MOTIVIC CLASSES OF FIXED-GENERATORS HILBERT SCHEMES OF UNIBRANCH CURVE SINGULARITIES AND IGUSA ZETA FUNCTIONS

ILARIA ROSSINELLI

ABSTRACT. This paper delves into the study of Hilbert schemes of unibranch plane curves whose points have a fixed number of minimal generators. Building on the work of Oblomkov, Rasmussen and Shende we provide a formula for their motivic classes and investigate the relationship with principal Hilbert schemes of the same given unibranch curve.

In addition, the paper specializes this study to the case of (p,q)-curves, where we obtain more structured results for the motivic classes of fixed-generators Hilbert schemes: their positivity and topological invariance, and an explicit relationship to one-generator schemes i.e. principal ideals in $\widehat{\mathcal{O}}_{C,0}$.

Finally, we focus on a special open component in the one-generator locus, whose motivic class is naturally related to the motivic measure on the arc scheme $\mathbb{A}^2_\infty$ of the plane introduced by Denef and Loeser as well as to the Igusa zeta function. We also provide an explicit formulation of these motivic classes in terms of an embedded resolution of the singularity, proving their polynomiality as well as making them an interesting topological invariant of the given curve.

CONTENTS

1. INTRODUCTION

Given a plane curve singularity, we are naturally interested in studying its geometric and topological properties. Algebraic links, originally presented in [17] as the topological data of the given curve, are in perfect correspondence with the geometry of the singularity as proved in [2, Chapter III.8]. For a given germ $(C,0)$ of an isolated reduced complex plane curve singularity, its link is defined as the nonsingular one-dimensional real submanifold of the sphere with radius $\epsilon << 1$

$$L = S^3_{\epsilon,0} \cap C.$$

Despite being easy to define, links turn out to be very hard to be handled directly, and the associated (co)homological invariants still lack an interpretation in terms of the given singularity predicted by the equivalence in [2].

Starting in [3] and consequently in the works [19], [18], and many more, Hilbert schemes have been considered as a possible geometric setting to answer these interpretation questions. Hilbert schemes whose points have a fixed number of minimal generators are defined as

$$\mathrm{Hilb}_{0,k}^n(C) = \{I \in \mathrm{Hilb}_0^n(C) \,|\, \dim_{\mathbb{K}} I/m_0 I = k\}$$

and are, in particular, the central object of interest in the conjecture proposed by Oblomkov, Rasmussen and Shende in [18]. This conjecture originates as a homological enhancement of the version presented in [19] and proved in [16], and relates the virtual Poincaré polynomials $\omega(\mathrm{Hilb}_{0,k}^l(C))$ with the superpolynomial $\mathcal{P}_L$ associated to the triply-graded homology of the link of the curve singularity.

In this paper, we focus on the former, studying the *motivic class* of $\mathrm{Hilb}_{0,k}^n(C)$, that can be later specialized to its weight polynomial.

We begin with Section 3 and our first result, which is the existence and uniqueness of some special representatives for the points of $\mathrm{Hilb}_{0,k}^n(C)$ using their t-adic valuations in $\widehat{\mathcal{O}}_{C,0} \subseteq \mathbb{K}[[t]]$ and the semigroup ideals of Γ_C.

Theorem 1.1. *Let $I \in \mathrm{Hilb}_{0,k}^n(C)$. Then, there exist a minimal set of generators $g_1, \ldots, g_k$ such that the values $val_t(g_i) = n_i$ form an independent k-uple in the sense of Definition 3.3, and this is the only possible independent k-uple of valuations arising from minimal sets of generators of I.*

The computation is carried out also in Section 3, where we explicitly present the motivic classes of the partition in terms of certain subvarieties of determinantal varieties arising from the syzygies encoding the $\widehat{\mathcal{O}}_{C,0}$-combinations corresponding to valuations in Γ-semigroup ideals. We obtain the following result.

Theorem 1.2. *Let $(C, 0)$ be a unibranch plane curve. We have*

$$[\mathrm{Hilb}_{0,k}^n(C)] = \sum_{\Delta} \mathbb{L}^{\sum_i |\Gamma_{>\nu_i} \setminus \Delta|} \sum_{\underline{n}} \mathbb{L}^{-\sum_i |\{n_i + H\} \cap \Delta|} \sum_{a=1\ldots m_\Delta} [S_a] \mathbb{L}^{-a}$$

for every $n, k \geq 1$ where

$$S_a = \{p \in Syz \,|\, rk(M_\Delta(p)) = a\}$$

and m_Δ is defined in Definition 2.56 and M_Δ is defined in Corollary 3.15.

In Section 4, we specialize the previous discussion to the simpler case of (p, q)-curves, as the semigroup ideals of Γ_C are easier to list, without involving the additional structure coming from non-trivial combinations in $\widehat{\mathcal{O}}_{C,0}$ as before. Consequently, we obtain the following expression for $[\mathrm{Hilb}_{0,k}^n(C)]$.

Theorem 1.3. *Let $(C, 0)$ be a (p, q)-curve. We have*

$$[\mathrm{Hilb}_{0,k}^n(C)] = \sum_{\underline{n} \in M_n} \left(\prod_i [\mathrm{Hilb}_{0,1}^{n_i}(C)] \right) \mathbb{L}^{-\sum_i |\{n_i + H\} \cap \Gamma_{\underline{n}}|} = \sum_{\underline{n} \in M_n} \mathbb{L}^{\sum_i |\{n_i + H\} \cap \Gamma_C| - \sum_i |\{n_i + H\} \cap \Gamma_{\underline{n}}|}$$

for every $n, k \geq 1$ where

$$\Gamma_{\underline{n}} = \coprod_i \{n_i + \Gamma_C\}.$$

The positivity (of the coefficients of $[\mathrm{Hilb}_{0,k}^n(C)]$ seen as a polynomial in $\mathbb{L}$, see [18, Proposition 6]), the topological invariance, and the relationship to the motivic class of the one-generator locus immediately follow from the theorem.

Corollary 1.4. *Let $(C, 0)$ be a (p, q)-curve. Then, $[\mathrm{Hilb}_{0,k}^n(C)]$ is a polynomial in $\mathbb{L}$ with positive coefficients for every $n, k \geq 1$. This polynomial is also a topological invariant of $(C, 0)$ since Γ_C is.*

Moreover, building on the work presented in [23], we aim to further develop the theory of motivic integration *in relationship to* Hilbert schemes, and specifically express the classic motivic Igusa zeta function

$$Z_C(s) = \sum_{n=1}^{\infty} \mu(\mathcal{X}_C^n) \mathbb{L}^{-ns}$$

defined in [6] in terms of motivic classes of principal Hilbert schemes. Precisely, we want to understand the motivic measure $\mu(\mathcal{X}_C^n)$, defined in [5] and [6], of the contact locus $\mathcal{X}_C^n \subseteq \mathcal{L}$ of a given unibranch plane curve C in connection with $\mathrm{Hilb}_{0,1}^n(C)$. This is accomplished in Section 5, where we construct a morphism

$$\Pi : \pi_{n+1}(\mathcal{X}_C^n) \to \mathrm{Im}(\Pi) \subseteq \mathrm{Hilb}_{0,1}^n(C)$$

realizing pairs of (truncated) arcs as (intersections of the given curve C with) unibranch plane curves. The morphism Π, unfortunately, is not surjective in general. However, the "unibranch" open component

$$\mathrm{UHilb}_{0,1}^n(C) := \mathrm{Im}(\Pi) \subseteq \mathrm{Hilb}_{0,1}^n(C)$$

is enough to retrieve the coefficients of the Igusa zeta function $Z_C(s)$.

To be more precise, the relationship between $\mathrm{UHilb}_{0,1}^n(C)$ and $Z_C(s)$ is obtained thanks to the explicit description of the fibers of Π, based on [12] and the so-called space of branches presented therein. In the section we will see how the morphism Π descends to a geometric morphism in the space of branches, directly leading to the description of $[\mathrm{UHilb}_{0,1}^n(C)]$ in terms of the (truncation of the) contact locus $[\pi_{n+1}(\mathcal{X}^n)]$ of the given curve.

Theorem 1.5. *Let $(C, 0)$ be a unibranch curve. We have*

$$[\mathrm{UHilb}_{0,1}^n(C)] = \frac{\mathbb{L}}{\mathbb{L}-1} \sum_k \mathbb{L}^k \frac{[\pi_{n+1}(\mathcal{X}^{n,k})]}{[F^{n,k}]}$$

for every $n \geq 1$ where

$$\mathcal{X}_C^{n,k} = \mathcal{X}_C^n \cap \pi_k^{-1}(0) \setminus \mathcal{X}_C^n \cap \pi_{k+1}^{-1}(0)$$

is a subset of the contact locus $\mathcal{X}_C^n$ defined in Definition 2.14 and $F^{n,k}$ is defined in Proposition 5.18.

As a consequence, combining each (truncation of) contact locus in their generating series, we obtain the classic Igusa zeta function $Z_C(s)$, and the coefficients of the series are now related to Hilbert schemes.

Corollary 1.6. *Let $(C, 0)$ be a unibranch curve and let $Z_C(s)$ be the Igusa zeta function of C. Then, we have*

$$Z_C(s) = (\mathbb{L}-1)\mathbb{L}^{-2} \sum_{n \geq 1} \mathbb{L}^{-n(s+1)} \sum_k \mathbb{L}^{-k} [\mathrm{Im}(\Pi_b^k)]$$

where Π_b denotes the S_0^{n+1}-quotient of the map Π and $[\mathrm{Im}(\Pi_b^k)]$ is related to $[\mathrm{UHilb}_{0,1}^n(C)]$ in Corollary 5.19.

In the same section, we also observe that the components $\mathcal{X}^{n,k}$ in the n-th contact locus and their motivic measure are completely determined by an embedded resolution of $(C, 0)$. This leads to an explicit relationship of $[\mathrm{UHilb}_{0,1}^n(C)]$ with an embedded resolution, hence becoming an interesting topological polynomial invariant.

Corollary 1.7. *Let $(C, 0)$ be a unibranch curve. We have*

$$[\mathrm{UHilb}_{0,1}^n(C)] = \frac{1}{(\mathbb{L}-1)\mathbb{L}^n} \sum_k \frac{\mathbb{L}^k}{[F^{n,k}]} \sum_{\substack{J \subseteq I, J \neq \emptyset, j \in J \\ k_j \geq 1 \mid k = \sum m_j k_j}} (\mathbb{L}-1)^{|J|-1}[E_J^o] \left(\sum_{j \in J, \sum k_j N_j = n} \mathbb{L}^{-\sum_{j \in J} k_j \nu_j} \right)$$

for every $n \geq 1$, where m_j is defined in [23, Definition 3.21].

Corollary 1.8. *Let $(C, 0)$ be a unibranch curve. Then, $[\mathrm{UHilb}_{0,1}^n(C)]$ is a polynomial in $\mathbb{L}$ for every $n \geq 1$. This polynomial is also a topological invariant of $(C, 0)$ since the resolution is.*

Finally, we also prove the existence of a threshold from which the open $\mathrm{UHilb}_{0,1}^n(C)$ is non-empty, making it an interesting open to study not only in relationship to the Igusa zeta function.

Theorem 1.9. *Let $(C, 0)$ be a unibranch plane curve and let $\rho : (Y, E) \to (\mathbb{A}^2, C)$ be an embedded resolution of the pair $(\mathbb{A}^2, C)$ as in Definition 2.18. Let N_i denote the multiplicity of E_i as in Definition 2.22 and let $N = \max_i N_i$. Then,*

$$\mathrm{UHilb}_{0,1}^n(C) \neq \emptyset$$

for all $n \geq N$.

This paper is organized as follows. We recall the main notions and results of the theory of motivic integration, unibranch plane curve singularities, and Hilbert schemes in Section 2. Then, the first part of Section 3 is dedicated to constructing a partition of the Hilbert scheme of a given unibranch plane curve with a fixed number of generators, determined by t-adic valuations of the generators in $\widehat{\mathcal{O}}_{C,0}$ and semigroup ideals. Later, we study the relationship of motivic classes of fixed-generators Hilbert schemes to certain determinantal subvarieties to prove Theorem 1.2. In Section 4 we then specialize the results of Section 3 to the case of (p, q)-curves, and factoring in some specific properties of these simpler plane curves we obtain Theorem 1.3. Finally, the first part of Section 5 is devoted to studying the existence of irreducible representatives for points of principal Hilbert schemes of a given unibranch

plane curve, leading to Theorem 1.9. We conclude by relating the motivic classes of unibranch opens of the one-generator locus to the motivic measure of contact loci of the given unibranch curve in Theorem 1.5.

2. Background

We denote by $\mathbb{K}$ an algebraically closed field of characteristic zero. We denote by $\mathrm{Sch}_{\mathbb{K}}$ the category of Noetherian schemes over $\mathbb{K}$ and by $\mathrm{Var}_{\mathbb{K}}$ the category of varieties i.e. separated integral schemes of finite type over the same field $\mathbb{K}$. Throughout this paper, when referring in general to a scheme or a variety we imply objects of these categories and over $\mathbb{K}$. Moreover, by plane curve or simply curve, we will always mean an hypersurface in $\mathbb{A}^2_{\mathbb{K}} = \mathbb{A}^2$, that we will assume to be reduced and irreducible throughout this paper.

2.1. **Motivic measure and contact loci.** In this section, we recall some basic facts about the construction of the motivic measure and introduce the notion of contact loci. We refer to [4] or [6] as references on the subject.

We start with the fundamental notions of k-jet scheme and arc scheme, and their truncation maps, which will have a central role in the definition of motivic integrals.

Definition 2.1. Let X, Y be schemes. For any $k > 0$ we denote by $X_k(Y)$ the set

$$X_k(Y) = \mathrm{Hom}_{\mathrm{Sch}_{\mathbb{K}}}(Y \times_{\mathbb{K}} \mathrm{Spec}(\mathbb{K}[t]/(t^k)), X)$$

and the k-jet scheme is defined as the scheme representing the corresponding functor $X_k : \mathrm{Sch}_{\mathbb{K}}^{op} \to \mathrm{Set}$ defined on objects by $Y \mapsto X_k(Y)$.

