Finding Equilibria for Bimatrix Games

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1. Introduction

In this report, we consider the problem of finding an equilibrium for a bimatrix game. Bimatrix games are used to study strategic two-entity decision making contexts when a decision of one player has a significant impact on the response behavior of the other player. There is extensive literature on the problem of finding an equilibrium of such a game, and the field traces its origins to the 1950s in John Nash's proof that for any bimatrix game, there exists at least one pair of strategies such that when played simultaneously, neither player has an incentive to play a different strategy. Such a pair is an equilbirium, but the initial proof was non-constructive. Constructive proofs similar to the simplex method for LPs emerged fifteen years later. In this document, we study one historical method for the construction of an equilibrium before turning to a more recent approach for finding approximate an equilibrium.

The structure of the document is as follows. We first review the necessary bimatrix game formulations in Section 2. In the remainder of the report, we explore exact solutions to the bimatrix game from a historical pivoting perspective in the context of Linear Complementarity Problems in Sections 3-4 before turning to the notion of approximate-equilibria from a more recent perspective in Section 5. In the former, we introduce the pivoting schemes known generally as "complementary pivoting methods" and in the latter, we consider a recent convex relaxation of the Nash equilibrium problem based on semidefinite programming. Finally, we study some of the numerical properties of the two methods in Section 6.

2. Bimatrix games

2.1. Formulation

We consider a bimatrix game having players say I and II. Player-I and II has a set of m and n strategies respectively, to choose from. Further, denote the cost incurred by player I and II after selecting pure strategy i and j be a_{ij} and b_{ij} respectively. A mixed (or randomized) strategy say $x \in \mathbb{R}^m$ for Player-I, where x_i denotes the probability of choosing strategy i, satisfies $x \geq 0$ and $\sum_{i=1}^{m} x_i = 1$. Similarly, this holds for strategy $y \in \mathbb{R}^n$ by Player-II. Clearly, the aim of each of the players is to choose a strategy which minimizes their individual cost with respect to other player's strategy.

Define $m-$dimensional and $n-$dimensional simplices

$$\Delta_m = \left\{ x \in \mathbb{R}^m \mid x_i \geq 0 \; \forall i, \; \sum_{i=1}^{m} x_i = 1 \right\},$$
$$\Delta_n = \left\{ y \in \mathbb{R}^n \mid y_i \geq 0 \; \forall i, \; \sum_{i=1}^{n} y_i = 1 \right\}. \tag{1}$$

A Nash equilibrium $(x^*, y^*) \in \Delta_m \times \Delta_n$ for game (A, B) exists, if

$$x^{*\top} A y^* \leq x^\top A y^*, \quad \forall x \in \Delta_m, \tag{2a}$$
$$x^{*\top} B y^* \leq x^{*\top} B y, \quad \forall y \in \Delta_n, \tag{2b}$$

where $A, B \in \mathbb{R}^{m \times n}$, $[A]_{ij} = a_{ij}$ and $[B]_{ij} = b_{ij}$. The problem of finding (x^*, y^*) satisfying (2) can be formulated as a LCP,

$$u = -\mathbf{1}_m + Ay \geq 0, \qquad \forall x \geq 0, \;\; x^\top u = 0, \tag{3a}$$
$$v = -\mathbf{1}_n + B^\top x \geq 0, \qquad \forall y \geq 0, \;\; y^\top v = 0 \tag{3b}$$

where $\mathbf{1}_m$ denote the $m-$dimensional vector of all ones. The LCP is obtained by first dividing both relationships in (2) by the value of the game $\nu := x^{*\top} A y^* > 0$ and absorbing ν into a renormalized strategy $x' := (x/\nu)/[e_m^\top (x/\nu)]$ as follows

$$(2) \iff 1 \leq x'^\top A y* \iff \mathbf{1}_m^\top x' \leq x'^\top A y^* \iff A y^* \geq \mathbf{1}_m \tag{4}$$

and similarly for the relaionship involving B.

If (x', y') is a solution to (3), then the Nash equilibrium (x^*, y^*) is given by

$$x^* = x'/\mathbf{1}_m^\top x' \quad \text{and} \quad y^* = y'/\mathbf{1}_n^\top y'. \tag{5}$$

The vector q and matrix M defining the LCP (3) are given by

$$q = \begin{bmatrix} -\mathbf{1}_m \\ -\mathbf{1}_n \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & A \\ B^\top & 0 \end{bmatrix}$$

Further $w^\top = \begin{bmatrix} u & v \end{bmatrix}^\top$ and $z^\top = \begin{bmatrix} x & y \end{bmatrix}^\top$. Note $w \geq 0$, $z \geq 0$ and $w \circ z = 0$.

3. Relation to LCPs

3.1. LCP formulation

The linear complementarity problem (LCP) can be stated as: find nonnegative vectors $w, z \in \mathbb{R}^n_+$ such that for a particular linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$ and a particular vector $q \in \mathrm{Range}\,(M) \subseteq \mathbb{R}^n$, the vectors w and z are complementary, meaning that $\langle w \,, z \rangle = 0$. This can be expressed succinctly as the following feasibility problem

$$w = Mz + q \tag{6a}$$

$$w \geq 0 \tag{6b}$$
$$z \geq 0 \tag{6c}$$
$$w_i z_i = 0 \ \ i = 1, 2, \ldots, n. \tag{6d}$$

It is often useful to rewrite the affine-linear relationship as $w - Mz = q$ so that it resembles a simplex dictionary where w can be interpreted as a nonnegative slack variable for the inequality $(-M)z \leq q$. This interpretation lends itself nicely to extending the simplex method to solve LCPs.

3.2. Intuition

To build geometric intuition for the LCP (following [1]) and motivate some solution approaches, first consider the simplest, one-dimensional case of finding scalar variables x and y that satisfy

$$x \geq 0 \tag{7a}$$
$$y \geq 0 \tag{7b}$$
$$x \perp y \iff x \cdot y = 0. \tag{7c}$$

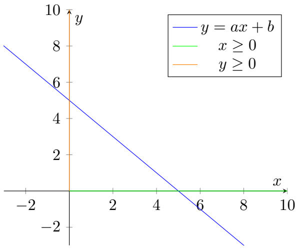
Without the complementarity constraint, it is immediate that the entire nonnegative orthant is feasible (so the feasible set is convex). The complementarity constraint introduces a combinatorial aspect that destroys convexity and restricts the feasible set to either the nonnegative x-axis or the nonnegative y-axis (or their intersection, the origin). Thus the full set of solutions to (7) is nonconvex, but given information that, say $x = 0$, the solution set becomes convex, namely the nonnegative y-axis. Given this intuition, we see that the problem of finding *a* solution may become easier when a "basis" for the problem is known.

Slightly generalizing system (7) to allow for coupling between x and y via an affine-linear relationship $y = ax + b$ for scalar parameters a and b, the problem becomes finding scalars x and y that satisfy

$$y = ax + b \tag{8a}$$
$$x \geq 0 \tag{8b}$$
$$y \geq 0 \tag{8c}$$
$$x \perp y \tag{8d}$$

which can be seen in Fig. 1a. Taking $a = 0$, the parameter b then parametrizes solutions along the nonnegative y-axis and the parameter a parametrizes solutions along the nonnegative x-axis. One complication, however, is that the affine-linear relationship is capable of expressing infeasible relationships between x and y, for example if $a < 0$ and $b < 0$. The number of solutions to system (8) is summarized in Fig. 1b and shows that the problem can be somewhat complex even in the simplest setup.

When looking to solve the LCP, it was clear that knowledge of the support of the solution (or basis of the solution, i.e., whether x or y was zero) helped identify a solution. From a point $p = (x, y)$ for $x > 0, y > 0$, a solution could be obtained by choosing $x = 0$ (or $y = 0$) and following the linear relationship to the y-axis to find a "basic" vertex solution. Such methods fall under the class of "pivoting" methods and will be discussed in detail later.



