Numerical Analysis : [MA214]
Lecture 4

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Recall:- Osculatory interpolation

Simplest and often occurring case

$$x_0, x_1, \cdots, x_k$$

$$f(x_0), f(x_1), f(x_2), \cdots, f(x_k)$$

$$f'(x_0), f'(x_1), f'(x_2), \cdots, f'(x_k)$$

Algorithm:- $z_{2i} = x_i$, and $z_{2i+1} = x_i$ for $i = 0, 1, \cdots, k$

$$\text{Set } f[z_i, z_{i+1}] = \left\{ \begin{array}{ll} f'(z_i) & \text{if } z_i = z_{i+1} \\ \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i} & \text{if } z_i \neq z_{i+1} \end{array} \right\}$$

$s \geq 2$

$$f[z_i, z_{i+1}, \cdots, z_{i+s}] = \frac{f[z_{i+1}, \cdots, z_{i+s}] - f[z_i, \cdots, z_{i+s-1}]}{z_{i+s} - z_i}$$

$$P_{2k+1}(x) = f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1)$$

$$+ \cdots + f[z_0, \cdots, z_{2k+1}] \prod^{2k} (x - z_i)$$

For interpolation problems it is good idea to create the Newton divided-difference table.

We then did the general Osculatory interpolating formula

Let $x_0, x_1, \cdots, x_m$ be not necessarily distinct points.

We say two function $f(x)$ and $g(x)$ agree at the points $x_0, x_1, \cdots, x_m$ if

$$f^{(j)}(z) = g^{(j)}(z) \text{ , for } j = 0, 1, \cdots, k-1$$

for every points z which occurs k times in the sequence $x_0, x_1, \cdots, x_m$.

Last time we proved that

if $f(x)$ has r continuous derivatives and no point in the $x_0, x_1, \cdots, x_m$ occurs more than r times then there exists exactly one polynomial $P_m(x)$ of degree $\leq m$ which agree with $f(x)$ at $x_0, x_1, \cdots, x_m$.

Uniqueness was an exercise, I hope you have done it.

Disadvantage of interpolation

Usually the number of data points $x_0, x_1, \cdots, x_m$ is very large.

Large degree polynomial oscillate a lot. It also create lot of round off errors.

To avoide this problem we study piecewise-polynomial interpolation

We did

1. piecewise linear interpolation
2. piecewise cubic interpolation
    1. piecewise-cubic Hermite
    2. piecewise-cubic Spline (we will do this today)

Piecewise cubic Hermite

$$a = x_1 < x_2 < x_3 < \cdots < x_{N+1} = b$$

in $[x_i, x_{i+1}]$, $g_i(x)$ is given by polynomial $P_i(x)$ which interpolate $f(x)$ at $x_i, x_i, x_{i+1}, x_{i+1}$, *i.e.*

$$P_i(x_i) = f(x_i) \text{ and } P_i(x_{i+1}) = f(x_{i+1})$$

$$P_i'(x_i) = f'(x_i) \text{ and } P_i'(x_{i+1}) = f'(x_{i+1})$$

$$\begin{aligned} P_i(x) &= f(x_i) + f[x_i, x_i](x - x_i) \\ &+ f[x_i, x_i, x_{i+1}](x - x_i)^2 + f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^2(x - x_{i+1}) \\ & x \in [x_i, x_{i+1}] \end{aligned}$$

The piecewise Hermite polynomial $g(x)$ is continuously differentiable in $[a, b]$

Disadvantage of piecewise Hermite polynomial

$f'(x)$ might not be available.

Cubic Spline Interpolation I

Definition

Given a function f defined on $[a, b]$ and a set of nodes

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$$

A cubic spline interpolation S for f is a function that satisfies the following conditions.