This definition relies on the representability result of the functor X_k, which one can find in [4, Proposition 2.1.3]. The scheme representing the functor X_k will also be denoted by X_k and it will be called the k-jet scheme, and we call k-jet a point of X_k. The $\mathbb{K}$-points of X_k are given by $X_k(\mathbb{K}) = \mathrm{Hom}_{\mathrm{Sch}_{\mathbb{K}}}(\mathrm{Spec}(\mathbb{K}[t]/(t^k)), X)$ and correspond to $\mathbb{K}[t]/(t^k)$-points of X.

Definition 2.2. Let $l \geq r > 0$ be two integers and consider the map $\pi_r^l : \mathbb{K}[t]/(t^l) \to \mathbb{K}[t]/(t^r)$ of reduction modulo t^r. We denote also by $\pi_r^l : X_l \to X_r$ the induced morphism of schemes.

Following [4, Section 3.3.1, Corollary 3.3.7] the truncation morphisms $(\rho_k^{k+1})_{k \geq 0}$ form a projective system which admits a limit in the category of functors over $\mathrm{Sch}_{\mathbb{K}}$. The limit functor is again representable and the scheme representing it can be taken as the definition of the arc scheme. In particular, we summarize the key points of the arc scheme construction in the next definition.

Definition 2.3. The arc scheme is defined as the scheme representing the limit functor $\varprojlim X_k = X_\infty : \mathrm{Sch}_{\mathbb{K}}^{op} \to \mathrm{Set}$. In particular, the functor on objects is defined as

$$X_\infty(Y) = \mathrm{Hom}_{\mathrm{Sch}_{\mathbb{K}}}(Y \times_{\mathbb{K}} \mathrm{Spec}(\mathbb{K}[[t]]), X).$$

The scheme representing the functor X_∞ will also be denoted by X_∞, and we call arc a point of X_∞. We observe that in this limit case as well the $\mathbb{K}$-points of X_∞ are given by $X_\infty(\mathbb{K}) = \mathrm{Hom}_{\mathrm{Sch}_{\mathbb{K}}}(\mathrm{Spec}(\mathbb{K}[[t]]), X)$ and correspond to $\mathbb{K}[[t]]$-points of X as explained in [4, Remark 3.3.9].

Definition 2.4. Let $r > 0$ be an integer and consider the map $\pi_r : \mathbb{K}[[t]] \to \mathbb{K}[t]/(t^r)$ of reduction modulo t^r. We denote also by $\pi_r : X_\infty \to X_r$ the induced morphism of schemes.

Example 2.5. We present the example of the affine space and more in general affine varieties since this ultimately is our local model of varieties. Let $X = \mathbb{A}^n$ with coordinates $x_1, \ldots, x_n$. We have that a morphism $\phi : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[t]/(t^k)$ corresponds to its images $\phi(x_i) = a^i \in \mathbb{K}[t]/(t^k)$, each given by a k-tuple of coefficients $(a_0^i, \ldots, a_{k-1}^i) \in \mathbb{K}^k$. Therefore, we have that

$$(\mathbb{A}^n)_k = (\mathbb{A}^k)^n = \mathbb{A}^{nk}.$$

Moreover, considering $Y \subseteq X$ an affine subvariety determined by $I = (f_1, \ldots, f_r)$ its k-jet scheme is then the affine subscheme of $(\mathbb{A}^n)_k(\mathbb{K})$ determined by the conditions $f_j(a^i) = 0$ modulo t^k for $j = 1, \ldots, r$.

Next, we turn our attention to the notion of motivic classes of varieties. These classes are fundamental to defining the motivic measure introduced by Denef and Loeser in [5] for varieties over $\mathbb{K}$ since it takes value into the ring of motivic classes of varieties.

Definition 2.6. The Grothendieck ring of varieties is defined as the quotient of the free abelian group generated on the set of isomorphism classes of varieties over $\mathbb{K}$ modulo the subgroup generated by the relations

$$[X] = [Z] + [X - Z] \text{ for any } X \in \mathrm{Var}_{\mathbb{K}} \text{ and } Z \subseteq X \text{ Zariski closed}$$

and then equipped by the multiplication defined as

$$[X][Y] = [X \times Y] \text{ for any } X, Y \in \mathrm{Var}_{\mathbb{K}}.$$

The elements of this ring are called motivic classes and we denote the ring by $\mathbf{K_0}(\mathrm{Var}_{\mathbb{K}})$ and by $\mathbb{L} = [\mathbb{A}^1]$ the motivic class of the affine line.

We recall the following well-known and useful facts, from [8, Proposition 5] and [1, Lemma 2.9].

Proposition 2.7. *Let X, Y be varieties and let $f : X \to Y$ be a fibration of fiber F which is locally trivial for the Zariski topology of Y. Then, $[X] = [F][Y]$.*

Proposition 2.8. *Let X, Y be varieties and let $f : X \to Y$ be a geometric bijection, that is a morphism which is also a bijection $f(\mathbb{K}) : X(\mathbb{K}) \to Y(\mathbb{K})$ of $\mathbb{K}$-points. Then, $[X] = [Y]$.*

The construction of the motivic measure of constructible subsets of the arc scheme is based on the result [5, Theorem 7.1] by Denef and Loeser, where the authors prove that the limit considered in the next definition exists. As required by their result, we need to work with the localization $\mathbf{M_0} = \mathbf{K_0}(\mathrm{Var}_{\mathbb{K}})[\mathbb{L}^{-1}]$ as we need $\mathbb{L}$ to be invertible and with the completion $\widehat{\mathbf{M_0}}$ of $\mathbf{M_0}$ with respect to the filtration $F^k \mathbf{M_0}$, where $F^k \mathbf{M_0}$ denotes the subgroup generated by classes of varieties $\dfrac{[X]}{\mathbb{L}^i}$ with $\dim X - i \leq -k$.

Definition 2.9. Let X be a variety. A constructible subset in X_k is a union of Zariski locally closed sets in X_k and a constructible subset in the arc scheme X_∞ is a subset of the form $A = \pi_k^{-1}(C_k)$ for $C_k \subset X_k$ constructible and some $k > 0$. Then, the limit

$$\mu(A) = \lim_{m \to \infty} \frac{[\pi_m(A)]}{\mathbb{L}^{m \dim X}}$$

exists in $\widehat{\mathbf{M_0}}$ and we call it the motivic measure of A.

Remark 2.10. As observed in [5, Definition-Proposition 3.2], in the case of a smooth variety X for $A = \pi_k^{-1}(C_k)$ a constructible set in the arc scheme X_∞ the definition of motivic measure simplifies to

$$\mu(A) = \frac{[\pi_k(A)]}{\mathbb{L}^{k \dim X}}.$$

Remark 2.11. As also discussed in the accompanying result [5, Definition-Proposition 3.2] the definition of μ extends from constructible subsets to a σ-additive measure μ on the Boolean algebra of constructible subsets of the arc scheme.

To conclude this section, we introduce the measure of certain special sets of arcs and their generating series, obtained as a parameteric motivic integral called Igusa zeta functions and taking values in the ring $\widehat{\mathbf{M_0}}[[T]]$ of formal power series with coefficients in $\widehat{\mathbf{M_0}}$. Motivic Igusa zeta functions were defined in [6].

Definition 2.12. Let W be a variety and let $V \subseteq W$ be a closed subvariety. We consider the integrable function $ord_V : W_\infty \backslash V_\infty \to \mathbb{N}$ returning the order of arcs along the non-zero coherent sheaf of ideals defining V. Concretely, if $\mathcal{I}$ denotes the latter and $p_\gamma = \gamma(\mathrm{Spec}(\mathbb{K})) \in V$ then

$$ord_V(\gamma) = \inf\{val_t(\gamma^*(f)) \mid f \in \mathcal{I}_{p_\gamma} - \{\mathbf{0}\}\}$$

where val_t denotes the t-adic valuation (or simply valuation).

We often refer to $ord_V(\gamma)$ as the valuation of the arc.

Remark 2.13. From now on, we will omit removing V_∞ and just write W_∞ since $\mu(V_\infty) = 0$, see [4, Chapter 7, Section 3, Subsection 3.3].

Definition 2.14. In the same setting as Definition 2.12, we define the n-th contact locus of V as

$$\mathcal{X}_V^n = ord_V^{-1}(n) \subseteq W_\infty$$

and its measure as

$$\mu(\mathcal{X}_V^n) = \frac{[\pi_{n+1}(\mathcal{X}_V^n)]}{\mathbb{L}^{(n+1) \dim X}}.$$

Definition 2.15. Let W be a variety and let $V \subseteq W$ be a closed subvariety. Let $A \subseteq W_\infty$ be a constructible subset. The motivic Igusa zeta function of V over A is defined as

$$Z_V^A(T) = \int_A T^{ord_V} \, d\mu = \sum_{n=0}^\infty \mu(\mathcal{X}_V^n) T^n \in \widehat{\mathbf{M}_0}[[T]].$$

The formal variable T is usually written as $\mathbb{L}^{-s}$ for s a formal parameter, and we usually omit the upper A when $A = \pi_1^{-1}(0)$.

2.2. **Embedded resolutions.** The study of embedded resolutions was extensively developed by Hironaka, in addition to many others, converging in his result regarding the existence of embedded resolutions of singular varieties over a field of characteristic zero. We refer the reader to [13] for any further details. Since our object of interest is plane curves, that is, the pair $(\mathbb{A}^2, C)$ for some curve C we will assume the ambient variety to be smooth.

Definition 2.16. Let D be a divisor of a smooth variety, namely a closed subscheme of pure codimension 1, and let I be the set indexing of its irreducible components. We denote by D_i the irreducible component indexed by $i \in I$. For $J \subseteq I$ we denote with

$$D_J = \bigcap_{j \in J} D_j$$

and with

$$D_J^o = D_J \setminus \bigcup_{j \notin J} D_j.$$

Definition 2.17. Let D be a divisor of a smooth variety. We say that D has simple normal crossing if the irreducible components of D are smooth and meet transversally.

Definition 2.18. Let W be a variety and let $V \subseteq W$ be a closed subscheme. Then, an embedded resolution of the pair (W, V) is a pair (Y, ρ) where $\rho : Y \to W$ is a proper birational morphism, Y is smooth, and $\rho^{-1}(V)$ is a simple normal crossing divisor on Y.

Definition 2.19. We call strict transform $\widetilde{V}$ of V the component of $\rho^{-1}(V)$ that corresponds to the closure of the dense subset of V where ρ is an isomorphism.

We usually denote by I the indexing set of the irreducible components of $\rho^{-1}(V)$, including the strict transform unless otherwise specified (explicitly writing of $\widetilde{V}$ and following the notation $E_i, i \in I$ for the components. We denote their union by

$$E = \bigcup_{i \in I} E_i.$$

We refer to a component E_i as an exceptional divisor.

Theorem 2.20 ([13, Main Theorem II(N)]). *Let $\mathbb{K}$ be a field of characteristic zero and (W, V) a variety and closed subvariety respectively over $\mathbb{K}$. Then, there exists an embedded resolution (Y, ρ) of the pair such that it is an isomorphism outside of $W_s \cup V$, where W_s denotes the singular locus of W. Moreover, the morphism ρ is a composition of blow-ups with smooth centers.*

Utilizing the transversality condition, we can deduce certain numerical information of the singularity as presented in [4, Chapter 7, Theorem 3.3.4]. Before that, we recall the notion of multiplicity of a non-zero coherent sheaf of ideals at a point.

Definition 2.21. We recall that given a variety X and a point $p \in X$, the multiplicity of a non-zero coherent sheaf of ideals $\mathcal{I}$ on X at p is defined as

$$\max\{\mu \,|\, \mathcal{I}_p \subseteq m_p^\mu\}$$

where $m_p \subseteq \mathcal{O}_{X,p}$ denotes the maximal ideal of p.

Definition 2.22. Let $\rho : (Y, E) \to (W, V)$ be an embedded resolution as in Definition 2.18 and let $\mathcal{I}$ be the non-zero coherent sheaf of ideals determining V. For every $i \in I$ we denote by N_i the multiplicity of $\mathcal{I}\mathcal{O}_Y$ along any smooth point of E_i and by ν_i the multiplicity of $K_{Y|W}$ along any smooth point of E_i. These are respectively called multiplicity and discrepancy of E_i.

Finally, we present a classic example of an embedded resolution, whose details can be found in [4, Example 3.1.9].

Example 2.23. We briefly recall the resolution of singularities of the cusp $f = x^2 - y^3 \in \mathbb{K}[x,y]$. One can check that three blowups are needed, thus obtaining three exceptional divisors in addition to the strict transform of the curve for which the multiplicities and discrepancies are $(N_i, \nu_i)_{i=1,2,3} = (2,1), (3,2), (6,4)$.

We conclude this section by presenting a formula for $[\pi_{n+1}(\mathcal{X}_V^n)]$, hence $\mu(\mathcal{X}_V^n)$, in terms of an embedded resolution of (W, V), this highlighting a connection between invariants of singularities and motivic integration, which was also our original reason of interest, as explained in Section 1. We refer to it as Denef's formula, as it was proven in [6, Proposition 3.2.1].

Theorem 2.24. *Let $\rho : (Y, E) \to (W, V)$ be an embedded resolution as in Definition 2.18. We have*

$$[\pi_n(\mathcal{X}_V^n)] = \mathbb{L}^{n \dim W} \sum_{J \subseteq I, J \neq \emptyset} (\mathbb{L} - 1)^{|J|-1}[E_J^o] \left(\sum_{k_j \geq 1, \, j \in J, \, \sum k_j N_j = n} \mathbb{L}^{-\sum_{j \in J} k_j \nu_j} \right)$$

for every $n \geq 1$.

2.3. **Puiseux parameterizations and unibranch curves.** Unless otherwise stated, $(C, 0)$ will always denote a plane curve locally defined as the zero locus of $f \in \mathbb{C}[[x,y]]$, $f(0) = 0$. We denote by f_d the d-th degree homogeneous component of f, and the smallest $d > 0$ such that $f_d \neq 0$ is called multiplicity of the singularity and denoted by $mult_C(0) = mult_C$.

Definition 2.25. Let $(C, 0)$ be a plane curve and let $f \in \mathbb{K}[[x,y]], f = \prod_i f_i$ be the equation locally defining it and its factorization in $\mathbb{K}[[x,y]]$ in irreducible factors allowing repetitions. We call branches of C the curves corresponding to each f_i, and we say that C is unibranch when it has only one branch i.e. f is irreducible in $\mathbb{K}[[x,y]]$.