(a) 1d LCP geometry.

$a \backslash b$	$-$	0	$+$
$-$	0	1	2
0	0	∞	1
$+$	1	1	1

(b) 1d LCP number of solutions.

Aternatively, depending on the sign of a, the explicit relationship between y and x suggests a different solution approach based on eliminating y and rewriting the feasibility problem solely in terms of x as

$$ax + b \geq 0 \tag{9a}$$
$$x \geq 0 \tag{9b}$$
$$x \cdot (ax + b) = 0. \tag{9c}$$

The first two conditions ensure that for any feasible x, a quadratic expression of x will be nonnegative, and hence the problem can be viewed in the following "optimization sense"

$$\begin{aligned} (P)\ \underset{x}{\text{minimize}} \quad & x \cdot (ax + b) \\ \text{subject to} \quad & x \geq 0, \\ & ax + b \geq 0. \end{aligned} \tag{10}$$

Since the objective is bounded below by 0 for any feasible x, we can recover the LCP. System (10) has the interpretation of trying to maintain primal and dual feasibility while driving complementarity to zero, in spirit similar to that of interior point methods (IPMs).

By considering KKT conditions of a relaxation of system (10), we see that the nonnegativity on $ax + b$ is necessary for the existence of a KKT solution, and that it need not be included explicitly in the primal problem. Specifically, suppose that

1. $ax + b$ is not constrained and
2. a is replaced with $\frac{a}{2}$

and consider the relaxation

$$\begin{aligned} (P')\ \underset{x}{\text{minimize}} \quad & x \cdot (\tfrac{a}{2}x + b) \\ \text{subject to} \quad & x \geq 0. \end{aligned} \tag{11}$$

Letting $L(x; \lambda)$ denote the Lagrangian of the system (11), the KKT conditions are

$$0 = \nabla_x L(x; y) = (ax + b) - y \iff ax + b = y \tag{12a}$$
$$0 \leq x \tag{12b}$$
$$0 \leq y \tag{12c}$$
$$0 = x \cdot y \tag{12d}$$

which we see corresponds to the original LCP in system (9). Thus, finding a KKT point for the relaxed system (11) will give a solution to the feasibility LCP in system (9), and vice versa.

A benefit of the optimization formulation is that it becomes easier to see when solutions should exist. In particular, if $a > 0$, then system (11) is convex, and since any $x > 0$ is strictly feasible, Slater's condition applies. This indicates that strong duality holds, and further, that if the value of the dual is finite, then it is attained. When $a > 0$, the dual function $d(y) = \frac{1}{2}ax^2 + (b - y)x = -\frac{1}{2a}(y - b)^2$ can be obtained by substituting directly the stationarity condition from Eq. (12) which gives

$$\begin{aligned} (D') \ \underset{y}{\text{maximize}} \quad & -\tfrac{1}{2a}(y-b)^2 \\ \text{subject to} \quad & y \geq 0. \end{aligned} \tag{13}$$

In this case, the duality gap

$$\Delta(x, y) = \tfrac{1}{2}ax^2 + bx + \tfrac{1}{2a}(y-b)^2 = \tfrac{1}{2}a(x + \tfrac{b}{a})^2 - \tfrac{b^2}{2a} + \tfrac{1}{2a}(y-b)^2$$

is seen to be zero when

1. $b < 0$, so that $x = -\frac{b}{a}$ and $y = 0$
2. $b > 0$, so that $x = 0$ and $y = b$.

3.3. Relation to linear programming

The LCP generalizes linear programming (LP). Consider the LP

$$\begin{aligned} (LP) \ \underset{x}{\text{minimize}} \quad & c^\top x \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0 \end{aligned} \tag{14}$$

in "symmetric form". Then by duality, we know that there must exist a primal slack variable $v \in \mathbb{R}^n$, dual variable $y \in \mathbb{R}^m$, and dual slack variable $u \in \mathbb{R}^m$ such that [2][1]

$$\begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 0 & -A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ -b \end{bmatrix} \tag{15a}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \geq 0, \quad \begin{bmatrix} x \\ y \end{bmatrix} \geq 0 \tag{15b}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}^\top \begin{bmatrix} x \\ y \end{bmatrix} = 0. \tag{15c}$$

We can write this as an LCP by gathering the slack variables into the vector w, the primal and dual variables into the vector z, and finding

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad z = \begin{bmatrix} x \\ y \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -A^\top \\ A & 0 \end{bmatrix}, \quad q = \begin{bmatrix} c \\ -b \end{bmatrix}. \tag{16}$$

This shows that a symmetric, zero-sum game can be solved by an LP, and hence can be solved in polynomial time [3].

[1] Equation 1.10 in Section 1.2.

4. Complementary Pivoting Methods

One way to solve a (nondegenerate[2]) LCP (q, M) problem instance is to work toward the dual goal of *(i)* identifying an appropriate basis to express the righthand side q as $q = w - Mz$ while preserving nonnegativity of w and z and *(ii)* obtaining a feasible basis that is also *complementary*. In what follows, to make the analogy to the simplex method clear, it is convenient to refer to a basis that satisfies *(i)* as a *feasible* basis and a basis that satisfies *(i)* and *(ii)* as an *optimal feasible* basis. Thus, a solution to the LCP will be a vertex of a polyhedron where exactly n of the $2n$ constraints are active.

An extreme point that satisfies both the affine-linear relationship and complementarity conditions but compromises on nonnegativity is the initial "dictionary" (in analogous terms to the simplex method) obtained from $q = w + (-M)z$ by taking $z \equiv 0$ as nonbasic variables and $w = q$ as basic variables. By construction, $\langle w\,, z\rangle = 0$, but the nonnegativity constraint $w = q + Mz \geq 0$ may not hold. In the case that $q \geq 0$, the solution $(w = q, z = 0)$ is immediate, but if $q \ngeq 0$, then there are two options for how to proceed:

1. compromise on non-negativity and attempt to make $w \geq 0$ while preserving complementarity
2. compromise on complementarity to satisfy $w \geq 0$ and attempt to regain complementarity while preserving $w, z \geq 0$.

The first case has not been studied, but the second case leads to the class of pivoting methods collectively known as *complementary pivoting methods*.

4.1. Initialization

The first step in a complementary pivoting method is to obtain a feasible (but not necessarily complementary) initial dictionary for the affine-linear equation

$$q = w - Mz \tag{17}$$

Recall that vectors w and z satisfying equation (17) can be interpreted as coordinates of q in the basis composed of vectors from the set of $2n$ columns of $(I, -M)$

$$w_1 I_{[:,1]} + w_2 I_{[:,2]} + \cdots + w_n I_{[:,n]} - z_1 M_{[:,1]} - z_2 M_{[:,2]} - \cdots - z_n M_{[:,n]} = q$$

where $A_{[:,j]}$ is matlab notation for the jth column of matrix A. Therefore

- a solution to (17) will be a linear combination of of the $2n$ columns of $(I, -M)$;
- a *complementary* solution to (17) will be a linear combination of n basis vectors where the jth vector is chosen from the jth *complementary vector pair* $\left\{I_{[:,j]}, -M_{[:,j]}\right\}$ along with its corresponding *complementary coordinate pair* $\{w_j, z_j\}$ for $j = 1, 2, \ldots, n$; and
- a *nonnegative, complementary* solution to (17) (which will solve the LCP) will be a conic combination of the complementary vectors given by nonnegative, complementary coordinates w and z.

