Minimum cost spanning r-arborescence

Combinatorial Optimization

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The problem

Problem data:

- a digraph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$,
- a node $r \in \mathcal{N}$,
- a cost function $c : \mathcal{A} \to \Re_+$.

Problem (Minimum Cost Spanning r-Arborescence Problem). Find a spanning r-arborescence of minimum cost.

A digraph $\mathcal{T} = (\mathcal{N}, \mathcal{A})$ is a spanning rooted arborescence (r-arborescence, for short) if and only if there is a unique directed path from its root node $r \in \mathcal{N}$ to all the other nodes in $\mathcal{N} \setminus \{r\}$ and no directed path from any node in $\mathcal{N} \setminus \{r\}$ to r.

Counter-example

The algorithms for the MSTP do not work.

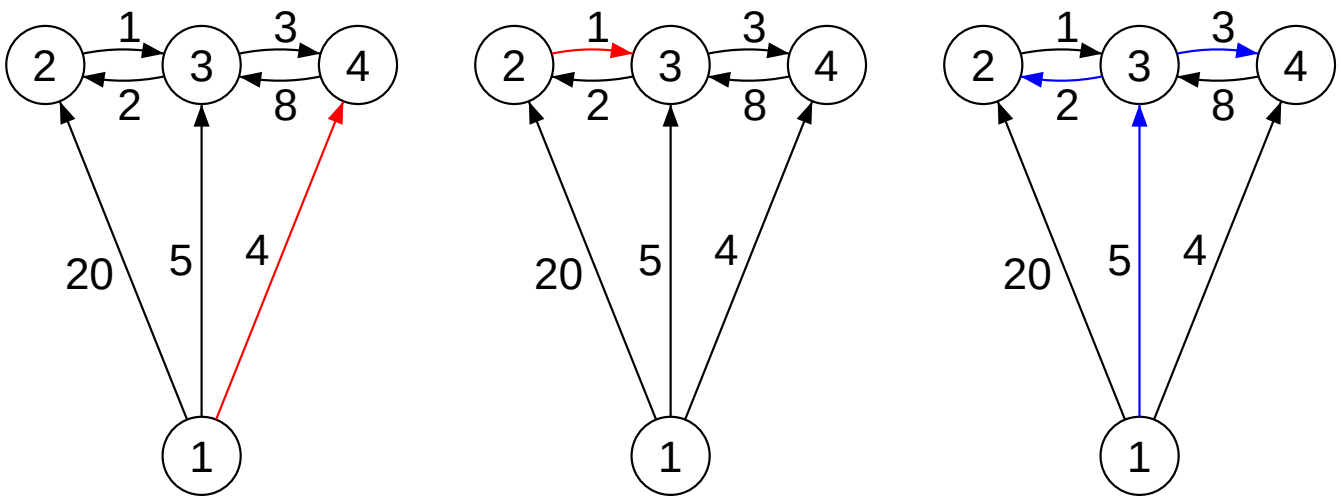


Figure: Prim algorithm would select $(1, 4)$ first. Kruskal algorithm would select $(2, 3)$ first. None of them belongs to the optimal solution.

A mathematical programming model

$$\min z = \sum_{a \in \mathcal{A}} c_a x_a$$

$$\text{s.t.} \sum_{a \in \delta(S)} x_a \geq 1 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\} \qquad (1)$$

$$x_a \in \{0,1\} \qquad \forall a \in \mathcal{A}$$

A binary variable x_a indicates whether each arc $a \in \mathcal{A}$ belongs to the solution or not.
We call r-cuts all arc subsets $\delta(S)$ corresponding to all node subsets S not containing r, made of the arcs entering S:

$$\delta(S) = \{(i,j) \in \mathcal{A} : i \notin \mathcal{A} \vee j \in \mathcal{A}\} \;\; \forall S \subset \mathcal{N} : r \notin S$$

Constraints (1) impose that all such subsets be reached by at least one entering arc.

Since the constraint matrix is totally unimodular and the rhs are integer, the integrality conditions are redundant.

A mathematical programming model

$$\min z = \sum_{a \in \mathcal{A}} c_a x_a$$

$$\text{s.t.} \sum_{a \in \delta(S)} x_a \geq 1 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

$$0 \leq x_a \leq 1 \qquad \forall a \in \mathcal{A}$$

The bounds $x_a \geq 1$ are redundant too, because no constraint may require any x variable to take value larger than 1, since all terms of the sums $\sum_{a \in \delta(S)} x_a$ are non-negative and the right-hand-side is equal to 1.

