Approximation of Copostive Cones

Akshit Goyal

Department of ISyE, University of Minnesota.

IE8534 Course Project

Outline

1. **Introduction**
2. **Facts about $\mathcal{C}$ and $\mathcal{C}^*$**
3. **Problem & Approximations**
   - SDP Inner Approximation
   - LP Inner Approximation
   - Simplex Partitions based Inner-Outer Approximation
4. **Standard Quadratic Program (SQP)**
   - Exact Reformulation as a COP
   - Numerical Experiments
     - Models from literature
     - Randomly generated matrices
5. **Extensions/Ongoing Research**

Introduction I

- A copositive cone $\mathcal{C}$ is defined as

$$\mathcal{C} = \{X \in \mathcal{S} : y^\top X y \geq 0 \ \forall y \in \mathbb{R}^n_+\}$$

where $\mathcal{S} \subset \mathbb{R}^{n \times n}$ is the set of symmetric matrices.

- The dual cone to $\mathcal{C}$ is the cone of completely positive matrices $\mathcal{C}^*$ defined as

$$\mathcal{C}^* = \mathrm{conv}\{yy^\top : y \in \mathbb{R}^n_+\}$$

- Both $\mathcal{C}$ and $\mathcal{C}^*$ are closed, convex, pointed, full dimensional, nonpolyhedral cones.
- In literature, several NP-hard problems shown to have completely positive (or equivalently copositive) representation:

Introduction II

- standard quadratic programs (QPs) Bomze et al. (2000), Bomze and De Klerk (2002),
- maximum stable set problem De Klerk and Pasechnik (2002),
- graph partitioning problem Povh and Rendl (2007),
- quadratic assignment problem Povh and Rendl (2009), and many more.
- Burer (2009) showed that any QP with binary variables and linear constraints also have such a representation.

- In this talk, discuss polynomially computable approximations of the cone of copositive matrices, $\mathcal{C}$.

Notations

- $\mathcal{S}^+$: Cone of symmetric positive semidefinite (psd) matrices.

$$\mathcal{S}^+ = \{X \in \mathcal{S} : y^\top X y \geq 0 \ \forall y \in \mathbb{R}^n\}$$

- $\mathcal{N}$: Cone of symmetric non-negative matrices.

$$\mathcal{N} = \{X \in \mathcal{S} : X_{ij} \geq 0 \ \forall i, j = 1, \ldots, n\}$$

- $\mathcal{D}$: Cone of symmetric doubly nonnegative matrices.

$$\mathcal{D} = \mathcal{S}^+ \cap \mathcal{N}$$

Outline

Facts about $\mathcal{C}$ and $\mathcal{C}^*$

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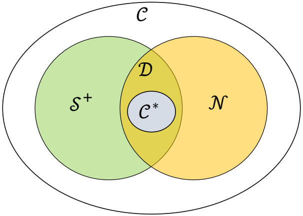

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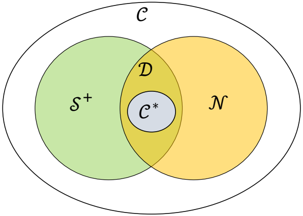
    - Berman and Shaked-Monderer (2003) for $n \times n$ matrices with $n \leq 4$, equality in above relations i.e., $\mathcal{C} = \mathcal{S}^+ + \mathcal{N}$ and $\mathcal{C}^* = \mathcal{D} = \mathcal{S}^+ \cap \mathcal{N}$. But for $n \geq 5$, the inclusions are strict in above relations.



Facts about $\mathcal{C}$ and $\mathcal{C}^*$

3. **Fact 3:** Criteria for copositivity can be restated as following.

$$X \in \mathcal{C} \iff P_X(z) = \sum_{i,j=1}^{n} z_i^2 X_{ij} z_j^2 \geq 0 \;\forall z \in \mathbb{R}^n$$

$P_X(z)$: homogeneous polynomial of degree 4. Provides some sufficient conditions for inner approximating $\mathcal{C}$.

