SHARP L^p ESTIMATES AND SIZE OF NODAL SETS FOR GENERALIZED STEKLOV EIGENFUNCTIONS

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ABSTRACT. We prove sharp L^p estimates for the Steklov eigenfunctions on compact manifolds with boundary in terms of their L^2 norms on the boundary. We prove it by establishing L^p bounds for the harmonic extension operators as well as the spectral projection operators on the boundary. Moreover, we derive lower bounds on the size of nodal sets for a variation of the Steklov spectral problem. We consider a generalized version of the Steklov problem by adding a non-smooth potential on the boundary but some of our results are new even without potential.

1. Introduction

Eigenfunction estimates have been recently considered in the case of Schrödinger operators with singular potentials (see e.g. [43], [1], [2], [30], [20], [31], [32], [33]). In the present paper, we investigate a generalization of the well-known Steklov problem with non-smooth potentials. For surveys on the Steklov problem, see e.g. [26], [12].

Let (Ω, h) be a smooth manifold with boundary (M, g), where dim $\Omega = n + 1 \ge 2$ and $h|_M = g$. The Steklov eigenvalue problem with potential V is

$$\begin{cases} \Delta_h e_{\lambda}(x) = 0, \ x \in \Omega \\ \partial_{\nu} e_{\lambda}(x) + V(x)e_{\lambda}(x) = \lambda e_{\lambda}(x), \ x \in \partial \Omega = M. \end{cases}$$

Here ν is an unit outer normal vector on M. Then the restriction of the eigenfunction $e_{\lambda}(x)$ (denote also by e_{λ} to simplify notations) to the boundary M is an eigenfunction of $\mathcal{D} + V$:

$$(\mathcal{D} + V)e_{\lambda}(x) = \lambda e_{\lambda}(x), \ x \in M.$$

Here \mathcal{D} is the Dirichlet-to-Neumann operator \mathcal{D} : $H^{\frac{1}{2}}(M) \to H^{-\frac{1}{2}}(M)$

$$\mathcal{D}f = \partial_{\nu} u|_{M},$$

where u is the harmonic extension of f:

(1.1)
$$\begin{cases} \Delta_h u(x) = 0, \ x \in \Omega, \\ u(x) = f(x), \ x \in \partial \Omega = M. \end{cases}$$

Such a type of Steklov problem with potential has been considered in [13] from the point of view of conformal geometry, where the potential V is the mean curvature on the boundary $\partial\Omega$. See e.g. [16], [17], [18], [38] for related works on Yamabe problem on compact manifolds with boundary. In the current paper, we derive estimates whenever the potential is merely bounded or Lipschitz.

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For $m \in \mathbb{R}$, we denote OPS^m the class of pseudodifferential operators of order m. It is known that $\mathcal{D} \in OPS^1$ and one can write (see e.g. [56, Proposition C.1])

$$\mathcal{D} = \sqrt{-\Delta_g} + P_0,$$

for some $P_0 \in OPS^0$. Therefore, up to a classical pseudo-differential operator of order zero, the problem of eigenfunction bounds (among other results) on the boundary M has been treated in our previous paper [30]. In this setting the model is related to relativistic matter (see e.g. [9, 15, 21, 22, 36, 37]).

In our first result below, we provide a control of the L^p norms of the Steklov eigenfunctions in the domain by their L^2 norms on the boundary.

Theorem 1. Let $V \in L^{\infty}(M)$. Then for $\lambda \geq 1$ we have

(1.2)
$$||e_{\lambda}||_{L^{p}(\Omega)} \lesssim \lambda^{-\frac{1}{p} + \sigma(p)} ||e_{\lambda}||_{L^{2}(M)}, \quad 2 \leq p \leq \infty,$$

where

$$\sigma(p) = \begin{cases} \frac{n-1}{2} (\frac{1}{2} - \frac{1}{p}), & 2 \le p < \frac{2(n+1)}{n-1} \\ \frac{n-1}{2} - \frac{n}{p}, & \frac{2(n+1)}{n-1} \le p \le \infty. \end{cases}$$

The previous result is new, even for $V \equiv 0$. Note that the estimate is sharp when $V \equiv 0$ and Ω is the unit ball $B(0,1) \subset \mathbb{R}^{n+1}$ with boundary $M = S^n$. In this case, the Steklov eigenfunction $e_{\lambda}(x) = r^k e_k(\omega)$ in the polar coordinate $r \in [0,1]$, $\omega \in S^n$, where $\lambda^2 = k(k+n-1)$, $k \in \mathbb{N}$ and $e_k(\omega)$ is a spherical harmonic of degree k, that is, the restriction to S^n of homogeneous harmonic polynomials of degree k. It is straightforward to see that

(1.3)
$$||e_{\lambda}||_{L^{p}(B(0,1))} \approx \lambda^{-\frac{1}{p}} ||e_{\lambda}||_{L^{p}(S^{n})}.$$

The following L^p estimates of the Laplacian eigenfunctions on the sphere S^n are sharp

$$(1.4) ||e_{\lambda}||_{L^{p}(S^{n})} \lesssim \lambda^{\sigma(p)} ||e_{\lambda}||_{L^{2}(S^{n})},$$

and they are saturated by zonal spherical harmonic for $p \ge \frac{2(n+1)}{n-1}$ and highest weight spherical harmonic for $p \le \frac{2(n+1)}{n-1}$ (see e.g. [47, 49]). Thus, combining (1.3) with (1.4), we see that (1.2) is sharp.

The motivation for this result is to investigate the feature that Steklov eigenfunctions concentrate near the boundary, and rapidly decay away from the boundary (see e.g. [29], [39], [14], [23]). Motivated by the elliptic inverse boundary value problems such as Calderon problem (see e.g. [8], [35]), Hislop-Lutzer [29] proved that for any compact set $K \subset \Omega$,

$$||e_{\lambda}||_{L^{2}(K)} \le C_{N} \lambda^{-N} ||e_{\lambda}||_{L^{2}(M)}, \ \forall N.$$

This bound reflects the fact that the Steklov eigenfunctions become highly oscillatory as the eigenvalue increases, hence they decay rapidly away from the boundary. Hislop-Lutzer [29] conjectured that the decay is actually of order $e^{-\lambda d_g(K,\partial\Omega)}$. One may see by examining the case of unit ball $B(0,1) \subset \mathbb{R}^{n+1}$ that the exponential decay is optimal. This is confirmed for real-analytic surfaces (n=1) by Polterovich-Sher-Toth [39] and the eigenfunction decay is a key feature in their main results on nodal length. They proved that for any real-analytic compact Riemannian surface Ω with boundary $M = \partial \Omega$, and any compact set $K \subset \Omega$, there exist constants C, c > 0 such that

$$\|e_{\lambda}\|_{L^{\infty}(K)} \leq C e^{-c\lambda d_g(K,\partial\Omega)} \|e_{\lambda}\|_{L^2(M)}.$$

Their methods are specific to the case of real-analytic surfaces. A different method of proving this bound has been communicated to them by M. Taylor. Recently, Hislop-Lutzer's conjecture is confirmed for higher dimensional real-analytic manifolds by Galkowski-Toth [23]. Furthermore, this interesting concentration feature is also related to the restriction estimates of eigenfunctions to submanifolds (see e.g. [5], [6], [48], [54])

In our second result, we prove the following lower bound on the measure of the nodal set

$$N_{\lambda} = \{ x \in M : e_{\lambda}(x) = 0 \}.$$

Theorem 2. If $V \in Lip^1(M)$ and zero is a regular value of e_{λ} , then

$$|N_{\lambda}| \gtrsim \lambda^{\frac{3-n}{2}}$$
.

When $V \equiv 0$, this result is due to Wang-Zhu [59], which follows from the idea in Sogge-Zelditch [51]. The Lipschitz assumption is used to ensure that the eigenfunctions is in C^1 , so that the restriction of ∇e_{λ} to the nodal sets makes sense. The assumption that zero is a regular value is used to ensure the validity of Gauss-Green theorem.

To prove the theorems, we will need the following key lemmas. Incidentally, one does not require Lipschitz potentials but only bounded ones.

Lemma 1. If $V \in L^{\infty}(M)$, then the following two eigenfunction estimates hold

(1.5)
$$||e_{\lambda}||_{L^{p}(M)} \lesssim \lambda^{\sigma(p)} ||e_{\lambda}||_{L^{2}(M)}, \ 2 \leq p \leq \infty.$$

Moreover,

(1.6)
$$||e_{\lambda}||_{L^{1}(M)} \gtrsim \lambda^{-\frac{n-1}{4}} ||e_{\lambda}||_{L^{2}(M)}.$$

For smooth V, (1.5) was proved by Seeger-Sogge [42]. Indeed, they obtained the eigenfunction estimates for self-adjoint elliptic pesudo-differential operators satisfying a convexity assumtion on the principal symbol. In the case of the pure power (i.e. $P_0 = 0$), (1.5) was stated in [30, Remark 1] by three of us. Both (1.5) and (1.6) are sharp on S^n . Indeed, they can be saturated by zonal spherical harmonic or highest weight spherical harmonic (see e.g. [47, 49]).

Lemma 2. If $V \in L^{\infty}(M)$, then

(1.7)
$$||e_{\lambda}||_{L^{p}(\Omega)} \lesssim \lambda^{-1/p} ||e_{\lambda}||_{L^{p}(M)}, \ 2 \leq p \leq \infty.$$

The endpoint $p=\infty$ follows from the maximum principle, since e_{λ} is harmonic in Ω . The other endpoint p=2 can be obtained from the trace theorem and standard regularity estimates. And then (1.7) is proved by an interpolation argument involving the harmonic extension operator on Ω . From (1.3), we see that the estimate (1.7) is sharp for $\Omega = B(0,1)$.

The paper is organized as follows. In Section 2, we prove sharp heat kernel estimates that will be used later. In Section 3, we prove Lemma 1. In section 4, we prove some kernel estimates for pseudo-differential operators on compact manifolds. In Section 5, we prove the interior eigenfunction estimates in Theorem 1. In Section 6, we prove the size estimates of the nodal sets in Theorem 2.

Throughout this paper, $X \lesssim Y$ (or $X \gtrsim Y$) means $X \leq CY$ (or $X \geq CY$) for some positive constant C independent of λ . This constant may depend on V and the domain Ω . $X \approx Y$ means $X \lesssim Y$ and $X \gtrsim Y$.

