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Department of Mathematical Sciences
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Spring 2017

MA 573

Computational Methods of Mathematical Finance

Assignment 5

due on Wednesday, March, in Bloomberg class, to Qingyun Ren

1. Let X be a standard normal random variable. We want to use Monte Carlo methods to estimate $\mathbb{P}(X \geq a)$ for fixed $a \in \mathbb{R}$ by sampling X and computing the sample mean for $Y = \mathbb{1}_{\{X \geq a\}}$. For large a this is a nontrivial task as there will not just be enough sample points that give a positive contribution. Control variates are the way out here. First we consider the control variate estimator

$$\mathbb{1}_{\{X \geq a\}} - b\big(X - \mathbb{E}[X]\big) \tag{1}$$

for some constant b.

   (a) Compute the optimal b^* and the variance reduction factor in using this estimator instead of the sample mean.

   (b) Implement the calculation for $a = 3$ and $a = 8$ and $N = 100,000$ (or larger, if needed).

2. As in problem 1, we want to use Monte Carlo methods to estimate $\mathbb{P}(X \geq a)$ for fixed $a \in \mathbb{R}$ by control variates. However we want now to use importance sampling, with X having under the measure $\tilde{\mathbb{P}}$ a normal distribution with mean μ and variance 1.

   (a) Calculate the optimal mean μ^* and use this estimator instead of the sample mean.

   (b) Implement the calculation for $a = 3$ and 8.

   (c) Compare the convergence of different variance reduction techniques (antithetic sampling, control variates, importance sampling) among eachother andwith the base case (without variance reduction).

3. A popular stochastic process to model interest rates, exchange rates, volatility or commodity prices is the Ornstein-Uhlenbeck (OU) process. It follow the dynamics

$$dX_t = \lambda(\kappa - X_t)\,dt + \sigma\,dW_t, \qquad X_0 = x.$$

Assume for the following the model parameters $\lambda = 2$, $\kappa = 120$ and $\sigma = 25$ and $x = 100$.

(a) Simulate 10 paths of the Ornstein-Uhlenbeck process, using a time horizon of 1 and 1000 time steps and plot them.

(b) Play around with the parameter and try to find out there intuitive meaning. Which properties of the process are they describing?

4. An interest rate swap is a financial product that exchanges the interest rate gains from the floating market rate r against those of a previously fixed rate r_{fix}. Thus it pays

$$N\left(e^{\int_0^T r_s\,ds} - e^{r_{\text{fix}}T}\right)$$

where N is the amount notional of the contract. Assume that the interest rate follows under the risk-neutral measure the Ornstein-Uhlenbeck dynamics

$$dr_t = \lambda(\kappa - r_t)\,dt + \sigma\,dW_t, \qquad r_0 = 0.02.$$

with parameters $\lambda = 0.7$, $\kappa = 0.05$ and $\sigma = 0.006$ and notional $N = \$10,000$.

(a) Assume that the fixed rate $r_{\text{fix}} = 4\%$, what is the expected payoff of an interest rate swap at maturity $T = 3$?

(b) If you want to issue an interest rate swap with maturity $T = 2$ trading *at par* (i.e., with value 0), which fixed rate r_{fix} do you have to choose?

Note: All programming problems should be either in Python 2.7 or Python 3.5. Matlab and R are accepted, but no support for these languages is provided. Please comment the programs *extensively* and send them in a .zip file with title **Lastname**_HW5.zip and suject line "MA 573 HW5 **Lastname**" to Qingyun Ren qren@wpi.edu before the due date of the homework (replacing the bold words by your actual last name). Please provide printouts of programs amd plots that one can comment on them.

6 points per problems