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MA 573

Computational Methods of Mathematical Finance

Assignment 7

due on Thursday, March 23, in class

1. Assume that stock prices follow under the risk-neutral measure the dynamics of Scott's exponential Ornstein-Uhlenbeck process model,

$$\begin{aligned} dS_t &= rS_t\,dt + e^{Y_t}S_t\,dW_t^1, & S_0 &= s\\ dY_t &= \lambda\big(\kappa - Y_t\big)\,dt + \xi\,dW_t^2, & Y_0 &= y\\ \mathbb{E}[W_t^1 W_t^2] &= \rho t \end{aligned}$$

with constant interest rate r. Describe the price of the European call option with payoff

$$\big(S_T - K\big)^+$$

at time $t \in [0,T]$ as solution of a Cauchy problem of a PDE.

2. Consider the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}f(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}f(x,t) + \frac{\partial}{\partial x}f(x,t) - f(x,t) = 0\\ f(x,T) = e^{-3x} \end{cases} \tag{1}$$

for $t \in [0,T]$ and and $x \in \mathbb{R}$.

(a) Write the probabilistic representation for $f(x,t)$ as a conditional expectation of a function of a solution of a stochastic differential equation you should define.

(b) Compute this conditional expectation.

(c) Verify that $f(x,t)$ you found in b) solves the PDE (1) and satisfies the terminal condition.

3. Consider the option pricing problem

$$\mathbb{E}\Big[e^{-\int_0^t r_s\,ds}\big(K - S_T\big)^+\Big]$$

where the underlying stock price follows under the risk-neutral measure a CEV model

$$dS_t = r_t S_t\,dt + \zeta S_t^\beta\,dW_t^1, \qquad S_0 = s$$

and the interest rate follows a CIR process

$$dr_t = \kappa\big(\theta - r_t\big)\,dt + \xi\sqrt{r_t}\,dW_t^2, \qquad r_0 = r$$

and the two Brownian motions are correlated, $\mathbb{E}[W_t^1 W_t^2] = \rho t$. Describe the option price via the solution of a Cauchy problem of a PDE.

4. While in general it is not possible to find a PDE formulation for option prices of path-dependent derivatives, this can be done sometimes by introducing auxiliary processes. If we can rephrase the option pricing problem in a way that the payoff depends only on the terminal value of the stock price and the auxiliary process. Specifically, the payoff of lookback options and barrier options can be rephrased in terms of the running maximum process

$$M_t = \max_{s\in[0,t]} S_s$$

and Asian options in terms of the integral process

$$A_t = \int_0^t S_s\,ds,$$

for instance

$$\begin{aligned}
\big(K - S_T\big)^+ \mathbb{1}_{\{S_t\le B \text{ for all } t\in[0,T]\}} &= \big(K - S_T\big)^+ \mathbb{1}_{\{M_T\le B\}} && \text{(up-and-out put)},\\
\max_{s\in[0,t]} S_s - S_T &= M_T - S_T && \text{(lookback put)},\\
\Big(\frac{1}{T}\int_0^T S_s\,ds - K\Big)^+ &= \Big(\frac{A_T}{T} - K\Big)^+ && \text{(Asian call)}.
\end{aligned}$$

One can proof Itô formulas for those processes, specifically we have for the stock dynamics

$$dS_t = b(S_t)\,dt + \sigma(S_t)\,dW_t, \qquad S_0 = s$$

that for sufficiently smooth (i.e., differentiable) f we have on the one hand

$$\begin{aligned}
f(t, S_t, M_t) = f(0, S_0, M_0) &+ \int_0^t f_t(t, S_t, M_t) + f_s(t, S_t, M_t)b(S_t) + \frac{1}{2}f_{ss}(t, S_t, M_t)\sigma^2(S_t)\,dt\\
&+ \int_0^t f_s(t, S_t, M_t)\sigma(S_t)\,dW_t + \int_0^t f_m(t, S_t, M_t)\,dM_t
\end{aligned}$$

and on the other

$$f(t,S_t,A_t) = f(0,S_0,A_0) + \int_0^t f_t(t,S_t,A_t) + f_s(t,S_t,A_t)b(S_t) + \frac{1}{2}f_{ss}(t,S_t,A_t)\sigma^2(S_t)\,dt$$
$$+ \int_0^t f_s(t,S_t,A_t)\sigma(S_t)\,dW_t + \int_0^t f_a(t,S_t,A_t)S_t\,dt.$$

Derive a PDE formulation for option pricing problems

$$\mathbb{E}^{\mathbb{Q}}\big[e^{-rT}g(T,S_T,A_T)\big], \qquad \mathbb{E}^{\mathbb{Q}}\big[e^{-rT}h(S_T,M_T)\big]$$

for some smooth payoff functions g, h by applying Itô's formula to the discounted payoff. *Note that for this purpose (as we focus on applications and not theory and rigorous proofs) you may assume that*

(a) The expectation of Itô integrals dW_s is zero;

(b) The expectation of a Riemann integrals ds is zero if and only if the integrand is zero;

(c) The expectation of a Stieltjes integral dM_s is zero if and only if the integrand is zero whenever M is increasing.