1. $S(x)$ is a cubic polynomial denoted by $S_j(x)$ on the sub-interval $[x_j, x_{j+1}]$, for each $j = 0, 1, \cdots, n-1$
2. $S_j(x_j) = f(x_j)$, $S_j(x_{j+1}) = f(x_{j+1})$, for each $j = 0, 1, \cdots, n-1$
3. for each $j = 0, 1, \cdots, n-2$
    1. $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$
    2. $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$
    3. $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$
4. One of the following set of boundary condition is satisfied

1. $S''(x_1) = S''(x_n) = 0$, (free boundary)
2. $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$ (clamped boundary)

Construction of cubic spline interpolant on $[x_j, x_{j+1}]$

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

$$S_j(x_j) = f(x_j). \text{ So } a_j = f(x_j)$$

Set $h_j = x_{j+1} - x_j$

$$a_{j+1} = S_j(x_{j+1}) = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \tag{1}$$

$$S_j'(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$$

$$S_j'(x_j) = b_j$$

$$b_{j+1} = S_{j+1}'(x_{j+1}) = S_j'(x_{j+1})$$

$$b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2 \tag{2}$$

Define $b_n = S'(x_n)$

$$S_j''(x) = 2c_j + 6d_j(x - x_j)$$

$$c_j = \frac{S_j''(x_j)}{2}$$

$$c_{j+1} = \frac{S_{j+1}''(x_{j+1})}{2} = \frac{S_j''(x_{j+1})}{2}$$

$$c_{j+1} = c_j + 3d_j h_j \tag{3}$$

Define $c_n = \frac{S''(x_n)}{2}$

By equation (3), $d_j = \frac{1}{3h_j}(c_{j+1} - c_j)$

by equation (1)

$$
\begin{aligned}
a_{j+1} &= a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \\
&= a_j + b_j h_j + c_j h_j^2 + \frac{1}{3}(c_{j+1} - c_j) h_j^2
\end{aligned}
$$

$$
a_{j+1} = a_j + b_j h_j + \frac{1}{3}(c_{j+1} + 2c_j) h_j^2 \tag{4}
$$

Plugging values of d_j in equation (2) we get

$$
b_{j+1} = b_j + 2c_j h_j + h_j(c_{j+1} - c_j)
$$

$$
b_{j+1} = b_j + h_j(c_{j+1} + c_j) \tag{5}
$$

By equation (4) we get

$$
b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(c_{j+1} + 2c_j) \tag{6}
$$

$$b_{j-1} = \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(c_j + 2c_{j-1})$$

plugging these values of b_j and b_{j-1} in equation (5)

$$\begin{aligned}\frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(c_{j+1} + 2c_j) &= \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(c_j + 2c_{j-1}) \\ &+ h_{j-1}(c_j + c_{j-1})\end{aligned}$$

Collecting terms of c_j, c_{j+1}, c_{j-1} we get

$$\begin{aligned}h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} &= \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_{j-1} - a_j) \\ & j = 1, 2, \cdots, n-1\end{aligned}$$

$n-1$ equations and $n+1$ unknowns $c_0, c_1, \cdots, c_n$

Case 1 : free boundary $S''(x_1) = S''(x_n) = 0$

$$S_j''(x) = 2c_j + 6d_j(x - x_j)$$

$$0 = S_j''(x_0) = 2c_0 + 6d_j(0)$$

So $c_0 = 0$ and $c_n = \frac{S''(x_n)}{2} = 0$.
So we have a system $Ax = b$, where A is $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & \cdots & 0 & 0 \\ 0 & h_1 & 2(h_1 + h_2) & h_2 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ & & & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$
b = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix}
$$

Question 1. Does $Ax = b$ have a solution?

Question 2. Any comments on the structure of A?

Definition

A matrix $T = (t_{ij})_{n\times n}$ is called strictly diagonally dominated if

$$|t_{ii}| > \sum_{j=1, j\neq i}^{n} |t_{ij}|$$

We will see later that strictly diagonally dominated matrix T is invertible.

Our matrix A is strictly diagonally dominated.

So we can solve $Ax = b$ to get $c_0, c_1, \cdots, c_n$.

Then we obtain $d_0, d_1, \cdots, d_n$ and $b_0, b_1, \cdots, b_n$

Example

Approximate $f(x) = e^x$ in the interval $[0, 3]$.

$$x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$$

$$f(x_0) = 1, f(x_1) = e, f(x_2) = e^2, f(x_3) = e^3$$

Find cubic spline with free boundary.