The primal-dual pair

The primal problem.

$$\min z = \sum_{a \in \mathcal{A}} c_a x_a$$

$$\text{s.t.} \sum_{a \in \delta(S)} x_a \geq 1 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

$$x_a \geq 0 \qquad \forall a \in \mathcal{A}$$

The dual problem.

$$\max w = \sum_{S \subseteq \mathcal{N} \setminus \{r\}} y_S$$

$$\text{s.t.} \sum_{S \subseteq \mathcal{N} \setminus \{r\} : a \in \delta(S)} y_S \leq c_a \qquad \forall a \in \mathcal{A}$$

$$y_S \geq 0 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

Complementary slackness conditions

Primal constraints and dual variables correspond to subsets S non including r, i.e. to r-cuts.

Primal C.S.C.: $x_a(c_a - \sum_{S \subseteq \mathcal{N} \setminus \{r\} : a \in \delta(S)} y_S) = 0 \quad \forall a \in \mathcal{A}$

Dual C.S.C.: $y_S(\sum_{a \in \delta(S)} x_a - 1) = 0 \quad \forall S \subseteq \mathcal{N} \setminus \{r\}$

The initial primal solution is $x_a = 0 \; \forall a \in \mathcal{A}$ and it is primal infeasible (and super-optimal).

The corresponding dual solution is $y_S = 0 \; \forall S \subseteq \mathcal{N} \setminus \{r\}$ and it is dual feasible (and sub-optimal).

Complementary slackness conditions

Owing to the constraints

$$\sum_{a \in \delta(S)} x_a \geq 1 \quad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

primal infeasibility is measured by the number of unreached node subsets (SCCs).

Owing to the primal C.S.C.

$$x_a (c_a - \sum_{S \subseteq \mathcal{N} \setminus \{r\} : a \in \delta^{in}(S)} y_S) = 0 \quad \forall a \in \mathcal{A}$$

the only arcs that can be used to traverse r-cuts are those with null reduced cost:

$$x_a = 1 \Rightarrow \overline{c}_a = 0$$

where $\overline{c}_a = c_a - \sum_{S \subseteq \mathcal{N} \setminus \{r\} : a \in \delta(S)} y_S$.
We call them admissible arcs.

Edmonds algorithm

This algorithm is due to Chu and Liu (1965), Edmonds (1967), Bock (1971). It is a primal-dual algorithm and iteratively it does the following.

- Dual iteration: a violated primal constraint is selected. It corresponds to an r-cut which does not contain any basic arc, so that its correspondent SCC S is not reachable from r through basic arcs. The corresponding dual variable y_S enters the dual basis: it is increased (dual ascent procedure) until it activates one or more dual constraints corresponding to one or more primal variables x_a (arcs). Such arcs get zero reduced cost and therefore they become admissible.
- Primal iteration: a primal variable x_a corresponding to an admissible arc enters the primal basis: it is increased in order to repair the infeasibility of the selected primal constraint. This means that an r-cut is now traversed by the selected basic arc a.

If a cut contains more than one admissible arc, any of them (arbitrarily chosen) can become basic.

Edmonds algorithm: correctness (1)

Dual iterations.

Dual feasibility requires $\sum_{S \subseteq N \setminus \{r\}: a \in \delta(S)} y_S \leq c_a \ \ \forall a \in A$. Therefore, when a dual variable y_S enters the basis it takes a value equal to the minimum reduced cost among those of the arcs in the r-cut $\delta(S)$.
In this way dual feasibility is always kept.

Primal iterations.

Primal feasibility requires $\sum_{a \in \delta(S)} x_a \geq 1 \ \ \forall S \subseteq \mathcal{N} \setminus \{r\}$. Therefore, when a primal variable enters the basis, it takes value 1, which is the minimum amount needed to satisfy at least one more primal constraint.
In this way primal feasibility is iteratively achieved.

Since the C.S.C. are satisfied at each iteration, when primal feasibility is achieved the dual solution is optimal.

Edmonds algorithm: correctness (2)

To find a violated primal constraint, we need to find a SCC, different from $\{r\}$, which is not yet reachable from r through basic arcs.

If some basic arcs form a SCC, then reaching any node in it makes all the others reachable.

