Tutorial 1

(a) $x = \dfrac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$ $\qquad x_0 = 1.31,\ y_0 = 3.24$

$\qquad\qquad\qquad\qquad\qquad\qquad x_1 = 1.93,\ y_1 = 4.76$

(b) $x = x_0 - \dfrac{(x_1 - x_0)\, y_0}{y_1 - y_0}$

<u>Case (a)</u> : $x_0 y_1 = 6.236 \qquad y_1 - y_0 = 1.52$

$x_1 y_0 = 6.253$

$x_0 y_1 - x_1 y_0 = -0.017 = -1.700E-2$

$x = \dfrac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = -0.01118 = -1.118 \times 10^{E-2}$

In fact, $x = \dfrac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = -0.01158$

<u>Case (b)</u> : $x_1 - x_0 = 0.62 \qquad x_0 = 1.31,\ y_0 = 3.24$

$(x_1 - x_0)\, y_0 = 2.0088$

$y_1 - y_0 = 1.52$

$\therefore \dfrac{(x_1 - x_0)\, y_0}{y_1 - y_0} = 1.322$

$x = x_0 - \dfrac{(x_1 - x_0)\, y_0}{y_1 - y_0} = -0.012 = -1.200E-2$

Actually, $x = x_0 - (x_1 - x_0)\, y_0 / (y_1 - y_0)$ $\cancel{\dfrac{x_0 y_1 - x_1 y_0}{y_1 - y_0}}$ $= -0.01158$

$= 1.158E-2$

Case (b) is better, case (a) involves subtraction of two nearly similar quantities.

2. $f : [a,b] \to \mathbb{R}$ is continuous.
$g \geq 0$ is integrable function on $[a,b]$.

To show: $\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$.
for $\xi \in [a,b]$

Apply extreme value theorem to f.
$\exists\, m, M \in \mathbb{R}$ s.t. $\forall x \in [a,b]$, $m \leq f(x) \leq M$.
As $g \geq 0$,
$\therefore\ m \underbrace{\int_a^b g(x)\,dx}_{I} \leq \int_a^b f(x)g(x)\,dx \leq M \int_a^b g(x)\,dx$

Case 1 : $I = 0$ $\therefore \int_a^b f(x)g(x)\,dx = 0 = f(\xi)\int_a^b g(x)\,dx$
$\forall$ ~~where~~ $\xi \in [a,b]$

Case 2 : $I \neq 0$

$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{I} \leq M$$

Apply intermediate value theorem to f.
$\therefore f$ takes on all values from m to M.

$\therefore \exists\, \xi \in [a,b]$ s.t. $\dfrac{\int_a^b f(x)g(x)\,dx}{I} = f(\xi)$

$\therefore \int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$

for some $\xi \in [a,b]$.

3. Lagrange Interpolation

x	$f(x)$
0	1
0.6	8.253 E−1
0.9	6.216 E−1

Find $f(0.4)$, $f(0.7)$ approximately.

$$\boxed{\begin{aligned} f(x) &= a_0 l_0(x) + a_1 l_1(x) + \dots + a_n l_n(x) \\ l_k(x) &= \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}, \quad k = 0 \text{ to } n \\ a_i &= f(x_i) \end{aligned}}$$

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$x_0 = 0$, $x_1 = 0.6$, $x_2 = 0.9$

$f(x_0) = 1$, $f(x_1) = 8.253E-1$, $f(x_2) = 6.216E-1$

$$l_0(0.4) = \left(\frac{0.4 - 0.6}{0 - 0.6}\right)\left(\frac{0.4 - 0.9}{0 - 0.9}\right) = 1.852E-1$$

$$l_0(0.7) = \left(\frac{0.7 - 0.6}{0 - 0.6}\right)\left(\frac{0.7 - 0.9}{0 - 0.9}\right) = -3.704E-2$$

$$l_1(0.4) = \left(\frac{0.4 - 0}{0.6 - 0}\right)\left(\frac{0.4 - 0.9}{0.6 - 0.9}\right) = 1.111$$

$$l_1(0.7) = \left(\frac{0.7 - 0}{0.6 - 0}\right)\left(\frac{0.7 - 0.9}{0.6 - 0.9}\right) = 7.778E-1$$

$$l_2(0.4) = \left(\frac{0.4 - 0}{0.9 - 0}\right)\left(\frac{0.4 - 0.6}{0.9 - 0.6}\right) = -2.963E-1$$

