Numerical Analysis : [MA214]
Lecture 2

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Recall I

1. Suppose x^* is an approximation to exact value x. Then

$$|x - x^*| = \text{ Absolute error}$$

$$\frac{|x - x^*|}{|x|} = \text{ relative error, (provided } x \neq 0)$$

2. x^* is said to be approximate x to t significant digits if

$$\left|\frac{x - x^*}{x}\right| \leq 5 \times 10^{-t}$$

3. Things which create loss of significant digits
   (a) Subtraction of nearly equal quantities
   (b) division by number which is close to zero
4. Once an error has been committed it contaminates subsequent results. Error propagation is studied in terms of two related concepts:
   1. condition

2. instability

5.

$$\begin{aligned}
\text{Condition} \quad &\leftrightarrow \quad \text{sensitivity of } f(x) \text{ to changes in } x \\
&= \quad max \left\{ \frac{\frac{|f(x)-f(x^*)|}{|f(x)|}}{\frac{|x-x^*|}{|x|}} : |x - x^*| \text{ small} \right\} \\
&\approx \quad \left| \frac{f'(x)x}{f(x)} \right|
\end{aligned}$$

Example

(a) $f(x) = \sqrt{x}$ is well conditioned.

(b) $f(x) = \frac{10}{1-x^2}$ is ill conditioned near 1.

(c) I gave an example where $f = f_1(f_2(f_3(f_4)))$ with f well conditioned but f_3 ill-conditioned. This also creates lot of error.

6. Last time I gave a spectacular example of instability.

Today we first discuss Polynomials

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

This is called power form. This may lead to loss of significant digits.

Example:- (We work with 4 sig digits)
Suppose p is a straight line such that $p(6000) = \frac{1}{3}$, $p(6001) = \frac{-2}{3}$.

$$p(x) = 6000 - x \text{ , in 4 sig digits}$$

This gives $p(6000) = 0$ and $p(6001) = -1$

Remedy:- (Shift of center)

$$p(x) = 3.333E{-1} - (x - 6000)$$

$p(6000) = 3.333E{-1}$ and $p(6001) = -6.667E{-1}$

$$p(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_n(x - c)^n$$

Computing directly requires more multiplications and additions. Better to compute it as

$$\begin{aligned}
p(x) &= a_0 + (x - c)\{a_1 + a_2(x - c) + \cdots + a_n(x - c)^{n-1}\} \\
&= a_0 + (x - c)\{a_1 + (x - c)\{a_2 + a_3(x - c) + \cdots + a_n(x - c)^{n-2}\}\} \\
&= a_0 + (x - c)\{a_1 + (x - c)\{a_2 + (x - c)\{a_3 + \cdots + a_n(x - c)^{n-3}\}\}\}
\end{aligned}$$

Bonus of nested form is preservation of significant digits

Example (4 Significant digits)

$$p(x) = x^3 - 6.1x^2 + 3.2x + 1.5$$

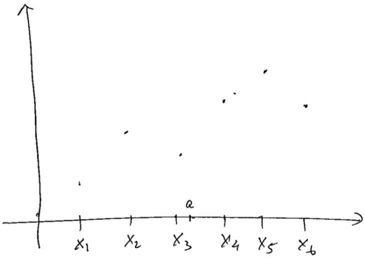
$$p(4.71) = -14.26 \text{ , (correct upto 4 sig digits)}$$

However if you directly compute

$$\begin{aligned} p(4.71) &= (4.71)^3 - 6.1(4.71)^2 + 3.2(4.71) + 1.5 \\ &= -14.23 \text{ , (correct upto 3 sig digits)} \end{aligned}$$

In nested form

$$\begin{aligned} p(x) &= x(x^2 - 6.1x + 3.2) + 1.5 \\ &= x(x(x - 6.1) + 3.2) + 1.5 \\ p(4.71) &\approx -14.26 \end{aligned}$$



We have $f(x_1), f(x_2), \cdots, f(x_6)$. We need to approximate $f(a)$.

Idea:- Fit a curve "passing through"

$$(x_1, f(x_1)), (x_2, f(x_2)), \cdots, (x_6, f(x_6))$$

and then approximate $f(a)$.

Question:- Which curve to fit ?

Weirstass Approximation Theorem

One classical and non-trivial result is the following.