A useful way of rephrasing this information is through the terminology of complementary cones. A *complementary cones* of M is defined by choosing one vector from the complementary vector pair $\left\{I_{[:,j]}, -M_{[:,j]}\right\}$ for each j and forming a conic combination of such

[2] Degeneracy and degenerate cases will not be discussed in detail; refer to [2] for more detail.

vectors. Thus the set of complementary cones of M has 2^n elements where each element C looks like

$$C = \left\{ v : v = \sum_{j=1}^{n} \alpha_j v_j,\ \alpha_j \geq 0,\ \alpha_j \in \{w_j, z_j\},\ v_j \in \left\{ I_{[:,j]}, -M_{[:,j]} \right\}\ j = 1, 2, \ldots, n \right\}.$$

Thus a complementary cone C containing q is a solution to the LCP.

Returning to the task of finding a feasible initial dictionary, one approach (that is analogous to Phase-I of the simplex method) is to introduce an artificial variable z_0 alongside a strictly positive vector d called a *covering vector* to create an augmented dictionary

$$Iw - dz_0 - Mz = q.$$

When z_0 is initialized as a nonbasic variable ($z_0 = 0$), it can be pivoted into the basis by determining the leaving variable w_{i^*} from a minimum ratio test

$$i^* = \arg\min_i \{q_i/d_i\}$$

to ensure that the new righthand side becomes strictly positive, since

$$\bar{q} \coloneqq q - (q_{i^*}/d_{i^*})d > 0 \quad \text{and} \quad z_0 \leftarrow [q - (q_{i^*}/d_{i^*})d]_{i^*} > 0.$$

Thus the new basis becomes $(w_1, w_2, \ldots, w_{i^*-1}, z_0, w_{i^*+1}, \ldots, w_n)$, and the problem becomes similar to the "immediate-solution" case where $q \geq 0$, except that z_0 pollutes the basis and the nonbasis contains one full set of complementary vectors, namely $\{w_{i^*}, z_{i^*}\}$. Nevertheless, the new dictionary provides a way of satisfying (17) while ensuring nonnegativity and complementarity for an augmented problem. It is then clear that a solution to the augmented problem is a solution to the original LCP if and only if z_0 drops out of the basis and becomes nonbasic again. The next task is to scan through the possible complementary cones while preserving complementarity and nonnegativity.

4.2. Pivoting

One approach for uniqely stepping through the exponential list of complementary bases is by *complementary pivoting*. Complementary pivoting uses the same basic Gaussian pivot operation as in the simplex method. Whereas the simplex method chose the entering variable to improve the objective function of an LP, an LCP does not have an objective function. However, once a leaving variable is determined by a minimum ratio test to ensure nonnegativity of the modified $\bar{q}$, complementarity can be preserved by selecting the entering variable to the be complementary variable to the leaving variable. Specifically, whenever a new variable w_k (respectively z_k) is chosen to leave the basis, its complement z_k (respectively w_k) must be selected to enter the basis. Pivoting in this way will ensure that at most one element of the kth coordinate pair $\{w_k, z_k\}$ will be in the basis, thereby ensuring complementarity for the augmented problem.

4.3. Termination

Simlar to the simplex method, there are three possibilities for the termination of the pivoting steps:

1. Solution: at some stage, z_0 is selected to leave the basis, and the resulting vectors w and z solve the LCP
2. Ray-termination: if the vector corresponding to the entering variable has no positive component, then can be increased infinitely (similar to an unbounded LP)
3. Degeneracy: the algorithm can cycle, similar to the simplex method. Geometrically, this corresponds to a vertex of the polyhedron that has three or more complementary edges and will lead to cycling. This issue can be resolved by the lexico-minimum ratio test, similar to what's used in the simplex method.

When M is a positive semidefinite matrix, then this pivoting method paired with the least index rule will terminate finitely. In this case, the artificial variable z_0 is monotonically decreasing, and the algorithm can be interpreted as minimizing $z_0 : z_0 \geq 0$. For matrices not satisfying this property, though, the least-index pivot rule does not guarantee finite termination; the reason for this was not known as of 1979 (see Section 2 in [4]; I am not sure whether it is known now.).

4.4. Algorithm

Finally, the complementary pivoting algorithm is summarized in Alg. 1 where

- `pivot!` performs a Gaussian elimination pivot at row corresponding to l_{i^*} and column corresponding to e_{i^*} in the new dictionary D
- $B[i^*]$ (respectively $B[i^*]^\complement$) is the variable in the basis corresponding to the result of the minimum ratio test (and its complement)
- $\bar{D}_i$ is the ith element of the vector corresponding to variable $B[i^*]$ in the dictionary.

Algorithm 1: Augmented complementary pivoting method

Input: $d > 0$
Result: w^*, z^*
$B \leftarrow [w_1, w_2, \ldots, w_n]$;
$i^* \leftarrow \arg\min_i \{q_i/d_i\}$;
$e_{i^*} \leftarrow z_0$; $l_{i^*} \leftarrow w_{i^*}$;
while $z_0 \in B$ **do**
    pivot!(e_{i^*}, l_{i^*});
    $B \leftarrow B \setminus \{l_{i^*}\} \cup \{e_{i^*}\}$;
    $i^* \leftarrow \arg\min_i \{\bar{q}_i/\bar{D}_i : \bar{D}_i > 0\}$;
    $l_{i^*} \leftarrow B[i^*]$; $e_{i^*} \leftarrow B[i^*]^\complement$
end

4.5. Example problem

As an example (from [2] 2.8) consider the LCP (q, M) where

$$q = \begin{bmatrix} 3 \\ 5 \\ -9 \\ -5 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}. \tag{18}$$

The complementary pivoting method produces the sequence of dictionaries summarized in Table 9 in the appendix.

4.6. Solving bimatrix games

Complementary pivoting methods can be used to solve bimatrix games. The most well-known algorithm is the Lemke-Howson (LH) procedure, which is a specialization of Alg. 1. LH finds a path along vertices in the dual representation of the polyhedron, and as such, the standard terminology used to describe the method differs from the above complementary pivoting method. To make the parallels between a Lemke-Howson type method and the simplex method more explicit, the following exposition is based on [2] and describes a method for using the complementary pivoting method to solve bimatrix games. References [5, 6, 7] provide more standard overviews of the LH method.

To apply complementary pivoting as in Alg. 1 to a bimatrix game, the dictionary must be initialized in a particular way to avoid secondary-ray termination. Consider the problem given by minimizing worst-case payoff for matrices

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} \tag{19}$$

where the w variables are initially selected as a basis. The bimatrix game is expressed as

$$\begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} - \begin{bmatrix} \cdot & \cdot & 2 & 2 & 1 \\ \cdot & \cdot & 1 & 2 & 2 \\ 1 & 2 & \cdot & \cdot & \cdot \\ 3 & 1 & \cdot & \cdot & \cdot \\ 2 & 3 & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad u, v \geq 0, \quad x, y \geq 0, \quad u^\top x = v^\top y = 0 \tag{20}$$

and the initial (infeasible) dictionary corresponding to the choice of w as basis can be written as $w = q + Mz$ as in Table 1.

Observe that the dictionary in Table (1) initially places zero weight on every pure strategy in x and y and hence is not even a possible strategy for the game. One way to obtain a non-identically-zero strategy pair is to arbitrarily select two strategies from $\{x_1, x_2, y_1, y_2, y_3\}$ to enter the basis. Selecting x_1 and x_2 (or likewise two strategies from y) will compromise the existing complementarity structure in addition to failing to provide an initial strategy for both players. Since it is clear that the two choices must include one strategy from x and one strategy from y, a greedy goal of the pivots is to also ensure that complementarity is maintained to the extent possible.

basis-0		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	ratio
u_1	$=$	-1	$\cdot$	$\cdot$	$+2y_1$	$+2y_2$	$+1y_3$	
u_2	$=$	-1	$\cdot$	$\cdot$	$+1y_1$	$+2y_2$	$+2y_3$	
v_1	$=$	-1	$+1x_1$	$+2x_2$	$\cdot$	$\cdot$	$\cdot$	-1/1
v_2	$=$	-1	$+3x_1$	$+1x_2$	$\cdot$	$\cdot$	$\cdot$	-1/3
v_3	$=$	-1	$+2x_1$	$+3x_2$	$\cdot$	$\cdot$	$\cdot$	-1/2

Table 1: Initial dictionary corresponding to basis w.
Old pivot: (entering $\rightarrow$ leaving) = ($- \rightarrow -$).
New pivot: (entering $\rightarrow$ leaving) = ($x_1 \rightarrow v_1$).
MINIMUM ratio test.

