112)

(4)

a) $U = x^y$

S.T $U_{xxy} = U_{xyx}$

$$\frac{\partial v}{\partial x} = V_x = y x^{y-1}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial y} , V_{yx} = y\left(x^{y-1} \log x\right) + x^{y-1} \ (1)$$

$$U_{yx} = y x^{y-1} \log x + x^{y-1}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial y \partial x}\right) = \frac{\partial^3 v}{\partial x \partial y \partial x} = U_{xyx}$$

$$= y\left[x^{y-1} \frac{1}{x} + (\log x)(y-1) x^{y-2}\right) + (y-1) x^{y-2}$$

$$U_{xyx} = y x^{y-2} + (y-1) x^{y-2}(1+\log x) - (1)$$

$$\frac{\partial v}{\partial y} = U_y = x^y \log x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y} = V_{xy}$$

$$= x^y\left(\frac{1}{x}\right) + \log (y x^{y-1})$$

$$\therefore V_{xy} = x^{y-1} + y x^{y-1} \log x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x \partial y}\right) = \frac{\partial^3 v}{\partial x \partial x \partial y} = V_{xxy}$$

$$\Rightarrow (y-1) x^{y-2} + y\left[(\log x)(y-1) x^{y-2}\right) + x^{y-1}\left(\frac{1}{x}\right)\Big)$$

$= (y-1)x^{y-2} + y(y-1)x^{y-2}\log x + yx^{y-2}$

$\therefore \quad u_{xxy} = yx^{y-2} + (y-1)x^{y-2}(1+y\log x)$ — (2)

(1) = (2)

$V_{xxy} = V_{xyx}$

(x,y) : (ii) $u = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$

Dif

let $v = f(s,t)$

$s = \frac{y-x}{xy} = \frac{y}{xy} - \frac{x}{xy} \Rightarrow s = \frac{1}{x} - \frac{1}{y}$

$t = \frac{z-x}{zx} \Rightarrow t = \frac{1}{x} - \frac{1}{z}$

W.K.T

$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial v}{\partial t} \cdot \frac{\partial t}{\partial s}$

$= \frac{\partial v}{\partial s}\left(\frac{-1}{x^2}\right) + \frac{\partial v}{\partial t}\left(-\frac{1}{x^2}\right)$

$x^2 \cdot \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial s} - \frac{\partial v}{\partial t}$ — (1)

Similary

$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial v}{\partial t} \cdot \frac{\partial t}{\partial y}$

$= \frac{\partial v}{\partial s} \cdot \left(\frac{1}{y^2}\right) + \frac{\partial v}{\partial t} \cdot (0)$

$y^2 \frac{\partial v}{\partial y} = \frac{\partial v}{\partial s}$ — (2)

Similarly $\frac{\partial v}{\partial z} = \frac{\partial v}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial v}{\partial t} \cdot \frac{\partial t}{\partial z} \Rightarrow \frac{\partial v}{\partial s}(0) + \frac{\partial v}{\partial t} - \frac{1}{z^2}$

$= z^2 \frac{\partial v}{\partial z} = \frac{\partial v}{\partial t}$ — (3)

(1) + (2) + (3) $= x^2\frac{\partial v}{\partial x} + y^2\frac{\partial v}{\partial y} + z^2\frac{\partial u}{\partial z} = \frac{\partial v}{\partial s} - \frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}$

$x^2\frac{\partial v}{\partial x} + y^2 \cdot \frac{\partial v}{\partial y} + z^2 \frac{\partial v}{\partial z} = 0$

Hence proved,,

The Maximum ... (-2,1)

(2)

a(i) Given $xy^2+2x-3y$

According to Taylor Series

$$f(x,y) = f(a,b) + \frac{1}{1!}\left(h f_x(a,b) + k f_y(a,b)\right) + \frac{1}{2!}\left(h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right) + \frac{1}{3!}\left(h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right)$$