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B. If $P_X(z)$ has non-negative coefficients, then $X \in \mathcal{C}$.
C. Higher order sufficient conditions by replacing $P_X(z)$ with $P_X^r(z)$ where

$$P_X^r(z) = P_X(z) \left(\sum_{k=1}^{n} z_k^2 \right)^r .$$

Facts about $\mathcal{C}$ and $\mathcal{C}^*$

4. **Fact 4:** Another criteria for copositivity is

$$X \in \mathcal{C} \iff y^\top X y \geq 0 \ \forall y \in \mathbb{R}^n_+ \text{ with } \|y\| = 1.$$

Facts about $\mathcal{C}$ and $\mathcal{C}^*$

4. **Fact 4:** Another criteria for copositivity is

$$X \in \mathcal{C} \iff y^\top X y \geq 0 \ \forall y \in \mathbb{R}^n_+ \text{ with } \|y\| = 1.$$

From Facts 1-2, restating a problem as a conic optimization over cones $\mathcal{C}$ and $\mathcal{C}^*$ does not resolve the difficulty of that problem.
Facts 3-4 give rise to different polynomially computable inner approximations of $\mathcal{C}$.

Outline

Problem

Often referred to as the dual problem[1], the **copositive program** is a standard conic optimization over $\mathcal{C}$, i.e.

$$\begin{aligned} \vartheta^* := \max\ & \langle C, X \rangle \\ \text{s.t.}\ & \langle A_i, X \rangle = b_i, \quad i = 1, \dots, m \\ & X \in \mathcal{C} \end{aligned} \tag{COP}$$

where $C \in \mathbb{R}^{n \times n}$, $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}$. Clearly, the difficulty of (COP) lies in the copositive cone constraint.

[1] Because COP is dual to the completely positive program (CPP) representation of original NP-hard problems.

Outline

SDP Inner Approximation I

- Using sufficient condition A from Fact 3 but with $P_X^r(z)$, Parrilo (2000) defines a hierarchy of cones $\mathcal{K}^r$ inner approximating $\mathcal{C}$

$$\mathcal{K}^r := \left\{ X \in \mathcal{S} : \; P_X^r(z) = P_X(z) \left(\sum_{k=1}^{n} z_k^2 \right)^r \text{ has an s.o.s. decomposition.} \right\} \tag{1}$$

where $\mathcal{K}^r \subseteq \mathcal{K}^{r+1}$.

- Hierarchy follows, if $X \in \mathcal{K}^r \; \exists \; f_\ell(z)$ such that $P_X^r(z) = \sum_\ell f_\ell(z)^2$ then

$$P_X^{r+1}(z) = P_X^r(z) \left(\sum_{k=1}^{n} z_k^2 \right) = \sum_{\ell,k} f_\ell(z)^2 z_k^2 \text{ or } \sum_{\ell,k} [f_\ell(z) z_k]^2.$$

Thus, if $X \in \mathcal{K}^r$ then $X \in \mathcal{K}^{r+1}$.

SDP Inner Approximation II

- Bomze and De Klerk (2002) provides complete characterization of $\mathcal{K}^r$ for any given r.
  1. **Case** $r = 0$: Parrilo (2000) showed that $X \in \mathcal{K}^0$ iff $\mathcal{K}^0 := \mathcal{S}^+ + \mathcal{N}$. Therefore have following SDP approximation (SDP-0) to (COP).

$$\begin{aligned} \vartheta_{\mathcal{K}^0} := \max_{X,Y,Z} \ & \langle C, X \rangle \\ \text{s.t.} \ & \langle A_i, X \rangle = b_i, \quad i = 1, \ldots, m \\ & X = Y + Z, \ Y \in \mathcal{S}^+, \ Z \in \mathcal{N} \end{aligned} \tag{SDP-0}$$

SDP Inner Approximation III

2. **Case** $r = 1$: Parrilo (2000), Bomze and De Klerk (2002) showed that $X \in \mathcal{K}^1$ iff $\exists X \in \mathcal{S}, M^{(i)} \in \mathcal{S},\ i = 1, \dots, n$ such that

$$X - M^{(i)} \in \mathcal{S}^+, \quad i = 1, \dots, n \tag{2a}$$

$$M^{(i)}_{ii} = 0, \quad i = 1, \dots, n \tag{2b}$$

$$M^{(j)}_{ii} + 2M^{(i)}_{ij} = 0, \quad i \neq j \tag{2c}$$

$$M^{(i)}_{jk} + M^{(j)}_{ik} + M^{(k)}_{ij} \geq 0, \quad i < j < k \tag{2d}$$

This gives following SDP approximation (SDP-1) to (COP).

