Approximation of Copostive Cones

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1 Introduction

In this report, we discuss various polynomially computable approximations to the cone of copositive matrices. A copositive cone $\mathcal{C}$ is defined as

$$\mathcal{C} = \{X \in \mathcal{S} : y^\top X y \geq 0 \ \forall y \in \mathbb{R}^n_+\}$$

where $\mathcal{S} \subset \mathbb{R}^{n \times n}$ is the set of symmetric matrices. The dual cone to $\mathcal{C}$ is the cone of completely positive matrices $\mathcal{C}^*$ defined as

$$\mathcal{C}^* = \text{conv}\{yy^\top : y \in \mathbb{R}^n_+\}$$

Both $\mathcal{C}$ and $\mathcal{C}^*$ are closed, convex, pointed, full dimensional, nonpolyhedral cones. It has been shown in the literature that many NP-hard problems have completely positive representation such as: standard quadratic programs (QPs) Bomze et al. (2000), Bomze and De Klerk (2002); maximum stable set problem De Klerk and Pasechnik (2002); quadratic assignment problem Povh and Rendl (2009); graph partitioning problem Povh and Rendl (2007). Moreover, Burer (2009) showed that any QP with binary variables and linear constraints also have such a representation.

2 Notations

- $\mathcal{S}^+$: Cone of symmetric positive semidefinite (psd) matrices defined as

$$\mathcal{S}^+ = \{X \in \mathcal{S} : y^\top X y \geq 0 \ \forall y \in \mathbb{R}^n\}$$

- $\mathcal{N}$: Cone of symmetric non-negative matrices defined as

$$\mathcal{N} = \{X \in \mathcal{S} : X_{ij} \geq 0 \ \forall i, j = 1, \ldots, n\}$$

- $\mathcal{D}$: Cone of symmetric doubly nonnegative matrices defined as

$$\mathcal{D} = \mathcal{S}^+ \cap \mathcal{N}$$

3 Organization

In section 4, we state some well-known facts about $\mathcal{C}$ and $\mathcal{C}^*$. In section 5, we focus on the dual problem of the completely positive representation of original NP-hard problems of interest. Different approximation schemes from the literature are discussed. In section 6, we shift our focus on standard QP problem and discuss why it is equivalent to solving a COP. The reason to choose standard QP is that it encompasses maximum stable set problem and many other well-studied models form literature on which we conduct our numerical experiments in section 6.1. In section 7, some ongoing works on generalized copositive cones are mentioned for interested readers. Generalized copositive cones is a generalization of $\mathcal{C}$ (which are copositive cones over $\mathbb{R}^n_+$) over general convex domain.

4 Facts about $\mathcal{C}$ and $\mathcal{C}^*$

1. Testing whether a given matrix is in $\mathcal{C}$ is co-NP complete Murty and Kabadi (1987).

   (a) Therefore it is reasonable that no polynomially computable self-concordant barriers are known for $\mathcal{C}$ and $\mathcal{C}^*$ Quist et al. (1998).

2. One can easily observe that $\mathcal{C} \supseteq \mathcal{S}^+ + \mathcal{N}$ and $\mathcal{C}^* \subseteq \mathcal{D} = \mathcal{S}^+ \cap \mathcal{N}$.

   (a) Berman and Shaked-Monderer (2003), Shaked-Monderer and Berman (2021) Interestingly, for $n \times n$ matrices of order $n \leq 4$, we have equality in above relations i.e., $\mathcal{C} = \mathcal{S}^+ + \mathcal{N}$ and $\mathcal{C}^* = \mathcal{D} = \mathcal{S}^+ \cap \mathcal{N}$.
   But for $n \geq 5$, the inclusions are strict in above relations. Interested readers can refer the book Shaked-Monderer and Berman (2021) Section 2.10: $\mathcal{COP}_5$ for detailed proofs.

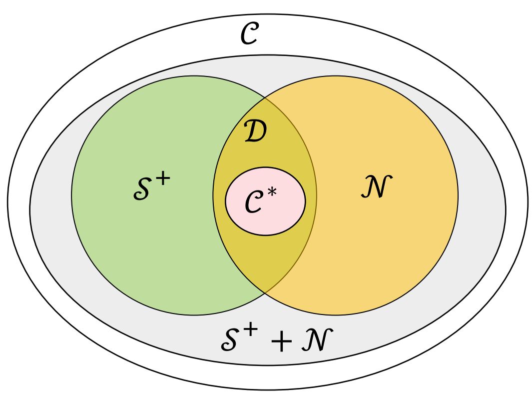

Figure 1: Illustration of inclusion of various cones.

3. It is easy to observe that the criteria for copositivity can be restated as following.

$$X \in \mathcal{C} \iff P_X(z) = \sum_{i,j=1}^{n} z_i^2 X_{ij} z_j^2 \geq 0 \ \forall z \in \mathbb{R}^n$$

$P_X(z)$ is a homogeneous polynomial of degree 4. From this criteria, we have some sufficient conditions on copositivity which are useful for inner approximating $\mathcal{C}$.