From now on we will always work in the unibranch case. We now recall some details about Puiseux parameterizations of unibranch curves. We refer the reader to [2] or [24] for any further details.

Definition 2.26. We recall the notation $\mathbb{K}[[t]]$ for the domain of formal power series with coefficients in the field $\mathbb{K}$ and indeterminate t, and we write its general element as

$$p = p(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Moreover, we denote by $\mathbb{K}[[t]]_0$ the domain of formal power series centered at 0, that is $\mathbb{K}[[t]]_0 = \mathbb{K}[[t]] \setminus \mathbb{K}[[t]]^\times$. This equivalently corresponds to formal power series with

$$a_0 = 0.$$

Definition 2.27. Following the notations of Definition 2.26, we say that $p \in \mathbb{K}[[t]]$ has valuation k if $a_k \neq 0$ is the smallest non-zero coefficient. We denote this by

$$k = val_t(p).$$

Remark 2.28. We observe that $\mathbb{K}[[t]]_0 = \{p \in \mathbb{K}[[t]] \mid val_t(p) > 0\}$.

Definition 2.29. Let $p, q \in \mathbb{K}[[t]]_0$ as in Definition 2.26 and let $(C, 0)$ be a unibranch plane curve as in Definition 2.25. We say that the pair (p, q) is a parameterization of C if

$$f(p, q) = 0$$

in $\mathbb{K}[[t]]$.

Definition 2.30. We say that two pairs (p, q), (p', q') are equivalent parameterizations when they are parameterizations for the same curve C, that is, there exists $\alpha \in \mathbb{K}[[t]]_0$ with $val_t(\alpha) = 1$ such that

$$p'(t) = p(\alpha(t)), q'(t) = q(\alpha(t)).$$

Remark 2.31. We say that a parameterization (p', q') is degenerate if there exist another parameterization (p, q) and $\alpha \in \mathbb{K}[[t]]_0$ with $val_t(\alpha) > 1$ such that $p'(t) = p(\alpha(t)), q'(t) = q(\alpha(t))$. Viewing $\mathbb{K}[[t]]$ as $(\mathbb{A}^1)_\infty$ and following [9, Definition, pg. 5], the parameterization action induced by any $\alpha \in \mathbb{K}[[t]]_0$ with $val_t(\alpha) > 1$ would lead to orbits in $\mathbb{K}[[t]]^2 \cong (\mathbb{A}^2)_\infty = \mathcal{L}$ of motivic measure zero. Accordingly to Definition 2.30, and for this reason, we will always assume our parameterizations to be non-degenerate.

Remark 2.32. We mention here that equivalence classes of parameterizations are also called branches since they all correspond to the same geometric locus, that is, the same set of zeroes in $\mathbb{A}^2$. Therefore, the branch of the unibranch curve C refers to the zero locus determined by f as well as the set of all parameterizations of C.

Remark 2.33. We observe that composing by α in Definition 2.30 induces an automorphism of $\mathbb{K}[[t]]$ that preserves val_t. This implies that all the parameterizations in a given class have the same valuations.

We now recall a classic result by Puiseux, guaranteeing the existence of parameterizations for polynomials in the indeterminate $y = t$ and coefficients in $\mathbb{K}[[x]]$. Finding a parameterization corresponds to finding a solution of the given polynomial in $P = \cup_{k=1}^{\infty} \mathbb{K}((t^{\frac{1}{k}}))$, and Puiseux's theorem ensures the existence of a solution by constructively proving that P is algebraically closed. The same result and proof can be extended to $f \in \mathbb{K}[[x,y]]$, namely power series instead of polynomials, as pointed out in [24, Section 2.1].

Theorem 2.34 ([21])**.** *Let $\mathbb{K}((t))$ denote the quotient field of $\mathbb{K}[[t]]$ and let $P = \cup_{k=1}^{\infty} \mathbb{K}((t^{\frac{1}{k}}))$. Then, P is an algebraically closed field.*

Remark 2.35 ([2, Pg. 406])**.** From the proof of Theorem 2.34 it also follows that in the equivalence class of Definition 2.36 there exists a representative of the form

$$(p_C, q_C) = \left(t^n, \sum_{m \geq n} a_m t^m \right)$$

with $n = mult_C$.

Definition 2.36. We refer to the parameterization (p_C, q_C) of Remark 2.35 as the Puiseux parameterization of $(C, 0)$.

The equivalence class of parameterizations of C is also equivalent to the data of Puiseux pairs, which we avoid defining and refer the reader can find in [2, Pg. 406]. We just limit ourselves to introducing the following definition.

Definition 2.37. Let $(C, 0)$ be a unibranch curve. We say that C is a (p, q)-curve when $(p_C, q_C) = (t^p, t^q)$ for some $p, q > 0$ with $gcd(p, q) = 1$.

We also recall this useful fact, which allows us to compute the intersection number of two curves using any of their equivalent parameterizations.

Lemma 2.38 ([24, Lemma 1.2.1])**.** *Let (p, q) be a parameterization of C as in Definition 2.29 and let $(D, 0)$ be another unibranch plane curve locally defined by $g \in \mathbb{K}[[x,y]]$, $g(0) = 0$. Then, the intersection number of the two curves can be computed as*

$$C.D = val_t(g(p, q)).$$

2.3.1. *Properties of the completion of the local coordinate ring of a unibranch plane curve.* Let $(C, 0)$ be a unibranch curve as in Definition 2.25 and let (p_C, q_C) denote its Puiseux parameterization as in Definition 2.36. We denote by $\widehat{\mathcal{O}}_{C,0}$ the (completion of the) local ring of the curve, that is

$$\widehat{\mathcal{O}}_{C,0} \cong \mathbb{K}[[x,y]]/(f) \cong \mathbb{K}[[p_C, q_C]].$$

We need to keep track of the valuations appearing in $\widehat{\mathcal{O}}_{C,0}$ since they will be crucial in later sections to get to our central results. To do so, we recall the notion of (numerical) semigroup of C.

Definition 2.39. We denote by

$$\Gamma_C = \{val_t(g) \,|\, g \in \widehat{\mathcal{O}}_{C,0}\}$$

and we refer to it as the semigroup of the curve. We simply denote it by Γ when it is clear which curve C we refer to. We denote by

$$H = \mathbb{N} \setminus \Gamma$$

the set of holes of the semigroup, and we refer to

$$|H| = \delta$$

as the delta invariant of the curve.

We recall that any semigroup, considered as a submonoid of $\mathbb{N}$, admits a unique minimal system of generators.

Theorem 2.40 ([22, Theorem 2.7])**.** *Every semigroup admits a unique minimal system of generators. This minimal system of generators is finite.*

We are interested in a particular class of semigroups, called semigroup ideals, as they will be the semigroups of valuations appearing from $\widehat{\mathcal{O}}_{C,0}$ in the Hilbert scheme.

Definition 2.41. We say that $S \subseteq \Gamma$ is a semigroup ideal if $S + \Gamma \subseteq S$.

Example 2.42. Let $J \subseteq \widehat{\mathcal{O}}_{C,0}$ be an ideal and let $\Delta = val_t(J) = \{val_t(g) \mid g \in J\}$ be the semigroup of the valuations appearing in J. It is clear that Δ is a semigroup ideal. However, the converse is false, as pointed out in [19, Remark, pg. 13].

Definition 2.43. Let $\Delta \subseteq \Gamma$ be a semigroup ideal. We denote by $c(\Delta) = min\{j \in \Delta \mid j + \mathbb{N} \subseteq \Delta\}$ the element from where there are no holes anymore, and we refer to it as conductor of Δ.

Many elements could have the same valuation in $\widehat{\mathcal{O}}_{C,0}$. However, each valuation $n \in \Gamma$ can be achieved by a special element, as proved in [19].

Lemma 2.44 ([19, Lemma 25]). *Let $n \in \Gamma$. Then, there exists a unique $\tau_n \in \widehat{\mathcal{O}}_{C,0}$ of the form*

$$\tau_n = t^n + \sum_{i \in H, i > n} c_{n,i} t^i.$$

2.4. **The Hilbert scheme of unibranch plane curves.** The Hilbert scheme of points on C is defined as the moduli space parameterizing zero-dimensional subschemes of C with fixed length, originally introduced in [11] for any variety X. We recall its definition starting with the construction as a functor of points and then present its representability in the category $\text{Sch}_{\mathbb{K}}$ of schemes.

Definition 2.45. Let X be a scheme. For any $n \geq 1$ we consider the functor $H_X^n : \text{Sch}_{\mathbb{K}}^{op} \to \text{Set}$ defined on objects of $\text{Sch}_{\mathbb{K}}$ as

$$S \mapsto H_X^n(S) = \{Z \subseteq X \times S \mid p|_Z : Z \to S \text{ finite, flat and surjective of degree } n\}$$

and on morphisms given $f : T \to S$ morphism in $\text{Sch}_{\mathbb{K}}$ as

$$H_X^n(f) : H_X^n(S) \to H_X^n(T), \, Z \mapsto (id_X, f)^{-1}(Z) = \text{ pullback with respect to } (id_X, f) : X \times T \to X \times S.$$

$$
\begin{array}{ccc}
Z \times_S T \subseteq X \times T \xrightarrow{\ p_2\ } T & \quad & (x,t) \longmapsto t \\
\Big\downarrow{\scriptstyle p_1} \qquad\qquad \Big\downarrow{\scriptstyle f} & \quad & \Big\downarrow \qquad\qquad \Big\downarrow \\
Z \subseteq X \times S \xrightarrow{\ p|_Z\ } S & \quad & z = (x,s) \longmapsto s = f(t)
\end{array}
$$

The representability of the so-called Hilbert functor H_X^n has been proved in [11, Theorem 3.2]. This is a key fact, as our primary focus will rely on the scheme representing it.

Definition 2.46. We denote by $\text{Hilb}^n(X)$ the scheme representing the functor H_X^n and we call it the Hilbert scheme of points of X of length n. The universal subscheme is the subscheme $\mathcal{Z} \in H_X^n(\text{Hilb}^n(X))$ corresponding to $id_{\text{Hilb}^n(X)}$ under representability.

Remark 2.47. Using the representability we have a correspondence

$$pt \mapsto \mathcal{Z} \cap (X \times \{pt\})$$

between closed points of $\text{Hilb}^n(X)$ and zero-dimensional subschemes $Z \subseteq X$ with finite length equal to $\dim \mathcal{O}_Z = n$. One can view these subschemes as points on the Hilbert scheme.

Definition 2.48. Given a zero-dimensional finite-length subscheme $Z \in \text{Hilb}^n(X)$ we call its support the set of points $|Z| = \{x_1, \ldots, x_k\}$ where the scheme is geometrically supported, and we define the cluster associated to Z as the 0-cycle $[Z] = \sum_{x_i \in |Z|} \dim \mathcal{O}_{Z, x_i} \, x_i$.

Constraining the support, we can define punctual schemes and the punctual Hilbert scheme of points as the ones with support concentrated in one point of higher multiplicity.

Definition 2.49. Let X be a scheme and let $p \in X$. We denote by

$$\text{Hilb}_p^n(X) = \{Z \in \text{Hilb}^n(X) \mid |Z| = \{p\}\}$$

the closed subscheme of $\text{Hilb}^n(X)$ made of the zero-dimensional subschemes of length n that are supported entirely at the point p.

From now on, we will omit the word "punctual" and we will always refer to the Hilbert scheme as the one of Definition 2.49.

Returning to our local case of interest, which is unibranch plane curves, the Hilbert scheme can be described in an even clearer way. Let $(C,0)$ be a unibranch plane curve and let $\widehat{\mathcal{O}}_{C,0}$ be (the completion of) its local ring of coordinates, as defined in Section 2.3.

Example 2.50. Given a unibranch plane curve $(C,0)$ the Hilbert scheme takes the form

$$\mathrm{Hilb}_0^n(C) = \{I \subseteq \widehat{\mathcal{O}}_{C,0}\ \text{ideal}\,|\,\dim_{\mathbb{K}} \widehat{\mathcal{O}}_{C,0}/I = n\} = \{J \subseteq \mathbb{K}[[x,y]]\ \text{ideal}\,|\,f \in J\,,\,J \subseteq (x,y)\,,\,\dim_{\mathbb{K}}\mathbb{K}[[x,y]]/J = n\}.$$

In this paper, we will be interested in points of the Hilbert scheme with a given number of minimal generators. We recall the following map, that appears in [19, Conjecture 2'].

Definition 2.51. Let $m : \mathrm{Hilb}_0^n(C) \to \mathbb{N}_{>0}$ be the map defined as

$$I \mapsto \dim_{\mathbb{K}} I/m_0 I$$

where m_0 denotes the maximal ideal of $\widehat{\mathcal{O}}_{C,0}$. Equivalently, by Nakayama's lemma we have that $m(I)$ corresponds to the minimal number of generators of I in $\widehat{\mathcal{O}}_{C,0}$.

Definition 2.52. We denote by $\mathrm{Hilb}_{0,k}^n(C) \subseteq \mathrm{Hilb}_0^n(C)$ the closed subscheme of schemes with k minimal generators i.e. $\mathrm{Hilb}_{0,k}^n(C) = m^{-1}(k)$ or more explicitly

$$\mathrm{Hilb}_{0,k}^n(C) = \{I \in \mathrm{Hilb}_0^n(C)\,|\,\dim_{\mathbb{K}} I/m_0 I = k\}.$$

In particular, when $k = 1$ we refer to $\mathrm{Hilb}_{0,1}^n(C)$ as principal Hilbert scheme or one-generator locus, and to its points as principal schemes.

Remark 2.53. As observed in [19, Proof of Theorem 13], the minimal number of generators of a point in $\mathrm{Hilb}_0(C)$ is bounded by $mult_C$: $\mathrm{Hilb}_{0,k}^n(C) = \emptyset$ for $k > mult_C$.

2.4.1. *Valuation semigroups, syzygies and Hilbert schemes.* From [19] and [18] we recall the following key results, which allow us to control the generators and semigroup ideals appearing in principal or fixed-generator subschemes. These results will be fundamental in Section 3.2 to control the motivic class of $\mathrm{Hilb}_{0,k}^n(C)$.

