Finding Equilibria for Bimatrix Games

Jake Roth and Akshit Goyal

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Outline

Overview of bimatrix games

A bimatrix game

- Consider the game of *Rock-Paper-Scissors* between two players
- The game can be formulated as a two-player, zero-sum game with payoff matrix for the row player given by

$$\begin{array}{cc} & \begin{array}{ccc} r & p & s \end{array} \\ \begin{array}{c} R \\ P \\ S \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

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- An *equilibrium* is a pair of strategies (X, x) where neither player has incentive to deviate
- The pair ($R =$ [1,0,0], $r =$ [1,0,0]) is not an equilibrium
- Nash: there always exists an equilibrium "mixture" of strategies (and sometimes pure ones, too)
- Proof uses Brouwer fixed point theorem; non-constructive

Finding equilibria

- How can we find an equilibrium for a given problem?

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  - Quick overview of two methods

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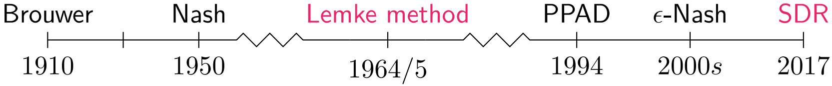

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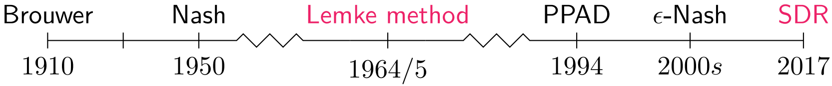

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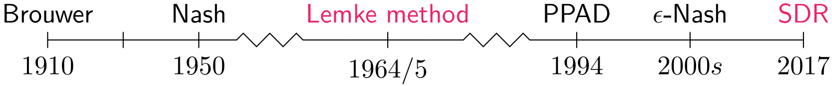

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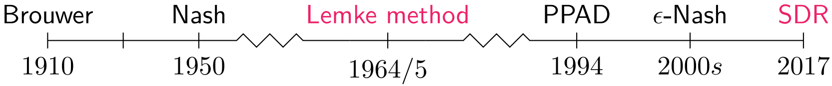

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  - Strategic decision making (e.g., advertising business, bidding process)

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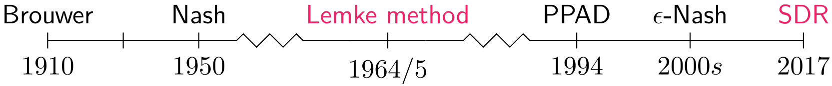

- Why are we interested in finding equilibria?
  - Model market equilibria in economics (e.g., utility companies and electricity)
  - Strategic decision making (e.g., advertising business, bidding process)
  - Interesting problem structure (motivates a generalization of LP)

A complementary pivoting method

Normal form

Notation: Δ_k : refers to the $k-$dimensional simplex, i.e.,

$$\Delta_k = \{x \in \mathbb{R}^k \mid x_i \geq 0 \; \forall i, \; \sum_{i=1}^{k} x_i = 1\}.$$

A Nash equilibrium $(x^*, y^*) \in \Delta_m \times \Delta_n$ for game (A, B) exists, if

$$x^{*\top} A y^* \leq x^\top A y^*, \;\; \forall x \in \Delta_m, \tag{1a}$$

$$x^{*\top} B y^* \leq x^{*\top} B y, \;\; \forall y \in \Delta_n, \tag{1b}$$

where $A, B \in \mathbb{R}^{m \times n}$

Note A_{ij} and B_{ij} is the cost incurred by player I and II after selecting pure strategy i and j respectively.

Express like an LP feasibility problem (+ a complementarity constraint)

$$u = -\mathbf{1}_m + Ay \geq 0, \qquad \forall x \geq 0, \;\; x^\top u = 0, \tag{2a}$$

$$v = -\mathbf{1}_n + B^\top x \geq 0, \qquad \forall y \geq 0, \;\; y^\top v = 0 \tag{2b}$$

Rearrange

$$q = \begin{bmatrix} -\mathbf{1}_m \\ -\mathbf{1}_n \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} 0 & A \\ B^\top & 0 \end{bmatrix}$$

Further $w^\top = \begin{bmatrix} u & v \end{bmatrix}^\top$ and $z^\top = \begin{bmatrix} x & y \end{bmatrix}^\top$. Note $w \geq 0$, $z \geq 0$ and $w \circ z = 0$.

return

(Generic) linear complementarity problems (LCPs)

For data $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, find vectors $w, z \in \mathbb{R}^n$ satisfying

$$w = q + Mz \tag{3a}$$

$$w \geq 0 \tag{3b}$$

$$z \geq 0 \tag{3c}$$

$$w_i z_i = 0 \quad i = 1, 2, \ldots, n. \tag{3d}$$

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For data $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, find vectors $w, z \in \mathbb{R}^n$ satisfying

$$w = q - (-M)z \tag{3a}$$
$$w \geq 0 \tag{3b}$$
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For data $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, find vectors $w, z \in \mathbb{R}^n$ satisfying