$$\begin{aligned} \vartheta_{\mathcal{K}^1} := \max_{X, M^{(i)}} \ & \langle C, X \rangle \\ \text{s.t.} \ & \langle A_i, X \rangle = b_i, \quad i = 1, \dots, m \\ & X \in \mathcal{S},\ M^{(i)} \in \mathcal{S}, \quad i = 1, \dots, n \\ & \text{(2a)} - \text{(2d)} \end{aligned} \tag{SDP-1}$$

Outline

LP Inner Approximation I

- Using sufficient condition B from Fact 3 but with $P_X^r(z)$, De Klerk and Pasechnik (2002) defines a hierarchy of cones $\mathcal{L}^r$ inner approximating $\mathcal{C}$,

$$\mathcal{L}^r := \left\{ X \in \mathcal{S} : \; P_X^r(z) = P_X(z) \left(\sum_{k=1}^{n} z_k^2 \right)^r \text{ has non-negative coefficients.} \right\} \tag{3}$$

  where $\mathcal{L}^r \subseteq \mathcal{L}^{r+1}$.
- Similar to $\mathcal{K}^r$, lets see characterization of $\mathcal{L}^r$ for $r = 0, 1$.

LP Inner Approximation II

1. **Case** $r = 0$: One can easily see that $\mathcal{L}^0 := \mathcal{N}$. Therefore have following LP approximation (LP-0) to (COP).

$$\begin{aligned} \vartheta_{\mathcal{L}^0} := \max_X \ & \langle C, X \rangle \\ \text{s.t.} \ & \langle A_i, X \rangle = b_i, \quad i = 1, \ldots, m \\ & X \in \mathcal{N} \end{aligned} \tag{LP-0}$$

2. **Case** $r = 1$: Bomze and De Klerk (2002) showed that $X \in \mathcal{L}^1$ iff $\exists X \in \mathcal{S}, M^{(i)} \in \mathcal{S}, \ i = 1, \ldots, n$ such that

$$X - M^{(i)} \in \mathcal{N}, \quad i = 1, \ldots, n \tag{4a}$$

$$\text{(2b)} - \text{(2d)} \tag{4b}$$

LP Inner Approximation III

This gives following LP approximation (LP-1) to (COP).

$$\begin{aligned} \vartheta_{\mathcal{L}^1} := \max_{X, M^{(i)}} \ & \langle C, X \rangle \\ \text{s.t.} \ & \langle A_i, X \rangle = b_i, \quad i = 1, \ldots, m \\ & X \in \mathcal{S}, \ M^{(i)} \in \mathcal{S}, \quad i = 1, \ldots, n \\ & (4a) - (4b) \end{aligned} \tag{LP-1}$$

Outline

Simplex Partition: Inner/Outer Approximation

- Using Fact 4 with 1-norm gives the criteria for copositivity

$$X \in \mathcal{C} \iff y^\top X y \geq 0 \ \forall y \in \Delta^S$$

where $\Delta^S = \{y \in \mathbb{R}^n : y_i \geq 0, \ \sum_{i=1}^n y_i = 1\}$ is the standard simplex.

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- Given general simplex Δ with vertices $\{v_1, \ldots, v_n\}$. For any $y \in \Delta$,

$$y = \sum_{i=1}^n \lambda_i v_i \quad \text{where} \quad \sum_{i=1}^n \lambda_i = 1$$

which implies that

$$y^\top X y = \sum_{i,j=1}^n \lambda_i \lambda_j v_i^\top X v_j.$$

Since $\lambda_i \geq 0$, then a sufficient condition for $y^\top X y \geq 0 \ \forall y \in \Delta$ is that $v_i^\top X v_j \geq 0, \ \forall i, j$ [Lemma 2.1, Bundfuss and Dür (2009)].