Theorem 2.54 ([19, Definition 26, Theorem 27]). *Let $V \subseteq \Gamma$ be a semigroup ideal and let $a_0, \ldots, a_l$ denote its generators as a semigroup ideal i.e.*

$$V = \cup_{i=0}^l \{a_i + \Gamma\}.$$

We denote by

$$\Sigma_i = \Gamma_{>a_i} \setminus V$$

and by

$$f_i = \tau_{a_i} + \sum_{j \in \Sigma_i} \lambda_j^i \tau_j$$

where τ_{a_i}, τ_j are defined as in Lemma 2.44. Let $J_V = (f_0, \ldots, f_l)$ and let

$$U_V \subset \mathrm{Spec}\,\mathbb{K}[\lambda_j^i\,|\,j \in \Sigma_i, i = 0, \ldots, l],\ U_V = \{s \in \mathrm{Spec}\,\mathbb{K}[\lambda_j^i\,|\,j \in \Sigma_i, i = 0, \ldots, l]\,|\,\dim_{\mathbb{K}} \widehat{\mathcal{O}}_{C,0}/(J_V|_s) = |\Gamma \setminus V|\}$$

where the notation $e|_s$ for $e \in U_V$ means taking $\lambda_j^i = s_{i,j} \in \mathbb{K}$ for all i, j (and for an ideal, it means evaluating all generators). Let $\mathrm{Hilb}_0^V(C)$ denote the subscheme of schemes I such that $val_t(I) = V$. Then, there exists a bijective morphism

$$U_V \to \mathrm{Hilb}_0^V(C),\ s \mapsto J_V|_s.$$

Remark 2.55. In particular, Theorem 2.54 takes explicit shape as follows for principal schemes. By definition, we have that $\mathrm{Hilb}_{0,1}^n(C) = \mathrm{Hilb}_0^V(C)$ for $V = n + \Gamma = a_0 + \Gamma$. This implies that $\mathrm{Hilb}_{0,1}^n(C) = U_V$ and all principal ideals are consequently generated by elements

$$f_n = \tau_n + \sum_{h \in H\,|\,n+h \in \Gamma} \lambda_h^n \tau_{n+h}$$

for $\lambda_h^n \in \mathbb{K}$ since

$$\{n + H\} \cap \Gamma = \{n + H\} \cap \Gamma_{>n} = \Gamma_{>n} \setminus \{n + \Gamma\} = \Sigma_0.$$

Therefore, we can write

$$\mathrm{Hilb}_{0,1}^n(C) = \{(f_n)\,|\,\lambda_h^n \in \mathbb{K}\}$$

and the bijection between $\mathrm{Hilb}_{0,1}^n(C)$ and $\mathbb{A}^{|\{n+H\}\cap\Gamma|}$ immediately follows. It extends to an isomorphism, following [20, Theorem 1].

Finally, let $\Delta \subseteq \Gamma$ be a semigroup ideal. We want to extract explicit conditions on the variables λ_j^i of Theorem 2.54 to ensure the ideal they generate has its semigroup equal to Δ. First, we start by recalling the following results from [18] that allow us to simplify generators of ideals in $\widehat{\mathcal{O}}_{C,0}$.

Definition 2.56 ([18, Section 3.1]). We denote by $\mathbb{K}[\Gamma] = \mathbb{K}[t^i \,|\, i \in \Gamma]$ and we choose a basis $\{\phi_i\}_{i \in \Gamma}$ of $\widehat{\mathcal{O}}_{C,0}$ that is also a basis of $\mathbb{K}[\Gamma]$, that we denote as

$$\phi_i = t^i + \sum_{j > i, \, j \in \Gamma} a_{ij} t^j.$$

Lemma 2.57 ([18, Section 3.1]). *Let $J \subseteq \widehat{\mathcal{O}}_{C,0}$ be an ideal such that $val_t(J) = \Delta$ and let $i \in \Delta$. Then, there exists a unique $f_i \in J$ of the form*

$$f_i = \phi_i + \sum_{k \in \Gamma_{>i} \setminus \Delta} \lambda_k^i \phi_k$$

with $\lambda_k^i \in \mathbb{K}$.

Remark 2.58. Roughly speaking, this choice of ring and basis (instead of the one provided by τ_i in Lemma 2.44) allows a simpler treatment, as with (p, q)-curves.

We can now define conditions on the coefficients λ_j^i of a minimal set of generators f_i of an ideal, following the notation introduced in Lemma 2.57, that determine the semigroup.

Definition 2.59 ([18, Equation (7)]). Let $\Delta \subseteq \Gamma$ be a semigroup ideal and let ν_i be the set of generators of $\Delta = \coprod_i \{\nu_i + \Gamma\}$ for $i = 1, \ldots, n$. Let γ_j be the set of generators of $\Gamma = \coprod_j \{\gamma_j + \Gamma\}$. Then, let $I_\Delta = (t^{\nu_1}, \ldots, t^{\nu_n}) \subseteq \mathbb{K}[\Gamma] = \mathbb{K}[t^{\gamma_j}]$. We consider the surjection

$$G : \mathbb{K}[\Gamma]^{\oplus n} \to I_\Delta, \, (0, \ldots, g_i, \ldots, 0) \mapsto g_i t^{\nu_i}$$

and the free resolution

$$\cdots \to \mathbb{K}[\Gamma]^{\oplus m_\Delta} \to \mathbb{K}[\Gamma]^{\oplus n} \to I_\Delta \to 0$$

that G is part of as per Hilbert's syzygy theorem [7, Theorem 1.1]. We denote by S the map $\mathbb{K}[\Gamma]^{\oplus m_\Delta} \to \mathbb{K}[\Gamma]^{\oplus n}$ of the resolution, and we refer to $\operatorname{Im}(S) = \ker(G)$ as syzygies of Δ.

Definition 2.60 ([18, Equation (7)]). The map S is represented by a $m_\Delta \times n$-matrix with entries in $\mathbb{K}[\Gamma]$. As proved in [20, Lemma 4], we can write $(S)_{ij} = c_{ij} t^{\sigma_i - \gamma_i}$ for some power σ_i and some constant $c_{ij} \in \mathbb{K}$. We call the powers σ_i orders of the syzygies.

Definition 2.61 ([18, pg. 10, pg. 12]). We denote by

$$Syz = \operatorname{Spec}(\mathbb{K}[x_{is}^{\sigma_i} \,|\, c(\Delta) > s > \sigma_i, i = 1, \ldots, m_\Delta])$$

the affine space of dimension $\sum_{i=1}^{m_\Delta} (c(\Delta) - \sigma_i - 1)$. We denote by

$$Gen = \operatorname{Spec}(\mathbb{K}[\lambda_k^{\nu_i} \,|\, k \in \Gamma_{>\nu_i} \setminus \Delta, i = 1, \ldots, n])$$

the affine space of dimension $\sum_{i=1}^n |\Gamma_{\nu_i} \setminus \Delta|$ of the coefficients of the generators f_{ν_i}.
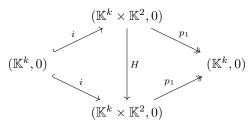
Theorem 2.62 ([18, Proposition 12]). *Let $\lambda \in Gen$ and let J_λ be the correspondent ideal. Then, $val_t(J_\lambda) = \Delta$ if and only if there exists $p \in Syz$ such that the equations*

$$\sum_j \left(u_i^j \phi_{\sigma_i - \nu_j} \phi_{\nu_j} + \sum_{c(\Delta) > s > \sigma_i} x_{is}^{\sigma_i} \phi_{s - \nu_j} \phi_{\nu_j} + \sum_{k \in \Gamma_{>\nu_j} \setminus \Delta} u_i^j \lambda_k^{\nu_j} \phi_{\sigma_i - \nu_j} \phi_k + H.O.T. \right) = o(t^{c(\Delta)})$$

for $i = 1, \ldots, m_\Delta$ are satisfied. Moreover, if such $p \in Syz$ exists, it is unique.

2.4.2. *Mather's contact equivalence for points of Hilbert schemes.* To conclude the background section we introduce the notion of $\mathcal{C}$-equivalence or contact equivalence, presented by Mather in his work [15]. It will be useful to control all the different representatives of any given ideal in $\widehat{\mathcal{O}}_{C,0}$.

Definition 2.63. We denote by $\mathcal{C}$ the group of C^∞-map-germs $H : (\mathbb{K}^k \times \mathbb{K}^2, 0) \to (\mathbb{K}^k \times \mathbb{K}^2, 0)$ such that the diagram

$$
\begin{array}{ccc}
 & (\mathbb{K}^k \times \mathbb{K}^2, 0) & \\
\nearrow^{i} \quad & \downarrow^{H} & \searrow^{p_1} \\
(\mathbb{K}^k, 0) & & (\mathbb{K}^k, 0) \\
\searrow_{i} \quad & \downarrow & \nearrow_{p_1} \\
 & (\mathbb{K}^k \times \mathbb{K}^2, 0) &
\end{array}
$$

is commutative. This group acts on C^∞-map germs $g : (\mathbb{K}^k, 0) \to (\mathbb{K}^2, 0)$ as

$$H \bullet g = p_2(H(id, g)).$$

Definition 2.64. Let $g_1, g_2 : (\mathbb{K}^k, 0) \to (\mathbb{K}^2, 0)$ be two C^∞-map-germs and let $g_i^j \in \mathbb{K}[[x, y]]$, $g_i^j(0) = 0$ be their coordinates for $j = 1, \ldots, k$. We say that g_1 is (contact, $\mathcal{C}$-) equivalent to g_2 if they are in the same $\mathcal{C}$-orbit.

Theorem 2.65 ([15, Section 2.3]). *Let $g_1, g_2 : (\mathbb{K}^k, 0) \to (\mathbb{K}^2, 0)$ be two C^∞-map-germs and let $g_i^j \in \mathbb{K}[[x, y]]$, $g_i^j(0) = 0$ be their coordinates for $j = 1, \ldots, k$. Then, g_1 is equivalent to g_2 if and only if*

$$(g_1^1, \ldots, g_1^k) = (g_2^1, \ldots, g_2^k)$$

as ideals in $\mathbb{K}[[x, y]]$. Moreover, this is also equivalent to the existence of an invertible matrix $U = (a_j^i)_{ij} \in GL_k(\mathbb{K}[[x, y]])$ such that

$$g_1^i = \sum_j a_j^i g_2^j$$

for all $i = 1, \ldots, k$.

Remark 2.66. Since we will work with ideals in $\widehat{\mathcal{O}}_{C,0} = \mathbb{K}[[p_C, q_C]] = \mathbb{K}[[x, y]]/(f)$ we will rather use their notation in just one parameter, substituting the two variables x, y as $p_C, q_C \in \mathbb{K}[[t]]$ respectively.

In particular, when two ideals are equal, it means that as points of the Hilbert scheme they are the same.

3. Motivic classes of fixed-generators Hilbert schemes of unibranch plane curves

To discuss motivic classes of $\mathrm{Hilb}_{0,k}^n(C)$ in general, we need to first simplify $\mathrm{Hilb}_{0,k}^n(C)$ as much as possible. The goal of this section is to construct some special representatives for the points of $\mathrm{Hilb}_{0,k}^n(C)$, that will lead to a partition of $\mathrm{Hilb}_{0,k}^n(C)$ into components whose motivic classes are easier to compute. We start with some notations we will use in the rest of the section.

Definition 3.1. Let $I \in \mathrm{Hilb}_{0,k}^n(C)$ be an ideal and let $g_1, \ldots, g_k$ denote a minimal set of generators. When considered in $\widehat{\mathcal{O}}_{C,0}$ we can write

$$g_i = \tau_{n_i} + \sum_{n_i < n \in \Gamma} \eta_n \tau_n$$

for $u_i \in \mathbb{K}[[t]]^\times$, τ_{n_i}, τ_n are defined as in Lemma 2.44, $\eta_n \in \mathbb{K}$ and

$$n_i = C_{g_i}.C$$

is the intersection number of C and the curve locally given by g_i by Lemma 2.38. We assume $n_1 \leq \cdots \leq n_k$ and we denote the k-uple they form by $\underline{n} = (n_1, \ldots, n_k)$.

The first thing we need to observe is that we can simplify the generators of the ideal in $\widehat{\mathcal{O}}_{C,0}$. Following the same reasoning as in Remark 2.55 and the same notations as in Theorem 2.54, we can consider the following simplified generators.

Remark 3.2. Let I be as in Definition 3.1. Then, taking $V = \coprod_i \{n_i + \Gamma\} \subseteq val_t(I)$ in Theorem 2.54 we obtain the representative

$$I = (f_1, \ldots, f_k)$$

with

$$f_i = \tau_{n_i} + \sum_{h \in H \mid n_i + h \in \Gamma} \eta_{n_i + h} \tau_{n_i + h}.$$

For now on we use this minimal set of generators, with a shift of the coefficients of f_i and denoting them as $\lambda_h^i = \eta_{n_i + h}$ for all $i = 1, \ldots, k$. We will also often omit the upper index i to ease the notation.

3.1. **Intersection numbers and** $\mathrm{Hilb}_{0,k}^n(C)$**.** In this section, we use the intersection numbers n_i of Definition 3.1 and Remark 3.2 to create the special representatives for points of $\mathrm{Hilb}_{0,k}^n(C)$ mentioned just before. This will also get us to a first, rough partition of $\mathrm{Hilb}_{0,k}^n(C)$. To being, we need to introduce a notion of independence for the n_i. The main idea behind such notion will be: we do not want the leading terms of generators in $\widehat{\mathcal{O}}_{C,0}$ to cancel, that is, we do not want the k-uple (which keeps track of the leading terms) to change when considering different representatives of the same ideal.

Definition 3.3. Let $I \in \mathrm{Hilb}_{0,k}^n(C)$ and let $g_1, \ldots, g_k$ be a minimal set of generators. Let $val_t(g_i) = n_i$ and let $\Delta_{g_i} = val_t(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_k)$ for every $i = 1, \ldots, k$. Then, we say that the valuations of this set of generators form an independent k-uple when

$$n_i \notin \Delta_{g_i}$$

for all $i = 1, \ldots, k$.

We now observe that for any given ideal $I \in \mathrm{Hilb}_{0,k}^n(C)$, out of all representatives of I and the corresponding minimal sets of generators and k-uples, there exists only one independent k-uple.

Theorem 3.4. Let $I \in \mathrm{Hilb}_{0,k}^n(C)$. Then, there exist a minimal set of generators $g_1, \ldots, g_k$ such that the valuations $val_t(g_i) = n_i$ form an independent k-uple in the sense of Definition 3.3, and this is the only possible independent k-uple of valuations arising from minimal sets of generators of I.

