CONTENTS

SPECIAL EQUATIONS

We have learnt about simultaneous equations in two and three unknowns. When we have two independent equations in two unknowns or three independent equations in 3 unknowns, we can solve for the variables. This type of equations are called "Determinate Equations". The variables in Determinate Equations have unique values. However, if we have only one equation in two unknowns, or two equations in three unknowns, such equations are called "Indeterminate Equations". The variables here do not have unique values but take more than one value - in general, infinite number of values.

If we impose certain other conditions on these variables, then such indeterminate equations also can yield unique values for the variables. We take such conditions also into account while solving such equations.

In our earlier chapter on equations it has been discussed that the system of equations where the equations are less than the unknowns is indeterminate, i.e., we cannot determine the values of all the unknowns uniquely. Rather, the system has an infinite set of solutions.

Consider $3x + y = 10$. This, being one equation in two unknowns, is indeterminate. Suppose we impose a condition that both x and y are positive integers. With this condition the possibilities are reduced to a finite number $x = 1, y = 7$; $x = 2, y = 4$; $x = 3, y = 1$. If we further impose the condition that x is greater than y, then there is a unique solution $x = 3$, $y = 1$. So, even though we have one equation, because of additional conditions, it may have finite or sometimes even a unique solution. The conditions that we have, could be explicitly mentioned as above or could be in built into the problem as we see in the following example.

Examples:

1.01. Amit purchased certain number of pencils and a certain number of pens spending ₹10 on the whole. If each pencil costs him ₹3 and each pen ₹1, then how many pencils and pens could he have bought?

Sol. Solving this problem is basically solving the equation $3x + y = 10$ where x and y respectively denote the number of pencils and pens purchased and hence from the context we know that x and y should be both positive integers and hence this equation has exactly the three solutions as above.

Hence we see that when certain special conditions are imposed on the variables, the indeterminate equations also can yield finite number of solutions and sometimes even a unique solution. The conditions that we normally come across are
- minimum values of the variables
- maximum values of the variables
- variables being positive integers
- limits on difference in the values of variables, etc.

1.02. Suppose we take a date and a month, multiply the date with 12 (which is the number of months in a year) and the month with 31 (which is the number of days in a month) and add up the two products. Suppose we are given that the sum is 555 and asked to find out the date and month.

Sol. If we denote the date as D and month as M, we have $12D + 31M = 555$. Here we have only one equation with two unknowns. This is an indeterminate equation. However we have the following additional information.
- D is a positive integer less than or equal to 31
- M is a positive integer less than or equal to 12

To solve this equation and infact to solve any equation in two variables, we shall
- first divide the entire equation by the least coefficient
- get all the fractional values on to one side, say left and all the whole values on to the other
- replace the whole of right-hand side by k, where k is an integer

The following is the sequence of steps:

$12D + 31M = 555$

$D + 31M/12 = 555/12$

$D + 2M + 7M/12 = 46 + 3/12$

$$\frac{7M-3}{12} = k$$

$$M = \frac{12k+3}{7}$$

We now try to find the value of k. As M has to be positive integer the value of k = 5, which gives M = 9. Further other values of k are not feasible. Using this we now find D which is 23. Thus, the date is 23rd September. Here we get a unique solution for the equation.

1.03. Nakul bought two varieties of pens; the first variety costing ₹12 each and the second variety costing ₹17 each, spending ₹157 in total. In how many different combinations he could have purchased the pens?

Sol. Let x be the number of pens of first variety while y is the number of second variety then $12x + 17y = 157$

Proceeding as above,

$$x + y + \frac{5y}{12} = 13 + \frac{1}{12}$$

$$\frac{5y-1}{12} = k \text{ (integer)}$$

$$y = \frac{12k+1}{5}$$

As y is a positive integer the values of k can be k = 2, 7, 12, ...

When k is 2, y is 5 and this is the only possible value of y [for, if we consider the next possible value of k i.e., k = 7 then y is 17 which means the amount spent on second variety itself is 17 x 17 which is much more than the total amount spent].

As there is a unique value possible for y, it means that he can buy these pens in exactly one way.

We can write this solution briefly by focussing only on the remainders $12x + 17y = 157$ proceeding as discussed in the above example. Dividing the equation by 12 we have Rem $\left(\frac{5y}{12}\right)=1$, y can be 5, 17, 29 etc. i.e., y values can be obtained by adding (or subtracting) 12 successively and from (A) when y = 5 then the x value is 6 and the remaining values of x can be obtained by (subtracting or adding) 17 to 6 successively i.e., (x, y) could be (6, 5), (–11, 17), (–28, 29) etc.....
As $x > 0$ and $y > 0$ only (6, 5) is the acceptable solution.

Let us take another example and look at the most general method of solving such problems (Please note that some of the steps given in the example below may not be required in some problems).

1.04. I bought two different varieties of toys - one costing ₹8 per piece and the other costing ₹15 per piece. If I paid a total amount of ₹153 for both varieties of toys together, how many of each variety did I buy?

Sol. If p is the number of toys costing ₹8 per piece and q is the number of toys costing ₹15 per piece, we have
$8p + 15q = 153$ ——— (1)
The various steps involved in solving this equation are explained below.

Step 1:
Divide the equation throughout by the smallest coefficient (among all the variables).
In this case, 8 is the smallest coefficient. Dividing the equation throughout by 8, we get

$$p + q + \frac{7q}{8} = 19 + \frac{1}{8}$$

Step 2:
Bring all the fractions to one side (say, left-hand side) and whole numbers to the other side (right-hand side of the equation).

$$\frac{7q-1}{8} = 19 - p - q$$

Step 3:
The right-hand side of the equation now consists only integers (because the variables p and q are positive integers and the constant 19 is also an integer) and the sum or difference of any number of integers will be an integer only. Hence the right hand side of the equation is an integer and can be represented by k_1 (we do not know the value of this integer as of now. We do not even know whether k_1 is positive or negative at this stage).

$$\frac{7q-1}{8} = k_1$$

Step 4:
Rewrite this relationship such that the coefficient of the variable (q in this case) is 1. To do this, multiply both sides of the relationship such that the coefficient of q will then be 1 more than a multiple of the denominator (the denominator is 8 in this case)
By observation, we can make out that 49 (which is a multiple of the coefficient 7) is 1 more than 48 (which is a multiple of the denominator 8). Hence, to get 49, multiply both sides of the relationship with 7.

$$\frac{49q-7}{8} = 7k_1 \Rightarrow \frac{48q}{8} + \frac{q}{8} - \frac{7}{8} = 7k_1$$

$$\Rightarrow \frac{q-7}{8} = 7k_1 - 6q$$

Here again, the right hand side is only sum or difference of integers and hence it will be an integer. We can call this k and thus we get

$$\frac{q-7}{8} = k$$

(Please note that this step has to be done mentally. However, we completely skipped this step in the two examples we took earlier on in this chapter).

Step 5:
Write the variable in the equation in terms of k.
In this case the variable q is written in terms of k as $q = 8k + 7$

Step 6:
Substitute this value of one variable (in terms of k) in the original equation to express the other variable also in terms of k.
Here, putting $q = 8k + 7$ in equation (1), we get $p = 6 - 15k$

Step 7:
On the basis of the values of the two variables (expressed in terms of k), identify what values can k take to ensure that the variables are positive integers.
$q = 8k + 7$ and $p = 6 - 15k$
Here, to ensure that p is a positive integer, k has to be less than or equal to zero (i.e., $k \leq 0$)
To ensure that q is a positive integer, k has to be greater than or equal to zero (i.e., $k \geq 0$)
Now the only value of k that satisfies both these conditions is k = 0.
For k = 0, we get p = 6 and q = 7 Thus, this problem has a unique solution.

Note: In step 7 above, if k can have more than one possible value, then it means that the problem has more than one solution. The problem has as many solutions as the number of values k can take.

Sometimes, some additional conditions will be given in the problem (particularly when k can take more than one value). Those conditions acting on the variables have to be taken into account in step 7 to see that the values of k selected will satisfy these additional conditions given in the problem. Typically, such additional conditions are given in the problem to eliminate the possibility of k having more than one value thus getting us a unique solution to the problem.

Concept Review Questions

Directions for questions 1 to 15: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. The number of positive integer solutions of the equation $2x + 3y = 15$ is
   (A) 0 (B) 1 (C) 2 (D) 3

2. If x and y are integers, which of the following is a possible (x , y) satisfying the equation $3x - 7y = 28$?
   (A) (3, –8) (B) (7, 1)
   (C) (8, 3) (D) (14, 2)

3. If $7x + 4y = 64$ where x and y are positive integers, then the values that x can take is/are
   (A) 4 (B) 8
   (C) 4 and 8 (D) 4, 8 and 12

4. The number of non-negative integral solutions of the equation $8x + 10y = 120$ is ☐

5. If p is a positive integer such that the remainder when 4p divided by 9 is 5, then a possible value of p is
   (A) 3 (B) 4 (C) 6 (D) 8

6. If x is a positive integer and the remainder when 4x divided by 5 is 4, then, one of the values of x is 1, of the
   (A) $5x + 1$ (B) $4x + 3$
   (C) $5x + 4$ (D) $4x - 3$

7. The reminder when 3x divided by 7 is 6, where x is a positive integer. When all the values of x are arranged in progressive order, the difference of any two consecutive values of x is ☐

8. If $3a + 7b = -72$, where a, and b are negative integers, which of the following is a value that b can take?
   (A) –2 (B) –4 (C) –5 (D) –6

9. The number of non-negative integral solutions of the equation $12x + 7y = 35$ is
   (A) 0 (B) 1 (C) 2 (D) 3

10. A student purchases gel pens, ballpoint pens and pencils by spending a total of ₹28. Each gel pen, ball point pen and pencil cost ₹15, ₹5 and ₹3 respectively. In how many combinations can he purchase them, if he buys at least one of each item?
    ☐

11. In the above problem, the number of ball point pens the student purchases is
    (A) 0 (B) 1 (C) 2 (D) 3

12. On a particular day, a salesman sold three types of toys. Each toy of the 3 varieties costs ₹100, ₹50 and ₹25 respectively. If the total sale on that day was ₹300 and the salesman sold at least one toy of each variety, find the maximum number of toys he could have sold.
    (A) 4 (B) 5 (C) 6 (D) 8

13. In the above problem, find the minimum number of toys the salesman could have sold.
    ☐

14. Seven times a number plus eleven times another is equal to 61. Then a possible value of one of the numbers is
    (A) 4 (B) 2 (C) 1 (D) 6

15. The sum of a two-digit number and the number formed by reversing its digits is equal to 88. The number of such numbers in which the tens digit is greater than the units digit is ☐

Exercise – I(a)

Directions for questions 1 to 30: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. Ramya told Krishna "12 times the date of my birth added to 31 times the month of my birth is 376". On which date was Ramya born?
(A) 24th (B) 23rd (C) 22nd (D) 21st

2. Nayana purchased some pastries and cookies, each cooky costing ₹3 and each pastry costing ₹8. In how many different combinations can she buy the items if she spends a total amount of ₹68?