Simplex Partition: Inner/Outer Approximation

- Now lets take $\Delta := \Delta^S = \text{conv}\{e_1, \ldots, e_n\}$ which gives following sufficient condition for copositivity.
  If $e_i^\top X e_j \geq 0,\ \forall i, j$, then $X \in \mathcal{C}$. $\equiv$ 'well-known' $X \in \mathcal{N} \implies X \in \mathcal{C}$.

Now, lets see how we can improve on this well-known inner approximation.

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Now, lets see how we can improve on this well-known inner approximation.

Definition

Simplical Partition. Let Δ be a simplex in $\mathbb{R}^n$. A family of simplices $\mathcal{P} = \{\Delta^1, \ldots, \Delta^m\}$ satisfying

$$\Delta = \bigcup_{i=1}^{m} \Delta^i \quad \text{and} \quad \text{int}(\Delta^i) \cap \text{int}(\Delta^j) = \emptyset \quad \text{for } i \neq j$$

is called a simplical partition of Δ.

Simplex Partition: Inner/Outer Approximation

- let $V_{\mathcal{P}}$ denote the set of all vertices of simplices in $\mathcal{P}$ and $E_{\mathcal{P}}$ the set of all edges of simplices in $\mathcal{P}$.

Simplex Partition: Inner/Outer Approximation

- let $V_{\mathcal{P}}$ denote the set of all vertices of simplices in $\mathcal{P}$ and $E_{\mathcal{P}}$ the set of all edges of simplices in $\mathcal{P}$.
- If $\mathcal{P}$ is a simplical partition of Δ^S then Theorem 2.3 Bundfuss and Dür (2009) gives another sufficient condition for copositivity,
  If $u^\top X v \geq 0, \ \forall \{u, v\} \in E_{\mathcal{P}}$ and $v^\top X v \geq 0, \ \forall v \in V_{\mathcal{P}}$ then $X \in \mathcal{C}$.

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- For given partition $\mathcal{P}$ of Δ^S, Bundfuss and Dür (2009) defines the following polyhedral cone **inner approximating** $\mathcal{C}$

$$\mathcal{I}_{\mathcal{P}} = \{X \in \mathcal{S} : u^\top X v \geq 0, \ \forall \{u, v\} \in E_{\mathcal{P}} \text{ and } v^\top X v \geq 0, \ \forall v \in V_{\mathcal{P}}\}$$

Simplex Partition: Inner/Outer Approximation

Definition

Refinement. Let $\mathcal{P}_1, \mathcal{P}_2$ be two simplical partitions of the same simplex. $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ if for all $\Delta^i \in \mathcal{P}_1$ there exists $\mathcal{P}_{\Delta^i} \subseteq \mathcal{P}_2$ such that $\mathcal{P}_{\Delta^i}$ is a simplical partition of Δ^i.

Simplex Partition: Inner/Outer Approximation

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- From Lemma 2.1(c) Bundfuss and Dür (2009), it follows that for $\mathcal{P}_1, \mathcal{P}_2$ denoting the simplical partitions of Δ^S: if $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ then $\mathcal{I}_{\mathcal{P}_1} \subseteq \mathcal{I}_{\mathcal{P}_2}$.

Simplex Partition: Inner/Outer Approximation

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- Let $\{\mathcal{P}_\ell\}$ be sequence of partitions of Δ^S, then there is heirarchy of cones $\mathcal{I}_{\mathcal{P}_\ell} \subset \mathcal{I}_{\mathcal{P}_{\ell+1}}$ with $\mathcal{P}_0 = \Delta^S$ and $\mathcal{I}_{\mathcal{P}_0} = \mathcal{N}$.