(a) If $P_X(z)$ has a sum-of-squares (s.o.s) decomposition, then $X \in \mathcal{C}$.

(b) If $P_X(z)$ has non-negative coefficients for all its monomials, then $X \in \mathcal{C}$.

We can have higher order sufficient conditions by replacing $P_X(z)$ above with $P_X^r(z)$ where

$$P_X^r(z) = P_X(z)\left(\sum_{k=1}^{n} z_k^2\right)^r.$$

4. Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Another criteria for copositivity is

$$X \in \mathcal{C} \iff y^\top X y \geq 0 \ \forall y \in \mathbb{R}_+^n \text{ with } \|y\| = 1.$$

From Facts 1-2, we can note that restating a problem as a conic optimization over cones $\mathcal{C}$ and $\mathcal{C}^*$ does not resolve the difficulty of that problem. Facts 3-4 give rise to different polynomially computable inner approximations of $\mathcal{C}$ discussed in the next section.

5 Problem & Approximations

Often referred to as the dual problem[1], the **copositive program** (COP) is a standard conic optimization over $\mathcal{C}$, i.e.

$$\begin{aligned} \vartheta^* := \max\ & \langle C, X\rangle \\ \text{s.t. } & \langle A_i, X\rangle = b_i, \quad i = 1, \dots, m \\ & X \in \mathcal{C} \end{aligned} \tag{COP}$$

where $C \in \mathbb{R}^{n\times n}$, $A_i \in \mathbb{R}^{n\times n}$, $b_i \in \mathbb{R}$. Clearly, the difficulty of (COP) lies in the copositive cone constraint.

5.1 SDP Inner Approximation

Recall sufficient condition 3a from Fact 3 in section 4 but with $P_X^r(z)$ instead of $P_X(z)$. This gives rise to hierarchy of cones $\mathcal{K}^r$ approximating $\mathcal{C}$ from inside defined by Parrilo (2000) as

$$\mathcal{K}^r := \left\{ X \in \mathcal{S} : \ P_X^r(z) = P_X(z)\left(\sum_{k=1}^{n} z_k^2\right)^r \text{ has an s.o.s. decomposition.} \right\} \tag{1}$$

where $\mathcal{K}^r \subseteq \mathcal{K}^{r+1}$. The hierarchy is easy to observe because if $X \in \mathcal{K}^r$ $\exists$ $f_\ell(z)$ such that $P_X^r(z) = \sum_\ell f_\ell(z)^2$ then

$$P_X^{r+1}(z) = P_X^r(z)\left(\sum_{k=1}^{n} z_k^2\right) = \sum_{\ell,k} f_\ell(z)^2 z_k^2 \text{ or } \sum_{\ell,k} [f_\ell(z) z_k]^2.$$

[1]Because COP is dual to the completely positive program (CPP) representation of original NP-hard problems.

Thus, if $X \in \mathcal{K}^r$ then $X \in \mathcal{K}^{r+1}$. Parrilo (2000) further showed that: $\text{int}(C) \subseteq \bigcup_{r\in\mathbb{N}} \mathcal{K}^r$ i.e., for any strictly copositive matrix $X \notin \mathcal{K}^0$, $\exists r^* \geq 1$ such that

$$\mathcal{K}^0 \subset \mathcal{K}^1 \subset \ldots \subset \mathcal{K}^{r^*} \ni X$$

Lets see some results for complete characterization of $\mathcal{K}^r$ for $r = 0, 1$ (Bomze and De Klerk (2002) provides characterization of $\mathcal{K}^r$ for any given r.).

1. **Case** $r = 0$: Parrilo (2000) showed that $X \in \mathcal{K}^0$ if and only if $\mathcal{K}^0 := \mathcal{S}^+ + \mathcal{N}$. Therefore we have following SDP approximation (SDP-0) to (COP) corresponding to $\mathcal{K}^0$.

$$\begin{aligned} \vartheta_{\mathcal{K}^0} := \max_{X,Y,Z} \ & \langle C, X\rangle \\ \text{s.t.} \ & \langle A_i, X\rangle = b_i, \quad i = 1, \ldots, m \\ & X = Y + Z, \ Y \in \mathcal{S}^+, \ Z \in \mathcal{N} \end{aligned} \tag{SDP-0}$$

2. **Case** $r = 1$: Parrilo (2000), Bomze and De Klerk (2002) showed that $X \in \mathcal{K}^1$ if and only if following system of linear matrix inequalities (LMIs) has a solution, i.e. $\exists X \in \mathcal{S}, M^{(i)} \in \mathcal{S}, \ i = 1, \ldots, n$ such that

$$X - M^{(i)} \in \mathcal{S}^+, \quad i = 1, \ldots, n \tag{2a}$$
$$M^{(i)}_{ii} = 0, \quad i = 1, \ldots, n \tag{2b}$$
$$M^{(j)}_{ii} + 2M^{(i)}_{ij} = 0, \quad i \neq j \tag{2c}$$
$$M^{(i)}_{jk} + M^{(j)}_{ik} + M^{(k)}_{ij} \geq 0, \quad i < j < k \tag{2d}$$