3. (i) Nishant takes up an assignment wherein for each working day he is paid ₹75 but for each day he reports late for work, ₹12 is deducted from his pay for that day. If at the end of the month, Nishant is paid an amount of ₹1815, then for how many days did Nishant report late?
(A) 5 (B) 10 (C) 15 (D) 2

(ii) In the above question, for how many days did Nishant report to the assignment?
(A) 21 (B) 25 (C) 23 (D) 24

4. Atal and Bharat have a collection of less than 50 stamps together. If Bharat gives Atal a certain number of stamps, then Atal has five times as many as Bharat has and instead if Atal gives the same number of stamps to Bharat, then Atal will now have three times as many as Bharat. The number of stamps with Atal can be
(A) 18 (B) 16 (C) 38 (D) 15

5. The average weight of a certain group of 'n' men is 75 kg. Three men whose weights are 80 kg, 76 kg and 74 kg join the group and one man whose weight is between 90 kg and 100 kg leaves the group. The average of the group has now come down by 2 kg. If the number of men initially is a perfect square, then the weight of the man who left is
(A) 98 kg (B) 96 kg
(C) 94 kg (D) 92 kg

6. In how many ways can 141 be divided into two parts such that one part is divisible by 5 and the remaining part is divisible by 9?

7. Sachin wants to buy some samosas, kachoris and curry puffs, spending a total of ₹50. If each samosa costs ₹6, kachori ₹5 and curry puff ₹3 and if Sachin wants to buy at least three eatables of each variety, then in how many different combinations can he purchase the snacks?
(A) 1 (B) 2 (C) 3 (D) 4

8. On a rainy day, a dealer sold some umbrellas, raincoats and caps with the respective prices being ₹100, ₹120 and ₹25. If he makes a total collection ₹560 at the end of the day, how many pieces did he sell totally?
(A) 4 (B) 3 (C) 8 (D) 7

9. In an organization, there are totally 38 employees belonging to three different departments – A, B and C with each department having more than 5 employees. The organisation decides to pay bonuses of ₹11,000, ₹6,000 and ₹3,000 to each of the employees of departments A, B and C respectively. If the total amount paid to the employees as bonuses is ₹3,32,000, then the number of employees in the department 'C' is ______.

Directions for questions 10 and 11: These questions are based on the data given below.

Arjun buys some CDs and cassettes spending a total amount of ₹2000 with each CD costing ₹50 and each cassette ₹10. Instead, if he now buys as many cassettes as the CDs and as many CDs as the cassettes that he bought earlier, he will now spend an amount less than half of what he spent earlier.

10. In how many different combinations can he buy the CDs and cassettes?
(A) 1 (B) 2 (C) 3 (D) 4

11. If he bought at least 10 of each variety, then how many more CDs did he buy than the cassettes?
(A) 28 (B) 26 (C) 37 (D) 31

12. In a triangle, the measures of all the angles are acute and 19 times the measure of an angle equals 15 times the measure of the other. What is the least possible measure (in degrees) of any of the angles in the triangle?

13. How many 3 digit numbers leave a remainder of 4 when divided by 21 and a remainder of 8 when divided by 13?
(A) 2 (B) 3 (C) 5 (D) 4

14. Eatwell Chocolates, sell three types of candies namely, eclairs, caramel and mint priced at ₹10, ₹2 and ₹1 for each candy. Rakshitha purchased these candies worth ₹100. She bought 4 times as many mints as caramel. What is the maximum number of eclairs she could have bought?

Directions for questions 15 and 16: These questions are based on the data given below.

Mungeri Lal of Zaveri and Zaveri Sons, bought three types of precious stones – emerald, jade and topaz, totaling 27 and costing totally ₹60,000. The prices of which being ₹1500, ₹2000 and ₹2500 in that order. He has bought more than 3 stones of each variety and he has not bought the same number of stones of any two varieties.

15. How many emeralds and jades together did Mungerilal buy?
(A) 10 (B) 15 (C) 11 (D) 23

For the Non-Multiple Choice Questions, write your answer in the box pr...

1. Two friends Bharathi and Pranathi had a certain number of 5-rupee coins. If Pranathi gives a certain number of coins to Bharathi then Bharathi will have ₹50 more than Pranathi. Instead, if Bharathi gives the same number of coins to Pranathi, then Pranathi will have ₹30 more than Bharathi. Find the total number of coins the two can have.
(A) 22 (B) 21 (C) 23 (D) 19

2. In how many ways can 249 be divided into two positive parts such that one part is divisible by 7 and the other part is divisible by 11?

3. Deepti told Bhavya "10 times the month of my birth added to 12 times the date of my birth is 388. In which month could Deepti have been born?
(A) August (B) September
(C) October (D) February

4. Taste Well, a confectioner, sells two types of chocolates- milky bars and cookies priced at ₹8 and ₹5 respectively. Shiny purchased some of these for a total of ₹110. In how many different combinations could she have bought the items?
(A) 1 (B) 3 (C) 2 (D) 4

5. Dhanunjay has 72 coins of two rupee, one-rupee and fifty paise denominations. The total value of the coins is ₹66. The number of coins of each denomination is a multiple of 6 greater than 10. Find the number of fifty paise coins he has.
(A) 12 (B) 18 (C) 36 (D) 24

6. Nikhil takes up an assignment. On each working day, he is given a target. For each day he meets the target he is paid ₹95 and for each day he does not meet the target he is paid ₹18 less. By the end of the month, (30 days) he is paid a total of ₹2724. For how many days did he meet the target?
(A) 21 (B) 22 (C) 23 (D) 18

7. In the above question, for how many days did Nikhil not meet the target?
(A) 12 (B) 14 (C) 8 (D) 7

8. A fruit vendor sells two varieties of fruits-mangoes, at ₹12 each and apples at ₹15 each. On a particular day he sold fruits worth ₹387. If he sold at least one of each variety, in how many combinations could he have sold the fruits?

9. In the question above, what is the maximum number of apples he could have sold?
(A) 20 (B) 25 (C) 17 (D) 21

10. In 5 seconds, a magician can triple pink flowers and quadruple blue flowers. Twenty seconds after he begins, the total number of flowers is 917. Find the difference between the number of the pink and the blue flowers, 10 seconds after he begins.

11. Bhanu purchased some pens and erasers for ₹126. Had he interchanged the number of pens and erasers, it wouldn't have cost him more than ₹63. If each pen and each eraser cost him ₹6 and ₹0.75 respectively, then the total number of pens and erasers he actually purchased can be
(A) 20 (B) 28 (C) 24 (D) 26

12. In an acute angled triangle, thirteen times an angle is equal to seventeen times another. The angles, measured in degrees, are integers. One angle is less than or equal to 30°. The difference between the other two angles is
(A) 16° (B) 12° (C) 22° (D) 20°

13. Mahta Tools Ltd, a manufacturer of tools produces a certain type of pliers. Adam, a worker in the packaging section, is supposed to pack less than 275 pliers into boxes. Adam finds that if he packs nine more pliers in each box, he can pack them in five boxes less. If he has to pack more than 10 boxes, find the number of pliers he can actually pack.

14. On a rainy day a person sells raincoats, umbrellas and caps for a total of ₹1065. He sells each raincoat at ₹250, each umbrella at ₹100 and each cap at ₹55 and he sells at least one of each item. In how many combinations, could he have sold the items?
(A) 1 (B) 2 (C) 3 (D) 4

15. Kumar plays a game in which a dice is cast. Whenever one shows up he receives ₹50. When three shows up, he receives ₹75 and when five shows up he has to pay ₹30. He played the game ten times and every time he played, only 1, 3 or 5 showed up. He got a total of ₹360. Find the number of times he received ₹75.
(A) 1 (B) 2 (C) 3 (D) 4

Directions for questions 16 and 17: These questions are based on the data given below.

Arun plays a game wherein he casts a dice and scores six points if an odd number turns up and 9 points if an even number turns up. A total of exactly 117 points is required to win the game.

16. In how many ways can Arun win the game?
(A) 7 (B) 3 (C) 4 (D) 6

17. Arun casts the dice exactly 16 times and wins. What is the maximum number of times a 2 or a 4 could have turned up?
(A) 9 (B) 7 (C) 8 (D) 6

18. A three-digit number leaves a remainder of 16 when divided by 25 and a remainder of 5 when divided by 21. How many such numbers exist?

Directions for questions 19 and 20: These questions are based on the data given below.

A certain number of administrators and technicians attend an industrial exhibition. The total number of persons who attend is 120. Among the technicians, 9 times the male members is equal to 13 times the female members.

19. Among the technicians who attend, find the maximum possible difference between the number of male and female members.
(A) 15 (B) 20 (C) 16 (D) 12

20. The ratio of the number of male administrators to the number of female administrators who attend is 9 : 7. Find the ratio of the total number of males to the total number of females.
(A) 7 : 5 (B) 5 : 9 (C) 9 : 7 (D) 5 : 4

21. The average marks of a group of n students in a subject is 65. Three students with marks 48, 66 and 56 leave the group and one student with marks between 55 and 65 join the group. As a result, the average of the group goes up by 3. If the number of students in the group initially is an odd perfect square then the marks of the student who joined the group are ☐.

22. In the previous question, the number of students in the group initially is
(A) 8 (B) 49 (C) 25 (D) 9

23. Saroja purchases three different varieties of pens – Add gel, Montex and Luxes - for a total of ₹285. She decides to buy twice as many Luxes pens as Addgel pens and more Montex pens than Addgel pens but at the most, 15 more Montex pens than Addgel pens. Montex pens are one and half times as expensive as Luxes pens and Addgel pens are two and half times as expensive as Luxes pens. The cost of each Luxes pen is ₹2 Find the maximum number of Montex pens she could have bought for the amount she spent.
(A) 30 (B) 35 (C) 32 (D) 38

Directions for questions 24 and 25: These questions are based on the following data.

Lal and Lal Jewellers want to buy 32 precious stones, comprising diamonds, rubies and emeralds, for a total of ₹79,000. The prices of the precious stones are ₹3500, ₹3000 and ₹1000 respectively. They want to buy more than five stones of each variety.

