Numerical Analysis : [MA214]
Lecture 3

Instructor: Prof. Tony J. Puthenpurakal

Department of Mathematics
Indian Institute of Technology Bombay

tputhen@math.iitb.ac.in

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function and let $x_0, x_1, \cdots, x_n$ be distinct points in $[a, b]$ Then there exists a unique polynomial $P_n(x)$ such that

$$P_n(x_i) = f(x_i) \ , \forall i = 0, 1, \cdots, n$$

$P_n(x) \rightarrow$ Interpolating polynomial of f ($w.r.t.\ x_0, x_1, \cdots, x_n$)

we had two forms of $P_n(x)$

Lagrange's form

$$\begin{aligned} l_k(x) &= \prod_{i=0, i \neq k}^{n} \frac{x - x_i}{x_k - x_i} \ , \ k = 0, 1, \cdots, n \\ P_n(x) &= \sum_{i=0}^{n} f(x_i) l_i(x) \end{aligned}$$

Advantage of Lagrange's form:- It is simple.

Disvantage of Lagrange's Form:-
There is no way of using Lagrange form of $P_n(x)$ to determine $P_{n+1}(x)$.

Newtons form of interpolating polynomial

$$\begin{aligned}
P_n(x) = f(x_0) &+ f[x_0, x_1](x - x_0) \\
&+ f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&+ \cdots \\
&\vdots \\
&+ f[x_0, x_1, x_2, \cdots, x_n] \prod_{i=0}^{n-1} (x - x_i)
\end{aligned}$$

$$P_n(x) = P_{n-1}(x) + f[x_0, x_1, \cdots, x_n] \prod_{i=0}^{n-1} (x - x_i)$$

Here $$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, x_1, x_2, \cdots, x_k] = \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots, x_{k-1}]}{x_k - x_0}$$

Recall:- Error of interpolating polynomial

$$
\begin{aligned}
e_n(x) &= f(x) - P_n(x) \\
e_n(\overline{x}) &= f[x_0, x_1, \cdots, x_n, \overline{x}] \prod_{i=0}^{n} (x - x_i) \\
f[x_0, x_1, \cdots, x_n, x] &= \frac{f^{(n+1)(\xi)}}{(n+1)!} \\
\Psi_{n+1}(x) &= \prod_{j=0}^{n} (x - x_j)
\end{aligned}
$$

Not in syllabus on how to *minimize* $\Psi_{n+1}(x)$

Osculatory interpolation

Sometimes we have

$$x_0, x_1, \cdots, x_k$$
$$f(x_0), f(x_1), f(x_2), \cdots, f(x_k)$$
$$f'(x_0), f'(x_1), f'(x_2), \cdots, f'(x_k)$$

We need a polynomial $p(x)$ such that

$$p(x_i) = f(x_i) \text{ and } p'(x_i) = f'(x_i) \text{ , for } i = 0, 1, \cdots, n$$

Note:- deg $p(x) \leq 2n + 1$

Example:-

$$\frac{dy}{dx} = g(x, y) \text{ , } y(x_0) = y_0$$

Note:- $y'(x_i) = g(x_i, y_i)$ can be readily computed.

Last time I had given an example to calculate $P(x)$

Algorithm for computing $P(x)$

$$\text{Set } z_0 = x_0 \text{ , and } z_1 = x_0$$
$$z_2 = x_1 \text{ , and } z_3 = x_1$$
$$\vdots$$
$$z_{2i} = x_i \text{ , and } z_{2i+1} = x_i$$
$$\vdots$$
$$z_{2n} = x_n \text{ , and } z_{2n+1} = x_n$$

$$\text{Set } f[z_i, z_{i+1}] = \left\{ \begin{array}{ll} f'(z_i) & \text{if } z_i = z_{i+1} \\ \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i} & \text{if } z_i \neq z_{i+1} \end{array} \right\}$$

For higher order computation

$$f[z_i, z_{i+1}, \cdots, z_{i+k}] = \frac{f[z_{i+1}, \cdots, z_{i+k}] - f[z_i, \cdots, z_{i+k-1}]}{z_{i+k} - z_i}$$

$$\begin{aligned} P_{2n+1}(x) \quad &= \quad f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) \\ &+ \cdots + f[z_0, \cdots, z_{2n+1}] \prod^{2n} (x - z_j) \end{aligned}$$

Problem

$$f(0) = 1, \text{ and } f(1) = 1.9, \; f'(0) = 2, \text{ and } f'(1) = 2.5$$
Approximate $f(0.4)$