Theorem (Weirstass Approximation Theorem)

Suppose $f : [a, b] \longrightarrow \mathbb{R}$ *is a continuous function. For each* $\epsilon > 0$, *there exists a polynomial* $p(x)$ *such that*

$$|f(t) - p(t)| < \epsilon \text{ , for all } t \in [a, b]$$

Remark:- The polynomial constructed for proving this theorem has slow convergence. So it is ineffective in practice.

We might be tempted to use Taylor polynomial

Example:- $f(x) = \frac{1}{x}$, $x \in [1,4]$

$$\begin{aligned} f'(x) &= -\frac{1}{x^2} \\ f''(x) &= (-1)^2 \frac{2}{x^3} \\ &\vdots \\ f^{(k)}(x) &= (-1)^k \frac{k!}{x^{k+1}} \end{aligned}$$

So n^{th} Taylor polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n}(-1)^k(x-1)^k$$

n	2	3	4	5	6	7
$T_n(3)$	3	-5	11	-21	43	-85

Problem

Given $n+1$ distinct points $x_0, x_1, \cdots, x_n$ in $[a, b]$ and a function $f : [a, b] \longrightarrow \mathbb{R}$, does there exists a polynomial $p(x)$ of degree $\leq n$ which interpolates $f(x)$ at the points $x_0, x_1, \cdots, x_n$, *i.e.* $p(x)$ satisfies

$$p(x_i) = f(x_i) \text{ , for } i = 0, 1, 2, \cdots, n$$

We prove that there exists a unique polynomial which does the job.

Lagrange Polynomials

Given $x_0, x_1, \cdots, x_n$ distinct points

$$l_k(x) = \prod_{i=0, i \neq k}^{n} \frac{x - x_i}{x_k - x_i}, \; k = 0, 1, \cdots, n$$

Example:-

$n = 2$ and x_0, x_1, x_2 are distinct points

$$\begin{aligned} l_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\ l_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\ l_2(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \end{aligned}$$

$x_0, x_1, \cdots, x_n$ are $(n+1)$ distinct points

$$l_k(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}\ ,\ k = 0, 1, \cdots, n$$

Properties:-

1. $l_k(x)$ is a degree n-polynomial, for all $k = 0, 1, \cdots, n$
2. $l_k(x_k) = 1$
3. $l_k(x_i) = 0$ for $i \neq k$

$$P_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$$

Then $P_n(x)$ is a polynomial of degree $\leq n$ and

$$P_n(x_i) = f(x_i), \text{for all } i = 0, 1, \cdots, n$$

Thus $P_n(x)$ is an interpolating polynomial.

Proposition

Let $p(x), q(x)$ interpolate $f(x)$ in $x_0, \cdots, x_n$. Then $p(x) = q(x)$

Proof.

$$\text{degree } p(x) \leq n$$

$$\text{degree } q(x) \leq n$$

So $h(x) = p(x) - q(x)$ is a polynomial of degree $\leq n$.

$$\begin{aligned} h(x_i) &= p(x_i) - q(x_i) \text{ , for } i = 0, 1, \cdots, n \\ &= f(x_i) - f(x_i) \\ &= 0 \end{aligned}$$

Thus $h(x)$ has $n+1$ zeros $x_0, \cdots, x_n$.
Therefore $h(x) = 0$. Thus $p(x) = q(x)$ □

Example

x	$f(x)$
2	$6.931E{-1}$
3	1.099
4	1.386

Approximate $f(3.2)$.

Ans.:- $P_2(x) = f(2)l_0(x) + f(3)l_1(x) + f(4)l_2(x)$ interpolates $f(x)$. Recall

$$\begin{aligned} l_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\ \text{Thus } l_0(3.2) &= \frac{(3.2 - 3)(3.2 - 4)}{(2 - 3)(2 - 4)} = -8E{-2} \end{aligned}$$

Similarly

$$l_1(3.2) = \frac{(3.2 - 2)(3.2 - 4)}{(3 - 2)(3 - 4)} = 9.6E{-1}$$

$$l_2(3.2) = \frac{(3.2 - 2)(3.2 - 3)}{(4 - 2)(4 - 3)} = 1.2E{-1}$$

$$
\begin{aligned}
f(3.2) &\approx (6.931E{-}1)(-8E{-}2) + (9.6E{-}1)(1.099) + (1.2E{-}1)(1.386) \\
&= 1.166
\end{aligned}
$$

The function

$$
\begin{aligned}
f(x) &= \log(x) \\
f(3.2) &= \log(3.2) \\
&= 1.163
\end{aligned}
$$

Thus our approximation is correct upto 3 significant digits.