24. If they want to purchase the least possible number of rubies, then how many diamonds and emeralds should they buy?
(A) 26 (B) 22 (C) 24 (D) 28

25. If the difference between the number of stones of the two varieties is one, then how many emeralds can they buy?
(A) 10 (B) 8 (C) 11 (D) 12

26. How many ordered pairs of positive integers (x, y) satisfy the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{18}$?

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27. How many ordered pairs of integers (x, y), where x > 0 satisfy the equation $\frac{1}{x} + \frac{7}{y} = \frac{1}{21}$?
(A) 14 (B) 24 (C) 16 (D) 12

28. How many ordered pairs of integers satisfy the equation $\frac{13}{x} - \frac{7}{y} = \frac{1}{4}$?

☐

29. How many integral values of (x, y) satisfy the equation $x^2 - y^2 = 627$?
(A) 20 (B) 10 (C) 16 (D) 8

30. How many integral values of (x, y) satisfy the equation $x^2 - y^2 = 220$?

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Key

Concept Review Questions

1. C	4. 4	7. 7	10. 1	13. 5
2. D	5. D	8. D	11. C	14. A
3. C	6. A	9. B	12. D	15. 4

Exercise – 1(a)

1. D	7. A	14. 7	21. 6	28. 39
2. 3	8. C	15. C	22. A	29. A
3. (i) A	9. 7	16. C	23. 20	30. 12
(ii) B	10. B	17. A	24. B	
4. C	11. A	18. B	25. 4	
5. D	12. 44	19. 208	26. 21	
6. 3	13. D	20. C	27. A	

Exercise – 1(b)

1. A	5. C	9. B	13. 270	17. B	21. 61	25. C	29. C
2. 4	6. C	10. 13	14. B	18. 2	22. D	26. 15	30. 8
3. C	7. D	11. B	15. D	19. B	23. B	27. C	
4. B	8. 7	12. D	16. A	20. A	24. A	28. 39	

PERMUTATIONS AND COMBINATIONS

Permutations and Combinations is one of the important areas in many exams because of two reasons. The first is that solving questions in this area is a measure of students' reasoning ability. Secondly, solving problems in areas like Probability requires thorough knowledge of Permutations and Combinations.

Before discussing Permutations and Combinations, let us look at what is called as the "fundamental rule"

"If one operation can be performed in 'm' ways and (when, it has been performed in any one of these ways), a second operation then can be performed in 'n' ways, the number of ways of performing the two operations will be m x n".

This can be extended to any number of operations.

If there are three cities A, B and C such that there are 3 roads connecting A and B and 4 roads connecting B and C, then the number of ways one can travel from A to C is 3 x 4, i.e., 12.

This is a very important principle and we will be using it extensively in Permutations and Combinations. Because we use it very extensively, we do not explicitly state every time that the result is obtained by the fundamental rule but directly write down the result.

PERMUTATIONS

Each of the arrangements which can be made by taking some or all of a number of items is called a Permutation. Permutation implies "arrangement" or that "order of the items" is important.

The permutations of three items a, b and c taken two at a time are ab, ba, ac, ca, cb and bc. Since the order in which the items are taken is important, ab and ba are counted as two different permutations. The words "permutation" and "arrangement" are synonymous and can be used interchangeably.

The number of permutations of n things taking r at time is denoted by ^nP_r (and read as "nPr")

COMBINATIONS

Each of the groups or selections which can be made by taking some or all of a number of items is called a Combination. In combinations, the order in which the items are taken is not considered as long as the specific things are included.

The combination of three items a, b and c taken two at a time are ab, bc and ca. Here, ab and ba are not considered separately because the order in which a and b are taken is not important but it is only required that a combination including a and b is what is to be counted. The words "combination" and "selection" are synonymous.

The number of combinations of n things taking r at time is denoted by ^nC_r (and read as "nCr")

When a problem is read, it should first be clear to you as to whether it is a permutation or combination that is being discussed. Sometimes the problem specifically states whether it is the number of permutations (or arrangements) or the number of combinations (or selections) that you should find out. The questions can be as follows:

For permutations, "Find the number of permutations that can be made ....." OR "Find the number of arrangements that can be made....." OR "Find the number of ways in which you can arrange....."

For combinations,
"Find the number of combinations that can be made ....."
OR "Find the number of selections that can be made....."
OR "Find the number of ways in which you can select....."

Sometimes, the problem may not explicitly state whether what you have to find out is a permutation or a combination but the nature of what is to be found out will dictate whether it is the number of permutations or the number of combinations that you have to find out. Let us look at the following two examples to clarify this.

"How many four digit numbers can be made from the digits 1, 2, 3 and 4 using each digit once?"

Here, since we are talking of numbers, the order of the digits matters and hence what we have to find out is permutations.

"Out of a group of five friends that I have, I have to invite two for dinner. In how many different ways can I do this?"

Here, if the five friends are A, B, C, D and E, whether the two friends that I call for dinner on a particular day are A and B **or** B and A, it does not make any difference, i.e., here the order of the "items" does not play any role and hence it is the number of combinations that we have to find out.

Now we will find out the number of permutations and combinations that can be made from a group of given items.

Initially, we impose two constraints (conditions) while looking at the number of permutations. They are

- all the n items are distinct or dissimilar (or no two items are of the same type)
- each item is used at most once (i.e., no item is repeated in any arrangement)

Number of linear permutations of 'n' dissimilar items taken 'r' at a time without repetition (^nP_r)

Consider r boxes each of which can hold one item. When all the r boxes are filled, what we have is an arrangement of r items taken from the given n items. So, each time we fill up the r boxes with items taken from the given n items, we have an arrangement of r items taken from the given n items without repetition. Hence the number of ways in which we can fill up the r boxes by taking things from the given n things is equal to the number of permutations of n things taking r at a time.

Boxes □ □ □ □ .... □

1 2 3 4 r

The first box can be filled in n ways (because any one of the n items can be used to fill this box). Having filled the first box, to fill the second box we now have only $(n-1)$ items; any one of these items can be used to fill the second box and hence the second box can be filled in $(n-1)$ ways; similarly, the third box in $(n-2)$ ways and so on the r^{th} box can be filled in $\{n-(r-1)\}$ ways, i.e. $[n-r+1]$ ways. Hence, from the Fundamental Rule, all the r boxes together be filled up in

$n \cdot (n-1) \cdot (n-2)$..................$(n-r+1)$ ways

So, ${}^nP_r = n \cdot (n-1) \cdot (n-2)$..............$(n-r+1)$

This can be simplified by multiplying and dividing the right hand side by $(n-r)(n-r-1)$.........3.2.1 giving us

${}^nP_r = n(n-1)(n-2)$ .... $[n-(r-1)]$

$$= \frac{(n-1)(n-2).....[n-(r-1)].(n-r).......3.2.1}{(n-r)..................3.2.1}$$

$$= \frac{n!}{(n-r)!}$$

The number of permutations of n distinct items taking r items at a time is

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

If we take n items at a time, then we get ${}^nP_n$. From a discussion similar to that we had for filling the r boxes above, we can find that ${}^nP_n$ is equal to n!

The first box can be filled in n ways, the second one in $(n-1)$ ways, the third one in $(n-2)$ ways and so on, then the n^{th} box in 1 way; hence, all the n boxes can be filled in $n(n-1)(n-2)$ ....3.2.1 ways, i.e., n! ways. Hence,

$$\boxed{{}^nP_n = n!}$$

But if we substitute $r = n$ in the formula for ${}^nP_r$, then we get ${}^nP_n = \frac{n!}{0!}$; since we already found that ${}^nP_n = n!$, we can conclude that $0! = 1$

Number of combinations of n dissimilar things taken r at a time

Let the number of combinations ${}^nC_r$ be x. Consider one of these x combinations. Since this is a combination, the order of the r items is not important. If we now impose the condition that order is required for these r items, we can get r! arrangements from this one combination. So each combination can give rise to r! permutations. x combinations will thus give rise to x . r! permutations. But since these are all permutations of n things taken r at a time, this must be equal to ${}^nP_r$. So,

$$x.r! = {}^nP_r = \frac{n!}{(n-r)!} \Rightarrow {}^nC_r = \frac{n!}{r!.(n-r)!}$$

The number of combinations of n dissimilar things taken all at a time is 1

Out of n things lying on a table, if we select r things and remove them from the table, we are left with (n-r) things on the table - that is, whenever r things are selected out of n things, we automatically have another selection of the $(n-r)$ things. Hence, the number of ways of making combinations taking r out of n things is the same as selecting $(n-r)$ things out of n given things, i.e.

$${}^nC_r = {}^nC_{n-r}$$

When we looked at ${}^nP_r$, we imposed two constraints which we will now release one by one and see how to find out the number of permutations.

Number of arrangements of n items of which p are of one type, q are of a second type and the rest are distinct

When the items are all not distinct, then we cannot talk of a general formula for ${}^nP_r$ for any r but we can talk of only ${}^nP_n$ (which is given below). If we want to find out ${}^nP_r$ for a specific value of r in a given problem, we have to work on a case to case basis (this has been explained in one of the solved examples).

The number of ways in which n things may be arranged taking them all at a time, when p of the things are exactly alike of one kind, q of them exactly alike of another kind, r of them exactly alike of a third kind, and the rest all distinct is

$$\frac{n!}{p!\,q!\,r!}$$

Number of arrangements of n distinct items where each item can be used any number of times (i.e., repetition allowed)

You are advised to apply the basic reasoning given while deriving the formula for ${}^nP_r$ to arrive at this result also. The first box can be filled up in n ways; the second box can be filled in again inn ways (even though the first box is filled with one item, the same item can be used for filling the second box also because repetition is allowed); the third box can also be filled in n ways and so on ... the r^{th} box can be filled in n ways. Now all the r boxes together can be filled in {n.n.n.n ... r times} ways, i.e., n^r ways.

The number of permutations of n things, taken r at a time when each item may be repeated once, twice, ... up to r times in any arrangement is n^r

What is important is not this formula by itself but the reasoning involved. So, even while solving problems of this type, you will be better off if you go from the basic reasoning and not just apply this formula.

Total number of combinations:

Out of n given things, the number of ways of selecting **one or more** things is where we can select 1 or 2 or 3 ... and so on n things at a time; hence the number of ways is ${}^nC_1 + {}^nC_2 + {}^nC_3 +$ ..... ${}^nC_n$

This is called "the total number of combinations" and is equal to $2^n - 1$ where n is the number of things.

The same can be reasoned out in the following manner also.

There are n items to select from. Let each of these be represented by a box.