$$l_2(0.7) = \left(\frac{0.7 - 0}{0.9 - 0}\right)\left(\frac{0.7 - 0.6}{0.9 - 0.6}\right) = 2.593E-1$$

$$f(0.4) = (1)(1.852E-1) + \overset{(8.253E-1)}{\cancel{(0.6)}}(1.111) + (6.216E-1)(-2.963E-1)$$

$$= 1.852E-1 + 9.169E-1 \cancel{+} - 1.842E-1$$

$P_3(x) = 1.542E\text{-}1\,x^3 - 4.438E\text{-}1\,x^2 + 9.941E\text{-}1$

$f(0.3) = -3.162E\text{-}2 + 2.945E\text{-}1$

$f(0.3) = 2.629E\text{-}1$ from $P_2(x)$

$f(0.3) = 4.163E\text{-}3 - 3.994E\text{-}2 + 2.982E$

$f(0.3) = 2.624E\text{-}1$ from $P_3(x)$

$f(0.5) = -8.783E\text{-}2 + 4.909E\text{-}1$

$f(0.5) = 4.031E\text{-}1$ from $P_2(x)$

$f(0.5) = 1.928E\text{-}2 - 1.1010E\text{-}1 + 4.971E\text{-}1$

$f(0.5) = 4.054E\text{-}1$ from $P_3(x)$

5. $P[x_0\ x_1 \ldots x_k]$ of degree $\leq k$.

$P_1(x)$

$\downarrow$

$P_k(x)$ $\quad P_k(x_i) = P(x_i)\ \forall i = 0, 1 \ldots k$ interpolates $P(x)$ at $x_0 \ldots x_k$

Since interpolating polynomial is unique,

$P_k(x) = P(x)$

$P[x_0\ x_1 \ldots x_k] =$ coefficient of x^k

$P_1(x) = a_0 + a_1 x + \ldots + a_k x^k$

$P^{(k)}(x) = a_k k!$

$\therefore P[x_0 \ldots x_k] = \dfrac{P^{(k)}(x)}{x!}$ independent of interpolation points

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Tutorial 2

2. $f(0) = 1$, $f'(0) = 0.5$, $f''(0) = 2$, $f'''(0) = 3$

Find a polynomial of degree ≤ 3 that agrees with $f(x)$ at 0, 0, 0, 0.

i.e. $p(0) = 1$, $p'(0) = 0.5$, $p''(0) = 2$, $p'''(0) = 3$

$$p(x) = \sum_{j=0}^{n} f^{(j)}(0) \frac{(x-0)^j}{j!}$$

$$p(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6}$$

$$p(x) = 1 + \frac{x}{2} + x^2 + \frac{x^3}{2}$$

$p(0) = 1$

$p'(x) = \frac{1}{2} + 2x + \frac{3x^2}{2}$, $p'(0) = \frac{1}{2}$

$p''(x) = 2 + 3x$, $p''(0) = 2$

$p'''(x) = 3$, $p'''(0) = 3$.

4. $f(z_1) = 0$ $\quad f(z_2) = 0$

$f'(z_1) = 0$ $\quad f'(z_2) = 0$

$f''(z_2) = 0$

Find $p(x)$ with degree ≤ 5 s.t.

$p(z_1) = 0$ $\quad p(z_2) = 0$ $\quad p(z_3) = f(z_3)$

$p'(z_1) = 0$ $\quad p'(z_2) = 0$

$p''(z_2) = 0$

$$p(x) = A(x - z_1)^2 (x - z_2)^3$$

$$p(z_3) = A(z_3 - z_1)^2 (z_3 - z_2)^3 = f(z_3)$$

$$\therefore\ p(x) = \frac{f(z_3)}{(z_3-z_1)^2(z_3-z_2)^3}(x-x_1)^2(x-x_2)^3$$

5. $f(0.2) = 1.987E-1$

$f'(0.2) = 9.801E-1$

$f''(0.2) = -1.987E-1$

$f(0.4) = 3.894E-1$.