For example, suppose that x_1 is first selected to enter the basis. It is not possible to make $\bar{q} > 0$ in one step due to the non-coupled structure of M, but in the following subsystem

$$v_1 = -1 + 1x_1 + 2x_1 \tag{21a}$$
$$v_2 = -1 + 3x_1 + 1x_1 \tag{21b}$$
$$v_3 = -1 + 2x_1 + 3x_2 \tag{21c}$$

the thee scenarios for leaving variable v_1, v_2, v_3

$$-1x_1 = -1 - 1v_1 + 2x_2, \text{ and } (21) \text{ or}$$
$$-3x_1 = -1 - 1v_1 + 1x_2, \text{ and } (21) \text{ or}$$
$$-2x_1 = -1 - 1v_1 + 3x_2, \text{ and } (21)$$

give the following three systems

$$\begin{array}{lll} x_1 = 1 + 1v_1 - 2x_1 & v_1 = -\frac{2}{3} + \frac{1}{3}v_2 + \frac{7}{3}x_2 & v_1 = -\frac{1}{2} + \frac{1}{2}v_3 + \frac{1}{2}x_2 \\ v_2 = 2 + 3v_1 - 5x_2 & x_1 = \frac{1}{3} + \frac{1}{3}v_2 + \frac{1}{3}x_2 & v_2 = \frac{1}{2} + \frac{3}{2}v_3 - \frac{1}{2}x_2 \\ v_3 = 1 + 2v_1 - 1x_2 & v_3 = -\frac{1}{3} + \frac{2}{3}v_2 + \frac{11}{3}x_2 & x_1 = \frac{1}{2} + \frac{1}{2}v_3 - \frac{3}{2}x_2 \end{array}$$

where it is evident that v_1 should leave the basis to ensure $\bar{q} > 0$ on the lower portion of the system. Generalizing this procedure, the minimum ratio test $i^* = \arg\min_i \bar{q}_i / D_{[i,x_1]}$ gives v_1 as the leaving variable (where $D_{[:,x_1]}$ is the column of the dictionary corresponding to variable x_1). Since the dictionary vector $D_{[:,x_1]}$ contains zeros in the entries corresponding to the w_1, w_2 components of the basis, the upper portion of the dictionary remains the same, and the following dictionary that is obtained after the first pivot is summarized in Table (2).

Given that v_1 has just left the basis, a natural choice is to use the complementary pivoting rule (to maintain the complementary structure between pairs $\{u, x\}$ and $\{v, y\}$), which indicates that y_1 should be the entering variable. Performing another minimum ratio test yields u_2 as the leaving variable, and thus will yield a feasible dictionary, as summarized in Table (3). Finally, the dictionary in Table (3) is feasible and "almost complementary" in the sense that the basis contains only one full complementary pair, namely (u_1, x_1).

basis-1		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	ratio
u_1	$=$	-1	$\cdot$	$\cdot$	$+2y_1$	$+2y_2$	$+1y_3$	-1/2
u_2	$=$	-1	$\cdot$	$\cdot$	$+1y_1$	$+2y_2$	$+2y_3$	-1/1
x_1	$=$	1	$+1v_1$	$-2x_2$	$\cdot$	$\cdot$	$\cdot$	
v_2	$=$	2	$+3v_1$	$-5x_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	1	$+2v_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	

Table 2: Dictionary after old pivot.
Old pivot: (entering $\rightarrow$ leaving) $= (x_1 \rightarrow v_1)$.
New pivot: (entering $\rightarrow$ leaving) $= (y_1 \rightarrow u_2)$.
MINIMUM ratio test.

basis-2		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	ratio
u_1	$=$	1	$\cdot$	$\cdot$	$+2u_2$	$-2y_2$	$-3y_3$	
y_1	$=$	1	$\cdot$	$\cdot$	$+1u_2$	$-2y_2$	$-2y_3$	
x_1	$=$	1	$+1v_1$	$-2x_2$	$\cdot$	$\cdot$	$\cdot$	1/-2
v_2	$=$	2	$+3v_1$	$-5x_2$	$\cdot$	$\cdot$	$\cdot$	2/-5
v_3	$=$	1	$+2v_1$	$-1x_2$	$\cdot$	$\cdot$	$\cdot$	1/-1

Table 3: Dictionary after old pivot.
Old pivot: (entering $\rightarrow$ leaving) $= (y_1 \rightarrow u_2)$.
New pivot: (entering $\rightarrow$ leaving) $= (x_2 \rightarrow v_2)$.
MAXIMUM ratio test.

basis-4		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	ratio
y_2	$=$	$\frac{1}{2}$	$\cdot$	$\cdot$	$-\frac{1}{2}u_1$	$+1u_2$	$-\frac{3}{2}y_3$	
y_1	$=$	0	$\cdot$	$\cdot$	$+1u_2$	$-2u_2$	$+1y_3$	
x_1	$=$	$\frac{1}{5}$	$-\frac{1}{5}v_1$	$+\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
x_2	$=$	$\frac{2}{5}$	$+\frac{3}{5}v_1$	$-\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	$\frac{3}{5}$	$+\frac{7}{5}v_1$	$+\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	

Table 5: Dictionary after old pivot.
Old pivot: (entering $\rightarrow$ leaving) $= (y_2 \rightarrow u_1)$.
DONE: u_1 (complement of initial choice x_1) dropped from basis.

Note that if u_1 had been selected to leave the basis in the second minimum ratio test in Table (2), then the algorithm would terminate successfully since the basis would contain no full set of complementary vectors while the two pivots just performed ensure that the dictionary (and basis) is feasible. Extending this idea, complementary pivoting method can be applied as in the previous section to find a unique scan of the feasible vertices, and if either u_1 or v_1 leave the basis, the algorithm terminates with a solution to the LCP and bimatrix game.

basis-3		$\bar{q}$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$	ratio
u_1	$=$	1	$\cdot$	$\cdot$	$+2u_2$	$-2y_2$	$-3y_3$	1/-2
y_1	$=$	1	$\cdot$	$\cdot$	$+1u_2$	$-2y_2$	$-2y_3$	1/-2
x_1	$=$	$\frac{1}{5}$	$-\frac{1}{5}v_1$	$+\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
x_2	$=$	$\frac{2}{5}$	$+\frac{3}{5}v_1$	$-\frac{1}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	
v_3	$=$	$\frac{3}{5}$	$+\frac{7}{5}v_1$	$+\frac{2}{5}v_2$	$\cdot$	$\cdot$	$\cdot$	

Table 4: Dictionary after old pivot.
Old pivot: (entering $\to$ leaving) = ($x_2 \to v_2$).
New pivot: (entering $\to$ leaving) = ($y_2 \to u_1$).
MAXIMUM ratio test.

5. Semidefinite Programming in Bimatrix Games

In this section, we explore the recent work [8] on the semidefinite relaxation (SDR) approach to solve bimatrix games. The concept of SDR with its application to signal processing problems has been well studied in [9]. The authors study SDR of nonconvex quadractically constrained quadratic programs (QCQPs) with providing guarantess on approximation accuracy for various problem types. At a high-level, the SDR approach relies on first convexifying a non-convex problem (NP-hard) so that it becomes tractable and then recover a feasible solution to original problem using different heuristics. The feasible solution is not necessarily optimal and hence one needs to establish theoretical approximation bounds for the worst-case.