This gives following SDP approximation (SDP-1) to (COP) corresponding to $\mathcal{K}^1$.

$$\begin{aligned} \vartheta_{\mathcal{K}^1} := \max_{X,M^{(i)}} \ & \langle C, X\rangle \\ \text{s.t.} \ & \langle A_i, X\rangle = b_i, \quad i = 1, \ldots, m \\ & X \in \mathcal{S}, \ M^{(i)} \in \mathcal{S}, \quad i = 1, \ldots, n \\ & (2a) - (2d) \end{aligned} \tag{SDP-1}$$

Obviously, $\vartheta_{\mathcal{K}^0} \leq \vartheta_{\mathcal{K}^1} \leq \vartheta^*$. Bomze and De Klerk (2002) remark that similar LMIs can be derived for $r \geq 2$ but the dimension d of psd cones increases rapidly with n as $d = \mathcal{O}(n^{r+2})$.

5.2 LP Inner Approximation

In this section, we use sufficient condition 3b from Fact 3 in section 4 but with $P^r_X(z)$ instead of $P_X(z)$. This again gives rise to hierarchy of cones $\mathcal{L}^r$ approximating $\mathcal{C}$ from inside defined by De Klerk and Pasechnik (2002), Bomze and De Klerk (2002) as

$$\mathcal{L}^r := \left\{ X \in \mathcal{S} : \ P^r_X(z) = P_X(z) \left(\sum_{k=1}^n z_k^2 \right)^r \text{ has non-negative coefficients.} \right\} \tag{3}$$

where $\mathcal{L}^r \subseteq \mathcal{L}^{r+1}$. Similar to results for $\mathcal{K}^r$, De Klerk and Pasechnik (2002), Bomze and De Klerk (2002) showed that: $\text{int}(C) \subseteq \bigcup_{r\in\mathbb{N}} \mathcal{L}^r$ i.e., for any strictly copositive matrix $X \notin \mathcal{L}^0$, $\exists r^* \geq 1$ such that

$$\mathcal{L}^0 \subset \mathcal{L}^1 \subset \ldots \subset \mathcal{L}^{r^*} \ni X$$

Lets see characterization of $\mathcal{L}^r$ for $r = 0, 1$.

1. **Case** $r = 0$: One can easily see that $\mathcal{L}^0 := \mathcal{N}$. Therefore we have following LP approximation (LP-0) to (COP) corresponding to $\mathcal{L}^0$.

$$\begin{aligned} \vartheta_{\mathcal{L}^0} := \max_{X}\ & \langle C, X\rangle \\ \text{s.t.}\ & \langle A_i, X\rangle = b_i, \quad i = 1,\ldots,m \\ & X \in \mathcal{N} \end{aligned} \tag{LP-0}$$

2. **Case** $r = 1$: Bomze and De Klerk (2002) showed that $X \in \mathcal{L}^1$ if and only if following system of linear matrix inequalities (LMIs) has a solution, i.e. $\exists X \in \mathcal{S}, M^{(i)} \in \mathcal{S},\ i = 1,\ldots,n$ such that

$$X - M^{(i)} \in \mathcal{N}, \quad i = 1,\ldots,n \tag{4a}$$

$$(2b) - (2d) \tag{4b}$$

This gives following LP approximation (LP-1) to (COP) corresponding to $\mathcal{L}^1$.

$$\begin{aligned} \vartheta_{\mathcal{L}^1} := \max_{X,M^{(i)}}\ & \langle C, X\rangle \\ \text{s.t.}\ & \langle A_i, X\rangle = b_i, \quad i = 1,\ldots,m \\ & X \in \mathcal{S},\ M^{(i)} \in \mathcal{S}, \quad i = 1,\ldots,n \\ & (4a) - (4b) \end{aligned} \tag{LP-1}$$

Obviously, $\vartheta_{\mathcal{L}^0} \leq \vartheta_{\mathcal{L}^1} \leq \vartheta^*$ and $\vartheta_{\mathcal{L}^0} \leq \vartheta_{\mathcal{K}^0}$. By using an alternative characterization of $\mathcal{K}^1$ and $\mathcal{L}^1$, Bomze and De Klerk (2002) showed $\mathcal{L}^1 \subset \mathcal{K}^1$. Thus, we also have $\vartheta_{\mathcal{L}^1} \leq \vartheta_{\mathcal{K}^1}$.

5.3 Simplex Partitions based Inner-Outer Approximation

Recall the Fact 4 from Section 4 but with 1-norm i.e. $\|y\|_1 = \sum_{i=1}^n y_i = 1$. The criteria for copositivity now becomes

$$X \in \mathcal{C} \iff y^\top X y \geq 0\ \forall y \in \Delta^S$$

where $\Delta^S = \{y \in \mathbb{R}^n : y_i \geq 0,\ \sum_{i=1}^n y_i = 1\}$ is the so-called standard simplex.