Approximate $f(0.3)$.

p agrees with f at $0.2, 0.2, 0.2, 0.4$

$z_0 = 0.2,\ z_1 = 0.2,\ z_2 = 0.2,\ z_3 = 0.4$

	z	$f(z)$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
0	0.2	1.987E-1	9.801E-1	-9.935E-2 ~~+1.987E-1~~	-1.683 ~~5.2485~~ E-1
1	0.2	1.987E-1	9.801E-1	-1.330E-1	
2	0.2	1.987E-1	9.535E-1		
3	0.4	3.894E-1			

$f(z_0) = f(0.2) = 1.987E-1$

$f[z_0,z_1] = f'(0.2) = 9.801E-1$

$f[z_1,z_2] = f'(0.2) = 9.801E-1$

$$f[z_2,z_3] = \frac{3.894E-1 - 1.987E-1}{0.4 - 0.2} = 9.535E-1$$

$$f[z_0,z_1,z_2] = \frac{f''(0.2)}{2} = \cancel{+1.987E-1}\ -9.935E-2$$

$$f[z_1,z_2,z_3] = \frac{f[z_2,z_3] - f[z_1,z_2]}{z_3 - z_1} = \frac{9.535E-1 - 9.801E-1}{0.4 - 0.2}$$

$$= -1.330E-1$$

$$f[z_0,z_1,z_2,z_3] = \frac{f[z_1,z_2,z_3] - f[z_0,z_1,z_2]}{z_3 - z_0}$$

$$= \frac{-1.330E-1 + 9.935E-2}{0.4 - 0.2} = -1.683E-1$$

$$p(x) = f(x_0) + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$p(x) = 1.987E-1 + 9.801E-1(x - 0.2)$$
$$- 9.935E-2(x - 0.2)^2 - 1.683E-1(x - 0.2)^3$$

$$p(0.3) = 1.987E-1 + 9.801E-1 \times 0.1$$
$$- 9.935E-2(0.1)^2 - 1.683E-1(0.1)^3$$

$$= 1.987E-1 + 9.801E-2 - 9.935E-4$$
$$- 1.683E-4$$

$$p(0.3) = 2.955E-1$$

$$\boxed{f(0.3) \approx 2.955E-1}$$

3. $f(0.2) = 1.823\text{ E}-1$

$f'(0.2) = 8.333\text{ E}-1$

$f(0.4) = 3.365\text{ E}-1$

$f'(0.4) = 7.143\text{ E}-1$

Approximate $f(0.3)$.

p agrees with f at 0.2, 0.2, 0.4, 0.4

$z_0 = 0.2$, $z_1 = 0.2$, $z_2 = 0.4$, $z_3 = 0.4$

	z	$f[z]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
0	0.2	1.823 E-1	8.333 E-1	-3.115 E-1	1.400 E-
1	0.2	1.823 E-1	7.710 E-1	-2.835 E-1	
2	0.4	3.365 E-1	7.143 E-1		
3	0.4	3.365 E-1			

$$f[z_0, z_1] = f'(0.2) = 8.333\text{ E}-1$$

$$f[z_1, z_2] = \frac{3.365\text{ E}-1 - 1.823\text{ E}-1}{0.4 - 0.2} = 7.710\text{ E}-1$$

$$f[z_2, z_3] = f'(0.4) = 7.143\text{ E}-1$$

$$f[z_0, z_1, z_2] = \frac{7.710\text{ E}-1 - 8.333\text{ E}-1}{0.4 - 0.2} = -3.115\text{ E}-1$$

$$f[z_1, z_2, z_3] = \frac{7.143\text{ E}-1 - 7.710\text{ E}-1}{0.4 - 0.2} = -2.835\text{ E}-1$$

$$f[z_0, z_1, z_2, z_3] = \frac{-2.835\text{ E}-1 + 3.115\text{ E}-1}{0.4 - 0.2} = 1.400\text{ E}-1$$

$p(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2]$
$(x-x_0)(x-x_1) + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$

$p(x) = 1.823E-1 + 8.333E-1(x-0.2)$
$- 3.115E-1(x-0.2)^2 + 1.400E-1(x-0.2)^2(x-0.4)$

$p(0.3) = 1.823E-1 + 8.333E-1 \times 0.1$
$-3.115E-1(0.1)^2 + 1.400E-1(0.1)^2(-0.1)$

$= 1.823E-1 + 8.333E-2 - 3.115E-3$
$-1.400E-4$

$p(0.3) = 2.624E-1$

$\boxed{f(0.3) \approx 2.624E-1}$

1. $x_0, x_1 \ldots x_m$ not necessarily distinct
$z_1, z_2 \ldots z_r$ distinct from $x_0, x_1 \ldots x_m$ where
each z_i is repeated c_i times.