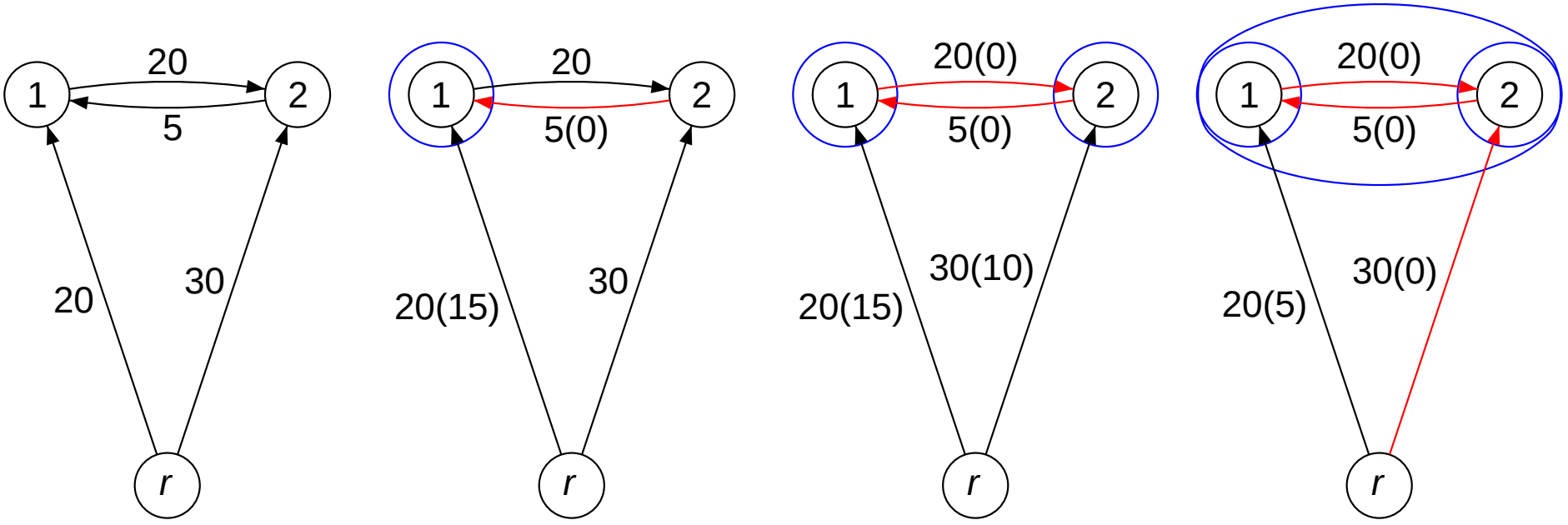
Therefore we consider the subgraph $\mathcal{D}_B = \{\mathcal{N}, \mathcal{B}\}$, where $\mathcal{B}$ is the set of admissible arcs.

If $\mathcal{D}_B$ contains a spanning r-arborescence $\mathcal{T}$, then $\mathcal{T}$ is a minimum cost spanning r-arborescence and the algorithm stops. Otherwise, there is a SCC S in $\mathcal{D}_B$ such that

- $r \notin S$
- $\overline{c}_a > 0 \quad \forall a \in \delta(S)$.

Any SCC satisfying these properties can be chosen for the next iteration.

Example



To retrieve a corresponding optimal primal solution we need to select some of the basic arcs (some others can be redundant).
In this example there are three basic arcs: arcs $(r, 2)$ and $(2, 1)$ are necessary; arc $(1, 2)$ is redundant.

Two approaches

In the scientific and educational literature Edmonds algorithm is presented in two different ways.

- Some authors describe the algorithm with no reference to duality theory, without explicitly building a dual data-structure, just using a clever sequence of calls to a graph search sub-routine (such as DFS or BFS) to identify SCCs not reachable from r; in these implementations the complexity is $O(mn)$ and the a priori knowledge of r is assumed.
- Some authors describe the algorithm as a two steps algorithm, where the first step builds the dual data-structure and the second step uses it to find the optimal solution. This is what happens although in many cases there is no explicit mention to duality and the dual data-structure is not explicitly identified as such. The construction of the dual data-structure is independent of r. In these implementations it is possible to obtain a better computational complexity, such as $O(m \log n)$, depending on the data-structures used. Very sophisticated implementations are possible.

Edmonds algorithm: complexity (1)

In this implementation the algorithm has complexity $O(nm)$, because it requires at most $2(n-1)$ iterations and each of them has complexity $O(m)$.

Let h be the number of SCCs of $\mathcal{D}_B$ (excluding $\{r\}$).
Let h_0 be the number of SCCs of $\mathcal{D}_B$ with no admissible entering arcs (excluding $\{r\}$).

At each iteration, the sum $h + h_0$ decreases by at least 1:

- if the selected SCC S remains a SCC, then it gets an entering admissible arc: therefore, h does not change and h_0 decreases;
- if S is merged with another SCC, then h decreases and h_0 does not increase.

Initially $h = n - 1$ and $h_0 = n - 1$. Eventually $h = h_0 = 0$. Therefore the iterations are at most $2(n-1)$.

Edmonds algorithm: complexity (2)

In $O(m)$ time it is possible to find a SCC S in $\mathcal{D}_B$, sorting the nodes in pre-topological order, starting with $Scan(r)$, so that the first node belongs to a SCC not reachable from r and with no admissible entering arcs.

Therefore each iteration has time complexity $O(m)$.

In this implementation

- the root r must be known since the beginning;
- more than one arc may become admissible in a same iteration;
- no dual data-structure is required;
- any graph search algorithm can be used as a sub-routine (DFS, BFS,...).