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Main ideas

Want to find a basis for consisting of *precisely one* vector from each pair $\{I_{[:,j]}, \tilde{M}_{[:,j]}\}_{j=1}^{n}$ to write $q = w + \tilde{M}z$ as

$$q = w_1 I_{[:,1]} + w_2 I_{[:,2]} + \cdots + w_n I_{[:,n]} + z_1 \tilde{M}_{[:,1]} + z_2 \tilde{M}_{[:,2]} + \cdots + z_n \tilde{M}_{[:,n]}$$

while ensuring that $w_j, v_j \geq 0$ for all $j = 1, 2, \ldots, n$

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Write dictionaries like simplex

$$\text{basic} = \bar{q} - \tilde{M}'\text{nonbasic}$$

and pivot to get a basis that is: (i) feasibile, then (ii) optimal

Example problem

▸ BMG

Example problem

Consider the $m = 2$, $n = 3$ problem [1] with cost matrices A and B

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} \tag{4}$$

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$$u, v \geq 0, \quad x, y \geq 0, \tag{6}$$

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$$u^\top x = v^\top y = 0. \tag{7}$$

▸ BMG

Example problem (continued: feasibility)

basis-0		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	-1	$\cdot$	$\cdot$	$-2y_1$	$-2y_2$	$-1y_3$	
u_2	$=$	-1	$\cdot$	$\cdot$	$-1y_1$	$-2y_2$	$-2y_3$	
v_1	$=$	-1	$-1x_1$	$-2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	-1	$-3x_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	-1	$-2x_1$	$-3x_2$	$\cdot$	$\cdot$	$\cdot$	

- Initial dictionary corresponding to basis w.
- Old pivot: ($e :=$ entering, $l :=$ leaving $= f(e)$) $= (e := -, l := -)$.
- New pivot: ($e :=$ entering, $l :=$ leaving $= f(e)$) $= (e := ?, l := ?)$.

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v_1	$=$	-1	$-1x_1$	$-2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	-1	$-3x_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	
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v_1	$=$	-1	$-1x_1$	$-2x_2$	$\cdot$	$\cdot$	$\cdot$	-1/-1
v_2	$=$	-1	$-3x_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	-1/-3
v_3	$=$	-1	$-2x_1$	$-3x_2$	$\cdot$	$\cdot$	$\cdot$	-1/-2

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- MAXIMUM ratio test: makes $\bar{q}$ positive.

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x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$+2v_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
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- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_2, l := ?)$.

Example problem (continued: feasibility)

basis-1		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	-1	$\cdot$	$\cdot$	$-2y_1$	$-2y_2$	$-1y_3$	
u_2	$=$	-1	$\cdot$	$\cdot$	$-1y_1$	$-2y_2$	$-2y_3$	
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$+2v_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_1, l := v_1)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_2, l := ?)$.
- Destroy existing (almost) "complementaryness"

Example problem (continued: feasibility)

basis-1		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	-1	$\cdot$	$\cdot$	$-2y_1$	$-2y_2$	$-1y_3$	
u_2	$=$	-1	$\cdot$	$\cdot$	$-1y_1$	$-2y_2$	$-2y_3$	
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$+2v_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_1, l := v_1)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_2, l := ?)$.

Example problem (continued: feasibility)

basis-1		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	-1	$\cdot$	$\cdot$	$-2y_1$	$-2y_2$	$-1y_3$	
u_2	$=$	-1	$\cdot$	$\cdot$	$-1y_1$	$-2y_2$	$-2y_3$	
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$+2v_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_1, l := v_1)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_1, l := ?)$.

Example problem (continued: feasibility)

basis-1		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	-1	$\cdot$	$\cdot$	$-2y_1$	$-2y_2$	$-1y_3$	-1/-2
u_2	$=$	-1	$\cdot$	$\cdot$	$-1y_1$	$-2y_2$	$-2y_3$	-1/-1
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$+2v_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_1, l := v_1)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_1, l := ?)$.

Example problem (continued: feasibility)

basis-1		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	-1	$\cdot$	$\cdot$	$-2y_1$	$-2y_2$	$-1y_3$	-1/-2
u_2	$=$	-1	$\cdot$	$\cdot$	$-1y_1$	$-2y_2$	$-2y_3$	-1/-1
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$+2v_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_1,\ l := v_1)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_1,\ l := u_2)$.
- MAXIMUM ratio test.