2. Heat kernel bounds

In this section, we prove the heat kernel estimates for the operators

$$H_V = (-\Delta_q)^{\alpha/2} + P_{\alpha-1} + V,$$

where $P_{\alpha-1}$ is a classical pseudo-differential operator of order $\alpha-1$, and the real-valued potential V belongs to the Kato class on the closed manifold (M,g). These generalize the results of Gimperlein-Grubb [25, Theorem 4.3]. When $P_{\alpha-1}=0$, the Euclidean version was proved in [10] and [52]. We give a detailed proof for this special case on compact manifolds by using Duhamel's principle and Picard iterations. And then we slightly modify this argument to obtain the upper bound of the heat kernel of H_V . Although the potentials in our main theorems are just bounded, we prove the heat kernel estimates under the minimal assumption so that they may be used for related reseach.

Definition 1. For $n \geq 2$ and $0 < \alpha < 2$, the potential V is said to be in the Kato class $\mathcal{K}_{\alpha}(M)$ if

(2.1)
$$\lim_{r \downarrow 0} \sup_{x \in M} \int_{B_r(x)} d_g(x, y)^{\alpha - n} |V(y)| dy = 0$$

where $d_g(\cdot,\cdot)$ denotes geodesic distance and $B_r(x)$ is the geodesic ball of radius r about x and dy denotes the volume element on (M,g). To define the Kato class for n=1 and $0 < \alpha < 2$, we replace the function $d_g(x,y)^{\alpha-n}$ in (2.1) by

$$w(x,y) = \begin{cases} d_g(x,y)^{\alpha-1}, & \alpha < 1\\ \log(2 + d_g(x,y)^{-1}), & \alpha = 1\\ 1, & \alpha > 1. \end{cases}$$

Since M is compact we have $\mathcal{K}_{\alpha}(M) \subset L^1(M)$, and for any $p > \frac{n}{\alpha}$, we have $L^p(M) \subset \mathcal{K}_{\alpha}(M)$ by Hölder's inequality. We recall that the assumption $V \in \mathcal{K}_{\alpha}(M)$ implies that the operators $H_V = (-\Delta_g)^{\alpha/2} + V$ are self-adjoint and bounded from below. See the proof of [30, Proposition 2]. The same argument is still valid to prove that $H_V = (-\Delta_g)^{\alpha/2} + P_{\alpha-1} + V$ is self-adjoint and bounded from below, whenever $P_{\alpha-1}$ is self-adjoint.

Proposition 1. Let $n \geq 1$, $0 < \alpha < 2$ and t > 0. Let $p^{V}(t, x, y)$ be the heat kernel of $H_{V} = (-\Delta_{q})^{\alpha/2} + V$, where $V \in \mathcal{K}_{\alpha}(M)$. Then for any $t \in (0, 1]$, $x, y \in M$

(2.2)
$$p^{V}(t, x, y) \approx q_{\alpha}(t, x, y)$$

where $q_{\alpha}(t,x,y) = \min\{t^{-n/\alpha}, td_q(x,y)^{-n-\alpha}\}$. Moreover, for any $t > 0, x, y \in M$

(2.3)
$$e^{-C_1 t} q_{\alpha}(t, x, y) \lesssim p^V(t, x, y) \lesssim e^{C_2 t} q_{\alpha}(t, x, y)$$

for some constants $C_1, C_2 > 0$.

Proposition 2. Let $n \ge 1$, $0 < \alpha < 2$ and t > 0. Let $p^V(t, x, y)$ be the heat kernel of $H_V = (-\Delta_g)^{\alpha/2} + P_{\alpha-1} + V$, where $V \in \mathcal{K}_{\alpha}(M)$ and $P_{\alpha-1}$ is a classical pseudo-differential operator of order $\alpha - 1$. Then for any t > 0, $x, y \in M$

$$(2.4) |p^{V}(t,x,y)| \lesssim e^{Ct} q_{\alpha}(t,x,y).$$

for some constant C > 0.

The following key lemma is called 3P-inequality in [3, Theorem 4] and [58, Proposition 2.4]. We remark that such 3P-inequality holds for all $\alpha \in (0,2)$ but fails to hold for the Gaussian kernel $(\alpha = 2)$.

Lemma 3. We have for any s, t > 0 and $x, y, z \in M$

$$q_{\alpha}(t,x,z)q_{\alpha}(s,z,y) \le Cq_{\alpha}(s+t,x,y)(q_{\alpha}(t,x,z) + q_{\alpha}(s,z,y)),$$

where C > 0 is a constant.

Proof. The proof is straightforward. Indeed, by using the fact that for A, B > 0

$$\min\{A, B\} \approx \frac{AB}{A+B},$$
$$(A+B)^{\frac{n}{\alpha}} \approx A^{\frac{n}{\alpha}} + B^{\frac{n}{\alpha}},$$

and the triangle inequality $d_q(x,y) \leq d_g(x,z) + d_g(z,y)$, we have

$$\begin{split} \frac{q_{\alpha}(t,x,z) + q_{\alpha}(s,z,y)}{q_{\alpha}(t,x,z)q_{\alpha}(s,z,y)} &= \frac{1}{q_{\alpha}(t,x,z)} + \frac{1}{q_{\alpha}(s,z,y)} \\ &\approx t^{\frac{n}{\alpha}} + t^{-1}d_{g}(x,z)^{n+\alpha} + s^{\frac{n}{\alpha}} + s^{-1}d_{g}(z,y)^{n+\alpha} \\ &\approx (t+s)^{\frac{n}{\alpha}} + t^{-1}d_{g}(x,z)^{n+\alpha} + s^{-1}d_{g}(z,y)^{n+\alpha} \\ &\geq (t+s)^{\frac{n}{\alpha}} + (s+t)^{-1}(d_{g}(x,z)^{n+\alpha} + d_{g}(z,y)^{n+\alpha}) \\ &\approx (t+s)^{\frac{n}{\alpha}} + (s+t)^{-1}(d_{g}(x,z) + d_{g}(z,y))^{n+\alpha} \\ &\geq (t+s)^{\frac{n}{\alpha}} + (s+t)^{-1}d_{g}(x,y)^{n+\alpha} \\ &\approx \frac{1}{q_{\alpha}(s+t,x,y)}. \end{split}$$

The implicit constants may depend on n and α . This completes the proof of Lemma 3.

Proof of Proposition 1. It is not hard to see that (2.3) follows from (2.2) and the semigroup property. So it suffices to prove (2.2).

Since (M, g) is a closed manifold, the heat kernel of $-\Delta_g$ satisfies the two-sided estimates (see Li-Yau [34], Sturm [53], Saloff-Coste [40])

$$t^{-n/2}e^{-C_1d_g(x,y)^2/t} \lesssim p_t(x,y) \lesssim t^{-n/2}e^{-C_2d_g(x,y)^2/t}, \ t > 0, \ x,y \in M$$

for some constants $C_1, C_2 > 0$. Moreover, it is well-known that the semigroups $e^{t\Delta_g}$ and $e^{-t(-\Delta_g)^{\alpha/2}}$ are related by subordination formulas (see e.g. [25, (4.8)], [27], [60]), which imply that the heat kernel of $H^0 = (-\Delta_g)^{\alpha/2}$ is continuous and satisfies the two-sided estimates (see e.g. [25, Theorem 4.2], [4, Theorem 3.1])

(2.5)
$$C^{-1}q_{\alpha}(t,x,y) \le p_0(t,x,y) \le Cq_{\alpha}(t,x,y), \ t > 0, \ x,y \in M.$$

The heat kernel $p_0(t, x, y)$ is the Schwartz kernel of $f \to e^{-tH^0} f = u^0(t, x)$, which solves the heat equation

(2.6)
$$\begin{cases} (\partial_t + H^0)u^0(t, x) = 0, & (t, x) \in (0, \infty) \times M, \\ u^0|_{t=0} = f. \end{cases}$$

Similarly, the heat kernel $p^{V}(t, x, y)$ is the Schwartz kernel of $f \to e^{-tH_{V}} f = u_{V}(t, x)$, which solves the heat equation

(2.7)
$$\begin{cases} (\partial_t + H_V)u_V(t, x) = 0, & (t, x) \in (0, \infty) \times M, \\ u_V|_{t=0} = f. \end{cases}$$

Note that (2.6) and (2.7) imply that

$$(\partial_t + H^0)(e^{-tH_V}f - e^{-tH^0}f) = -V(x)e^{-tH_V}f$$

and

$$(e^{-tH_V}f - e^{-tH^0}f)|_{t=0} = 0.$$

By Duhamel's principle for the heat equation, we have

$$e^{-tH_V}f - e^{-tH^0}f = -\int_0^t e^{-(t-r)H^0}(Ve^{-rH_V}f)dr$$

= $-\int_0^t \int_M \int_M p_0(t-r,x,z)V(z)p^V(r,z,y)f(y)dydzdr.$

where dy and dz denote the volume element on (M, g). So the heat kernel of H_V satisfies the integral equation

(2.8)
$$p^{V}(t,x,y) = p_{0}(t,x,y) - \int_{0}^{t} \int_{M} p_{0}(t-r,x,z) p^{V}(r,z,y) V(z) dz dr.$$

To prove (2.2), we use Picard iterations (see e.g. [3], [58]) to construct a solution to (2.8). For $t > 0, x, y \in M$, let

(2.9)
$$p_m(t,x,y) = p_0(t,x,y) - \int_0^t \int_M p_0(t-r,x,z) p_{m-1}(r,z,y) V(z) dz dr, \quad m \ge 1.$$

Moreover, let

$$\Theta_m(t, x, y) = p_m(t, x, y) - p_{m-1}(t, x, y), \quad m \ge 1$$

and $\Theta_0(t,x,y) = p_0(t,x,y)$. Clearly,

(2.10)
$$\Theta_m(t, x, y) = -\int_0^t \int_M p_0(t - r, x, z) V(z) \Theta_{m-1}(r, z, y) dz dr.$$

We claim that for some constant $c_0 > 0$ and c(t) > 0

$$(2.11) |\Theta_m(t,x,y)| \le (c_0c(t))^m p_0(t,x,y), \quad m \ge 0.$$

To prove the claim, we define

(2.12)
$$c(t) = \sup_{y \in M} \int_0^t \int_M q_\alpha(r, y, z) |V(z)| dz dr.$$

It is straightforward to see that V is in the Kato class implies that

$$\lim_{t \downarrow 0} c(t) = 0.$$

Indeed, for $n \geq 2$,

$$\int_0^t \int_M q_\alpha(r,y,z) |V(z)| dz dr \lesssim \int_{d_g(z,y) < t^{\frac{1}{2\alpha}}} d_g(z,y)^{\alpha-n} |V(z)| dz + \int_M t d_g(z,y)^{\alpha-n} |V(z)| dz,$$

which implies (2.13) by the definition (2.1). The case n = 1 is similar.