Iterative dual ascent

for $a \in \mathcal{A}$ **do**
  $\overline{c}_a \leftarrow c_a$
end for
VisitGraphFw
while $Order[n] \neq r$ **do**
  VisitGraphBw$(Order[n])$
  FindArc
  UpdateCost
  VisitGraphFw
end while

Upon termination, a vector of predecessors, *Pred*, identifies to the primal optimal solution.

Procedure *VisitGraphFw*

---

for $i \in \mathcal{N}$ **do**
  $Pred[i] \leftarrow nil$
end for
$Pred[r] \leftarrow r$
$k \leftarrow 0$
$DFS(r)$
for $i \in \mathcal{N}$ **do**
  if $Pred[i] \neq nil$ **then**
    $DFS(i)$
  end if
end for

The vector *Pred* indicates the predecessors when the graph is visited.
The vector *Order* indicates the inverse pre-topological order of the nodes.
k is the index of *Order*.

Procedure $DFS(i)$

for $j \in \delta^+(i)$ **do**
  if $(\overline{c}_{ij} = 0) \wedge (Pred[j] = nil)$ **then**
    $Pred[j] \leftarrow i$
    $DFS(j, k)$
  end if
end for
$k \leftarrow k + 1$
$Order[k] \leftarrow i$

DFS is a recursive procedure.

Procedure *VisitGraphBw*

---

for $i \in \mathcal{N}$ **do**
  $Succ[i] \leftarrow nil$
end for
$Succ[Order[n]] \leftarrow Order[n]$
$h \leftarrow 0$
$RevDFS(Order[n])$

The vector *Succ* indicates the successors when the graph is visited backward.
S is a vector representing a connected component.
h is its index.

Procedure *RevDFS*(i)

---

for $j \in \delta^-(i)$ **do**
  if $(\overline{c}_{ji} = 0) \wedge (Succ[j] = nil)$ **then**
    $Succ[j] \leftarrow i$
    RevDFS(j)
  end if
end for
$h \leftarrow h + 1$
$S[h] \leftarrow i$

RevDFS is a recursive procedure, similar to DFS, but with the reversed arcs.

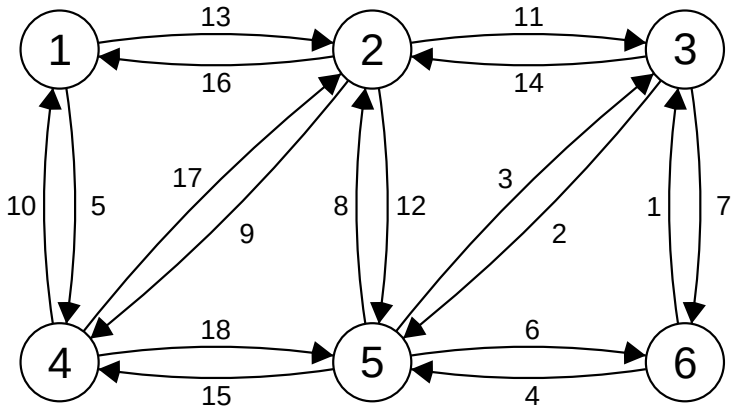
Procedure *FindArc*

/* Find a minimum reduced cost arc entering S */
for $i \in \mathcal{N}$ **do**
  $Flag[i] \leftarrow false$
end for
for $u = 1, \ldots, h$ **do**
  $Flag[S[u]] \leftarrow true$
end for
$\alpha \leftarrow \infty$
for $u = 1, \ldots, h$ **do**
  for $(i, S[u]) \in \delta^-(S[u])$ **do**
    if $(notFlag[i]) \wedge (\overline{c}_{iS[u]} < \alpha)$ **then**
      $\alpha \leftarrow \overline{c}_{iS[u]}$
    end if
  end for
end for

Procedure *UpdateCost*

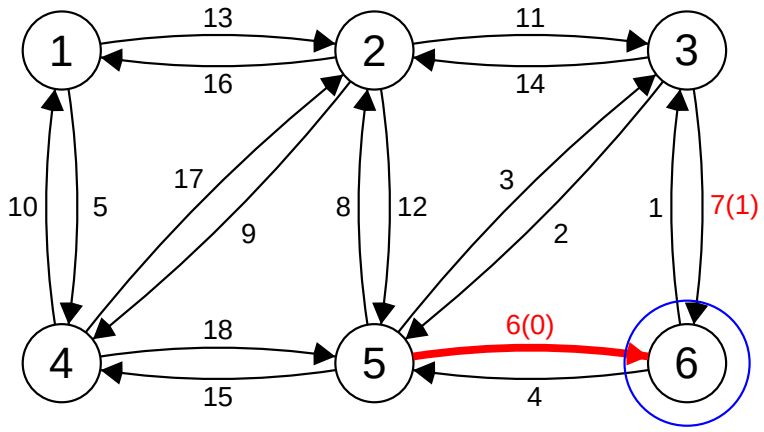
/* Update the reduced costs */
for $u = 1, \ldots, h$ **do**
  for $(i, S[u]) \in \delta^{-}(S[u])$ **do**
    if $(notFlag[i])$ **then**
      $\overline{c}_{iS[u]} \leftarrow \overline{c}_{iS[u]} - \alpha$
    end if
  end for
end for

Example



Node 1 is the root.