Example problem (continued: feasibility)

basis-1		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	-1	$\cdot$	$\cdot$	$-2y_1$	$-2y_2$	$-1y_3$	-1/-2
u_2	$=$	-1	$\cdot$	$\cdot$	$-1y_1$	$-2y_2$	$-2y_3$	-1/-1
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$+2v_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_1, l := v_1)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_1,)$.
- MAXIMUM ratio test.

Example problem (continued: feasibility)

basis-2		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$-2v_1$	$+1x_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_1, l := u_2)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_2, l := ?)$.

Example problem (continued: feasibility)

basis-2		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$-2v_1$	$+1x_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_1, l := u_2)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_2, l := ?)$.

Example problem (continued: feasibility)

basis-2		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	1/2
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	2/5
v_3	$=$	1	$-2v_1$	$+1x_2$	$\cdot$	$\cdot$	$\cdot$	1/1

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_1, l := u_2)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_2, l := ?)$.

Example problem (continued: feasibility)

basis-2		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	1/2
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	2/5
v_3	$=$	1	$-2v_1$	$+1x_2$	$\cdot$	$\cdot$	$\cdot$	1/1

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_1, l := u_2)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_2, l := ?)$.
- Minimum ratio test.

Example problem (continued: feasibility)

basis-2		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	
x_1	$=$	1	$-1v_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	1/2
v_2	$=$	2	$-3v_1$	$+5x_2$	$\cdot$	$\cdot$	$\cdot$	2/5
v_3	$=$	1	$-2v_1$	$+1x_2$	$\cdot$	$\cdot$	$\cdot$	1/1

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := y_1, l := u_2)$.
- New pivot: $(e :=$ entering, $l :=$ leaving $= f(e)) = (e := x_2, l := v_2)$.
- Minimum ratio test.

Example (continued: pivoting)

basis-3		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	
x_1	$=$	$\frac{1}{5}$	$+\frac{1}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
x_2	$=$	$\frac{2}{5}$	$-\frac{3}{5}v_1$	$+\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	$\frac{3}{5}$	$-\frac{7}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: ($e :=$ entering, $l :=$ leaving) $= (e := x_2,\ l := v_2)$.
- New pivot: ($e :=$ entering, $l :=$ leaving) $= (e := y_2,\ l := ?)$.

Example (continued: pivoting)

basis-3		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	
x_1	$=$	$\frac{1}{5}$	$+\frac{1}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
x_2	$=$	$\frac{2}{5}$	$-\frac{3}{5}v_1$	$+\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	$\frac{3}{5}$	$-\frac{7}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: ($e :=$ entering, $l :=$ leaving) $= (e := x_2, l := v_2)$.
- New pivot: ($e :=$ entering, $l :=$ leaving) $= (e := y_2, l := ?)$.

Example (continued: pivoting)

basis-3		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	1/2
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	1/2
x_1	$=$	$\frac{1}{5}$	$+\frac{1}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
x_2	$=$	$\frac{2}{5}$	$-\frac{3}{5}v_1$	$+\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	$\frac{3}{5}$	$-\frac{7}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving$) = (e := x_2,\ l := v_2)$.
- New pivot: $(e :=$ entering, $l :=$ leaving$) = (e := y_2,\ l := ?)$.

Example (continued: pivoting)

basis-3		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	$1/2$
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	$1/2$
x_1	$=$	$\frac{1}{5}$	$+\frac{1}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
x_2	$=$	$\frac{2}{5}$	$-\frac{3}{5}v_1$	$+\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	$\frac{3}{5}$	$-\frac{7}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: ($e :=$ entering, $l :=$ leaving) $= (e := x_2,\ l := v_2)$.
- New pivot: ($e :=$ entering, $l :=$ leaving) $= (e := y_2,\ l := ?)$.
- Minimum ratio test.

Example (continued: pivoting)

basis-3		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
u_1	$=$	1	$\cdot$	$\cdot$	$-2u_2$	$+2y_2$	$+3y_3$	$1/2$
y_1	$=$	1	$\cdot$	$\cdot$	$-1u_2$	$+2y_2$	$+2y_3$	$1/2$
x_1	$=$	$\frac{1}{5}$	$+\frac{1}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
x_2	$=$	$\frac{2}{5}$	$-\frac{3}{5}v_1$	$+\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	$\frac{3}{5}$	$-\frac{7}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving$) = (e := x_2, l := v_2)$.
- New pivot: $(e :=$ entering, $l :=$ leaving$) = (e := y_2, l := u_1)$.
- Minimum ratio test.