The claim (2.11) is clear for m=0. If the claim is true for m-1, then by (2.10) we have

$$\begin{aligned} |\Theta_m(t,x,y)| &\leq (c_0c(t))^{m-1} \int_0^t \int_M p_0(t-r,x,z) p_0(r,z,y) |V(z)| dz dr \\ &\leq C(c_0c(t))^{m-1} \int_0^t \int_M p_0(t,x,y) (p_0(t-r,x,z) + p_0(r,z,y)) |V(z)| dz dr \\ &\leq 2C^2 (c_0c(t))^{m-1} p_0(t,x,y) \sup_{y \in M} \int_0^t \int_M q_\alpha(r,y,z) |V(z)| dz dr \\ &= \frac{2C^2}{c_0} (c_0c(t))^m p_0(t,x,y) \end{aligned}$$

where we use Lemma 3 and the upper bound in (2.5). Here C > 0 is a constant independent of m, s, t, x, y, z. So we may fix $c_0 \ge 2C^2$, and the claim (2.11) is proved by induction.

By (2.13), there is $0 < t_0 < 1$ so that for any $t \in (0, t_0]$, we have $c_0 c(t) \leq \frac{1}{3}$. Let

$$p^{V}(t, x, y) = \sum_{m=0}^{\infty} \Theta_{m}(t, x, y), \quad t \in (0, t_{0}], \ x, y \in M.$$

This series is uniformly convergent, and

$$|p^{V}(t,x,y) - p_0(t,x,y)| \le \sum_{m=1}^{\infty} |\Theta_m(t,x,y)| \le \frac{c_0c(t)}{1 - c_0c(t)} p_0(t,x,y) \le \frac{1}{2} p_0(t,x,y).$$

Combining this with (2.5), we have

(2.14)
$$p^{V}(t, x, y) \approx q_{\alpha}(t, x, y), \quad t \in (0, t_{0}].$$

By letting $m \to \infty$ in (2.9), we get (2.8) for $t \in (0, t_0]$.

Moreover, when $t \in (0, t_0]$, $p^V(t, x, y)$ is the unique solution to the integral equation (2.8) satisfying (2.14). Indeed, let $\tilde{p}(t, x, y)$ be another solution satisfying (2.14), and $\Theta = p^V - \tilde{p}$. Note that $|\Theta(t, x, y)| \leq Cp_0(t, x, y)$ for some constant C > 0. Then by the same induction argument above we obtain

$$|\Theta(t, x, y)| \le C(c_0 c(t))^m p_0(t, x, y), \ \forall m \ge 0.$$

By letting $m \to \infty$ we get $\Theta(t, x, y) = 0$ for $t \in (0, t_0]$.

For $t > t_0$, we recursively define

$$p^{V}(t,x,y) = \int_{M} p^{V}(t/2,x,z)p^{V}(t/2,z,y)dz.$$

Then $p^V(t, x, y)$ is extended to be a jointly continuous function on $(0, \infty) \times M \times M$. Moreover, the estimate (2.14) can be recursively extended to

$$p^V(t, x, y) \approx q_{\alpha}(t, x, y), \quad t \in (0, 1].$$

This completes the proof of Proposition 1.

Proof of Proposition 2. The proof is similar to Proposition 1. It suffices to prove (2.4) for $t \in (0,1]$ by the semigroup property. Then the argument above is still valid for (2.4), if we replace (2.5) by the heat kernel bounds of $H^0 = (-\Delta_q)^{\alpha/2} + P_{\alpha-1}$ (see [25, Theorem 4.3])

$$|p_0(t, x, y)| \le Cq_\alpha(t, x, y), \quad t \in (0, 1], \ x, y \in M.$$

3. Global eigenfunction estimates: proof of Lemma 1

To prove Lemma 1, we begin with the following resolvent estimate.

Proposition 3. For $\lambda \geq 1$, we have

where

(3.2)
$$\sigma(p) = \begin{cases} \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{p}\right), & 2 \le p < \frac{2(n+1)}{n-1} \\ \frac{n-1}{2} - \frac{n}{p}, & \frac{2(n+1)}{n-1} \le p \le \infty. \end{cases}$$

Proof. For $k \in \mathbb{N}$, let $\chi_{[k,k+1)}$ denote the spectral projection operators for $\sqrt{-\Delta_g}$ corresponds to the spectral interval [k,k+1), and let $\chi_{[2[\lambda],\infty)}$ spectral projection operator onto the interval $[2[\lambda],\infty)$, where $[\lambda]$ denotes the largest integer that is smaller than λ . Then for any function f, by Cauchy-Schwarz inequality

$$(\sqrt{-\Delta_g} - (\lambda + i))^{-1} f = \sum_{k < 2[\lambda]} \frac{1}{k - (\lambda + i)} (k - (\lambda + i)) (\sqrt{-\Delta_g} - (\lambda + i))^{-1} \chi_{[k, k+1)} f$$

$$+ \chi_{[2[\lambda], \infty)} (\sqrt{-\Delta_g} - (\lambda + i))^{-1} f$$

$$\lesssim (\sum_{k < 2[\lambda]} \left| (k - (\lambda + i)) (\sqrt{-\Delta_g} - (\lambda + i))^{-1} \chi_{[k, k+1)} f \right|^2)^{\frac{1}{2}}$$

$$+ \left| \chi_{[2[\lambda], \infty)} (\sqrt{-\Delta_g} - (\lambda + i))^{-1} f \right|$$

Thus, by Minkowski's inequality

(3.4)
$$\|(\sqrt{-\Delta_g} - (\lambda + i))^{-1}f\|_{L^p} \le (\sum_{k < 2[\lambda]} \|(k - (\lambda + i))(\sqrt{-\Delta_g} - (\lambda + i))^{-1}\chi_{[k,k+1)}f\|_{L^p}^2)^{\frac{1}{2}} + \|\chi_{[2[\lambda],\infty)}(\sqrt{-\Delta_g} - (\lambda + i))^{-1}f\|_{L^p}$$

To handle the first term on the right, note that $\chi_{[k,k+1)} = \chi_{[k,k+1)} \circ \chi_{[k,k+1)}$, and by the classical results in [44],

(3.5)
$$\|\chi_{[k,k+1)}f\|_{L^p} \lesssim (1+k)^{\sigma(p)} \|f\|_{L^2} \lesssim \lambda^{\sigma(p)} \|f\|_{L^2}, \text{ if } k < 2[\lambda].$$

Thus,

$$\sum_{k<2[\lambda]} \|(k-(\lambda+i))(\sqrt{-\Delta_g} - (\lambda+i))^{-1}\chi_{[k,k+1)}f\|_{L^p}^2)^{\frac{1}{2}}
\lesssim \lambda^{\sigma(p)} \sum_{k<2[\lambda]} \|(k-(\lambda+i))(\sqrt{-\Delta_g} - (\lambda+i))^{-1}\chi_{[k,k+1)}f\|_{L^2}^2)^{\frac{1}{2}}
\lesssim \lambda^{\sigma(p)} \sum_{k<2[\lambda]} \|\chi_{[k,k+1)}f\|_{L^2}^2)^{\frac{1}{2}}
\lesssim \lambda^{\sigma(p)} \|f\|_{L^2},$$
(3.6)

where in the second inequality we used the fact that by spectral theorem,

(3.7)
$$\|(k - (\lambda + i))(\sqrt{-\Delta_g} - (\lambda + i))^{-1}\chi_{[k,k+1)}f\|_{L^2} \lesssim \|\chi_{[k,k+1)}f\|_{L^2}, \ \forall k \in \mathbb{N}.$$

To handle the second term, we use Sobolev estimates to see that

(3.8)
$$\|\chi_{[2[\lambda],\infty)}(\sqrt{-\Delta_g} - (\lambda+i))^{-1}f\|_{L^p}$$

$$\lesssim \|\chi_{[2[\lambda],\infty)}(\sqrt{-\Delta_g})^{n(\frac{1}{2}-\frac{1}{p})}(\sqrt{-\Delta_g} - (\lambda+i))^{-1}f\|_{L^2}.$$

When $2 , it is straightforward to check that <math>n(\frac{1}{2} - \frac{1}{n}) < 1$, thus by spectral theorem,

which is better than the desired bound in (3.1).

Now we shall prove Lemma 1, this follows from similar strategies as in [2]. Recall that $\mathcal{D} = \sqrt{-\Delta_g} + P_0$, by using the second resolvent formula, we have

$$(3.10) \quad (\mathcal{D} + V - (\lambda + i))^{-1} = (\sqrt{-\Delta_g} - (\lambda + i))^{-1} - (\sqrt{-\Delta_g} - (\lambda + i))^{-1} (P_0 + V)(\mathcal{D} + V - (\lambda + i))^{-1}.$$

Since $P_0 \in OPS^0$ and the eigenvalues of $\mathcal{D} + V$ are real, by spectral theorem, we have

$$(3.11) ||P_0(\mathcal{D} + V - (\lambda + i))^{-1}||_{L^2 \to L^2} \lesssim ||(\mathcal{D} + V - (\lambda + i))^{-1}||_{L^2 \to L^2} \lesssim 1.$$

Similarly, since $V \in L^{\infty}(M)$, we have

$$||V(\mathcal{D} + V - (\lambda + i))^{-1}||_{L^2 \to L^2} \lesssim 1.$$

Thus, (3.10), (3.11), (3.12) and (3.1) yield that

(3.13)
$$\|(\mathcal{D} + V - (\lambda + i))^{-1}\|_{L^2 \to L^p} \lesssim \lambda^{\sigma(p)}, \ 2$$

If we let $\chi_{[\lambda,\lambda+1)}^V$ denote the spectral projection operator associated with $\sqrt{-\Delta_g} + P_0 + V$ for the interval $[\lambda,\lambda+1)$, then (3.13) implies the following

Corollary 1. Let $V \in L^{\infty}(M)$, we have

(3.14)
$$\|\chi_{[\lambda,\lambda+1)}^V f\|_{L^p} \lesssim \lambda^{\sigma(p)} \|f\|_{L^2}, \ 2$$

Note that if we take $f = e_{\lambda}$ in (3.14), and use the fact that $\chi_{[\lambda, \lambda+1)}^{V} e_{\lambda} = e_{\lambda}$, we obtain (1.5).