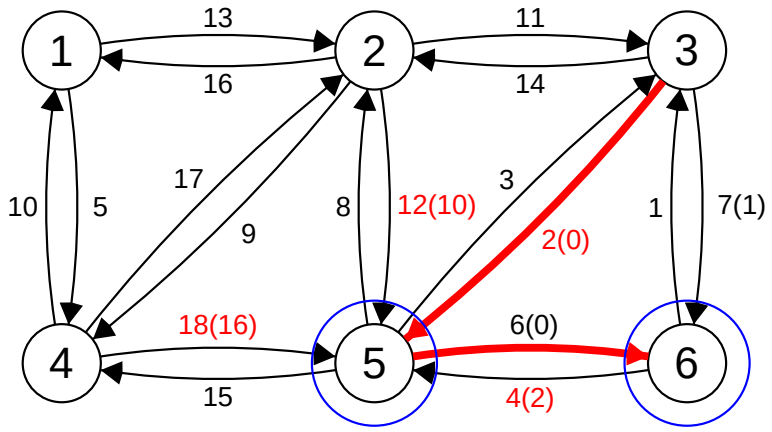
Iteration 1



DFS : 1 2 3 4 5 6
Order : 1 2 3 4 5 6
$S = \{6\}$
$\delta^-(S) = \{(3,6),(5,6)\}$
$\alpha = 6$

$y_6 = 6$

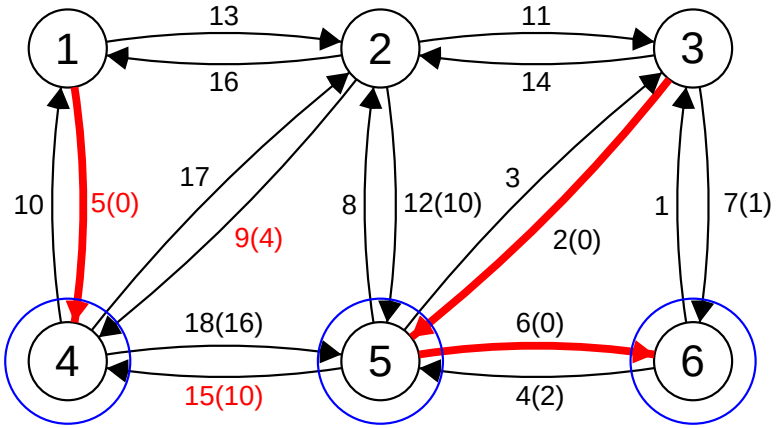
Iteration 2



DFS : 1 2 3 4 5 6
Order : 1 2 3 4 6 5
$S = \{5\}$
$\delta^-(S) = \{(2,5),(3,5),(4,5),(6,5)\}$
$\alpha = 2$

$y_6 = 6$ $y_5 = 2$

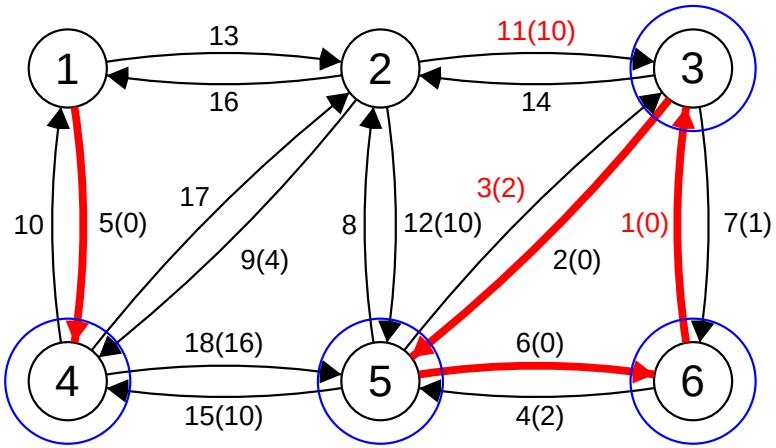
Iteration 3



DFS : 1 2 3 5 6 4
Order : 1 2 6 5 3 4
$S = \{4\}$
$\delta^-(S) = \{(1,4),(2,4),(5,4)\}$
$\alpha = 5$