Example (continued: pivoting)

basis-4		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\bar{q}/\tilde{M}'_{[:,e]}$
y_2	$=$	$\frac{1}{2}$	$\cdot$	$\cdot$	$+\frac{1}{2}u_1$	$-1u_2$	$+\frac{3}{2}y_3$	
y_1	$=$	0	$\cdot$	$\cdot$	$-1u_2$	$+2u_2$	$-1y_3$	
x_1	$=$	$\frac{1}{5}$	$+\frac{1}{5}v_1$	$-\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
x_2	$=$	$\frac{2}{5}$	$-\frac{3}{5}v_1$	$+\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	$\frac{3}{5}$	$-\frac{7}{5}v_1$	$-\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	

- Dictionary after old pivot.
- Old pivot: $(e :=$ entering, $l :=$ leaving$) = (e := y_2,\ l := u_1)$.
- The pair (x_1, u_1) from the initial basis is now complementary.
- Lexicographic rule needed to avoid cycling.

Example (continued: discussion)

Observations

1. Idea

Implications

Example (continued: discussion)

Observations

1. Idea
   - First get a feasible dictionary

Implications

Example (continued: discussion)

Observations

1. Idea
   - First get a feasible dictionary
   - Then generate pivots in search of an optimal dictionary

Implications

Example (continued: discussion)

Observations

1. Idea
   - First get a feasible dictionary
   - Then generate pivots in search of an optimal dictionary
2. As long as the entering variable is selected as the complement of the leaving variable, "almost complementary"ness is preserved

Implications

Example (continued: discussion)

Observations

1. Idea
   - First get a feasible dictionary
   - Then generate pivots in search of an optimal dictionary
2. As long as the entering variable is selected as the complement of the leaving variable, "almost complementary"ness is preserved
3. Stop when one of the initializing variables (or its complement) leaves the basis

Implications

Example (continued: discussion)

Observations

1. Idea
   - First get a feasible dictionary
   - Then generate pivots in search of an optimal dictionary
2. As long as the entering variable is selected as the complement of the leaving variable, "almost complementary"ness is preserved
3. Stop when one of the initializing variables (or its complement) leaves the basis

Implications

1. Cycling: guaranteed finite termination with lexico-minimum ratio test

Example (continued: discussion)

Observations

1. Idea
   - First get a feasible dictionary
   - Then generate pivots in search of an optimal dictionary
2. As long as the entering variable is selected as the complement of the leaving variable, "almost complementary"ness is preserved
3. Stop when one of the initializing variables (or its complement) leaves the basis

Implications

1. Cycling: guaranteed finite termination with lexico-minimum ratio test
2. Exponential examples exist

Example (continued: discussion)

Observations

1. Idea
   - First get a feasible dictionary
   - Then generate pivots in search of an optimal dictionary
2. As long as the entering variable is selected as the complement of the leaving variable, "almost complementary"ness is preserved
3. Stop when one of the initializing variables (or its complement) leaves the basis

Implications

1. Cycling: guaranteed finite termination with lexico-minimum ratio test
2. Exponential examples exist
3. Parameter of model is initial entering variable

A Semidefinite Relaxation (SDR) approach

Nash Equilibrium conditions

- Instead of minimizing the costs (as discussed previously), we will be maximizing the individual payoffs for players A, B w.r.t each other's strategy. The Nash Equilibrium can be restated as

$$
\begin{aligned}
{x^*}^\top A y^* &\geq x^\top A y^*, \ \ \forall x \in \Delta_m \\
{x^*}^\top B y^* &\geq {x^*}^\top B y, \ \ \forall y \in \Delta_n
\end{aligned} \tag{8}
$$

Nash Equilibrium conditions

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{x^*}^\top A y^* &\geq x^\top A y^*, \ \ \forall x \in \Delta_m \\
{x^*}^\top B y^* &\geq {x^*}^\top B y, \ \ \forall y \in \Delta_n
\end{aligned} \tag{8}
$$

- We can formulate the Nash Equilibria conditions using payoff matrices (A, B) as Nash Equilibria conditions using cost matrices $(\tilde{A}, \tilde{B})$ where

$$
\begin{aligned}
\tilde{A} &= c\mathbf{1}_{m \times n} - eA \\
\tilde{B} &= d\mathbf{1}_{m \times n} - fB
\end{aligned}
$$

$c, d \in \mathbb{R}$ and $e, f > 0$.

$$
(8) \iff \begin{aligned}
{x^*}^\top \tilde{A} y^* &\leq x^\top \tilde{A} y^*, \ \ \forall x \in \Delta_m \\
{x^*}^\top \tilde{B} y^* &\leq {x^*}^\top \tilde{B} y, \ \ \forall y \in \Delta_n.
\end{aligned}
$$

Nash Equilibrium conditions (contd.)

For any $x \in \Delta_m$, $x^\top A y^*$ is a convex combination of pure-strategy payoffs $e_i^\top A y^*$, $i = 1, \ldots, m$

$$x^\top A y^* = \sum_{i=1}^{m} x_i e_i^\top A y^*.$$

Similarly for any $y \in \Delta_n$,

$$x^{*\top} B y = \sum_{j=1}^{n} x^{*\top} B e_j y_j.$$

Nash Equilibrium conditions (contd.)