Proof of Corollary 1. If $2 , this follows from (3.13) by letting <math>f = \chi^V_{[\lambda,\lambda+1)} f$ there along with the fact that

(3.15)
$$\|(\mathcal{D} + V - (\lambda + i))\chi^{V}_{[\lambda, \lambda + 1)}\|_{L^{2} \to L^{2}} \lesssim 1.$$

If $p > \frac{2(n+1)}{n-1}$, we shall use the heat kernel bounds in Proposition 2. More explicitly, let $H_V = \sqrt{-\Delta_g} + P_0 + V$, note that if $V \in L^{\infty}(M)$, then (2.1) holds with $\alpha = 1$, thus $V \in \mathcal{K}_1(M)$, which, by Proposition 2, implies that we have the kernel estimate (2.4) for e^{-tH_V} . As a result, by (2.4) and Young's inequality, we have the following:

(3.16)
$$\|e^{-tH_V}\|_{L^p(M) \to L^q(M)} \lesssim t^{-n(\frac{1}{p} - \frac{1}{q})}, \text{ if } 0 < t \le 1, \text{ and } 1 \le p \le q \le \infty.$$

If we fix $t = \lambda^{-1}$ and $p_c = \frac{2(n+1)}{n-1}$, and apply the above bound, we have for $p > \frac{2(n+1)}{n-1}$,

(3.17)
$$\|\chi_{[\lambda,\lambda+1)}^{V}f\|_{L^{p}} \lesssim \lambda^{n(\frac{1}{p_{c}}-\frac{1}{p})} \|e^{\lambda^{-1}H_{V}}\chi_{[\lambda,\lambda+1)}^{V}f\|_{L^{p_{c}}}$$

$$= \lambda^{n(\frac{1}{p_{c}}-\frac{1}{p})} \|\chi_{[\lambda,\lambda+1)}^{V}e^{\lambda^{-1}H_{V}}\chi_{[\lambda,\lambda+1)}^{V}f\|_{L^{p_{c}}}$$

$$\lesssim \lambda^{n(\frac{1}{p_{c}}-\frac{1}{p})}\lambda^{\frac{n-1}{2}-\frac{n}{p_{c}}} \|e^{\lambda^{-1}H_{V}}\chi_{[\lambda,\lambda+1)}^{V}f\|_{L^{2}}$$

$$\lesssim \lambda^{\frac{n-1}{2}-\frac{n}{p}} \|f\|_{L^{2}},$$

where in the third line we applied (3.14) at $p=p_c$ and in the last line we applied spectral theorem. Since $\frac{n-1}{2} - \frac{n}{p} = \sigma(p)$ when $p \ge p_c$, the proof of Corollary 1 is complete.

To prove Lemma 1 it remains to prove (1.6). By using the arguments from Sogge-Zelditch [51], we note that (1.6) can be obtained from Hölder's inequality and (1.5)

$$\|e_{\lambda}\|_{L^{2}(M)}^{\frac{1}{\theta}} \leq \|e_{\lambda}\|_{L^{1}(M)} \|e_{\lambda}\|_{L^{p}(M)}^{\frac{1}{\theta}-1} \lesssim \|e_{\lambda}\|_{L^{1}(M)} (\lambda^{\sigma(p)}\|e_{\lambda}\|_{L^{2}(M)})^{\frac{1}{\theta}-1} = \|e_{\lambda}\|_{L^{1}(M)} \lambda^{\frac{n-1}{4}} \|e_{\lambda}\|_{L^{2}(M)}^{\frac{1}{\theta}-1}.$$
Here $2 , and $\theta = \frac{p}{p-1}(\frac{1}{2} - \frac{1}{p})$.$

4. Kernels of Pseudo-differential operators

In this section, we prove a useful lemma concerning the kernel estimates of the pseudo-differential operators on compact manifolds.

Lemma 4. Let $\mu \in \mathbb{R}$, and $m \in C^{\infty}(\mathbb{R})$ belong to the symbol class S^{μ} , that is, assume that

$$(4.1) |\partial_t^{\alpha} m(t)| \le C_{\alpha} (1 + |t|)^{\mu - \alpha}, \quad \forall \alpha.$$

If $P = \sqrt{-\Delta_g}$, then m(P) is a pseudo-differential operator of order μ . Moreover, if $R \ge 1$, then the kernel of the operator m(P/R) satisfies for all $N \in \mathbb{N}$

$$(4.2) |m(P/R)(x,y)| \lesssim \begin{cases} R^n \left(Rd_g(x,y)\right)^{-n-\mu} \left(1 + Rd_g(x,y)\right)^{-N}, & n+\mu > 0\\ R^n \log(2 + (Rd_g(x,y))^{-1}) \left(1 + Rd_g(x,y)\right)^{-N}, & n+\mu = 0\\ R^n (1 + Rd_g(x,y))^{-N}, & n+\mu < 0. \end{cases}$$

See [45, Theorem 4.3.1] for the proof of the fact that m(P) is a pseudo-differential operator of order μ . The kernel bounds (4.2) can be viewed as the rescaled version on compact manifolds compared to the Euclidean estimates in [55, Proposition 1 on page 241]. We mean that the bounds hold near the diagonal (so that $d_g(x, y)$ is smaller than the injectivity radius of M) and that outside the neighborhood of the diagonal they are $O(R^{-N})$. Roughly speaking, modulo lower order terms, m(P/R)(x, y) equals

$$(2\pi)^{-n} \int_{\mathbb{R}^n} m(|\xi|/R) e^{id_g(x,y)\xi_1} d\xi$$

near the diagonal, which satisfies the bounds in (4.2), while outside of a fixed neighborhood of the diagonal m(P/R)(x,y) is $O(R^{-N})$. For completeness, we give a detailed proof by using the Hadamard parametrix.

Proof of (4.2). Since the spectrum of $P = \sqrt{-\Delta_g}$ is nonnegative, we may assume that m(t) is an even function on \mathbb{R} . Let $\delta > 0$ be smaller than the injectivity radius of (M, g). Let $\rho \in C_0^{\infty}(-1, 1)$

be even and satisfy $\rho \equiv 1$ on $\left(-\frac{\delta}{2}, \frac{\delta}{2}\right)$. So we can write

$$(4.3) m(P/R) = \frac{R}{2\pi} \int_{\mathbb{R}} \hat{m}(tR) \cos(tP) dt$$
$$= \frac{R}{2\pi} \int \rho(t) \hat{m}(tR) \cos(tP) dt + \frac{R}{2\pi} \int (1 - \rho(t)) \hat{m}(tR) \cos(tP) dt.$$

To handle the first term in (4.3), we need to use the Hadamard parametrix (see e.g. [46, Section 1.2 and Theorem 3.1.5]). For $0 < t < \delta$ and $N_0 > n + 3$, we have

(4.4)
$$\cos t P(x,y) = \sum_{\nu=0}^{N_0} \omega_{\nu}(x,y) \partial_t E_{\nu}(t, d_g(x,y)) + R_{N_0}(t,x,y)$$

where the leading term

(4.5)
$$\partial_t E_0 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{id_g(x,y)\xi_1} \cos(t|\xi|) d\xi$$

and E_{ν} satisfies $2\partial_t E_{\nu} = tE_{\nu-1}$, and $\partial_t E_{\nu}(t/R,r) = R^{n-2\nu}\partial_t E_{\nu}(t,Rr)$ for any R > 0. Here $\omega_{\nu} \in C^{\infty}(M \times M)$, and $\omega_0(x,x) = 1, \forall x \in M$. For $\nu \geq 1$, we have the following explicit formula (see e.g. [46, Section 1.2])

$$E_{\nu} = \nu! (2\pi)^{-n} \int_{0 \le s_1 \le \dots \le s_{\nu} \le t} \int_{\mathbb{R}^n} e^{id_g(x,y)\xi_1} \frac{\sin(t-s_{\nu})|\xi|}{|\xi|} \frac{\sin(s_{\nu}-s_{\nu-1})|\xi|}{|\xi|} \cdots \frac{\sin(s_2-s_1)|\xi|}{|\xi|} \frac{\sin s_1|\xi|}{|\xi|} d\xi ds_1 \dots ds_{\nu}.$$

So for $\nu \geq 1$ we can obtain (see e.g. [46, Section 1.2])

(4.6)
$$\partial_t E_{\nu} = \frac{1}{2} t E_{\nu-1} = \int e^{id_g(x,y)\xi_1} a_{\nu}(t,|\xi|) d\xi$$
$$= \sum_{\pm} \sum_{j=0}^{\nu-1} a_{j\nu}^{\pm} \int e^{id_g(x,y)\xi_1 \pm it|\xi|} t^{j+1} |\xi|^{-2\nu+1+j} d\xi,$$

where $a_{j\nu}^{\pm}$ are constants, and $a_{\nu} \in C^{\infty}$. The remainder kernel $R_{N_0} \in C^{N_0-n-3}$ satisfies

$$(4.7) |\partial_{t,x,y}^{\alpha} R_{N_0}(t,x,y)| \lesssim |t|^{2N_0 + 2 - n - |\alpha|}, \quad |\alpha| \le N_0 - n - 2.$$

Then we plug (4.4) into the first term of (4.3). We first handle the contribution of the leading term in (4.4). By (4.5), we can write

$$\frac{R}{2\pi} \iint \rho(t)\hat{m}(tR)\cos(t|\xi|)e^{id_g(x,y)\xi_1}dtd\xi = \int m(|\xi|/R)e^{id_g(x,y)\xi_1}d\xi + \frac{R}{2\pi} \iint (1-\rho(t))\hat{m}(tR)\cos(t|\xi|)e^{id_g(x,y)\xi_1}dtd\xi := I_1 + I_2.$$

Using the property (4.1) and integration by parts, we see that for any $N \in \mathbb{N}$

(4.8)
$$|I_1| \lesssim \begin{cases} R^n \left(Rd_g(x,y) \right)^{-n-\mu} \left(1 + Rd_g(x,y) \right)^{-N}, & n+\mu > 0 \\ R^n \log(2 + \left(Rd_g(x,y) \right)^{-1}) \left(1 + Rd_g(x,y) \right)^{-N}, & n+\mu = 0 \\ R^n (1 + Rd_g(x,y))^{-N}, & n+\mu < 0 \end{cases}$$

and

$$|I_{2}| \lesssim \left| R \iiint (1 - \rho(t))(tR)^{-N} m^{(N)}(s) e^{-itRs} \cos(t|\xi|) e^{id_{g}(x,y)\xi_{1}} ds dt d\xi \right|$$

$$\lesssim R^{-N+1} \iint (1 + ||\xi| - R|s||)^{-N_{1}} (1 + |s|)^{-N+\mu} ds d\xi$$

$$\lesssim R^{-N} \int (1 + |\xi|/R)^{-N+\mu} d\xi$$

$$\lesssim R^{-N+n}.$$
(4.9)

Here we choose $N_1 > N > n + \mu$.