	1	2	3	4		n
No. of ways of dealing with the boxes	□	□	□	□	....	□
	2	2	2	2		2

The first box can be dealt with in two ways. In any combination that we consider, this box is **either** included or not included. These are the two ways of dealing with the first box. Similarly, the second box can be dealt with in two ways, the third one in two ways and so on, the n^{th} box in two ways. By the Fundamental Rule, the number of ways of dealing with all the boxes together in $2 \cdot 2 \cdot 2 \cdot$ .... n times ways, i.e., in 2^n ways. But out of these, there is one combination where we "do not include the first box, do not include the second box, do not include the third box and so on, do not include the n^{th} box." That means, no box is included. But this is not allowed because we have to select **one or more** of the items (i.e., at least one item). Hence this combination of no box being included is to be subtracted from the 2^n ways to give the result of

Number of ways of selecting one or more items from n given items is $2^n - 1$

Dividing given items into groups:

Dividing (p + q) items into two groups of p and q items respectively

Out of (p + q) items, if we select p items (which can be done in ${}^{p+q}C_p$ ways), then we will be left with q items, i.e., we have two groups of p and q items respectively. So, the number of ways of dividing (p + q) items into two groups of p and q items respectively is equal to ${}^{p+q}C_p$ which is equal to $\dfrac{(p+q)!}{p! \cdot q!}$

The number of ways of dividing (p + q) items into two groups of p and q items respectively is $\dfrac{(p+q)!}{p!\,q!}$

If p = q, i.e., if we have to divide the given items into two EQUAL groups, then two cases arise
(i) when the two groups have distinct identity and
(ii) when the two groups do not have distinct identity.

In the first case, we just have to substitute p = q in the above formula which then becomes

The number of ways of dividing 2p items into two equal groups of p each is $\dfrac{(2p)!}{(p!)^2}$ where the two groups have distinct identity.

In the second case, where the two groups do not have distinct identity, we have to divide the above result by 2!, i.e., it then becomes

The number of ways of dividing 2p items into two equal groups of p each is $\dfrac{(2p)!}{2!.(p!)^2}$ where the two groups do not have distinct identity.

Dividing (p + q + r) items into three groups consisting of p, q and r items respectively

The number of ways in which (p + q + r) things can be divided into three groups containing p, q and r things respectively is $\dfrac{(p+q+r)!}{p!q!r!}$

If p = q = r, i.e., if we have to divide the given items into three EQUAL groups, then we have two cases where the three groups are distinct and where the groups are not distinct.

When the three groups are distinct, the number of ways is $\dfrac{(3p)!}{(p!)^3}$

When the three groups are not distinct, then the number of ways is $\dfrac{(3p)!}{3!\,(p!)^3}$

Circular Permutations:

When n distinct things are arranged in a straight line taking all the n items, we get n! permutations. However, if these n items are arranged in a circular manner, then the number of arrangements will not be n! but it will be less than that. This is because in a straight line manner, if we have an arrangement ABCDE and if we move every item one place to the right (in cyclic order), the new arrangement that we get EABCD is not the same as ABCDE and this also is counted in the n! permutations that we talked of. However, if we have an arrangement ABCDE in a circular fashion, by shifting every item by one place in the clockwise direction, we still get the same arrangement ABCDE. So, if we now take n! as the number of permutations, we will be counting the same arrangement more than once.

The number of arrangements in circular fashion can be found out by first fixing the position of one item. Then the remaining (n – 1) items can be arranged in (n – 1)! ways. Now even if we move these (n – 1) items by one place in the clockwise direction, then the arrangement that we get will not be the same as the initial arrangement because one item is fixed and it does not move.

Hence, the number of ways in which n distinct things can be arranged in a circular arrangement is (n – 1)!

If we take the case of five persons A, B, C, D and E sitting around a table, then the two arrangements ABCDE (in clockwise direction) and AEDCB (the same order but in anticlockwise direction) will be different and distinct. Here we say that the clockwise and anticlockwise arrangements are different. However, if we consider the circular arrangement of a necklace made of five precious stones A, B, C, D and E, the two arrangements talked of above will be the same because we take one arrangement and turn the necklace around (front to back), then we get the other arrangement. Here, we say that there is no difference between the clockwise and anticlockwise arrangements. In this case the number of arrangements will be half of what it is in the case where the clockwise and anticlockwise arrangements are different.

> The number of **circular arrangements** of n **distinct items** is
> **(n – 1)!** if there is **DIFFERENCE** between clockwise and anticlockwise arrangements and
> **(n – 1)!/2** if there is **NO DIFFERENCE** between clockwise and anticlockwise arrangements

Sum of all numbers formed from given digits:

If n distinct digits are used to make all the possible n-digit numbers, we get n! numbers. We now want to find out the sum if all these n! numbers are added. Let us take an example and understand how it is to be done and then look it as a formula.

To find the sum of all the four digit numbers formed using the digits 2, 3, 4 and 5 without repetition:

We can form a total of 4! or 24 numbers. When we add all these numbers, let us look at the contribution of the digit 2 to the sum.

When 2 occurs in the thousands place in a particular number, its contribution to the total will be 2000. The number of numbers that can be formed with 2 in the thousands place is 3!, i.e., 6 numbers. Hence, when 2 is in the thousands place, its contribution to the sum is 3! x 2000.

Similarly, when 2 occurs in the hundreds place in a particular number, its contribution to the total will be 200 and since there are 3! numbers with 2 in the hundreds place, the contribution 2 makes to the sum when it comes in the hundreds place is 3! x 200.

Similarly, when 2 occurs in the tens and units place respectively, its contribution to the sum is 3! x 20 and 3! x 2 respectively. Thus the total contribution of 2 to the sum is 3! x 2000 + 3! x 200 + 3! x 20 + 3! x 2, i.e., 3! x 2222. This takes care of the digit 2 completely.

In a similar manner, the contribution of 3, 4 and 5 to the sum will respectively be 3! x 3333, 3! x 4444 and 3! x 5555 respectively.

The sum can now be obtained by adding the contributions of these four digits. Hence the sum of the numbers formed by using the four digits is 3! x (2222 + + 3333 + 4444 + 5555), i.e.,
3! x (2 + 3 + 4 + 5) x 1111

We can now generalize the above as

> **If all the possible n-digit numbers using n distinct digits are formed, the sum of all the numbers so formed is equal to (n-1)! x {sum of the n digits} x {11111 ....} n times**

Rank of a word:

Finding the rank of a given word is basically finding out the position of the word when all possible words have been formed using all the letters of this word exactly once and arranged in alphabetical order as in the case of dictionary. Let us understand this by taking an example

Let us look at the word "POINT". The letters involved here, when taken in alphabetical order are I, N, O, P, T.

To arrive at the word "POINT", initially we have to go through the words that begin with I, then all those that begin with N, those that begin with O which are 4! in each case. Then we arrive at words that begin with PI, PN which are 3! in each case. Then we arrive at the word POINT.

There are $3 \times 4! + 2 \times 3! = 84$ words that precede the word POINT i.e., POINT is the 85th word. Hence rank of 'POINT' is 85.

The number of diagonals in an n-sided regular polygon

An n-sided regular polygon has n vertices. Joining any two vertices we get a line of the polygon which are ${}^nC_2$ in number. Of these ${}^nC_2$ lines, n of them are sides. Hence diagonals are

$${}^nC_2 - n = \frac{n(n-3)}{2}$$

Number of integral solution of the equation $x_1 + x_2 + \ldots\ldots + x_n = s$

Consider the equation

$x_1 + x_2 + x_3 = 10$
If we consider all possible integral solutions of this equation, there are infinitely many. But the number of positive (or non-negative) integral solutions is finite.

We would like the number of positive integral solutions of this equation, i.e., values of (x_1, x_2, x_3) such that each $x_i > 0$.

We imagine 10 identical objects arranged on a line. There are 9 gaps between these 10 objects. If we choose any two of these gaps, we are effectively splitting the 10 identical objects into 3 parts of distinct identity. Conversely, every split of these 10 objects corresponds to a selection of 2 gaps out of the 9 gaps.

Therefore, the number of positive integral solutions is ${}^9C_2$. In general, if $x_1 + x_2 + \ldots\ldots + x_n = s$ where $s \geq n$, the number of positive integral solutions is ${}^{s-1}C_{n-1}$.

If we need the number of non negative integral solutions, we proceed as follows. Let $a_1, a_2, \ldots\ldots$ be a non-negative integral solution. Than $a_1 + 1, a_2 + 1, \ldots\ldots a_n + 1$ is a positive integral solution of the equation $x_1 + x_2 + \ldots\ldots + x_n = s + n$. Therefore, the number of non-negative integral solutions of the given equation is equal to the number of positive integral solutions of $x_1 + x_2 + \ldots\ldots + x_n = s + n$, which is ${}^{s+n-1}C_{n-1}$.

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Number of integral solution of the equation $x_1 + x_2 + \dots\dots + x_n = s$

Consider the equation

$x_1 + x_2 + x_3 = 10$

If we consider all possible integral solutions of this equation, there are infinitely many. But the number of positive (or non-negative) integral solutions is finite.

We would like the number of positive integral solutions of this equation, i.e., values of (x_1, x_2, x_3) such that each $x_i > 0$.

We imagine 10 identical objects arranged on a line. There are 9 gaps between these 10 objects. If we choose any two of these gaps, we are effectively splitting the 10 identical objects into 3 parts of distinct identity. Conversely, every split of these 10 objects corresponds to a selection of 2 gaps out of the 9 gaps.

Therefore, the number of positive integral solutions is ${}^9C_2$.
In general, if $x_1 + x_2 + \dots\dots + x_n = s$ where $s \geq n$, the number of positive integral solutions is ${}^{s-1}C_{n-1}$.

If we need the number of non negative integral solutions, we proceed as follows. Let $a_1, a_2, \dots\dots$ be a non-negative integral solution. Than $a_1 + 1$, $a_2 + 1$, ... $a_n + 1$ is a positive integral solution of the equation $x_1 + x_2 + \dots\dots + x_n = s + n$. Therefore, the number of non-negative integral solutions of the given equation is equal to the number of positive integral solutions of $x_1 + x_2 + \dots\dots + x_n = s + n$, which is ${}^{s+n-1}C_{n-1}$.