$y_6 = 6\ y_5 = 2\ y_4 = 5$

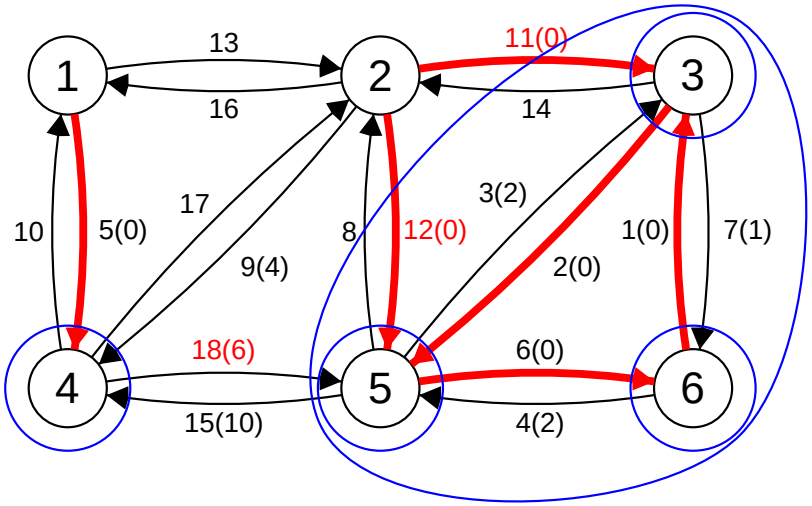
Iteration 4



DFS : 1 4 2 3 5 6
Order : 4 1 2 6 5 3
$S = \{3\}$
$\delta^-(S) = \{(2,3),(5,3),(6,3)\}$
$\alpha = 1$

$y_6 = 6\ y_5 = 2\ y_4 = 5\ y_3 = 1$

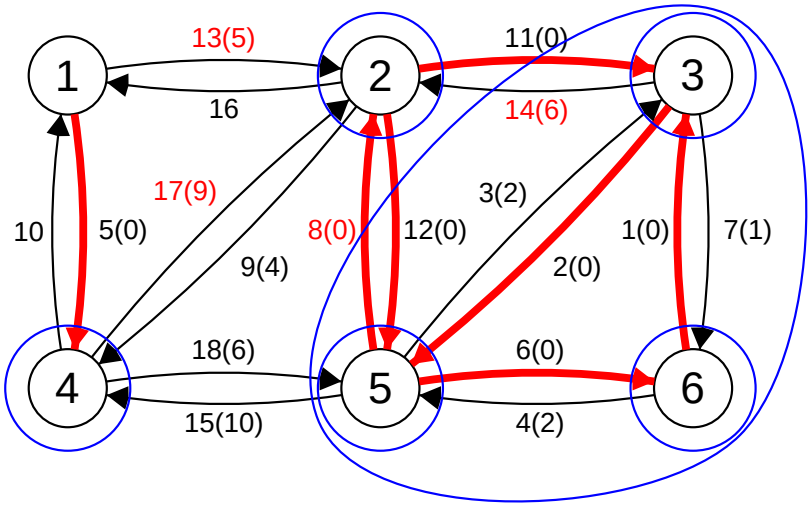
Iteration 5



DFS : 1 4 2 3 5 6
Order : 4 1 2 6 5 3
$S = \{3, 5, 6\}$
$\delta^-(S) = \{(2, 3), (2, 5), (4, 5)\}$
$\alpha = 10$

$y_6 = 6$ $y_5 = 2$ $y_4 = 5$ $y_3 = 1$ $y_{356} = 10$

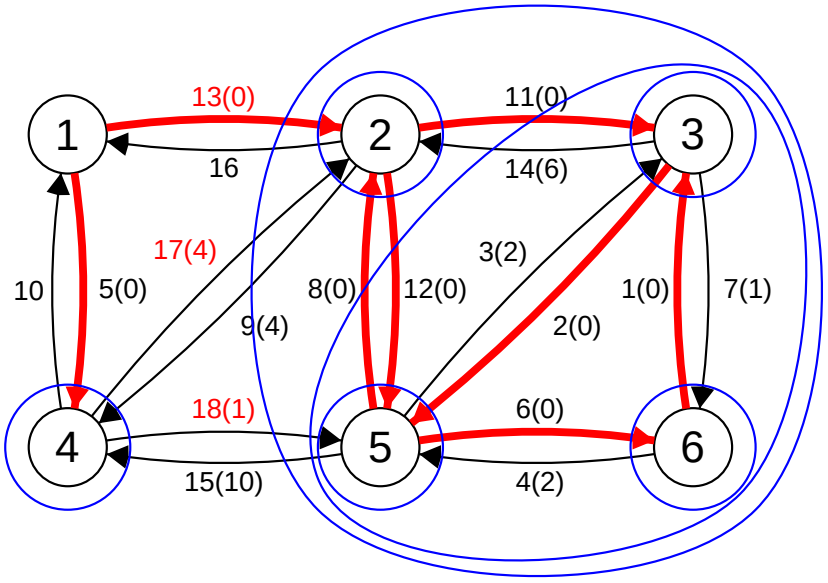
Iteration 6



DFS : 1 4 2 3 5 6
Order : 4 1 6 5 3 2
$S = \{2\}$
$\delta^-(S) = \{(1,2),(3,2),(4,2),(5,2)\}$
$\alpha = 8$

