Numerical Analysis : [MA214]
Lecture 10

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Recall

Last time we discussed some iteration methods to find roots.

1. **Bisection method:** a_n, b_n such that $f(a_n)f(b_n) < 0$

$$w = \frac{a_n + b_n}{2}$$

   If $f(a_n)f(w) < 0$, then set $a_{n+1} = a_n$ and $b_{n+1} = w$
   Otherwise set $a_{n+1} = w$ and $b_{n+1} = b_n$
   Then root lies in the interval $[a_{n+1}, b_{n+1}]$

2. Regula falsi method
3. Modified Regula falsi method
4. Secant method
5. Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0$

Newton's method converges if x_0 is "close" to the root.

Fixed point iteration:-

Newton's method is a form of fixed point iteration.

$$g : I \to I, \quad I = [a, b]$$

$$x_0 \in I, \quad x_{n+1} = g(x_n)$$

Conditions for convergence for fixed point iteration

1. $g : I \to I, \quad I = [a, b]$.
2. g is continuous.

   (1) and (2) ensure that g has a fixed point but still iteration may not converge.
3. $|g'(x)| \leq K$, for some $K < 1$, for all $x \in I$.

(3) ensures convergence of fixed point convergence.

It also ensures that there is a unique fixed point for g

Linear convergence of fixed point convergence

$$g : I \to I, \quad I = [a, b]$$

g is continuous. $|g'(x)| \leq K < 1$, for all $x \in I$.

ξ is a unique fixed point of g. Assume $g'(\xi) \neq 0$. $x_0 \in I$ be any point.

$$x_1, x_2, \cdots, x_n, \cdots \quad x_n = g(x_{n-1})$$

$$\begin{aligned}
e_n &= \xi - x_n \\
e_{n+1} &= \xi - x_{n+1} \\
&= g(\xi) - g(x_n) \\
&= g'(\eta_n)(\xi - x_n), \quad \eta_n \text{ between } \xi \text{ and } x_{n+1} \\
e_{n+1} &= g'(\eta_n) e_n
\end{aligned}$$

$$\lim_{n\to\infty} x_n = \xi \quad \Longrightarrow \quad \lim_{n\to\infty} \eta_n = \xi$$

$$\text{So} \quad \lim_{n\to\infty} g'(\eta_n) = g'(\xi)$$

$$g'(\eta_n) \approx g'(\xi) \qquad \text{for } n\text{-large}$$

$$\text{So} \quad e_{n+1} \approx g'(\xi) e_n \qquad \text{for } n\text{-large}$$

Thus the error in the $(n+1)^{st}$ iterate depends (more or less) linearly on the error e_n in the n^{th} iterate.

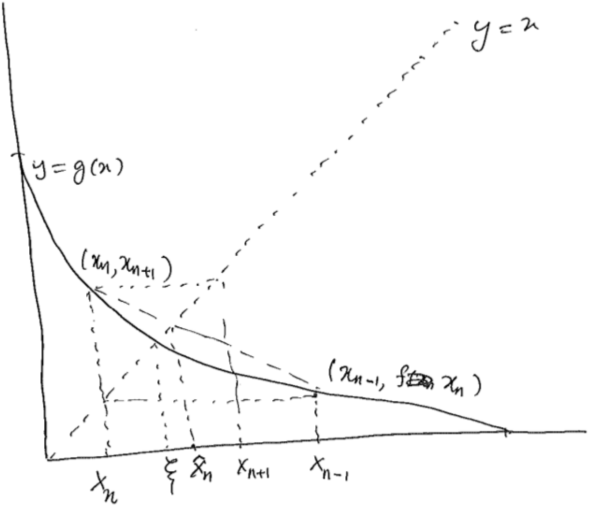
Accelerating fixed point convergence

Fixed point iteration at times requires many iterations.

There is a standard method to accelerate linear convergence sequence.

