced or unlocked to satisfy joint equilibrium. Once distributed in this manner, the moments at each joint are summed, yielding the final results shown on the bottom line of the table in Fig. 12–5*e*. Notice that joint B is now in equilibrium. Since M_{BC} is negative, this moment is applied to span BC in a counterclockwise sense as shown on free-body diagrams of the beam spans in Fig. 12–5*f*. With the end moments known, the end shears have been computed from the equations of equilibrium applied to each of these spans.

Consider now the same beam, except the support at C is a rocker, Fig. 12–6*a*. In this case only *one member* is at joint C, so the distribution factor for member CB at joint C is

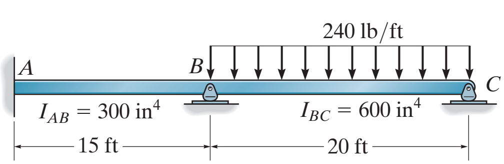
$$\mathrm{DF}_{CB} = \frac{4E(30)}{4E(30)} = 1$$



(f)

Fig. 12–5

The other distribution factors and the FEMs are the same as computed previously. They are listed on lines 1 and 2 of the table in Fig. 12–6*b*. Initially, we will assume joints *B* and *C* are locked. We begin by unlocking joint *C* and placing an equilibrating moment of $-8000\ \text{lb}\cdot\text{ft}$ at the joint. The entire moment is distributed in member *CB* since $(1)(-8000)\ \text{lb}\cdot\text{ft} = -8000\ \text{lb}\cdot\text{ft}$. The arrow on line 3 indicates that $\frac{1}{2}(-8000)\ \text{lb}\cdot\text{ft} = -4000\ \text{lb}\cdot\text{ft}$ is carried over to joint *B* since joint *C* has been allowed to rotate freely. Joint *C* is now *relocked*. Since the total moment at *C* is *balanced,* a line is placed under the -8000-lb·ft moment. We will now consider the unbalanced $-12\,000$-lb·ft moment at joint *B*. Here for equilibrium, a $+12\,000$-lb·ft moment is applied to *B* and this joint is unlocked such that portions of the moment are distributed into *BA* and *BC*, that is, $(0.4)(12\,000) = 4800\ \text{lb}\cdot\text{ft}$ and $(0.6)(12\,000) = 7200\ \text{lb}\cdot\text{ft}$ as shown on line 4. Also note that $+\frac{1}{2}$ of these moments must be carried over to the fixed wall *A* and roller *C* since joint *B* has rotated. Joint *B* is now *relocked.* Again joint *C* is unlocked and the unbalanced moment at the roller is distributed as was done previously. The results are on line 5. Successively locking and unlocking joints *B* and *C* will essentially diminish the size of the moment to be balanced until it becomes negligible compared with the original moments, line 14. Each of the steps on lines 3 through 14 should be thoroughly understood. Summing the moments, the final results are shown on line 15, where it is seen that the final moments now satisfy joint equilibrium.



(a)

Joint	*A*	*B*		*C*	
Member	*AB*	*BA*	*BC*	*CB*	
DF	0	0.4	0.6	1	1
FEM			−8000	8000	2
			−4000	−8000	3
	2400	4800	7200	3600	4
			−1800	−3600	5
	360	720	1080	540	6
			−270	−540	7
	54	108	162	81	8
			−40.5	−81	9
	8.1	16.2	24.3	12.2	10
			−6.1	−12.2	11
	1.2	2.4	3.6	1.8	12
			−0.9	−1.8	13
		0.4	0.5		14
ΣM	2823.3	5647.0	−5647.0	0	15

(b)

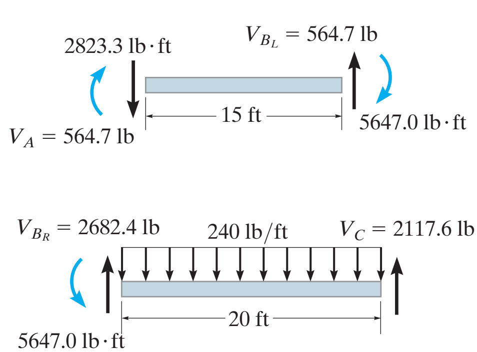
Fig. 12–6

Rather than applying the moment distribution process successively to each joint, as illustrated here, it is also possible to apply it to all joints at the *same time*. This scheme is shown in the table in Fig. 12–6*c*. In this case, we start by fixing all the joints and then balancing and distributing the fixed-end moments at both joints *B* and *C*, line 3. Unlocking joints *B* and *C* simultaneously (joint *A* is always fixed), the moments are then carried over to the end of each span, line 4. Again the joints are relocked, and the moments are balanced and distributed, line 5. Unlocking the joints once again allows the moments to be carried over, as shown in line 6. Continuing, we obtain the final results, as before, listed on line 24. By comparison, this method gives a slower convergence to the answer than does the previous method; however, in many cases this method will be more efficient to apply, and for this reason we will use it in the examples that follow. Finally, using the results in either Fig. 12–6*b* or 12–6*c*, the free-body diagrams of each beam span are drawn as shown in Fig. 12–6*d*.

Although several steps were involved in obtaining the final results here, the work required is rather methodical since it requires application of a series of arithmetical steps, rather than solving a set of equations as in the slope deflection method. It should be noted, however, that the

Joint	*A*	*B*		*C*	
Member	*AB*	*BA*	*BC*	*CB*	
DF	0	0.4	0.6	1	1
FEM			−8000	8000	2
Dist.		3200	4800	−8000	3
CO	1600		−4000	2400	4
Dist.		1600	2400	−2400	5
CO	800		−1200	1200	6
Dist.		480	720	−1200	7
CO	240		−600	360	8
Dist.		240	360	−360	9
CO	120		−180	180	10
Dist.		72	108	−180	11
CO	36		−90	54	12
Dist.		36	54	−54	13
CO	18		−27	27	14
Dist.		10.8	16.2	−27	15
CO	5.4		−13.5	8.1	16
Dist.		5.4	8.1	−8.1	17
CO	2.7		−4.05	4.05	18
Dist.		1.62	2.43	−4.05	19
CO	0.81		−2.02	1.22	20
Dist.		0.80	1.22	−1.22	21
CO	0.40		−0.61	0.61	22
Dist.		0.24	0.37	−0.61	23
ΣM	2823	5647	−5647	0	24

(c)



(d)

Fig. 12–6

fundamental process of moment distribution follows the same procedure as any displacement method. There the process is to establish load-displacement relations at each joint and then satisfy joint equilibrium requirements by determining the correct angular displacement for the joint (compatibility). Here, however, the equilibrium and compatibility of rotation at the joint is satisfied *directly,* using a "moment balance" process that incorporates the load-deflection relations (stiffness factors). Further simplification for using moment distribution is possible, and this will be discussed in the next section.

Procedure for Analysis

The following procedure provides a general method for determining the end moments on beam spans using moment distribution.

Distribution Factors and Fixed-End Moments

The joints on the beam should be identified and the stiffness factors for each span at the joints should be calculated. Using these values the distribution factors can be determined from $\text{DF} = K/\Sigma K$. Remember that $\text{DF} = 0$ for a fixed end and $\text{DF} = 1$ for an *end* pin or roller support.

The fixed-end moments for each loaded span are determined using the table given on the inside back cover. Positive FEMs act clockwise on the span and negative FEMs act counterclockwise. For convenience, these values can be recorded in tabular form, similar to that shown in Fig. 12–6*c*.

Moment Distribution Process

Assume that all joints at which the moments in the connecting spans must be determined are *initially locked*. Then:

1. Determine the moment that is needed to put each joint in equilibrium.

2. Release or "unlock" the joints and distribute the counterbalancing moments into the connecting span at each joint.

3. Carry these moments in each span over to its other end by multiplying each moment by the carry-over factor $+\frac{1}{2}$.

By repeating this cycle of locking and unlocking the joints, it will be found that the moment corrections will diminish since the beam tends to achieve its final deflected shape. When a small enough value for the corrections is obtained, the process of cycling should be stopped with no "carry-over" of the last moments. Each column of FEMs, distributed moments, and carry-over moments should then be added. If this is done correctly, moment equilibrium at the joints will be achieved.

EXAMPLE 12.1

Determine the internal moments at each support of the beam shown in Fig. 12–7*a*. *EI* is constant.

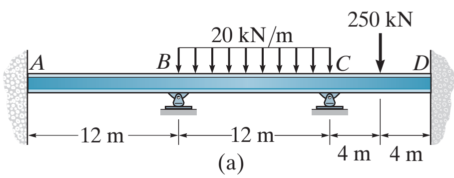


Fig. 12–7

SOLUTION

The distribution factors at each joint must be computed first.* The stiffness factors for the members are

$$K_{AB} = \frac{4EI}{12} \qquad K_{BC} = \frac{4EI}{12} \qquad K_{CD} = \frac{4EI}{8}$$

Therefore,

$$\text{DF}_{AB} = \text{DF}_{DC} = 0 \qquad \text{DF}_{BA} = \text{DF}_{BC} = \frac{4EI/12}{4EI/12 + 4EI/12} = 0.5$$

$$\text{DF}_{CB} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.4 \quad \text{DF}_{CD} = \frac{4EI/8}{4EI/12 + 4EI/8} = 0.6$$

The fixed-end moments are

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = \frac{-20(12)^2}{12} = -240 \text{ kN}\cdot\text{m} \qquad (\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{20(12)^2}{12} = 240 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CD} = -\frac{PL}{8} = \frac{-250(8)}{8} = -250 \text{ kN}\cdot\text{m} \qquad (\text{FEM})_{DC} = \frac{PL}{8} = \frac{250(8)}{8} = 250 \text{ kN}\cdot\text{m}$$

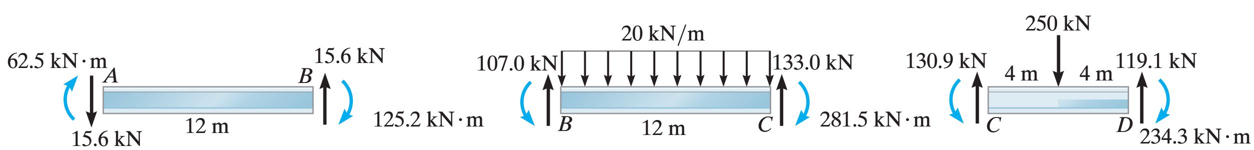
Starting with the FEMs, line 4, Fig. 12–7*b*, the moments at joints *B* and *C* are distributed *simultaneously*, line 5. These moments are then carried over *simultaneously* to the respective ends of each span, line 6. The resulting moments are again simultaneously distributed and carried over, lines 7 and 8. The process is continued until the resulting moments are diminished an appropriate amount, line 13. The resulting moments are found by summation, line 14.

Placing the moments on each beam span and applying the equations of equilibrium yields the end shears shown in Fig. 12–7*c* and the bending-moment diagram for the entire beam, Fig. 12–7*d*.

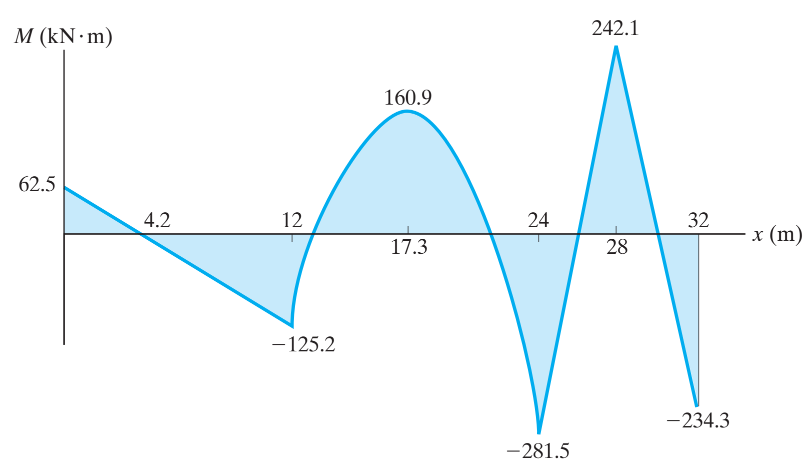
*Here we have used the stiffness factor 4*EI*/*L*; however, the relative stiffness factor *I*/*L* could also have been used.

Joint	A	B		C		D	1
Member	AB	BA	BC	CB	CD	DC	2
DF	0	0.5	0.5	0.4	0.6	0	3
FEM			−240	240	−250	250	4
Dist.		120	120	4	6		5
CO	60		2	60		3	6
Dist.		−1	−1	−24	−36		7
CO	−0.5		−12	−0.5		−18	8
Dist.		6	6	0.2	0.3		9
CO	3		0.1	3		0.2	10
Dist.		−0.05	−0.05	−1.2	−1.8		11
CO	−0.02		−0.6	−0.02		−0.9	12
Dist.		0.3	0.3	0.01	0.01		13
ΣM	62.5	125.2	−125.2	281.5	−281.5	234.3	14

(b)



(c)

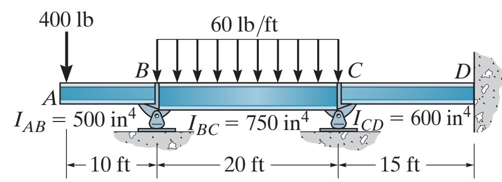


(d)

12

EXAMPLE 12.2

Determine the internal moment at each support of the beam shown in Fig. 12–8*a*. The moment of inertia of each span is indicated.



(a)

Fig. 12–8

SOLUTION

In this problem a moment does not get distributed in the overhanging span AB, and so the distribution factor $(\text{DF})_{BA} = 0$. The stiffness of span BC is based on $4EI/L$ since the pin rocker is not at the far end of the beam. The stiffness factors, distribution factors, and fixed-end moments are computed as follows:

$$K_{BC} = \frac{4E(750)}{20} = 150E \qquad K_{CD} = \frac{4E(600)}{15} = 160E$$

$$\text{DF}_{BC} = 1 - (\text{DF})_{BA} = 1 - 0 = 1$$

$$\text{DF}_{CB} = \frac{150E}{150E + 160E} = 0.484$$

$$\text{DF}_{CD} = \frac{160E}{150E + 160E} = 0.516$$

$$\text{DF}_{DC} = \frac{160E}{\infty + 160E} = 0$$

Due to the overhang,

$$(\text{FEM})_{BA} = 400\text{ lb}(10\text{ ft}) = 4000\text{ lb}\cdot\text{ft}$$

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{60(20)^2}{12} = -2000\text{ lb}\cdot\text{ft}$$

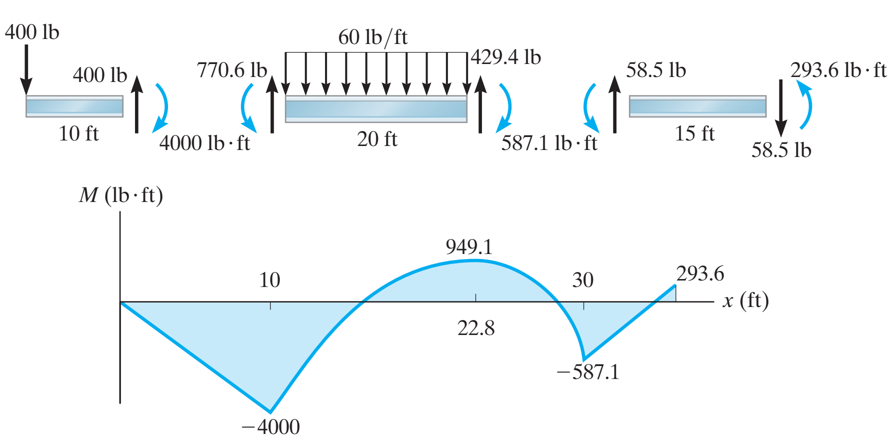
$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{60(20)^2}{12} = 2000\text{ lb}\cdot\text{ft}$$

These values are listed on the fourth line of the table, Fig. 12–8*b*. The overhanging span requires the internal moment to the left of B to be $+4000\text{ lb}\cdot\text{ft}$. Balancing at joint B requires an internal moment of $-4000\text{ lb}\cdot\text{ft}$ to the right of B. As shown on the fifth line of the table $-2000\text{ lb}\cdot\text{ft}$ is added to BC in order to satisfy this condition. The distribution and carry-over operations proceed in the usual manner as indicated.