Proof. Let $\Delta = val_t(I)$. Given two k-uples of valuations $\underline{n}, \underline{n}'$ we say that $\underline{n} < \underline{n}'$ when there exists a $0 \le j \le k$ such that $n_i = n_i'$ for $1 \le i < j$ and $n_j < n_j'$. Then, let $\underline{m}$ be the maximal k-uple for this order and let $h_1, \ldots, h_k$ denote a minimal set of generators associated with $\underline{m}$. We now assume that $\underline{m}$ is not independent, that is, there exists a $1 \le r \le k$ such that $m_r \in \Delta_{h_r}$. This means

$$\sum_{i \ne r} a_i h_i = \tau^{m_r} + H.O.T.$$

for some $a_i \in \widehat{\mathcal{O}}_{C,0}$. Then, we can consider the new generator $h_r' = h_r - \sum_{i \ne r} a_i h_i$ with $val_t(h_r') = m_r' > m_r = val_t(h_r)$. We still have that $I = (h_1, \ldots, h_r', \ldots, h_k)$ and

$$(m_1, \ldots, m_r', \ldots, m_k) > \underline{m}$$

which is absurd. Therefore, $\underline{m}$ is an independent k-uple for I and the existence is proved.

Given the existence, we are now left with proving the uniqueness of the independent k-uple amongst all appearing from different representatives of I. Let $\underline{n}, \underline{n}'$ be two independent k-uples, say with $n_1 \le n_1'$, let $I \in \mathrm{Hilb}_{0,k}^n(C)$ and let $g_1, \ldots, g_k$ be a minimal set of generator such that $val_t(g_i) = n_i$. We want to prove that if we can find another minimal set of generators $g_1', \ldots, g_k'$ of I such that $val_t(g_i') = n_i'$, then $\underline{n} = \underline{n}'$. By Theorem 2.65, the change of representative is equivalent to the existence of an invertible matrix

$$U = \begin{pmatrix} a_1^1 & \ldots & a_1^k \\ & \ldots & \\ a_k^1 & \ldots & a_k^k \end{pmatrix} \in GL_k(\widehat{\mathcal{O}}_{C,0})$$

such that

(1) $$g_i = \sum_j a_j^i g_j'$$

for all $i = 1, \ldots, k$. We proceed by induction on i to prove the equality $\underline{n} = \underline{n}'$.

From Equation (1) for $i = 1$, it immediately follows for degree reasons that

$$n_1 = n_1'.$$

Then, we assume $n_i = n_i'$ for all $i = 1, \ldots l$ and we can assume $n_{l+1} \le n_{l+1}'$. We consider Equation (1) for $i = l + 1$. We observe that for all $i = 1, \ldots l$ by inductive hypothesis we have $n_i' = n_i < n_{l+1}$. Therefore, the equation implies

$$n_{l+1} \in val_t(g_1, \ldots, g_l) \subseteq \Delta_{g_{l+1}} \quad \text{or} \quad a_j^{l+1} = 0 \text{ for all } j = 1, \ldots l.$$

Since the first condition would go against the independence hypothesis, it follows that $a_j^{l+1} = 0$ for all $j = 1, \ldots, l$. The same equation now becomes

$$g_{l+1} = a_{l+1}^{l+1} g_{l+1}' + H.O.T.$$

and, since $n'_{l+1} \geq n_{l+1}$, for degree reasons it follows that

$$n'_{l+1} = n_{l+1}.$$

$\square$

From now on, we will always assume to work with independent k-uples.

Remark 3.5. Applying this proposition to the set of generators of Remark 3.2, we obtain a presentation of the generators f_i further simplified, since $\coprod_{j \neq i} n_j + \Gamma \subseteq \Delta_{f_i}$. This generating set is also compatible with the one of Lemma 2.57.

This allows us to define the following components.

Definition 3.6. Let $\underline{n}$ be an independent k-uple as in Definition 3.3. Following the notation of Definition 3.1, we denote by

$$\mathrm{Hilb}_{0,k}^{n,\underline{n}}(C) = \{I \in \mathrm{Hilb}_{0,k}^{n}(C) \mid I = (g_1, \ldots, g_k), val_t(g_i) = n_i\}.$$

To conclude, we observe that thanks to Theorem 3.4, the components we just defined lead to the partition

$$\mathrm{Hilb}_{0,k}^{n}(C) = \coprod_{\underline{n}} \mathrm{Hilb}_{0,k}^{n,\underline{n}}(C).$$

Many of these components will be empty as random k-uples $\underline{n}$ will not necessarily appear for a fixed length n. In the next section, we study what contributes to a given length n.

3.2. **Semigroup ideals appearing in** $\mathrm{Hilb}_{0,k}^{n,\underline{n}}(C)$**.** The goal of this section is to understand what k-uples contribute to a given length n. To do so, we require the additional information provided by the semigroup ideal structure of Γ, since we will later realize that the values appearing in the k-uple, that is, the t-adic valuation of the corresponding set of generators, are not enough. We start by introducing the following map.

Definition 3.7. Let $n, k \geq 1$ and let $\underline{n} = (n_1, \ldots, n_k)$ be an independent k-uple as in Definition 3.3. We define

$$\Psi_{\underline{n}} : \prod_{i=1}^{k} \mathrm{Hilb}_{0,1}^{n_i}(C) \to \coprod_n \mathrm{Hilb}_{0,k}^{n,\underline{n}}(C) \, , \ ((f_1), \ldots, (f_k)) \mapsto (f_1, \ldots, f_k)$$

where we follow the notations of Remark 3.2.

We use principal Hilbert schemes $\mathrm{Hilb}_{0,1}^{n_i}(C)$ as their points exactly correspond to elements of valuation n_i. Then, to determine the correct image of $\Psi_{\underline{n}}$, namely the length n, it is not enough to only consider the k-uple of valuations $\underline{n}$.

Definition 3.8. Let $\Delta \subseteq \Gamma$ be a semigroup ideal such that $|\Gamma \setminus \Delta| = n$. We denote by

$$\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C) = \{I \in \mathrm{Hilb}_{0,k}^{n,\underline{n}}(C) \mid val_t(I) = \Delta\}.$$

This is because, in general, given a semigroup ideal Δ in Definition 3.8, we only have the inclusion

$$\coprod_i \{n_i + \Gamma\} \subseteq \Delta$$

and not the equality. This means that the set of non-negative integers n_i might be just a proper subset of the generating set of the semigroup ideal Δ and the equality is obtained through non-trivial $\widehat{\mathcal{O}}_{C,0}$-combinations of f_i. In other words, we need to take into account the syzygies of Δ to determine the correct length. This will be done in the next section, where we substitute the constraint $val_t(I) = \Delta$ determining the component $\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)$ with the equation arising from the syzygies of Δ.

Finally, using these new components we obtain the partition

$$\mathrm{Hilb}_{0,k}^{n}(C) = \coprod_{\underline{n},\Delta} \mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)$$

which was our initial goal.

We now introduce some morphisms that will support us in studying the map $\Psi_{\underline{n}}$ and explicating the relationship between one-generator and fixed-generators loci.

Definition 3.9. Let $n \geq 1$ and let $\Delta \subseteq \Gamma$ be a semigroup ideal such that $n \notin \Delta$. Regarding $\mathrm{Hilb}_{0,1}^n(C)$ as an affine space, as per Remarks 2.55, we define the morphism

$$\pi_\Delta : \mathrm{Hilb}_{0,1}^n(C) \to \mathrm{Hilb}_{0,1}^n(C) , \ I = (\lambda_i)_{i \in \{n+H\} \cap \Gamma} \mapsto (\mu_i)_{i \in \{n+H\} \cap \Gamma}$$

with

$$\mu_i = \begin{cases} \lambda_i & \text{for } i \in \{n+H\} \cap \Gamma \setminus \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.10. We observe that the generators $\pi_\Delta(f_i)$ exactly correspond to the generators considered in Lemma 2.57.

The morphisms π_Δ allow us to express $\Psi_{\underline{n}}$ in a more informative way, making us able to prove the following result.

Theorem 3.11. *Let $\Delta \subseteq \Gamma$ be a semigroup ideal and let $I \in \mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)$. Let $S_i = \{n_i + H\} \cap \Delta$ and let $F_j = \mathbb{A}^{|S_i|}$. The map $\Psi_{\underline{n}}$ is a trivial fibration with fiber*

$$\prod_i F_i = \mathbb{A}^{\sum_i |S_i|}$$

that is,

$$\Psi_{\underline{n}}^{-1}(\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)) \cong \mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C) \times \mathbb{A}^{\sum_i |S_i|}.$$

Proof. We recall that $\Psi_{\underline{n}}$ is defined as (see Remark 3.10)

$$((f_1), \dots, (f_n)) \mapsto (f_1, \dots, f_k) = (\pi_\Delta((f_1)), \dots, \pi_\Delta((f_k))).$$

We denote by $I_j = \{n_j + H\} \cap \Delta$ and we construct the morphism

$$\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C) \times F_1 \times \cdots \times F_k \to \Psi_{\underline{n}}^{-1}(\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)),$$

$$(f_1, \dots, f_k), (\lambda_i^1)_{i \in I_1}, \dots, (\lambda_i^k)_{i \in I_k} \mapsto$$

$$\left(\pi_\Delta(f_1) + \sum_{i \in I_1} \lambda_i^1 \tau_i \right), \dots, \left(\pi_\Delta(f_k) + \sum_{i \in I_k} \lambda_i^k \tau_i \right)$$

and the inverse (up to change of U_Δ-representatives, see Remark 3.10) is

$$\Psi_{\underline{n}}^{-1}(\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)) \to \mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C) \times F_1 \times \cdots \times F_k,$$

$$(f_1), \dots, (f_k) \mapsto (\pi_\Delta(f_1), \dots, \pi_\Delta(f_k)), (\lambda_i^1)_{i \in I_1}, \dots, (\lambda_i^k)_{i \in I_k}.$$

$\square$

We translate the isomorphism of Theorem 3.11 into an equality of motivic classes.

Corollary 3.12. *Let $(C, 0)$ be a unibranch curve. We have*

$$[\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)] = [\Psi_{\underline{n}}^{-1}(\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C))]\mathbb{L}^{-\sum_{i=1}^n |\{n_i+H\} \cap \Delta|}$$

for every $n \geq 1, \underline{n}, \Delta$.

This way, we obtain a first description for the motivic class $[\mathrm{Hilb}_{0,k}^n(C)]$.

Corollary 3.13. *Let $(C, 0)$ be a unibranch curve. We have*

$$[\mathrm{Hilb}_{0,k}^n(C)] = \sum_{\underline{n},\Delta} [\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)] = \sum_{\underline{n},\Delta} [\Psi_{\underline{n}}^{-1}(\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C))]\mathbb{L}^{-\sum_{i=1}^n |\{n_i+H\} \cap \Delta|}$$

for every $n, k \geq 1$.

We are left with studying more in detail $[\Psi_{\underline{n}}^{-1}(\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C))]$. To do this, as mentioned at the end of Section 3.2, we will use conditions arising from the syzygies of Δ, which will substitute the general condition $val_t(I) = \Delta$ defining $[\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)]$ and explicitly describe $\Psi_{\underline{n}}^{-1}(\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C))$.

Theorem 2.62 provides us with equivalent, simpler conditions to describe the syzygies of a semigroup ideal: they take form as certain hyperplane conditions on the coefficients of the generators of the points of $\mathrm{Hilb}_{0,1}^{n_i}(C)$. Applying this theorem, we obtain the following result.

Corollary 3.14. *Let $(C, 0)$ be a unibranch curve. We have*

$$[\mathrm{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)] = \mathbb{L}^{\sum_{i=1}^n |\Gamma_{>\nu_i} \setminus \Delta| - \sum_{i=1}^n |\{n_i + H\} \cap \Delta|} \sum_{a=1\ldots m_\Delta} [S_a] \mathbb{L}^{-a}$$

for every $n \geq 1, \underline{n}, \Delta$ where

$$(M_\Delta(p))_{i(jk)} = u_i^j \phi_{\sigma_i - \nu_j} \phi_k \,, \quad S_a = \{p \in Syz \mid rk(M_\Delta(p)) = a\}$$

is the stratum along which the motivic class of hyperplanes for $p \in S_a$ is constantly equal to

$$\mathbb{L}^{\sum_{i=1}^n |\Gamma_{>\nu_i} \setminus \Delta| - a}.$$

Proof. We follow the notations of Definition 2.61. For any $p = (x_{is}^{\sigma_i})_{is} \in Syz$, from Theorem 2.62 we obtain the hyperplane equations

$$F_p^i(\lambda) = \sum_j \left(u_i^j \phi_{\sigma_i - \nu_j} \phi_{\nu_j} + \sum_{c(\Delta)>s>\sigma_i} x_{is}^{\sigma_i} \phi_{s - \nu_j} \phi_{\nu_j} + \sum_{k \in \Gamma_{>\nu_j} \setminus \Delta} u_i^j \lambda_k^{\nu_j} \phi_{\sigma_i - \nu_j} \phi_k \right)$$

for $i = 1, \ldots, m_\Delta$. We can then consider the $m_\Delta \times \dim(Gen)$-matrix of the coefficients of the equations F_p^i, that is,

$$(M_\Delta(p))_{i(jk)} = u_i^j \phi_{\sigma_i - \nu_j} \phi_k.$$

Finally, the rank is enough to determine the generic hyperplane arrangement: if $rk(M_\Delta(p)) = a$ then

$$\dim(F_p^1 \cap \cdots \cap F_p^{m_\Delta}) = \dim(Gen) - a = \sum_{i=1}^n |\Gamma_{>\nu_i} \setminus \Delta| - a.$$

This gives us a partition of Syz in strata along which the motivic class of the hyperplane arrangement in Gen is constantly equal to

$$\mathbb{L}^{\sum_i |\Gamma_{>\nu_i} \setminus \Delta| - a}$$

and we are done. □

Eventually, this leads to the final description of $[\mathrm{Hilb}_{0,k}^n(C)]$. This extends the results of [18, Section 3.1] since no formula is given for $[\mathrm{Hilb}_{0,k}^n(C)]$.