Ans.

$$n = 3, h_0 = h_1 = h_2 = 1$$

$$a_0 = 1, a_1 = e, a_2 = e^2, a_3 = e^3$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 3(e^2 - 2e + 1) \\ 3(e^3 - 2e^2 + e) \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

free boundary. So $c_0 = c_3 = 0$

$$\begin{aligned} 4c_1 + c_2 &= 3(e^2 - 2e + 1) \\ c_1 + 4c_2 &= 3(e^3 - 2e^2 + e) \end{aligned}$$

$$c_1 = 0.7569 \qquad c_2 = 5.83$$

$$
\begin{aligned}
d_1 &= \frac{1}{3h_0}(c_1 - c_0) = \frac{1}{3}c_1 = 0.2523 \\
d_2 &= \frac{1}{3h_1}(c_2 - c_1) = 1.691 \\
d_3 &= \frac{1}{3h_2}(c_3 - c_2) = -1.943 \\
b_0 &= \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1) \\
&= a_1 - a_0 - \frac{1}{3}c_1 = 1.466 \\
b_1 &= \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(2c_1 + c_2) \\
&= 2.223 \\
b_2 &= \frac{1}{h_2}(a_3 - a_2) - \frac{h_2}{3}(2c_2 + c_3) \\
&= 8.81
\end{aligned}
$$

$$S(x) = \begin{cases} 1 + 1.466x + 0x^2 + 0.2523x^3, & 0 \leq x \leq \\ 2.718 + 2.223(x-1) + 0.7569(x-1)^2 + 1.691(x-1)^3, & 1 \leq x \leq \\ 7.389 + 8.81(x-2) + 5.83(x-2)^2 - 1.943(x-2)^3, & 2 \leq x \leq \end{cases}$$

Case 2 : Clamped boundary

$$S'(x_0) = f'(x_0) \text{ and } S'(x_n) = f'(x_n), S'(x_0) = f'(a) = b_0$$
$$f'(a) = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1)$$

$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a)$$

$$f'(b) = b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n)$$

So equation 6 with $j = n - 1$ gives

$$f'(b) = \frac{1}{h_{n-1}}(a_n - a_{n-1}) - \frac{h_{n-1}}{3}(c_n + 2c_{n-1}) + h_{n-1}(c_{n-1} + c_n)$$

Simplify to get,

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1})$$

Thus we obtain $Ax = b$ where

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & \cdots & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & \cdots & 0 & h_{n-1} & 2h_n \end{bmatrix}$$

$$x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$b = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{bmatrix}$$

The matrix A is again strictly diagonally dominated. So A is invertible.

Thus we can solve $Ax = b$ to get $c_0, \cdots, c_n$.

Then we find d and b to get the final answer.

Reasons for Numerical Integration

1. We can not solve $\int_a^b f(x)dx$ exactly.
   Examples:-
   1. $\int_0^1 sin(x^2)dx$
   2. $\int_0^1 e^{-x^2}dx$
   3. $\int_0^1 \sqrt{1+cos^4 x}dx$
2. And is not known explicitly. Only values of f at some points are known
   Example:-

$$\frac{dy}{dx} = sin(x^2+y^2) \ , \ y(0) = 1$$

   find $\int_0^1 y(t)dt$

Note that we do not know y exactly.

By Numerical Solution to ODE's we can approximate $y(0.1), y(0.2), \cdots, y(1)$

Derivation of Numerical Integration formulae

Let $P_k(x)$ be the polynomial which interpolates $f(x)$ at points $x_1, x_2, \cdots, x_k$.