Since the internal moments are known, the moment diagram for the beam can be constructed (Fig. 12–8*c*).

Joint	*B*		*C*		*D*
Member		*BC*	*CB*	*CD*	*DC*
DF	0	1	0.484	0.516	0
FEM	4000	−2000	2000		
Dist.		−2000	−968	−1032	
CO		−484	−1000		−516
Dist.		484	484	516	
CO		242	242		258
Dist.		−242	−117.1	−124.9	
CO		−58.6	−121		−62.4
Dist.		58.6	58.6	62.4	
CO		29.3	29.3		31.2
Dist.		−29.3	−14.2	−15.1	
CO		−7.1	−14.6		−7.6
Dist.		7.1	7.1	7.6	
CO		3.5	3.5		3.8
Dist.		−3.5	−1.7	−1.8	
CO		−0.8	−1.8		−0.9
Dist.		0.8	0.9	0.9	
CO		0.4	0.4		0.4
Dist.		−0.4	−0.2	−0.2	
CO		−0.1	−0.2		−0.1
Dist.		0.1	0.1	0.1	
ΣM	4000	−4000	587.1	−587.1	−293.6

(b)



(c)

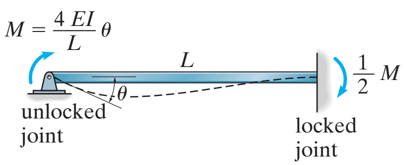


Fig. 12–9

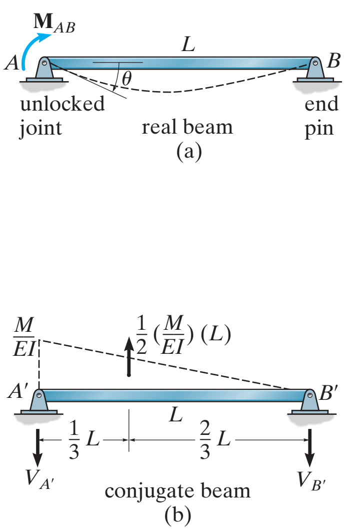


Fig. 12–10

12.3 Stiffness-Factor Modifications

In the previous examples of moment distribution we have considered each beam span to be constrained by a fixed support (locked joint) at its far end when distributing and carrying over the moments. For this reason we have computed the stiffness factors, distribution factors, and the carry-over factors based on the case shown in Fig. 12–9. Here, of course, the stiffness factor is $K = 4EI/L$ (Eq. 12–1), and the carry-over factor is $+\frac{1}{2}$.

In some cases it is possible to modify the stiffness factor of a particular beam span and thereby simplify the process of moment distribution. Three cases where this frequently occurs in practice will now be considered.

Member Pin Supported at Far End. Many indeterminate beams have their far end span supported by an end pin (or roller) as in the case of joint B in Fig. 12–10*a*. Here the applied moment $\mathbf{M}$ rotates the end A by an amount θ. To determine θ, the shear in the conjugate beam at A' must be determined, Fig. 12–10*b*. We have

$$\circlearrowright + \Sigma M_{B'} = 0; \qquad V'_A(L) - \frac{1}{2}\left(\frac{M}{EI}\right)L\left(\frac{2}{3}L\right) = 0$$

$$V'_A = \theta = \frac{ML}{3EI}$$

or

$$M = \frac{3EI}{L}\theta$$

Thus, the stiffness factor for this beam is

$$\boxed{K = \frac{3EI}{L} \quad \text{Far End Pinned or Roller Supported}} \qquad (12\text{–}4)$$

Also, note that *the carry-over factor is zero,* since the pin at B does not support a moment. By comparison, then, *if the far end was fixed supported, the stiffness factor $K = 4EI/L$ would have to be modified by $\frac{3}{4}$ to model the case of having the far end pin supported*. If this modification is considered, the moment distribution process is simplified since the end pin does *not* have to be unlocked–locked successively when distributing the moments. Also, since the end span is pinned, the fixed-end moments for the span are computed using the values in the right column of the table on the inside back cover. Example 12–4 illustrates how to apply these simplifications.

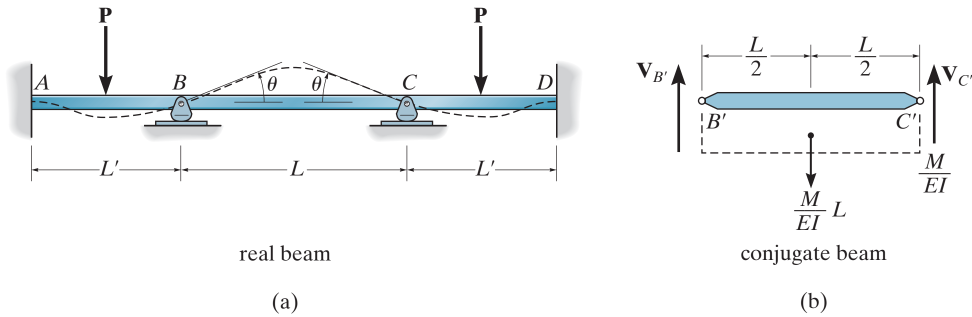


Fig. 12–11

Symmetric Beam and Loading. If a beam is symmetric with respect to both its loading and geometry, the bending-moment diagram for the beam will also be symmetric. As a result, a modification of the stiffness factor for the center span can be made, so that moments in the beam only have to be distributed through joints lying on either half of the beam. To develop the appropriate stiffness-factor modification, consider the beam shown in Fig. 12–11*a*. Due to the symmetry, the internal moments at B and C are equal. Assuming this value to be $\mathbf{M}$, the conjugate beam for span BC is shown in Fig. 12–11*b*. The slope θ at each end is therefore

$$\circlearrowright + \Sigma M_{C'} = 0; \qquad -V_{B'}(L) + \frac{M}{EI}(L)\left(\frac{L}{2}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{2EI}$$

or

$$M = \frac{2EI}{L}\theta$$

The stiffness factor for the center span is therefore

$$K = \frac{2EI}{L} \qquad (12\text{–}5)$$

Symmetric Beam and Loading

Thus, moments for only half the beam can be distributed provided the stiffness factor for the center span is computed using Eq. 12–5. *By comparison, the center span's stiffness factor will be one half that usually determined using* $K = 4EI/L$.

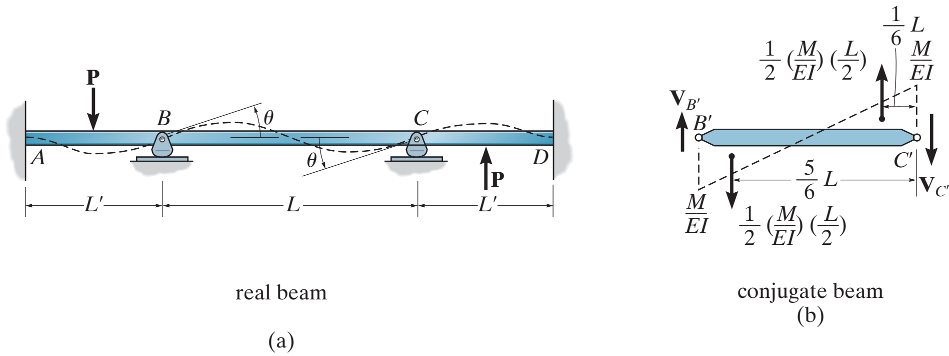


Fig. 12–12

Symmetric Beam with Antisymmetric Loading. If a symmetric beam is subjected to antisymmetric loading, the resulting moment diagram will be antisymmetric. As in the previous case, we can modify the stiffness factor of the center span so that only one half of the beam has to be considered for the moment-distribution analysis. Consider the beam in Fig. 12–12*a*. The conjugate beam for its center span BC is shown in Fig. 12–12*b*. Due to the antisymmetric loading, the internal moment at B is equal, but opposite to that at C. Assuming this value to be $\mathbf{M}$, the slope θ at each end is determined as follows:

$$\circlearrowleft+\Sigma M_{C'} = 0; \quad -V_{B'}(L) + \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{5L}{6}\right) - \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{6EI}$$

or

$$M = \frac{6EI}{L}\theta$$

The stiffness factor for the center span is, therefore,

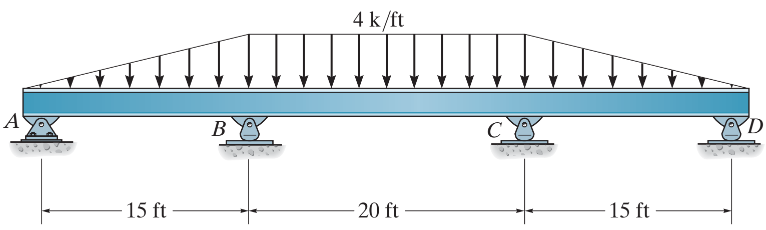
$$\boxed{K = \frac{6EI}{L}} \qquad (12\text{–}6)$$

Symmetric Beam with Antisymmetric Loading

Thus, when the stiffness factor for the beam's center span is computed using Eq. 12–6, the moments in only half the beam have to be distributed. *Here the stiffness factor is one and a half times as large as that determined using $K = 4EI/L$.*

EXAMPLE 12.3

Determine the internal moments at the supports for the beam shown in Fig. 12–13*a*. *EI* is constant.



(a)

Fig. 12–13

SOLUTION

By inspection, the beam and loading are symmetrical. Thus, we will apply $K = 2EI/L$ to compute the stiffness factor of the center span BC and therefore use only the left half of the beam for the analysis. The analysis can be shortened even further by using $K = 3EI/L$ for computing the stiffness factor of segment AB since the far end A is pinned. Furthermore, the distribution of moment at A can be skipped by using the FEM for a triangular loading on a span with one end fixed and the other pinned. Thus,

$$K_{AB} = \frac{3EI}{15} \qquad \text{(using Eq. 12–4)}$$

$$K_{BC} = \frac{2EI}{20} \qquad \text{(using Eq. 12–5)}$$

$$\text{DF}_{AB} = \frac{3EI/15}{3EI/15} = 1$$

$$\text{DF}_{BA} = \frac{3EI/15}{3EI/15 + 2EI/20} = 0.667$$

$$\text{DF}_{BC} = \frac{2EI/20}{3EI/15 + 2EI/20} = 0.333$$

$$(\text{FEM})_{BA} = \frac{wL^2}{15} = \frac{4(15)^2}{15} = 60 \text{ k}\cdot\text{ft}$$

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{4(20)^2}{12} = -133.3 \text{ k}\cdot\text{ft}$$

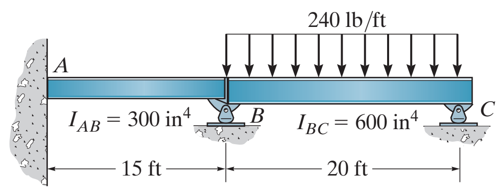
These data are listed in the table in Fig. 12–13*b*. Computing the stiffness factors as shown above considerably reduces the analysis, since only joint B must be balanced and carry-overs to joints A and C are not necessary. Obviously, joint C is subjected to the same internal moment of $108.9 \text{ k}\cdot\text{ft}$.

Joint	A	B	
Member	AB	BA	BC
DF	1	0.667	0.333
FEM Dist.		60 48.9	−133.3 24.4
ΣM	0	108.9	−108.9

(b)

EXAMPLE 12.4

Determine the internal moments at the supports of the beam shown in Fig. 12–14*a*. The moment of inertia of the two spans is shown in the figure.



(a)

Fig. 12–14

SOLUTION

Since the beam is roller supported at its far end C, the stiffness of span BC will be computed on the basis of $K = 3EI/L$. We have

$$K_{AB} = \frac{4EI}{L} = \frac{4E(300)}{15} = 80E$$

$$K_{BC} = \frac{3EI}{L} = \frac{3E(600)}{20} = 90E$$

Thus,

$$\mathrm{DF}_{AB} = \frac{80E}{\infty + 80E} = 0$$

$$\mathrm{DF}_{BA} = \frac{80E}{80E + 90E} = 0.4706$$

$$\mathrm{DF}_{BC} = \frac{90E}{80E + 90E} = 0.5294$$

$$\mathrm{DF}_{CB} = \frac{90E}{90E} = 1$$

Further simplification of the distribution method for this problem is possible by realizing that a *single* fixed-end moment for the end span BC can be used. Using the right-hand column of the table on the inside back cover for a uniformly loaded span having one side fixed, the other pinned, we have

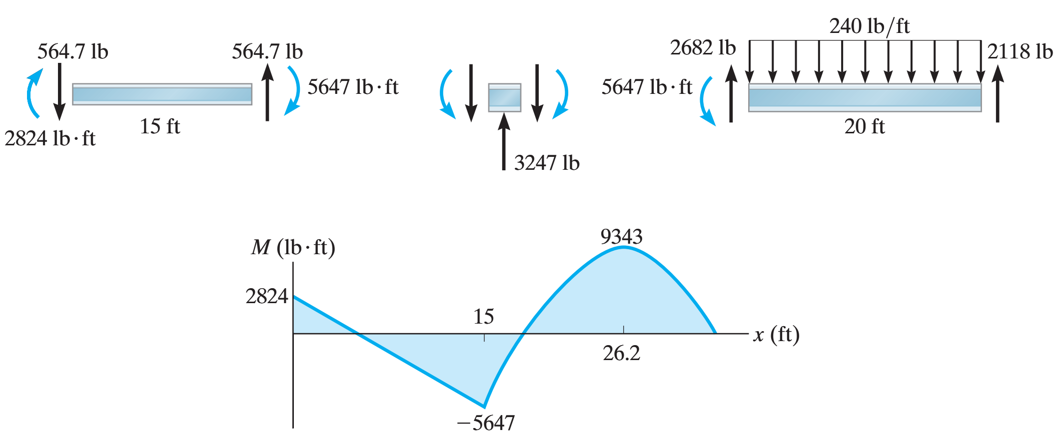
$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{8} = \frac{-240(20)^2}{8} = -12\,000 \text{ lb}\cdot\text{ft}$$

The foregoing data are entered into the table in Fig. 12–14*b* and the moment distribution is carried out. By comparison with Fig. 12–6*b*, this method considerably simplifies the distribution.

Using the results, the beam's end shears and moment diagrams are shown in Fig. 12–14*c*.

Joint	*A*	*B*		*C*
Member	*AB*	*BA*	*BC*	*CB*
DF	0	0.4706	0.5294	1
FEM			−12 000	
Dist.		5647.2	6352.8	
CO	2823.6			
ΣM	2823.6	5647.2	−5647.2	0

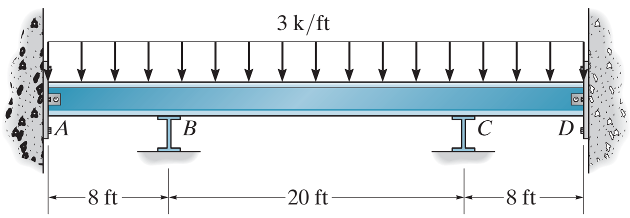
(b)



(c)

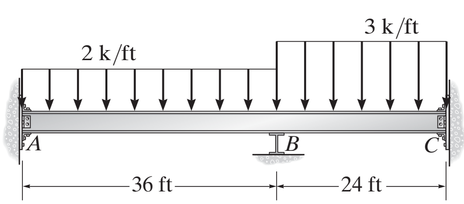
PROBLEMS

12–1. Determine the moments at B and C. EI is constant. Assume B and C are rollers and A and D are pinned.



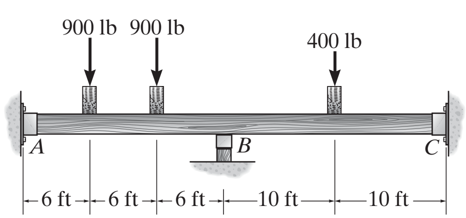
Prob. 12–1

12–2. Determine the moments at A, B, and C. Assume the support at B is a roller and A and C are fixed. EI is constant.



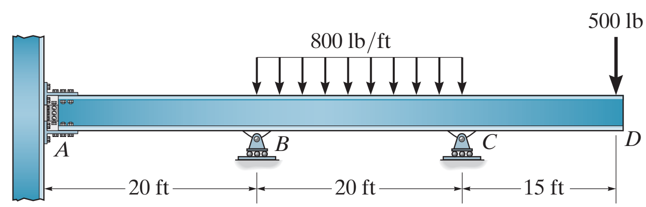
Prob. 12–2

12–3. Determine the moments at A, B, and C, then draw the moment diagram. Assume the support at B is a roller and A and C are fixed. EI is constant.



