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## MA 573

# Computational Methods of Mathematical Finance

## Assignment 9

due on Thursday, April 13, in class

1. (a) Write a Python code to implement the solver for a tridiagonal matrix-vector system of linear equations,  $Ax = b$ .
- (b) Modify the solving algorithm such that it can handle also the generalized tridiagonal matrices of the form

$$\begin{pmatrix} * & * & * & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & * & * & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & * & * & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & * & * & * & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \cdots & * & * & * \end{pmatrix}$$

as they arise when implementing boundary conditions, either linearly or from non-central difference quotients.

*Context:* The solver should be written in a separate file, such that the program takes as input the matrix tridiagonal/generalized tridiagonal  $A$  and the vector  $b$ . Best implement a check that the algorithm runs only if the matrix has the correct structure and produces an error message otherwise. As many remaining problems will use this program, please test it extensively (e.g., by checking  $Ax = b$  by multiplication) to make sure not to have errors in.

2. Price a European call option  $(S_T - K)^+$  by using an explicit finite difference scheme for the Black-Scholes PDE

$$\begin{cases} -v_t(t, s) + \frac{1}{2}\sigma^2 s^2 v_{ss}(t, s) + rsv_s(t, s) = rv(t, s) \\ v(0, s) = (s - K)^+ \end{cases}$$

for  $S_0 = 100$ ,  $\sigma = 20\%$ ,  $r = 3\%$ ,  $T = 1$  and  $K = 115$ . Use 1000 space discretization point on the interval  $[0, 200]$  and implement the explicit boundary condition at 0 and a linearity boundary condition at 200. Calculate the price for (a) 20, (b) (50), (c) 100, (d) 500 time discretization steps and compare it to the price explicitly calculated by the Black-Scholes formula.

3. Exactly the same as problem 2., but use now the Crank-Nicolson scheme instead of the explicit one.
4. Consider the finite difference scheme

$$\frac{3v(t, x) - 4v(t - \Delta t, x) + v(t - 2\Delta t, x)}{2\Delta t}$$

(similar to HW 8). Assume that you want to use it to solve the system of ODEs

$$\frac{\partial}{\partial t}v = Av$$

numerically (ignoring the issue for the first step, which can e.g. be implemented by an implicit scheme).

- Find the shift polynomial associate to the finite difference scheme.
- Find the roots of the shift polynomial.
- Calculate the amplification factors  $\Lambda_{im}$  for all eigenvalues  $\lambda_i$  of  $A$ .
- Assuming that the eigenvalues are all in the interval  $[\lambda_{\min}, \lambda_{\max}]$ , under which condition on  $\Delta t$  is the finite difference scheme stable?