We approximate $I(f) = \int_a^b f(x)dx$ by $I(P_k) = \int_a^b P_k(x)dx$

$$f(x) = P_k(x) + f[x_0, x_1, \cdots, x_k, x]\Psi_k(x)$$

$$\Psi_k(x) = \prod_{j=0}^{k}(x - x_j)$$

Error in computing integral $= E(f) = I(f) - I(P_k)$

$$E(f) = \int_a^b f[x_0, x_1, \cdots, x_k, x]\Psi_k(x)dx$$

Simplification of error terms

Case 1 $\Psi_k(x)$ is of one sign on (a, b).
Then by MVT for integral

$$
\begin{aligned}
E(f) &= \int_a^b f[x_0.x_1, \cdots, x_k, x]\Psi_k(x)dx \\
&= f[x_0, x_1, \cdots, x_k, \xi] \int_a^b \Psi_k(x)dx
\end{aligned}
$$

If in addition $f(x)$ is $k+1$ times differentiable then

$$
E(f) = \frac{f^{(k+1)}(\eta)}{(k+1)!} \int_a^b \Psi_k(x)dx
$$

for some $\eta \in (a, b)$

Case 2 $\int_a^b \Psi_k(x)dx = 0$. We use the identity

$$f[x_0, x_1, \cdots, x_k, x] = f[x_0, x_1, \cdots, x_k, x_{k+1}] + f[x_0, x_1, \cdots, x_{k+1}, x](x - x_{k+1})$$

$$\begin{aligned} E(f) &= \int_a^b f[x_0, x_1, \cdots, x_k, x]\Psi_k(x)dx \\ &= \int_a^b f[x_0, x_1, \cdots, x_k, x_{k+1}]\Psi_k(x)dx \\ &+ \int_a^b f[x_0, x_1, \cdots, x_{k+1}, x](x - x_{k+1})\Psi_k(x)dx \\ &= \int_a^b f[x_0, x_1, \cdots, x_{k+1}, x]\Psi_{k+1}(x)dx \end{aligned}$$

$$\Psi_{k+1}(x) = (x - x_{k+1})\Psi_k(x) = \prod_{j=0}^{k+1}(x - x_j)$$

$$E(f) = \int^b f[x_0, x_1, \cdots, x_{k+1}, x]\Psi_{k+1}(x)dx$$

If we can choose x_{k+1} such that $\Psi_{k+1}(x)$ is of one sign on (a, b), then by MVT of integrals

$$E(f) = f[x_0, x_1, \cdots, x_{k+1}, \xi] \int_a^b \Psi_{k+1}(x)dx$$

If $f(x)$ is $k+2$ times continuously differentiable then

$$E(f) = \frac{f^{(k+2)}(\eta)}{(k+2)!} \int_a^b \Psi_{k+1}(x)dx$$
$$\text{for some } \eta \in (a, b)$$

Examples

Let $k = 0$

$$P_0 = f(x_0)$$

$$f(x) = f(x_0) + f[x_0, x](x - x_0)$$

$$I = \int_a^b f(x)dx = \int_a^b f(x_0)dx + \int_a^b f[x_0, x](x - x_0)dx$$

$$I(P_0(x)) = (b - a)f(x_0)$$

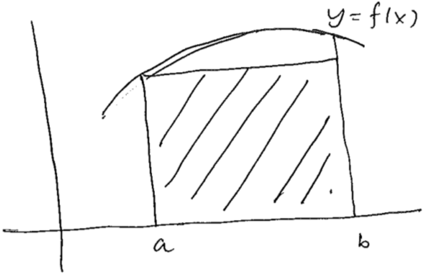
Case 1 $x_0 = a$

$$I(f) \approx R = (b - a)f(a)$$

$\Psi_0(x) = x - a$ has one sign in $[a, b]$.

$$E^R = f'(\eta)\int_a^b (x - a)dx = f'(\eta)\frac{(b - a)^2}{2}$$

$R \leftarrow$ Rectangle rule



Case 2 $x_0 = \frac{a+b}{2}$

$$\Psi_0(x) = x - x_0, \text{ not of one sign}$$

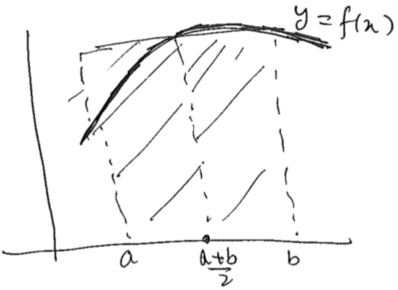
However $\int_a^b (x - \frac{a+b}{2})dx = 0$