5.3.1 Sufficient condition for copositivity

Given any general simplex Δ with vertices set $\{v_1,\ldots,v_n\}$, then any $y \in \Delta$ can be represented as

$$y = \sum_{i=1}^n \lambda_i v_i \quad \text{where} \quad \sum_{i=1}^n \lambda_i = 1$$

which implies that

$$y^\top X y = \sum_{i,j=1}^{n} \lambda_i \lambda_j v_i^\top X v_j.$$

Since $\lambda_i \geq 0$, then a sufficient condition for $y^\top X y \geq 0 \; \forall y \in \Delta$ is that $v_i^\top X v_j \geq 0, \; \forall i, j$ [Lemma 2.1, Bundfuss and Dür (2009)]. Now lets take $\Delta := \Delta^S = \text{conv}\{e_1, \ldots, e_n\}$ which gives following sufficient condition for copositivity.

- If $e_i^\top X e_j \geq 0, \; \forall i, j$, then $X \in \mathcal{C}$. This is equivalent to $X \in \mathcal{N} \implies X \in \mathcal{C}$ which is well-known.

Now, lets see how we can improve on this well-known inner approximation.

Definition 5.1. *Simplical Partition. Let Δ be a simplex in $\mathbb{R}^n$. A family of simplices $\mathcal{P} = \{\Delta^1, \ldots, \Delta^m\}$ satisfying*

$$\Delta = \bigcup_{i=1}^{m} \Delta^i \quad \textit{and} \quad \textit{int}(\Delta^i) \cap \textit{int}(\Delta^j) = \emptyset \quad \textit{for } i \neq j$$

is called a simplical partition of Δ.

For example, given any $\Delta = \text{conv}\{v_1, \ldots, v_n\}$: let $w \in \Delta \setminus \{v_1, \ldots, v_n\}$ represented by

$$w = \sum_{i=1}^{n} \lambda_i v_i \quad \text{with} \quad \lambda_i \geq 0, \; \sum_{i=1}^{n} \lambda_i = 1$$

then for $\forall \lambda_i > 0$, define $\Delta^i = \text{conv}\{v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_n\}$. The collection of these Δ^i's provides a simplical partition of Δ.
For convenience, let $V_\mathcal{P}$ denote the set of all vertices of simplices in $\mathcal{P}$ and $E_\mathcal{P}$ the set of all edges of simplices in $\mathcal{P}$. If $\mathcal{P}$ is a simplical partition of Δ^S then from Theorem 2.3 Bundfuss and Dür (2009) we have another sufficient condition for copositivity,

- If $u^\top X v \geq 0, \; \forall \{u, v\} \in E_\mathcal{P}$ and $v^\top X v \geq 0, \; \forall v \in V_\mathcal{P}$ then $X \in \mathcal{C}$.

5.3.2 Inner Approximation

For given partition $\mathcal{P}$ of Δ^S, Bundfuss and Dür (2009) defines the following polyhedral cone approximating $\mathcal{C}$ from inside

$$\mathcal{I}_\mathcal{P} = \{X \in \mathcal{S} : u^\top X v \geq 0, \; \forall \{u, v\} \in E_\mathcal{P} \text{ and } v^\top X v \geq 0, \; \forall v \in V_\mathcal{P}\}$$

Definition 5.2. *Refinement. Let $\mathcal{P}_1, \mathcal{P}_2$ be two simplical partitions of the same simplex. $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ if for all $\Delta^i \in \mathcal{P}_1$ there exists $\mathcal{P}_{\Delta^i} \subseteq \mathcal{P}_2$ such that $\mathcal{P}_{\Delta^i}$ is a simplical partition of Δ^i.*

From Lemma 2.1(c) Bundfuss and Dür (2009), it follows that for $\mathcal{P}_1, \mathcal{P}_2$ denoting the simplical partitions of Δ^S: if $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ then $\mathcal{I}_{\mathcal{P}_1} \subseteq \mathcal{I}_{\mathcal{P}_2}$. Let $\{\mathcal{P}_\ell\}$ be sequence of partitions of Δ^S, then we have heirarchy of cones $\mathcal{I}_{\mathcal{P}_\ell}$ given as $\mathcal{I}_{\mathcal{P}_\ell} \subset \mathcal{I}_{\mathcal{P}_{\ell+1}}$ where $\mathcal{P}_0 = \Delta^S$ and therefore $\mathcal{I}_{\mathcal{P}_0} = \mathcal{N}$. Bundfuss and Dür (2009) further showed that if diameter $\delta(\mathcal{P}_\ell) \to 0$ then $\mathrm{int}(C) \subseteq \bigcup_{\ell\in\mathbb{N}} \mathcal{I}_{\mathcal{P}_\ell}$ where $\delta(\mathcal{P}) = \max_{\{u,v\}\in E_{\mathcal{P}}} \|u - v\|$. This gives following inner LP Approximation (ILP$_\ell$) to (COP).