It is called **Aitken's $\triangle^2$-process**.

Graphical justification



$\hat{X}_n$ is closer to ξ, than X_n, X_{n-1}, X_{n+1}

Determination of $\hat{X}_n$

Parametric form of line segment joining (x_n, x_{n+1}) and (x_{n-1}, x_n).

$$r(t) = (x_n, x_{n+1}) + t(x_{n-1} - x_n, x_n - x_{n+1}), \quad t \in [0, 1]$$

$$\text{determine } t, \quad \text{when } r(t) = (\hat{x}_n, \hat{x}_n)$$

$$x_n + t(x_{n-1} - x_n) = x_{n+1} + t(x_n - x_{n+1})$$

$$x_n - x_{n+1} = t(2x_n - x_{n+1} - x_{n-1})$$

$$t = \frac{x_n - x_{n+1}}{2x_n - x_{n+1} - x_{n-1}}$$

$$t = \frac{x_{n+1} - x_n}{x_{n+1} - 2x_n + x_{n-1}}$$

So point

$$\hat{X}_n = x_{n+1} - \frac{(x_{n+1} - x_n)^2}{x_{n+1} - 2x_n + x_{n-1}}$$

For any sequence $\{y_n\}$

$$\begin{aligned} \triangle y_n &= y_{n+1} - y_n \\ \triangle^2 y_n &= \triangle y_{n+1} - \triangle y_n \\ &= y_{n+2} - 2y_{n+1} + y_n \end{aligned}$$

$$\text{Thus} \quad \hat{X}_n = X_{n+1} - \frac{(\triangle X_n)^2}{\triangle^2 X_{n-1}}$$

This is called **Aitken's $\triangle^2$-process**.

This can be applied to any linear convergent sequence and not just coming from fixed point iteration.

Algorithm:-
Given a sequence $x_0, x_1, x_2, \cdots$ converging to ξ, calculate $\hat{x_1}, \hat{x_2}, \cdots, \hat{x_n}, \cdots$ by

$$\hat{x_n} = x_{n+1} - \frac{(\triangle x_n)^2}{\triangle^2 x_{n-1}}$$

If the sequence $x_0, x_1, x_2, \cdots, x_n, \cdots$ converges linearly to ξ that is if

$$\xi - x_{n+1} = K(\xi - x_n) + \theta(\xi - x_n), \quad \text{for some } K \neq 0$$

$$\text{then} \quad \hat{x_n} = \xi + O(\xi - x_n)$$

$$\left[i.e. \quad \frac{\hat{x_n} - \xi}{x_n - \xi} \longrightarrow 0 \right]$$

Example:- $g(x) = \frac{1}{2}e^{\frac{x}{2}}, \quad x \in [0,1]$

$$\begin{array}{rcl|rcl}
x_0 &=& 0 & & & \\
x_1 &=& 0.5 & \hat{x}_1 &=& 0.698349 \\
x_2 &=& 0.642013 & \hat{x}_2 &=& 0.712809 \\
x_3 &=& 0.689257 & \hat{x}_3 &=& 0.714556 \\
x_4 &=& 0.705733 & \hat{x}_4 &=& 0.714772 \\
x_5 &=& 0.711570 & \hat{x}_5 &=& 0.714804 \\
x_6 &=& 0.713651 & \hat{x}_6 &=& 0.714806 \\
x_7 &=& 0.714393 & & & \\
x_8 &=& 0.714658 & & & \\
x_9 &=& 0.714753 & & & \\
x_{10} &=& 0.714787 & & & \\
x_{11} &=& 0.714799 & & & \\
x_{12} &=& 0.714804 & & & \\
x_{13} &=& 0.714805 & & & \\
x_{14} &=& 0.714806 & & & \\
x_n &=& x_{14} \quad \text{for } n \geq 14 & & &
\end{array}$$

For fixed point iteration

If $\hat{x}_n$ is a better approximation to ξ than x_n then it is certainly wasteful to continue generating x_{n+2}, x_{n+3}.