$y_6 = 6$ $y_5 = 2$ $y_4 = 5$ $y_3 = 1$ $y_{356} = 10$ $y_2 = 8$

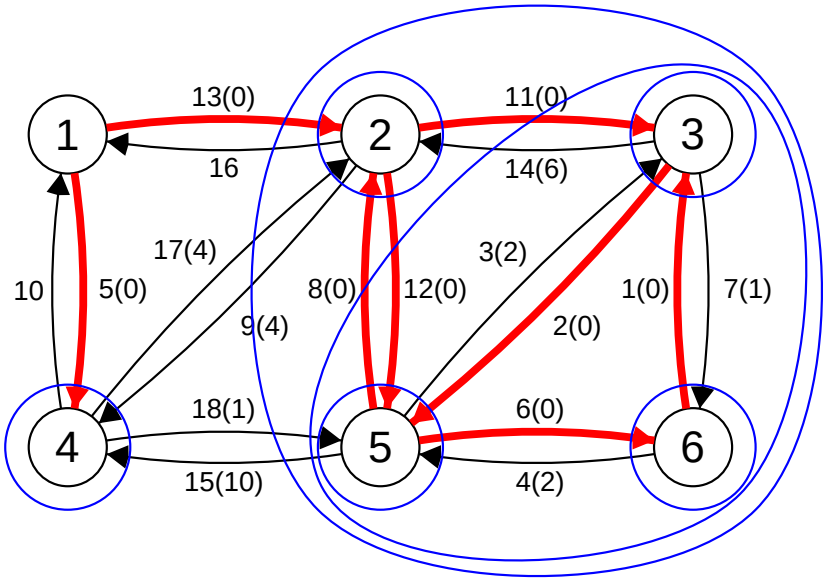
Iteration 7



DFS : 1 4 2 3 5 6
Order : 4 1 6 5 3 2
$S = \{2, 3, 5, 6\}$
$\delta^-(S) = \{(1, 2), (4, 2), (4, 5)\}$
$\alpha = 5$

$y_6 = 6$ $y_5 = 2$ $y_4 = 5$ $y_3 = 1$ $y_{356} = 10$ $y_2 = 8$ $y_{2356} = 5$

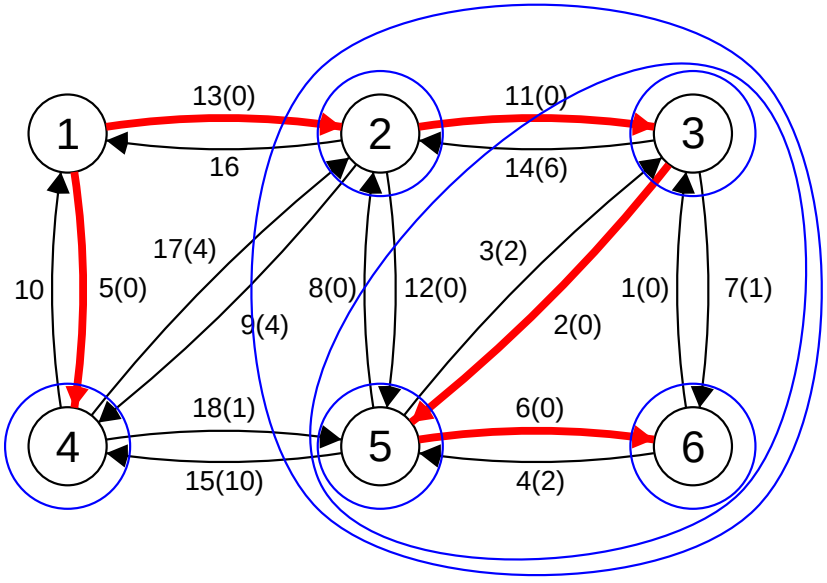
Iteration 8



DFS : 1 4 2 3 5 6
Order : 4 6 5 3 2 **1**
Node 1 is the root: STOP!

$y_6 = 6$ $y_5 = 2$ $y_4 = 5$ $y_3 = 1$ $y_{356} = 10$ $y_2 = 8$ $y_{2356} = 5$

Optimal solution



$$y_6 = 6\ y_5 = 2\ y_4 = 5\ y_3 = 1\ y_{356} = 10\ y_2 = 8\ y_{2356} = 5$$

$$w = \sum_S y_S = \mathbf{37}$$

Implementation 2

In this second approach, Edmonds algorithm is implemented using both primal and dual data-structures.
The implementation does not make use of graph search sub-routines and assumes that the digraph is strongly connected.