$$\begin{aligned} \vartheta_{\mathcal{I}_{\mathcal{P}_\ell}} := \max_X \;& \langle C, X\rangle \\ \text{s.t.} \;& \langle A_i, X\rangle = b_i, \quad i = 1, \dots, m \\ & X \in \mathcal{I}_{\mathcal{P}_\ell} \end{aligned} \tag{ILP$_\ell$}$$

5.3.3 Outer Approximation

An outer approximation to $\mathcal{C}$ can be derived by observing the following necessary condition. Let $\mathcal{P}$ be a simplical partition of Δ^S.

- If $X \in \mathcal{C}$, then $v^\top X v \geq 0$, $\forall\ v \in V_{\mathcal{P}}$.

Again let $\{\mathcal{P}_\ell\}$ be sequence of partitions of Δ^S, Bundfuss and Dür (2009) defines the hierarchy of polyhedral cones

$$\mathcal{O}_{\mathcal{P}_\ell} = \{X \in \mathcal{S} : v^\top X v \geq 0,\ \forall v \in V_{\mathcal{P}_\ell}\}$$

where $\mathcal{O}_{\mathcal{P}_\ell} \supset \mathcal{O}_{\mathcal{P}_{\ell+1}}$. Similar to inner approximation, they showed that if $\delta(\mathcal{P}_\ell) \to 0$ then $\mathcal{C} = \bigcap_{\ell\in\mathbb{N}} \mathcal{O}_{\mathcal{P}_\ell}$. This gives following outer LP Approximation (OLP$_\ell$) to (COP).

$$\begin{aligned} \vartheta_{\mathcal{O}_{\mathcal{P}_\ell}} := \max_X \;& \langle C, X\rangle \\ \text{s.t.} \;& \langle A_i, X\rangle = b_i, \quad i = 1, \dots, m \\ & X \in \mathcal{O}_{\mathcal{P}_\ell} \end{aligned} \tag{OLP$_\ell$}$$

Obviously, $\vartheta_{\mathcal{I}_{\mathcal{P}_\ell}} \leq \vartheta^* \leq \vartheta_{\mathcal{O}_{\mathcal{P}_\ell}}$.

5.3.4 Implementation & Key Features

Implementation:

Algorithm 1: ϵ-Approximation Algorithm for (COP).

Initialize: $\ell = 0,\ \mathcal{P}_0 = \Delta^S$.

Result: $X_\ell^{\mathcal{I}}$

while *True* **do**

  Solve (ILP$_\ell$), (OLP$_\ell$) using $\mathcal{P}_\ell$ and obtain respective optimal solutions $X_\ell^{\mathcal{I}}$, $X_\ell^{\mathcal{O}}$;

  tol $= \frac{\langle C, X_\ell^{\mathcal{O}}\rangle - \langle C, X_\ell^{\mathcal{I}}\rangle}{1+|\langle C, X_\ell^{\mathcal{O}}\rangle| + |\langle C, X_\ell^{\mathcal{I}}\rangle|}$;

  if $tol < \epsilon$ **then**

    STOP: $X_\ell^{\mathcal{I}}$ is an ϵ-optimal solution of (COP).

  end

  Refine $\mathcal{P}_\ell$ to $\mathcal{P}_{\ell+1}$;

  $\ell \leftarrow \ell + 1$

end

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1. The *Refine* step in the algorithm essentially involves first choosing a $\Delta \in \mathcal{P}_\ell$ and bisecting $\Delta = \Delta^1 \cup \Delta^2$. Then $\mathcal{P}_{\ell+1} \leftarrow \mathcal{P}_\ell \setminus \{\Delta\} \cup \{\Delta^1, \Delta^2\}$.

2. Since *Refine* step provides some flexibility, Bundfuss and Dür (2009) suggests to choose a longest active edge $\{u, v\} \in E_{\mathcal{P}_\ell}$, i.e. choose u, v such that $u^\top X_\ell^{\mathcal{I}} v = 0$ in (ILP$_\ell$). As Bundfuss and Dür (2009) prove that there always exist an active edge, this implies there is hope of improving the objective value by bisecting on such an edge. This actually help adapt the approximation according to objective function.

3. An edge $\{u, v\} \in E_{\mathcal{P}_\ell}$ chosen for bisection can be a part of more than one simplices. In such a case, all simplices containing the chosen edge must be bisected. Otherwise, objective value wouldn't improve in the subsequent iteration.

Key features:

1. The number of decision variables in linear subproblems (ILP$_\ell$), (OLP$_\ell$) is constant as compared to SDP approximations: (SDP-0), (SDP-1) and LP approximations: (LP-0), (LP-1).