$$\sum_{i=1}^{r} c_i = m+1$$

$p(x) = f(x) - g(x)$ has degree $\leq m$.
$p(z_1) = 0 \ldots p^{(c_1-1)}(z_1) = 0 \implies (x-z_1)^{c_1} \mid P(x)$
$\vdots \qquad\qquad\qquad\qquad \vdots$
$p(z_r) = 0 \ldots p^{(c_r-1)}(z_r) = 0 \quad (x-z_r)^{c_r} \mid P(x)$

z_i are distinct
$\therefore (x-z_1)^{c_1}(x-z_2)^{c_2} \ldots (x-z_r)^{c_r} \mid P(x)$
$r(x) \mid P(x)$ where $\deg(r(x)) = m+1$
But $\deg(P(x)) \leq m \therefore P(x) = 0 \quad \therefore f(x) = g(x)$

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eg $\int_0^1 e^{-x^2}dx$ $h = 0.25$ $a = x_0 = 0$ $b = x_N = 1$ $f(x) = e^{-x^2}$ $N = \frac{1-0}{0.25} = 4$

(1) $T_N = \frac{0.25}{2}[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)]$

$= 1.250E-1[1.000E0 + 1.879E0 + 1.558E0 + 1.140E0 + 3.679E-1]$

$= 1.250E-1 \times 5.945E0$

$T_N = 7.431E-1$

(2) $S_N = \frac{0.25}{6}[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + 4f(0.125) + 4f(0.375) + 4f(0.625) + f(1)]$

$= 4.167E-2[1.000E0 + 1.879E0 + 1.558E0 + 1.140E0 + 3.938E0 + 3.475E0 + 2.707E0 + 3.679E-1]$

$= 6.694E-1$

(3) $x_0 = -0.7746$ $c_0 = 0.5556$ $f(x_0) = 5.488E-1$

$x_1 = 0$ $c_1 = 0.8889$ $f(x_1) = 1.000E0$

$x_2 = 0.7746$ $c_2 = 0.5556$ $f(x_2) = 5.488E-1$

$\int_0^1 e^{-x^2}dx = \frac{1}{2}\int_{-1}^1 e^{-x^2}dx = \frac{1}{2}[f(-0.7746)\times 0.5556 + f(0)\times 0.8889 + f(0.7746)\times 0.5556]$

$\int_0^1 e^{-x^2}dx = 8.121E-1$

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* How many subdivisions for a given error tolerance?

eg $I = \int_0^1 \sin(x^3)\,dx$

(a) How many subdivisions of $[0,1]$ so that trapezoidal rule gives an error of 10^{-4} or less?

$|E_N^T| \leq 10^{-4}$ $\qquad h = \frac{1}{N}$

$$E_N^T = \left| \frac{f''(\eta)\,h^2(b-a)}{12} \right| = \left| \frac{f''(\eta)\,h^2}{12} \right| \leq 10^{-4}$$

$f(x) = \sin(x^3)$

$f'(x) = 3x^2 \cos(x^3)$

$f''(x) = 6x\cos(x^3) - 9x^4 \sin(x^3)$

$|f''(x)| \leq 15 \quad \forall x \in [0,1]$

$$\left| \frac{f''(\eta)\,h^2}{12} \right| = |f''(\eta)| \frac{h^2}{12} \leq 10^{-4}$$

$$|f''(\eta)| \frac{h^2}{12} \leq \frac{15}{12N^2} \leq 10^{-4}$$

$N^2 \geq 12500$

$N \geq 111.8$

$\therefore N = 112$ will give error $\leq 10^{-4}$

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(b) How many subdivisions of the interval $[0,1]$ for Simpson's rule to give error of 10^{-4} or less?