For any $x \in \Delta_m$, $x^\top A y^*$ is a convex combination of pure-strategy payoffs $e_i^\top A y^*$, $i = 1, \ldots, m$

$$x^\top A y^* = \sum_{i=1}^{m} x_i e_i^\top A y^*.$$

Similarly for any $y \in \Delta_n$,

$$x^{*\top} B y = \sum_{j=1}^{n} x^{*\top} B e_j y_j.$$

Nash Equilibrium conditions (contd.)

For any $x \in \Delta_m$, $x^\top A y^*$ is a convex combination of pure-strategy payoffs $e_i^\top A y^*$, $i = 1, \ldots, m$

$$x^\top A y^* = \sum_{i=1}^{m} x_i e_i^\top A y^*.$$

Similarly for any $y \in \Delta_n$,

$${x^*}^\top B y = \sum_{j=1}^{n} {x^*}^\top B e_j y_j.$$

One can easily prove that

$$(8) \iff {x^*}^\top A y^* \geq e_i^\top A y^*, \ \forall i \in \{1, \ldots, m\}$$

$$\text{and } {x^*}^\top B y^* \geq {x^*}^\top B e_j, \ \forall j \in \{1, \ldots, n\}.$$

ϵ-Nash Equilibrium

- Often, interested in computing ϵ-Nash Equilibria rather than exact. It is defined as

$$x^{*\top} A y^* \geq x^\top A y^* - \epsilon, \ \ \forall x \in \Delta_m$$
$$x^{*\top} B y^* \geq x^{*\top} B y - \epsilon, \ \ \forall y \in \Delta_n$$

OR

$$x^{*\top} A y^* \geq e_i^\top A y^* - \epsilon, \ \ \forall i \in \{1, \ldots, m\}$$
$$x^{*\top} B y^* \geq x^{*\top} B e_j - \epsilon, \ \ \forall j \in \{1, \ldots, n\}.$$

- Note:

$$\epsilon = \max\{\max_i \ e_i^\top A y^* - x^{*\top} A y^*, \ \max_j \ x^{*\top} B e_j - x^{*\top} B y^*\}$$

- For ϵ to make sense, the entries of A and B should be normalized between 0 and 1 (since Nash Equilibria is invariant to certain affine transformation to A, B).

ϵ-Nash Equilibrium

- Often, interested in computing ϵ-Nash Equilibria rather than exact. It is defined as

$$x^{*\top} A y^* \geq x^\top A y^* - \epsilon, \;\; \forall x \in \Delta_m$$
$$x^{*\top} B y^* \geq x^{*\top} B y - \epsilon, \;\; \forall y \in \Delta_n$$

OR

$$x^{*\top} A y^* \geq e_i^\top A y^* - \epsilon, \;\; \forall i \in \{1, \ldots, m\}$$
$$x^{*\top} B y^* \geq x^{*\top} B e_j - \epsilon, \;\; \forall j \in \{1, \ldots, n\}.$$

- Note:

$$\epsilon = \max\{\max_i \; e_i^\top A y^* - x^{*\top} A y^*, \; \max_j \; x^{*\top} B e_j - x^{*\top} B y^*\}$$

- For ϵ to make sense, the entries of A and B should be normalized between 0 and 1 (since Nash Equilibria is invariant to certain affine transformation to A, B).

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QP feasibility problem

- The problem of finding exact Nash Equilibria can be posed as the following quadratic programming (QP) feasibility problem-

$$
\begin{aligned}
\min_{x\in\mathbb{R}^m,\, y\in\mathbb{R}^n} \quad & 0 \\
\text{subject to} \quad & x^\top A y \geq e_i^\top A y, \ \ \forall i \in \{1,2,\ldots,m\} && \text{(9a)}\\
& x^\top B y \geq x^\top B e_j, \ \ \forall j \in \{1,2,\ldots,n\} && \text{(9b)}\\
& x_i \geq 0, \ \ \forall i \in \{1,2,\ldots,m\} && \text{(9c)}\\
& y_j \geq 0, \ \ \forall j \in \{1,2,\ldots,n\} && \text{(9d)}\\
& \sum_{i=1}^{m} x_i = 1, && \text{(9e)}\\
& \sum_{j=1}^{n} y_j = 1. && \text{(9f)}
\end{aligned}
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$$x^\top B y \geq x^\top B e_j, \ \ \forall j \in \{1, 2, \ldots, n\} \tag{9b}$$

$$x_i \geq 0, \ \ \forall i \in \{1, 2, \ldots, m\} \tag{9c}$$

$$y_j \geq 0, \ \ \forall j \in \{1, 2, \ldots, n\} \tag{9d}$$

$$\sum_{i=1}^{m} x_i = 1, \tag{9e}$$

$$\sum_{j=1}^{n} y_j = 1. \tag{9f}$$

- Any feasible solution to (9) gives a Nash Equilibria $(x^*, y^*) \in \Delta_m \times \Delta_n$.