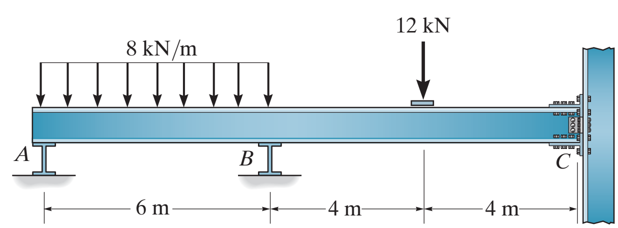
Prob. 12–3

***12–4.** Determine the reactions at the supports and then draw the moment diagram. Assume A is fixed. EI is constant.



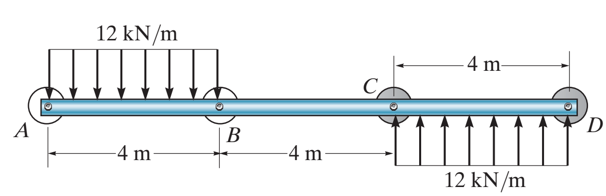
Prob. 12–4

12–5. Determine the moments at B and C, then draw the moment diagram for the beam. Assume C is a fixed support. EI is constant.



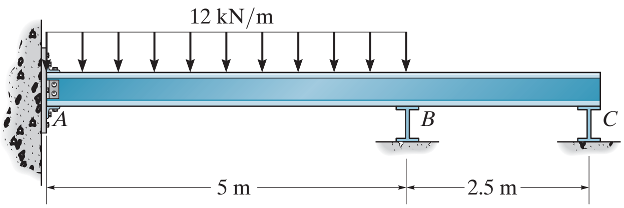
Prob. 12–5

12–6. Determine the moments at B and C, then draw the moment diagram for the beam. All connections are pins. Assume the horizontal reactions are zero. EI is constant.



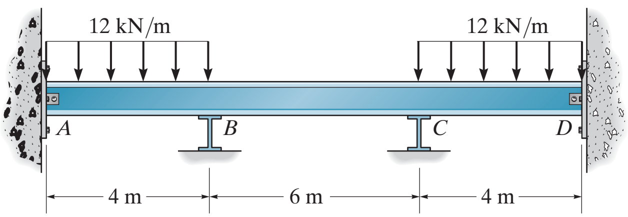
Prob. 12–6

12–7. Determine the reactions at the supports. Assume A is fixed and B and C are rollers that can either push or pull on the beam. EI is constant.



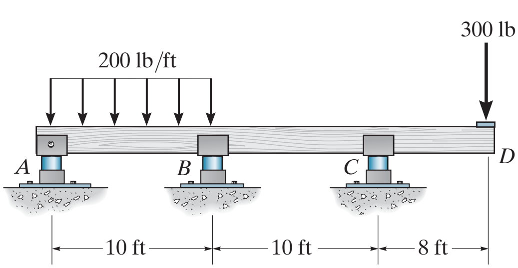
Prob. 12–7

*__12–8.__ Determine the moments at B and C, then draw the moment diagram for the beam. Assume the supports at B and C are rollers and A and D are pins. EI is constant.



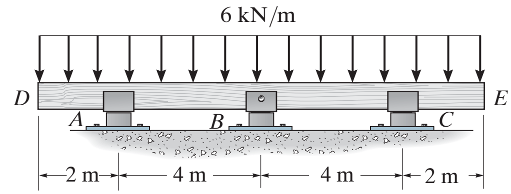
Prob. 12–8

12–9. Determine the moments at B and C, then draw the moment diagram for the beam. Assume the supports at B and C are rollers and A is a pin. EI is constant.



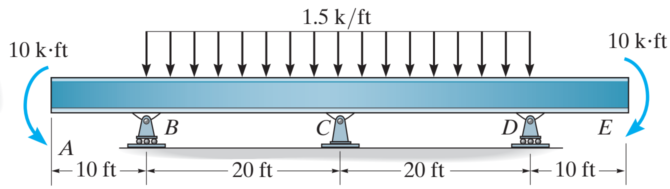
Prob. 12–9

12–10. Determine the moment at B, then draw the moment diagram for the beam. Assume the supports at A and C are rollers and B is a pin. EI is constant.



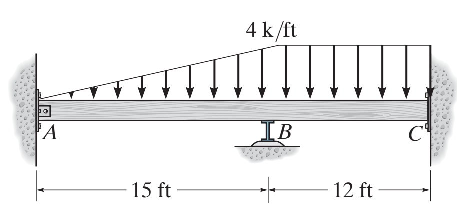
Prob. 12–10

12–11. Determine the moments at B, C, and D, then draw the moment diagram for the beam. EI is constant.



Prob. 12–11

*__12–12.__ Determine the moment at B, then draw the moment diagram for the beam. Assume the support at A is pinned, B is a roller and C is fixed. EI is constant.



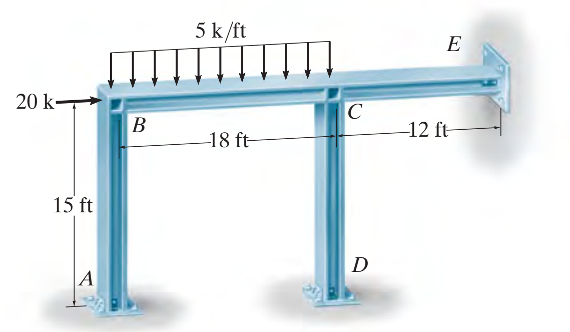
Prob. 12–12

12.4 Moment Distribution for Frames: No Sidesway

Application of the moment-distribution method for frames having no sidesway follows the same procedure as that given for beams. To minimize the chance for errors, it is suggested that the analysis be arranged in a tabular form, as in the previous examples. Also, the distribution of moments can be shortened if the stiffness factor of a span can be modified as indicated in the previous section.

EXAMPLE 12.5

Determine the internal moments at the joints of the frame shown in Fig. 12–15*a*. There is a pin at *E* and *D* and a fixed support at *A*. *EI* is constant.



(a)

Joint	A	B		C			D	E
Member	AB	BA	BC	CB	CD	CE	DC	EC
DF	0	0.545	0.455	0.330	0.298	0.372	1	1
FEM			−135	135				
Dist.		73.6	61.4	−44.6	−40.2	−50.2		
CO	36.8		−22.3	30.7				
Dist.		12.2	10.1	−10.1	−9.1	−11.5		
CO	6.1		−5.1	5.1				
Dist.		2.8	2.3	−1.7	−1.5	−1.9		
CO	1.4		−0.8	1.2				
Dist.		0.4	0.4	−0.4	−0.4	−0.4		
CO	0.2		−0.2	0.2				
Dist.		0.1	0.1	−0.1	0.0	−0.1		
ΣM	44.5	89.1	−89.1	115	−51.2	−64.1		

(b)

Fig. 12–15

SOLUTION

By inspection, the pin at E will prevent the frame from sidesway. The stiffness factors of CD and CE can be computed using $K = 3EI/L$ since the far ends are pinned. Also, the 20-k load does not contribute a FEM since it is applied at joint B. Thus,

$$K_{AB} = \frac{4EI}{15} \qquad K_{BC} = \frac{4EI}{18} \qquad K_{CD} = \frac{3EI}{15} \qquad K_{CE} = \frac{3EI}{12}$$

$$\mathrm{DF}_{AB} = 0$$

$$\mathrm{DF}_{BA} = \frac{4EI/15}{4EI/15 + 4EI/18} = 0.545$$

$$\mathrm{DF}_{BC} = 1 - 0.545 = 0.455$$

$$\mathrm{DF}_{CB} = \frac{4EI/18}{4EI/18 + 3EI/15 + 3EI/12} = 0.330$$

$$\mathrm{DF}_{CD} = \frac{3EI/15}{4EI/18 + 3EI/15 + 3EI/12} = 0.298$$

$$\mathrm{DF}_{CE} = 1 - 0.330 - 0.298 = 0.372$$

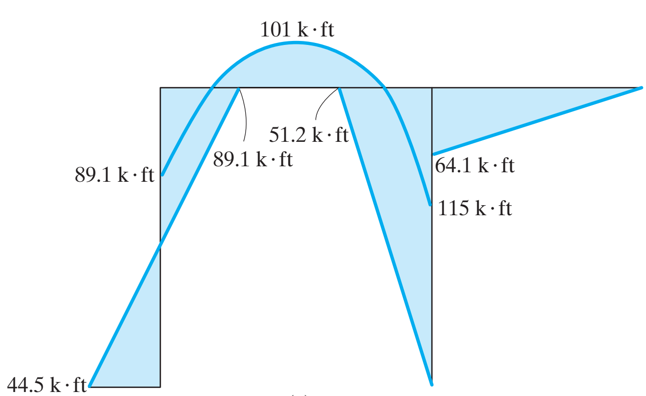
$$\mathrm{DF}_{DC} = 1 \qquad \mathrm{DF}_{EC} = 1$$

$$(\mathrm{FEM})_{BC} = \frac{-wL^2}{12} = \frac{-5(18)^2}{12} = -135 \text{ k}\cdot\text{ft}$$

$$(\mathrm{FEM})_{CB} = \frac{wL^2}{12} = \frac{5(18)^2}{12} = 135 \text{ k}\cdot\text{ft}$$

The data are shown in the table in Fig. 12–15*b*. Here the distribution of moments successively goes to joints B and C. The final moments are shown on the last line.

Using these data, the moment diagram for the frame is constructed in Fig. 12–15*c*.



(c)

12.5 Moment Distribution for Frames: Sidesway

It has been shown in Sec. 11–5 that frames that are nonsymmetrical or subjected to nonsymmetrical loadings have a tendency to sidesway. An example of one such case is shown in Fig. 12–16*a*. Here the applied loading **P** will create unequal moments at joints B and C such that the frame will deflect an amount Δ to the right. To determine this deflection and the internal moments at the joints using moment distribution, we will use the principle of superposition. In this regard, the frame in Fig. 12–16*b* is first considered held from sidesway by applying an artificial joint support at C. Moment distribution is applied and then by statics the restraining force **R** is determined. The equal, but opposite, restraining force is then applied to the frame, Fig. 12–16*c*, and the moments in the frame are calculated. One method for doing this last step requires first *assuming* a numerical value for one of the internal moments, say $\mathbf{M}'_{BA}$. Using moment distribution and statics, the deflection Δ' and external force $\mathbf{R}'$ corresponding to the assumed value of $\mathbf{M}'_{BA}$ can then be determined. Since linear elastic deformations occur, the force $\mathbf{R}'$ develops moments in the frame that are *proportional* to those developed by **R**. For example, if $\mathbf{M}'_{BA}$ and $\mathbf{R}'$ are known, the moment at B developed by **R** will be $M_{BA} = M'_{BA}(R/R')$. Addition of the joint moments for both cases, Fig. 12–16*b* and *c*, will yield the actual moments in the frame, Fig. 12–16*a*. Application of this technique is illustrated in Examples 12–6 through 12–8.

Multistory Frames. Quite often, multistory frameworks may have several *independent* joint displacements, and consequently the moment distribution analysis using the above techniques will involve more computation. Consider, for example, the two-story frame shown in Fig. 12–17*a*. This structure can have two independent joint displacements, since the sidesway Δ_1 of the first story is independent of any displacement

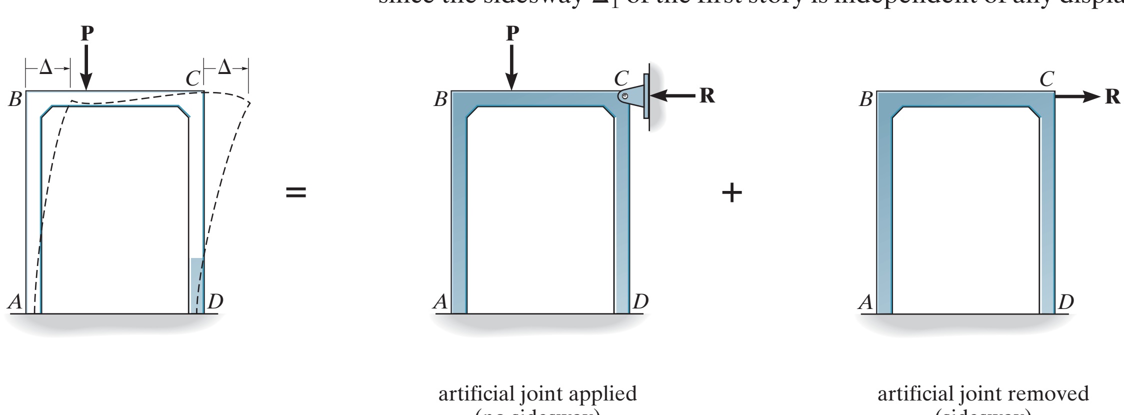


Fig. 12–16

Δ_2 of the second story. Unfortunately, these displacements are not known initially, so the analysis must proceed on the basis of superposition, in the same manner as discussed previously. In this case, two restraining forces $\mathbf{R_1}$ and $\mathbf{R_2}$ are applied, Fig. 12–17*b*, and the fixed-end moments are determined and distributed. Using the equations of equilibrium, the numerical values of $\mathbf{R_1}$ and $\mathbf{R_2}$ are then determined. Next, the restraint at the floor of the first story is removed and the floor is given a displacement Δ'. This displacement causes fixed-end moments (FEMs) in the frame, which can be assigned specific numerical values. By distributing these moments and using the equations of equilibrium, the associated numerical values of $\mathbf{R}'_1$ and $\mathbf{R}'_2$ can be determined. In a similar manner, the floor of the second story is then given a displacement Δ'', Fig. 12–17*d*. Assuming numerical values for the fixed-end moments, the moment distribution and equilibrium analysis will yield specific values of $\mathbf{R}''_1$ and $\mathbf{R}''_2$. Since the last two steps associated with Fig. 12–17*c* and *d* depend on *assumed* values of the FEMs, correction factors C' and C'' must be applied to the distributed moments. With reference to the restraining forces in Fig. 12–17*c* and 12–17*d*, we require equal but opposite application of $\mathbf{R_1}$ and $\mathbf{R_2}$ to the frame, such that

$$R_2 = -C'R'_2 + C''R''_2$$
$$R_1 = +C'R'_1 - C''R''_1$$

Simultaneous solution of these equations yields the values of C' and C''. These correction factors are then multiplied by the internal joint moments found from the moment distribution in Fig. 12–17*c* and 12–17*d*. The resultant moments are then found by adding these corrected moments to those obtained for the frame in Fig. 12–17*b*.

Other types of frames having independent joint displacements can be analyzed using this same procedure; however, it must be admitted that the foregoing method does require quite a bit of numerical calculation. Although some techniques have been developed to shorten the calculations, it is best to solve these types of problems on a computer, preferably using a matrix analysis. The techniques for doing this will be discussed in Chapter 16.