Simplex Partition: Inner/Outer Approximation

- This gives following inner LP Approximation (ILP$_\ell$) to (COP).

$$\begin{aligned} \vartheta_{\mathcal{I}_{\mathcal{P}_\ell}} := \max_X \; & \langle C, X \rangle \\ \text{s.t.} \; & \langle A_i, X \rangle = b_i, \quad i = 1, \ldots, m \\ & X \in \mathcal{I}_{\mathcal{P}_\ell} \end{aligned} \tag{ILP$_\ell$}$$

Simplex Partition: Inner/Outer Approximation

- An outer approximation to $\mathcal{C}$ can be derived by observing the following necessary condition given $\mathcal{P}$ a simplical partition of Δ^S.

$$\text{If } X \in \mathcal{C}, \text{ then } v^\top X v \geq 0, \; \forall \; v \in V_{\mathcal{P}}.$$

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- Again let $\{\mathcal{P}_\ell\}$ be sequence of partitions of Δ^S, Bundfuss and Dür (2009) defines the hierarchy of polyhedral cones

$$\mathcal{O}_{\mathcal{P}_\ell} = \{X \in \mathcal{S} : v^\top X v \geq 0,\ \forall v \in V_{\mathcal{P}_\ell}\}$$

  where $\mathcal{O}_{\mathcal{P}_\ell} \supset \mathcal{O}_{\mathcal{P}_{\ell+1}}$.

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- This gives following outer LP Approximation (OLP$_\ell$) to (COP).

$$\begin{aligned} \vartheta_{\mathcal{O}_{\mathcal{P}_\ell}} := \max_X \ & \langle C, X \rangle \\ \text{s.t.} \ & \langle A_i, X \rangle = b_i, \quad i = 1, \ldots, m \\ & X \in \mathcal{O}_{\mathcal{P}_\ell} \end{aligned} \tag{OLP$_\ell$}$$

Approximation Algorithm

Algorithm 1: ϵ-Inner/Outer Approximation Algorithm.

Initialize: $\ell = 0,\ \mathcal{P}_0 = \Delta^S$.

Result: $X_\ell^{\mathcal{I}}$

while *True* **do**

  Solve (ILP$_\ell$), (OLP$_\ell$) using $\mathcal{P}_\ell$ and obtain respective optimal solutions $X_\ell^{\mathcal{I}}$, $X_\ell^{\mathcal{O}}$;

  tol $= \frac{\langle C, X_\ell^{\mathcal{O}} \rangle - \langle C, X_\ell^{\mathcal{I}} \rangle}{1 + |\langle C, X_\ell^{\mathcal{O}} \rangle| + |\langle C, X_\ell^{\mathcal{I}} \rangle|}$;

  if $tol < \epsilon$ **then**

    STOP: $X_\ell^{\mathcal{I}}$ is an ϵ-optimal solution of (COP).

  end

  Refine $\mathcal{P}_\ell$ to $\mathcal{P}_{\ell+1}$;

  $\ell \leftarrow \ell + 1$

end

Outline

Why Standard QP $\equiv$ COP? I

- Consider (NP-hard) standard QP problem

$$\begin{aligned} \vartheta^*_{\mathsf{SQP}} := \min\ & x^\top Q x \\ \text{s.t.}\ \ & e^\top x = 1,\ x \geq 0 \end{aligned} \tag{SQP}$$

  where $Q \in \mathcal{S}$ and e is vector of all ones. Denote by $E = ee^\top$.

- One can prove that

$$\begin{aligned} \vartheta^*_{\mathsf{SQP}} := \min\ & \mathrm{Tr}(QX) \\ \text{s.t.}\ \ & \mathrm{Tr}(EX) = 1,\ X \in \mathcal{C}^* \end{aligned} \tag{CPP-SQP}$$

  The dual of above (CPP-SQP) is given by the following COP

$$\begin{aligned} \max\ & \lambda \\ \text{s.t.}\ \ & Q - \lambda E \in \mathcal{C},\ \lambda \in \mathbb{R} \end{aligned} \tag{COP-SQP}$$

Why Standard QP $\equiv$ COP? II

- Observe that (COP-SQP) has a feasible interior point by taking $\lambda < 0$ and $|\lambda|$ large enough so that $Q - \lambda E \in \mathcal{N}$.
- Less obvious is to find a feasible interior point of (CPP-SQP). Proposition 1 Bomze et al. (2000) shows the following cone $\mathcal{C}_+^*$ is strictly smaller than $\mathcal{C}^*$.

$$\mathcal{C}_+^* = \{X \in \mathcal{S}^+ : X^{1/2} \text{ has no negative entries}\}$$

By taking $\hat{X} = \frac{1}{n} I \in \mathcal{C}_+^*$ in (CPP-SQP), we can claim strong duality between (CPP-SQP) and (COP-SQP) using Slater's condition.