Set $z_0 = 0, \; z_1 = 0, \; z_2 = 1, \; z_3 = 1$

z	$f(z)$	$f[,]$	$f[,,]$	$f[,,,]$
0	1	2	-1.1	2.7
0	1	0.9	1.6	
1	1.9	2.5		
1	1.9			

Here

$$
\begin{aligned}
f[0,0] &= f'(0) = 2 \\
f[1,1] &= f'(1) = 2.5 \\
f[0,1] &= \frac{f(1) - f(0)}{1 - 0} = \frac{1.9 - 1}{1} = 0.9 \\
f[0,0,1] &= \frac{f[0,1] - f[0,0]}{1 - 0} = 0.9 - 2 = -1.1 \\
f[0,1,1] &= \frac{f[1,1] - f[0,1]}{1 - 0} = 2.5 - 0.9 = 1.6
\end{aligned}
$$

finally

$$
\begin{aligned}
f[0,0,1,1] &= \frac{f[0,1,1] - f[0,0,1]}{1 - 0} = 1.6 - (-1.1) = 2.7 \\
P_3(x) &= 1 + 2(x - 0) + (-1.1)(x - 0)^2 + 2.7(x - 0)^2(x - 1) \\
&= 1 + 2x - 1.1x^2 + 2.7x^2(x - 1) \\
f(0.4) &\approx P_3(0.4) = 1.365
\end{aligned}
$$

Convention:-

Let $x_0, x_1, \cdots, x_m$ be not necessarily distinct points.

We say two function $f(x)$ and $g(x)$ agree at the points $x_0, x_1, \cdots, x_m$ if

$$f^{(j)}(z) = g^{(j)}(z) \text{ , for } j = 0, 1, \cdots, k-1$$

for every points z which occurs k times in the sequence $x_0, x_1, \cdots, x_m$.

Example:-

$$x = 1, 2, 2, 1, 3, 2$$

$f(x)$ and $g(x)$ agree at the points $1, 2, 2, 1, 3, 2$. If

$$f(1) = g(1)$$
$$f'(1) = g'(1)$$
$$f(2) = g(2)$$
$$f'(2) = g'(2)$$
$$f''(2) = g''(2)$$
$$f(3) = g(3)$$

Problem:-

Given $x_0, x_1, \cdots, x_m$ not necessarily distinct points and $f : [a, b] \to \mathbb{R}$.

We need a polynomial $p(x)$ of degree $\leq m$ such that $p(x)$ and $f(x)$ agree at $x_0, x_1, \cdots, x_m$.

Exercise:- Show that if two polynomials of degree $\leq m$ agree at $x_0, x_1, \cdots, x_m$, then they are equal.

So it makes sense to talk about the polynomial of degree $\leq m$ which agrees with $f(x)$ at the $m+1$ points $x_0, x_1, \cdots, x_m$.

Theorem

If $f(x)$ has r continuous derivatives and no point in the $x_0, x_1, \cdots, x_m$ occurs more than r times then there exists exactly one polynomial $P_m(x)$ of degree $\leq m$ which agree with $f(x)$ at $x_0, x_1, \cdots, x_m$.

Uniqueness-Exercise

Existence:-
Assume $x_0 \leq x_1 \leq \cdots \leq x_m$
for $m = 0$ there is nothing to show.

Assume the statement is true for $m = k - 1$ and consider it for $m = k$. There are two cases

Case 1 $x_0 = x_k$. Then $x_0 = x_1 = x_2 = \cdots = x_k$.
So $r \geq k$. By assumption f has at least k continuous derivatives.

Then the Taylor polynomial $P_k(x)$ for $f(x)$ around the center $c = x_0$ does the job.

Remark:- Note that its leading coefficient is $\frac{f^{(k)}(x_0)}{k!}$

Case 2 $x_0 < x_k$

Then by induction hypothesis we can find polynomial $P_{k-1}(x)$ of degree $\leq k-1$ which agree with $f(x)$ at $x_0, x_1, \cdots, x_k$ and a polynomial $q_{k-1}(x)$ of degree $\leq k-1$ which agree with $f(x)$ at $x_1, x_2, \cdots, x_k$.

Verify

$$P_k(x) = \frac{x - x_0}{x_k - x_0} q_{k-1}(x) + \frac{x_k - x}{x_k - x_0} P_{k-1}(x)$$

does the job.

[Slightly tricky to show. See textbook Conte and de Boor page 64.]

Convention

$x_0, x_1, \cdots, x_m$ not necessarily distinct points

$P_m(x) =$ unique polynomial which agree with $f(x)$ at $x_0, x_1, \cdots, x_m$.