As suggested in [8], the problem of computing Nash Equilibrium in bimatrix games can be formulated as a quadratic programming (QP) feasibility problem (23). Note that the cost matrices A, B introduced in Section 2 are considered as payoff matrices in this section. So, instead of minimizing the costs, each player need to pick strategies to maximize their individual payoffs. The Nash equilibrium conditions can be restated as,

$$x^{*\top} A y^* \geq x^\top A y^*, \ \ \forall x \in \Delta_m \quad \text{and} \quad x^{*\top} B y^* \geq x^{*\top} B y, \ \ \forall y \in \Delta_n. \tag{22}$$

We can formulate the Nash Equilibria conditions using payoff matrices (A, B) as Nash Equilibria conditions using cost matrices $(\tilde{A}, \tilde{B})$ where

$$\tilde{A} = c\mathbf{1}_{m\times n} - eA$$
$$\tilde{B} = d\mathbf{1}_{m\times n} - fB$$

$c, d \in \mathbb{R}$ and $e, f > 0$.

$$(22) \iff \begin{aligned} x^{*\top} \tilde{A} y^* &\leq x^\top \tilde{A} y^*, \ \ \forall x \in \Delta_m \\ x^{*\top} \tilde{B} y^* &\leq x^{*\top} \tilde{B} y, \ \ \forall y \in \Delta_n. \end{aligned}$$

Moreover, Nash Equilibria is invariant to certain affine transformations to A, B such as

$$A' = c\mathbf{1}_{m\times n} + eA$$

$$B' = d\mathbf{1}_{m\times n} + fB$$

for $c, d \in \mathbb{R}$, $e, f > 0$, i.e.,

$$(22) \iff \begin{aligned} &{x^*}^\top A' y^* \geq x^\top A' y^*, \;\; \forall x \in \Delta_m \\ &{x^*}^\top B' y^* \geq {x^*}^\top B' y, \;\; \forall y \in \Delta_n. \end{aligned}$$

For any $x \in \Delta_m$, we can see that the payoff $x^\top A y^*$ can be seen as a convex combination of payoffs given by $e_i^\top A y^*$, $i = 1, \dots, m$ where e_i is the vector of zeros with only i^{th} entry as 1. That is, $x^\top A y^* = \sum_{i=1}^m x_i e_i^\top A y^*$. Similarly, for any $y \in \Delta_n$, ${x^*}^\top B y = \sum_{j=1}^n {x^*}^\top B e_j y_j$. (Note that the dimension of e_i should be clear from the context in which it is used.) One can easily prove that,

$$(22) \iff {x^*}^\top A y^* \geq e_i^\top A y^*, \;\; \forall i \in \{1, \dots, m\} \;\; \text{and} \;\; {x^*}^\top B y^* \geq {x^*}^\top B e_j, \;\; \forall j \in \{1, \dots, n\}.$$

The QP feasibility problem is formulated as the following nonconvex QCQP,

$$\begin{aligned} \min_{x\in\mathbb{R}^m, y\in\mathbb{R}^n} \quad & 0 \\ \text{subject to} \quad & x^\top A y \geq e_i^\top A y, \;\; \forall i \in \{1, 2, \dots, m\} && (23a) \\ & x^\top B y \geq x^\top B e_j, \;\; \forall j \in \{1, 2, \dots, n\} && (23b) \\ & x_i \geq 0, \;\; \forall i \in \{1, 2, \dots, m\} && (23c) \\ & y_j \geq 0, \;\; \forall j \in \{1, 2, \dots, n\} && (23d) \\ & \sum_{i=1}^m x_i = 1, && (23e) \\ & \sum_{j=1}^n y_j = 1. && (23f) \end{aligned}$$

Clearly, any feasible solution to (23) is a Nash Equilibrium $(x^*, y^*) \in \Delta_m \times \Delta_n$.

5.1. ϵ-Nash Equilibrium

Often, interested in computing ϵ-Nash Equilibria rather than exact. It is defined as

$$\begin{aligned} {x^*}^\top A y^* &\geq x^\top A y^* - \epsilon, \;\; \forall x \in \Delta_m \\ {x^*}^\top B y^* &\geq {x^*}^\top B y - \epsilon, \;\; \forall y \in \Delta_n \end{aligned}$$

OR

$$\begin{aligned} {x^*}^\top A y^* &\geq e_i^\top A y^* - \epsilon, \;\; \forall i \in \{1, \dots, m\} \\ {x^*}^\top B y^* &\geq {x^*}^\top B e_j - \epsilon, \;\; \forall j \in \{1, \dots, n\}. \end{aligned}$$

Note:

$$\epsilon \geq \max\{\max_i \; e_i^\top A y^* - {x^*}^\top A y^*, \; \max_j \; {x^*}^\top B e_j - {x^*}^\top B y^*\}$$

For ϵ to make sense, the entries of A and B should be normalized between 0 and 1 (since Nash Equilibria is invariant to certain affine transformation to A, B).

5.2. SDR

Define a matrix $\mathcal{M}$ as

$$\mathcal{M} = \begin{bmatrix} X & P \\ Z & Y \end{bmatrix}$$

and an augmented matrix $\mathcal{M}'$ as

$$\mathcal{M}' = \begin{bmatrix} X & P & x \\ Z & Y & y \\ x & y & 1 \end{bmatrix}$$

where $X \in \mathbb{S}^{m\times m}$, $Y \in \mathbb{S}^{n\times n}$, $Z \in \mathbb{R}^{n\times m}$ and $P = Z^\top$ ($\mathbb{S}^{k\times k}$ is the set of $k \times k$ symmetric matrices).

The SDP relaxation can be expressed as given in [8],

$$\min_{\mathcal{M}' \in \mathbb{S}^{(m+n+1)\times(m+n+1)}} \quad 0 \tag{SDP1}$$

$$\text{subject to} \quad \operatorname{Tr}(AZ) \geq e_i^\top Ay, \quad \forall i \in \{1, 2, \ldots, m\} \tag{24a}$$

$$\operatorname{Tr}(BZ) \geq x^\top Be_j, \quad \forall j \in \{1, 2, \ldots, n\} \tag{24b}$$

$$\sum_{i=1}^{m} x_i = 1, \tag{24c}$$

$$\sum_{j=1}^{n} y_j = 1, \tag{24d}$$

$$\mathcal{M}' \geq 0, \tag{24e}$$

$$\mathcal{M}'_{m+n+1,m+n+1} = 1, \tag{24f}$$

$$\mathcal{M}' \succeq 0. \tag{24g}$$

Consider (x, y) a feasible solution[3] to (23) and construct

$$\mathcal{M}' = \begin{bmatrix} xx^\top & xy^\top & x \\ yx^\top & yy^\top & y \\ x & y & 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^\top,$$

which is a rank-1 feasible solution to (SDP1). But (SDP1) has no constraints on rank of $\mathcal{M}'$ and hence it is a relaxation. We can see $\operatorname{rank}(\mathcal{M}') = 1$ is a desirable solution to (SDP1) towards solving the original problem.

Theorem 1 in [8] shows that the above relaxation is weak. Consider (x, y) feasible to (23). We can construct

$$\mathcal{M}' = \gamma \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^\top,$$

[3] There is always a solution to (23) since Nash equilibrium always exists for any given A, B.

which is feasible to (SDP1) for any $\gamma > 0$. Therefore, [8] proposes to add some valid inequalities to (SDP1) in order to tighten the relaxation so that it favors a Nash Equilibrium solution.

Further if $\mathcal{M}'$ is rank-1 then it should be of the form,

$$\mathcal{M}' = \begin{bmatrix} xx^\top & xy^\top & x \\ yx^\top & yy^\top & y \\ x & y & 1 \end{bmatrix}.$$

Therefore, finding the rank-1 solution to (SDP1) is equivalent to solving (23).