Similarly, we can handle the contributions of the remaining terms in (4.4). For each $\nu \geq 1$, we can write

$$\frac{R}{2\pi} \int \rho(t) \hat{m}(tR) \partial_t E_{\nu}(t, d_g(x, y)) dt = \frac{R}{2\pi} \int \hat{m}(tR) \partial_t E_{\nu}(t, d_g(x, y)) dt - \frac{R}{2\pi} \int (1 - \rho(t)) \hat{m}(tR) \partial_t E_{\nu}(t, d_g(x, y)) dt := I_3 + I_4.$$

Using the scaling property $\partial_t E_{\nu}(t/R, r) = R^{n-2\nu} \partial_t E_{\nu}(t, Rr)$ and the formula (4.6), we can integrate by parts to see that

$$|I_{3}| = (2\pi)^{-1}R^{n-2\nu} \Big| \int \hat{m}(t)\partial_{t}E_{\nu}(t,Rd_{g}(x,y))dt \Big|$$

$$= (2\pi)^{-1}R^{n-2\nu} \Big| \sum_{\pm} \sum_{j=0}^{\nu-1} a_{j\nu}^{\pm} \int \int e^{iRd_{g}(x,y)\xi_{1}\pm it|\xi|} \hat{m}(t)t^{j+1} |\xi|^{-2\nu+1+j} dt d\xi \Big|$$

$$= R^{n-2\nu} \Big| \sum_{\pm} \sum_{j=0}^{\nu-1} i^{-j-1} a_{j\nu}^{\pm} \int e^{iRd_{g}(x,y)\xi_{1}} m^{(j+1)} (\pm |\xi|) |\xi|^{-2\nu+1+j} d\xi \Big|$$

$$(4.10) \qquad \lesssim R^{n-2\nu} (1 + Rd_{g}(x,y))^{-N} + \sum_{j=0}^{\nu-1} \Big| \int e^{iRd_{g}(x,y)\xi_{1}} m^{(j+1)} (|\xi|) |\xi|^{-2\nu+1+j} \varphi(|\xi|) d\xi \Big|$$

$$\leq \begin{cases} R^{n-2\nu} (Rd_{g}(x,y))^{-n-\mu} (1 + Rd_{g}(x,y))^{-N}, & n+\mu > 0 \\ R^{n-2\nu} \log(2 + (Rd_{g}(x,y))^{-1}) (1 + Rd_{g}(x,y))^{-N}, & n+\mu = 0 \\ R^{n-2\nu} (1 + Rd_{g}(x,y))^{-N}, & n+\mu < 0 \end{cases}$$

where $\varphi \in C^{\infty}$ vanishes near the origin but equals one near infinity. The first term in (4.10) follows from the smoothness of a_{ν} in (4.6) near $\xi = 0$ and integration by parts. Moreover,

$$|I_{4}| \lesssim \sum_{\pm} \sum_{j=0}^{\nu-1} \left| R \iiint (1 - \rho(t))(tR)^{-N} m^{(N+j+1)}(s) e^{-itRs} e^{id_{g}(x,y)\xi_{1} \pm it|\xi|} \phi_{j\nu}(|\xi|) d\xi ds dt \right|$$

$$\lesssim R^{-N+1} \iint (1 + ||\xi| - R|s||)^{-N_{1}} (1 + |s|)^{-N+\mu} ds d\xi$$

$$\lesssim R^{-N} \int (1 + |\xi|/R)^{-N+\mu} d\xi$$

$$(4.12) \qquad \lesssim R^{-N+n}.$$

The remainder term R_{N_0} in (4.4) is easy to handle. Indeed, for $n + \mu < N \le N_0 - n - 2$, using (4.7) we integrate by parts to obtain

$$\left| \frac{R}{2\pi} \int \rho(t) \hat{m}(tR) R_{N_0}(t, x, y) dt \right| \lesssim R^{-N+1} \left| \iint \rho(t) t^{-N} R_{N_0}(t, x, y) m^{(N)}(s) e^{-itRs} ds dt \right|
\lesssim R^{-N+1} \int (1 + R|s|)^{-N} (1 + |s|)^{\mu - N} ds
\lesssim R^{-N+1}.$$
(4.13)

To handle the second term in (4.3), we notice that for $\lambda \geq 0$

$$\left| \frac{R}{2\pi} \int (1 - \rho(t)) \hat{m}(tR) \cos(t\lambda) dt \right| \lesssim \left| R \iint (1 - \rho(t)) (tR)^{-N} m^{(N)}(s) e^{-itRs} \cos(t\lambda) dt ds \right|$$
$$\lesssim R^{-N+1} \int (1 + |\lambda - R|s||)^{-N_1} (1 + |s|)^{-N+\mu} ds$$
$$\lesssim R^{-N} (1 + \lambda/R)^{-N+\mu}.$$

Thus, we obtain

$$\left| \frac{R}{2\pi} \int (1 - \rho(t)) \hat{m}(tR) \cos(tP)(x, y) dt \right| \lesssim R^{-N} \sum_{j} (1 + \lambda_j / R)^{-N+\mu} |e_j(x) e_j(y)|$$

$$\lesssim R^{-N} \sum_{k} (1 + k / R)^{-N+\mu} \sum_{\lambda_j \in [k, k+1)} |e_j(x) e_j(y)|$$

$$\lesssim R^{-N} \sum_{k} (1 + k / R)^{-N+\mu} (1 + k)^{n-1}$$

$$\lesssim R^{-N+n}.$$
(4.14)

Here we used the L^{∞} bound of Laplace eigenfunctions (see e.g. [45, Lemma 4.2.4])

$$\sum_{\lambda_j \in [k,k+1)} |e_j(x)e_j(y)| \lesssim \sup_{x \in M} \sum_{\lambda_j \in [k,k+1)} |e_j(x)|^2 \lesssim (1+k)^{n-1}.$$

Combining the bounds (4.8), (4.9), (4.11), (4.12), (4.13), (4.14), we complete the proof.

5. Interior Eigenfunction estimates

In this section, we prove the eigenfunction estimates in Theorem 1. We just need to prove Lemma 2, and then Theorem 1 follows from the L^p bounds in Lemma 1. To proceed, we shall use the following lemma.

Lemma 5. For any $f \in H^{1/2}(\partial\Omega)$, let $u \in H^1(\Omega)$ be the weak solution to the Dirichlet boundary value problem (1.1). Then there exists a constant C > 0 such that

$$||u||_{L^{2}(\Omega)} \leq C||f||_{H^{-1/2}(\partial\Omega)}.$$

This lemma was proved in [14, Proposition 2.17]. It follows from the trace theorem and standard regularity estimates (see e.g. [28, Theorem 1.5.1.2, Theorem 1.5.1.3, Corollary 2.2.2.4, Corollary 2.2.2.6]).

Lemma 6. Let $Q \in OPS^0$. Then Q is bounded on L^p for 1 , i.e.

$$||Qf||_{L^p} \leq C||f||_{L^p}.$$

Here the L^p norm can be taken on \mathbb{R}^n and compact manifolds. See e.g. [45, Theorem 3.1.6, Theorem 4.3.1] for the proofs.

Proof of Theorem 1. It suffices to consider two cases, $p = \infty$ and $p < \infty$.

Case 1: $p = \infty$. In this case, from the maximal principle (see e.g. [24, Theorem 8.1]), since e_{λ} is harmonic in Ω . We get

And since $V \in L^{\infty}(M)$, by Lemma 1, we have

$$||e_{\lambda}||_{L^{\infty}(\partial\Omega)} \lesssim \lambda^{\frac{n-1}{2}} ||e_{\lambda}||_{L^{2}(\partial\Omega)},$$

which yields (1.2) for the case $p = \infty$.

Case 2: $p < \infty$. In this case, let us fix a Littlewood-Paley bump function $\beta \in C_0^{\infty}((1/2,2))$ satisfying

$$\sum_{\ell=-\infty}^{\infty} \beta(2^{-\ell}s) = 1, \quad s > 0.$$

And define

$$\beta_0(s) = 1 - \sum_{\ell>0} \beta(2^{-\ell}|s|), \ \beta_{\ell}(s) = \beta(2^{-\ell}|s|), \ \text{for } \ell>0.$$

Let $P = \sqrt{-\Delta_q}$. Then we have for $\ell \geq 0$,

(5.3)
$$\|\beta_{\ell}(P)f\|_{L^{p}(\partial\Omega)} \lesssim \|f\|_{L^{p}(\partial\Omega)}, \quad 1 \leq p \leq \infty.$$

The implicit constant is independent of ℓ . Indeed, by Lemma 4 we have the kernel estimates

$$|\beta_{\ell}(P)(x,y)| \lesssim 2^{n\ell} (1 + 2^{\ell} d_g(x,y))^{-N}.$$

Then (5.3) follows from Young's inequality.