$f[x_0, x_1, \cdots, x_m] =$ leading coefficient of $P_m(x) =$ coefficient of x^m in $P_m(x)$.

We have

$$P_m(x) = P_{m-1}(x) + f[x_0, x_1, \cdots, x_m] \prod_{i=0}^{m-1} (x - x_i)$$

Proof:-

$$P_m(x) - f[x_0, x_1, \cdots, x_m] \prod_{i=0}^{m-1} (x - x_i)$$

has degree $\leq m - 1$ and agrees with $f(x)$ at $x_0, x_1, \cdots, x_m$.

So by uniqueness of interpolating polynomial the result follows.

Thus we can write $P_m(x)$ as

$$P_m(x) = f(x_0) + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \cdots, x_m] \prod_{i=0}^{m-1} (x - x_i)$$

Note we are assuming $x_0 \leq x_1 \leq \cdots \leq x_m$.

Case 1 $x_0 = x_1 = x_2 = \cdots = x_m$.

$$\begin{aligned} P_m(x) &= \text{Taylor polynomial with center } x_0 \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \cdots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m \end{aligned}$$

So $$f[x_0, x_0, \cdots, x_0 \text{ (} m+1 \text{ times) }] = \frac{f^{(m)}(x_0)}{m!}$$

Case 2 $x_m \neq x_0$. Then

$$f[x_0, x_1, x_2, \cdots, x_m] = \frac{f[x_1, x_2, \cdots, x_m] - f[x_0, x_1, \cdots, x_{m-1}]}{x_m - x_0}$$

Example:- $f(0) = 1, f'(0) = 0, f''(0) = 1, f(0.1) = 9.95E{-1}$.
Approximate $f(0.05)$.

z	$f(z)$	$f[,]$	$f[,,]$	$f[,,,]$
0	1	0	0.5	-10
0	1	0	$-5E{-1}$	
0	1	$-5E{-2}$		
0.1	$9.95E{-1}$			

$$\begin{aligned} P(x) &= 1 + 0(x-0) + (0.5)(x-0)^2 + (-10)(x-0)^3 \\ &= 1 + 0.5x^2 - 10x^3 \\ P(0.05) &\approx 1 + 0.5(0.05)^2 - 10(0.05)^3 = 1 \end{aligned}$$

Calculations

$$z_0 = 0, z_1 = 0, z_2 = 0, z_3 = 0.1$$

$$\begin{aligned}
f[z_0, z_1] &= f'(0) = 0 \\
f[z_1, z_2] &= f'(0) = 0 \\
f[z_2, z_3] &= \frac{f(z_3) - f(z_2)}{z_3 - z_2} = -5E{-}2 \\
f[z_0, z_1, z_2] &= \frac{f''(z_0)}{2} = 0.5 \\
f[z_1, z_2, z_3] &= \frac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1} = \frac{-5E{-}2 - 0}{0.1} = -5E{-}1 \\
f[z_0, z_1, z_2, z_3] &= \frac{f[z_1, z_2, z_3] - f[z_0, z_1, z_2]}{z_3 - z_0} = -10
\end{aligned}$$

Examples where Osculatory interpolation is useful

1. $\frac{dy}{dx} = f(x, y)$, and $y(x_0) = y_0$

Note that numerical methods for solving O.D.Es yield

x	y
x_1	$y(x_1)$
x_2	$y(x_2)$
$\vdots$	$\vdots$
x_n	$y(x_n)$

Note that $y'(x_i) = f(x_i, y_i)$ can be calculated easily.

2. $f(x) = \int_0^x g(t)dt$

Numerical integration techniques yield $f(x_0), f(x_1), \cdots, f(x_n)$.
Note $f'(x) = g(x)$, $f'(x_i) = g(x_i)$ can be calculated easily.

As the following example will be used often I give a direct proof.

Example:- a, b distinct points. We know $f(a)$, $f(b)$, $f'(a)$, $f'(b)$.

$$P_3(x) = f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b)$$

We prove by direct computation that $P_3(x)$ agree with $f(x)$ at a, a, b, b

$$\begin{aligned}
P_3(a) &= f(a) \\
P_3'(a) &= f[a,a] = f'(a) \\
f[a,b] &= \frac{f(b)-f(a)}{b-a} \\
f[a,a,b] &= \frac{f[a,b]-f[a,a]}{b-a} \\
&= \frac{\frac{f(b)-f(a)}{b-a} - f'(a)}{b-a} \\
&= \frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2}
\end{aligned}$$

$$
\begin{aligned}
P_3(b) &= f(a) + f'(a)(b-a) + \frac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}(b-a)^2 \\
&= f(b)
\end{aligned}
$$