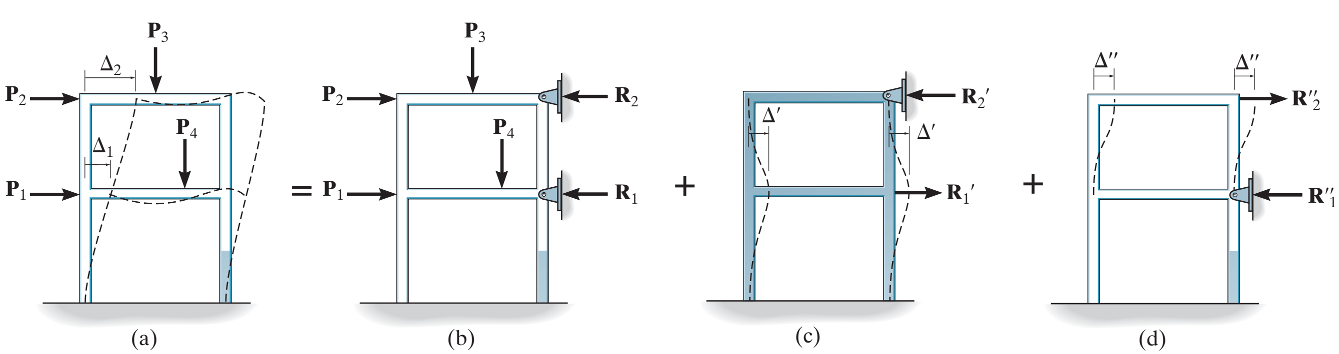
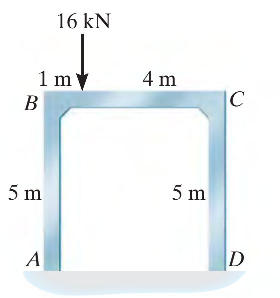


Fig. 12–17

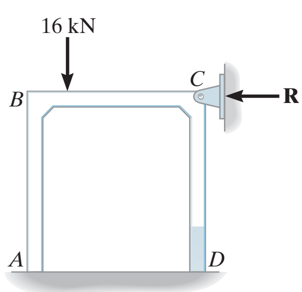
12

EXAMPLE 12.6



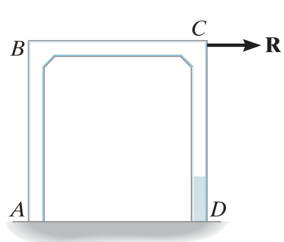
(a)

=



(b)

+



(c)

Fig. 12–18

Determine the moments at each joint of the frame shown in Fig. 12–18*a*. *EI* is constant.

SOLUTION

First we consider the frame held from sidesway as shown in Fig. 12–18*b*. We have

$$(\text{FEM})_{BC} = -\frac{16(4)^2(1)}{(5)^2} = -10.24 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{16(1)^2(4)}{(5)^2} = 2.56 \text{ kN}\cdot\text{m}$$

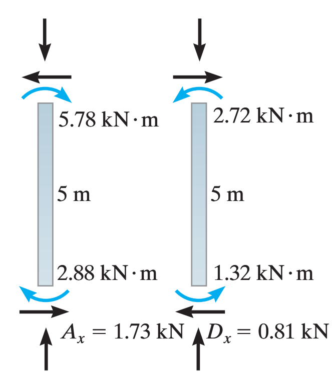
The stiffness factor of each span is computed on the basis of $4EI/L$ or by using the relative-stiffness factor I/L. The DFs and the moment distribution are shown in the table, Fig. 12–18*d*. Using these results, the equations of equilibrium are applied to the free-body diagrams of the columns in order to determine $\mathbf{A}_x$ and $\mathbf{D}_x$, Fig. 12–18*e*. From the free-body diagram of the entire frame (not shown) the joint restraint **R** in Fig. 12–18*b* has a magnitude of

$$\Sigma F_x = 0; \qquad R = 1.73 \text{ kN} - 0.81 \text{ kN} = 0.92 \text{ kN}$$

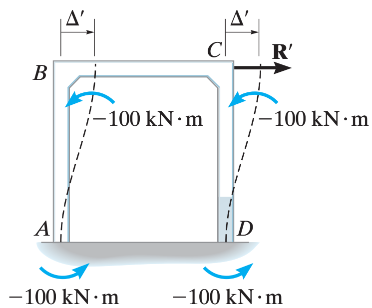
An equal but opposite value of $R = 0.92$ kN must now be applied to the frame at *C* and the internal moments computed, Fig. 12–18*c*. To solve the problem of computing these moments, we will assume a force $\mathbf{R}'$ is applied at *C*, causing the frame to deflect Δ' as shown in Fig. 12–18*f*. Here the joints at *B* and *C* are *temporarily restrained from rotating*, and as a result the fixed-end moments at the ends of the columns are determined from the formula for deflection found on the inside back cover, that is,

Joint	*A*	*B*		*C*		*D*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
DF	0	0.5	0.5	0.5	0.5	0
FEM			−10.24	2.56		
Dist.		5.12	5.12	−1.28	−1.28	
CO	2.56		−0.64	2.56		−0.64
Dist.		0.32	0.32	−1.28	−1.28	
CO	0.16		−0.64	0.16		−0.64
Dist.		0.32	0.32	−0.08	−0.08	
CO	0.16		−0.04	0.16		−0.04
Dist.		0.02	0.02	−0.08	−0.08	
ΣM	2.88	5.78	−5.78	2.72	−2.72	−1.32

(d)



(e)



(f)

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.5	0.5	0.5	0.5	0
FEM	−100	−100			−100	−100
Dist.		50	50	50	50	
CO	25		25	25		25
Dist.		−12.5	−12.5	−12.5	−12.5	
CO	−6.25		−6.25	−6.25		−6.25
Dist.		3.125	3.125	3.125	3.125	
CO	1.56		1.56	1.56		1.56
Dist.		−0.78	−0.78	−0.78	−0.78	
CO	−0.39		−0.39	−0.39		−0.39
Dist.		0.195	0.195	0.195	0.195	
ΣM	−80.00	−60.00	60.00	60.00	−60.00	−80.00

(g)

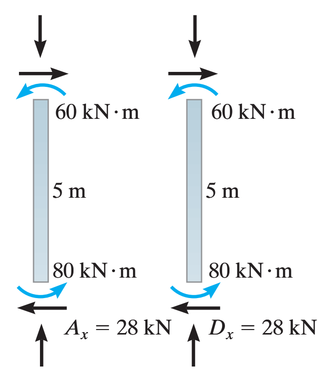
$$M = \frac{6EI\Delta}{L^2}$$

Since *both* B and C happen to be displaced the same amount Δ', and AB and DC have the *same* E, I, and L, the FEM in AB will be the *same* as that in DC. As shown in Fig. 12–18*f*, we will arbitrarily *assume* this fixed-end moment to be

$$(\text{FEM})_{AB} = (\text{FEM})_{BA} = (\text{FEM})_{CD} = (\text{FEM})_{DC} = -100 \text{ kN}\cdot\text{m}$$

A *negative sign* is necessary since the moment must act *counterclockwise* on the column for deflection Δ' to the right. The value of $\mathbf{R}'$ associated with this -100 kN·m moment can now be determined. The moment distribution of the FEMs is shown in Fig. 12–18*g*. From equilibrium, the horizontal reactions at A and D are calculated, Fig. 12–18*h*. Thus, for the entire frame we require

$$\Sigma F_x = 0; \qquad R' = 28 + 28 = 56.0 \text{ kN}$$



(h)

Hence, $R' = 56.0$ kN creates the moments tabulated in Fig. 12–18*g*. Corresponding moments caused by $R = 0.92$ kN can be determined by proportion. Therefore, the resultant moment in the frame, Fig. 12–18*a*, is equal to the *sum* of those calculated for the frame in Fig. 12–18*b* plus the proportionate amount of those for the frame in Fig. 12–18*c*. We have

$$M_{AB} = 2.88 + \tfrac{0.92}{56.0}(-80) = 1.57 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BA} = 5.78 + \tfrac{0.92}{56.0}(-60) = 4.79 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BC} = -5.78 + \tfrac{0.92}{56.0}(60) = -4.79 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{CB} = 2.72 + \tfrac{0.92}{56.0}(60) = 3.71 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{CD} = -2.72 + \tfrac{0.92}{56.0}(-60) = -3.71 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{DC} = -1.32 + \tfrac{0.92}{56.0}(-80) = -2.63 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$

12

EXAMPLE 12.7

Determine the moments at each joint of the frame shown in Fig. 12–19*a*. The moment of inertia of each member is indicated in the figure.

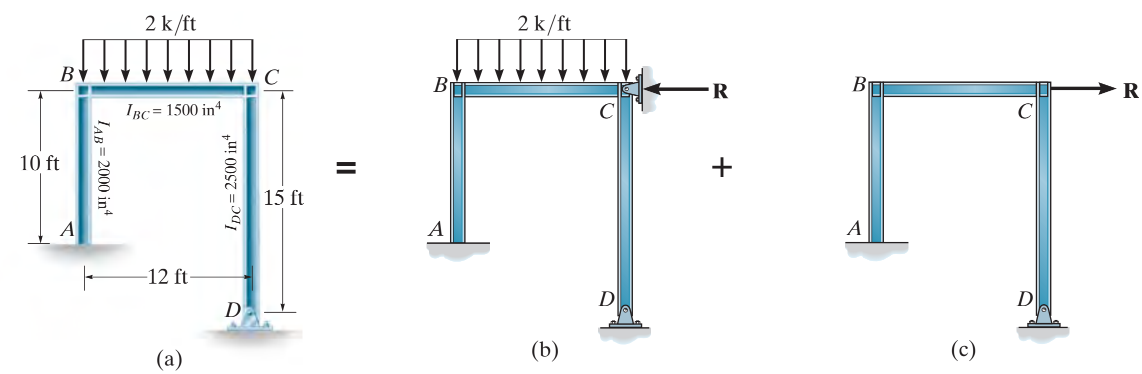


Fig. 12–19

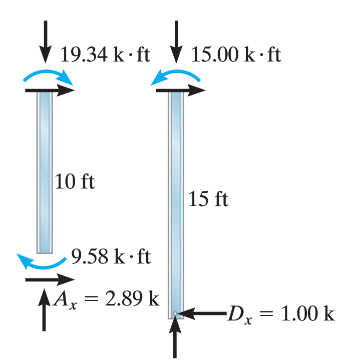
SOLUTION

The frame is first held from sidesway as shown in Fig. 12–19*b*. The internal moments are computed at the joints as indicated in Fig. 12–19*d*. Here the stiffness factor of CD was computed using $3EI/L$ since there is a pin at D. Calculation of the horizontal reactions at A and D is shown in Fig. 12–19*e*. Thus, for the entire frame,

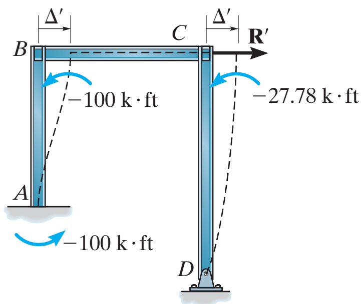
$$\Sigma F_x = 0; \qquad R = 2.89 - 1.00 = 1.89 \text{ k}$$

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.615	0.385	0.5	0.5	1
FEM			−24	24		
Dist.		14.76	9.24	−12	−12	
CO	7.38		−6	4.62		
Dist.		3.69	2.31	−2.31	−2.31	
CO	1.84		−1.16	1.16		
Dist.		0.713	0.447	−0.58	−0.58	
CO	0.357		−0.29	0.224		
Dist.		0.18	0.11	−0.11	−0.11	
ΣM	9.58	19.34	−19.34	15.00	−15.00	0

(d)



(e)



(f)

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.615	0.385	0.5	0.5	1
FEM	−100	−100			−27.78	
Dist.		61.5	38.5	13.89	13.89	
CO	30.75		6.94	19.25		
Dist.		−4.27	−2.67	−9.625	−9.625	
CO	−2.14		−4.81	−1.34		
Dist.		2.96	1.85	0.67	0.67	
CO	1.48		0.33	0.92		
Dist.		−0.20	−0.13	−0.46	−0.46	
ΣM	−69.91	−40.01	40.01	23.31	−23.31	0

(g)

The opposite force is now applied to the frame as shown in Fig. 12–19*c*. As in the previous example, we will consider a force $\mathbf{R}'$ acting as shown in Fig. 12–19*f*. As a result, joints B and C are displaced by the same amount Δ'. The fixed-end moments for BA are computed from

$$(\text{FEM})_{AB} = (\text{FEM})_{BA} = -\frac{6EI\Delta}{L^2} = -\frac{6E(2000)\Delta'}{(10)^2}$$

However, from the table on the inside back cover, for CD we have

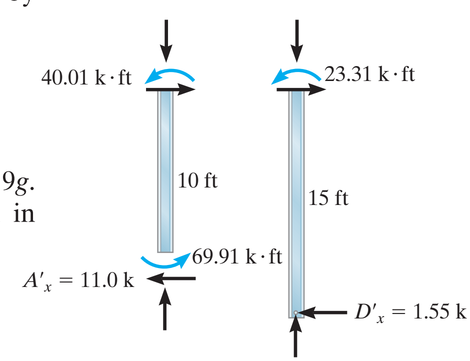
$$(\text{FEM})_{CD} = -\frac{3EI\Delta}{L^2} = -\frac{3E(2500)\Delta'}{(15)^2}$$

Assuming the FEM for AB is $-100\ \text{k}\cdot\text{ft}$ as shown in Fig. 12–19*f*, the *corresponding* FEM at C, causing the *same* Δ', is found by comparison, i.e.,

$$\Delta' = -\frac{(-100)(10)^2}{6E(2000)} = -\frac{(\text{FEM})_{CD}(15)^2}{3E(2500)}$$
$$(\text{FEM})_{CD} = -27.78\ \text{k}\cdot\text{ft}$$

Moment distribution for these FEMs is tabulated in Fig. 12–19*g*. Computation of the horizontal reactions at A and D is shown in Fig. 12–19*h*. Thus, for the entire frame,

$$\Sigma F_x = 0; \qquad R' = 11.0 + 1.55 = 12.55\ \text{k}$$



(h)

The resultant moments in the frame are therefore

$$M_{AB} = 9.58 + \left(\tfrac{1.89}{12.55}\right)(-69.91) = -0.948\ \text{k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BA} = 19.34 + \left(\tfrac{1.89}{12.55}\right)(-40.01) = 13.3\ \text{k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{BC} = -19.34 + \left(\tfrac{1.89}{12.55}\right)(40.01) = -13.3\ \text{k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CB} = 15.00 + \left(\tfrac{1.89}{12.55}\right)(23.31) = 18.5\ \text{k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CD} = -15.00 + \left(\tfrac{1.89}{12.55}\right)(-23.31) = -18.5\ \text{k}\cdot\text{ft} \qquad \textit{Ans.}$$

12

EXAMPLE 12.8

Determine the moments at each joint of the frame shown in Fig. 12–20*a*. *EI* is constant.

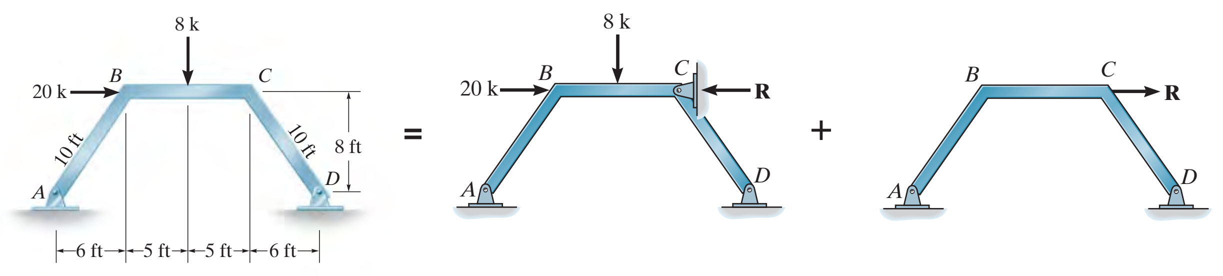


Fig. 12–20

SOLUTION

First sidesway is prevented by the restraining force **R**, Fig. 12–20*b*. The FEMs for member *BC* are

$$(\mathrm{FEM})_{BC} = -\frac{8(10)}{8} = -10\ \mathrm{k\cdot ft} \qquad (\mathrm{FEM})_{CB} = \frac{8(10)}{8} = 10\ \mathrm{k\cdot ft}$$

Since spans *AB* and *DC* are pinned at their ends, the stiffness factor is computed using $3EI/L$. The moment distribution is shown in Fig. 12–20*d*.