Let T_H be the harmonic extension operator from $\partial\Omega$ to Ω . Then by Lemma 5, we have

(5.4)
$$||T_H(\beta_\ell(P)f)||_{L^2(\Omega)} \lesssim ||\beta_\ell(P)f||_{H^{-1/2}(\partial\Omega)} \lesssim 2^{-\ell/2} ||f||_{L^2(\partial\Omega)}.$$

And from the maximal principle and (5.3), we have

$$(5.5) ||T_H(\beta_\ell(P)f)||_{L^{\infty}(\Omega)} \lesssim ||\beta_\ell(P)f||_{L^{\infty}(\partial\Omega)} \lesssim ||f||_{L^{\infty}(\partial\Omega)}.$$

By (5.4), (5.5) and interpolation, we have the following L^p estimate of the frequency-localized harmonic extension operator

(5.6)
$$||T_H(\beta_\ell(P)f)||_{L^p(\Omega)} \lesssim 2^{-\frac{\ell}{p}} ||f||_{L^p(\partial\Omega)}, \ 2 \le p \le \infty.$$

Thus, if $2^{\ell} \gtrsim \lambda$, we have

$$(5.7) ||T_H(\sum_{2^{\ell} \gtrsim \lambda} \beta_{\ell}(P)e_{\lambda})||_{L^p(\Omega)} \lesssim \sum_{2^{\ell} \gtrsim \lambda} 2^{-\frac{\ell}{p}} ||e_{\lambda}||_{L^p(\partial\Omega)} \lesssim \lambda^{-1/p} ||e_{\lambda}||_{L^p(\partial\Omega)}.$$

So it remains to consider $2^{\ell} \lesssim \lambda$. Let $\tilde{\beta} \in C_0^{\infty}$ with $\tilde{\beta} \equiv 1$ in a neighborhood of (1/2, 2) and define $\tilde{\beta}_{\ell}(s) = \tilde{\beta}(2^{-\ell}|s|)$. Then by (5.6)

$$(5.8) ||T_H(\beta_\ell(P)e_\lambda)||_{L^p(\Omega)} = ||T_H(\beta_\ell(P)\tilde{\beta}_\ell(P)e_\lambda)||_{L^p(\Omega)} \lesssim 2^{-\frac{\ell}{p}} ||\tilde{\beta}_\ell(P)e_\lambda||_{L^p(\partial\Omega)}.$$

Moreover, for $2 \le p < \infty$

$$\|\tilde{\beta}_{\ell}(P)e_{\lambda}\|_{L^{p}(\partial\Omega)}$$

$$(5.9) = (1+\lambda)^{-1} \|\tilde{\beta}_{\ell}(P)(1+\sqrt{-\Delta_{g}}+P_{0}+V)e_{\lambda}\|_{L^{p}(\partial\Omega)} \lesssim (1+\lambda)^{-1} \|\tilde{\beta}_{\ell}(P)(1+\sqrt{-\Delta_{g}})e_{\lambda}\|_{L^{p}(\partial\Omega)} + (1+\lambda)^{-1} \|\tilde{\beta}_{\ell}(P)(P_{0}+V)e_{\lambda}\|_{L^{p}(\partial\Omega)} \lesssim (1+\lambda)^{-1} 2^{\ell} \|e_{\lambda}\|_{L^{p}(\partial\Omega)} + (1+\lambda)^{-1} \|e_{\lambda}\|_{L^{p}(\partial\Omega)}$$

where we used (5.3), Lemma 6, and the fact that $V \in L^{\infty}$. Using (5.8) and (5.9), we have

$$(5.10) ||T_H(\sum_{2^{\ell} \lesssim \lambda} \beta_{\ell}(P)e_{\lambda})||_{L^p(\Omega)} \lesssim \sum_{2^{\ell} \lesssim \lambda} (1+\lambda)^{-1} 2^{\ell} 2^{-\frac{\ell}{p}} ||e_{\lambda}||_{L^p(\partial\Omega)} \lesssim \lambda^{-1/p} ||e_{\lambda}||_{L^p(\partial\Omega)}.$$

So we obtain (1.7) in Lemma 2. Using the L^p bounds in Lemma 1, we complete the proof of Theorem 1.

6. Measure of nodal set

In this section, we prove the nodal set estimates in Theorem 2.

First, we establish some general results for Sobolev spaces on compact manifolds. These results will be used to prove the regularity of eigenfunctions. They are likely to be useful for future research, so we give detailed proofs for them.

Let s > 0 and 1 . We can define the Sobolev norm on M by local coordinates

(6.1)
$$||f||_{W^{s,p}(M)} = \sum_{\nu} ||(I - \Delta)^{s/2} f_{\nu}||_{L^{p}(\mathbb{R}^{n})}.$$

where $f_{\nu} = (\phi_{\nu} f) \circ \kappa_{\nu}^{-1}$, and $\{\phi_{\nu}\}$ is a partition of unity subordinate to a finite covering $M = \cup \Omega_{\nu}$, and $\kappa_{\nu} : \Omega_{\nu} \to \tilde{\Omega}_{\nu} \subset \mathbb{R}^{n}$ is the coordinate map. For simplicity, we sometimes do not distinguish between Ω_{ν} and $\tilde{\Omega}_{\nu}$, f_{ν} and $\phi_{\nu} f$, since they are identical up to the coordinate map.

Moreover, we can also define another Sobolev norm by pseudo-differential operators

(6.2)
$$||f||_{H^{s,p}(M)} = ||(I - \Delta_g)^{s/2} f||_{L^p(M)}.$$

By [45, Theorem 4.3.1], we see that $(I - \Delta_g)^{s/2}$ is an invertible pseudo-differential operator of order s with elliptic principal symbol $(\sum g^{jk}(x)\xi_j\xi_k)^{s/2}$. Moreover, if we replace $(I - \Delta_g)^{s/2}$ in (6.2) by any invertible pseudo-differential operator of order s, then it still gives a comparable norm, by Lemma 6.

We prove that these two Sobolev norms are equivalent.

Proposition 4. For s > 0 and 1 , we have

$$||f||_{W^{s,p}(M)} \approx ||f||_{H^{s,p}(M)}.$$

The implicit constants are independent of f.

As a corollary, different partitions of unity and such coordinate atlases in the definition (6.1) give comparable norms. When p=2, Proposition 4 follows from Plancherel theorem and the L^2 -boundedness of zero order pseudo-differential operators, see e.g. [46, section 4.2]. The case $p \neq 2$ is more complicated, and it is very difficult to find good references. To prove this on our own, we start with the following key lemma. Roughly speaking, this lemma establishes a "linear relation" between any two pseudo-differential operators of the same order.

Lemma 7. Let s > 0. Let V_1 , V, Ω be open sets such that $\bar{V}_1 \subset V \subset \Omega$. Let P_1 , $P \in OPS^s$ with symbols supported in V_1 , V respectively. If the principal symbol $\bar{p}(x,\xi)$ of P is elliptic on \bar{V}_1 , i.e., for any $x \in \bar{V}_1$,

$$\bar{p}(x,\xi) \neq 0, \ \forall \xi \neq 0,$$

then there is a $Q \in OPS^0$ with symbol supported in V_1 such that

$$(6.3) P_1 - QP \in OPS^0.$$

Proof. Let $p_1(x,\xi)$ be the symbols of P_1 on Ω . Since $\bar{p}(x,\xi)$ is elliptic on the support of $p_1(x,\xi)$, we have

(6.4)
$$\frac{\varphi(\xi)p_1(x,\xi)}{\bar{p}(x,\xi)} \in S^0$$

where $\varphi \in C^{\infty}$ vanishes near the origin but equals one near infinity. Denote the associated zero order pseudo-differential operator by Q_0 . Let $R_{-1} = P_1 - Q_0 P$. Then by the Kohn-Nirenberg theorem (see e.g. [45, Theorem 3.1.1]), we have $R_{-1} \in OPS^{s-1}$. The symbol of R_{-1} is supported in V_1 . If $s \leq 1$, then we are done by setting $Q = Q_0$, since $P_1 - Q_0 P \in OPS^{s-1} \subset OPS^0$.

Next, it remains to consider s > 1. Let $k = \lceil s \rceil \ge 2$. We need to construct $Q_{-i} \in OPS^{-i}$, $R_{-i-1} \in OPS^{s-i-1}$ recursively for $1 \le i \le k-1$. If $r_i(x,\xi)$ is the symbol of R_{-i} , and Q_{-i} has the symbol

(6.5)
$$\frac{\varphi(\xi)r_i(x,\xi)}{\bar{p}(x,\xi)} \in S^{-i},$$

then using the Kohn-Nirenberg theorem we have $R_{-i-1} = R_{-i} - Q_{-i}P \in OPS^{s-i-1}$. The symbol of R_{-i-1} is supported in V_1 . Let

$$Q = \sum_{i=1}^{k-1} Q_{-i}.$$

The symbol of Q is supported in V_1 . Then $P_1 - QP = R_{-k} \in OPS^{s-k} \subset OPS^0$.

Proof of Proposition 4. The basic idea is to verify these two equivalences

(6.6)
$$||(I - \Delta_g)^{s/2} f||_{L^p(M)} \approx \sum_{\nu} ||(I - \Delta_g)^{s/2} f_{\nu}||_{L^p(M)} \approx \sum_{\nu} ||(I - \Delta)^{s/2} f_{\nu}||_{L^p(\mathbb{R}^n)}.$$

The first equivalence is straightforward. Indeed, The relation \lesssim follows from Minkowski inequality. And for the other direction, we use Lemma 6 to see that

(6.7)
$$\|(I - \Delta_g)^{s/2} f_{\nu}\|_{L^p(M)} = \|(I - \Delta_g)^{s/2} M_{\phi_{\nu}} (I - \Delta_g)^{-s/2} ((I - \Delta_g)^{s/2} f)\|_{L^p(M)}$$

$$\lesssim \|(I - \Delta_g)^{s/2} f\|_{L^p(M)},$$

where $M_{\phi_{\nu}}$ stands for the operator of multiplying by $\phi_{\nu}(x)$. Summing up of (6.7) over ν we obtain the first equivalence in (6.6).

To prove the second equivalence in (6.6), it suffices to show that for each ν

(6.8)
$$||(I - \Delta_g)^{s/2} f_{\nu}||_{L^p(M)} \approx ||(I - \Delta)^{s/2} f_{\nu}||_{L^p(\mathbb{R}^n)}.$$

For each Ω_{ν} , $\phi_{\nu} \in C_0^{\infty}(\Omega_{\nu})$ in (6.1), we can find open subsets V_{ν} , U_{ν} , W_{ν} of Ω_{ν} , and cutoff functions $\psi_{\nu} \in C_0^{\infty}(V_{\nu})$, $\psi_{\nu 1} \in C_0^{\infty}(U_{\nu})$, $\psi_{\nu 2} \in C_0^{\infty}(W_{\nu})$, $\eta_{\nu} \in C_0^{\infty}(\Omega_{\nu})$ such that

$$\operatorname{supp} \phi_{\nu} \subset\subset U_{\nu} \subset V_{\nu} \subset\subset W_{\nu}$$

and $\psi_{\nu} \equiv 1$ on \bar{U}_{ν} , $\psi_{\nu 2} \equiv 1$ on \bar{V}_{ν} , $\eta_{\nu} \equiv 1$ on \bar{W}_{ν} .