Corollary 3.15. *Let $(C, 0)$ be a unibranch curve. We have*

$$[\mathrm{Hilb}_{0,k}^n(C)] = \sum_\Delta \mathbb{L}^{\sum_{i=1}^n |\Gamma_{>\nu_i} \setminus \Delta|} \sum_{\underline{n}} \mathbb{L}^{-\sum_{i=1}^n |\{n_i + H\} \cap \Delta|} \sum_{a=1\ldots m_\Delta} [S_a] \mathbb{L}^{-a}$$

for every $n, k \geq 1$.

We now present an example of application of Corollary 3.15, to further develop the computations presented in [19, Section 7] and [20, Proposition 10]. The ideals cannot be read from [19, Section 7], and the number of generators is not controlled in [19, Section 7].

Example 3.16. We consider the ring

$$\widehat{\mathcal{O}}_{C,0} = \mathbb{K}[[t^4, t^6 + t^7]]$$

with

$$\Gamma = \{0, 4, 6, 8, 10, 12, 13, 14, 16, \ldots\}.$$

The holes are $H = 1, 2, 3, 5, 7, 9, 11, 15$ and $\delta = |H| = 8$.

We compute motivic classes and Euler characteristics of $\mathrm{Hilb}_{0,k}^4(C)$ and match the Euler characteristics with the ones computed in [19, Section 7]. We start with $n = 4$ because it is the first "interesting" length, since $\mathrm{Hilb}_{0,k}^2(C)$ and $\mathrm{Hilb}_{0,k}^3(C)$ just correspond to the same schemes but for $\mathbb{A}^2$.

In the partition of $\mathrm{Hilb}_{0,k}^4(C)$, it is enough to consider the index $\underline{n}$ as the syzygies are very easy to control, as presented in [20, Proposition 10]. The only independent k-uples appearing for $n = 4$ are the following:

- $k = 1$. We only have $n_1 = 4$ and this corresponds to $\mathrm{Hilb}_{0,1}^4(C)$. We have that

$$[\mathrm{Hilb}_{0,1}^4(C)] = \mathbb{L}^3$$

(as confirmed by the Alexander polynomial or by the expression of Remark 2.55) and $\chi(\mathrm{Hilb}_{0,1}^4(C)) = 1$.

- $k = 2$. We have the 2-tuples $n_1 = 6, n_2 = 17$ and $n_1 = 8, n_2 = 13$. We have that

$$[\Psi_{\underline{n}}(\mathrm{Hilb}_{0,1}^8(C), \mathrm{Hilb}_{0,1}^{13}(C))] = \mathbb{L}^3$$

$$[\Psi_{\underline{n}}(\mathrm{Hilb}_{0,1}^6(C), \mathrm{Hilb}_{0,1}^{17}(C))] = \mathbb{L}.$$

In total,

$$[\mathrm{Hilb}_{0,2}^4(C)] = \mathbb{L}^3 + \mathbb{L}$$

and $\chi(\mathrm{Hilb}_{0,2}^4(C)) = 2$.
- $k = 3$. We have the 3-tuples $n_1 = 8, n_2 = 10, n_3 = 19$ and $n_1 = 10, n_2 = 12, n_3 = 13$. We have that

$$[\Psi_{\underline{n}}(\mathrm{Hilb}_{0,1}^8(C), \mathrm{Hilb}_{0,1}^{10}(C), \mathrm{Hilb}_{0,1}^{19}(C))] = \mathbb{L}^2$$

$$[\Psi(\mathrm{Hilb}_{0,1}^{10}(C), \mathrm{Hilb}_{0,1}^{12}(C), \mathrm{Hilb}_{0,1}^{13}(C))] = 1.$$

In total,

$$[\mathrm{Hilb}_{0,3}^4(C)] = \mathbb{L}^2 + 1$$

and $\chi(\mathrm{Hilb}_{0,3}^4(C)) = 2$.

These computations match the ones in [19, Section 7]. Indeed, we recall the formula for the generating series of the Euler characteristics in the special case of $\widehat{\mathcal{O}}_{C,0} = \mathbb{K}[[t^4, t^6 + t^7]]$ of [19, Equation (14)]. We have

$$\sum_{l,k} q^{2l}(1 - a^2)^{k-1}\chi(\mathrm{Hilb}_{0,k}^l(C)) = \sum_{i,\Delta,k} q^{2(i-(8-|\mathbb{N}\backslash\Delta|))=2l}(1 - a^2)^{k-1}\chi(U_{i+\Delta,k})$$

since on the components $U_{i+\Delta} \subseteq \mathrm{Hilb}_0^l(C)$ the number of generators might not be constant. According to [19, Table 1] these are the only i, Δ contributing to $n = l = 4$ and the number of generators and Euler characteristics can be extracted from [19, Section 7].

- $i = 4, \Delta = 0$: it corresponds to our $n_1 = 4$. We have $\chi(\mathrm{Hilb}_{0,1}^4(C)) = 1$.
- $i = 6, \Delta = (0, 11)$: it corresponds to our $n_1 = 6, n_2 = 17$. We have that $\chi(U_\Delta) = 1$ from [19, Corollary 31].
- $i = 8, \Delta = (0, 5)$: it corresponds to our $n_1 = 8, n_2 = 13$. We have $\chi(U_\Delta) = 1$ from [19, Corollary 31].
- In total, we have $\chi(\mathrm{Hilb}_{0,2}^4(C)) = 2$.
- $i = 8, \Delta = (0, 2, 9, 11)$: it corresponds to our $n_1 = 8, n_2 = 10, n_3 = 19$. We have $\chi(U_\Delta) = 1$ from [19, Pg. 18].
- $i = 10, \Delta = (0, 2, 3)$: it corresponds to our $n_1 = 10, n_2 = 12, n_3 = 13$. We have $\chi(U_\Delta) = 1$ from [19, Corollary 31].
- In total, we have $\chi(\mathrm{Hilb}_{0,3}^4(C)) = 2$.

One could think that also $i = 6, \Delta = (0, 2)$ contributes to $n = l = 4$. However, that is not the case since $(6, 8)$ is not a semigroup ideal coming from any actual ideal. Hence, the lists above are complete.

We conclude by observing that the polynomiality and positivity of $[\mathrm{Hilb}_{0,k}^n(C)]$ are determined by the same properties for $[S_a]$. We mention here that the variety S_a is just a subvariety of the rank a-determinantal variety, as not all matrices will appear as $M_\Delta(p)$ for some $p \in Syz$.

4. The case of (p, q)-curves

This section is dedicated to the case of (p, q)-curves, since all the results of Section 3 can be further simplified in this case. This is mainly because given $I \in \mathrm{Hilb}_{0,k}^{n,n}(C)$ we can immediately say that

$$(2) \qquad\qquad val_t(I) = \coprod_i \{n_i + \Gamma\}$$

since the $\widehat{\mathcal{O}}_{C,0}$-linear combinations of the f_i exactly correspond to Γ-combinations of the n_i.

This is proved in the following theorem, also implying that the length is fully determined by the independent k-uple of Proposition 3.6. We also remark here that this statement follows by [19, Corollary 23] using Equation (2).

Theorem 4.1. Let $I \subseteq \mathrm{Hilb}_{0,k}^{\star,n}(C)$. We denote by

$$\Gamma_{\underline{n}} = \coprod_i \{n_i + \Gamma\}.$$

Then, we have

$$\star = n_1 + \delta + |\Gamma_{\underline{n}}| = n_1 - \delta + |\mathbb{N} \setminus \coprod_i \{(n_i - n_1) + \Gamma\}|$$

where δ notes the delta invariant of the curve $(C,0)$.

Proof. Let $Y = \mathcal{O}_C/I$ and $X = \mathbb{C}[[t]]/(t^{n_1})$. Let

$$K = \ker(\mathcal{O}_C/I \xrightarrow{f} \mathbb{K}[[t]]/(t^{n_1})) = \ker(Y \xrightarrow{f} X) \text{ and } Q = \operatorname{coker}(\mathcal{O}_C/I \xrightarrow{f} \mathbb{K}[[t]]/(t^{n_1})) = \operatorname{coker}(Y \xrightarrow{f} X)$$

where the map $f : Y \to X$ is induced to quotients by the inclusion $i : \mathcal{O}_C \hookrightarrow \mathbb{K}[[t]]$ noticing that $i(I) = (t^{n_1})$. Using the exact sequence of finite $\mathbb{K}$-modules

$$0 \to K \to Y \to X \to Q \to 0$$

we want to find $\star = \dim_{\mathbb{K}} \mathcal{O}_C/I = \dim_{\mathbb{K}} Y$. We have

(3) $$\star = \dim_{\mathbb{K}} X + \dim_{\mathbb{K}} K - \dim_{\mathbb{K}} Q = n_1 + \dim_{\mathbb{K}} K - \dim_{\mathbb{K}} Q.$$

Then, consider the commutative diagram with exact sequences as rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \lhook\joinrel\longrightarrow & \mathcal{O}_C & \longtwoheadrightarrow & \mathcal{O}_C/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & (t^{n_1}) & \lhook\joinrel\longrightarrow & \mathbb{K}[[t]] & \longtwoheadrightarrow & \mathbb{K}[[t]]/(t^{n_1}) & \longrightarrow & 0.
\end{array}
$$

By the snake lemma, we get the exact sequence of finite $\mathbb{K}$-modules

$$0 \to K \to (t^{n_1})/I \to \mathbb{K}[[t]]/\mathcal{O}_C \to Q \to 0.$$

We recall that $\delta = \dim_{\mathbb{K}} \mathbb{K}[[t]]/\mathcal{O}_C$. Therefore, from the exact sequence we have

$$\dim_{\mathbb{K}} Q = \dim_{\mathbb{K}} K - \dim_{\mathbb{K}}(t^{n_1})/I + \dim_{\mathbb{K}} \mathbb{K}[[t]]/\mathcal{O}_C = \dim_{\mathbb{K}} K - \dim_{\mathbb{K}}(t^{n_1})/I + \delta.$$

Combining this into Equation (3) we get

$$\star = n_1 + \dim_{\mathbb{K}} K - (\dim_{\mathbb{K}} K - \dim_{\mathbb{K}}(t^{n_1})/I + \delta) = n_1 + \dim_{\mathbb{K}}(t^{n_1})/I - \delta.$$

Finally, we are just left with computing $\dim_{\mathbb{K}}(t^{n_1})/I$. We recall that $I = (t^{n_1}u_1, \ldots, t^{n_k}u_k)$ with $n_1 \leq \cdots \leq n_k$. Up to a t^{n_1}-shift, the quotient $(t^{n_1})/I$ is isomorphic to

$$(1)/(u_1, \ldots, t^{n_k - n_1}u_k) = \mathbb{K}[[t]]/\widehat{\mathcal{O}}_{C,0} + \cdots + t^{n_k - n_1} \cdot \widehat{\mathcal{O}}_{C,0}$$

as $\mathbb{K}$-modules. In terms of powers of powers of t i.e. the basis, the quotient corresponds to removing $\{\Gamma \cup \ldots \cup (n_k - n_1) + \Gamma\}$-powers from $\mathbb{N}$-powers. $\qquad\square$

Remark 4.2. We observe that Theorem 4.1 in the principal case recovers the usual intersection number, or also [19, Lemma 9]. Indeed, $|\mathbb{N} \setminus \coprod_i \{(n_i - n_1) + \Gamma\}| = |\mathbb{N} \setminus \Gamma| = \delta$ thus $n = n_1 - \delta + \delta = n_1$.

As for Theorem 3.11 and its proof, the map $\Psi_{\underline{n}}$ remains a trivial fibration.

Corollary 4.3. *In the same setting of Theorem 4.1, let $S_i = \{n_i + H\} \cap \Gamma_{\underline{n}}$ and let $F_j = \mathbb{A}^{|S_i|}$. The map $\Psi_{\underline{n}}$ is a trivial fibration with fiber*

$$\prod_i F_i = \mathbb{A}^{\sum_i |S_i|}$$

that is,

$$\Psi_{\underline{n}}^{-1}(\operatorname{Hilb}_{0,k}^{n,\underline{n},\Delta}(C)) \cong \operatorname{Hilb}_{0,k}^{n,\underline{n}}(C) \times \mathbb{A}^{\sum_i |S_i|}.$$

Finally, in the case of (p,q)-curves, we obtain a stronger structural result for $[\operatorname{Hilb}_{0,k}^n(C)]$, since from Remark 2.55 we know that $[\operatorname{Hilb}_{0,1}^{n_i}(C)]$ is a power of $\mathbb{L}$.

We denote by

$$d(\underline{n}) = n_1 + \delta + |\Gamma_{\underline{n}}|$$

the formula provided by Theorem 4.1 and by $M_n = \{\underline{n} \text{ independent} \mid d(\underline{n}) = n\}$.

Corollary 4.4. *Let $(C,0)$ be a (p,q)-curve. We have*

$$[\mathrm{Hilb}^n_{0,k}(C)] = \sum_{\underline{n} \in M_n} [\mathrm{Hilb}^{n;\underline{n}}_{0,k}(C)] =$$

$$\sum_{\underline{n} \in M_n} \left(\prod_i [\mathrm{Hilb}^{n_i}_{0,1}(C)] \right) \mathbb{L}^{-\sum_i |\{n_i + H\} \cap \Gamma_{\underline{n}}|} = \sum_{\underline{n} \in M_n} \mathbb{L}^{\sum_i |\{n_i + H\} \cap \Gamma| - \sum_i |\{n_i + H\} \cap \Gamma_{\underline{n}}|}$$

for every $n, k \geq 1$.

Remark 4.5. As a side note, we observe that M_n also corresponds to the set of all possible sets of semigroup ideal generators, for semigroup ideals such that

$$\left| \Gamma \setminus \coprod_i \{n_i + \Gamma\} \right| = n.$$

Since the semigroup is equivalent to the topology as explained in [2, Theorem 21], the polynomials $[\mathrm{Hilb}^n_{0,k}(C)]$ also admit a topological meaning for curve singularities.