Numerical Experiments

1. Maximum stable set problem, Population genetics, Portfolio optimization
2. Randomly generated matrices

Maximum stable set problem

- De Klerk and Pasechnik (2002) The maximum stable set number $\alpha(G)$ of graph G with adjacency matrix A can be found by solving

$$\frac{1}{\alpha(G)} := \min\ x^\top (A+I)x \quad \text{s.t.}\quad e^\top x = 1,\ x \geq 0 \tag{5}$$

By taking $Q := A + I$ in (COP-SQP), we have the equivalent COP.

Example 1 (Max. Stable Set)

$$Q_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\vartheta^*_{\mathrm{SQP}} = \frac{1}{2}$$

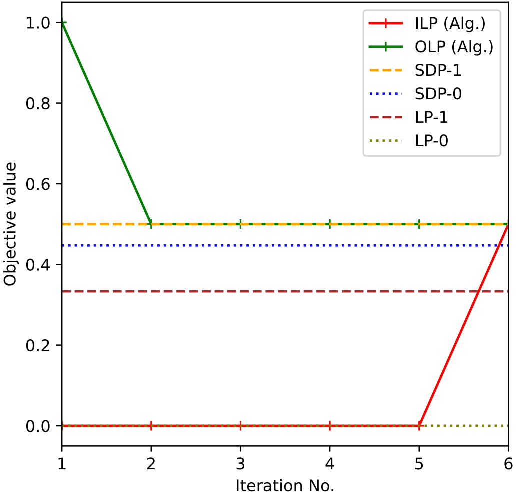

Figure 1: Comparing approximations (Alg. tol=10^{-6}).

Example 2 (Max. Stable Set)

$$Q_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\vartheta^*_{\text{SQP}} = \frac{1}{3}$$

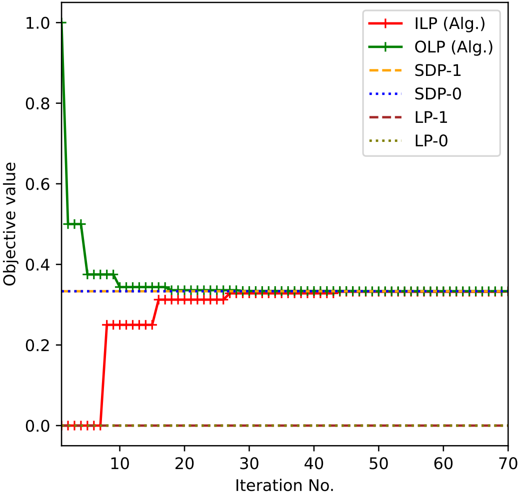

Figure 2: Comparing approximations (Alg. tol=$3e-4$).

Example 3 (Population Genetics)

$$Q_3 =$$

$$\begin{pmatrix} -14 & -15 & -16 & 0 & 0 \\ -15 & -14 & -12.5 & -22.5 & -15 \\ -16 & -12.5 & -10 & -26.5 & -16 \\ 0 & -22.5 & -26.5 & 0 & 0 \\ 0 & -15 & -16 & 0 & -14 \end{pmatrix}$$

$$\vartheta^*_{\text{SQP}} = -16\tfrac{1}{3}$$

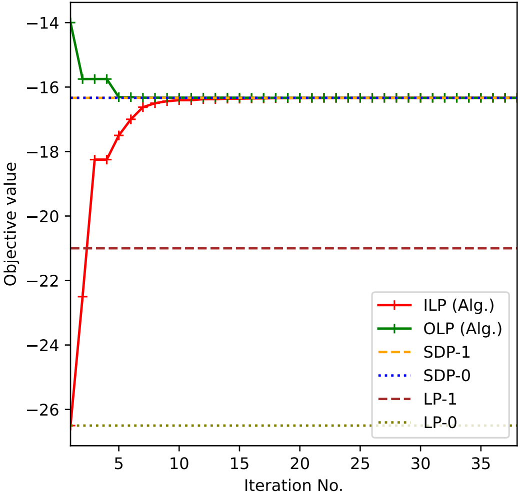

Figure 3: Comparing approximations (Alg. tol=10^{-6}).