Semidefinite Relaxation (SDR)

- Problem (9) cannot be solved directly since it is NP-hard. For tractability, we need to convexify it first.
- Recall constraints (9a) and (9b) causing non-convexity. Approach is to relax them through SDR.
- Define a matrix $\mathcal{M}$ as

$$\mathcal{M} = \begin{bmatrix} X & P \\ Z & Y \end{bmatrix}$$

and an augmented matrix $\mathcal{M}'$ as

$$\mathcal{M}' = \begin{bmatrix} X & P & x \\ Z & Y & y \\ x & y & 1 \end{bmatrix}$$

where $X \in \mathbb{S}^{m\times m}$, $Y \in \mathbb{S}^{n\times n}$, $Z \in \mathbb{R}^{n\times m}$ and $P = Z^\top$.

SDR (contd.)

- The SDR of (9) is expressed as

$$\min_{\mathcal{M}' \in \mathbb{S}^{(m+n+1)\times(m+n+1)}} \quad 0 \tag{SDP1}$$

subject to

$$\mathrm{Tr}(AZ) \geq e_i^\top Ay, \quad \forall i \in \{1, 2, \ldots, m\} \tag{10a}$$

$$\mathrm{Tr}(BZ) \geq x^\top Be_j, \quad \forall j \in \{1, 2, \ldots, n\} \tag{10b}$$

$$\sum_{i=1}^{m} x_i = 1, \tag{10c}$$

$$\sum_{j=1}^{n} y_j = 1, \tag{10d}$$

$$\mathcal{M}' \geq 0, \tag{10e}$$

$$\mathcal{M}'_{m+n+1,m+n+1} = 1, \tag{10f}$$

$$\mathcal{M}' \succeq 0. \tag{10g}$$

SDR (contd.)

- Why (SDP1) is a relaxation?
  Consider (x, y) a feasible solution[1] to (9) and construct

$$\mathcal{M}' = \begin{bmatrix} xx^\top & xy^\top & x \\ yx^\top & yy^\top & y \\ x & y & 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^\top,$$

  which is a rank-1 feasible solution to (SDP1).
  But (SDP1) has no constraints on rank of $\mathcal{M}'$ and hence it is a relaxation.
- We can see $\text{rank}(\mathcal{M}') = 1$ is a desirable solution to (SDP1) towards solving the original problem.

[1] There is always a solution to (9) since Nash equilibrium always exists for any given A, B.

SDR (contd.)

- Supposes somehow we find rank-1 solution but (SDP1) is a weak relaxation. Why?
  Consider (x, y) feasible to (9). We can construct

$$\mathcal{M}' = \gamma \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^\top,$$

  which is feasible to (SDP1) for any $\gamma > 0$.
- Therefore, [2] proposes to add some valid inequalities to (SDP1) in order to tighten the relaxation so that it favors a Nash Equilibrium solution.

Tightened formulation

- Further, (SDP1) can be equivalently written as

$$\min_{\mathcal{M} \in \mathbb{S}^{(m+n)\times(m+n)}} \quad 0 \tag{SDP2}$$

$$\text{subject to} \quad \mathcal{M} \succeq 0 \tag{11}$$

$$\mathcal{M} \geq 0 \tag{12}$$

$$\sum_{j=1}^{m} X_{i,j} = \sum_{j=1}^{n} P_{i,j} = x_i, \ \forall i \in [m] \tag{13}$$

$$\sum_{j=1}^{m} Y_{i,j} = \sum_{j=1}^{n} Z_{i,j} = y_i, \ \forall i \in [n] \tag{14}$$

$$\sum_{j=1}^{n} A_{i,j} P_{i,j} \geq \sum_{j=1}^{n} A_{k,j} P_{i,j} \ \ \forall i, k \in [m] \tag{15}$$

$$\sum_{j=1}^{m} B_{j,i} P_{j,i} \geq \sum_{j=1}^{m} B_{j,k} P_{j,i} \ \ \forall i, k \in [n] \tag{16}$$

Objective functions for minimizing rank$(\mathcal{M})$

The authors in [2] consider the following two nonconvex objective functions which are bounded below and the bound is tight if and only if $\mathcal{M}$ is rank-1.