Definition

$$P_n(x) = \sum_{k=0}^{n} f(x_k) l_k(x)$$

is called "Lagrange" form of interpolating polynomial.

Problem with Lagrange form of interpolating polynomial

Suppose we have found $P_n(x)$ by interpolating $f(x)$ at points $x_0, x_1, \cdots, x_n$.

Suppose we also know $f(x_{n+1})$. Then we can form $P_{n+1}(x)$ by interpolating $f(x)$ at $x_0, x_1, \cdots, x_n, x_{n+1}$.

There is no obvious relation between the Lagrange form of $P_n(x)$ and $P_{n+1}(x)$.

Newton form of interpolating polynomial

We write

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$$

Important remark:- $x - x_n$ does not appear in the last term above.

Set

$$\begin{aligned}
q(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\
&+ a_{n-1}(x - x_0)\cdots(x - x_{n-2}) \\
P_n(x) &= q(x) + a_n(x - x_0)\cdots(x - x_{n-1}) \\
\text{Note } q(x_i) &= P_n(x_i) = f(x_i), \ i = 0, 1, \cdots, n-1
\end{aligned}$$

Also degree $q(x) \leq n - 1$. By uniqueness of interpolating polynomial

$$q(x) = P_{n-1}(x)$$

So

$$P_n(x) = P_{n-1}(x) + a_n(x - x_0)\cdots(x - x_{n-1})$$

Note : a_n = coefficient of x^n in $P_n(x)$

Definition

$$f[x_0, x_1, \cdots, x_n] =: a_n$$

$f[x_0, x_1, \cdots, x_n]$ is called the n^{th} divided difference of $f(x)$ at the points $x_0, x_1, \cdots, x_n$.

we write

$$\begin{aligned} P_n(x) = f(x_0) &+ f[x_0, x_1](x - x_0) \\ &+ f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \cdots \\ &\vdots \\ &+ f[x_0, x_1, x_2, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1}) \end{aligned}$$

Thus to determine $P_n(x)$ we only have to find

$$f[x_0, x_1], f[x_0, x_1, x_2], \cdots, f[x_0, x_1, x_2, \cdots, x_n]$$

Determining $f[x_0, x_1, x_2, \cdots, x_n]$

$$\begin{aligned} f[x_0] &= f(x_0) \\ P_1(x_1) &= f(x_1) \\ f(x_1) &= f(x_0) + f[x_0, x_1](x_1 - x_0) \\ \text{So } f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \end{aligned}$$

Proposition

$$f[x_0, x_1, x_2, \cdots, x_k] = \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots, x_{k-1}]}{x_k - x_0}$$

Proof.

Let $P_i(x) =$ polynomial of degree $\leq i$ which agrees with $f(x)$ at the points $x_0, x_1, x_2, \cdots, x_i$.
Let $q_{k-1}(x) =$ polynomial of degree $\leq k-1$ which agrees with $f(x)$ at the points $x_1, x_2, \cdots, x_k$.

Proof Continue

Set

$$p(x) = \frac{x - x_0}{x_k - x_0} q_{k-1}(x) + \frac{x_k - x}{x_k - x_0} P_{k-1}(x)$$

Note: degree $p(x) \leq k$

$$p(x_0) = f(x_0) \text{ and } p(x_k) = f(x_k) \text{ and for } 1 \leq i \leq k-1$$

$$p(x_i) = \frac{x_i - x_0}{x_k - x_0} f(x_i) + \frac{x_k - x_i}{x_k - x_0} f(x_i) = f(x_i)$$

By uniqueness of interpolating polynomial

$$p(x) = P_k(x)$$

$$\begin{aligned}
f[x_0, x_1, \cdots, x_{k-1}] &= \text{Coeff of } x^k \text{ in } P_k(x) \\
&= \frac{\text{Coeff of } x^{k-1} \text{ in } q_{k-1}(x)}{x_k - x_0} - \frac{\text{Coeff of } x^{k-1} \text{ in } P_{k-1}}{x_k - x_0} \\
&= \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots, x_{k-1}]}{x_k - x_0}
\end{aligned}$$