Let $P_{\nu} = \psi_{\nu}(I - \Delta)^{s/2}$, $P_{\nu 1} = \psi_{\nu 1}(I - \Delta_g)^{s/2}M_{\eta_{\nu}}$. We see that $M_{\eta_{\nu}} \in OPS^0$, and $P_{\nu}, P_{\nu 1} \in OPS^s$. Note that the principal symbol of P_{ν} is $\psi_{\nu}(x)|\xi|^s$, which is elliptic on \bar{U}_{ν} . By Lemma 7, we can find $Q_{\nu 1} \in OPS^0$ supported in U_{ν} such that

$$P_{\nu 1} - Q_{\nu 1} P_{\nu} \in OPS^0.$$

Then by Lemma 6 we obtain the local estimate

$$(6.9) \quad \|P_{\nu 1}(f_{\nu})\|_{L^{p}(\Omega_{\nu})} = \|(P_{\nu 1} - Q_{\nu 1}P_{\nu})(f_{\nu}) + Q_{\nu 1}P_{\nu}(f_{\nu})\|_{L^{p}(\Omega_{\nu})} \lesssim \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|P_{\nu}f_{\nu}\|_{L^{p}(\Omega_{\nu})}.$$

Moreover, if $P_{\nu 2} = \psi_{\nu 2} (I - \Delta_g)^{s/2} M_{\eta_{\nu}}$, then $P_{\nu 2}$ has the principal symbol $\psi_{\nu 2}(x) (\sum g^{jk}(x) \xi_j \xi_k)^{s/2}$, which is elliptic on \bar{V}_{ν} . Similarly, by applying Lemma 7 to P_{ν} and $P_{\nu 2}$, we obtain the local estimate

Next, we handle the nonlocal part. We write

$$(6.11) \qquad (1 - \psi_{\nu})(I - \Delta)^{s/2} f_{\nu} = (1 - \psi_{\nu})(I - \Delta)^{s/2} (\phi_{\nu} \eta_{\nu} f) = (1 - \psi_{\nu})(I - \Delta)^{s/2} M_{\phi_{\nu}}(\eta_{\nu} f).$$

Since dist(supp $(1 - \psi_{\nu})$, supp ϕ_{ν}) = $\delta_{\nu} > 0$, using integration by parts, we see that the kernel of $(1 - \psi_{\nu})(I - \Delta)^{s/2}M_{\phi_{\nu}}$ satisfies

$$\left| \int_{\mathbb{R}^n} (1 - \psi_{\nu}(x)) e^{i(x-y)\cdot\xi} \phi_{\nu}(y) (1 + |\xi|^2)^{s/2} d\xi \right| \lesssim (1 + |x-y|)^{-N}, \ \forall N.$$

By Young's inequality, we get

(6.12)
$$\|(1-\psi_{\nu})(I-\Delta)^{s/2}(f_{\nu}))\|_{L^{p}(\mathbb{R}^{n})} \lesssim \|\eta_{\nu}f\|_{L^{p}(\mathbb{R}^{n})} = \|f_{\nu}\|_{L^{p}(\Omega_{\nu})}.$$

Similarly, using the fact that the kernel of pseudo-differential operators on compact manifolds is smooth away from diagonal, we have (6.13)

$$\|(1-\psi_{\nu 1})(I-\Delta_g)^{s/2}(f_{\nu})\|_{L^p(M)} = \|(1-\psi_{\nu 1})(I-\Delta_g)^{s/2}(\phi_{\nu}\eta_{\nu}f)\|_{L^p(M)} \lesssim \|\eta_{\nu}f\|_{L^p(M)} = \|f_{\nu}\|_{L^p(\Omega_{\nu})}$$
 and

(6.14)
$$\|(1 - \psi_{\nu 2})(I - \Delta_g)^{s/2}(f_{\nu})\|_{L^p(M)} \lesssim \|f_{\nu}\|_{L^p(\Omega_{\nu})}.$$

Combining (6.9) with the nonlocal estimates (6.12) and (6.13), we obtain

$$\|(I - \Delta_{g})^{s/2} f_{\nu}\|_{L^{p}(M)} \lesssim \|(1 - \psi_{\nu 1})(I - \Delta_{g})^{s/2} f_{\nu}\|_{L^{p}(M)} + \|\psi_{\nu 1}(I - \Delta_{g})^{s/2} f_{\nu}\|_{L^{p}(M)}$$

$$\lesssim \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|\psi_{\nu 1}(I - \Delta_{g})^{s/2} f_{\nu}\|_{L^{p}(\Omega_{\nu})}$$

$$= \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|P_{\nu 1} f_{\nu}\|_{L^{p}(\Omega_{\nu})}$$

$$\lesssim \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|P_{\nu} f_{\nu}\|_{L^{p}(\Omega_{\nu})}$$

$$= \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|(I - \Delta)^{s/2} f_{\nu} - (1 - \psi_{\nu})(I - \Delta)^{s/2} f_{\nu}\|_{L^{p}(\Omega_{\nu})}$$

$$\lesssim \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|(I - \Delta)^{s/2} f_{\nu}\|_{L^{p}(\mathbb{R}^{n})}$$

$$\lesssim \|(I - \Delta)^{s/2} f_{\nu}\|_{L^{p}(\mathbb{R}^{n})}.$$

Here in the last step we apply Lemma 6 to $(I - \Delta)^{-s/2} \in OPS^0$.

Similarly, combining (6.10) with the nonlocal estimates (6.12) and (6.14), we have

$$\|(I - \Delta)^{s/2} f_{\nu}\|_{L^{p}(\mathbb{R}^{n})} \lesssim \|(1 - \psi_{\nu})(I - \Delta)^{s/2} f_{\nu}\|_{L^{p}(\mathbb{R}^{n})} + \|\psi_{\nu}(I - \Delta)^{s/2} f_{\nu}\|_{L^{p}(\mathbb{R}^{n})}$$

$$\lesssim \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|\psi_{\nu}(I - \Delta)^{s/2} f_{\nu}\|_{L^{p}(\Omega_{\nu})}$$

$$= \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|P_{\nu} f_{\nu}\|_{L^{p}(\Omega_{\nu})}$$

$$\lesssim \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|P_{\nu} f_{\nu}\|_{L^{p}(\Omega_{\nu})}$$

$$= \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|(I - \Delta_{g})^{s/2} f_{\nu} - (1 - \psi_{\nu})(I - \Delta_{g})^{s/2} f_{\nu}\|_{L^{p}(\Omega_{\nu})}$$

$$\lesssim \|f_{\nu}\|_{L^{p}(\Omega_{\nu})} + \|(I - \Delta_{g})^{s/2} f_{\nu}\|_{L^{p}(M)}$$

$$\lesssim \|(I - \Delta_{g})^{s/2} f_{\nu}\|_{L^{p}(M)} .$$

In the last step we used Lemma 6 for $(I - \Delta_g)^{-s/2} \in OPS^0$. So we finish the proof of (6.8). Thus, the proof of Proposition 4 is complete.

Let $[\mathcal{D}, V] = \mathcal{D}V - V\mathcal{D}$. We need to following commutator estimate.

Lemma 8. Let $1 . Given <math>P \in OPS^1$,

$$||[P, f]u||_{L^p} \le C||f||_{\text{Lip}^1}||u||_{L^p}.$$

Here $||f||_{\text{Lip}^1}$ is the Lipschitz norm of f.

Here the L^p norm can be taken on \mathbb{R}^n and compact manifolds. See Proposition 1.3 in Taylor [57]. The result was proven in Calderón [7] for classical first-order pseudodifferential operators and by Coifman-Meyer [11] for OPS^1 .

Lemma 9. If $V \in Lip^1(M)$, then $e_{\lambda} \in C^{1,\alpha}(M)$, for any $0 < \alpha < 1$.

Proof. By Sobolev imbedding (see e.g. [19]), we only need to show $||e_{\lambda}||_{W^{2,p}(M)} < \infty$ for any $1 . Indeed, using the commutator estimate in Lemma 8 and the equation <math>(\mathcal{D}+V)e_{\lambda} = \lambda e_{\lambda}$, we have

$$\|\mathcal{D}(Ve_{\lambda})\|_{L^{p}(M)} \leq \|V(\mathcal{D}+V)e_{\lambda}\|_{L^{p}(M)} + \|V^{2}e_{\lambda}\|_{L^{p}(M)} + \|[\mathcal{D},V]e_{\lambda}\|_{L^{p}(M)}$$

$$\lesssim \lambda \|V\|_{L^{\infty}} \|e_{\lambda}\|_{L^{p}(M)} + \|V\|_{L^{\infty}}^{2} \|e_{\lambda}\|_{L^{p}(M)} + \|V\|_{\operatorname{Lip}^{1}} \|e_{\lambda}\|_{L^{p}(M)}$$

$$\lesssim (1+\lambda)\|e_{\lambda}\|_{L^{p}(M)}.$$

So by Proposition 4, we obtain

$$\begin{aligned} \|e_{\lambda}\|_{W^{2,p}(M)} &\approx \|(1+\mathcal{D})^{2}e_{\lambda}\|_{L^{p}(M)} \\ &\lesssim \|(1+\mathcal{D})(1+\mathcal{D}+V)e_{\lambda}\|_{L^{p}(M)} + \|\mathcal{D}(Ve_{\lambda})\|_{L^{p}(M)} + \|V\|_{L^{\infty}}\|e_{\lambda}\|_{L^{p}(M)} \\ &\lesssim (1+\lambda)(\|(1+\mathcal{D})e_{\lambda}\|_{L^{p}(M)} + \|e_{\lambda}\|_{L^{p}(M)}) \\ &\leq (1+\lambda)(\|(1+\mathcal{D}+V)e_{\lambda}\|_{L^{p}(M)} + \|V\|_{L^{\infty}}\|e_{\lambda}\|_{L^{p}(M)} + \|e_{\lambda}\|_{L^{p}(M)}) \\ &\lesssim (1+\lambda)^{2}\|e_{\lambda}\|_{L^{p}(M)}. \end{aligned}$$