Corollary 4.6. *Let $(C,0)$ be a (p,q)-curve. Then, $[\mathrm{Hilb}^n_{0,k}(C)]$ is a polynomial in $\mathbb{L}$ with positive coefficients for every $n, k \geq 1$. Moreover, this polynomial is a topological invariant since Γ is.*

We conclude this section by noting that Corollaries 4.4 and 4.6 generalize the affine cell decomposition proposed by [20, Theorem 12], as well as the results of [10, Theorem 2.11] for the cells dimension. The corollaries also extend the results of [18, Section 3.2] since the polynomiality can be deduced from [18, Proposition 6], but no formula is directly given for $[\mathrm{Hilb}^n_{0,k}(C)]$. Moreover, the positivity of the coefficients and the topological invariance, as well as the relationship to principal Hilbert schemes, are new to the best of our knowledge.

5. Contact loci and motivic classes of unibranch Hilbert schemes of plane curves

This section aims to relate the contact locus $\mathcal{X}^n_C = \mathcal{X}^n$ of order n of a given unibranch curve $(C,0)$, introduced in Definition 2.14, to the principal Hilbert schemes of the given curve. Thinking of points of $\mathcal{L}$ as parameterizations of unibranch plane curves in $\mathbb{A}^2$, we naturally have a map to principal Hilbert schemes, opposite to the map $Z : \widehat{\mathcal{O}}_{C,0} \to \coprod_k S^k \mathcal{L}$ introduced in [9].

Remark 5.1. We mention here that no result in this section makes strict use of the unibranch condition, which could then be removed. We just keep this condition to work in the same setting used throughout the rest of this paper.

Remark 5.2. By definition, the points of $\mathcal{X}^n \subseteq \mathcal{L}$ are pairs of formal power series $p, q \in \mathbb{K}[[t]]$ such that $val_t(f(p,q)) = n$. From Lemma 2.38, up to excluding the measure-zero set of degenerate pairs in $\mathcal{X}^n$ (see Remark 2.31) this means the curve parameterized by (p,q) has intersection number n with the curve C. Moreover, we observe that for every $I = (g,f) \in \mathrm{Hilb}^n_{0,1}(C)$ we have $(x,y)^n \subseteq I$. Therefore, different arcs γ, γ' such that $\pi_{n+1}(\gamma) = \pi_{n+1}(\gamma')$ correspond to the same point of $\mathrm{Hilb}^n_{0,1}(C)$.

Remark 5.3. Following the notation of Definition 2.36, let $i \in I$ index the set of coefficients $a_{m_i} \neq 0$. Then, the degenerate condition translates to the existence of some $k > 1$ such that $gcd(n, m_i) = k$ for all $i \in I$. From the proof of Theorem 2.34, it also follows that Puiseux parameterizations are non-degenerate.

Definition 5.4. Let $n \geq 1$. We define the map

$$\Pi : \pi_{n+1}(\mathcal{X}^n) \to \mathrm{Hilb}^n_{0,1}(C), \ \gamma \mapsto \mathrm{Im}(\gamma) = \mathrm{Im}(\gamma) \times_{\mathbb{A}^2} C$$

where Im denotes the scheme-theoretic image.

Theorem 5.5. *The map Π of Definition 5.4 is a morphism.*

Proof. The proof follows the same construction as [23, Theorem 3.14]. More in detail, we immediately observe that Π is a well-defined map since $\mathrm{Im}(\gamma)$ is a finite length scheme on C. Moreover, working locally the map Π is defined as $\gamma \mapsto (\ker(\gamma), f)$ where $f \in \mathbb{K}[[x,y]]$ locally defines C.

Then, we have to prove that Π is a morphism. It is not straightforward to deduce it directly from its definition, so we will use the representability of the more general Hilbert functor of Definition 2.45, constructing a morphism that will correspond to the same map on the $\mathbb{K}$-points of $\pi_{n+1}(\mathcal{X}^n)$ and $\mathrm{Hilb}^n_{0,1}(C)$. We denote by $\mathcal{Z}$ the universal subscheme, corresponding under representability to id_C and consider the subvariety

$$Z_J \subseteq C \times \pi_{n+1}(\mathcal{X}^n), \ Z_J = \{(supp(\phi) \cap m^k, \phi)\}.$$

The projection p is clearly finite, flat and surjective of degree n - having as fibers the n points of the support of the corresponding parameterized subscheme of length n at m. Then, we consider by representability the morphism Π such that $Z_J = (H(\Pi))(\mathcal{Z}) = \mathcal{Z} \times_{\mathrm{Hilb}_{0,1}^n(C)} \pi_{n+1}(\mathcal{X}^n)$.

$$
\begin{array}{ccc}
& & \mathcal{Z} \\
& & \downarrow \\
Z_J \hookrightarrow C \times \pi_{n+1}(\mathcal{X}^n) \xrightarrow{id_C \times \widetilde{\Pi}} C \times \mathrm{Hilb}_{0,1}^n(C) \\
\downarrow p & & \downarrow \pi_2 \\
\pi_{n+1}(\mathcal{X}^n) \xrightarrow{\qquad \widetilde{\Pi} \qquad} \mathrm{Hilb}_{0,1}^n(C)
\end{array}
$$

By definition of Z_J, the morphism $\widetilde{\Pi}$ is then defined as associating to an arc the corresponding scheme as a point of the principal Hilbert scheme $\mathrm{Hilb}_{0,1}^n(C)$, corresponding to Π. Finally, on $\mathbb{K}$-points it corresponds to the explicit Π considered above by construction. $\qquad\square$

Definition 5.6. We say that a point $I \in \mathrm{Hilb}_{0,1}^n(C)$ is when $I \in \mathrm{Im}(\Pi)$. This equivalently means the principal ideal $I \subseteq \widehat{\mathcal{O}}_{C,0}$ admits a generator $\bar{g} \in \widehat{\mathcal{O}}_{C,0} = \mathbb{K}[[x,y]]/(f)$ such that $g \in \mathbb{K}[[x,y]]$ is irreducible. We denote by

$$\mathrm{UHilb}_{0,1}^n(C)$$

the open component of $\mathrm{Hilb}_{0,1}^n(C)$ determined by the image of Π.

The rest of the section will be dedicated to studying the image and the fibers of Π. In particular, the interest in the open component $\mathrm{UHilb}_{0,1}^n(C) = \mathrm{Im}(\Pi)$ in the one-generator locus will be motivated by the Igusa zeta function, as its motivic class $[\mathrm{UHilb}_{0,1}^n(C)]$ will be related to the coefficients of $Z_C(s)$, and a threshold that makes it always non-empty therefore meaningful. However, this is not all: we will also conjecture the surjectivity of Π from the same mentioned threshold, hence conjecturing the equality $\mathrm{UHilb}_{0,1}^n(C) = \mathrm{Hilb}_{0,1}^n(C)$ of the unibranch open and the entire one-generator locus.

We conclude the first part of this section with an example that shows a limitation to the surjectivity of the map Π.

Example 5.7. Consider the cusp $(C,0)$ of Example 2.23. It is easy to see that

$$\mathcal{X}^5 = \emptyset$$

while

$$\mathrm{Hilb}_{0,1}^5(C) \cong \mathbb{P}^1.$$

This implies that Π cannot be surjective for $n = 5$, since $\mathrm{Im}(\Pi) = \emptyset \subseteq \mathrm{Hilb}_{0,1}^5(C)$.

5.1. **Holes of the Igusa zeta function and unibranch Hilbert scheme.** The goal of this section is to show that the irreducibility condition of Definition 5.6 is not too strict, as there exists a threshold N from which $\mathrm{UHilb}_{0,1}^n(C) \neq \emptyset$ for $n \geq N$, hence defining an interesting open of $\mathrm{Hilb}_{0,1}^n(C)$. Moreover, we will see that the threshold is entirely determined by (the topology of) the curve C.

From the definition of Π, it follows that $\mathcal{X}^n = \emptyset$ if and only if $\mathrm{UHilb}_{0,1}^n = \mathrm{Im}(\Pi) = \emptyset$. We already presented an example of this behaviour in Example 5.7. Showing that there exists a threshold from which the contact loci become non-empty is the central argument leading to the following result.

Proposition 5.8. *Let $\rho : (Y, E) \to (\mathbb{A}^2, C)$ be an embedded resolution of the pair $(\mathbb{A}^2, C)$ as in Definition 2.18. Let N_i denote the multiplicity of E_i as in Definition 2.22 and let $N = \max_i N_i$. Then,*

$$\mathrm{UHilb}_{0,1}^n(C) \neq \emptyset$$

for all $n \geq N$.

Proof. We want to show that $\mathcal{X}^n$ is non-empty for $n \geq N$, since this would imply that $\mathrm{UHilb}_{0,1}^n(C)$ is also non-empty. Let i' be the index of the divisor corresponding to N. Equivalently, it corresponds to the last divisor obtained by the last blow-up of ρ. By Theorem 2.24, we know that $\mathcal{X}^n \neq \emptyset$ when

$$E_J^o \neq \emptyset \quad \text{and} \quad \sum_{j \in J} k_j N_j = n$$

for some $J \subseteq I$ and $k_j > 0$. Since ρ is the embedded resolution of a plane curve, it is known that we can only have three cases:

- Consider $E_J = E_i$: then, we have contributions to degrees kN_j for all $k > 0$. In particular, we have N of the divisor $E_{i'}$.
- Consider $E_J = E_i \cap E_j$ for $E_i \cap E_j \neq \emptyset$: then, we have contributions to degrees $hN_i + eN_j$ for all $k, e > 0$.
- Consider $E_J = E_{i'} \cap \tilde{C}$ (as the last divisor is the only one intersecting the strict transform): then, we have contributions to degrees $hN + e \times 1$ for all $k, e > 0$. In particular, we have $N + e$ for every $e > 0$.

$\square$

Remark 5.9. Proposition 5.8 and its proof are stated in the setting of unibranch curves, as that is the case of interest for this paper. However, the proof admits an immediate generalization to the case of multibranch curves. Indeed, given $C = \coprod_{l=1}^{b} C_l$ a multibranch curve, it is enough to consider all the contributions $E_J = E_{i'} \cap \tilde{C}_l$ for every $l = 1, \ldots, b$.

We conclude this part with a conjecture, supported by computations in some examples that we omit for the sake of brevity. We expect the open component $\mathrm{UHilb}_{0,1}^n(C)$ not only to be non-empty starting from the threshold $n \geq N$ but actually to retrieve the entire one-generator locus if n is big enough.

Conjecture 5.10. *Let $\rho : (Y, E) \to (\mathbb{A}^2, C)$ be an embedded resolution of the pair $(\mathbb{A}^2, C)$ as in Definition 2.18. We have*

$$\mathrm{UHilb}_{0,1}^n(C) = \mathrm{Hilb}_{0,1}^n(C)$$

for all $n >> 0$.

5.2. **The space of branches and unibranch Hilbert schemes.** The goal of this section is to use Π and the smooth reparameterizations of Definition 2.30 to relate $[\mathrm{UHilb}_{0,1}^n(C)]$, hence conjecturally $[\mathrm{Hilb}_{0,1}^n(C)]$ and $[\mathrm{Hilb}_{0,k}^n(C)]$ in the case of (p, q)-curves thanks to Corollary 4.4, to the motivic measure $\mu(\mathcal{X}^n) = \frac{[\pi_{n+1}(\mathcal{X}^n)]}{\mathbb{L}^{2(n+1)}}$ of the n-th contact locus of C. We start by introducing the action of smooth reparameterizations on (truncated) arcs, originally considered in [12], and understand how it relates to Hilbert schemes.

Definition 5.11. We denote by S_0^{n+1} the group of automorphisms of $\mathbb{K}[[t]]/(t^{n+1})$ induced by elements of valuation 1 in $\mathbb{K}[[t]]/(t^{n+1})$, as per Definition 2.30 under the truncation map π_{n+1}. On $\mathcal{L}$ we consider the action induced by S_0^{n+1} and the composition, defined as

$$\bullet : S_0^{n+1} \times \pi_{n+1}(\mathcal{X}^n) \to \pi_{n+1}(\mathcal{X}^n), \ (\alpha, \phi) \mapsto \phi \bullet \alpha = \phi \circ \mathrm{Spec}(\alpha).$$

Remark 5.12. This is equivalent to the $\mathrm{Aut}_{\mathbb{K}}$-action of [9, Section 2] and [12, Section 2], as well as [23, Definition 3.4]. Moreover, with the quotient

$$\mathcal{B} = \mathcal{L}/S_0^{n+1}, \ p_b : \mathcal{L} \to \mathcal{B}$$

by the S_0^{n+1}-action we obtain the space of branches $\mathcal{B}$ in the sense of [9, Section 2, Definition] and [12, Section 2, pg. 7]. Given $\gamma \in \mathcal{L}$ we will denote by

$$b_\gamma = [\gamma] = S_0^{n+1} \bullet \gamma$$

its corresponding class in $\mathcal{B}$. The motivic measure μ and the truncation maps π_k also descend to the branch space, as described in [12].

Remark 5.13. We now want to relate the motivic measure $\mu(\mathcal{X}^n) = \frac{[\pi_{n+1}(\mathcal{X}^n)]}{\mathbb{L}^{2(n+1)}}$ to $[\pi_{n+1}(\mathcal{B}^n)]$, as the latter will correspond to $[\mathrm{UHilb}_{0,1}^n(C)]$.

We observe that the S_0^{n+1}-action is not free, therefore directly proving that p_b is a Zariski locally trivial fibration can be hard. We will follow instead the reasoning of [12, Definitions and Proposition 2, pg. 66-67]. We start by simplifying $\mathcal{X}^n$ with the partition

$$\mathcal{X}^n = \coprod_k \mathcal{X}^{n,k}, \ \mathcal{X}^{n,k} = \mathcal{X}^n \cap \pi_k^{-1}(0) \setminus \mathcal{X}^n \cap \pi_{k+1}^{-1}(0).$$

Then, given $b_\gamma \in \mathcal{X}^{n,k}$ we have that

(4)
$$\mu(S_0^{n+1} \bullet \gamma) = \mu(p_b^{-1}(b_\gamma)) = \mathbb{L}^{-k-1}(\mathbb{L} - 1)$$

hence not depending on γ. Therefore, we have

(5)
$$[\pi_{n+1}(\mathcal{X}^{n,k})] = \mathbb{L}^{-k-1}(\mathbb{L} - 1)\mathbb{L}^{n+1}[\pi_{n+1}(\mathcal{B}^{n,k})]$$

where $p_b(\mathcal{X}^{n,k}) = \mathcal{B}^{n,k} \subseteq \mathcal{B}$ denotes the image of $\mathcal{X}^{n,k}$ in the branch space.

