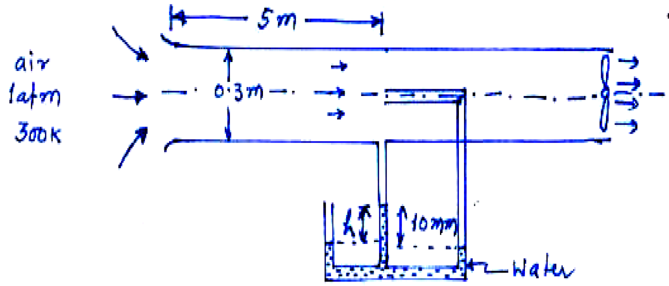
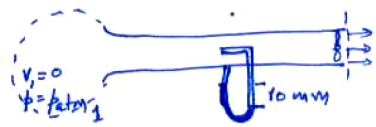
1. A fan draws air at 1 atm. pressure and 300K from a room through a 0.3m diam tube and discharges it at 1 atm. pressure outside. Measurements made give the value as shown in the figure. Friction losses are negligible.
   a) Calculate the average velocity inside the tube.
   b) The value of h
   c) The mass flow rate
   d) the rating of the fan if the fan efficiency is 0.5



Solution:

(a) Pitot tube side manometer gives the dynamic pressure directly (10 mm H_2O column)



$\frac{1}{2}\rho V_{2\,avg}^2 = \Delta p = \rho_w \cdot 9.81 \cdot (0.01) = 98.1$; $\rho_{air} \approx 1.18$

[Change is expected to be very small).

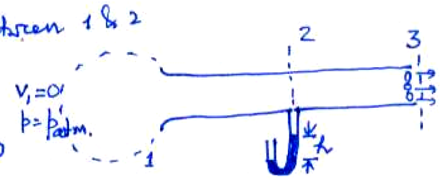
$V_2 = 12.895$ m/s

(b) Now apply Bernoulli eqn. between 1 & 2

we know: $\hat{W}_s = 0$, $\hat{w}_{fr} = 0$ (given)

$z_2 = z_1$, $V_1 \approx 0$.

Assume incompressible: $\frac{\Delta p}{p} \ll 0$



$$\frac{p_2 - p_1}{\rho_{air}} + \frac{V_2^2}{2} = 0$$

$$p_2 - p_1 = \frac{\rho_{air} V_2^2}{2} = 98.1 = \rho_w \times 9.81 \times h \Rightarrow \underline{h = 10\,mm}$$

[In a frictionless flow, total pressure $p + \frac{1}{2}\rho V^2$ remains constant].

(c) Mass flow rate $= \frac{\pi d^2}{4} \cdot \bar{V}_2 \cdot \rho_2 \approx 1.0756$ kg/s.

Apply Bernoulli's eqn. between 1 & 3 : $p_3 = p_1 = p_{atmospheric}$

$$\frac{V_3^2}{2} - \frac{V_1^2}{2} + \frac{p_3 - p_1}{\rho} + \hat{W}_s = 0 \Rightarrow -\hat{W}_s = \frac{V_3^2}{2} = \frac{V_2^2}{2} = 83.14$$

$$-W_s = -\hat{W}_s \cdot \dot{m} = 83.14 \times 1.0756 = 89.425\,W$$

$$\text{Rating of the fan} = \frac{-W_s}{\eta} = \frac{89.425}{0.5} = \underline{178.9\,W}$$

(d) If the pressure at '2' is still fairly close to atmospheric pressure, (verify)

$\rho_{air} \approx 1.18$, & V_2 remains at 12.895.

Apply Bernoulli between ① and ②:

$$\frac{V_2^2}{2} - \frac{V_1^2}{2} + \frac{p_2 - p_1}{\rho_{av}} + \hat{w}_{fr} = 0 \Rightarrow \frac{p_1 - p_2}{\rho_{av}} = \frac{12.895^2}{2} + 8 \times 5 = 123.14$$

$$p_1 - p_2 = 123.14 \times 1.18 \approx 145.3 = \rho_w \cdot 9.81 \cdot h \Rightarrow \underline{h = 14.8\,mm}$$

[$\rho_2 \approx \rho_1$ is valid].

2.4 FULLY DEVELOPED FLOW THROUGH A CIRCULAR TUBE

In this section we derive the momentum balance for steady flow through a long cylindrical tube for a Newtonian fluid and then for a non-Newtonian fluid, using an empirical equation that is often applied to polymeric melts.

2.4.1 Newtonian Fluids

Consider the fully developed flow of a fluid in a long tube of length L and radius R; we specify fully developed flow so that end effects are negligible. Since we are dealing with a pipe, it is convenient to work with cylindrical coordinates. Therefore the shell in Fig. 2.3 is cylindrical, of thickness Δr and length L.

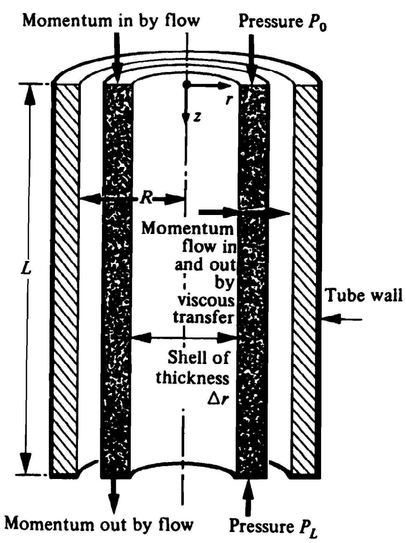


Fig. 2.3 Cylindrical shell chosen for momentum balance in tubes.