1. Square root: $\sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}}$

$$\sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}} \geq \sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i = 2$$

2. Diagonal gap: $\mathrm{Tr}(\mathcal{M}) - x^\top x - y^\top y$

$$\mathrm{Tr}(\mathcal{M}) - \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Linearization Algorithms

The two functions cannot be directly minimized by off-the-shelf solvers. Therefore, we focus on their first-order Taylor series expansion given as,

$$\sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}} \simeq \sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}^{(k-1)}} + \frac{1}{2\sqrt{\mathcal{M}_{i,i}^{(k-1)}}} \left(\mathcal{M}_{i,i} - \mathcal{M}_{i,i}^{(k-1)}\right)$$

Algorithm 1: Square Root Minimization

Initialize: $x^0 = \mathbf{1}_m,\ y^0 = \mathbf{1}_n,\ k = 1.$
Result: $x^k,\ y^k$
while *!convergence* **do**
  Solve (SDP2) with objective function
  $\sum_{i=1}^m \frac{1}{\sqrt{(x_i^{(k-1)}}} X_{i,i} + \sum_{i=1}^n \frac{1}{\sqrt{(y_i^{(k-1)}}} Y_{i,i}$
  $x^k \leftarrow \mathsf{diag}(X^*), \quad y^k \leftarrow \mathsf{diag}(Y^*)$
  $k \leftarrow k + 1;$
end

Linearization Algorithms (contd.)

$$\mathrm{Tr}(\mathcal{M}) - \begin{bmatrix} x \\ y \end{bmatrix}^{\top} \begin{bmatrix} x \\ y \end{bmatrix} \simeq \mathrm{Tr}(\mathcal{M}) - \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)\top} \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)} - 2 \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)\top} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)} \right)$$

Algorithm 2: Diagonal Gap Minimization

Initialize: $x^0 = 0,\ y^0 = 0,\ k = 1.$

Result: $x^k,\ y^k$

while *!convergence* **do**

Solve (SDP2) with objective function

$$\mathrm{Tr}(X) + \mathrm{Tr}(Y) - 2 \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)\top} \begin{bmatrix} x \\ y \end{bmatrix}$$

$x^k \leftarrow P^* \mathbf{1}_n, \quad y^k \leftarrow P^{*\top} \mathbf{1}_m$

$k \leftarrow k + 1;$

end

Property of Algorithms

The authors in [2] proved the monotonicity of iterates for both Algorithms 1 and 2, i.e.,

$$\sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}^{(k)}} \leq \frac{1}{2}\left(\sum_{i=1}^{m+n} \frac{\mathcal{M}_{i,i}^{(k)}}{\sqrt{\mathcal{M}_{i,i}^{(k-1)}}} + \sqrt{\mathcal{M}_{i,i}^{(k-1)}}\right)$$

$$\leq \frac{1}{2}\left(\sum_{i=1}^{m+n} \frac{\mathcal{M}_{i,i}^{(k-1)}}{\sqrt{\mathcal{M}_{i,i}^{(k-1)}}} + \sqrt{\mathcal{M}_{i,i}^{(k-1)}}\right) = \sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}^{(k-1)}}$$

and,

$$\operatorname{Tr}\left(\mathcal{M}^{(k)}\right) - \begin{bmatrix} x \\ y \end{bmatrix}^{(k)\top} \begin{bmatrix} x \\ y \end{bmatrix}^{(k)} \leq \operatorname{Tr}\left(\mathcal{M}^{(k-1)}\right) - \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)\top} \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)}.$$

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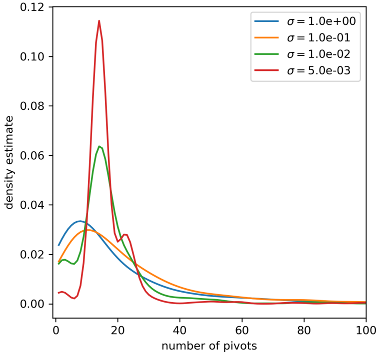
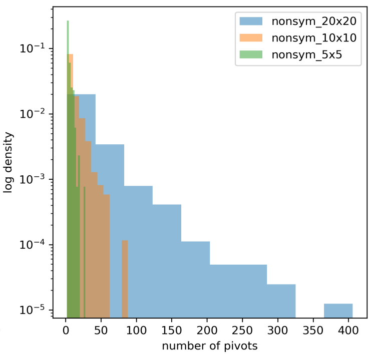
Experiments

Pivoting scheme results for exact Nash Equilibrium

Problem instances generated with entries a_{ij}, $b_{ij} \in [0, 1]$.



(a) Pivots for 1,000 instances of 15×20 perturbed problem.



(b) Pivots for 100 instances of randomly generated problems.

SDR results for ϵ-Nash Equilibrium

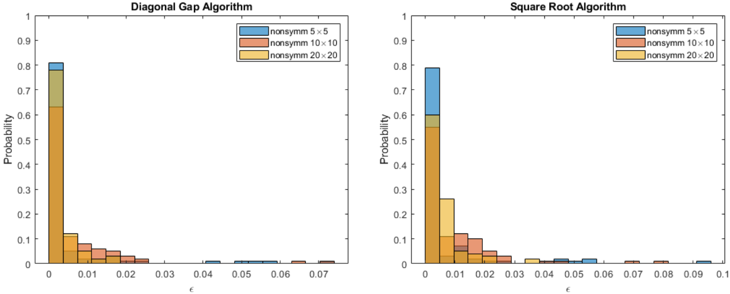


Figure 2: Distribution of ϵ after running 20 iterations of Algorithm 1 (right) and 2 (left), for 100 instances each of different sizes of (A, B).

