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Spring 2017

MA 573

Computational Methods of Mathematical Finance

Assignment 6

due on Thursday, March 16, in class

1. Let $(W_t^1)_{t\geq 0}$, and $(Z)_{t\geq 0}$ be Brownian motions and define the stochastic process $(W_t^2)_{t\geq 0}$ by

$$W_t^2 = \rho W_t^1 + \sqrt{1-\rho^2} Z_t$$

   (a) Show that $(W_t^2)_{t\geq 0}$ is itself a Brownian motion.
   (b) Calculate the covariance and the correlation between W_t^1 and W_t^2.
   (c) Plot one sample path for each of the two Brownian motions for the cases where $(i)\rho = 0.9$, $(ii)\rho = 0.3$, $(iii)\rho = 0$, $(iv)\rho = -0.3$, $(v)\rho = -0.9$ (one plot per case).

   Context: Correlated Brownian motions are often used in modeling assets with stochastic volatility. Specifically, equities like stock show usually a highly negative correlation between asset and volatility.

2. Assume that stock prices follow under the risk-neutral measure the dynamics of Scott's exponential Ornstein-Ulenbeck process model,

$$\begin{aligned} dS_t &= rS_t\,dt + e^{Y_t} S_t\,dW_t^1, & S_0 &= s \\ dY_t &= \lambda\big(\kappa - Y_t\big)\,dt + \xi\,dW_t^2, & Y_0 &= y \\ \mathbb{E}[W_t^1 W_t^2] &= \rho t \end{aligned}$$

   with parameters $s = 100$, $y = -1$, $\lambda = 5$, $\kappa = -1.5$, $\rho = -0.2$ and $\xi = 0.25$ and the market contains a money market account with constant interest rate $r = 3\%$. Calculate the price of a lookback put option

$$\max_{t\in[0,T]} S_t - S_T$$

   with maturity $T = 0.3$ using the Milstein scheme to create the sample paths.

3. Assume that stock prices follow under the risk-neutral measure the CEV model,

$$dS_t = rS_t\,dt + \sigma S_t^{\beta} S_t\,dW_t^1, \qquad\qquad S_0 = s$$

with parameters $s = 100$ and $\sigma = 0.2$ and that the market contains a money market account with constant interest rate $r = 3\%$.

   (a) Calculate the price of European call options with maturity $T = 1$ and strikes K every \$5 between \$85 and \$125 for $\beta = 0, -0.3, -0.5, -0.7, -1$.

   (b) Calculate the implied volatility of the options of (a). Describe the role of β on the volatility smile.

4. Assume that stock prices follow under the risk-neutral measure the Heston model,

$$\begin{aligned} dS_t &= rS_t\,dt + \sqrt{Y_t}S_t\,dW_t^1, && S_0 = s \\ dY_t &= \lambda\big(\kappa - Y_t\big)\,dt + \xi\sqrt{Y_t}\,dW_t^2, && Y_0 = y \\ \mathbb{E}[W_t^1 W_t^2] &= \rho t \end{aligned}$$

with parameters $s = 100$, $\lambda = 3$, $\kappa = 0.1$ $\xi = 0.25$, $y = 0.08$, $\rho = -0.8$ and that the market contains a money market account with constant interest rate $r = 3\%$.

   (a) Calculate the price of European call options with maturity $T = 1$ and strikes K every \$5 between \$85 and \$125.

   (b) Calculate the implies volatility of these options. Check how the volatility changes with changes in different parameters, notably ρ and ξ.

Context: One of the main reason to develop more sophisticated models then Black-Scholes is that the Black-Scholes model is unable to capture the volatility smile/skew of the markets. The two problems above investigate how well CEV model and Heston model are capable to model the implied volatility and which flexibility they offer.

5. We have seen in class that in a numerical implementation of the CIR process via Euler scheme, the process may become negative. Instead of simply cut it off at zero, there exist refined techniques. One of them should be developed in the following steps: Let X_t be the solution of the SDE, $dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t$, then one can define the generalized Stratonovich Integral

$$\int_0^T f(X_t) \circ_1 dW_t = \lim_{n\to\infty} \sum_{\substack{i=0 \\ t_i\in[0,T]}}^{n-1} f(X_{t_{i+1}})\big(W_{t_{i+1}} - W_{t_i}\big)$$

and one can show that it relates to the Itô integral as follows,

$$\int_0^T f(X_t) \circ_1 dW_t = \int_0^T f(X_t)\,dW_t + \int_0^T f'(X_t)\sigma(X_t)\,dt$$

(a) Use the concept of generalized Stratonovich integration to derive an implicit recursion scheme for the SDE

$$dY_t = a(b - Y_t)\,dt + \xi\sqrt{Y_t}\,dW_t; \qquad Y_0 = y > 0; \qquad a,\, b,\, \xi > 0.$$

Finally you should get for the partition $t_i = iT/n$ of $[0, T]$ the scheme

$$Y_{t_{i+1}} = Y_{t_i} + (ab - \frac{\xi^2}{2} - aY_{t_{i+1}})\frac{T}{n} + \xi\sqrt{Y_{t_{i+1}}}(W_{t_{i+1}} - W_{t_i}).$$

(b) Derive out of the implicit scheme an explicit representation of $Y_{t_{i+1}}$. This should be possible at least in the case that $\xi^2 \leq 2ab$.

(c) Simulate ten sample paths of the process (Y_t) on the interval $[0, 1]$ with step size 0.01 and plot the result.

6. Calibrate the Heston Stochastic volatility model

$$\begin{aligned} dS_t &= rS_t\,dt + \sqrt{Y_t}S_t\,dW_t^1, && S_0 = s \\ dY_t &= \lambda\big(\kappa - Y_t\big)\,dt + \xi\sqrt{Y_t}\,dW_t^2, && Y_0 = y \\ \mathbb{E}[W_t^1 W_t^2] &= \rho t \end{aligned}$$

to market data (choose a realistic risk-free rate from data). Use as data options on Apple stock with maturity May 19, 2017. Discard all strikes that have a trading volume of less then 10 and use always the marked mid-price (average between bid and ask). Calibrate $\Theta = \{\lambda, \kappa, \xi, y, \rho\}$ by minimizing the sum of the squared errors

$$\underset{\Theta}{\operatorname{argmin}} \sum_i \big|C_i^\Theta - C_i\big|^2.$$

This can be done numerically, using a numerical optimization procedure (you might use as initial data $\lambda = 3$, $\kappa = 0.1$ $\xi = 0.25$, $y = 0.08$, $\rho = -0.8$) Do the procedure twice, once using put and once using call data and compare the results.

Note: All programming problems should be either in Python 2.7 or Python 3.5. Matlab and R are accepted, but no support for these languages is provided. Please comment the programs *extensively* and send them in a .zip file with title **Lastname**_HW6.zip and suject line "MA 573 HW6 **Lastname**" to Qingyun Ren qren@wpi.edu before the due date of the homework (replacing the bold words by your actual last name). Please provide printouts of programs amd plots that one can comment on them.

4 points per problems