Choose $x_1 = x_0$

$$\Psi_1(x) = (x - x_0)^2, \text{ is of one sign}$$

$$
\begin{aligned}
I(f) &\approx M = (b-a) f\left(\frac{a+b}{2}\right), \text{ Midpoint rule} \\
E^M &= \frac{f''(\eta)}{2} \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \\
&= \frac{f''(\eta)}{2} \frac{(b-a)^3}{24}, \text{ for some } \eta \in [a, b]
\end{aligned}
$$

Midpoint rule is better than rectangle rule



Now let $k = 1$

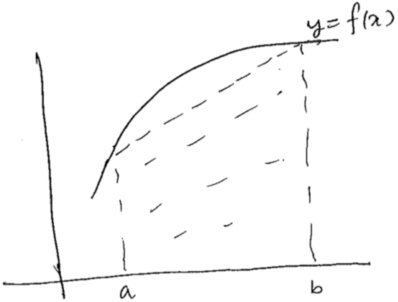
$$f(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x]\Psi_1(x)$$

Where $\Psi_1(x) = (x - x_0)(x - x_1)$, $x_0 = a$, $x_1 = b$

$$\Psi_1(x) = (x - a)(x - b) \leq 0 \text{ on } [a, b]$$

$$\begin{aligned} I(f) &= \int_a^b [f(x_0) + f[a, b](x - a)]dx \\ &+ \frac{1}{2} f''(\eta) \int_a^b (x - a)(x - b)dx \\ I(f) &\approx T = \frac{1}{2}(b - a)[f(a) + f(b)] \text{ , Trapezoidal rule} \end{aligned}$$

$$E^T = -\frac{f''(\eta)(b - a)^3}{12}, \text{ for some } \eta \in [a, b]$$



Now let $k = 2$

$$f(x) = P_2(x) + f[x_0, x_1, x_2, x]\Psi_2(x)$$

For distinct points x_0, x_1, x_2

$$\Psi_2(x) = (x - x_0)(x - x_1)(x - x_2) \text{ is not of } \textbf{one} \text{ sign on } [a, b]$$

However for $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$

$$\begin{aligned}\int_a^b \Psi_2(x)dx &= \int_a^b (x-a)\left(x-\frac{a+b}{2}\right)(x-b)dx \\ &\qquad \text{By using } u = x - \frac{a+b}{2} \\ &= \int_{-\frac{a+b}{2}}^{\frac{a+b}{2}} \left(u+\frac{b-a}{2}\right) u \left(u-\frac{b-a}{2}\right) du \\ &= 0\end{aligned}$$

Put $x_3 = x_1 = \frac{a+b}{2}$

$$\Psi_3(x) = (x-a)\left(x-\frac{a+b}{2}\right)^2 (x-b) \leq 0 \text{ on } [a,b]$$

$$
\begin{aligned}
I(f) &= I(P_2) + \frac{1}{4!} f^{(4)}(\eta) \int_a^b \Psi_3(x) dx
\end{aligned}
$$

One calculate

$$
\begin{aligned}
\int_a^b \Psi_3(x) dx &= \int_a^b (x-a)\left(x - \frac{a+b}{2}\right)^2 (x-b) dx \\
&= \frac{-4}{15}\left(\frac{b-a}{2}\right)^5 \\
E^S(f) &= -\frac{1}{90} f^{(4)}(\eta) \left(\frac{b-a}{2}\right)^5 \ , \ \eta \in [a, b] \\
I(f) &\approx I(P_2)
\end{aligned}
$$

$$
P_2(x) = f(a) + f[a, b](x-a) + f\left[a, b, \frac{a+b}{2}\right](x-a)(x-b)
$$

$$
\begin{aligned}
\int_a^b P_2(x)dx &= f(a)(b-a) + f[a,b]\frac{(b-a)^2}{2} \\
&+ f\left[a,b,\frac{a+b}{2}\right]\int_a^b (x-a)(x-b)dx
\end{aligned}
$$

