Defn **Binary Operation**

S: set (non empty)

A law of composition (binary or 2-ary opn) is a map

$$* : S \times S \to S$$
$$(a, b) \mapsto a * b$$

Eg

$$S \times S \to S$$
$$(a, b) \mapsto a * b$$
$$a * b := a$$
$$:= b$$
$$:= x_0 \qquad x_0 \in S$$

eg $S = \mathbb{R}$

$$+, -, \times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$(a, b) \mapsto \cos a$$

eg: $S = M_2(\mathbb{R}) = \{2\times 2 \text{ matrices}\}$

$a * b \mapsto a$

Usually $a, b$ cannot be recovered from image. Image doesn't come from unique elements.

$$\begin{tikzcd} S\times S\times S \arrow[r, "*\times id"] \arrow[d, "id\times *"'] & S\times S \arrow[d, "*"] \\ S\times S \arrow[r, "*"'] & S \end{tikzcd}$$

$$\begin{tikzcd} (a,b,c) \arrow[r] \arrow[d] & (a*b, c) \arrow[d] \\ (a, b*c) \arrow[r] & (a*b)*c \\ & a*(b*c) \end{tikzcd}$$

Def Law of composition $*$ on $S$ is associative if $\forall a, b, c \in S$

$$(a*b)*c = a*(b*c)$$

$-$ is not associative.
($-$ on $\mathbb{R}\times\mathbb{R}$)

Def Commutativity
$*$ is commutative if $a*b = b*a$

Eg: $T$: set
$Map(T, T) = \{f: T \to T\}$

Define $*$ by composition
$T \xrightarrow{f} T \xrightarrow{g} T$

$f*g = g\circ f$ — always asso

$(Maps(T,T), *)$ an associative law of composition.

$f(g\,h)\,f \qquad f*(g*h)$

$g*h$

$$T \xrightarrow{f} T \xrightarrow{g} T \xrightarrow{h} T$$

$f*g$

$g\circ f$

$h(g\,f)$

$(f*g)*h$

$a \to a$
$T: a \mapsto a$

$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x)) \xrightarrow{h} h(g(f(x)))$$

**Thm** $*$ is not commutative if $|T| > 1$

$$\begin{matrix}1\\2\end{matrix} \xrightarrow{f} \begin{matrix}1\\2\end{matrix} \xrightarrow{g} \begin{matrix}1\\2\end{matrix}$$

$f*g =$

$g*f =$

$T = \{a, b\}$

$$\mathrm{Map}(T, T) = \begin{cases} id \\ f: a \mapsto b \\ \quad\; b \mapsto a \\ g: a, b \mapsto a \\ h: a, b \mapsto b \end{cases}$$

Compose Table — Non commutative

| $*$ | id | f | g | h |
|---|---|---|---|---|
| id | id | f | g | h |
| f | f | id | g | h |
| g | g | h | g | h |
| h | h | g | g | h |

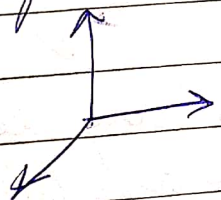
$\begin{array}{c|c} * & f \\ \hline g & g*f \end{array}$

Def[n] $(S, *)$ : associative law of comp[n]
An identity element $e \in S$ is identity of $S$ w.r.t $*$
if
$$e * a = a = a * e \qquad \forall\ a \in S$$

(1) If such an element exists then it has to be unique
~~$e_1 * a = a$~~ $\longrightarrow$ $e_1 * e_2 = e_2 = e_1$

Cas

(2) Write an example where id element doesn't exist
~~{M: m×n matrices}~~ Cross product of vectors
~~left id or right id.~~



direc[n] of any non zero $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}, \vec{w}$

Also + semigrp of +ve ~~Natural~~ no.s.