5.3. Valid Inequalities

The (SDP1) can be tightened by adding necessary constraints for $\mathcal{M}'$ to be rank-1. More inequalities can be derived as necessary conditions for a Nash Equilibrium. The following are the valid inequalities given in [8],

$$\sum_{j=1}^{m} X_{i,j} = \sum_{j=1}^{n} P_{i,j} = x_i, \ \forall i \in \{1, 2, \dots, m\} \tag{25}$$

$$\sum_{j=1}^{m} Y_{i,j} = \sum_{j=1}^{n} Z_{i,j} = y_i, \ \forall i \in \{1, 2, \dots, n\} \tag{26}$$

$$\sum_{j=1}^{n} A_{i,j} P_{i,j} \geq \sum_{j=1}^{n} A_{k,j} P_{i,j} \ \ \forall i, k \in \{1, 2, \dots, m\} \tag{27}$$

$$\sum_{j=1}^{m} B_{j,i} P_{j,i} \geq \sum_{j=1}^{m} B_{j,k} P_{j,i} \ \ \forall i, k \in \{1, 2, \dots, n\} \tag{28}$$

5.4. Simplifiying SDP

$$\begin{aligned} \min_{\mathcal{M}' \in \mathbb{S}^{(m+n+1)\times(m+n+1)}} \quad & 0 \\ \text{subject to} \quad & (24c) - (24g), \ (25) - (28) \end{aligned} \tag{SDP1$'$}$$

There is one-to-one correspondence between solutions to (SDP1′) and those to (SDP2).

$$\min_{\mathcal{M} \in \mathbb{S}^{(m+n)\times(m+n)}} \quad 0 \tag{SDP2}$$

$$\text{subject to} \quad \mathcal{M} \succeq 0 \tag{29}$$

$$\mathcal{M} \geq 0 \tag{30}$$

$$\sum_{j=1}^{m} X_{i,j} = \sum_{j=1}^{n} P_{i,j} = x_i, \ \forall i \in \{1, 2, \dots, m\} \tag{31}$$

$$\sum_{j=1}^{m} Y_{i,j} = \sum_{j=1}^{n} Z_{i,j} = y_i, \; \forall i \in \{1, 2, \ldots, n\} \tag{32}$$

$$\sum_{j=1}^{n} A_{i,j} P_{i,j} \geq \sum_{j=1}^{n} A_{k,j} P_{i,j} \;\; \forall i, k \in \{1, 2, \ldots, m\} \tag{33}$$

$$\sum_{j=1}^{m} B_{j,i} P_{j,i} \geq \sum_{j=1}^{m} B_{j,k} P_{j,i} \;\; \forall i, k \in \{1, 2, \ldots, n\} \tag{34}$$

5.5. Objective functions for minimizing rank

Recall that we need to find a rank-1 solution, $\mathcal{M}'$ to (SDP1′) or equivalently $\mathcal{M}$ to (SDP2). The authors in [8] consider the following two nonconvex objective functions which are bounded below and the bound is tight if and only if $\mathcal{M}$ is rank-1.

1. Square root: $\sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}}$

$$\sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}} \geq \sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i = 2$$

2. Diagonal gap: $\mathrm{Tr}(\mathcal{M}) - x^\top x - y^\top y$

$$\mathrm{Tr}(\mathcal{M}) - \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

5.6. Algorithms

The algorithms we discuss in this section are for minimizing the rank of matrix $\mathcal{M}$ in (SDP2). From section 5.5, we have two functions which cannot be directly minimized by off-the-shelf solvers. Therefore, we focus on their first-order Taylor series expansion given as,

$$\sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}} \simeq \sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}^{(k-1)}} + \frac{1}{2\sqrt{\mathcal{M}_{i,i}^{(k-1)}}} \left(\mathcal{M}_{i,i} - \mathcal{M}_{i,i}^{(k-1)} \right) \tag{35a}$$

$$\mathrm{Tr}(\mathcal{M}) - \begin{bmatrix} x \\ y \end{bmatrix}^\top \begin{bmatrix} x \\ y \end{bmatrix} \simeq \mathrm{Tr}(\mathcal{M}) - \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)\top} \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)} - 2 \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)\top} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)} \right) \tag{35b}$$

The linearized functions are iteratively minimized giving rise to non-increasing sequence of iterates of original objective functions, i.e., $\sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}^{(k)}}$ or $\mathrm{Tr}\left(\mathcal{M}^{(k)}\right) - \begin{bmatrix} x \\ y \end{bmatrix}^{(k)\top} \begin{bmatrix} x \\ y \end{bmatrix}^{(k)}$.

Algorithm 2: Square Root Minimization

Initialize: $x^0 = \mathbf{1}_m,\ y^0 = \mathbf{1}_n,\ k = 1.$

Result: $x^k,\ y^k$

while *!convergence* **do**

  Solve (SDP2) with objective function $\sum_{i=1}^m \frac{1}{\sqrt{(x_i^{(k-1)}}} X_{i,i} + \sum_{i=1}^n \frac{1}{\sqrt{(y_i^{(k-1)}}} Y_{i,i}$;

  $x^k \leftarrow \mathrm{diag}(X^*), \quad y^k \leftarrow \mathrm{diag}(Y^*)$;

  $k \leftarrow k+1$;

end

Algorithm 3: Diagonal Gap Minimization

Initialize: $x^0 = 0,\ y^0 = 0,\ k = 1.$

Result: $x^k,\ y^k$

while *!convergence* **do**

  Solve (SDP2) with objective function $\mathrm{Tr}(X) + \mathrm{Tr}(Y) - 2 \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)\top} \begin{bmatrix} x \\ y \end{bmatrix}$;

  $x^k \leftarrow P^* \mathbf{1}_n, \quad y^k \leftarrow P^{*\top} \mathbf{1}_m$;

  $k \leftarrow k+1$;

end

The authors in [8] proved the monotonicity of iterates for both Algorithms 2 and 3, i.e.,

$$\begin{aligned} \sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}^{(k)}} &\leq \frac{1}{2} \left(\sum_{i=1}^{m+n} \frac{\mathcal{M}_{i,i}^{(k)}}{\sqrt{\mathcal{M}_{i,i}^{(k-1)}}} + \sqrt{\mathcal{M}_{i,i}^{(k-1)}} \right) \\ &\leq \frac{1}{2} \left(\sum_{i=1}^{m+n} \frac{\mathcal{M}_{i,i}^{(k-1)}}{\sqrt{\mathcal{M}_{i,i}^{(k-1)}}} + \sqrt{\mathcal{M}_{i,i}^{(k-1)}} \right) = \sum_{i=1}^{m+n} \sqrt{\mathcal{M}_{i,i}^{(k-1)}} \end{aligned}$$

and,

$$\mathrm{Tr}\left(\mathcal{M}^{(k)}\right) - \begin{bmatrix} x \\ y \end{bmatrix}^{(k)\top} \begin{bmatrix} x \\ y \end{bmatrix}^{(k)} \leq \mathrm{Tr}\left(\mathcal{M}^{(k-1)}\right) - \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)\top} \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)}.$$