Verify

$$
\begin{aligned}
f[a,b,b] &= \frac{(b-a)f'(b) - (f(b) - f(a))}{(b-a)^2} \\
f[a,a,b,b] &= \frac{(b-a)(f'(b) + f'(a)) - 2(f(b) - f(a))}{(b-a)^3} \\
P_3'(b) &= f'(a) + 2f[a,a,b](b-a) + f[a,a,b,b](b-a)^2 \\
&= f'(b) \text{ (Check ?)}
\end{aligned}
$$

Continuity of divided differences

Theorem

$f[x_0, x_1, \cdots, x_n]$ is a continuous function of $x_0, x_1, \cdots, x_n$. (Assume f has n continuous derivatives)
i.e. for each r, $x_0^{(r)}, \cdots, x_n^{(r)}$ are $n+1$ points in $[a, b]$ and

$$\lim_{r\to\infty} x_i^{(r)} = y_i \text{ for } i = 0, 1, \cdots, n$$

Then

$$\lim_{r\to\infty} f[x_0^{(r)}, x_1^{(r)}, \cdots, x_n^{(r)}] = f[y_0, y_1, \cdots, y_n]$$

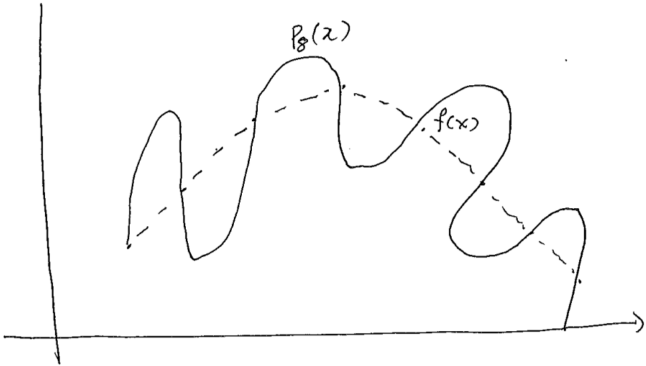
Proof.

See textbook Conte and de Boor page 65. □

Disadvantages of Interpolation

Note that if $x_0, x_1, \cdots, x_k$ are points in $[a, b]$ then the interpolating polynomial has degree k.

In practice k is large. Furthermore a polynomial of degree k with k large Oscillates a lot



For example if there are 101 points then it is not advisable to work with a degree 100 interpolating polynomial as this also creates lot of round-off error.

Strategy:-
Use piecewise-polynomial approximation

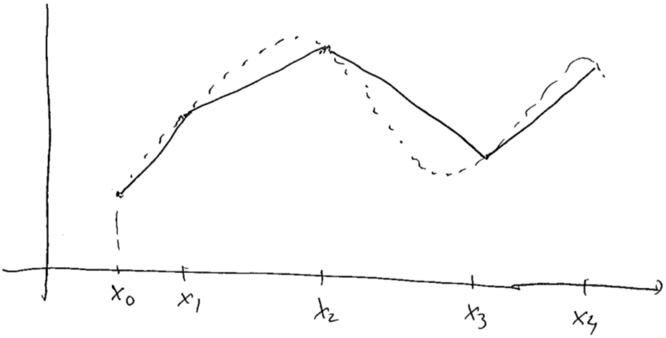
Simplest case Piecewise-liner interpolation:-

$$a = x_1 < x_2 < x_3 < \cdots < x_n = b$$

$f(\overline{x})$ is approximated at a point $\overline{x}$ by first locating the interval $[x_k, x_{k+1}]$ containing $\overline{x}$ and then taking

$$p(\overline{x}) = f(x_k) + f[x_k, x_{k+1}](\overline{x} - x_k)$$

Graphical representation



Note that if N is large and so $|x_i - x_{i+1}|$ is small then this is a good approximation of $f(x)$.

If we use higher degree (say cubic) piecewise-polynomial approximation then we get better approximation.

Construction of piecewise-cubic function at the points $x_1, \cdots, x_{N+1}$ where

$$a = x_1 < x_2 < x_3 < \cdots < x_{N+1} = b$$

On each $[x_i, x_{i+1}]$ we construct $g_3(x)$ as a cubic polynomial $P_i(x)$, $i = 1, 2, \cdots, N$

$$P_i(x) \quad = \quad c_{1,i} + c_{2,i}(x - x_i) + c_{3,i}(x - x_i)^2 + c_{4,i}(x - x_i)^3$$

for $i = 1, 2, \cdots, N$. Since $g_3(x_i) = f(x_i)$, for $i = 1, 2, \cdots, N+1$
We have

$$P_i(x_i) = f(x_i) \text{ and } P_i(x_{i+1}) = f(x_{i+1})$$

for $i = 1, 2, \cdots, N$

In particular

$$P_{i-1}(x_i) = P_i(x_i) = f(x_i), \text{ for } i = 1, 2, \cdots, N$$

So $g_3(x)$ is continuous on $[a, b]$.