Example 4 (Portfolio Optimization)

$$Q_4 = \begin{pmatrix} 0.9044 & 0.1054 & 0.5140 & 0.3322 & 0 \\ 0.1054 & 0.8715 & 0.7385 & 0.5866 & 0.9751 \\ 0.5140 & 0.7385 & 0.6936 & 0.5368 & 0.8086 \\ 0.3322 & 0.5866 & 0.5368 & 0.5633 & 0.7478 \\ 0 & 0.9751 & 0.8086 & 0.7478 & 1.2932 \end{pmatrix}$$

$$\vartheta^*_{\text{SQP}} = 0.4839$$

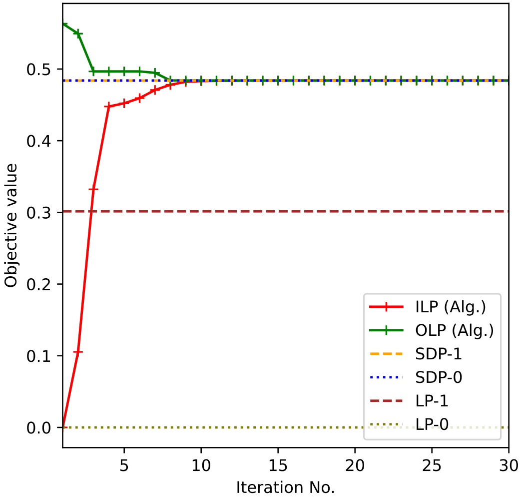

Figure 4: Comparing approximations (Alg. tol=10^{-6}).

Randomly generated matrices

- For each $n \in \{5, 10, 15, 20\}$, 10 random instances of Q are generated with each entry Q_{ij} sampled uniformly over $[-n, n]$.
- Average runtime across 10 instances with increasing n is compared.

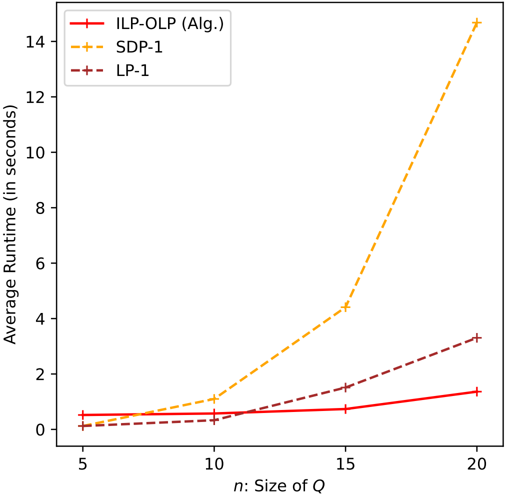

Figure 5: Comparing Runtimes (Alg. tol=10^{-6}).

Outline

Extensions/Ongoing Research

1. Think about copositivity over general convex domains. Several useful convex domains for practical applications are polyhedron, second-order cones (SOCs).
   Recent work Tian et al. (2013), Lu et al. (2014) provide adaptive approximation of non-negative quadratic forms over SOCs.
2. Since copositive cones hard to deal, can focus on specific applications. The hope is tighter approximations can be obtained by adding valid inequalities derived using problem structure.
3. Another possible route is to bypass COP and deal with the semi-infinite constraints as done by Zhen et al. (2022) in the context of robust optimization.

References I

Berman, A. and Shaked-Monderer, N. (2003). *Completely positive matrices*. World Scientific.

Bomze, I. M. and De Klerk, E. (2002). Solving standard quadratic optimization problems via linear, semidefinite and copositive programming. *Journal of Global Optimization*, 24(2):163–185.

Bomze, I. M., Dür, M., De Klerk, E., Roos, C., Quist, A. J., and Terlaky, T. (2000). On copositive programming and standard quadratic optimization problems. *Journal of Global Optimization*, 18(4):301–320.

Bundfuss, S. and Dür, M. (2009). An adaptive linear approximation algorithm for copositive programs. *SIAM Journal on Optimization*, 20(1):30–53.

