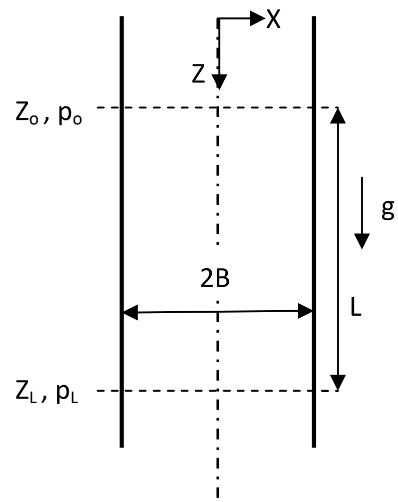
1. Two *large* vertical plates are parallel to each other and there is a fluid in between. The fluid is Newtonian and incompressible. The gap between the plates is 2B.
   If a pressure difference p_o-p_L is applied in the Z- direction between two plates in the fully developed (no end or side effects), steady flow region, a distance L apart (in the other two directions there is no pressure difference), perform a z-momentum (or force) balance and develop expressions for



   a) Differential equation for the velocity, V_z
   b) boundary conditions
   c) shear stress profile : $\tau_{xz}(x)$
   d) velocity profile : $V_z(x)$
   e) average velocity : $\overline{V_z}$

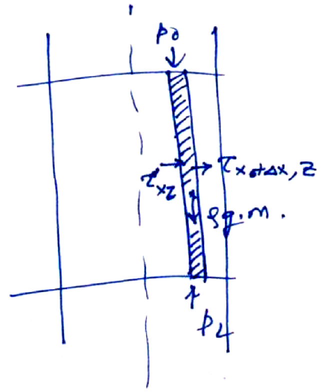
(1+1+1+1+1 =5 marks)

[Suggested: Take a shell of thickness Δx, height L and width W at a distance x from the axis plane as the control volume. Note that the natural coordinates for this system is Cartesian]

BONUS question IA: if additionally the left plate is moving at steady velocity of −V (up) and the right plate is moving at steady velocity of +V (down), write down the expression for the velocity profile. Only the expression is needed: derivations not essential. **(2 marks)**

Two Methods: ~~So~~

1. control volume of thickness Δx at a distance x from midplane.



Steady, fully developed, incompressible.

Force balance: z-direction

Pressure force: $\Delta x \cdot W \cdot p_0 - \Delta x \cdot W \cdot p_L$

Weight : $(\Delta x \cdot W \cdot L)\rho g$

Shear ~~Force~~ : $(L \cdot W) \cdot \tau_{xz}\big|_x - L \cdot W \, \tau_{xz}\big|_{x+\Delta x}$

(momentum flux)

Adding, and equating the sum to zero:

$$\Delta x \cdot W \cdot L \left(\frac{p_0 - p_L}{L} + \rho g\right) = W \cdot L \cdot \left(\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x\right)$$

Dividing by $(\Delta x \cdot W \cdot L)$, and taking limits as $\Delta x \to 0$

$$(1) - \frac{\partial \tau_{xz}}{\partial x} = \left(\frac{p_0 - p_L}{L} + \rho g\right) = \frac{P_0 - P_L}{L} \text{ ; since } P = p - \rho g z.$$

Newtonian fluid: $\tau_{xz} = \left(\frac{p_0 - p_L}{L} + \rho g\right)x + C_1 = \left(\frac{P_0 - P_L}{L}\right)x + C_1$

a).

$$(2) - \; -\mu \frac{\partial v_z}{\partial x} = \left(\frac{P_0 - P_L}{L}\right)x + C_1$$

OR, if you substitute Newton's law into (1),

$$(3) \; -\mu \frac{\partial^2 v_z}{\partial x^2} = \frac{P_0 - P_L}{L}$$

(b) $\rightarrow$ ~~Integrating~~ B.C.: $x = B,\ v_z = 0$ —(i)

$x = -B,\ v_z = 0$. —(ii)

Integrating

$$(4) \quad v_z = -\frac{P_0 - P_L}{2\mu L}x^2 + C_1 x + C_2$$

Applying B.Cs.: $-\frac{P_0 - P_L}{2\mu L}B^2 + C_1 B + C_2 = 0$

$-\frac{P_0 - P_L}{2\mu L}B^2 - C_1 B + C_2 = 0$

Hence $C_1 = 0$, $C_2 = \frac{P_0 - P_L}{2\mu L}B^2$

(c) $\tau_{xz} = \frac{P_0 - P_L}{L}x$; (d) $v_z = \left(\frac{P_0 - P_L}{2\mu L}\right)B^2\left[1 - \left(\frac{x}{B}\right)^2\right]$

e. Average velocity:

$$W \cdot 2B \cdot \bar{V}_z = \int_{-B}^{B} W \cdot \bar{V}_z \, dx = W \cdot \left(\frac{P_0 - P_L}{2\mu L}\right) B^2 \left(x - \frac{x^3}{3B^2}\right)_{-B}^{+B}$$