2. The inner and outer approximation can be guided through objective function as compared to LP and SDP approximations (discussed in previous sections) which approximate $\mathcal{C}$ uniformly.

6 Standard QP via COP

The problem of minimizing a homogeneous quadratic function over a standard simplex is the standard QP (SQP) problem and is known to be NP-hard.

$$\begin{aligned} \vartheta_{\text{SQP}} := \min\ & x^\top Q x \\ \text{s.t.}\ \ & e^\top x = 1,\ x \geq 0 \end{aligned} \tag{SQP}$$

where $Q \in \mathcal{S}$ and e is the vector of all ones in $\mathbb{R}^n$. Denote by $E = ee^\top$ the matrix of all ones. To avoid trivial cases, assume Q and E are linearly independent so that the objective function is not constant over the considered feasible region. Note that (SQP) covers minimizing a non-homogeneous quadratic function $f(x) = x^\top Ax + 2c^\top x$ over the standard simplex by setting $Q := A + ec^\top + ce^\top$.

$$\begin{aligned}
\vartheta_{\text{SQP}} &= \min \left\{ x^\top Qx \;\text{ s.t. } e^\top x = 1,\; x \geq 0 \right\} \\
&= \min \left\{ \operatorname{Tr}(QX) \;\text{ s.t. } X = xx^\top,\; e^\top x = 1,\; x \geq 0 \right\} \\
&= \min \left\{ \operatorname{Tr}(QX) \;\text{ s.t. } X = xx^\top,\; \operatorname{Tr}(EX) = 1,\; x \geq 0 \right\} \\
&\quad \left(\text{For any feasible } \hat{x}, \hat{X},\; \operatorname{Tr}(E\hat{X}) = (e^\top \hat{x})^2 = 1,\; \because \hat{x} \geq 0 \implies e^\top \hat{x} = 1 \right) \\
&\geq \min \left\{ \operatorname{Tr}(QX) \;\text{ s.t. } \operatorname{Tr}(EX) = 1,\; X \in \mathcal{C}^* \right\} \qquad \text{(CPP)} \\
&\quad \left(\because \mathcal{C}^* = \operatorname{conv}\{xx^\top : x \geq 0\} \right)
\end{aligned}$$

One can easily prove that last inequality will hold with equality. Say x^* is an optimal solution to (SQP) and X^* optimal to (CPP). Then

$$x^{*\top} Qx^* \geq \operatorname{Tr}(QX^*) \tag{5}$$

From the feasibility requirements of (CPP)

$$X^* = \sum_i x_i x_i^\top, \quad \operatorname{Tr}(EX^*) = \sum_i (e^\top x_i)^2 = 1 \;\text{ where } x_i \gneqq 0 \;\forall i$$

For each i define $u_i = \frac{1}{e^\top x_i} x_i$ then u_i is feasible to (SQP). This further implies

$$x^{*\top} Qx^* \leq u_i^\top Qu_i, \forall i \implies x^{*\top} Qx^* \sum_i (e^\top x_i)^2 \leq \sum_i x_i^\top Qx_i \implies x^{*\top} Qx^* \leq \operatorname{Tr}(QX^*) \tag{6}$$

Combining (5) and (6) we can conclude

$$\begin{aligned}
\vartheta_{\text{SQP}} = \min\; & \operatorname{Tr}(QX) \\
\text{s.t.}\; & \operatorname{Tr}(EX) = 1,\; X \in \mathcal{C}^*
\end{aligned} \tag{CPP-SQP}$$

The dual of above (CPP-SQP) is given by the following COP

$$\begin{aligned}
&\max \lambda \\
&\text{s.t. } \; Q - \lambda E \in \mathcal{C},\; \lambda \in \mathbb{R}
\end{aligned} \tag{COP-SQP}$$

One can observe that (COP-SQP) has a feasible interior point by taking $\lambda < 0$ and $|\lambda|$ large enough so that $Q - \lambda E \in \mathcal{N}$. Less obvious is to find a feasible interior point of (CPP-SQP). Proposition 1 Bomze et al. (2000) shows the following cone $\mathcal{C}_+^*$ is strictly smaller than $\mathcal{C}^*$.

$$\mathcal{C}_+^* = \{X \in \mathcal{S}^+ : X^{1/2} \text{ has no negative entries}\}$$

By taking $\hat{X} = \frac{1}{n} I \in \mathcal{C}_+^*$ in (CPP-SQP), we can claim strong duality between (CPP-SQP) and (COP-SQP) using Slater's condition.

6.1 Numerical Experiments

6.1.1 Maximum stable set problem

From De Klerk and Pasechnik (2002) and references therein, the problem of finding maximum stable set number $\alpha(G)$ of graph G with adjacency matrix A can be cast as the following SQP.

$$\begin{aligned} \frac{1}{\alpha(G)} &:= \min\ x^\top (A+I)x \\ &\quad \text{s.t.}\quad e^\top x = 1,\ x \geq 0 \end{aligned} \tag{7}$$

By taking $Q := A + I$ in (COP-SQP), we have the equivalent COP reformulation. We run experiments to test the strength and computational efficiency of various approximations of $\mathcal{C}$ discussed in section 5.