Using these results, the *horizontal reactions* at *A* and *D* must be determined. This is done using an equilibrium analysis of *each member,* Fig. 12–20*e*. Summing moments about points *B* and *C* on each leg, we have

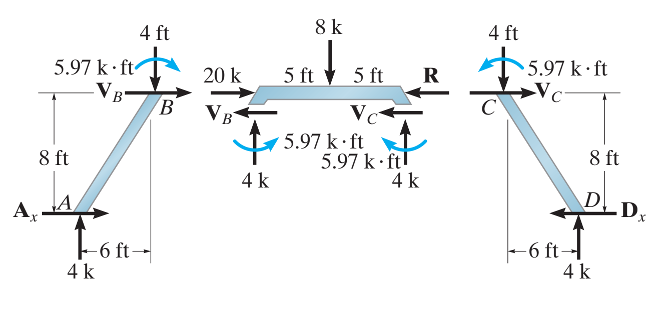
$$\circlearrowright + \Sigma M_B = 0; \qquad -5.97 + A_x(8) - 4(6) = 0 \qquad A_x = 3.75\ \mathrm{k}$$
$$\circlearrowright + \Sigma M_C = 0; \qquad 5.97 - D_x(8) + 4(6) = 0 \qquad D_x = 3.75\ \mathrm{k}$$

Thus, for the entire frame,

$$\Sigma F_x = 0; \qquad R = 3.75 - 3.75 + 20 = 20\ \mathrm{k}$$

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	1	0.429	0.571	0.571	0.429	1
FEM			−10	10		
Dist.		4.29	5.71	−5.71	−4.29	
CO			−2.86	2.86		
Dist.		1.23	1.63	−1.63	−1.23	
CO			−0.82	0.82		
Dist.		0.35	0.47	−0.47	−0.35	
CO			−0.24	0.24		
Dist.		0.10	0.13	−0.13	−0.10	
ΣM	0	5.97	−5.97	5.97	−5.97	0

(d)



(e)

The opposite force **R** is now applied to the frame as shown in Fig. 12–20*c*. In order to determine the internal moments developed by **R** we will first consider the force **R**′ acting as shown in Fig. 12–20*f*. Here the dashed lines do not represent the distortion of the frame members; instead, they are constructed as straight lines extended to the final positions B' and C' of points B and C, respectively. Due to the symmetry of the frame, the displacement $BB' = CC' = \Delta'$. Furthermore, these displacements cause BC to rotate. The vertical distance between B' and C' is $1.2\Delta'$, as shown on the displacement diagram, Fig. 12–20*g*. Since each span undergoes end-point displacements that cause the spans to rotate, fixed-end moments are induced in the spans. These are: $(\text{FEM})_{BA} = (\text{FEM})_{CD} = -3EI\Delta'/(10)^2$, $(\text{FEM})_{BC} = (\text{FEM})_{CB} = 6EI(1.2\Delta')/(10)^2$.

Notice that for BA and CD the moments are *negative* since clockwise rotation of the span causes a *counterclockwise* FEM.

If we arbitrarily assign a value of $(\text{FEM})_{BA} = (\text{FEM})_{CD} = -100\text{ k}\cdot\text{ft}$, then equating Δ' in the above formulas yields $(\text{FEM})_{BC} = (\text{FEM})_{CB} = 240\text{ k}\cdot\text{ft}$. These moments are applied to the frame and distributed, Fig. 12–20*h*. Using these results, the equilibrium analysis is shown in Fig. 12–20*i*. For each leg, we have

$$\circlearrowright + \Sigma M_B = 0; \quad -A'_x(8) + 29.36(6) + 146.80 = 0 \qquad A'_x = 40.37\text{ k}$$
$$\circlearrowright + \Sigma M_C = 0; \quad -D'_x(8) + 29.36(6) + 146.80 = 0 \qquad D'_x = 40.37\text{ k}$$

Thus, for the entire frame,

$$\Sigma F_x = 0; \qquad R' = 40.37 + 40.37 = 80.74\text{ k}$$

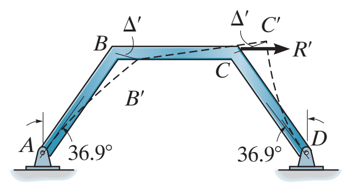
The resultant moments in the frame are therefore

$$M_{BA} = 5.97 + \left(\frac{20}{80.74}\right)(-146.80) = -30.4\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

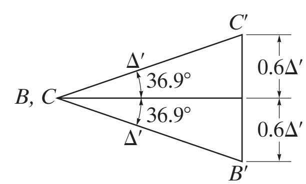
$$M_{BC} = -5.97 + \left(\frac{20}{80.74}\right)(146.80) = 30.4\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

$$M_{CB} = 5.97 + \left(\frac{20}{80.74}\right)(146.80) = 42.3\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$

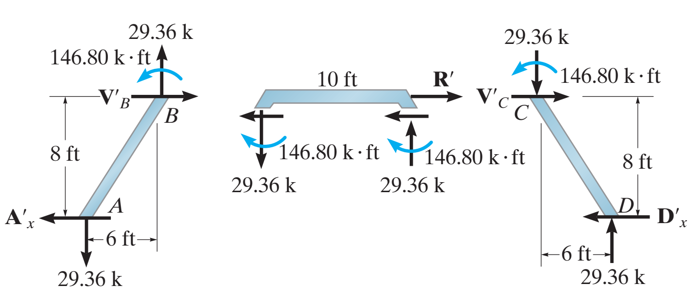
$$M_{CD} = -5.97 + \left(\frac{20}{80.74}\right)(-146.80) = -42.3\text{ k}\cdot\text{ft} \qquad \textit{Ans.}$$



(f)



(g)



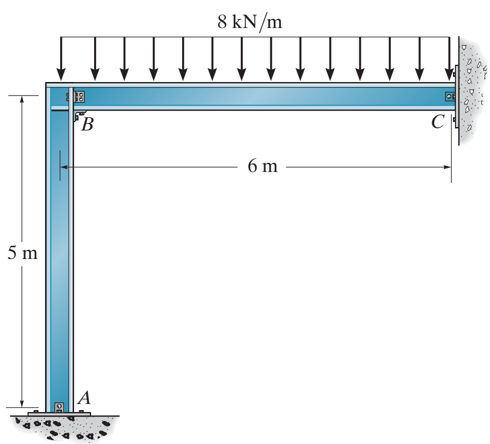
(i)

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	1	0.429	0.571	0.571	0.429	1
FEM		−100	240	240	−100	
Dist.		−60.06	−79.94	−79.94	−60.06	
CO			−39.97	−39.97		
Dist.		17.15	22.82	22.82	17.15	
CO			11.41	11.41		
Dist.		−4.89	−6.52	−6.52	−4.89	
CO			−3.26	−3.26		
Dist.		1.40	1.86	1.86	1.40	
CO			0.93	0.93		
Dist.		−0.40	−0.53	−0.53	−0.40	
ΣM	0	−146.80	146.80	146.80	−146.80	0

(h)

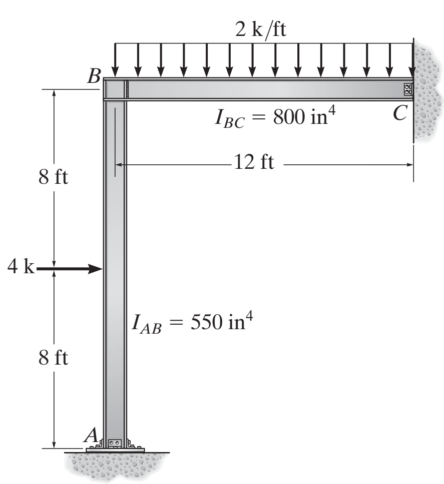
PROBLEMS

12–13. Determine the moment at B, then draw the moment diagram for each member of the frame. Assume the supports at A and C are pins. EI is constant.



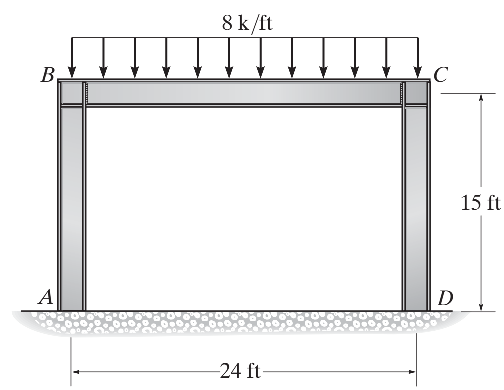
Prob. 12–13

12–14. Determine the moments at the ends of each member of the frame. Assume the joint at B is fixed, C is pinned, and A is fixed. The moment of inertia of each member is listed in the figure. $E = 29(10^3)$ ksi.



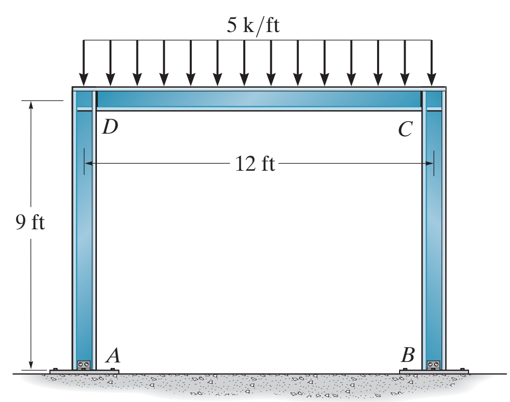
Prob. 12–14

12–15. Determine the reactions at A and D. Assume the supports at A and D are fixed and B and C are fixed connected. EI is constant.



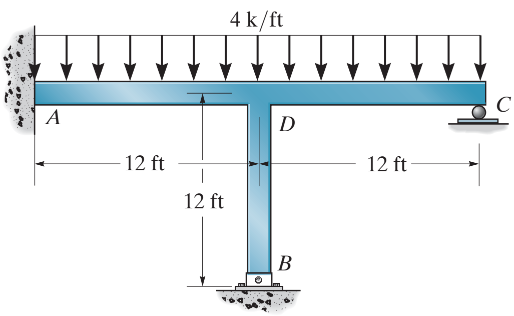
Prob. 12–15

***12–16.** Determine the moments at D and C, then draw the moment diagram for each member of the frame. Assume the supports at A and B are pins and D and C are fixed joints. EI is constant.



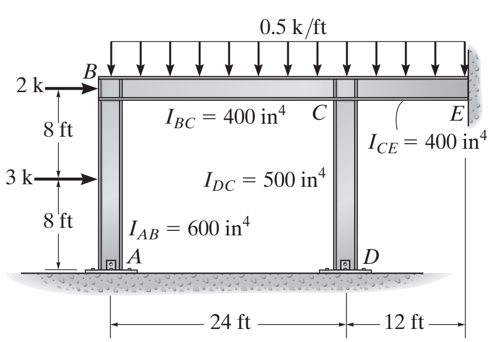
Prob. 12–16

12–17. Determine the moments at the fixed support A and joint D and then draw the moment diagram for the frame. Assume B is pinned.



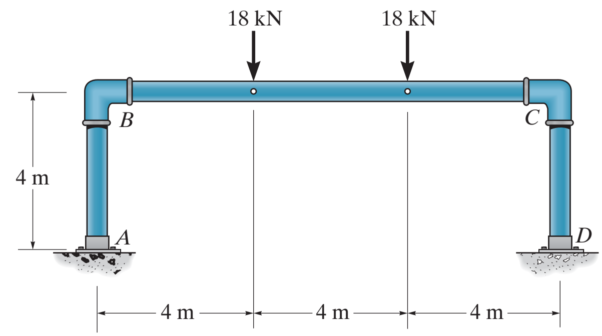
Prob. 12–17

12–18. Determine the moments at each joint of the frame, then draw the moment diagram for member BCE. Assume B, C, and E are fixed connected and A and D are pins. $E = 29(10^3)$ ksi.



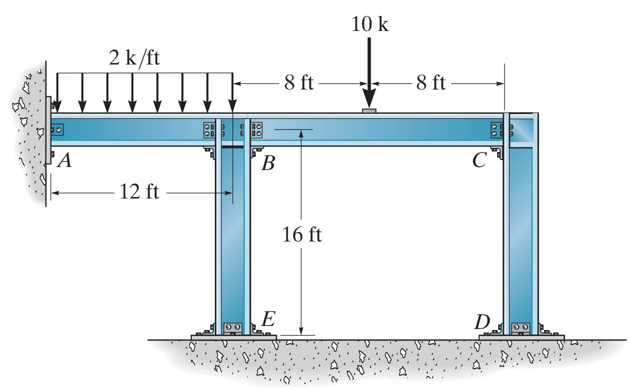
Prob. 12–18

12–19. The frame is made from pipe that is fixed connected. If it supports the loading shown, determine the moments developed at each of the joints. EI is constant.



Prob. 12–19

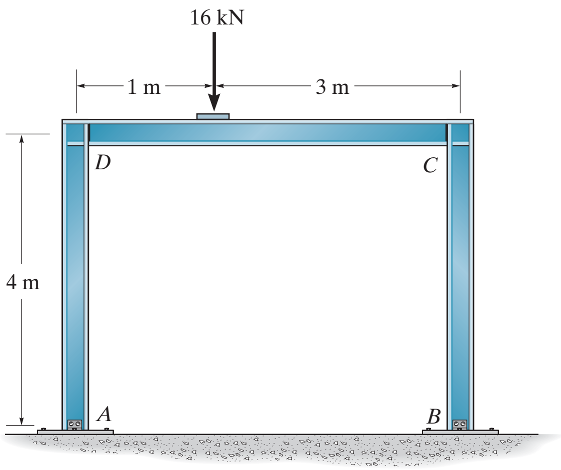
***12–20.** Determine the moments at B and C, then draw the moment diagram for each member of the frame. Assume the supports at A, E, and D are fixed. EI is constant.



Prob. 12–20

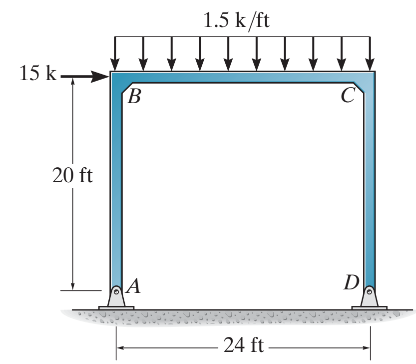
12

12–21. Determine the moments at D and C, then draw the moment diagram for each member of the frame. Assume the supports at A and B are pins. EI is constant.



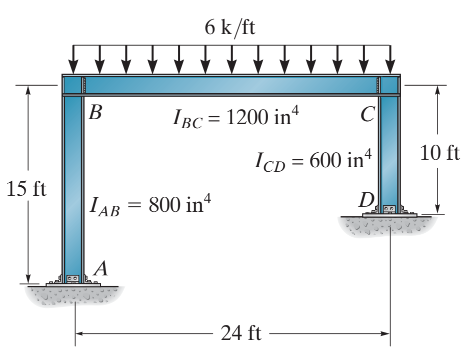
Prob. 12–21

12–23. Determine the moments acting at the ends of each member of the frame. EI is the constant.



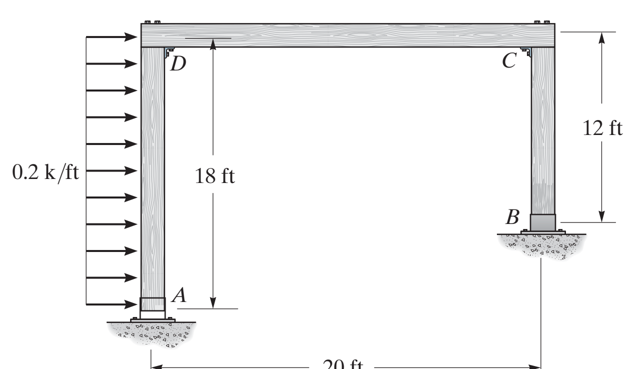
Prob. 12–23

12–22. Determine the moments acting at the ends of each member. Assume the supports at A and D are fixed. The moment of inertia of each member is indicated in the figure. $E = 29(10^3)$ ksi.



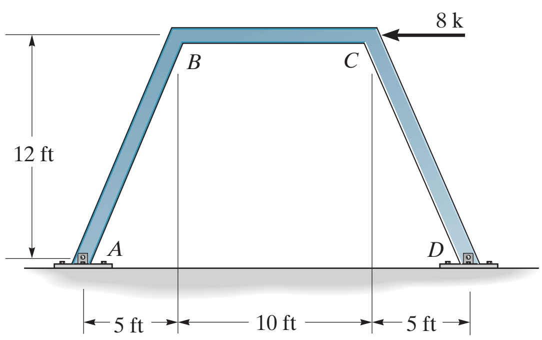
Prob. 12–22

***12–24.** Determine the moments acting at the ends of each member. Assume the joints are fixed connected and A and B are fixed supports. EI is constant.