For $x_1 + x_2 + x_3 + \ldots\ldots + x_n = s$ where $s \geq 0$, the number of **positive integral solutions** (when $s \geq n$) is ${}^{s-1}C_{n-1}$ and the number of **non-negative integral solutions** is ${}^{n+s-1}C_{n-1}$

Some additional points:

1. Suppose there are n letters and n corresponding addressed envelopes. The numbers of ways of placing these letters into the envelopes such that no letter is placed in its corresponding envelope is often referred as derangements. The number of derangements of n objects is given by

$$D(n) = n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}\right]$$

For example, when n = 3, the number of derangements is

$$D(3) = 3!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right]=2 \text{ and when } n = 4,$$

$$D(4) = 4!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right]=9$$

2. The total number of ways in which a selection can be made by taking some or all out of p + q + r + ... things where p are alike of one kind, q alike of a second kind, r alike of a third kind and so on is $[\{(p+1)(q+1)(r+1)\ldots\}-1]$.

3. ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$ and ${}^nP_r = r\,{}^{n-1}P_{r-1} + {}^{n-1}P_r$

Examples:

9.01. Find the number of permutations and combinations which can be made by taking four items at a time from six given distinct items, without repetition.

Sol. The number of permutations is

$${}^6P_4 = \frac{6!}{2!} = 360 \quad \left[{}^nP_r = \frac{n!}{(n-r)!}\right]$$

The number of combinations is ${}^6C_4$

$$= \frac{6!}{4!\,2!} = 15 \quad \left[{}^nC_r = \frac{n!}{(n-r)!\,r!}\right]$$

9.02. In how many ways can three persons be seated in five chairs?

Sol. Since each person occupies exactly one chair, it is a question of arrangements without repetition. The first person can be seated in 5 ways (into any of the five chairs). The second person can be seated in 4 ways (into any of the remaining 4 chairs). Similarly the third person can be seated in 3 ways. Hence the number of ways in which these 3 persons can arrange themselves in 5 chairs is 5 x 4 x 3 = 60.

9.03. Fairplay Athletics club has six coaches, viz. A, B, C, D, E and F. A panel of coaches comprising of three members has to be formed.

(i) In how many ways can the panel be formed?

(ii) How many of these panels always include coach E?

(or)

In how many ways can the panel be formed if coach E has to be definitely included?

(iii) In how many of these panels (as discussed in (i) above) will coach C be excluded?

In how many ways can the panel be formed, if coach C is to be excluded?

(iv) In how many ways can the panel be formed if coach C and coach F should be there together if at all any one of them is there?

(v) In how many ways can the panel be formed if coach E and coach B cannot be together on the panel?

(vi) In how many ways can the panel be formed if exactly one among A and B should be included?

(vii) In how many ways can the panel be formed if it is known that coach D will not be on the panel if coach A is there on it?

Sol. (i) The panel can be formed in ${}^6C_3$ ways i.e. 20 ways.

(ii) If coach E has to always be there on the panel, we need to select only 2 more coaches from the remaining 5 coaches. This can be done in ${}^5C_2$ i.e., 10 ways.

(iii) If coach C should be excluded, then out of the remaining five coaches, we need to select three coaches, which can be done in ${}^5C_3$ i.e., 10 ways. Alternately, as discussed in the previous case, if coach C is there in 10 out of 20 ways, in the remaining 20-10 i.e. 10 ways, coach C will not be there.

(iv) Here, we need to consider 2 cases- one where both coaches C and F are included on the panel and the second where neither of the two coaches are included.

In the 1st case, out of 3 persons, 2 (coaches C and F) are directly decided and the remaining one can be any one of the other FOUR coaches, which can be done in 4 ways.

In the 2nd case, we need to select 3 coaches from 4 coaches, this can be done in ${}^4C_3$ i.e. 4 ways.

∴Total number of ways = 4 + 4 = 8 ways

(v) If coaches B and E cannot be together, we will find out in how many ways they can be together on the panel and then subtract this figure from the total of 20. If coaches B and E have to be included on the panel, we need one more from the remaining four persons, this can be done in 4 ways. Hence, the required figure is 20 – 4 = 16 ways.

(vi) The given condition can be broken down into 2 distinct conditions. (i) coach A is included on the panel and coach B is NOT and (ii) coach A is NOT included on the panel and coach B is included on the panel".

Let us look at the 1st case, "when coach A is included on the panel and coach B is NOT". There are 6 ways of doing this [as done in (v) above]. Similarly, the 2nd case also can be done in 6 ways.

∴Total number of ways = 6 + 6 = 12

(vii) Let us consider the two possibilities – one where coach A is NOT included on the panel and the second where coach A is included on the panel. In the 1st case, where 3 coaches from the remaining 5 coaches, can be selected in ${}^5C_3$ i.e.

10 ways. If coach A is on the panel, then coach D is NOT on the panel. Then the other 2 coaches can be selected out of the remaining four coaches, in ${}^4C_2$ i.e. 6 ways.

∴ Total number of ways = 10 + 6 = 16

9.04. From ten persons waiting in a queue, in how many ways can a selection of six be made so that
(i) a specified person is always included?
(ii) a specified person is always excluded?

Sol. (i) Since one specified person is included we have to choose 5 out of remaining 9 which can be done in

$${}^9C_5 = \frac{9!}{5!\,4!} = \frac{(9\times8\times7\times6)}{(4\times3\times2\times1)} = 126 \text{ ways}$$

(ii) Since one specified person is excluded, we have to choose 6 out of 9 which can be done in

$${}^9C_6 = \frac{9!}{6!\,3!} = \frac{9\times8\times7}{3\times2} = 84 \text{ ways}$$

Note: On the basis of the above example, we can generalize the following two rules:

> The number of combinations of 'n' things taken 'r' at a time in which 'p' particular things will always occur is ${}^{n-p}C_{r-p}$
>
> The number of combinations of 'n' things taken 'r' at a time in which 'p' particular things never occur is ${}^{n-p}C_r$

9.05. A cultural committee of eight is to be formed from nine Asians and five Africans. In how many ways can it be done when the committee consists of
(i) exactly 3 Africans
(ii) at least 3 Africans

Sol. (i) We have to choose 3 out of 5 Africans and 5 out of 9 Asians which can be done in

$${}^5C_3 \times {}^9C_5 = \frac{5!}{3!\,2!} \times \frac{9!}{4!\,5!} = 1260 \text{ ways}$$

(ii) The committee may have 3, 4 or 5 Africans. We compute the number of ways in each case and add them up. Hence, total number of ways
$= {}^5C_3 \times {}^9C_5 + {}^5C_4 \times {}^9C_4 + {}^5C_5 \times {}^9C_3$
$= 1260 + 630 + 84 = 1974$ ways

9.06. Out of seven different pens that I have, each day I take two to the office. If the combinations of the two pens on any given day is not the same as that on any other day, find the number of days that are required to exhaust all such combinations. On how many days will I be taking one particular pen to the office during this period?

Sol. Taking 2 out of 7 pens each day is same as selecting 2 out of 7 pens which can be done in ${}^7C_2 = 21$ ways.
As these selections are distinct, it would require 21 days to exhaust all such possible combinations. Let us fix one pen. Then one more pen from the remaining 6 can be chosen in ${}^6C_1$ i.e. 6 ways. Thus I can take one specified pen on 6 days in this 21 days span.

9.07. Out of six consonants and three vowels, how many words can be made so that each word contains two consonants and three vowels?

Sol. We have to choose 2 consonants and 3 vowels, and arrange them into words. The 2 consonants can be selected in ${}^6C_2$ and the 3 vowels in ${}^3C_3$ ways. Each such set thus selected can be arranged into words in 5! ways. Hence the total number of words is

$${}^6C_2 \times {}^3C_3 \times 5! = \frac{6\times5}{1\times2} \times 1 \times 5! = 1800$$

9.08. How many different signals can be made by waving five different coloured flags one along the other when one or more of them can be waved at a time?

Sol. There could be 5, 4, 3, 2 or 1 flags. If we consider a selection of 'r' flags we get ${}^5C_r$ such selections, where r = 5, 4, 3, 2, 1.
In each of these selections, we can again arrange them depending on their colours. If the selection has 'r' flags, these 'r' flags can be arranged in r! ways.
∴ Total number of signals
$= ({}^5C_5 \times 5!) + ({}^5C_4 \times 4!) + ({}^5C_3 \times 3!) + ({}^5C_2 \times 2!) + ({}^5C_1 \times 1!)$
$= 120 + (5 \times 24) + (10 \times 6) + (10 \times 2) + 5 = 325$

9.09. How many different permutations can be made out of the letters of the word, 'ASSISTANTS' taken all together?

Sol. In this word, there are 10 letters composed of 4S's, 2A's, 2T's, 1N and 1I. Hence, the number of permutations

$$= \frac{10!}{4!\;2!\;2!}$$

9.10. In how many ways can one divide twelve books (i) into four equal bundles (ii) equally among four boys.

Sol. When the books have to be made into four equal BUNDLES, the four groups are not distinct whereas if they have to be distributed equally among four BOYS, then the four groups are distinct.

(i) When the 4 groups do not have a distinct identity, then the number of ways in which the books can be divided into 4 equal groups of 3 each is $\frac{12!}{4!\,(3!)^4}$.

(ii) When the groups have distinct identity, then the number of ways in which the books can be divided into 4 equal groups of 3 each is $\frac{12!}{(3!)^4}$.

Directions for examples 9.11 to 9.19: All the letters of the word 'GYRATION' are permuted to form various word patterns, with no letter being repeated. We first note that 8! permutations are possible. In each of the questions from 5.11 to 5.19, some conditions are given. We will find out the number of permutations subject to these conditions.

9.11. How many of these words begin with G?

Sol. If we fix G, in the first place, the remaining 7 places can be filled in 7! ways.
Hence we have 7! such words.

9.12. How many of these words end with T?

Sol. By similar logic as in example 5.11, if we fix T in the last place, the other 7 places can be filled up by the other 7 letters in 7! ways.

9.13. How many of these words do not start with G?

Sol. There are 2 ways of answering this question:

(i) Out of the total of 8! words that can be formed, 7! words start with G. Hence the number of words that do not start with G are 8! – 7!
= (8 x 7!) – 7! = 7 x 7!

(ii) Since G cannot go into the 1st place, any of the other 7 letters can go into the 1st place. So, the 1st place can be filled in 7 ways. The other 7 places can be filled up by the remaining 7 letters in 7! ways. Hence there are 7 x 7! words that do not start with G.

9.14. How many words that start with G and end with T can be formed?

Sol. If we fix G in the first place and T in the last place, the other 6 places can be filled up by the remaining 6 letters in 6! ways.

9.15. How many words start with G but do not end with N?

Sol. We fix G in the first place. Out of the other 7 letters, N cannot go into the last place. So the last place can be filled up by any of the other 6 letters. Having filled the 1st and the last place, the remaining 6 places can be filled up by the remaining 6 letters in 6! ways. So the number of words that can be formed is 6 x 6!