6.1.2 Models from literature

- **Maximum Stable Set:** We choose following adjacency matrices for the numerical experiments. Matrix Q_1 corresponds to pentagon. Matrix Q_2 is a triangle with one of the vertices connected additionally to two isolated nodes. The optimal value of corresponding optimization problems are $\frac{1}{2}$ and $\frac{1}{3}$ respectively or maximum stable set numbers are 2 and 3 respectively.

$$Q_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

- **Population Genetics:** The following matrix arises in population genetics Bomze and De Klerk (2002). The corresponding optimal value is $-16\frac{1}{3}$.

$$Q_3 = \begin{pmatrix} -14 & -15 & -16 & 0 & 0 \\ -15 & -14 & -12.5 & -22.5 & -15 \\ -16 & -12.5 & -10 & -26.5 & -16 \\ 0 & -22.5 & -26.5 & 0 & 0 \\ 0 & -15 & -16 & 0 & -14 \end{pmatrix}$$

- **Portfolio Optimization:** This matrix deals with portfolio optimization Bomze and De Klerk (2002). Its optimal value is 0.4839.

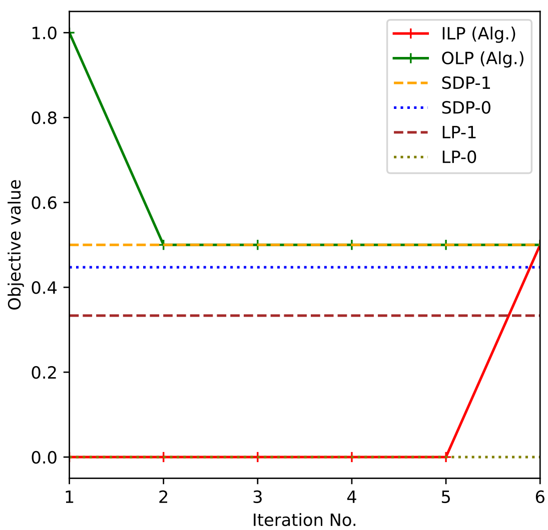
$$Q_4 = \begin{pmatrix} 0.9044 & 0.1054 & 0.5140 & 0.3322 & 0 \\ 0.1054 & 0.8715 & 0.7385 & 0.5866 & 0.9751 \\ 0.5140 & 0.7385 & 0.6936 & 0.5368 & 0.8086 \\ 0.3322 & 0.5866 & 0.5368 & 0.5633 & 0.7478 \\ 0 & 0.9751 & 0.8086 & 0.7478 & 1.2932 \end{pmatrix}$$

In Figures 2-4, we compare various approximations discussed in section 5 after running computations on all the four matrices mentioned above. Note all these matrices are of $n = 5$. The following observations can be made:

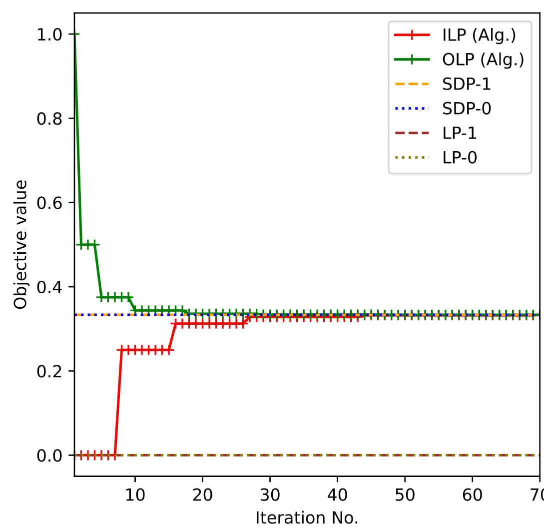
1. (SDP-0) provides a very tight approximation for all cases except for Q_1. For Q_1 (the case of pentagon), the objective value provided by (SDP-0) is 0.4472 whereas the optimal value is 0.5. Since (SDP-1) provides the optimal objective value, this example demonstrates the strict containment i.e., $\mathcal{S}^+ + \mathcal{N} \subset \mathcal{C}$ for $n = 5$ as stated in Fact 2.

2. For all cases, both (LP-0) and (LP-1) provides a very weak bound, whereas the LP approximation using simplex partitions (ILP$_\ell$), (OLP$_\ell$) closes the gap in reasonable time.

3. Instance corresponding to Q_2 is harder to solve using LP approximation. Both (LP-0), (LP-1) give trivial lower bound of 0 and using simplex partition algorithm we couldn't achieve tolerance beyond 3×10^{-4}.



(a) Q_1 (Alg. tol=10^{-6})



(b) Q_2 (Alg. tol=3×10^{-4})

Figure 2: Comparing different approximations for Maximum Stable Set.