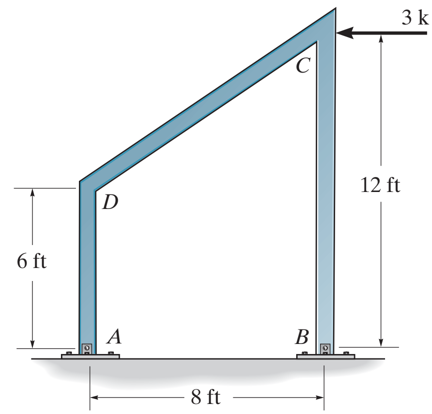
Prob. 12–24

12–25. Determine the moments at joints B and C, then draw the moment diagram for each member of the frame. The supports at A and D are pinned. EI is constant.



Prob. 12–25

12–26. Determine the moments at C and D, then draw the moment diagram for each member of the frame. Assume the supports at A and B are pins. EI is constant.



Prob. 12–26

CHAPTER REVIEW

Moment distribution is a method of successive approximations that can be carried out to any desired degree of accuracy. It initially requires locking all the joints of the structure. The equilibrium moment at each joint is then determined, the joints are unlocked and this moment is distributed onto each connecting member, and half its value is carried over to the other side of the span. This cycle of locking and unlocking the joints is repeated until the carry-over moments become acceptably small. The process then stops and the moment at each joint is the sum of the moments from each cycle of locking and unlocking.

The process of moment distribution is conveniently done in tabular form. Before starting, the fixed-end moment for each span must be calculated using the table on the inside back cover of the book. The distribution factors are found by dividing a member's stiffness by the total stiffness of the joint. For members having a far end fixed, use $K = 4EI/L$; for a far-end pinned or roller supported member, $K = 3EI/L$; for a symmetric span and loading, $K = 2EI/L$; and for an antisymmetric loading, $K = 6EI/L$. Remember that the distribution factor for a fixed end is DF $= 0$, and for a pin or roller-supported end, DF $= 1$.

The use of variable-moment-of-inertia girders has reduced considerably the deadweight loading of each of these spans.

Beams and Frames Having Nonprismatic Members

13

In this chapter we will apply the slope-deflection and moment-distribution methods to analyze beams and frames composed of nonprismatic members. We will first discuss how the necessary carry-over factors, stiffness factors, and fixed-end moments are obtained. This is followed by a discussion related to using tabular values often published in design literature. Finally, the analysis of statically indeterminate structures using the slope-deflection and moment-distribution methods will be discussed.

13.1 Loading Properties of Nonprismatic Members

Often, to save material, girders used for long spans on bridges and buildings are designed to be nonprismatic, that is, to have a variable moment of inertia. The most common forms of structural members that are nonprismatic have haunches that are either stepped, tapered, or parabolic, Fig. 13–1. Provided we can express the member's moment of inertia as a function of the length coordinate x, then we can use the principle of virtual work or Castigliano's theorem as discussed in Chapter 9 to find its deflection. The equations are

$$\Delta = \int_0^l \frac{Mm}{EI}dx \qquad \text{or} \qquad \Delta = \int_0^l \frac{\partial M}{\partial P}\frac{M}{EI}dx$$

If the member's geometry and loading require evaluation of an integral that cannot be determined in closed form, then Simpson's rule or some other numerical technique will have to be used to carry out the integration.

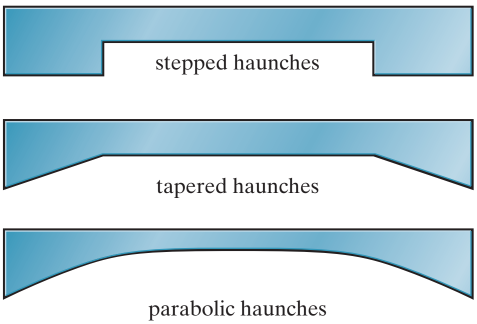


Fig. 13–1

If the slope deflection equations or moment distribution are used to determine the reactions on a nonprismatic member, then we must first calculate the following properties for the member.

Fixed-End Moments (FEM). The end moment reactions on the member that is assumed fixed supported, Fig. 13–2*a*.

Stiffness Factor (K). The magnitude of moment that must be applied to the end of the member such that the end rotates through an angle of $\theta = 1$ rad. Here the moment is applied at the pin support, while the other end is assumed fixed, Fig. 13–2*b*.

Carry-Over Factor (COF). Represents the numerical fraction (C) of the moment that is "carried over" from the pin-supported end to the wall, Fig. 13.2*c*.

Once obtained, the computations for the stiffness and carry-over factors can be checked, in part, by noting an important relationship that exists between them. In this regard, consider the beam in Fig. 13–3 subjected to the loads and deflections shown. Application of the Maxwell-Betti reciprocal theorem requires the work done by the loads in Fig. 13–3*a* acting through the displacements in Fig. 13–3*b* be equal to the work of the loads in Fig. 13–3*b* acting through the displacements in Fig. 13–3*a*, that is,

$$U_{AB} = U_{BA}$$

$$K_A(0) + C_{AB}K_A(1) = C_{BA}K_B(1) + K_B(0)$$

or

$$C_{AB}K_A = C_{BA}K_B \qquad (13\text{–}1)$$

Hence, once determined, the stiffness and carry-over factors must satisfy Eq. 13–1.

The tapered concrete bent is used to support the girders of this highway bridge.

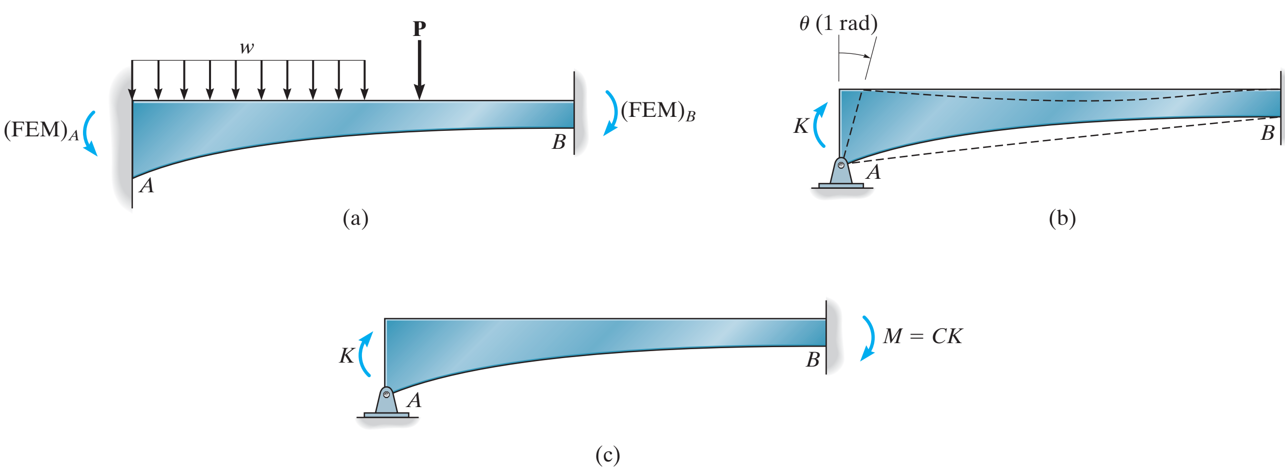


Fig. 13–2

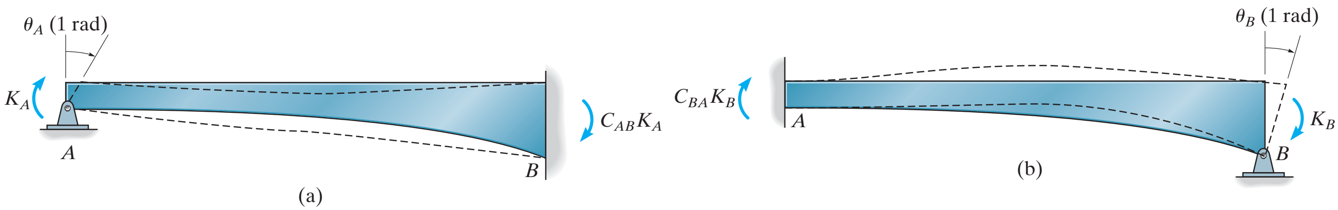


Fig. 13–3

These properties can be obtained using, for example, the conjugate beam method or an energy method. However, considerable labor is often involved in the process. As a result, graphs and tables have been made available to determine this data for common shapes used in structural design. One such source is the *Handbook of Frame Constants*, published by the Portland Cement Association.* A portion of these tables, taken from this publication, is listed here as Tables 13–1 and 13–2. A more complete tabular form of the data is given in the PCA handbook along with the relevant derivations of formulas used.

The nomenclature is defined as follows:

a_A, a_B = ratio of the length of haunch at ends A and B to the length of span.

b = ratio of the distance from the concentrated load to end A to the length of span.

C_{AB}, C_{BA} = carry-over factors of member AB at ends A and B, respectively.

h_A, h_B = depth of member at ends A and B, respectively.

h_C = depth of member at minimum section.

I_C = moment of inertia of section at minimum depth.

k_{AB}, k_{BA} = stiffness factor at ends A and B, respectively.

L = length of member.

M_{AB}, M_{BA} = fixed-end moment at ends A and B, respectively; specified in tables for uniform load w or concentrated force P.

r_A, r_B = ratios for rectangular cross-sectional areas, where $r_A = (h_A - h_C)/h_C$, $r_B = (h_B - h_C)/h_C$.

As noted, the fixed-end moments and carry-over factors are found from the tables. The absolute stiffness factor can be determined using the tabulated stiffness factors and found from

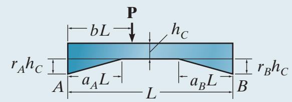
$$K_A = \frac{k_{AB}EI_C}{L} \qquad K_B = \frac{k_{BA}EI_C}{L} \tag{13–2}$$

Application of the use of the tables will be illustrated in Example 13–1.

Timber frames having a variable moment of inertia are often used in the construction of churches.

***Handbook of Frame Constants*. Portland Cement Association, Chicago, Illinois.

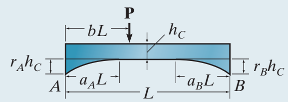
TABLE 13–1 Straight Haunches—Constant Width



Note: All carry-over factors are negative and all stiffness factors are positive.

Right Haunch		Carry-over Factors		Stiffness Factors		Unif. Load FEM Coef. $\times wL^2$		Concentrated Load FEM—Coef. $\times PL$										Haunch Load at			
								b										Left		Right	
								0.1		0.3		0.5		0.7		0.9		FEM Coef. $\times w_A L^2$		FEM Coef. $\times w_B L^2$	
a_B	r_B	C_{AB}	C_{BA}	k_{AB}	k_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}
$a_A = 0.3$ $a_B =$ *variable* $r_A = 1.0$ $r_B =$ *variable*																					
	0.4	0.543	0.766	9.19	6.52	0.1194	0.0791	0.0935	0.0034	0.2185	0.0384	0.1955	0.1147	0.0889	0.1601	0.0096	0.0870	0.0133	0.0008	0.0006	0.0058
	0.6	0.576	0.758	9.53	7.24	0.1152	0.0851	0.0934	0.0038	0.2158	0.0422	0.1883	0.1250	0.0798	0.1729	0.0075	0.0898	0.0133	0.0009	0.0005	0.0060
0.2	1.0	0.622	0.748	10.06	8.37	0.1089	0.0942	0.0931	0.0042	0.2118	0.0480	0.1771	0.1411	0.0668	0.1919	0.0047	0.0935	0.0132	0.0011	0.0004	0.0062
	1.5	0.660	0.740	10.52	9.38	0.1037	0.1018	0.0927	0.0047	0.2085	0.0530	0.1678	0.1550	0.0559	0.2078	0.0028	0.0961	0.0130	0.0012	0.0002	0.0064
	2.0	0.684	0.734	10.83	10.09	0.1002	0.1069	0.0924	0.0050	0.2062	0.0565	0.1614	0.1645	0.0487	0.2185	0.0019	0.0974	0.0129	0.0013	0.0001	0.0065
	0.4	0.579	0.741	9.47	7.40	0.1175	0.0822	0.0934	0.0037	0.2164	0.0419	0.1909	0.1225	0.0856	0.1649	0.0100	0.0861	0.0133	0.0009	0.0022	0.0118
	0.6	0.629	0.726	9.98	8.64	0.1120	0.0902	0.0931	0.0042	0.2126	0.0477	0.1808	0.1379	0.0747	0.1807	0.0080	0.0888	0.0132	0.0010	0.0018	0.0124
0.3	1.0	0.705	0.705	10.85	10.85	0.1034	0.1034	0.0924	0.0052	0.2063	0.0577	0.1640	0.1640	0.0577	0.2063	0.0052	0.0924	0.0131	0.0013	0.0013	0.0131
	1.5	0.771	0.689	11.70	13.10	0.0956	0.1157	0.0917	0.0062	0.2002	0.0675	0.1483	0.1892	0.0428	0.2294	0.0033	0.0953	0.0129	0.0015	0.0008	0.0137
	2.0	0.817	0.678	12.33	14.85	0.0901	0.1246	0.0913	0.0069	0.1957	0.0750	0.1368	0.2080	0.0326	0.2455	0.0022	0.0968	0.0128	0.0017	0.0006	0.0141
$a_A = 0.2$ $a_B =$ *variable* $r_A = 1.5$ $r_B =$ *variable*																					
	0.4	0.569	0.714	7.97	6.35	0.1166	0.0799	0.0966	0.0019	0.2186	0.0377	0.1847	0.1183	0.0821	0.1626	0.0088	0.0873	0.0064	0.0001	0.0006	0.0058
	0.6	0.603	0.707	8.26	7.04	0.1127	0.0858	0.0965	0.0021	0.2163	0.0413	0.1778	0.1288	0.0736	0.1752	0.0068	0.0901	0.0064	0.0001	0.0005	0.0060
0.2	1.0	0.652	0.698	8.70	8.12	0.1069	0.0947	0.0963	0.0023	0.2127	0.0468	0.1675	0.1449	0.0616	0.1940	0.0043	0.0937	0.0064	0.0002	0.0004	0.0062
	1.5	0.691	0.691	9.08	9.08	0.1021	0.1021	0.0962	0.0025	0.2097	0.0515	0.1587	0.1587	0.0515	0.2097	0.0025	0.0962	0.0064	0.0002	0.0002	0.0064
	2.0	0.716	0.686	9.34	9.75	0.0990	0.1071	0.0960	0.0028	0.2077	0.0547	0.1528	0.1681	0.0449	0.2202	0.0017	0.0975	0.0064	0.0002	0.0001	0.0065
	0.4	0.607	0.692	8.21	7.21	0.1148	0.0829	0.0965	0.0021	0.2168	0.0409	0.1801	0.1263	0.0789	0.1674	0.0091	0.0866	0.0064	0.0002	0.0020	0.0118
	0.6	0.659	0.678	8.65	8.40	0.1098	0.0907	0.0964	0.0024	0.2135	0.0464	0.1706	0.1418	0.0688	0.1831	0.0072	0.0892	0.0064	0.0002	0.0017	0.0123
0.3	1.0	0.740	0.660	9.38	10.52	0.1018	0.1037	0.0961	0.0028	0.2078	0.0559	0.1550	0.1678	0.0530	0.2085	0.0047	0.0927	0.0064	0.0002	0.0012	0.0130
	1.5	0.809	0.645	10.09	12.66	0.0947	0.1156	0.0958	0.0033	0.2024	0.0651	0.1403	0.1928	0.0393	0.2311	0.0029	0.0950	0.0063	0.0003	0.0008	0.0137
	2.0	0.857	0.636	10.62	14.32	0.0897	0.1242	0.0955	0.0038	0.1985	0.0720	0.1296	0.2119	0.0299	0.2469	0.0020	0.0968	0.0063	0.0003	0.0005	0.0141

TABLE 13–2 Parabolic Haunches—Constant Width



Note: All carry-over factors are negative and all stiffness factors are positive.