$h = \frac{1}{N}$, $b-a=1$

$$|E_N^S| \le 10^{-4}$$

$$E_N^S = -\frac{f^{(4)}(\eta)(b-a)}{180}\left(\frac{h}{2}\right)^4$$

$$|E_N^S| = \frac{|f^{(4)}(\eta)|}{2880} \times \frac{1}{N^4}$$

$f(x) = \sin(x^3)$

$f'(x) = 3x^2\cos(x^3)$

$f''(x) = 6x\cos(x^3) - 9x^4\sin(x^3)$

$f'''(x) = 6\cos(x^3) - 18x^3\sin(x^3) - 36x^3\sin(x^3) - 27x^6\cos(x^3)$

$f^{(4)}(x) = -18x^2\sin(x^3) - 54x^2\sin(x^3) - 54x^5\cos(x^3) - 108x^5\cos(x^3) - 108x^2\sin(x^3) - 162x^5\cos(x^3) + 81x^8\sin(x^3)$

$$|f^{(4)}(x)| \le 585 \;\forall x \in [0,1]$$

$$\frac{|f^{(4)}(x)|}{2880} \times \frac{1}{N^4} \le \frac{585}{2880} \times \frac{1}{N^4} \le 10^{-4}$$

$$\therefore N^4 \ge \frac{585 \times 10^4}{2880}$$

$$N \ge 6.713$$

$N=7$ does the job

14 * *Taylor Series Method*

eg. $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$

$g(x) = \cos x$

$P_4(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$

$$\int_0^1 \frac{P_4(x)}{\sqrt{x}}\,dx = \int_0^1 \left(\frac{1}{\sqrt{x}} - \frac{x^{\frac{3}{2}}}{2} + \frac{x^{\frac{7}{2}}}{24}\right)dx$$

$$= \left[2\sqrt{x} - \frac{x^{\frac{5}{2}}}{5} + \frac{x^{\frac{9}{2}}}{108}\right]_0^1$$

$$= 2 - \frac{1}{5} + \frac{1}{108} = 1.809$$

$G(x) = \dfrac{1}{\sqrt{x}}\left(\cos x - P_4(x)\right)$ CS; $N = 2, h = 0.5$

x	$G(x) = \dfrac{\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}}{\sqrt{x}}$
0	0
0.25	~~[illegible]~~ $-6.774E-7$
0.50	$-3.055E-5$
0.75	~~[illegible]~~ $-2.826E-4$
1.00	$-1.364E-3$

$$S_2 = \frac{0.5}{6}\left[f(0) + 2f(0.5) + 4f(0.25) + f(1)\right]$$

$$= 8.333E-2\left[0 - 6.110E-5 - 2.710E-6 - 1.364E-3\right]$$

$S_2 = -1.190E-4$

$$I = \int_0^1 \frac{\cos x}{\sqrt{x}} = 1.809 - 1.190E-4 = \boxed{1.809}$$

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* We need to show that the exact value of an integral lies between its mid-point approximation and its composite trapezoidal approximation given $f''(x) \geq 0 \ \forall \ x \in [a,b]$

$I = \int_a^b f(x)\,dx$ lies between $\alpha = M$

$\beta = T_N$

$$E^M = \frac{f''(\eta)}{2} \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx = \frac{f''(\eta)(b-a)^3}{24}$$

$\forall \eta \in [a,b], \ f''(\eta) \geq 0$

If $b > a$, $E^M \geq 0$

$I(f) = I(P_k) + E(f)$

$I(f) - \alpha \geq 0$

$\therefore I(f) \geq \alpha$

$$E_N^T = -\frac{f''(\eta) h^2 (b-a)}{12}$$

$\forall \eta \in [a,b], \ f''(\eta) \geq 0$

If $b > a$, $E_N^T \leq 0$

$I(f) = I(P_k) + E(f)$

$I - \beta < 0$

$I \leq \beta$

$\therefore \alpha \leq I \leq \beta$

If $b < a$, $\alpha \geq I \geq \beta$

$\therefore I$ is between α and β.

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* Gaussian Quadrature with n nodes is exact for a polynomial of degree $\leq 2n-1$.

Let us consider $f(x) = x^4$. $\deg(f(x)) = 4$

Let us try to approximate $\int_{-1}^{1} f(x)\,dx$ by

Gaussian Quadrature of ~~deg~~ order 2.

Exact: $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x^4\,dx = \left[\frac{x^5}{5}\right]_{-1}^{1} = \frac{2}{5}$

$= 0.4$

Approximate: $\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$= 2.222E-1$ (error!)

Now consider Gaussian Quadrature of order 3.

$\int_{-1}^{1} f(x)\,dx \approx f(-0.7746) \times 0.5556 + f(0) \times 0.8889$
$+ f(0.7746) \times 0.5556$
$= 0.4$ (exact!)