9.16. How many words can be formed such that either G is in the first place or T is in the last place?

Sol. We can form 7! words with G in the 1st place and 7! words with T in the last place. From the 7! + 7! (= 2 x 7!) words, we should subtract the number of words that start with G and end with T. As it is counted twice – once in the number of words that start with G and second time in the number of words that end with T. This we know as 6! (as per example 5.14).
Hence the number of words that either start with G or end with T = (2 x 7!) – 6!
= (2 x 7 x 6!) – 6! = 13 x 6!
This can also be arrived as follows:
number of words that either start with G or end with T = number of words that start with G but not end with T + number of words that do not start with G but end with T + number of words that start with G and end with T = 6 x 6! + 6 x 6! + 6! = 13 x 6!

9.17. How many words can be formed which neither start with G nor end with T?

Sol. We will consider the following 2 cases

Case 1: Words that start with T:
Since T is in the first place it takes care of both conditions given. Here the other 7 places can be filled up by the remaining 7 letters in 7! ways.
Case 2: Words that do not start with T:
Here we have to ensure G does not come in the first place. So any of the other 6 (except G and T) can fill the first place in 6 ways. Now out of 7 letters, T cannot go into the last place. Hence any of the other 6 letters can go into the last place in 6 ways. The other 6 places (except the first and the last) can be filled in 6! ways. Hence, the number of words = 6 x 6 x 6! = 36 x 6!
Combining both the cases, the total number of ways
= 7! + 36 x 6!
= (7 x 6!) + 36 x 6! = 43 x 6!

Alternative solution:

From Ex 5.16, we know the number of words that either start with G or end with T is 13 x 6!. If we subtract this from total number of words, we will get number of words that neither start with G nor end with T.
∴ The required number = 8! – 13 x 6!
= 8 x 7 x 6! – 13 x 6! = 43 x 6!

9.18. How many words can be formed so that vowels go only into even places other than the second?

Sol. Out of the 8 places available, 4 are even places. Among these, the second place is to be ignored, which means there are effectively 3 even places. The word 'GYRATION' has 3 vowels and 5 consonants. The 3 vowels can go into 3 even places in 3! ways.
The 5 consonants can occupy the remaining 5 places in 5! ways. So, the total number of words that can be formed is 5! X 3! = 720.

9.19. How many words can be formed such that all the 5 consonants are together?

Sol. All the five consonants have to be treated as one single group. This group along with the other 3 letters i.e., the vowels gives us 4 units. These can be arranged in 4! ways. Now the group of 5 consonants can arrange themselves in 5! ways. Thus the required number is 4! x 5!.

9.20. How many seven lettered patterns without repetition can be formed using all the letters of the word 'MISTAKE' so that the vowels come only in the even places?

Sol. In a seven lettered pattern, there will be 3 even places. There are 3 vowels in 'MISTAKE'. So they can occupy the 3 even places in 3! ways. The 4 consonants will then occupy the other 4 places in 4! ways. Hence, a total of 3! x 4! patterns can be formed.

9.21. How many seven lettered words without repetition can be formed out of the letters of 'MISTAKE' such that the vowels go only into odd places?

Sol. There are 4 odd places and 3 vowels. We can arrange the vowels in ${}^4P_3$ ways. The other 4 places can be then filled by the remaining 4 letters in 4! ways. Thus ${}^4P_3$ x 4! = 576 words.

9.22. How many seven lettered words without repetition can be formed out of the letters of 'MISTAKE' such that only consonants go into odd places?

Sol. Only consonants in odd places is the same as only vowels into even places, same as Ex.5.20 which is 3! x 4! ways.

9.23. How many five lettered words (without repetition) can be formed using the letters of the word 'CRANE' so that the vowels are never together?

Sol. We know that in all 5! words can be made using the letters of CRANE. Out of these, the vowels are together in (4! x 2!) occasions. [refer to Ex.5.19]
∴Number of words where the two vowels are never together = 5! – 4! x 2! = 120 – 48 = 72

9.24. How many seven lettered words without repetition can be formed using the letters of the word 'MISTAKE' so that no two vowels are together?

Sol. Here, we cannot obtain the required figure like in the previous example, as there will be an additional case where "two vowels are together and the third is separated". Hence we will solve it in the manner which is the most general method for problems of this type.
First we arrange the items of the other type – in this case the 4 consonants, which can be arranged in 4! ways.
Now, the items that cannot come together should be arranged using the above items already arranged as 'separators'. In this example, having arranged the 4 from the 7 distinct letters in $^7C_2$ ways. Hence required number of ways are 4 x $^7C_2$ = 84
∴total combinations are
70 + 7 + 6 + 84 = 167

Permutations

For permutations, we find the arrangements for each of the above combinations and add them up
(a) Number of arrangements are
70 x 4! = 1680 ways
(b) Number of arrangements is 7 x 4!/3!
= 28 (since 3 out of 4 are similar)
(c) Since there are 2 groups of 2 similar letters, number of arrangements
= 6 x 4!/2! 2! = 36
(d) Since one pair is alike, number of arrangements
= 112 x 4!/2! = 1344 ways
Total number of arrangements

consonants we now have 5 places in which we can place the 3 vowels as shown below (where 'C' represents the consonants and 'V' the vowels)
V C V C V C V C V
The vowels can be arranged in the5 places that are marked with 'V' in $^5P_3$ ways. On compounding both the tasks, the required number of words equals 4! x $^5P_3$ = 1440.

9.25. Find the number of ways in which
(i) a selection
(ii) an arrangement of 4 letters can be made from the letters of the word "DISTILLATIONS".

Sol. There are 13 letters of 8 different sorts I, I, I, S, S, T, T, L, L, A, O, N, D
In finding groups of 4, the following are the possibilities to be considered.
(a) All 4 distinct
(b) 3 alike, one different
(c) 2 alike, 2 other alike
(d) 2 alike, the other 2 distinct

Combinations

(a) 4 different letters can be selected in
$$^8C_4 = \frac{8x7x6x5}{1x2x3x4} = 70 \text{ ways.}$$
(b) With 3 letters of I, the 4th letter can be selected from the remaining 7 in 7 ways.
(c) The 2 pairs can be selected in $^4C_2$ = 6 ways, as we have to select from I, S, T, L
(d) The alike pair can be selected in 4 ways. While the other 2 distinct letters can now be selected

= 1680 + 28 + 36 + 1344 = 3088 ways

9.26. A question paper has five questions having internal choice of two questions. If a student has to answer all five questions, in how many ways can the paper be attempted?

Sol. The first question out of the given 5 can be attempted in 2 ways by answering either the first or second of the internal choice. Similarly the second question also can be answered in 2 different ways. Hence the first, second together can be answered in 2 x 2 or 2^2 ways.

Similarly since the third, fourth and fifth - each of the questions can be attempted in 2 ways, all 5 questions can be together answered in 2^5 or 32 ways.

Concept Review Questions

Directions for questions 1 to 30: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. (i) Find the value of $^{120}C_{119}$.
(A) 119! (B) 120!
(C) 120 (D) 119

(ii) Find the value of $^{35}P_{34}$.
(A) 35! (B) 35 (C) 34! (D) 34

2. If $^nC_5 = {^nC_7}$, then $^{2n+1}C_2$ is
(A) 250 (B) 300 (C) 240 (D) 280

3. If $^nC_r = {^nP_r}$ then the value of r can be
(A) 0 (B) 1 (C) 2
(D) More than one of the above

4. $\frac{^nP_r}{^nC_r}$ = ______.
(A) 1 (B) $\frac{1}{r}$ (C) r! (D) $\frac{1}{r!}$

5. If $^7C_3 + {^7C_4} = {^NC_4}$, then N is $\boxed{}$

6. If $^NP_4 = 5040$, then find $^NC_4$ $\boxed{}$

7. The number of ways of forming a committee of four from a group of 5 men and 3 women is $\boxed{}$.

8. A committee of five is to be formed from a group of 4 men and 3 women. In how many ways can the committee be formed such that it has more men than women?

$\boxed{}$

9. In how many ways can 4 letters be posted into 3 letters boxes?
(A) 32 (B) 64 (C) 27 (D) 81

10. In how many ways can 8 letters be posted into 5 letter boxes?
(A) $^8P_5$ (B) 5^8 (C) 8^5 (D) $^8C_5$

11. How many words can be formed using all the letters of the word GINGER?
(A) 720 (B) 240 (C) 380 (D) 360

12. How many different words, which begin with N can be formed using all the letters of the word COUNTRY?
(A) 720 (B) 120 (C) 24 (D) 5040

13. How many different words can be formed using the letters of the word MARKET so that they begin with K and end with R?
(A) 16 (B) 24 (C) 12 (D) 64

14. How many different words can be formed using all the letters of the word TRINETRA?
(A) 20160 (B) 40320
(C) 5040 (D) 10080

15. In how many ways can four letters be selected from the word EDUCATION?
(A) $^9C_8$ (B) $^9C_7$ (C) $^9C_4$ (D) $^9C_5$

16. A five lettered word is formed using some of the letters {b, c, l, j, q, s, t}. How many such words, not necessarily meaningful, are the same as those when read backwards?

$\boxed{}$

17. How many three-digit numbers can be formed using the digits {1, 2, 3, 4, 5}, so that each digit is repeated any number of times?
(A) 150 (B) 200
(C) 25 (D) 125

18. How many odd numbers can be formed using the digits {0, 2, 4, 6}?
(A) 0 (B) 192 (C) 18 (D) 20

19. All possible four-digit numbers, with distinct digits are formed, using the digits {1, 3, 4, 5, 6}. How many of them are divisible by 5?
(A) 8 (B) 12 (C) 24 (D) 20

20. In how many ways can 4 boys be seated in 6 chairs?
(A) 180 (B) 720 (C) 360 (D) 240

21. The number of ways of arranging 6 persons in a row is
(A) 6! (B) 5! (C) 6 (D) $\frac{6!}{2}$

22. In how many ways can 7 boys and 6 girls be seated in a row of chairs such that the boys and the girls sit alternately?
(A) 2(7!)(6!) (B) (7!)(6!)
(C) $(7!)^2$ (D) $(6!)^2$

23. In how many ways can three boys and two girls be seated in a row, so that all girls sit together?
(A) 12 (B) 24 (C) 84 (D) 48

24. In how many ways can five men and three women be seated around a circular table?
(A) 720 (B) 5040 (C) 4020 (D) 2520

25. If n books can be arranged on an ordinary shelf in 720 ways, then in how many ways can these books be arranged on a circular shelf?
(A) 120 (B) 720 (C) 360 (D) 60

26. In how many ways can two consonants be selected from the English alphabet?
(A) 420 (B) 105 (C) 210 (D) 300

27. The number of ways of selecting four members from a group of ten members so that one particular member is always included is
(A) 63 (B) 72 (C) 84 (D) 56

28. In how many way can a cricket team of 11 be formed from 14 players such that a particular player is a part of team and another particular player is not a part of the team?