Def^n: $(S, *)$ : associative with identity $e$

Fix $x \in S$

$y \in S$ is a left inverse of $x$ if $y * x = e$

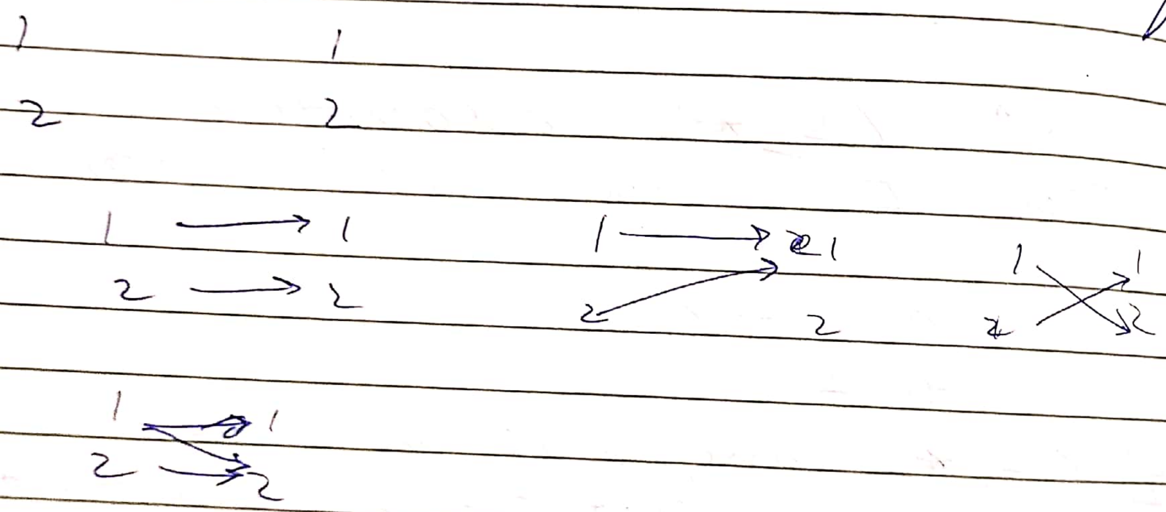
Similarly $z \in S$ is right inverse of $x$ if

$$x * z = e$$

$x$ is invertible ~~and~~ if it has a left inverse and right inverse

→ If $x \in S$ is invertible, then
1. its left and right inverse coincide
2. left and right inverse are unique, denote by $x^{-1}$

Sol^n →



19. $(S, \cdot)$ associative law of compos^n

1) If identity elt $e$ exists, ie. $e \cdot x = x$, $x \cdot e = x$ $\forall x \in S$ Then $e$ is unique.

2) Assume identity element $e$ exists then $x \in S$ is invertible if

$\exists\, y, z \in S$ s.t. $xy = 1$, $zx = 1$

If $x$ is invertible then any left inverse and any right inverse of $x$ coincide, further they are unique

$l_1 \underset{e \cdot x}{=} x_1 = l_2 = x$

INV

Not$^n$: 1 will denote identity elt of $(S, \cdot)$ ~ $x^{-1}$
0 " " " $(S, +)$ ~ $-x$

Def $(S, \cdot)$ : associative law of comp$^n$.

This is called a group if this has an identity element 1 and every element of $S$ is invertible.

Ex

| set | group | id elt | inverse of $x$ |
|---|---|---|---|
| $(\mathbb{Z}, +)$ | ✓ | 0 | $-x$ |
| $(\mathbb{Z}, \cdot)$ | ✗ | 1 | 0 ? |
| $(\mathbb{Q}/\mathbb{R}/\mathbb{C}, +)$ | ✓ | 0 | $-x$ |
| $(\mathbb{Q}^n/\mathbb{R}^n/\mathbb{C}^n, +)$ → componentwise add$^n$ | ✓ | $\vec{0} = (0, \ldots, 0)$ | $-x$ |
| $M_n(\mathbb{R}/\mathbb{C}/\mathbb{Q}), +$ | ✓ | 0 - matrix | $-M$ |
| $GL_n(\mathbb{R}/\mathbb{Q}), \cdot$ | ✓ | 1 - Id | $x^{-1} = \frac{(\text{adj } x)}{\det x}$ |
| $X$: set $\{x \to \mathbb{R}\}, +$ | ✓ | const func $0 : x \to 0$ | $-f$ |

$(f+g)(x) \mapsto f(x) + g(x)$

In general $G$: group

| | | |
|---|---|---|
| $\{X \to G\}, \cdot$ | const func to $id_G$ $x \mapsto 1_G$ | $f^{-1} : x \to (f(x))^{-1}$ |

$(f \cdot g)(x) \mapsto f(x) \cdot g(x)$