It seems more reasonable to start fixed point iteration afresh with $\hat{x}_n$ as the initial given.

Steffensen iteration:-

Given the iteration function $g(x)$ and a point y_0

For $n = 0, 1, 2, \cdots$ until satisfied do

$$x_0 = y_0$$

$$\begin{array}{ll} \text{Calculate} & x_1 = g(x_0), \quad x_2 = g(x_1) \\ \text{Calculate} & \triangle x_1 \quad \text{and} \quad \triangle^2 x_0 \\ \text{Calculate} & y_{n+1} = x_2 - \dfrac{(\triangle x_1)^2}{\triangle^2 x_0} \end{array}$$

Example:- $g(x) = \frac{1}{2}e^{\frac{x}{2}}, \quad x \in [0,1]$

$$\begin{aligned} y_0 &= 0 \\ x_0 &= 0 \\ x_1 &= 0.5 \\ x_2 &= 0.642013 \\ y_1 &= 0.698349 \\ x_0 &= 0.698349 \\ x_1 &= 0.708948 \\ x_2 &= 0.712715 \\ y_2 &= 0.714792 \\ x_0 &= 0.714792 \\ x_1 &= 0.714801 \\ x_2 &= 0.714804 \\ y_3 &= 0.714806 \end{aligned}$$

(We have done 6 function calculation and 3 Aitken $\triangle^2$ computation.)

Suppose $f(x)$ is continuously twice differentiable and $f(\xi) = 0$, $f'(\xi) \neq 0$

Iteration function of Newton's Method

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ g'(x) &= \frac{f(x)f''(x)}{(f'(x))^2} \\ g'(\xi) &= 0 \end{aligned}$$

Note there exists $\epsilon > 0$ such that on $|x - \xi| \leq \epsilon$

$$|g'(x)| \leq K < 1$$

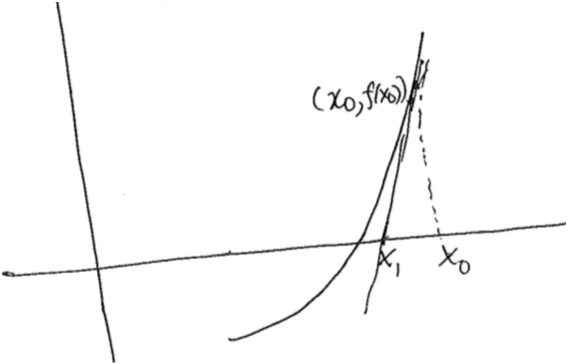
$$I = [\xi - \epsilon, \xi + \epsilon], \quad |x - \xi| \leq \epsilon$$

$$\begin{aligned} |g(x) - \xi| &= |g(x) - g(\xi)| \\ &= |g'(\eta)||x - \xi|, \quad \eta \text{ between } x \text{ and } \xi \\ &= K|x - \xi| \\ &\leq \epsilon \\ g(x) &\in I \\ g : I \longrightarrow I & \text{ is continuous} \\ I &= [\xi - \epsilon, \xi + \epsilon] \\ |g'(x)| \leq K < 1, \quad \text{for all } x &\in I \end{aligned}$$

So fixed point iteration converge.