Right Haunch		Carry-over Factors		Stiffness Factors		Unif. Load FEM Coef. × wL^2		Concentrated Load FEM—Coef. × PL										Haunch Load at			
								b										Left		Right	
								0.1		0.3		0.5		0.7		0.9		FEM Coef. × $w_A L^2$		FEM Coef. × $w_B L^2$	
a_B	r_B	C_{AB}	C_{BA}	k_{AB}	k_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}	M_{AB}	M_{BA}
$a_A = 0.2$ $a_B =$ *variable* $r_A = 1.0$ $r_B =$ *variable*																					
	0.4	0.558	0.627	6.08	5.40	0.1022	0.0841	0.0938	0.0033	0.1891	0.0502	0.1572	0.1261	0.0715	0.1618	0.0073	0.0877	0.0032	0.0001	0.0002	0.0030
	0.6	0.582	0.624	6.21	5.80	0.0995	0.0887	0.0936	0.0036	0.1872	0.0535	0.1527	0.1339	0.0663	0.1708	0.0058	0.0902	0.0032	0.0001	0.0002	0.0031
0.2	1.0	0.619	0.619	6.41	6.41	0.0956	0.0956	0.0935	0.0038	0.1844	0.0584	0.1459	0.1459	0.0584	0.1844	0.0038	0.0935	0.0032	0.0001	0.0001	0.0032
	1.5	0.649	0.614	6.59	6.97	0.0921	0.1015	0.0933	0.0041	0.1819	0.0628	0.1399	0.1563	0.0518	0.1962	0.0025	0.0958	0.0032	0.0001	0.0001	0.0032
	2.0	0.671	0.611	6.71	7.38	0.0899	0.1056	0.0932	0.0044	0.1801	0.0660	0.1358	0.1638	0.0472	0.2042	0.0017	0.0971	0.0032	0.0001	0.0000	0.0033
	0.4	0.588	0.616	6.22	5.93	0.1002	0.0877	0.0937	0.0035	0.1873	0.0537	0.1532	0.1339	0.0678	0.1686	0.0073	0.0877	0.0032	0.0001	0.0007	0.0063
	0.6	0.625	0.609	6.41	6.58	0.0966	0.0942	0.0935	0.0039	0.1845	0.0587	0.1467	0.1455	0.0609	0.1808	0.0057	0.0902	0.0032	0.0001	0.0005	0.0065
0.3	1.0	0.683	0.598	6.73	7.68	0.0911	0.1042	0.0932	0.0044	0.1801	0.0669	0.1365	0.1643	0.0502	0.2000	0.0037	0.0936	0.0031	0.0001	0.0004	0.0068
	1.5	0.735	0.589	7.02	8.76	0.0862	0.1133	0.0929	0.0050	0.1760	0.0746	0.1272	0.1819	0.0410	0.2170	0.0023	0.0959	0.0031	0.0001	0.0003	0.0070
	2.0	0.772	0.582	7.25	9.61	0.0827	0.1198	0.0927	0.0054	0.1730	0.0805	0.1203	0.1951	0.0345	0.2293	0.0016	0.0972	0.0031	0.0001	0.0002	0.0072
$a_A = 0.5$ $a_B =$ *variable* $r_A = 1.0$ $r_B =$ *variable*																					
	0.4	0.488	0.807	9.85	5.97	0.1214	0.0753	0.0929	0.0034	0.2131	0.0371	0.2021	0.1061	0.0979	0.1506	0.0105	0.0863	0.0171	0.0017	0.0003	0.0030
	0.6	0.515	0.803	10.10	6.45	0.1183	0.0795	0.0928	0.0036	0.2110	0.0404	0.1969	0.1136	0.0917	0.1600	0.0083	0.0892	0.0170	0.0018	0.0002	0.0030
0.2	1.0	0.547	0.796	10.51	7.22	0.1138	0.0865	0.0926	0.0040	0.2079	0.0448	0.1890	0.1245	0.0809	0.1740	0.0056	0.0928	0.0168	0.0020	0.0001	0.0031
	1.5	0.571	0.786	10.90	7.90	0.1093	0.0922	0.0923	0.0043	0.2055	0.0485	0.1818	0.1344	0.0719	0.1862	0.0035	0.0951	0.0167	0.0021	0.0001	0.0032
	2.0	0.590	0.784	11.17	8.40	0.1063	0.0961	0.0922	0.0046	0.2041	0.0506	0.1764	0.1417	0.0661	0.1948	0.0025	0.0968	0.0166	0.0022	0.0001	0.0032
	0.4	0.554	0.753	10.42	7.66	0.1170	0.0811	0.0926	0.0040	0.2087	0.0442	0.1924	0.1205	0.0898	0.1595	0.0107	0.0853	0.0169	0.0020	0.0042	0.0145
	0.6	0.606	0.730	10.96	9.12	0.1115	0.0889	0.0922	0.0046	0.2045	0.0506	0.1820	0.1360	0.0791	0.1738	0.0086	0.0878	0.0167	0.0022	0.0036	0.0152
0.5	1.0	0.694	0.694	12.03	12.03	0.1025	0.1025	0.0915	0.0057	0.1970	0.0626	0.1639	0.1639	0.0626	0.1970	0.0057	0.0915	0.0164	0.0028	0.0028	0.0164
	1.5	0.781	0.664	13.12	15.47	0.0937	0.1163	0.0908	0.0070	0.1891	0.0759	0.1456	0.1939	0.0479	0.2187	0.0039	0.0940	0.0160	0.0034	0.0021	0.0174
	2.0	0.850	0.642	14.09	18.64	0.0870	0.1275	0.0901	0.0082	0.1825	0.0877	0.1307	0.2193	0.0376	0.2348	0.0027	0.0957	0.0157	0.0039	0.0016	0.0181

13.2 Moment Distribution for Structures Having Nonprismatic Members

Once the fixed-end moments and stiffness and carry-over factors for the nonprismatic members of a structure have been determined, application of the moment-distribution method follows the same procedure as outlined in Chapter 12. In this regard, recall that the distribution of moments may be shortened if a member stiffness factor is modified to account for conditions of end-span pin support and structure symmetry or antisymmetry. Similar modifications can also be made to nonprismatic members.

Beam Pin Supported at Far End. Consider the beam in Fig. 13–4*a*, which is pinned at its far end B. The absolute stiffness factor K'_A is the moment applied at A such that it rotates the beam at $A, \theta_A = 1$ rad. It can be determined as follows. First assume that B is temporarily fixed and a moment K_A is applied at A, Fig. 13–4*b*. The moment induced at B is $C_{AB}K_A$, where C_{AB} is the carry-over factor from A to B. Second, since B is not to be fixed, application of the opposite moment $C_{AB}K_A$ to the beam, Fig. 13–4*c*, will induce a moment $C_{BA}C_{AB}K_A$ at end A. By superposition, the result of these two applications of moment yields the beam loaded as shown in Fig. 13–4*a*. Hence it can be seen that the absolute stiffness factor of the beam at A is

$$\boxed{K'_A = K_A(1 - C_{AB}C_{BA})} \qquad (13\text{–}3)$$

Here K_A is the absolute stiffness factor of the beam, assuming it to be fixed at the far end B. For example, in the case of a prismatic beam, $K_A = 4EI/L$ and $C_{AB} = C_{BA} = \frac{1}{2}$. Substituting into Eq. 13–3 yields $K'_A = 3EI/L$, the same as Eq. 12–4.

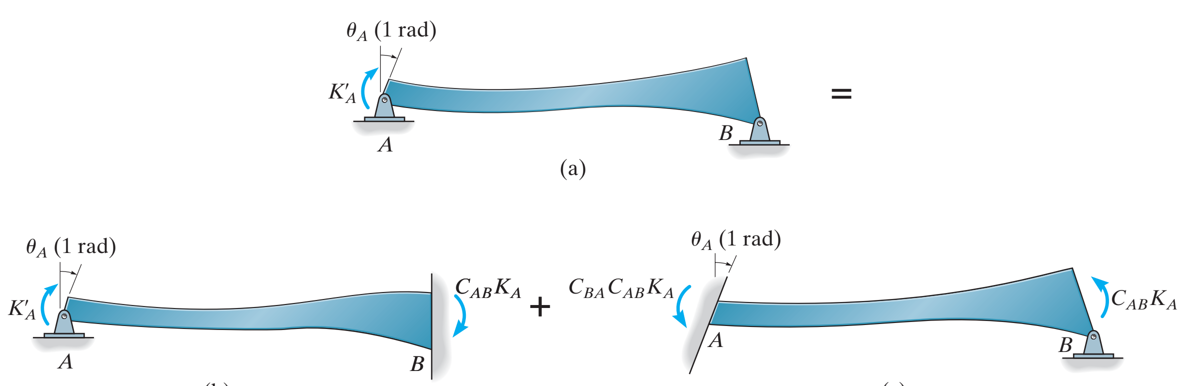


Fig. 13–4

Symmetric Beam and Loading. Here we must determine the moment K'_A needed to rotate end A, $\theta_A = +1$ rad, while $\theta_B = -1$ rad, Fig. 13–5*a*. In this case we first assume that end B is fixed and apply the moment K_A at A, Fig. 13–5*b*. Next we apply a negative moment K_B to end B assuming that end A is fixed. This results in a moment of $C_{BA}K_B$ at end A as shown in Fig. 13–5*c*. Superposition of these two applications of moment at A yields the results of Fig. 13–5*a*. We require

$$K'_A = K_A - C_{BA}K_B$$

Using Eq. 13–1 ($C_{BA}K_B = C_{AB}K_A$), we can also write

$$\boxed{K'_A = K_A(1 - C_{AB})} \qquad (13\text{–}4)$$

In the case of a prismatic beam, $K_A = 4EI/L$ and $C_{AB} = \frac{1}{2}$, so that $K'_A = 2EI/L$, which is the same as Eq. 12–5.

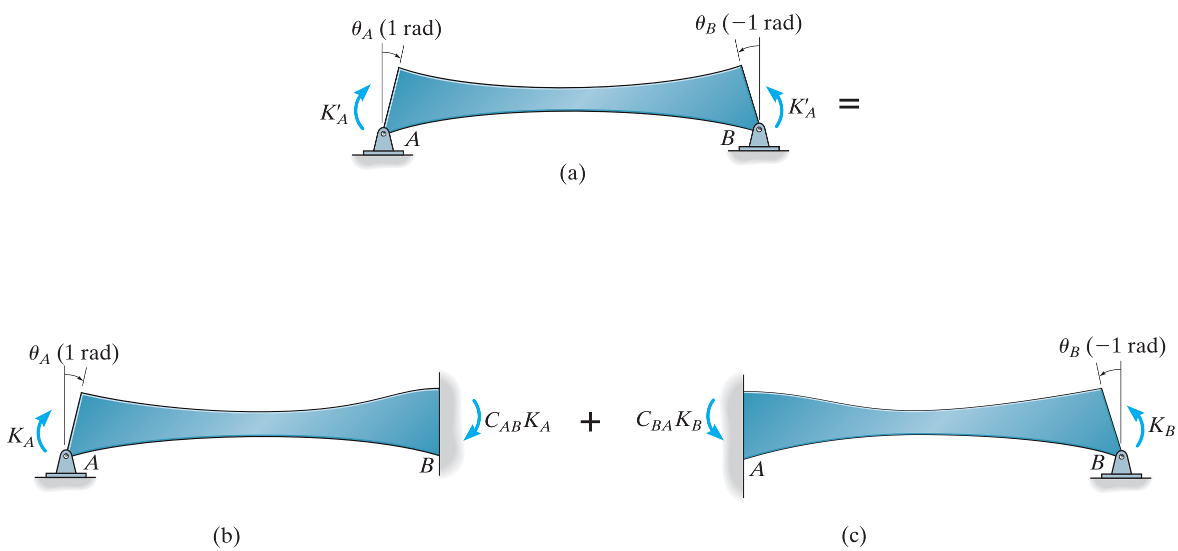


Fig. 13–5

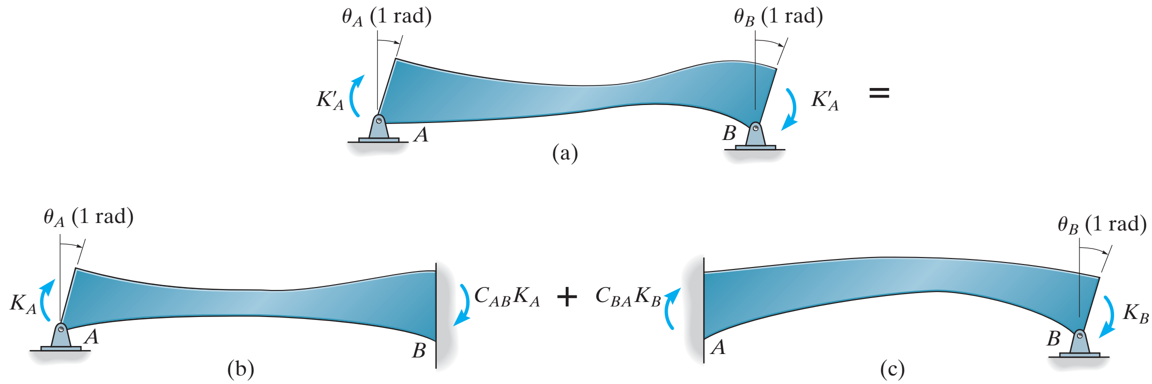


Fig. 13–6

Symmetric Beam with Antisymmetric Loading.

In the case of a symmetric beam with antisymmetric loading, we must determine K'_A such that equal rotations occur at the ends of the beam, Fig. 13–6*a*. To do this, we first fix end B and apply the moment K_A at A, Fig. 13–6*b*. Likewise, application of K_B at end B while end A is fixed is shown in Fig. 13–6*c*. Superposition of both cases yields the results of Fig. 13–6*a*. Hence,

$$K'_A = K_A + C_{BA}K_B$$

or, using Eq. 13–1 ($C_{BA}K_B = C_{AB}K_A$), we have for the absolute stiffness

$$\boxed{K'_A = K_A(1 + C_{AB})} \qquad (13\text{–}5)$$

Substituting the data for a prismatic member, $K_A = 4EI/L$ and $C_{AB} = \frac{1}{2}$, yields $K'_A = 6EI/L$, which is the same as Eq. 12–6.

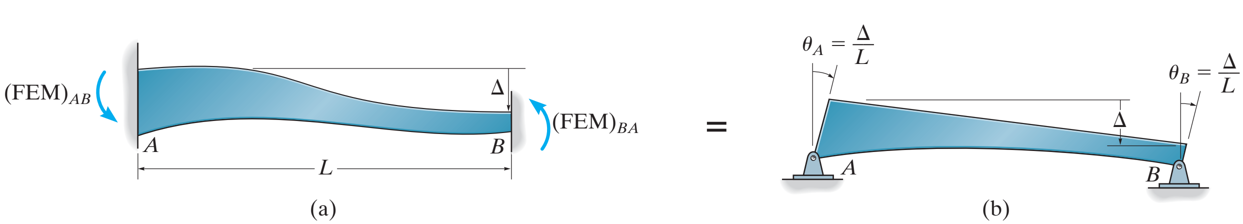


Fig. 13–7

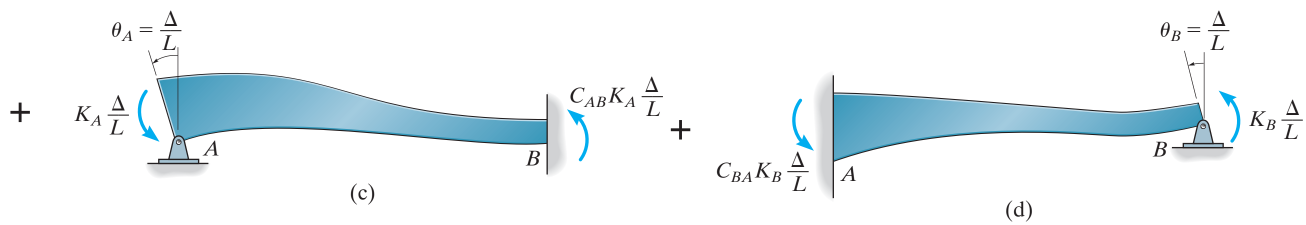


Fig. 13–7

Relative Joint Translation of Beam.