We now want to understand the sets $\mathcal{B}^{n,k}$ in the space of branches, and what are the points of $\mathrm{UHilb}_{0,1}^n(C)$ that correspond to points of $\mathcal{B}^{n,k}$. They correspond to "easier" valuation problems to solve.

Proposition 5.14. *The points of $\mathcal{B}^{n,k}$ are in one-to-one correspondence with curves with intersection number n with C and Puiseux parameterization $(t^k, t^\star + H.O.T.)$ where $\star$ is determined by n, k, C.*

Proof. Let $b_\gamma \in \mathcal{B}^{n,k}$ for some $\gamma \in \mathcal{X}^{n,k} \subseteq \mathcal{X}^n$. Then, by definition $\gamma = (t^k, t^m u)$ with $m \geq k$ and $u \in 1 + t\mathbb{K}[[t]]$. Moreover, $\gamma \in \mathcal{X}^n$ additionally implies that

$$f(\gamma) = f(t^k, t^m u) = t^n v$$

for some $v \in \mathbb{K}[[t]]^\times$. The (leading term of the) equation will determine $m = \star$. $\square$

Remark 5.15. We will now see that all these curves are identified if they differ by a smooth parameterization, or in degree $\geq n$. This means that $\pi_r(t^m + H.O.T)$ determines the corresponding point of the Hilbert scheme for $r = \lfloor \frac{n+2\delta-1}{d_C} \rfloor$ where $d_C = val_t(q_C)$ (we use the notation of Definition 2.36), since any coefficient u_i of $t^m u$ for $i > \lfloor \frac{n+2\delta-1}{d_C} \rfloor$ appears in terms of valuation $> n + 2\delta - 1$ in $\widehat{\mathcal{O}}_{C,0}$, hence not effectively contributing to $\mathrm{Hilb}_{0,1}^n(C)$. This is the optimal bound.

We immediately see that the fibers of Π are completely determined by the truncations of Remarks 5.2 and 5.15, and the action introduced in Definition 5.11. The points of the unibranch open will therefore coincide with some points of the branch space $\mathcal{B}$.

Proposition 5.16. *Let $I \in \mathrm{UHilb}_{0,1}^n(C)$. Let $g \in \widehat{\mathcal{O}}_{C,0}$ be any generator of I and let $\gamma_g \in \pi_{n+1}(\mathcal{X}^{n,k})$ denote the Puiseux parameterization of the corresponding curve $V(g) \subseteq \mathbb{A}^2$ for some k. Then, we have*

$$[\Pi^{-1}(I)] = (\mathbb{L} - 1)\mathbb{L}^{-k-1}[F^{n,k}]$$

where

$$[F^{n,k}] = \pi_{n+1}(p_2(\mathcal{B}^{n,k})) \setminus \pi_r(p_2(\mathcal{B}^{n,k})).$$

Proof. We already observed in Remark 5.15 that if

$$\pi_r(p_2(\gamma_g)) = \pi_r(p_2(\gamma))$$

then

$$(g) = (g_\gamma)$$

as points of $\mathrm{Hilb}_{0,1}^n(C)$. Then, by Lemma 2.38 we have that $val_t(C(\gamma_g)) = n$ and by definition

$$\Pi^{-1}(I) = \{\gamma \in \pi_{n+1}(\mathcal{X}^n) \mid \mathrm{Im}(\gamma) = V(g)\} =$$

$$\{\gamma_g \circ \alpha \in \pi_{n+1}(\mathcal{X}^n) \mid \alpha \in \mathrm{Aut}_{\mathbb{K},0}^{n+1}\} = S_0^{n+1} \bullet \gamma_g$$

since all equivalent parameterizations of the same curve are described in Definition 2.30. Finally, using the $\mathcal{C}$-equivalence presented in Definition 2.64, it is immediate to see that in the principal case any other representative of I is of the form

$$g_1 = ug$$

with $u \in \widehat{\mathcal{O}}_{C,0}^\times$. Therefore, we have $\gamma_g = \gamma_{g_1}$ and, using Equation (4), we are done. $\square$

Remark 5.17. With the previous proof, we equivalently observed that Π is S_0^{n+1}-invariant. This leads to the map

$$\Pi_b : \pi_{n+1}(\mathcal{B}^n) \to \mathrm{UHilb}_{0,1}^n(C), \ b_\gamma = p_b(\gamma) \mapsto \Pi(\gamma)$$

where $\mathcal{B}^n = p_b(\mathcal{X}^n)$ denotes the quotient of $\mathcal{X}^n$ by the S_0^{n+1}-action, as introduced in Remark 5.12, and p_b commutes with π_{n+1} as explained in [12].

Knowing the fibers of the morphism Π, hence of Π_b, we are now finally able to use it to construct a geometric morphism that, thanks to Proposition 2.8, will connect the motivic class of $\mathrm{UHilb}_{0,1}^n(C)$ with the n-th contact locus $\mathcal{X}^n$.

Proposition 5.18. *The restriction $\Pi_b^k : \pi_{n+1}(\mathcal{B}^{n,k}) \to \mathrm{UHilb}_{0,1}^n(C)$ of the map Π_b defined in Remark 5.17 is a geometric morphism up to the fiber $F^{n,k}$ introduced by Proposition 5.16, implying*

$$[\mathrm{Im}(\Pi_b^k)][F^{n,k}] = [\pi_{n+1}(\mathcal{B}^{n,k})]$$

for every $n, k \geq 1$.

Proof. We reiterate the steps of [23, Theorem 3.14]. We already proved in Theorem 5.5 that Π is a morphism, and with Proposition 5.16 we have the description of the fiber of Π_b^k. Finally, it is surjective by definition since $\mathrm{UHilb}_{0,1}^n(C) = \mathrm{Im}(\Pi)$. $\qquad\qquad\square$

We finally have all the information needed to relate contact loci with the unibranch open. Indeed, putting together Proposition 5.18 and Remark 5.13, we obtain the following chain of relationships.

Corollary 5.19. *Let $(C,0)$ be a unibranch curve. Let $\mathrm{Im}(\Pi_b^k) \subseteq \mathrm{UHilb}_{0,1}^n(C)$ denote the image of the restriction $\Pi_b^k : \pi_{n+1}(\mathcal{B}^{n,k}) \to \mathrm{UHilb}_{0,1}^n(C)$ of the map Π_b introduced in Remark 5.17. We have*

$$[\pi_{n+1}(\mathcal{X}^{n,k})] = \mathbb{L}^{n-k}(\mathbb{L}-1)[\mathrm{Im}(\Pi_b^k)][F^{n,k}]$$

$$[\mathrm{UHilb}_{0,1}^n(C)] = \frac{\mathbb{L}}{\mathbb{L}-1}\sum_k \mathbb{L}^k \frac{[\pi_{n+1}(\mathcal{X}^{n,k})]}{[F^{n,k}]}$$

for every $n,k \geq 1$.

This also means we just obtained the coefficients of the Igusa zeta function

$$Z_C(s) = \int_{\pi_1^{-1}(0)} (\mathbb{L}^{-ord_C})^s d\mu = \sum_{n\geq 1} \mu(\mathcal{X}_C^n)\mathbb{L}^{-ns} = \sum_{n\geq 1} \frac{[\pi_{n+1}(\mathcal{X}^n)]}{\mathbb{L}^{2(n+1)}}\mathbb{L}^{-ns}$$

from the unibranch open in the one-generator locus of the Hilbert scheme of the given curve.

Corollary 5.20. *Let $(C,0)$ be a unibranch curve and let $Z_C(s)$ be the Igusa zeta function of C. Then, we have*

$$Z_C(s) = (\mathbb{L}-1)\mathbb{L}^{-2}\sum_{n\geq 1}\mathbb{L}^{-n(s+1)}\sum_k \mathbb{L}^{-k}[\mathrm{Im}(\Pi_b^k)]$$

and $[\mathrm{Im}(\Pi_b^k)]$ is related to $[\mathrm{UHilb}_{0,1}^n(C)]$ in Corollary 5.19.

Remark 5.21. We mention here that the relationship between principal Hilbert schemes and motivic integration was also explored in [14]. However, we still do not know the connection between our and that work, and our connection between $\mathrm{UHilb}_{0,1}^n(C)$ and $\mathrm{Hilb}_{0,1}^n(C)$ is just conjectural.

Finally, combining Theorem 2.24 with [23, Proposition 3.24] it follows that $[\pi_{n+1}(\mathcal{X}^{n,k})]$ is fully determined by an embedded resolution of $(C,0)$. As a consequence, we obtain an explicit expression in terms of certain divisors for $[\mathrm{UHilb}_{0,1}^n(C)]$.

Corollary 5.22. *Let $(C,0)$ be a unibranch curve. Let $\rho : (Y,E) \to (\mathbb{A}^2, C)$ be an embedded resolution as in Definition 2.18. Then, we have*

$$[\mathrm{UHilb}_{0,1}^n(C)] = \frac{1}{(\mathbb{L}-1)\mathbb{L}^n}\sum_k \frac{\mathbb{L}^k}{[F^{n,k}]}\sum_{\substack{J\subseteq I, J\neq\emptyset, j\in J \\ k_j\geq 1 \,|\, k=\sum m_j k_j}}(\mathbb{L}-1)^{|J|-1}[E_J^o]\left(\sum_{j\in J,\,\sum k_j N_j=n}\mathbb{L}^{-\sum_{j\in J}k_j\nu_j}\right)$$

for every $n \geq 1$, where m_j is defined in [23, Definition 3.21].

Proof. We can prove the polynomiality in $\mathbb{L}$ of $[\mathrm{UHilb}_{0,1}^n(C)]$ by proving it for all the motivic classes involved in the formula. From Proposition 5.16, it follows that $[F^{n,k}]$ is a power of $\mathbb{L}$. Then, since divisors can only be $\mathbb{P}^1$, E_J^o is a $\mathbb{P}^1$ with a finite number k_J of points removed, or a point. Therefore, we have

$$[E_J^o] = (\mathbb{L}+1) - k_i \text{ or } [E_J^o] = 1.$$

$\qquad\qquad\square$

Finally, the equivalence between the geometric and topological data comes from [2, Theorem 21].

Corollary 5.23. *Let $(C,0)$ be a unibranch curve. Then, $[\mathrm{UHilb}_{0,1}^n(C)]$ is a polynomial in $\mathbb{L}$ for every $n \geq 1$. This polynomial is also a topological invariant of $(C,0)$ since the resolution is (see [2, Theorem 21]).*

From this corollaries, Conjecture 5.10 would lead to an expression for $[\mathrm{Hilb}_{0,1}^n(C)]$, and in the case of (p,q)-curves for $[\mathrm{Hilb}_{0,k}^n(C)]$, in terms of the resolution. This would be in perfect agreement with the topological invariance obtained in Remark 2.55 and Corollary 4.6 respectively.

6. References

[1] T. Bridgeland. "An introduction to motivic Hall algebras". In: *Advances in Mathematics 229 (1)* (2012).

[2] E. Brieskorn and H. Knörrer. *Plane Algebraic Curves: Translated by John Stillwell.* Modern Birkhäuser Classics, Basel, 2012.

[3] A. Campillo, F. Delgado, and S. M. Gusein-Zade. "The Alexander polynomial of a plane curve singularity via the ring of functions on it". In: *Duke Math. J. 117 (1)* (2002).

[4] A. Chambert-Loir, J. Nicaise, and J. Sebag. *Motivic Integration.* Progress in Mathematics 325. Birkhäuser, New York, 2018.

[5] J. Denef and F. Loeser. "Germs of arcs on singular algebraic varieties and motivic integration". In: *J. Algebraic Geom. 7 (3)* (1998).

[6] J. Denef and F. Loeser. "Motivic Igusa zeta functions". In: *Invent. Math. 135* (1999).

[7] D. Eisenbud. *The Geometry of Syzygies.* Graduate Texts in Mathematics. Springer New York, NY, 2005.

[8] H. Gillet and C. Soule. "Descent, Motives and K-theory". In: *J. für die reine und angewandte Math., 478* (1996).

[9] E. Gorsky. "On the motivic measure on the space of functions". In: *Rev. Mat. Complut. 20* (2007).

[10] E. Gorsky and M. Mazin. "Compactified Jacobians and -Catalan numbers, I". In: *J. of Combinat. Th. 120(1)* (2013).

[11] A. Grothendieck. "Techniques de construction et théorèmes d'existence en géométrie algébrique IV : les schémas de Hilbert". In: *Séminaire Bourbaki: années 1960/61, exposé n. 221* (1961).

[12] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. "Integration over spaces of nonparametrized arcs and motivic versions of the monodromy zeta function". In: *Proc. of the Steklov Institute of Math. 252* (2006).

[13] H. Hironaka. "Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero: I". In: *Annals of Math. 79 (1)* (1964).

[14] O. Kivinen, A. Oblomkov, and D. Wyss. "Discriminants and motivic integration". In: (2025).

[15] J. N. Mather. "Stability of C^∞-mappings, III. Finitely determined map-germs". In: *Publications mathématiques de l'I.H.É.S., tome 35* (1968).

[16] D. Maulik. "Stable pairs and the HOMFLY polynomial". In: *Invent. Math. 204* (2016).

[17] J. Milnor. *Singular Points of Complex Hypersurfaces.* Annals of Mathematics Studies 61. Princeton University Press, 1968.

[18] A. Oblomkov, J. Rasmussen, and V. Shende. "The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link". In: *Geom. Topol. 22 (2)* (2012).

[19] A. Oblomkov and V. Shende. "The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link". In: *Duke Math. J. 161* (2021).

[20] J. Piontkowski. "Topology of the Compactified Jacobians of Singular Curves". In: *Math. Z. 255* (2007).

[21] V. A. Puiseux. "Recherches sur les fonctions algébriques". In: *J. Math. Pures Appl. 15* (1850).

[22] J. C. Rosales and P. A. García-Sánchez. *Numerical Semigroups.* Developments in Mathematics 20. Springer New York, NY, 2009.

[23] I. Rossinelli. "Motivic Classes of Curvilinear Hilbert Schemes and Igusa Zeta Functions". In: *Int. Math. Res. Not. 24* (2024).

[24] C. T. C. Wall. *Singular Points of Plane Curves.* London Mathematical Society Student Texts 63. Cambridge University Press, 2004.

Department of Mathematics, EPFL Lausanne

Email address: ilaria.rossinelli@epfl.ch