Rate of momentum in across surface at r (due to viscosity)	$(2\pi r L \tau_{rz})\|_r$

Note that here we include the area factor $(2\pi rL)$ in parentheses. This is because the area as well as the shear stress is a function of r.

Rate of momentum out across surface at $r + \Delta r$ (due to viscosity)	$(2\pi r L \tau_{rz})\|_{r+\Delta r}$

Since we are considering fully developed flow, the momentum fluxes due to flow are equal; hence these terms are omitted.

Gravity force acting on the cylindrical shell	$(2\pi r \Delta r L)\rho g$

Pressure force acting on surface at $z = 0$ $(2\pi r\Delta r)P_0$

Pressure force acting on surface at $z = L$ $-(2\pi r\Delta r)P_L$

We now add up the contributions to the momentum balance:

$$\left(2\pi rL\tau_{rz}\right)\big|_r - \left(2\pi rL\tau_{rz}\right)\big|_{r+\Delta r} + 2\pi r\Delta rL\rho g + 2\pi r\Delta r\left(P_0 - P_L\right) = 0. \tag{2.24}$$

Note that all terms contain the factor r; however, since r is a variable, it should not be used as a common divisor. By dividing through by $2\pi L\Delta r$ and taking the limit as Δr goes to zero, we develop the differential equation

$$\frac{d}{dr}\left(r\tau_{rz}\right) = \left[\frac{P_0 - P_L}{L} + \rho g\right] r. \tag{2.25}$$

Integration yields

$$\tau_{rz} = \left[\frac{P_0 - P_L}{L} + \rho g\right]\frac{r}{2} + \frac{C_1}{r}. \tag{2.26}$$

At $r = 0$, the velocity gradient (hence, the shear stress) equals zero; this can be realized because of the symmetry of flow.

Thus for this case,

$$\text{B.C. 1} \qquad \text{at } r = 0, \qquad \tau_{rz} = 0. \tag{2.27}$$

Therefore, $C_1 = 0$, and the momentum flux is given by

$$\tau_{rz} = \left[\frac{P_0 - P_L}{L} + \rho g\right]\frac{r}{2}. \tag{2.28}$$

Substituting Newton's law of viscosity

$$\tau_{rz} = -\eta\frac{dv_z}{dr}, \tag{2.29}$$

and noting

$$\text{B.C. 2} \qquad \text{at } r = R, \qquad v_z = 0, \tag{2.30}$$

we obtain the solution for the velocity distribution:

$$v_z = \left[\frac{P_0 - P_L}{L} + \rho g\right]\left[\frac{R^2}{4\eta}\right]\left[1 - \left[\frac{r}{R}\right]^2\right]. \tag{2.31}$$

As before:

i) The maximum velocity is at $r = 0$, and is given by

$$V_z^{\max} = \left[\frac{P_0 - P_L}{L} + \rho g\right]\frac{R^2}{4\eta}. \tag{2.32}$$

ii) The average velocity is

$$\bar{v}_z = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R v_z r \, dr \, d\theta = \left[\frac{P_0 - P_L}{L} + \rho g \right] \frac{R^2}{8\eta}. \tag{2.33}$$

iii) The volume flow rate is

$$Q = \left[\frac{P_0 - P_L}{L} + \rho g \right] \left[\frac{\pi R^4}{8\eta} \right]. \tag{2.34}$$

This latter result, which is commonly referred to as the *Hagen-Poiseuille law*, is valid for laminar steady-state flow of incompressible fluids in tubes having sufficient length to make end effects negligible. An entrance length given by $L_e = 0.035\,D\mathrm{Re}$ is required before we can establish fully developed parabolic velocity distribution.

Example 2.2 Water at 290 K flows through a horizontal tube of diameter 1.6 mm with a pressure drop of 900 N m^{-3}. Find the mass flow rate through the tube.

Solution. In this situation, the force of gravity does not act on the fluid in the direction of flow, so according to Eq. (2.34) the volume flow rate is

$$Q = \left[\frac{P_0 - P_L}{L} \right] \frac{\pi R^4}{8\eta}.$$

The viscosity and density of water at 290 K are 1.080×10^{-3} N s m^{-2} and 10^3 kg m^{-3}, respectively. Substituting in values, we obtain:

$$Q = \frac{900 \text{ N}}{\text{m}^3} \left| \frac{\pi}{8} \right| \frac{(0.8 \times 10^{-3})^4 \text{ m}^4}{} \left| \frac{\text{m}^2}{1.080 \times 10^{-3} \text{ N s}} \right. = 1.34 \times 10^{-7} \text{ m}^3 \text{ s}^{-1}$$

Thus we see that the mass flow rate is

$$\rho Q = (10^3)(1.34 \times 10^{-7}) = 1.34 \times 10^{-4} \text{ kg s}^{-1}$$

We should then check if the flow is laminar, by evaluating the Reynolds number. As mentioned in Chapter 1, the criterion is Re < 2100:

$$\mathrm{Re} = \frac{D\bar{V}}{\nu} = \frac{D\bar{V}\rho}{\eta}.$$

Also

$$\rho \bar{V} = \frac{\rho Q}{\pi D^2/4}.$$