Fixed-end moments are developed in a nonprismatic member if it has a relative joint translation Δ between its ends A and B, Fig. 13–7*a*. In order to determine these moments, we proceed as follows. First consider the ends A and B to be pin connected and allow end B of the beam to be displaced a distance Δ such that the end rotations are $\theta_A = \theta_B = \Delta/L$, Fig. 13–7*b*. Second, assume that B is fixed and apply a moment of $M'_A = -K_A(\Delta/L)$ to end A such that it rotates the end $\theta_A = -\Delta/L$, Fig. 13–7*c*. Third, assume that A is fixed and apply a moment $M'_B = -K_B(\Delta/L)$ to end B such that it rotates the end $\theta_B = -\Delta/L$, Fig. 13–7*d*. Since the total sum of these three operations yields the condition shown in Fig. 13–7*a*, we have at A

$$(\text{FEM})_{AB} = -K_A\frac{\Delta}{L} - C_{BA}K_B\frac{\Delta}{L}$$

Applying Eq. 13–1 ($C_{BA}K_B = C_{AB}K_A$) yields

$$\boxed{(\text{FEM})_{AB} = -K_A\frac{\Delta}{L}(1 + C_{AB})} \qquad (13\text{–}6)$$

A similar expression can be written for end B. Recall that for a prismatic member $K_A = 4EI/L$ and $C_{AB} = \frac{1}{2}$. Thus $(\text{FEM})_{AB} = -6EI\Delta/L^2$, which is the same as Eq. 11–5.

If end B is pinned rather than fixed, Fig. 13–8, the fixed-end moment at A can be determined in a manner similar to that described above. The result is

$$\boxed{(\text{FEM})'_{AB} = -K_A\frac{\Delta}{L}(1 - C_{AB}C_{BA})} \qquad (13\text{–}7)$$

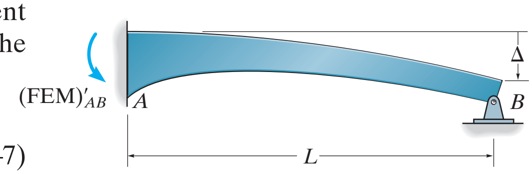


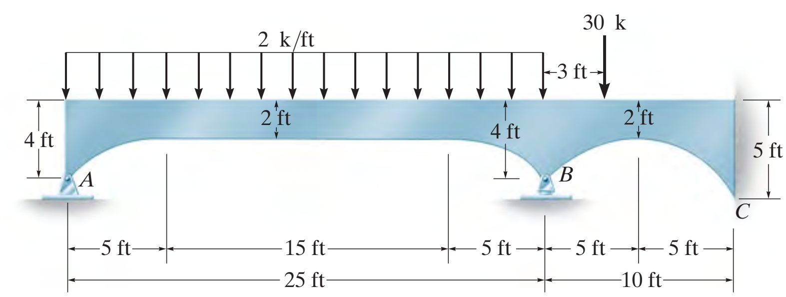
Fig. 13–8

Here it is seen that for a prismatic member this equation gives $(\text{FEM})'_{AB} = -3EI\Delta/L^2$, which is the same as that listed on the inside back cover.

The following example illustrates application of the moment-distribution method to structures having nonprismatic members. Once the fixed-end moments and stiffness and carry-over factors have been determined, and the stiffness factor modified according to the equations given above, the procedure for analysis is the same as that discussed in Chapter 12.

EXAMPLE 13.1

Determine the internal moments at the supports of the beam shown in Fig. 13–9*a*. The beam has a thickness of 1 ft and E is constant.



(a)

Fig. 13–9

SOLUTION

Since the haunches are parabolic, we will use Table 13–2 to obtain the moment-distribution properties of the beam.

Span *AB*

$$a_A = a_B = \frac{5}{25} = 0.2 \qquad r_A = r_B = \frac{4-2}{2} = 1.0$$

Entering Table 13–2 with these ratios, we find

$$C_{AB} = C_{BA} = 0.619$$
$$k_{AB} = k_{BA} = 6.41$$

Using Eqs. 13–2,

$$K_{AB} = K_{BA} = \frac{kEI_C}{L} = \frac{6.41E\left(\frac{1}{12}\right)(1)(2)^3}{25} = 0.171E$$

Since the far end of span BA is pinned, we will modify the stiffness factor of BA using Eq. 13–3. We have

$$K'_{BA} = K_{BA}(1 - C_{AB}C_{BA}) = 0.171E[1 - 0.619(0.619)] = 0.105E$$

Uniform load, Table 13–2,

$$(\text{FEM})_{AB} = -(0.0956)(2)(25)^2 = -119.50 \text{ k} \cdot \text{ft}$$
$$(\text{FEM})_{BA} = 119.50 \text{ k} \cdot \text{ft}$$

Span *BC*

$$a_B = a_C = \frac{5}{10} = 0.5 \qquad r_B = \frac{4-2}{2} = 1.0$$

$$r_C = \frac{5-2}{2} = 1.5$$

From Table 13–2 we find

$$C_{BC} = 0.781 \qquad C_{CB} = 0.664$$

$$k_{BC} = 13.12 \qquad k_{CB} = 15.47$$

Thus, from Eqs. 13–2,

$$K_{BC} = \frac{kEI_C}{L} = \frac{13.12E\left(\frac{1}{12}\right)(1)(2)^3}{10} = 0.875E$$

$$K_{CB} = \frac{kEI_C}{L} = \frac{15.47E\left(\frac{1}{12}\right)(1)(2)^3}{10} = 1.031E$$

Concentrated load,

$$b = \frac{3}{10} = 0.3$$

$$(\text{FEM})_{BC} = -0.1891(30)(10) = -56.73\ \text{k}\cdot\text{ft}$$

$$(\text{FEM})_{CB} = 0.0759(30)(10) = 22.77\ \text{k}\cdot\text{ft}$$

Using the foregoing values for the stiffness factors, the distribution factors are computed and entered in the table, Fig. 13–9*b*. The moment distribution follows the same procedure outlined in Chapter 12. The results in k · ft are shown on the last line of the table.

Joint	*A*	*B*		*C*
Member	*AB*	*BA*	*BC*	*CB*
K	0.171*E*	0.105*E*	0.875*E*	1.031*E*
DF	1	0.107	0.893	0
COF	0.619	0.619	0.781	0.664
FEM	−119.50	119.50	−56.73	22.77
Dist.	119.50	−6.72	−56.05	
CO		73.97		−43.78
Dist.		−7.91	−66.06	
CO				−51.59
ΣM	0	178.84	−178.84	−72.60

(b)

Fig. 13–9

13

13.3 Slope-Deflection Equations for Nonprismatic Members

The slope-deflection equations for prismatic members were developed in Chapter 11. In this section we will generalize the form of these equations so that they apply as well to nonprismatic members. To do this, we will use the results of the previous section and proceed to formulate the equations in the same manner discussed in Chapter 11, that is, considering the effects caused by the loads, relative joint displacement, and each joint rotation separately, and then superimposing the results.

Loads. Loads are specified by the fixed-end moments $(\mathrm{FEM})_{AB}$ and $(\mathrm{FEM})_{BA}$ acting at the ends A and B of the span. Positive moments act clockwise.

Relative Joint Translation. When a relative displacement Δ between the joints occurs, the induced moments are determined from Eq. 13–6. At end A this moment is $-[K_A\Delta/L](1 + C_{AB})$ and at end B it is $-[K_B\Delta/L](1 + C_{BA})$.

Rotation at *A*. If end A rotates θ_A, the required moment in the span at A is $K_A\theta_A$. Also, this induces a moment of $C_{AB}K_A\theta_A = C_{BA}K_B\theta_A$ at end B.

Rotation at *B*. If end B rotates θ_B, a moment of $K_B\theta_B$ must act at end B, and the moment induced at end A is $C_{BA}K_B\theta_B = C_{AB}K_A\theta_B$.

The total end moments caused by these effects yield the generalized slope-deflection equations, which can therefore be written as

$$M_{AB} = K_A\left[\theta_A + C_{AB}\theta_B - \frac{\Delta}{L}(1 + C_{AB})\right] + (\mathrm{FEM})_{AB}$$

$$M_{BA} = K_B\left[\theta_B + C_{BA}\theta_A - \frac{\Delta}{L}(1 + C_{BA})\right] + (\mathrm{FEM})_{BA}$$

Since these two equations are similar, we can express them as a single equation. Referring to one end of the span as the near end (N) and the other end as the far end (F), and representing the member rotation as $\psi = \Delta/L$, we have

$$\boxed{M_N = K_N(\theta_N + C_N\theta_F - \psi(1 + C_N)) + (\mathrm{FEM})_N} \qquad (13\text{–}8)$$

Here

M_N = internal moment at the near end of the span; this moment is positive clockwise when acting on the span.

K_N = absolute stiffness of the near end determined from tables or by calculation.

θ_N, θ_F = near- and far-end slopes of the span at the supports; the angles are measured in *radians* and are *positive clockwise.*

ψ = span cord rotation due to a linear displacement, $\psi = \Delta/L$; this angle is measured in *radians* and is *positive clockwise.*

$(\text{FEM})_N$ = fixed-end moment at the near-end support; the moment is *positive* clockwise when acting on the span and is obtained from tables or by calculations.

Application of the equation follows the same procedure outlined in Chapter 11 and therefore will not be discussed here. In particular, note that Eq. 13–8 reduces to Eq. 11–8 when applied to members that are prismatic.

Light-weight metal buildings are often designed using frame members having variable moments of inertia.

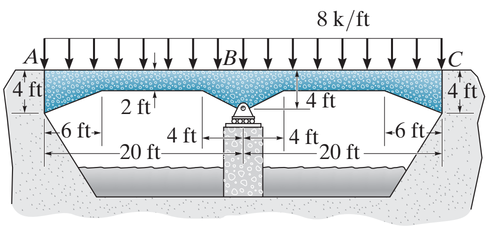
A continuous, reinforced-concrete highway bridge.

PROBLEMS

13

13–1. Determine the moments at A, B, and C by the moment-distribution method. Assume the supports at A and C are fixed and a roller support at B is on a rigid base. The girder has a thickness of 4 ft. Use Table 13–1. E is constant. The haunches are straight.

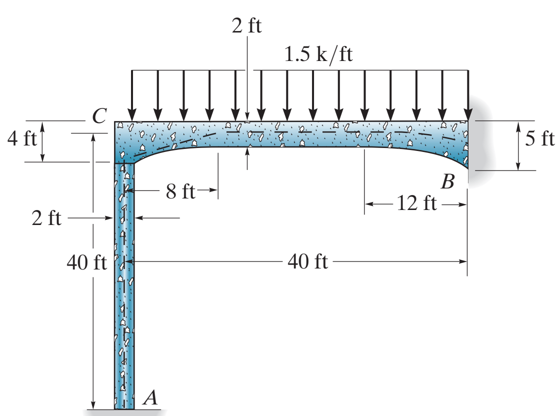
13–2. Solve Prob. 13–1 using the slope-deflection equations.



Probs. 13–1/13–2

13–3. Apply the moment-distribution method to determine the moment at each joint of the parabolic haunched frame. Supports A and B are fixed. Use Table 13–2. The members are each 1 ft thick. E is constant.

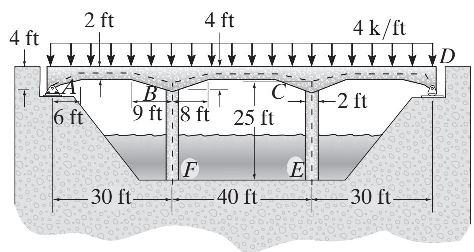
***13–4.** Solve Prob. 13–3 using the slope-deflection equations.



Probs. 13–3/13–4

13–5. Use the moment-distribution method to determine the moment at each joint of the symmetric bridge frame. Supports at F and E are fixed and B and C are fixed connected. Use Table 13–2. Assume E is constant and the members are each 1 ft thick.

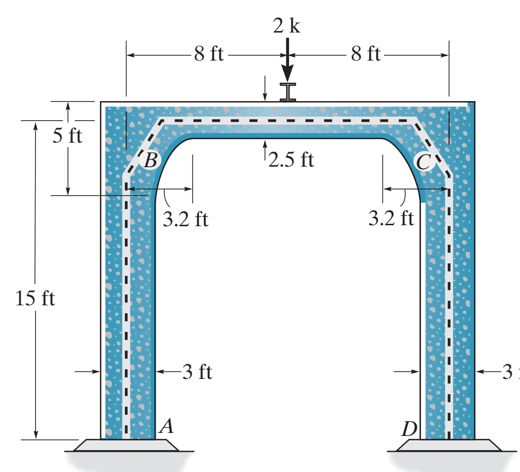
13–6. Solve Prob. 13–5 using the slope-deflection equations.



Probs. 13–5/13–6

13–7. Apply the moment-distribution method to determine the moment at each joint of the symmetric parabolic haunched frame. Supports A and D are fixed. Use Table 13–2. The members are each 1 ft thick. E is constant.

***13–8.** Solve Prob. 13–7 using the slope-deflection equations.



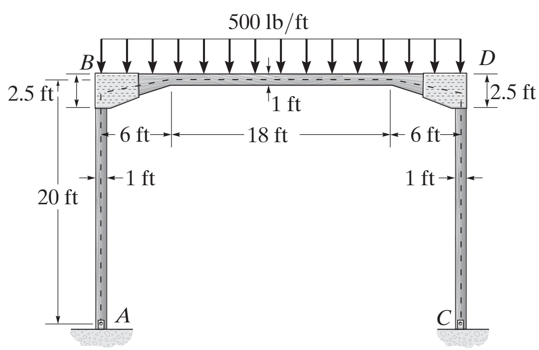
Probs. 13–7/13–8

13–9. Use the moment-distribution method to determine the moment at each joint of the frame. The supports at A and C are pinned and the joints at B and D are fixed connected. Assume that E is constant and the members have a thickness of 1 ft. The haunches are straight so use Table 13–1.

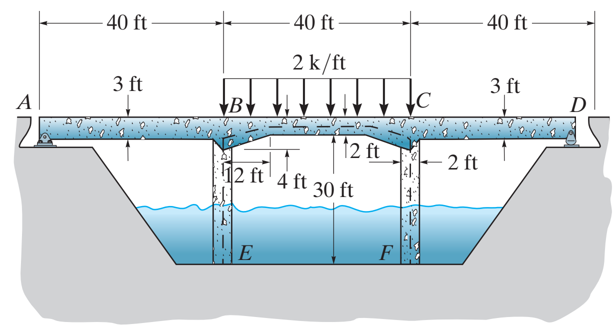
13–10. Solve Prob. 13–9 using the slope-deflection equations.

13–11. Use the moment-distribution method to determine the moment at each joint of the symmetric bridge frame. Supports F and E are fixed and B and C are fixed connected. The haunches are straight so use Table 13–2. Assume E is constant and the members are each 1 ft thick.

***13–12.** Solve Prob. 13–11 using the slope-deflection equations.



Probs. 13–9/13–10



Probs. 13–11/13–12

CHAPTER REVIEW

Non-prismatic members having a variable moment of inertia are often used on long-span bridges and building frames to save material.

A structural analysis using non-prismatic members can be performed using either the slope-deflection equations or moment distribution. If this is done, it then becomes necessary to obtain the fixed-end moments, stiffness factors, and carry-over factors for the member. One way to obtain these values is to use the conjugate beam method, although the work is somewhat tedious. It is also possible to obtain these values from tabulated data, such as published by the Portland Cement Association.

If the moment distribution method is used, then the process can be simplified if the stiffness of some of the members is modified.

The space-truss analysis of electrical transmission towers can be performed using the stiffness method.

Truss Analysis Using the Stiffness Method

14

In this chapter we will explain the basic fundamentals of using the stiffness method for analyzing structures. It will be shown that this method, although tedious to do by hand, is quite suited for use on a computer. Examples of specific applications to planar trusses will be given. The method will then be expanded to include space-truss analysis. Beams and framed structures will be discussed in the next chapters.

14.1 Fundamentals of the Stiffness Method

There are essentially two ways in which structures can be analyzed using matrix methods. The stiffness method, to be used in this and the following chapters, is a displacement method of analysis. A force method, called the flexibility method, as outlined in Sec. 9–1, can also be used to analyze structures; however, this method will not be presented in this text. There are several reasons for this. Most important, the stiffness method can be used to analyze both statically determinate and indeterminate structures, whereas the flexibility method requires a different procedure for each of these two cases. Also, the stiffness method yields the displacements and forces directly, whereas with the flexibility method the displacements are not obtained directly. Furthermore, it is generally much easier to formulate the necessary matrices for the computer operations using the stiffness method; and once this is done, the computer calculations can be performed efficiently.