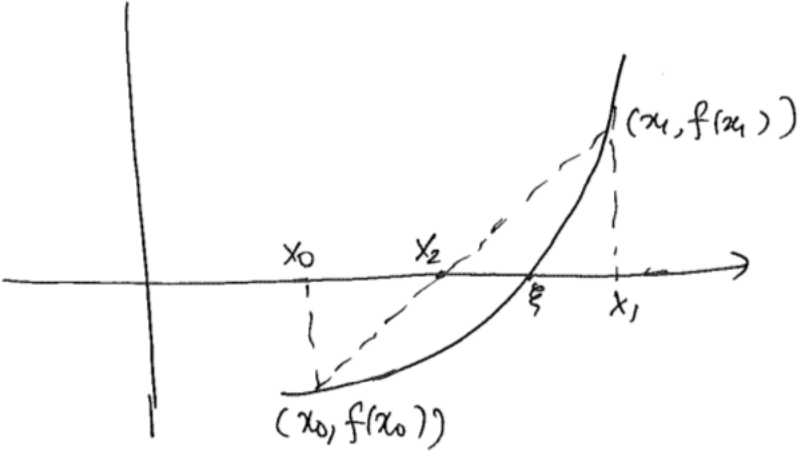
Thus Newton's method converges for points sufficiently close to the root.

Newton's Method



We draw tangent at $(x_0, f(x_0))$ and see where it hits x-axis.

Secant Method



Both interpolates $f(x)$ at two points α and β by a straight line

$$p(x) = f(\alpha) + f[\alpha, \beta](x - \alpha)$$

$$\text{whose zero} \quad \hat{\xi} = \alpha - \frac{f(\alpha)}{f[\alpha, \beta]}$$

is taken as the next approximation to the actual zero of $f(x)$.

In Secant method we take $\alpha = x_n$, $\beta = x_{n-1}$ and then produce $\hat{\xi} = x_{n+1}$

In Newton's method we take $\alpha = \beta = x_n$ and so $\hat{\xi} = x_{n+1}$

We know that

$$f(x) = f(\alpha) + f[\alpha, \beta](x - \alpha) + f[\alpha, \beta, x](x - \alpha)(x - \beta)$$

This holds for all x

So for $x = \xi$, $f(\xi) = 0$

$$
\begin{aligned}
0 = f(\xi) &= f(\alpha) + f[\alpha,\beta](\xi-\alpha) + f[\alpha,\beta,\xi](\xi-\alpha)(\xi-\beta) \\
f[\alpha,\beta](\xi-\alpha) &= -f(\alpha) - f[\alpha,\beta,\xi](\xi-\alpha)(\xi-\beta) \\
\xi &= \alpha - \frac{f(\alpha)}{f[\alpha,\beta]} - \frac{f[\alpha,\beta,\xi]}{f[\alpha,\beta]}(\xi-\alpha)(\xi-\beta) \\
\hat{\xi} &= \alpha - \frac{f(\alpha)}{f[\alpha,\beta]} \\
\text{So } \ \xi &= \hat{\xi} - \frac{f[\alpha,\beta,\xi]}{f[\alpha,\beta]}(\xi-\alpha)(\xi-\beta)
\end{aligned}
$$

This equation can now be used to obtain error bounds for Newton and Secant Method.

Error bound for Newton's Method

$$\alpha = \beta = x_n, \quad \hat{\xi} = x_{n+1}$$

$$e_j = \xi - x_j \;\; \text{error at the } j^{th} \text{ iteration}$$

We obtain

$$e_{n+1} = -\frac{f[x_n, x_n, \xi]}{f[x_n, x_n]} e_n^2$$

Recall $f[x_n, x_n = f'(x_n)]$ and

$$f[x_n, x_n, \xi] = \frac{1}{2} f''(\eta_n)$$

for some η_n between x_n and ξ.
Thus

$$e_{n+1} = -\frac{1}{2} \frac{f''(\eta_n)}{f'(x_n)} e_n^2$$

This shows that Newton's Method converges quadratically.

