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Spring 2017

MA 573

Computational Methods of Mathematical Finance

Assignment 4

due on Thursday, February 23, in class

1. Assume a financial markets where stock and money market account follow the dynamics

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sigma S_t\,dW_t, && S_0 = 100; \\ dB_t &= rB_t\,dt, && B_0 = 1, \end{aligned}$$

   with $\mu = 0.05$, $\sigma = 0.35$ and $r = 0.05$.

   Calculate the price of a European call option with maturity $T = 2$ and strike $K = 120$,

   (a) Using Black–Scholes formula.

   (b) Using direct Monte Carlo integration with sample size 5000.

   (c) Using Monte Carlo integration with sample size 5000 and antithetic sampling of the normal random variable as variance reduction technique.

   Plot the stepwise convergence of both Monte Carlo methods to the Black-Scholes value in one diagram.

2. Assume a financial markets where stock and money market account follow the dynamics

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sigma S_t\, dW_t, & S_0 &= 100; \\ dB_t &= r B_t\, dt, & B_0 &= 1, \end{aligned}$$

with $\mu = 0.1$, $\sigma = 0.2$ and $r = 0.02$. You want to calculate the price of an Asian call option with maturity $T = 5$ and strike $K = 120$ using Monte-Carlo integration.

(a) The underlying stock and a European put option with maturity $T = 0.5$ and $K = 95$.

(b) The underlying stock and a European call option with maturity $T = 0.5$ and $K = 95$.

(c) The underlying stock and an arithmetic average Asian put option with maturity $T = 0.5$ and $K = 95$ and daily samples for the average.

(d) A European put option with maturity $T = 0.5$ and $K = 95$ and an arithmetic average Asian put option with maturity $T = 0.5$ and $K = 95$ and daily samples for the average.

(e) A European put option with maturity $T = 0.5$ and $K = 95$ and a geometric average Asian put option with maturity $T = 0.5$ and $K = 95$ and daily samples for the average.

(f) A geometric average Asian put with maturity $T = 0.5$ and $K = 95$ and an arithmetic average Asian put option with maturity $T = 0.5$ and $K = 95$ and daily samples for the average.

3. Assume a financial markets where stock and money market account follow the dynamics

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sigma S_t\, dW_t, & S_0 &= 100; \\ dB_t &= r B_t\, dt, & B_0 &= 1, \end{aligned}$$

with $\mu = 0.3$, $\sigma = 0.3$ and $r = 0.03$. You want to calculate the price of an Asian call option with maturity $T = 5$ and strike $K = 120$ using Monte-Carlo integration.

(a) Determine the distribution of the random variable $Z = \sum_{i=1}^{N} W_{t_i}$. (*Hint*: Remember that the sum of *independent* normal random variables is again normal.)

(b) Use the result of (a) to find a closed form solution for the geometric average Asian call option with payoff

$$\left(\frac{1}{T}\sum_{i=1}^{N} S_{t_i} - K\right)^+, \qquad t_i = \frac{iT}{N}.$$

(c) Use the result of (b) as control variate to calculate the price of the regular (arithmetic average) Asian call option

$$\left(\frac{1}{T}\sum_{i=1}^{N} S_{t_i} - K\right)^+, \qquad t_i = \frac{iT}{N}$$

with $N = 1260$ and 10,000 paths.

(d) Compare the calculation of (c) with the direct Monte Carlo calculation (without variance reduction techniques) by plotting the stepwise convergence of both methods.

(e) Compare the prices of the geometric average Asian option, the arithmetic average Asian option and the regular call.

4. In credit risk, the time to the default of a company is often modeled by an exponentially distributed random variable τ. Assume that its parameter $\lambda = 2$ and a stock price follows the dynamics

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \qquad\qquad S_0 = 100;$$

with $\mu = 0.1$, $\sigma = 0.2$ (where the Brownian motion is independent of the default time. You want to calculate the average price at default (assuming a time horizon of 10 years – if the stock has not defaulted by then, use the the price at this terminal time), using a stratified sampling technique and a sample of 1000.

(a) Create the stratified sample for the exponential distribution, using 10 stratification intervals with a sample of 100 in each. Plot the histogram and compare it with a non-stratified sample as well as the density function of the exponential distribution.

(b) Create 1000 paths of the stock price using 1000 steps per year. Using the sample of the default time from (a), calculate the expected price at default $\mathbb{E}\big[S_{\tau \wedge 10}\big]$.

Bonus Question: Can you come up with an analytical method to calculate this expectation?

Note: All programming problems should be either in Python 2.7 or Python 3.5. Matlab and R are accepted, but no support for these languages is provided. Please comment the programs *extensively* and send them in a .zip file with title **Lastname**_HW4.zip and suject line "MA 573 HW4 **Lastname**" to Qingyun Ren qren@wpi.edu before the due date of the homework (replacing the bold words by your actual last name). Please provide printouts of programs amd plots that one can comment on them.

6 points per problems