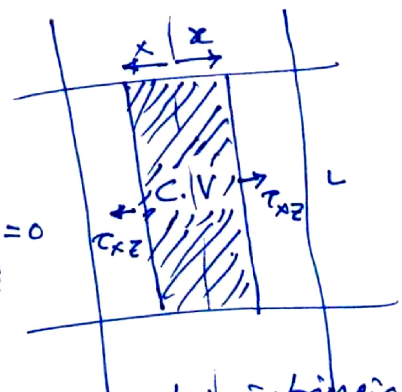
$$= W \cdot \frac{P_0 - P_L}{2\mu L} B^2 \left[2B - \frac{2B}{3}\right]$$

$$= W \cdot \frac{(P_0 - P_L)}{2\mu L} B^2 \frac{4B}{3}$$

$$\boxed{\bar{V}_z = \frac{P_0 - P_L}{3\mu L} B^2}$$

Method II:

Control volume: variable thickness slit ~~of~~ ~~of this~~ between $-x$ & $+x$



Force balance gives:

$$(2x \cdot W)(P_0 - P_L) + (2x \cdot W \cdot L)\rho g - 2W \cdot L \cdot \tau_{xz}\big|_x = 0$$

(c) $$\tau_{xz} = \left(\frac{P_0 - P_L}{L} + \rho g\right) x = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right) x$$

(Note:- No C_1) symmetry is intrinsic to selection of C.V.

(a) $$-\mu \frac{\partial v_z}{\partial x} = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}\right) x$$

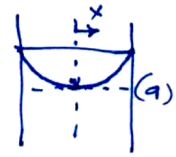
(b) B.C.: $x = B$, $V_z = 0$
$x = -B$, $V_z = 0$

(d)
e } as before.

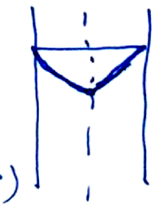
Comments:

- Note that in method I, we used the B.Cs at the two plates, which signifies symmetry, but not used explicitly.

We could have alternatively ~~&~~ used symmetry directly and said: At $x=0$, $\tau_{xz} = -\mu \frac{\partial v_z}{\partial x} = 0$ (as shown in fig(a))



This however is not the only situation where symmetry is applicable. The situation shown in fig(b) also is symmetric: At $x=0$, $\left.\frac{\partial v_z}{\partial x}\right|_{x=0^+} = -\left.\frac{\partial v_z}{\partial x}\right|_{x=0^-}$

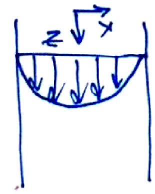


- This however leads to momentum flux (finite) away from the mid-plane surface on both sides. This is not possible, unless one generates momentum at the mid-plane surface. Momentum generation is by pressure and gravitational forces, which can only act on a volume.
- This becomes evident in method ~~2~~ II, where C_1 does not appear. And the solution say $\left.\tau_{xz}\right|_{x=0} = 0$.

<u>BONUS question</u>:

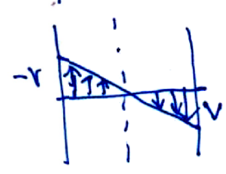
Since the equations are linear ODEs, one can split the problem into two and superimpose.

(A) Situations as in (A): Pressure & gravitation forces exist, but plates are stationary.



$$v_z = \frac{P_0 - P_L}{2L} B^2 \left(1 - \left(\frac{x}{B}\right)^2\right)$$

(B) No external forces (no gravitation, no pr. gradient). But B.Cs are:

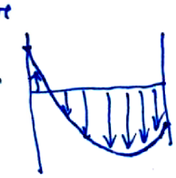


$$x = B,\ v_z = V,\quad x = -B,\ v_z = -V$$

$$\frac{\partial \tau_{xz}}{\partial x} = 0,\quad \tau_{xz} = C_1,\quad v_z = -\frac{C_1}{\mu}x + C_2$$

$$C_2 = 0;\quad \frac{C_1}{\mu} = \frac{\cancel{x}}{B}.V;\quad v_z = \cancel{\frac{x}{B}} V.\frac{x}{B}$$ Could have written directly.

The solution therefore is



Only this answer needed.

$$\left| v_z = \frac{P_0 - P_L}{2L} B^2 \left[1 - \left(\frac{x}{B}\right)^2\right] + V.\frac{x}{B} \right.$$