Table of divided differences

x	$f(x)$	$f[,]$	$f[,,]$	$f[,,,]$
x_0	$f(x_0)$	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$	$f[x_0,x_1,x_2,x_3]$
x_1	$f(x_1)$	$f[x_1,x_2]$	$f[x_1,x_2,x_3]$	
x_2	$f(x_2)$	$f[x_2,x_3]$		
x_3	$f(x_3)$			

Example:- Fill the table and approximate $f(3.2)$

x	$f(x)$	$f[,]$	$f[,,]$	$f[,,,]$
2	$6.931E{-1}$	$4.059E{-1}$	$-5.945E{-2}$	$9.15E{-3}$
3	1.099	$2.87E{-1}$	$-3.2E{-2}$	
4	1.386	$2.23E{-1}$		
5	1.609			

$$\begin{aligned}
P_2(x) &= f(2) + f[2,3](x-2) + f[2,3,4](x-2)(x-3) \\
&= 6.931E{-1} + 4.059E{-1}(x-2) - 5.945E{-2}(x-2)(x-3) \\
P_2(3.2) &= 1.166
\end{aligned}$$

Exact value $f(3.2) = \log(3.2) = 1.163$

$$P_3(x) = P_2(x) + 9.15E{-3}(x-2)(x-3)(x-4)$$

$$P_3(3.2) = 1.166 + 9.15E{-3}(1.2)(0.2)(-0.8) = 1.164$$

Example 2:- $f(x) = \int_0^x sin(t^2)dt$

x	$f(x)$	$f[,]$	$f[,,]$	$f[,,,]$
0.8	$1.657E{-1}$	$6.62E{-1}$	$6.1E{-1}$	$-2.5E{-1}$
0.9	$2.319E{-1}$	$7.84E{-1}$	$5.35E{-1}$	
1.0	$3.103E{-1}$	$8.91E{-1}$		
1.1	$3.994E{-1}$			

$$\begin{aligned} P_2(x) &= f(0.8) + f[0.8, 0.9](x-0.8) + f[0.8, 0.9, 1.0](x-0.8)(x-0.9 \\ &= 1.657E{-1} + 6.62E{-1}(x-0.8) + 6.1E{-1}(x-0.8)(x-0.9) \\ P_2(0.85) &= 1.973E{-1} \\ f(0.85) &= 1.974E{-1} \end{aligned}$$

$$
\begin{aligned}
P_3(x) &= P_2(x) + f[0.8, 0.9, 1.0, 1.1](x-0.8)(x-0.9)(x-1.0) \\
P_3(0.85) &= P_2(0.85) - 2.5E{-}1(0.05)(-0.05)(-0.15) \\
P_3(0.85) &= 1.973E{-}1
\end{aligned}
$$

Remarks:-

1. In the above example error of $P_3(x)$ was same as that $P_2(x)$.
2. It is possible that interpolating error to increase if we increase number of points.

$$\text{error } e_n(x) = f(x) - P_n(x)$$

$i.e.$ it is possible that

$$max_{x\in[a,b]}|e_{n+1}(x)| > max_{x\in[a,b]}|e_n(x)|$$

See example 2.4 page 44 of your textbook **Conte & de Boor**

The error of the interpolating polynomial

Let $x_0, x_1, x_2, \cdots, x_n$ be $n+1$ points and $f(x_0), f(x_1), f(x_2), \cdots, f(x_n)$ be function values

$$f : [a, b] \longrightarrow \mathbb{R}.$$

$P_n(x) =$ polynomial which interpolates $f(x)$ at $x_0, x_1, x_2, \cdots, x_n$

$$\text{error} \quad e_n(x) = \quad f(x) - P_n(x)$$

Let $\overline{x} \in [a, b]$ be distinct from $x_0, x_1, x_2, \cdots, x_n$.

We need to estimate $e_n(x)$.