6.1.3 Randomly generated matrices

For each $n \in \{5, 10, 15, 20\}$, we generate 10 random instances of $Q \in \mathbb{R}^{n \times n}$ where every entry Q_{ij} is sampled uniformly over the interval $[-n, n]$. We run all the three approaches discussed and compare their average runtimes with increasing value of n. The results are shown in Figure 5. The following observations are immediate

- The SDP approximation takes longer time as compared to the LP approximations.

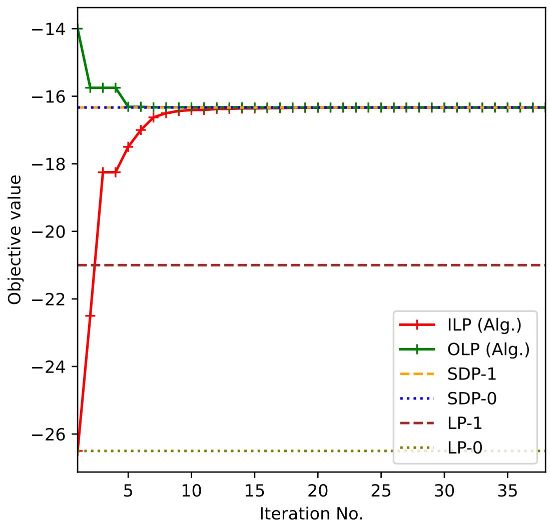


Figure 3: Comparing different approximations for Q_3 (Alg. tol=10^{-6}).

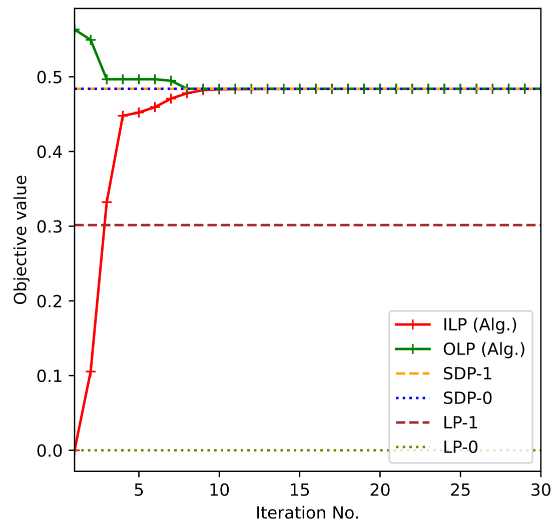


Figure 4: Comparing different approximations for Q_4 (Alg. tol=10^{-6}).

- The runtime for the simplex partitions algorithm increases only slightly as compared to the LP-1 approximation over the considered values of n. This highlight the advantage of partitioning approach as the # of decision variables remains constant.
- In terms of objective value, we have similar observations as discussed in the previous section.

7 Extensions/Ongoing Research

In this section, some possible extensions alongwith ongoing research are discussed.

1. We can think about copositivity over general convex domains. Several useful convex domains for practical applications such as in robust optimization are polyhedron, second-order cones. Recent work by Tian et al. (2013), Lu et al. (2014) provide adaptive computable approximation of non-negative quadratic forms over second order cones.
2. Since copositive cones are hard to deal in general, we can focus on developing application-specific solution approaches. The hope is tighter approximations of $\mathcal{C}$ can be obtained by utilizing valid inequalities derived using problem structure.
3. Another possible route is to bypass the copositive reformulation and deal with the semi-infinite constraints by making additional assumptions as done by Zhen et al. (2022) in the context of robust optimization

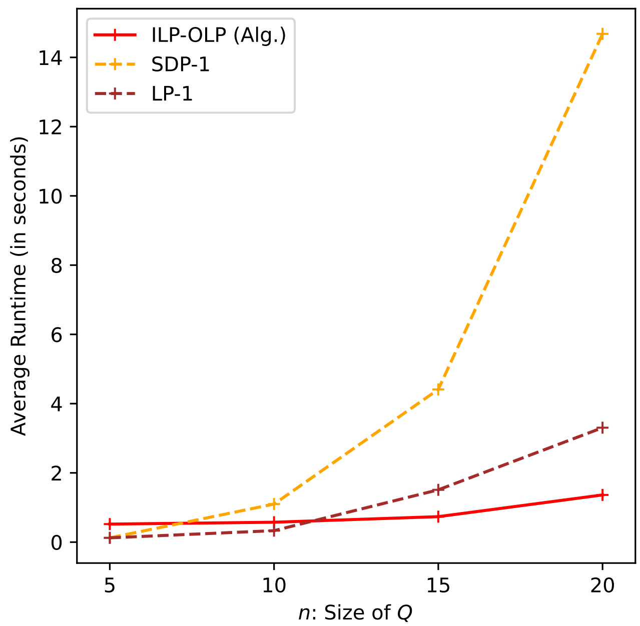


Figure 5: Comparing runtimes (Alg. tol=10^{-6}).

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