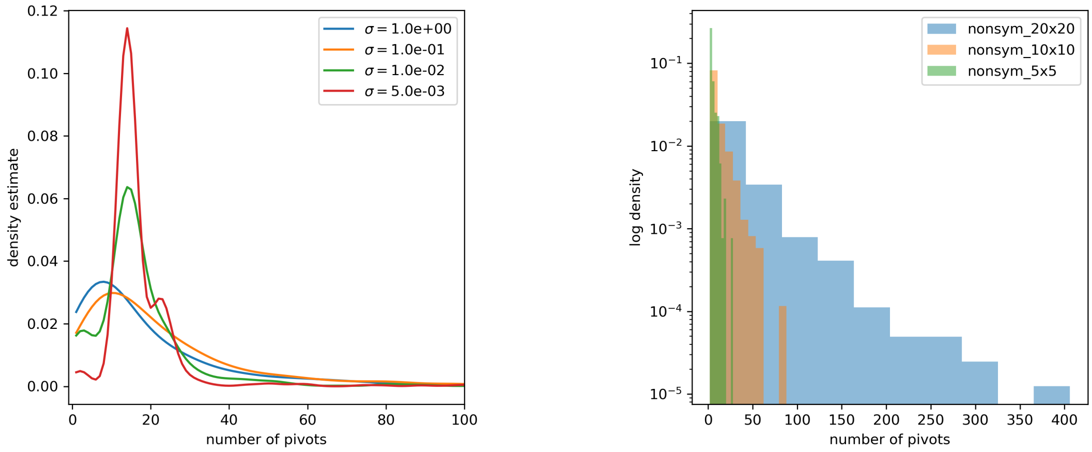
6. Computational studies

6.1. Complementary pivoting method

To study the effectiveness of the complementary pivoting scheme for generic solving bimatrix games, we compute equilibria for randomly generated problem instances and plot the number of pivots required to obtain a solution. Our experiments follow two directions:

- How sensitive to data corruption is an equilibrium for a random problem?
- How well does the complementary pivoting scheme perform on a random problem in general?

For the first question, a problem instance is generated with entries in $[0, 1]$ which are then corrupted by Gaussian noise with variannce σ^2. The complementary pivoting scheme is



(a) Pivots for 1,000 instances of 15×20 perturbed problem.

(b) Pivots for 100 instances of randomly generated problems.

Figure 2: Pivoting method computational performance.

then applied to obtain an equilibrium. For the second question 100 randomly generated problem instances of varying sizes per [8] are used as test cases for determining the distribution of number of pivots. Further, for each problem instance, m (not necessarily unique) equilibria are found, corresponding to m initial choices of the entering variablex_i for $i = 1, 2, \ldots, m$.[4]

Numerical results in Fig. (2) indicate that an "average" problem instance is easy to solve. Numerical results in Fig. (2a) indicate that the pivoting scheme is sensitive to data corruption, providing evidence in support of the claim that even for an exponentially hard case, a small perturbation of it may be an "easy" problem. On the other hand, there exist outlier cases as shown in Fig. (2b) where the number of pivots is large, but none appear to give the exponential behavior as in the known worst-case examples [10], and the majority of cases are solved easily.

6.2. SDR results for ϵ-Nash Equilibrium

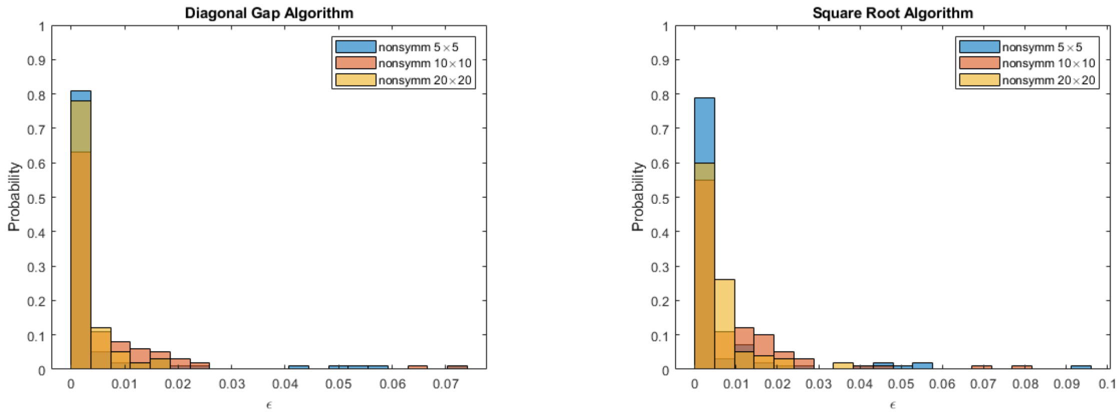
We ran 20 iterations each of Algorithms 2 and 3 for each of the 100 instances of varying sizes of matrices (as mentioned in the previous section). We make the following observations:

- The results in Figure 3 indicate that for a fixed number of iterations, with increasing size of matrices A, B, the probability of recovering exact Nash Equilibria decreases. This can also be seen by the lower median value of ϵ in Table 6 for 5×5 games as compared to 10×10 or 20×20 in Tables 7, 8.
- From Table 6, larger mean value for 5×5 games is supported by the large standard deviation in ϵ values.
- From Figure 4, we see in case of diagonal gap algorithm the mean reduction for

[4] This should be extended to allow for y_j where $j = 1, 2, \ldots, n$, but this extension has not been implemented.

20×20 games is more as compared to 5×5 or 10×10. This is also supported by comparing the mean value of ϵ across Tables 6, 7, 8.

- Overall, the diagonal gap algorithm outperforms square root in terms of the quality of ϵ-Nash equilibrium solution. For each of three sizes of games, the mean ϵ-value for diagonal gap is significantly lower than for square root.



(a) (b)

Figure 3: Distribution of ϵ after running 20 iterations of Algorithm 2 (right) and 3 (left), for 100 instances each of different sizes of (A, B).

Algorithm	Max	Mean	Median	Standard Deviation
Diagonal gap	0.0722	0.0045	2.4e-06	0.0127
Square root	0.0952	0.0059	3.9e-06	0.0152

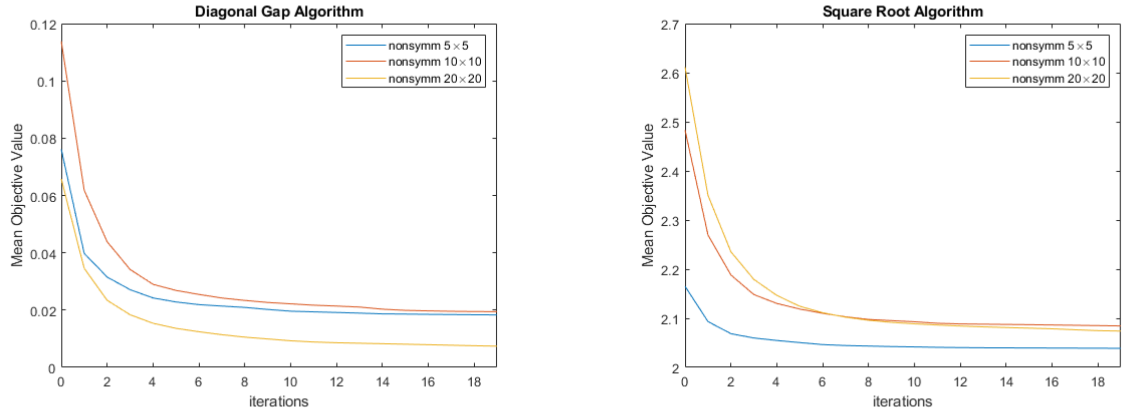
Table 6: Statisitcs on ϵ for nonsymmetric 5×5 games after 20 iterations.

Algorithm	Max	Mean	Median	Standard Deviation
Diagonal gap	0.0726	0.0056	0.0015	0.0110
Square root	0.0786	0.0087	0.0034	0.0131

Table 7: Statisitcs on ϵ for nonsymmetric 10×10 games after 20 iterations.