$$
\int_a^b (x-a)(x-b)dx = -\frac{(b-a)^3}{6}
$$

Note:-

$$
f\left[a,b,\frac{a+b}{2}\right] = f\left[a,\frac{a+b}{2},b\right], \text{ (why ?)}
$$

$$
\begin{aligned}
f\left[a,\frac{a+b}{2},b\right](b-a)^2 &= \left(f\left[\frac{a+b}{2},b\right] - f\left[a,\frac{a+b}{2}\right]\right)(b-a) \\
&= \left\{\frac{f(b)-f(\frac{a+b}{2})}{\frac{b-a}{2}} - \frac{f(\frac{a+b}{2})-f(a)}{\frac{b-a}{2}}\right\}(b-a) \\
&= 2(f(b) - 2f\left(\frac{a+b}{2}\right) + f(a))
\end{aligned}
$$

$$
\begin{aligned}
I(P_2) &= \int_a^b P_2(x)dx \\
&= f(a)(b-a) + (f(b) - f(a))\left(\frac{b-a}{2}\right) \\
&- \ 2\left(f(b) - 2f\left(\frac{a+b}{2}\right) + f(a)\right)\frac{b-a}{6} \\
&= \frac{b-a}{6}\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}, \text{ (Simpson's Rule)}
\end{aligned}
$$

$$
\begin{aligned}
\int_a^b f(x)dx &\approx \frac{b-a}{6}\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\} \\
E^S &= -\frac{1}{90}f^{(4)}(\eta)\left(\frac{b-a}{2}\right)^5
\end{aligned}
$$

$$f(x) = P_3(x) + f[x_0, x_1, x_2, x_3, x]\Psi_3(x)$$

$$x_0 = x_1 = a, x_2 = x_3 = b$$

$$\Psi_3(x) = (x-a)^2(x-b)^2 \text{ is of one sign in } [a, b]$$

$$E(f) = \frac{1}{4!} f^{(4)}(\eta) \int_a^b (x-a)^2(x-b)^2 dx = \frac{f^{(4)}(\eta)(b-a)^5}{720}$$

$$\begin{aligned}
P_3(x) &= f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2 \\
&= f[a,a,b,b](x-a)^2(x-b) \\
\int_a^b P_3(x)dx &= f(a)(b-a) + f'(a)\frac{(b-a)^2}{2} \\
&+ f[a,a,b]\frac{(b-a)^3}{3} + f[a,a,b,b]\left\{\frac{(b-a)^4}{4} - \frac{(b-a)^4}{3}\right\}
\end{aligned}$$

$$f[a,a,b] = \frac{f[a,b] - f'(a)}{b-a}$$

$$f[a,a,b,b] = \frac{f'(b) - 2f[a,b] + f'(a)}{(b-a)^2}$$

$$\begin{aligned}\int_a^b P_3(x)dx &= f(a)(b-a) + f'(a)\frac{(b-a)^2}{2} \\ &+ \{f[a,b] - f'(a)\}\frac{(b-a)^2}{3} \\ &- \{f'(b) - 2f[a,b] + f'(a)\}\frac{(b-a)^2}{12}\end{aligned}$$

replace $f[a,b]$ by $\frac{f(b)-f(a)}{b-a}$. So we get

$$I(f) \approx CT = \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{12}(f'(a) - f'(b))$$

Corrected Trapezoidal rule

$$E^{CT} = \frac{f^{(4)}(\eta)(b-a)^5}{720}$$

Rules for Numerical Integration

$$
\begin{aligned}
\text{Rectangle rule } I &\approx (b-a)f(a)\\
\text{Midpoint rule } I &\approx (b-a)f\left(\frac{a+b}{2}\right)\\
\text{Trapezoidal rule } I &\approx \frac{1}{2}(b-a)[f(a)+f(b)]\\
\text{Simpson's rule } I &\approx \frac{b-a}{6}\left\{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right\}
\end{aligned}
$$

Corrected Trapezoidal rule

$$
I \approx \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{12}(f'(a)-f'(b)
$$