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*Step*1 : *DualAscent*
*Step*2 : *ComputeSolution*(r)

- The aim of step 1 is to build a sequence of nested SCCs in the digraph of admissible arcs, until a unique SCC is found containing all the nodes. The information about which node is the root is ignored. This step can be implemented in $O(m \log n)$.
- The aim of step 2 is to find an optimal solution given a root r and the information stored in the dual data-structure built in step 1. This step can be implemented in $O(n)$.

Step 1: dual ascent

SCCs with no basic entering arcs are iteratively found, so that they form a path. Each node in the path is a basic SCC.

A minimum cost arc $(u, v) \in \mathcal{A}$ is found among those entering the first node in the path, i.e. a certain SCC S; the dual variable y_S is updated accordingly (dual ascent).

If u belongs to a SCC S' already in the path, then all components in the path between S and S' are merged into a single SCC.
Otherwise a new SCC including only node u is created and appended at the beginning of the path.

Besides concatenating basic SCCs in a path, the algorithm also records how the basic SCC are nested in one another forming a tree of basic SCCs.

Step 1: data structures

For every SCC S made of a single node, we define a priority queue $P(S)$ with the arcs entering it. When SCCs S' and S'' are merged into a larger one S, the corresponding priority queues $P(S')$ and $P(S'')$ are merged into a single priority queue $P(S)$.

A union-find data-structure *Comp* is used to find the SCCs to which specific nodes belong.

A vector *Arc* records the (unique) basic arc for each basic SCC.
A vector *y* records the values of the basic dual variables.
A vector *Next* indicates the next SCC for each SCC in the path.
A vector *Parent* indicates the parent SCC for each SCC in the tree.
A list *Children* records the information symmetric to that of *Parent*.

All these vectors have maximum size $2n$.

Step 1: pseudo-code

InitializePQs
$k \leftarrow 0$; *NewComponent*(*Random*($\mathcal{N}$))
$s \leftarrow 1$
while $P(s) \neq \emptyset$ **do**
  repeat
    $(u, v) \leftarrow$ *ExtractMin*($P(s)$)
  until $Comp[u] \neq s$
  $Arc[s] \leftarrow (u, v)$
  $y[s] \leftarrow c(u, v)$
  ReduceCosts($P(s), y[s]$)
  if $Comp[u] = nil$ **then**
    NewComponent(u)
    $Next[k] \leftarrow s$
  else
    Merge($Comp[u]$)
  end if
  $s \leftarrow k$
end while

InitializePQs

for $i \in \mathcal{N}$ **do**
  $P(i) \leftarrow nil$
  $Comp[i] \leftarrow nil$
end for
for $(u, v) \in \mathcal{A}$ **do**
  $Insert(u, c(u, v), P(v))$
end for

CreateNewSCC

$k \leftarrow k + 1$
$Arc[k] \leftarrow nil$
$y[k] \leftarrow 0$
$Parent[k] \leftarrow nil$
$Children[k] \leftarrow nil$

NewComponent(v)

CreateNewSCC
$Comp[v] \leftarrow k$
$Leaf[v] \leftarrow k$
$P[k] \leftarrow P[v]$

Merge(t)

CreateNewSCC
$P[k] \leftarrow nil$
$Next[k] \leftarrow Next[t]$
while $s \neq t$ **do**
  $Parent[s] \leftarrow k$
  $Insert(s, Children(k))$
  $P[k] \leftarrow MergePQ(P[s], P[k])$
  $UpdateComp(s, k)$
  $s \leftarrow Next[s]$
end while

Computational complexity of Step 1

The number of calls to *Insert* is $O(m)$ (all arcs are inserted in one PQ).
The number of calls to *ExtractMin* is $O(m)$ (each arc can be extracted at most once).
The number of calls to *MergePQ* is $O(n)$ (at most $2n$ SCCs become basic).
The number of calls to *ReduceCosts* is $O(n)$ (see above).

If all operations are implemented to take $O(\log n)$, then the overall complexity is $O(m \log n)$.

UpdateComp requires a Union-Find data-structure.

Step 2

$T \leftarrow \emptyset$
$RootList \leftarrow \emptyset$
$Dismantle(r)$
while $RootList \neq \emptyset$ **do**
  $s \leftarrow Extract(RootList)$
  $(u, v) \leftarrow Arc[s]$
  $T \leftarrow T \cup \{(u, v)\}$
  $Dismantle(v)$
end while
return T

Dismantle(v)

$s \leftarrow Leaf[v]$
while $Parent[s] \neq nil$ **do**
  for $t \in Children[Parent[s]] : t \neq s$ **do**
    $Parent[t] \leftarrow nil$
    $Insert(t, RootList)$
  end for
  $s \leftarrow Parent[s]$
end while