Let $P_{n+1}(x) =$ polynomial which interpolates $f(x)$ at the points $x_0, x_1, x_2, \cdots, x_n, \overline{x}$ ($n+2$ points)

$$
\begin{aligned}
P_{n+1}(x) &= P_n(x) + f[x_0, x_1, \cdots, x_n, \overline{x}] \prod_{j=0}^{n} (x - x_j) \\
f(\overline{x}) &= P_{n+1}(\overline{x}) \text{ by definition} \\
e_n(\overline{x}) &= f(\overline{x}) - P_n(\overline{x}) \\
&= P_{n+1}(\overline{x}) - P_n(\overline{x}) \\
&= f[x_0, x_1, \cdots, x_n, \overline{x}] \prod_{j=0}^{n} (x - x_j)
\end{aligned}
$$

Thus to estimate error we need to approximate

(a) $f[x_0, x_1, \cdots, x_n, \overline{x}]$

(b) $\prod_{j=0}^{n} (x - x_j)$

Theorem

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous and k times differentiable in (a,b). If $x_0, x_1, \cdots, x_k$ are $k+1$ distinct points in $[a,b]$, then there exists $\xi \in (a,b)$ such that

$$f[x_0, x_1, \cdots, x_k] = \frac{f^{(k)(\xi)}}{k!}$$

Proof.

$k = 1$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi) \text{ by M.V.T.}$$

$$e_k(x) = f(x) - P_k(x)$$

has $k+1$ zeros $x_0, x_1, \cdots, x_k$. So $e_k^{(k)}(x)$ will have a zero say ξ.

$$\begin{aligned} 0 &= e_k^{(k)}(\xi) = f^{(k)}(\xi) - P_k^{(k)}(\xi) \\ 0 &= f^{(k)}(\xi) - k! f[x_0, x_1, \cdots, x_k] \end{aligned}$$

$$\text{So} \qquad f[x_0, x_1, \cdots, x_k] = \frac{f^{(k)(\xi)}}{k!}$$

Corollary

$$\begin{aligned} e_n(\overline{x}) &= f(\overline{x}) - P_n(\overline{x}) \\ &= \frac{f^{(n+1)(\xi)}}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \end{aligned}$$

Estimating:-

$$\Psi_{n+1}(x) = \prod_{j=0}^{n} (x - x_j)$$

It is possible to choose $x_0, x_1, \cdots, x_n$ in $[a, b]$ such that $|\Psi_{n+1}(x)|$ is as small as possible.

This choice of points are called Chebyshev points of $[a, b]$.

(Unfortunately it is not in syllabus)

Osculatory interpolation

Sometimes we have the following situation

We have $x_0, x_1, \cdots, x_k$ and

$$f(x_0), f(x_1), f(x_2), \cdots, f(x_k)$$

$$f'(x_0), f'(x_1), f'(x_2), \cdots, f'(x_k)$$

We need a polynomial $p(x)$ such that

$$p(x_i) = f(x_i) \text{ and } p'(x_i) = f'(x_i) \text{ , for } i = 0, 1, \cdots, n$$

Note:- $\deg p(x) \leq 2n + 1$

Example where this situation arise

$$\frac{dy}{dx} = g(x, y) \text{ , } y(x_0) = y_0$$

x_0	$y(x_0)$	$y'(x_0) = g(x_0, y_0)$	Remark
x_1	$y(x_1)$	$y'(x_1) = g(x_1, y_1)$	$y(x_i)$ is
$\vdots$	$\vdots$	$\vdots$	calculated by some
x_n	$y(x_n)$	$y'(x_n) = g(x_n, y_n)$	Numerical Method

Remark:- $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$. So $\lim_{x_1 \to x_0} f[x_0, x_1] = f'(x_0)$

Definition

$$f[x_0, x_0] = f'(x_0)$$

Example:- $f(1) = 0,\ f'(1) = 1,\ f(2) = 6.931E{-1},\ f'(2) = 0.5.$

We need a cubic polynomial $P_3(x)$ such that $P_3(1) = f(1)$, $P_3'(1) = f'(1)$, $P_3(2) = f(2)$, $P_3'(2) = f'(2)$.

n	$f(x)$	$f[,]$	$f[,,]$	$f[,,,]$
$y_0 = 1$	0	1	-0.3069	0.1137
$y_1 = 1$	0	$6.931E{-1}$	-0.1931	
$y_2 = 2$	$6.931E{-1}$	0.5		
$y_3 = 2$	$6.931E{-1}$			

$$P_3(x) = 0 + 1(x-1) - 0.3069(x-1)^2 + 0.1137(x-1)^2(x-2)$$