Error bound for Secant Method

Set $\alpha = x_n, \ \beta = x_{n-1}, \ e_j = \xi - x_j$ error at the j^{th} stage.

$$e_{n+1} = -\frac{f[x_{n-1}, x_n, \xi]}{f[x_{n-1}, x_n]} e_n e_{n-1}$$

Error in the $(n+1)^{st}$ stage is proportional to the product of the error in the n^{th} and $(n-1)^{st}$ stage.

$$f[x_{n-1}, x_n, \xi] = \frac{1}{2} f''(\delta_n)$$

$$f[x_{n-1}, x_n,] = f'(\eta_n)$$

for some η_n, δ_n between x_n, x_{n-1} and ξ.

$$e_{n+1} \approx -\frac{1}{2}\frac{f''(\xi)}{f'(\xi)} e_n e_{n-1}, \quad \text{for} \quad n >> 0$$

Definition (Order of Convergence)

Let $x_0, x_1, \cdots, x_n, \cdots$ be a sequence which converges to a number ξ and set $e_n = \xi - x_n$.
If there exists a number P and a constant $C \neq 0$ such that

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^P} = C$$

then P is called **order of convergence** and C is called **asymptotic error constant**.

Examples:-

1. Fixed point iteration

Let ξ be fixed point of g. Assume $g'(\xi) \neq 0$

$$e_{n+1} \approx e_n g'(\xi)$$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = |g'(\xi)|$$

So order of convergence is 1 and the asymptotic error constant is $|g'(\xi)|$.

2. For Newton's Method

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \frac{1}{2}\frac{|f''(\xi)|}{|f'(\xi)|}$$

Provided $f'(\xi) \neq 0$.

So order of convergence is 2 and the asymptotic error constant is $\frac{1}{2}\frac{|f''(\xi)|}{|f'(\xi)|}$.
3. Secant Method We get from our previous calculation

$$|e_{n+1}| = C_n |e_n||e_{n+1}| \tag{1}$$

with

$$\lim_{n\to\infty} C_n = C_\infty = \frac{1}{2}\frac{|f''(\xi)|}{|f'(\xi)|}$$

(We are assuming $f'(\xi) \neq 0$)

We seek a number P such that

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^P} = C$$

for some non-zero constant C.
From equation (1)

$$\frac{|e_{n+1}|}{|e_n|^P} = C_n|e_n|^{1-P}|e_{n-1}| = C_n\left(\frac{|e_n|}{|e_{n-1}|^P}\right)^{\alpha} \tag{2}$$

Provided $\alpha = 1 - P$ and $\alpha P = -1$, $i.e.$ provided $P - P^2 = -1$

The equation $P^2 - P - 1 = 0$ has the simple positive root

$$P = \frac{1+\sqrt{5}}{2} = 1.618$$

With this choice of P and of $\alpha = 1 - P$ equation (2) defines a "fixed-point like iteration".

$$y_{n+1} = C_n y_n^{-\frac{1}{P}}$$

$$y_{n+1} = \frac{|e_{n+1}|}{|e_n|^P} \text{ and } \lim_{n\to\infty} C_n = C_\infty$$

It follows that y_n converges to the fixed point of the equation

$$x = C_\infty x^{-\frac{1}{P}} \implies x^{1+\frac{1}{P}} = C_\infty$$

$$\text{So } x = C_\infty^{\frac{1}{P}}, \qquad \text{since } 1 + \frac{1}{P} = P$$

$$\text{Thus } \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^P} = \left|\frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\right|^{1/P}$$

Thus order of convergence of such method $= P =$ positive root of $P^2 - P - 1 = 1.618$ and

asymptotic error constant $\left|\frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\right|^{1/P}$

Conergence of Newton's Method at double roots

Recall that when we said that Newton's method converges quadratically, we assumed $f'(\xi) \neq 0$, ($i.e.$ ξ is a simple root of $f(x)$).