29. In how many ways can three blue balls be selected from a bag which contains four white balls six blue balls?

(A) 20 (B) 10 (C) 120 (D) 210

30. A bag contains three white balls, four green balls and five red balls. In how many ways two balls can be selected?

(A) 132 (B) 66 (C) 33 (D) 76

31. Rahul has six friends. In how many ways can he invite five or more friends for dinner?

(A) 1 (B) 6 (C) 7 (D) 8

32. Ten points are selected on a plane, such that no three of them are collinear. How many different straight lines can be formed by joining these points?

(A) 54 (B) 45 (C) 90 (D) 108

33. Find the number of triangles that can be formed by joining 10 points on a plane, no three of which are collinear.

34. When a dice is rolled for n times, then the number of total outcomes is

(A) 6 (B) 6^{n-1} (C) 6^n (D) 6^{n+1}

35. When a coin is tossed for (n – 1) times, then the number of total outcomes is

(A) 2 (B) 2^n (C) 2^{n+1} (D) 2^{n-1}

36. When a coin is tossed for n times then the number of ways of getting exactly 'r' heads is

(A) 2^r (B) ^nC_r (C) ^nP_r (D) 2^n

37. When two coins are tossed and a cubical dice is rolled, then the total outcomes for the compound event is

(A) 42 (B) 24 (C) 28 (D) 10

38. If the number of diagonals of a polygon is five times the number of sides, the polygon is

(A) 13 (B) 20 (C) 15 (D) 17

39. The number of ways of arranging 8 books in a shelf such that two particular books are together is

40. There are 5 letters and 5 corresponding envelopes. If each letter is placed randomly in a different envelope, the number of ways in which exactly one letter is places in an envelope not corresponding to that

Exercise – 9(a)

Directions for questions 1 to 35: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. How many words can be formed using all the letters of the word "SPECIAL" without repetition such that the vowels occupy the even places?
(A) 144 (B) 720 (C) 360 (D) 1520

2. How many permutations are possible for the letters of the word "SATURDAY" such that the first three letters are S, A and T in that order and the vowels occupy the even places?

3. In how many ways can the letters of the word "DOUBLE" be arranged such that no two vowels are together?
(A) 696 (B) 576 (C) 144 (D) 714

4. (i) Among the arrangements that can be made by using all the letters of the word "EXAMINATION", in how many arrangements A's come together?

(A) 11! (B) $\frac{11!}{2!}$

(C) $\frac{10!}{2!2!}$ (D) 11! – 10! 2!

(ii) In how many arrangements of the above word A's do not come together?

(A) 11! – 10! 2! (B) $\frac{9 \times 10!}{8}$

(C) $\frac{11! \times 10}{2}$ (D) $\frac{10!}{8}$

5. (i) How may five-digit numbers can be formed using the digits 0 to 8 if no digit is to occur more than once in any number?
(A) 1680 (B) 6720 (C) 13440 (D) 1344

(ii) How many of the numbers part (a) are divisible by 5?

6. How many numbers between 50000 and 60000 can be formed using the digits 2 to 7 when any digit can occur any number of times?
(A) 1296 (B) 625 (C) 7776 (D) 2520

7. How many four-digit numbers, that are divisible by 4 can be formed, using the digits 0 to 7 if no digit is to occur more than once in each number?
(A) 520 (B) 370 (C) 345 (D) 260

8. How many numbers exceeding 999 and not exceeding 5000 can be formed such that each digit is any of the first six whole numbers?

9. How many numbers can be formed using all the digits 5, 4, 7, 6, 1, 4, 5, 4, 1 such that the even digits always occupy the even places?

(A) $\frac{9!}{2!\,2!\,2!}$ (B) $\frac{9!}{4!\,4!}$

(C) $\frac{4!\,5!}{3!\,2!\,2!}$ (D) $\frac{4!\,5!}{2!\,2!}$

10. In how many ways can the crew of a ten oared boat be arranged, when of the 10 persons available, two of whom can row only on the bow side and three of whom can row only on the stroke side?

(A) $\frac{10!}{2!\,3!}$ (B) $\frac{10!}{8!\,7!}$ (C) $\frac{5!}{3!\,2!}$ (D) $\frac{(5!)^3}{3!2!}$

11. In how many ways can 21 differently coloured beads be strung on a necklace?

(A) (21)! (B) (20)! (C) $\frac{(21)!}{2}$ (D) $\frac{(20)!}{2}$

12. Twelve friends go out for a dinner to a restaurant where they find two circular tables, one with 7 chairs and the other with 5 chairs. In how many ways can the group settle down themselves for the dinner?

(A) $\frac{12!}{7!\,5!}$ (B) $\frac{12!}{35}$ (C) 12! (D) 12! 5! 7!

13. In how many ways can five students and five teachers sit around a circular table so that no two teachers sit together?
(A) $(4!)^2$ (B) $(5!)^2$ (C) 4! 5! (D) 5! x $^6P_5$

14. In how many ways can the members of three couples be seated around a circular table such that persons of the same gender do not sit in adjacent positions and the members of exactly one of the three couples sit in adjacent positions?

15. Let k be a positive integer such that the sum of the digits of k is 4 and $10^5 < k < 10^6$. How many values does 'k' have?

16. In how many ways can 4 prizes each having 1st, 2nd and 3rd positions be given to 3 boys, if each boy is eligible to receive one prize for each event?
(A) $^{12}P_3$ (B) 6^4 (C) 4^3 (D) 4^6

17. A double decked bus can accommodate 110 passengers, 50 in the upper deck and 60 in the lower deck. In how many ways can the passengers be accommodated if 15 refuse to be in the upper deck while 10 others refuse to be in the lower deck?

(A) $\frac{85!\,50!\,60!}{40!\,45!}$ (B) $\frac{85!}{40!\,45!}$

(C) $\frac{110!}{50!\,60!}$ (D) $\frac{110!\,50!\,60!}{40!\,45!}$

18. In how many ways can 15 books be divided equally
(i) among 3 boys?

(A) $(5)^3$ (B) $\frac{15!}{(5!)^3}$ (C) $\frac{15!}{3!(5!)^3}$ (D) $\frac{15!}{(3!)^5}$

(ii) into 3 parcels?

(A) $(3!)^5$ (B) $(5!)^3$

(C) $\frac{(15)!}{(5!)^3 \times 3!}$ (D) $\frac{(15)!}{(3!)^5 \times 5!}$

19. A man has 8 friends whom he wants to invite for dinner. The number of ways in which he can invite

(i) at least one of them is ______.

(A) 256 (B) 255 (C) 8! (D) 8! – 1

(ii) at least 4 of them is ______.

(A) 70 (B) 48 (C) 126 (D) 163

20. There are 5 different books of subject X, 4 different books of subject Y and 3 different books of subject Z. The number of ways in which at least one book can be given away is ______.

(A) $(2^5 - 1)(2^4 - 1)(2^3 - 1)$

(B) 2^{12}

(C) $2^{12} - 1$

(D) $(2^5 - 1)(2^4 - 1)(2^3 - 1) - 1$

21. There are 5 copies of a book of subject P, 4 copies of a book of subject Q and 3 copies of a book of subject R. The number of ways in which one or more books can be given away is $\boxed{}$.

22. (i) Find the number of selections that can be made by taking 4 letters from the word "ENTRANCE".

(A) 70 (B) 36 (C) 35 (D) 40

(ii) In the above word, the number of arrangements by taking 4 letters are ______.

(A) 1680 (B) 840 (C) 625 (D) 606

23. If all the letters of the word RATE are taken and permuted and arranged in alphabetical order as in a dictionary, what is the rank of the word TEAR?

(A) 20 (B) 23 (C) 22 (D) 21

24. Find the sum of all numbers that can be formed by taking all the digits at a time from 4, 5, 6, 7, 8 without repetition.

$\boxed{}$

25. In how many ways can a representation of 12 students consisting of 8 boys and 4 girls be selected from 15 boys and 10 girls, if a particular boy A and a particular girl B are never together in the representation?

(A) $^{14}C_8\ (^9C_4)$

(B) $^{15}C_8\ (^{10}C_4) - (^{14}C_8)\ (^9C_4)$

(C) $(^{15}C_8)\ (^{10}C_4) - (^{14}C_7)\ (^9C_3)$

(D) $(^{14}C_7)\ (^9C_3)$

26. There are 15 points in a plane of which 8 of them are on a straight line. How many

(i) straight lines can be formed?

$\boxed{}$

(ii) triangles can be formed?

(A) 399 (B) 400 (C) 234 (D) 235

27. The number of positive integral solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 10$ is ______.

(A) 60 (B) 130 (C) 126 (D) 90

28. The number of non-negative integer solutions for the equation $x + y + z + t = 5$ is $\boxed{}$.

29. In a bag there are four numbered cards marked 1, 2, 3 and 4 respectively. A person draws a card and notes down the number on it and replaces it in the bag. He carries out this process for a total of 10 times. The number of times he draws a card marked 1 is (a), marked 2 is (b), marked 3 is (c) and marked 4 is (d). Which of the following gives the total number of possible values of (a, b, c, d)?

(A) 220 (B) 245 (C) 286 (D) 250

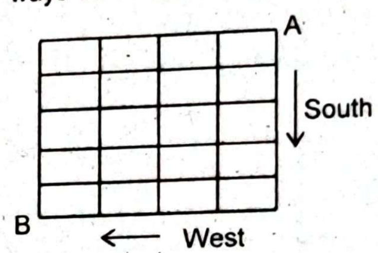
30. Four letters are to be placed into four addressed envelopes. If the letters are placed into the envelopes randomly, in how many ways can the letters be placed so that none of the letters is placed in its corresponding envelope?

$\boxed{}$

31. In a convex decagon all the diagonals are drawn. These diagonals intersect each other at p points inside the decagon, q points on the decagon and r points outside the decagon. Find the maximum possible value of p.

$\boxed{}$

32. In the grid below, the lines represent the one way roads allowing cars to travel only west or south. In how many ways can a car travel from the point A to point B?



(A) 160 (B) 128 (C) 150 (D) 126

Directions for questions 33 and 34: These questions are based on the following data.