Burer, S. (2009). On the copositive representation of binary and continuous nonconvex quadratic programs. *Mathematical Programming*, 120(2):479–495.

De Klerk, E. and Pasechnik, D. V. (2002). Approximation of the stability number of a graph via copositive programming. *SIAM Journal on Optimization*, 12(4):875–892.

Lu, C., Jin, Q., Fang, S.-C., Wang, Z., and Xing, W. (2014). Adaptive computable approximation to cones of nonnegative quadratic functions. *Optimization*, 63(6):955–980.

References II

Murty, K. G. and Kabadi, S. N. (1987). Some np-complete problems in quadratic and nonlinear programming. *Mathematical Programming*, 39(2):117–129.

Parrilo, P. A. (2000). *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*. California Institute of Technology.

Povh, J. and Rendl, F. (2007). A copositive programming approach to graph partitioning. *SIAM Journal on Optimization*, 18(1):223–241.

Povh, J. and Rendl, F. (2009). Copositive and semidefinite relaxations of the quadratic assignment problem. *Discrete Optimization*, 6(3):231–241.

Quist, A. J., de Klerk, E., Roos, C., and Terlaky, T. (1998). Copositive realxation for genera quadratic programming. *Optimization methods and software*, 9(1-3):185–208.

Tian, Y., Fang, S.-C., Deng, Z., and Xing, W. (2013). Computable representation of the cone of nonnegative quadratic forms over a general second-order cone and its application to completely positive programming. *Journal of Industrial & Management Optimization*, 9(3):703.

Zhen, J., de Ruiter, F. J., Roos, E., and den Hertog, D. (2022). Robust optimization for models with uncertain second-order cone and semidefinite programming constraints. *INFORMS Journal on Computing*, 34(1):196–210.

Thank you! Questions?

Additional slides I

$$
\begin{aligned}
\vartheta^*_{\mathsf{SQP}} &= \min\ \left\{x^\top Q x\ \text{ s.t. }\ e^\top x = 1,\ x \ge 0\right\} \\
&= \min\ \left\{\mathrm{Tr}(QX)\ \text{ s.t. }\ X = xx^\top,\ e^\top x = 1,\ x \ge 0\right\} \\
&= \min\ \left\{\mathrm{Tr}(QX)\ \text{ s.t. }\ X = xx^\top,\ \mathrm{Tr}(EX) = 1,\ x \ge 0\right\} \\
&\quad \Big(\text{For any feasible } \hat{x}, \hat{X},\ \mathrm{Tr}(E\hat{X}) = (e^\top \hat{x})^2 = 1,\ \because \hat{x} \ge 0 \\
&\qquad\qquad\qquad\qquad \implies e^\top \hat{x} = 1\Big) \\
&\ge \min\ \left\{\mathrm{Tr}(QX)\ \text{ s.t. }\ \mathrm{Tr}(EX) = 1,\ X \in \mathcal{C}^*\right\} \qquad \text{(CPP)} \\
&\quad \Big(\because \mathcal{C}^* = \mathrm{conv}\{xx^\top : x \ge 0\}\Big)
\end{aligned}
$$

Additional slides II

One can easily prove that last inequality will hold with equality. Say x^* is an optimal solution to (SQP) and X^* optimal to (CPP). Then

$$x^{*\top} Q x^* \geq \mathrm{Tr}(QX^*) \tag{6}$$

From the feasibility requirements of (CPP)

$$X^* = \sum_i x_i x_i^\top, \quad \mathrm{Tr}(EX^*) = \sum_i (e^\top x_i)^2 = 1 \text{ where } x_i \gneqq 0 \ \forall i$$

For each i define $u_i = \frac{1}{e^\top x_i} x_i$ then u_i is feasible to (SQP). This further implies

$$\begin{aligned} x^{*\top} Q x^* \leq u_i^\top Q u_i, \forall i \implies x^{*\top} Q x^* \sum_i (e^\top x_i)^2 &\leq \sum_i x_i^\top Q x_i \\ \implies x^{*\top} Q x^* &\leq \mathrm{Tr}(QX^*) \end{aligned} \tag{7}$$

Additional slides III

Combining (6) and (7), the equality is established.