Algorithm	Max	Mean	Median	Standard Deviation
Diagonal gap	0.0180	0.0027	0.0013	0.0037
Square root	0.0358	0.0053	0.0034	0.0064

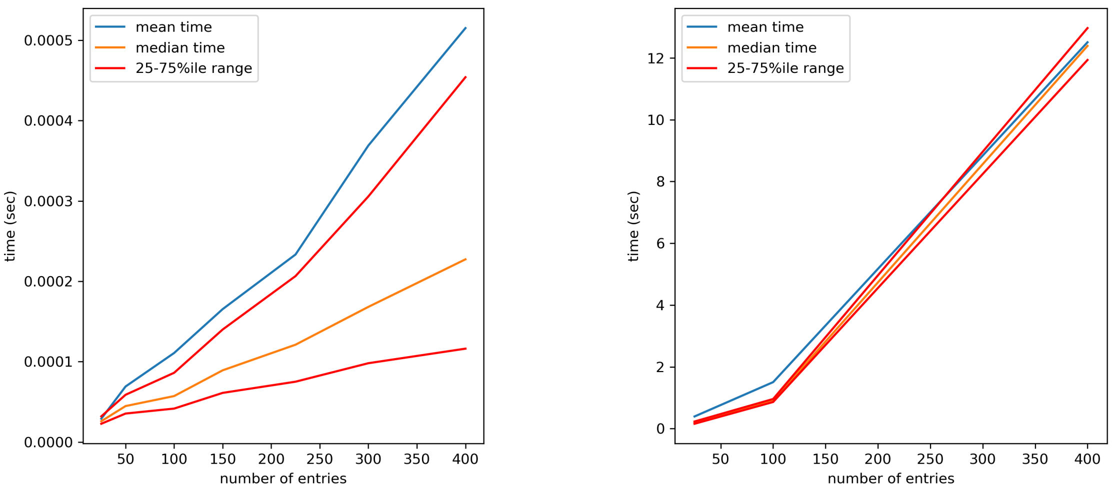
Table 8: Statisitcs on ϵ for nonsymmetric 20×20 games after 20 iterations.



(a) (b)

Figure 4: Mean of objective values of 100 instances of each of the different sizes (A, B) for each iteration of Algorithm 2 (right) and 3 (left).

6.3. Comparison of computational times



(a) Complementary pivoting computation time (until solution).

(b) SDR computation time (20 iterations).

The timing results indicate that that while the SDR method is a polynomial time algorithm, it is limited in its practical efficiency. On the other hand, for most problems, the complementary pivoting scheme is computationally efficient. One has to be cautious to avoid cycling, though, and as such the lexico-minimum ratio test should be used. In the simulations, we have avoided cases that cycle, though such cases can easily be generated by forcing particular entries of the A and B game matrices to be repeated.

7. Future directions

- Finding exact Nash equilibria is (essentially) NP-complete.
- Could try a suite of methods:
  - exact: pivoting
  - ϵ-heuristic: IPM / path-following
  - ϵ-heuristic: Newton for LCP
  - ϵ-heuristic: variational inequality formulation
  - ϵ-relaxation: Convex relaxation
- Can complementary pivoting handle a guess from an approximate basis?
- Use pivoting to refine an ϵ-solution?
- Variational inequality / first order minimax methods: can you say how many steps it will take before it WON'T converge?[5]

[5] Prof Zhang's suggestion.

References

[1] K. Erleben, "Numerical methods for linear complementarity problems in physics-based animation," February 2013. http://image.diku.dk/kenny/download/erleben.13.siggraph.course.notes.pdf.

[2] K. G. Murty, "Linear Complementarity, Linear and Nonlinear Programming (Internet Edition)," p. 660, 1997.

[3] I. Adler, "The equivalence of linear programs and Zero-Sum games," *International Journal of Game Theory*, vol. 42, p. 165–177, 2013.

[4] Yow-Yieh Chang, "Least-index resolution of degeneracy in linear complementarity problems," October 1979. https://apps.dtic.mil/dtic/tr/fulltext/u2/a080113.pdf.

[5] D. Koller, N. Megiddo, and B. von Stengel, "Efficient computation of equilibria for extensive two person games," 1995. http://theory.stanford.edu/~megiddo/pdf/lemke.pdf.

[6] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*. SIAM, 2009.

[7] B. Codenotti, S. D. Rossi, and M. Pagan, "An experimental analysis of lemke-howson algorithm," 2008. https://arxiv.org/abs/0811.3247.

[8] A. A. Ahmadi and J. Zhang, "Semidefinite programming and nash equilibria in bimatrix games," *INFORMS J. Comput.*, Sept. 2020.

[9] Z.-Q. Luo, W.-K. Ma, A. M. C. So, Y. Ye, and S. Zhang, "Semidefinite relaxation of quadratic optimization problems," *IEEE Signal Process. Mag.*, vol. 27, pp. 20–34, June 2010.

[10] B. von Stengel, "Computing Equilibria for Two-Person Games," *Handbook of Game Theory with Economic Applications*, 2002.

Appendices

A. Example: augmented Lemke

An example of the sequence of pivots generated by the augmented Lemke complementary pivoting scheme is given below.

basis	*w1	*w2	*w3	*w4	z1	z2	z3	z4	z0	q
w1	1.00	-	-	-	-1.00	1.00	1.00	1.00	-1.00	3.00
w2	-	1.00	-	-	1.00	-1.00	1.00	1.00	-1.00	5.00
w3	-	-	1.00	-	-1.00	-1.00	-2.00	-	**-1.00**	-9.00
w4	-	-	-	1.00	-1.00	-1.00	-	-2.00	-1.00	-5.00
basis	***w1**	***w2**	**w3**	***w4**	**z1**	**z2**	**z3**	**z4**	***z0**	**q**
w1	1.00	-	-1.00	-	-	2.00	3.00	1.00	-	12.00
w2	-	1.00	-1.00	-	2.00	-	3.00	1.00	-	14.00
z0	-	-	-1.00	-	1.00	1.00	2.00	-	1.00	9.00
w4	-	-	-1.00	1.00	-	-	**2.00**	-2.00	-	4.00
basis	***w1**	***w2**	**w3**	**w4**	**z1**	**z2**	***z3**	**z4**	***z0**	**q**
w1	1.00	-	0.50	-1.50	-	2.00	-	**4.00**	-	6.00
w2	-	1.00	0.50	-1.50	2.00	-	-	4.00	-	8.00
z0	-	-	-	-1.00	1.00	1.00	-	2.00	1.00	5.00
z3	-	-	-0.50	0.50	-	-	1.00	-1.00	-	2.00
basis	**w1**	***w2**	**w3**	**w4**	**z1**	**z2**	***z3**	***z4**	***z0**	**q**
z4	0.25	-	0.13	-0.38	-	0.50	-	1.00	-	1.50
w2	-1.00	1.00	-	-	**2.00**	-2.00	-	-	-	2.00
z0	-0.50	-	-0.25	-0.25	1.00	-	-	-	1.00	2.00
z3	0.25	-	-0.38	0.13	-	0.50	1.00	-	-	3.50
basis	**w1**	**w2**	**w3**	**w4**	***z1**	**z2**	***z3**	***z4**	***z0**	**q**
z4	0.25	-	0.13	-0.38	-	0.50	-	1.00	-	1.50
z1	-0.50	0.50	-	-	1.00	-1.00	-	-	-	1.00
z0	-	-0.50	-0.25	-0.25	-	**1.00**	-	-	1.00	1.00
z3	0.25	-	-0.38	0.13	-	0.50	1.00	-	-	3.50
basis	**w1**	**w2**	**w3**	**w4**	***z1**	***z2**	***z3**	***z4**	**z0**	**q**
z4	0.25	0.25	0.25	-0.25	-	-	-	1.00	-0.50	1.00
z1	-0.50	-	-0.25	-0.25	1.00	-	-	-	1.00	2.00
z2	-	-0.50	-0.25	-0.25	-	1.00	-	-	1.00	1.00
z3	0.25	0.25	-0.25	0.25	-	-	1.00	-	-0.50	3.00

Table 9: Lemke method with $d = \mathbb{1}$ for problem (18).