SDR results for mean objective values

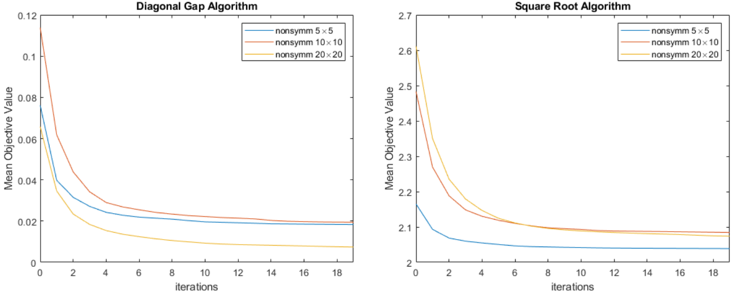


Figure 3: Mean of objective values of 100 instances of each of the different sizes (A, B) for each iteration of Algorithm 1 (right) and 2 (left).

Statistics on ϵ

Algorithm	Max	Mean	Median	Standard Deviation
Diagonal gap	0.0722	0.0045	2.4e-06	0.0127
Square root	0.0952	0.0059	3.9e-06	0.0152

Table 1: For nonsymmetric 5×5 games after 20 iterations.

Algorithm	Max	Mean	Median	Standard Deviation
Diagonal gap	0.0726	0.0056	0.0015	0.0110
Square root	0.0786	0.0087	0.0034	0.0131

Table 2: For nonsymmetric 10×10 games after 20 iterations.

Algorithm	Max	Mean	Median	Standard Deviation
Diagonal gap	0.0180	0.0027	0.0013	0.0037
Square root	0.0358	0.0053	0.0034	0.0064

Table 3: For nonsymmetric 20×20 games after 20 iterations.

Number of pivots: hard-to-solve

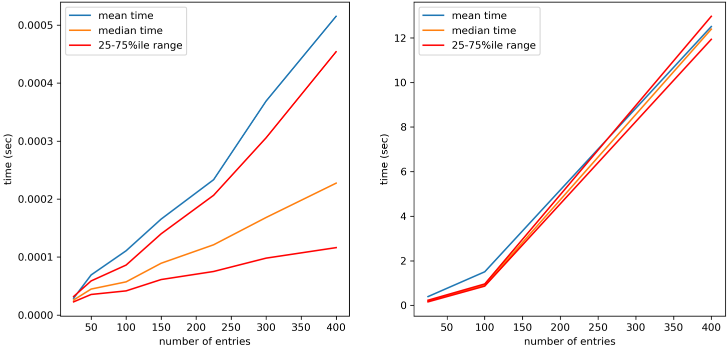
Exponential example [3]:

$$A = \begin{bmatrix} -180 & 72 & -333 & 297 & -153 & 270 \\ -30 & 17 & -33 & 42 & -3 & 20 \\ -81 & 36 & -126 & 126 & -36 & 90 \\ 90 & -36 & 126 & -126 & 36 & -81 \\ 20 & -3 & 42 & -33 & 17 & -30 \\ 270 & -153 & 297 & -333 & 72 & -180 \end{bmatrix}$$

6×6 but takes 88 pivots.

Can easily generate cases that cycle by forcing some entries of A and B to be repeated.

Timing comparison



(a) Pivoting timing.

(b) SDPR timing (Diagonal Gap Algorithm).

Figure 4: Computational time comparison.

Future directions

Future directions

- Finding exact Nash equilibria is (essentially) NP-complete
- Could try a suite of methods:
  - exact: pivoting
  - ϵ-heuristic: IPM / path-following
  - ϵ-heuristic: Newton for LCP
  - ϵ-heuristic: variational inequality formulation
  - ϵ-relaxation: Convex relaxation
- Can complementary pivoting handle a guess from an approximate basis?
- Use pivoting to refine an ϵ-solution?
- Variational inequality / first order minimax methods: can you say how many steps it will take before it WON'T converge?[2]

[2] Prof Zhang's suggestion.

References I

[1] K. G. Murty, "Linear Complementarity, Linear and Nonlinear Programming (Internet Edition)," p. 660, 1997.

[2] A. A. Ahmadi and J. Zhang, "Semidefinite programming and nash equilibria in bimatrix games," *INFORMS J. Comput.*, Sept. 2020.

[3] B. von Stengel, "Computing Equilibria for Two-Person Games," *Handbook of Game Theory with Economic Applications*, 2002.