Only constraints for $P_i(x)$ is

$$P_i(x_i) = f(x_i) \text{ and } P_i(x_{i+1}) = f(x_{i+1})$$

So we have some freedom in choosing $P_i(x)$

We study 2 cases

1. Piecewise-Cubic Hermite interpolation
2. Cubic-Spline interpolation

Piecewise-Cubic Hermite interpolation

One determine $P_i(x)$ so as to interpolate $f(x)$ at $x_i, x_i, x_{i+1}, x_{i+1}$, *i.e.* we also have

$$P_i'(x_i) = f'(x_i) \text{ and } P_i'(x_{i+1}) = f'(x_{i+1})$$

By Newtone formula

$$\begin{aligned}
P_i(x) &= f(x_i) + f[x_i, x_i](x - x_i) + f[x_i, x_i, x_{i+1}](x - x_i)^2 \\
&\quad + f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^2(x - x_{i+1}) \\
\text{Write } x - x_{i+1} &= (x - x_i) + (x_i - x_{i+1}) \\
P_i(x) &= f(x_i) + f'(x_i)(x - x_i) \\
&+ (f[x_i, x_i, x_{i+1}] - f[x_i, x_i, x_{i+1}, x_{i+1}]\triangle x_i)(x - x_i)^2 \\
&+ f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^3
\end{aligned}$$

Algorithim

For $i = 1, 2, \cdots, N+1$

$$f_i = f x_i$$
$$\triangle x_i = x_{i+1} - x_i$$
$$s_i = f'(x_i),\ c_{1,i} = f_i,\ c_{2,i} = s_i$$

$$\begin{aligned}
c_{3,i} &= f[x_i, x_i, x_{i+1}] - f[x_i, x_i, x_{i+1}, x_{i+1}]\triangle x_i \\
&= \frac{f[x_i, x_{i+1}] - s_i}{\triangle x_i} - c_{4,i}\triangle x_i \\
c_{4,i} &= f[x_i, x_i, x_{i+1}, x_{i+1}] \\
&= \frac{f[x_i, x_{i+1}, x_{i+1}] - f[x_i, x_i, x_{i+1}]}{\triangle x_i} \\
&= \frac{s_{i+1} + s_i - 2f[x_i, x_{i+1}]}{\triangle x_i}
\end{aligned}$$

Example

$$\frac{dy}{dx} = y - x^2 + 1, \text{ and } y(0) = 0.5, 0 \leq x \leq 1$$

x_i	$y(x_i)$	$y'(x_i)$
0	0.5	1.5
0.2	0.826	1.786
0.4	1.207	2.047
0.6	1.637	2.277
0.8	2.110	2.470
1.0	2.618	2.618

Find $y(0.7)$, $y(0.9)$

Note:- $y(x_i)$ is found using a Numerical method

Remarks:- Usual oscillatory polynomial has degree 11.

So we use piecewise Hermite interpolation

x	$y(x)$	$f[,]$	$f[,,]$	$f[,,,]$
0.6	1.637	2.277	$4.4E{-1}$	$4.25E{-1}$
0.6	1.637	2.365	$5.25E{-1}$	
0.8	2.110	2.470		
0.8	2.110			

$$
\begin{aligned}
P_4(x) &= 1.637 + 2.277(x-0.6) + 4.4E{-1}(x-0.6)^2 \\
&+ 4.25E{-1}(x-0.6)^2(x-0.8) \\
P_4(0.7) &= 1.869
\end{aligned}
$$

x	$y(x)$	$f[,]$	$f[,,]$	$f[,,,]$
0.8	2.110	2.47	0.35	0.2
0.8	2.110	2.54	0.39	
1.0	2.618	2.618		
1.0	2.2.618			

$$
\begin{aligned}
P_4(x) &= f[0.8] + f[0.8, 0.8](x-0.8) + f[0.8, 0.8, 1](x-0.8)^2 \\
&+ f[0.8, 0.8, 1, 1](x-0.8)^2(x-1) \\
&= 2.11 + 2.47(x-0.8) + 0.35(x-0.8)^2 + 0.2(x-0.8)^2(x-1) \\
P_5(0.9) &= 2.360 \text{ in 4 sig digits}
\end{aligned}
$$