What happens if f is a double root?

$$i.e., \quad f(\xi) = 0 = f'(\xi), \quad f''(\xi) \neq 0$$

Let $g(x) = x - \frac{f(x)}{f'(x)}$ be the iteration function of Newton's Method

$$g'(x) = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$
\begin{aligned}
\lim_{x\to\xi} g'(x) &= \lim_{x\to\xi} \frac{f(x)}{(f'(x))^2} \lim_{x\to\xi} f''(x) \\
&= \left(\lim_{x\to\xi} \frac{f(x)}{(f'(x))^2} \right) f''(\xi) \\
&= \quad \uparrow \\
&= \text{Apply L'Hospitals rule} \\
&= \left(\lim_{x\to\xi} \frac{f'(x)}{2f'(x)f''(x)} \right) f''(\xi) \\
&= \frac{1}{2f''(\xi)} f''(\xi) \\
&= \frac{1}{2}
\end{aligned}
$$

So

$$
g'(\xi) = \frac{1}{2} \neq 0
$$

Thus the fixed point iteration is linear.

Recall that Newton's method will converge if the initial approximation x_0 is "close-enough" to the root ξ.

The phrase "close-enough" is useful and many times Newton's iteration will not converge or it will converge to another zero than the one being sought.

It would be desirable to establish condition's which guarantee convergence of Newton's method for any choice of the initial iterate in a given interval.

One such set if condition is contained in the following theorem (see next slide).

Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ be twice continuously differentiable on the interval $[a, b]$ and let the following conditions be satisfied

1. $f(a)f(b) < 0$
2. $f'(x) \neq 0,$ *for all* $x \in [a, b]$
3. $f''(x)$ *is either* ≥ 0 *or* ≤ 0 *for all* $x \in [a, b]$
4. *At the end points*

$$\frac{|f(a)|}{|f'(a)|} \leq b - a, \quad \frac{|f(b)|}{|f'(b)|} \leq b - a$$

Then the Newton's method converges to the unique solution ξ of $f(x) = 0$ in $[a, b]$ for any choice $x_0 \in [a, b]$.

(1) ensures that f has a zero in $[a, b]$

(2) ensures that f has a unique zero in $[a, b]$

(3) $\implies$ graph of $f(x)$ is either concave from above or concave from below.

Furthermore together with (2) implies that $f'(x)$ is monotone.

(4) states that tangent to the curve at either end point intersects the x-axis within the interval $[a, b]$.

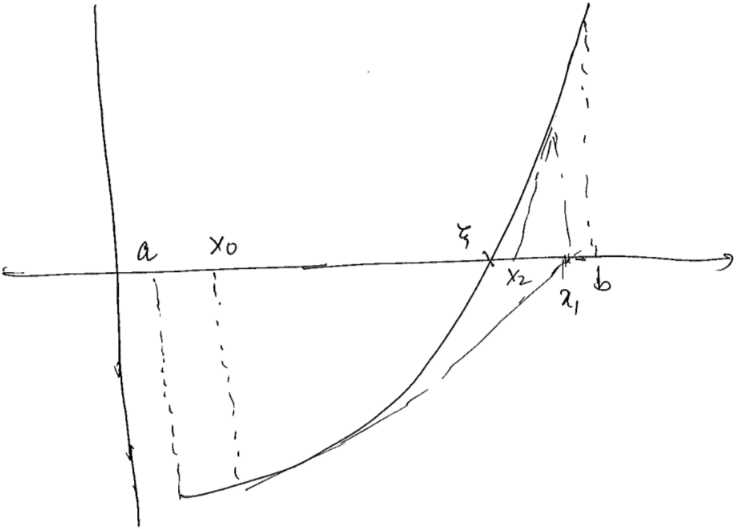
Sketch of proof of theorem

We assume without loss of generality $f(a) < 0$.

We then distinguish two cases
case (i) $f''(x) \geq 0$
case (ii) $f''(x) \leq 0$

Case(ii) reduces to case (i) if we replace f by $-f$.

It therefore suffices to consider case (i). Graph of $f(x)$ looks like



From the graph it is evident that for $x_0 > \xi$ the resulting iterate decrease monotonically to ξ.

While for $a \leq x_0 \leq \xi$, x_1 falls between ξ and b.

Then the subsequent iterate converges monotonically to ξ.