Next, we prove the nodal set estimates. Let

$$N_{\lambda} = \{ x \in M : e_{\lambda}(x) = 0 \},$$

 $D_{+} = \{ x \in M : e_{\lambda}(x) > 0 \},$

$$D_{-} = \{x \in M : e_{\lambda}(x) < 0\}.$$

We have $\partial D_{\pm} = N_{\lambda}$. We first express the manifold M as a (essentially) disjoint union

$$M = \bigcup_{j>1} D_{j,+} \cup \bigcup_{j>1} D_{j,-} \cup N_{\lambda}$$

where $D_{j,+}$ and $D_{j,-}$ are are the positive and negative nodal domains of e_{λ} , i.e, the connected components of the sets D_{+} and D_{-} . For simplicity, we assume that there are only two nodal domains D_{+} and D_{-} . Since ∇e_{λ} is continuous by Lemma 9 and we are assuming that zero is a regular value of e_{λ} , we can apply Gauss-Green theorem on each nodal domain D_{\pm} with boundary ∂D_{\pm} . We have

$$\int_{D_{+}} div(f \nabla e_{\lambda}) dV_{g} = \int_{N_{\lambda}} \langle f \nabla e_{\lambda}, \nu_{-} \rangle dS = -\int_{N_{\lambda}} f |\nabla e_{\lambda}| dS$$

$$\int_{D_{-}} div(f \nabla e_{\lambda}) dV_{g} = \int_{N_{\lambda}} \langle f \nabla e_{\lambda}, \nu_{+} \rangle dS = \int_{N_{\lambda}} f |\nabla e_{\lambda}| dS.$$

(6.17)
$$2\int_{N_{\lambda}} f|\nabla e_{\lambda}| = \int_{D_{-}} div(f\nabla e_{\lambda}) - \int_{D_{+}} div(f\nabla e_{\lambda}).$$

Note that by Cauchy-Schwarz

$$\int_{N_{\lambda}} |\nabla e_{\lambda}| \lesssim \left(\int_{N_{\lambda}} |\nabla e_{\lambda}|^2 \right)^{\frac{1}{2}} |N_{\lambda}|^{\frac{1}{2}}.$$

So to estimate the lower bound of $|N_{\lambda}|$, it suffices to estimate $\int_{N_{\lambda}} |\nabla e_{\lambda}|$ and $\int_{N_{\lambda}} |\nabla e_{\lambda}|^2$.

Lemma 10. If $V \in Lip^1(M)$, then

$$\int_{N_{\lambda}} |\nabla e_{\lambda}| \ge \frac{\lambda^2}{4} ||e_{\lambda}||_{L^1(M)}.$$

Lemma 11. If $V \in Lip^1(M)$, then

$$\int_{N_{\lambda}} |\nabla e_{\lambda}|^2 \lesssim \lambda^3 ||e_{\lambda}||_{L^2(M)}.$$

Using the these two lemmas and the eigenfunction estimate (1.6), we get the lower bound of the nodal set in Theorem 2

$$|N_{\lambda}| \gtrsim \lambda^{\frac{3-n}{2}}.$$

6.1. **Proof of Lemma 10.** We set f = 1 in (6.17). So

$$2\int_{N_{\lambda}} |\nabla e_{\lambda}| = \int_{D_{-}} \Delta_{g} e_{\lambda} - \int_{D_{+}} \Delta_{g} e_{\lambda}.$$

Since $\sqrt{-\Delta_g} = \mathcal{D} - P_0$, we have

$$-\Delta_g = (\mathcal{D} + V)^2 - (\mathcal{D}V - V\mathcal{D}) - 2V(\mathcal{D} + V) + V^2 - 2P_0(\mathcal{D} + V) + 2P_0V + Q_0,$$

where $Q_0 = P_0 \mathcal{D} - \mathcal{D} P_0 + P_0^2 \in OPS^0$. Thus,

$$2\int_{N_{\lambda}} |\nabla e_{\lambda}| = \int_{D_{+}} -\int_{D_{-}} (\lambda^{2} e_{\lambda} - [\mathcal{D}, V] e_{\lambda} - 2\lambda V e_{\lambda} + V^{2} e_{\lambda} - 2\lambda P_{0} e_{\lambda} + 2P_{0} V e_{\lambda} + Q_{0} e_{\lambda})$$

$$\geq \lambda^{2} \|e_{\lambda}\|_{L^{1}(M)} - \|[\mathcal{D}, V] e_{\lambda}\|_{L^{1}(M)} - 2\lambda \|V e_{\lambda}\|_{L^{1}(M)} - \|V^{2} e_{\lambda}\|_{L^{1}(M)}$$

$$- 2\lambda \|P_{0} e_{\lambda}\|_{L^{1}(M)} - 2\|P_{0} V e_{\lambda}\|_{L^{1}(M)} - \|Q_{0} e_{\lambda}\|_{L^{1}(M)}.$$

By Hölder's inequality and (1.6), we have

$$||e_{\lambda}||_{L^{1+\varepsilon}(M)} \lesssim \lambda^{\frac{(n-1)\varepsilon}{2(1+\varepsilon)}} ||e_{\lambda}||_{L^{1}(M)}, \ 0 < \varepsilon < 1.$$

Combining this estimate with Lemma 8, we have

$$\|[\mathcal{D}, V]e_{\lambda}\|_{L^{1}(M)} \lesssim \|[\mathcal{D}, V]e_{\lambda}\|_{L^{1+\varepsilon}(M)} \lesssim \|V\|_{\operatorname{Lip}^{1}} \|e_{\lambda}\|_{L^{1+\varepsilon}(M)} \lesssim \lambda \|V\|_{\operatorname{Lip}^{1}} \|e_{\lambda}\|_{L^{1}(M)},$$

if $\varepsilon > 0$ is small enough. Moreover, if $\varepsilon > 0$ is small enough, then by Lemma 6 we have

$$\lambda \|Ve_{\lambda}\|_{L^{1}(M)} \lesssim \lambda \|V\|_{L^{\infty}} \|e_{\lambda}\|_{L^{1}(M)}$$

$$\|V^{2}e_{\lambda}\|_{L^{1}(M)} \lesssim \|V\|_{L^{\infty}}^{2} \|e_{\lambda}\|_{L^{1}(M)}$$

$$\lambda \|P_{0}e_{\lambda}\|_{L^{1}(M)} \lesssim \lambda \|P_{0}e_{\lambda}\|_{L^{1+\varepsilon}(M)} \lesssim \lambda \|e_{\lambda}\|_{L^{1+\varepsilon}(M)} \lesssim \lambda^{\frac{3}{2}} \|e_{\lambda}\|_{L^{1}(M)}$$

$$\|P_{0}Ve_{\lambda}\|_{L^{1}(M)} \lesssim \|P_{0}Ve_{\lambda}\|_{L^{1+\varepsilon}(M)} \lesssim \|Ve_{\lambda}\|_{L^{1+\varepsilon}(M)} \lesssim \lambda \|V\|_{L^{\infty}} \|e_{\lambda}\|_{L^{1}(M)}$$

$$\|Q_{0}e_{\lambda}\|_{L^{1}(M)} \lesssim \|Q_{0}e_{\lambda}\|_{L^{1+\varepsilon}(M)} \lesssim \|e_{\lambda}\|_{L^{1+\varepsilon}(M)} \lesssim \lambda \|e_{\lambda}\|_{L^{1}(M)}.$$

So we finish the proof Lemma 10.

6.2. **Proof of Lemma 11.** We set $f = \sqrt{1 + |\nabla e_{\lambda}|^2}$ in (6.17). And then

$$\begin{split} 2\int_{N_{\lambda}}\sqrt{1+|\nabla e_{\lambda}|^{2}}\;|\nabla e_{\lambda}| &= \int_{D_{-}}div(\sqrt{1+|\nabla e_{\lambda}|^{2}}\nabla e_{\lambda}) - \int_{D_{+}}div(\sqrt{1+|\nabla e_{\lambda}|^{2}}\nabla e_{\lambda})\\ &\lesssim \int_{M}|div(\sqrt{1+|\nabla e_{\lambda}|^{2}}\nabla e_{\lambda})|\\ &\lesssim \int_{M}\sqrt{1+|\nabla e_{\lambda}|^{2}}\;|\nabla^{2}e_{\lambda}|\\ &\lesssim (\|e_{\lambda}\|_{L^{2}(M)}+\|\nabla e_{\lambda}\|_{L^{2}(M)})\|\nabla^{2}e_{\lambda}\|_{L^{2}(M)}\\ &\lesssim \lambda^{3}\|e_{\lambda}\|_{L^{2}(M)}. \end{split}$$

Here we use the Sobolev estimates of eigenfunctions in the last step. Indeed, we have the following Sobolev estimates

$$\begin{split} \|\nabla e_{\lambda}\|_{L^{2}(M)} &\lesssim \|\mathcal{D}e_{\lambda}\|_{L^{2}(M)} + \|e_{\lambda}\|_{L^{2}(M)} \\ &\leq \|(\mathcal{D} + V)e_{\lambda}\|_{L^{2}(M)} + \|Ve_{\lambda}\|_{L^{2}(M)} + \|e_{\lambda}\|_{L^{2}(M)} \\ &\lesssim \lambda \|e_{\lambda}\|_{L^{2}(M)} + \|V\|_{L^{\infty}} \|e_{\lambda}\|_{L^{2}(M)} \\ &\lesssim \lambda \|e_{\lambda}\|_{L^{2}(M)}, \end{split}$$

and similarly, we may exploit Lemma 8 to obtain

$$\begin{split} \|\nabla^{2}e_{\lambda}\|_{L^{2}(M)} &\lesssim \|\mathcal{D}^{2}e_{\lambda}\|_{L^{2}(M)} + \|\mathcal{D}e_{\lambda}\|_{L^{2}(M)} + \|e_{\lambda}\|_{L^{2}(M)} \\ &\lesssim \|(\mathcal{D}+V)^{2}e_{\lambda}\|_{L^{2}(M)} + \|[\mathcal{D},V]e_{\lambda}\|_{L^{2}(M)} + \|V(\mathcal{D}+V)e_{\lambda}\|_{L^{2}(M)} + \|V^{2}e_{\lambda}\|_{L^{2}(M)} + \lambda \|e_{\lambda}\|_{L^{2}(M)} \\ &\lesssim \lambda^{2}\|e_{\lambda}\|_{L^{2}(M)} + \lambda \|V\|_{\operatorname{Lip}^{1}}\|e_{\lambda}\|_{L^{2}(M)} + \|V\|_{L^{\infty}}^{2}\|e_{\lambda}\|_{L^{2}(M)} \\ &\lesssim \lambda^{2}\|e_{\lambda}\|_{L^{2}(M)}. \end{split}$$

So Lemma 11 is proved.

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