X is the set of all pairs (p, q) where $1 \le p < q \le n$. If two distinct members of X have one constituent of the pairs in common, they are called "mates" otherwise they are called "non-mates". For example, if N = 4, X = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)} (1, 2) and (1, 3) are mates (1, 2), (2, 3) are also mates, but (1, 4), (2, 3) are non-mates.

33. Find the number of non-mates that each member of X has for n = 7.

(A) 10 (B) 36 (C) 28 (D) 21

34. If two members of X are mates, how many other members of X will be common mates of both these members for n = 9?

(A) 5 (B) 7 (C) 11 (D) 10

35. $P_1, P_2, \ldots, P_N$ are N points, where N is even, on the circumference of a circle. The minimum number of triangles which can be formed, each of whose vertices are among $P_1, P_2, \ldots, P_N$ and the centre of the circle is 80. N = $\boxed{}$.

Exercise – 9(b)

Directions for questions 1 to 40: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. The number of ways of selecting 20 objects at a time from 35 objects is ______.
(A) ${}^{35}C_{10}$ (B) ${}^{35}C_{15}$ (C) ${}^{35}C_{5}$ (D) ${}^{35}C_{12}$

2. In how many different ways can 9 persons sit in a row of 12 chairs?
(A) ${}^{12}P_4$ (B) ${}^{12}P_{11}$ (C) ${}^{12}P_8$ (D) ${}^{12}P_9$

3. In how many ways can 20 different coloured beads be strung on a necklace?
(A) $\frac{19!}{2}$ (B) 20! (C) 18! (D) 19!

4. In how many ways can 6 letters be posted into 4 letter boxes?
(A) 6^4 (B) 4^6 (C) 24 (D) ${}^6P_4$

5. In a polygon, the number of diagonals is three times the number of its sides. The number of sides of the polygon is ☐.

6. Ten friends go out for dinner to a restaurant where they find two circular tables, one with 6 chairs and the other with 4 chairs. In how many ways can the group sit down for the dinner?
(A) $\frac{10!}{6!4!}$ (B) 6! 4! (C) $\frac{10!}{24}$ (D) $\frac{9!}{5!}$

7. There are 7 copies of an R.K. Narayan novel, 6 copies of Sidney Sheldon novel and 5 copies of the complete works of Milton. The number of ways in which one or more books can be given away is ☐.

8. In how many ways can 12 books be divided equally
(i) among 3 boys?
(A) $\frac{12!}{(4!)^3}$ (B) $\frac{12!}{3!(4!)^3}$ (C) $\frac{12!}{4!}$ (D) 4! 4! 4!

(ii) into 3 parcels?
(A) $\frac{12!}{4!}$ (B) $\frac{12!}{4!3!}$ (C) $\frac{12!}{(4!)^3}$ (D) $\frac{12!}{3!(4!)^3}$

9. A man has 7 friends whom he wants to invite for dinner. The number of ways in which he can invite
(i) at least one of them is ______.
(A) 127 (B) 128 (C) 126 (D) 100

(ii) at least 3 of them is ______.
(A) 98 (B) 100 (C) 99 (D) 92

10. In how many ways can prizes for 3 events, each having 1st, 2nd and 3rd prize, be given to 3 boys, if each boy is eligible to receive one prize for each event?
☐

11. Find the number of selections that can be made by taking four letters from the word SENTENCE.
(A) 21 (B) 22 (C) 17 (D) 10

12. In the above word, the number of arrangements that can be made by taking four letters is ______.
(A) 166 (B) 270 (C) 280 (D) 286

13. How many five-digit numbers can be formed using the digits 0 to 7 if no digit occurs more than once in any number?
☐

14. How many of the numbers in the above problem are divisible by 4?
☐

15. How many numbers between 60000 and 80000 can be formed using the digits 3 to 7, when any digit can be used any number of times?
(A) 250 (B) 625 (C) 1250 (D) 725

16. In how many ways can the letters of the word VICTORY be arranged so that consonants appear only in the even places?
(A) 180 (B) 360 (C) 240 (D) 0

17. (i) How many words can be formed using all the letters of the word PROBLEM without repetition such that the vowels occupy the even places?
(A) 1440 (B) 360 (C) 720 (D) 120

(ii) How many numbers can be formed using all the digits 6, 3, 5, 3, 6, 5, 2, 4, 3, such that odd digits occupy the odd places?
(A) 120 (B) 240 (C) 60 (D) 180

18. In how many ways can the letters of the word MALBORNE be arranged so that the vowels are always together?
☐

19. How many different arrangements can be made using all the letters of the word INSTITUTE, so that all Is come together?
(A) 8064 (B) 6720 (C) 40320 (D) 3360

20. The number of arrangements that can be made using all the letters of the word QUARTZ which begin with A but do not end with R is ☐.

21. In how many ways can the letters of the word TROUBLE be arranged such that no two vowels are together?
(A) 24 (B) 1440 (C) 60 (D) 720

22. In how many ways can the crew of a ten-oared boat be positioned, if of the 10 persons available, 3 can row only on the bow side and 4 can row only on the stroke side?
(A) 6(5!5!) (B) 2(5!5!) (C) 3(5!5!) (D) 4(5!5!)

23. Seven boxes numbered 1 to 7 are arranged in a row. Each is to be filled by either a black or blue coloured ball such that no two adjacent boxes contain blue coloured balls. In how many ways can the boxes be filled with the balls?

24. A question paper consists of 10 sections with each section having two questions. In how many ways can a candidate attempt one or more questions choosing not more than one question per section?
(A) 3^{10} (B) $3^{10}-1$ (C) $2^{10}-1$ (D) 2^{10}

25. How many different words can be formed using all the letters of the word "COMBINATION", such that the vowels as well as the consonants appear in alphabetical order?

26. There are 18 points in a plane of which 10 are on a straight line. Except for the triplets taken exclusively from these 10 points, no other set of three points are collinear. How many
(i) straight lines can be drawn by joining these points?
(A) 109 (B) 108 (C) 153 (D) 45
(ii) triangles can be drawn, by joining these points?
(A) 696 (B) 120 (C) 816 (D) 697

27. If all possible four-digit numbers are formed using the digits 3, 5, 6, 9 without repetition and arranged in ascending order of magnitude, then the position of the number 6953 is
(A) 20 (B) 16 (C) 18 (D) 15

28. (i) If all words that can be formed from the letters of the word GARNE are taken and arranged in alphabetical order, then the rank of the word RANGE is
(A) 101 (B) 100 (C) 102 (D) 78
(ii) If all the letters of the word "BANANA" are taken and rearranged in all possible ways, and the resulting words formed (not necessarily meaningful) are arranged in alphabetical order as in a dictionary, what is the rank of the word ANAANB.

29. In how many ways con six different chocolates be distributed among 3 children such that each child receives at least one chocolate?

30. Eight evenly spaced points lie on the circumference of a circle cantered at O. The number of triangles which can be formed, each of whose vertices are among the eight points and O is ______.

31. An advertisement board is to be designed with six vertical stripes using some or all of the colours: red, green, blue, black and orange. In how many ways can the board be designed such that no two adjacent stripes have the same colour?

32. Find the sum of all the numbers that can be formed by taking all the digits at a time from 3, 4, 6, 7 and 9 without repetition.
(A) 7733652 (B) 7733256
(C) 7373256 (D) 7373652

33. In how many ways can 7 identical balls be placed into four boxes P, Q, R, and S such that the two boxes P and Q have at least one ball each?

34. The number of positive integral solutions of the equation $a + b + c + d + e = 30$ is ______.
(A) 25173 (B) 23517 (C) 25731 (D) 23751

35. How many non-negative integral solutions does the equation $x_1 + x_2 + x_3 = 15$ have?

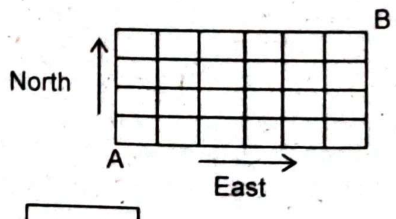
36. In how many ways can a committee of 5 consisting of 3 men and 2 women be formed from 7 men and 6 women, if a particular man A and a particular woman B never serve together in the committee?

37. There are three groups of persons A B and C. Group A has 12 persons and group B has 15 persons. Each person in one group shakes hands with each person in the other groups exactly once. No person in any group shook hands with any other person in that group the total number of handshakes is 666. Find the number of persons that group c has.

38. From the numbers 4, 5, 6, ...20, nine numbers are selected such that the greatest is 15. In how many ways can these 9 numbers be permuted?
(A) (33)10! (B) 165(9!)
(C) (33)9! (D) 145 (10!)

39. Nine villages in a district are divided into three zones with three villages per zone. The telephone department of the district intends to connect the villages with telephone lines such that every two villages in the same zone are connected with four direct lines and every two villages in different zones are connected with two direct lines. How many direct lines are required?

40. In the grid shown, the lines represent the one-way roads allowing cars to travel only eastwards or northwards. In how many ways can a car travel from the point A to the point B?



Key

Concept Review Questions

1. (i) C
   (ii) A
2. B
3. D
4. C
5. 8
6. 210
7. 70
8. 15
9. D
10. B
11. D
12. A
13. B
14. D
15. C
16. 343
17. D
18. A
19. C
20. C
21. A
22. B
23. D
24. B
25. A
26. C
27. C
28. 66
29. A
30. B
31. C
32. B
33. 120
34. C
35. D
36. B
37. B
38. A
39. 10080
40. 0

Exercise – 9(a)

1. A
2. 36
3. C
4. (i) C
   (ii) B
5. (i) C
   (ii) 3150
6. A
7. B
8. 865
9. C
10. D
11. D
12. B
13. C
14. 0
15. 56
16. B
17. B
18. (i) B
    (ii) C
19. (i) B
    (ii) D
20. C
21. 119
22. (i) B
    (ii) D
23. D
24. 7999920
25. C
26. (i) 78
    (ii) A
27. C
28. 56
29. C
30. 9
31. 210
32. D
33. A
34. B
35. 8

Exercise – 9(b)

1. B
2. D
3. A
4. B
5. 9
6. C
7. 335
8. (i) A
   (ii) D
9. (i) A
   (ii) C
10. 216
11. B
12. D
13. 5880
14. 1480
15. C
16. D
17. (i) C
    (ii) A
18. 4320
19. B
20. 96
21. B
22. C
23. 34
24. B
25. 462
26. (i) A
    (ii) A
27. C
28. (i) C
    (ii) 20
29. 540
30. 80
31. 5120
32. B
33. 56
34. D
35. 136
36. 360
37. 18
38. B
39. 90
40. 210