here $h=x-a$ $\quad k=y-b$ $\quad x+2=0$

$x=2$ $\quad y=1$ $\quad y-1=0$

$f(a,b) = (-2)(1)^2 + 2(-2) - 3(1) \Rightarrow -2-4-3 = -9$

$f_x = y^2+2$ $\quad f_x(a,b) = 1^2+2 = 3$

$f_y = 2xy-3$ $\quad f_y\ (-2)(2)(1)-3 = -7$

$f_{xx} = 0$ $\quad f_{xx} = 0$

$f_{xy} = 2y$ $\quad f_{xy} = 2$

$f_{yy} = 2x$ $\quad f_{yy} = -4$

$f_{xyy} = 2$ $\quad f_{xyy} = 2$

$f_{xxx} = 0$ $\quad f_{xxx} = 0$

$f_{xxy} = 0$ $\quad f_{xxy} = 0$

$f_{yyy} = 0$ $\quad f_{yyy} = 0$

$h = x-a$ $\quad k = y-b$

$h = x-(-2)$ $\quad k = y-1$

$h = x+2$

$f(x,y) = -9 + \frac{1}{1!}(x+2)3 + (y-1)(-7) + \frac{1}{2!}(x+2)^2(0) + 2(x+2)(y-1)2 + (-4)(y-1)^2) + \frac{1}{3!}(0(x+2)^3 + 3(x+2)^2(y-1)(0)) + 3(x+2)(y-1)^2(2) + (y-1)\cdot 3\ 0)$

$= -9 + (3x+6-7y+7) + \frac{1}{2}(4(xy-x+2y-2) - 4(y-1)^2) + \frac{1}{3!}(2\times 3(x+2)(y-1)^2)$

$f(x,y) =$

$-9 + (3x-7y+13) + 2(xy-x+2y-2) - 2(y-1)^2) + (x+2)(y-1)^2$

(12)

a(ii)

$$f(x,y) = x^3 + y^3 + 12x - 3y + 20 \quad —— ①$$

Diff ① w.r.t 'x' and 'y'

$$\frac{\partial f}{\partial x} = 3x^2 - 12 \longrightarrow ②$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3 \quad —— ③$$

diff ② partially w.r.t. 'x'

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

Diff ③ partially w.r.t. 'y' and 'x'

$$t = \frac{\partial^2 f}{\partial y^2} = 6y \qquad \Bigg| \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

let

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

$$3x^2 - 12 = 0 \qquad\qquad 3y^2 - 3 = 0$$

$$3x^2 = 12 \qquad\qquad 3y^2 = 3$$

$$x^2 = 4 \qquad\qquad \boxed{y = \pm 1}$$

$$\boxed{x = \pm 2}$$

The stationary points are.

$(2,1)$ $(-2,-1)$. $(2,-1)$ $(-2,1)$.

(13)
(a)

$$I = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x^2} dx \times \int_0^\infty x^2 e^{-x^4} dx$$

$$I = I_1 \times I_2$$

Here :-

$$I_1 = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x^2} dx \ , \ I_2 = \int_0^\infty x^2 e^{-x^4} dx$$

$x^2 = t$
$x = \sqrt{t}$

$$\frac{dt}{dx} = 2x$$

$$dx = \frac{1}{2x} dt$$

$x = 0 \Rightarrow t = 0$
$x = \infty \ , \ t = \infty$

$$= \frac{1}{2\sqrt{t}} dt$$

$$\int_0^\infty \frac{1}{t^{1/4}} e^{-t} \frac{1}{2t^{1/2}} dt$$

$$\frac{1}{2} \int_0^\infty t^{-\frac{1}{4} - \frac{1}{2}} e^{-t} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} t^{-\frac{3}{4}} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} . t^{\frac{1}{4} - 1} dt$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{4}\right)$$

Now,

$$I_2 = \int_0^\infty x^2 e^{-x^4} dx$$

Put

$t = x^4 \Rightarrow x = t^{1/4}$

$x^2 = t^{1/2}$

$$\frac{dt}{dx} = 4x^3 \qquad x = 0 \Rightarrow t = 0$$

$$dx = \frac{dt}{4x^3} \qquad x = \infty \Rightarrow t = \infty$$

$$\int_0^{\infty} t^{1/2} e^{-t} \, dx$$

$$\int_0^{\infty} e^{-t} \, t^{1/2} \cdot \frac{1}{4t^{3/4}} \, dt$$

$$\frac{1}{4} \int_0^{\infty} e^{-t} \, t^{1/2} \, t^{-3/4} \, dt$$

$$\frac{1}{4} \int_0^{\infty} e^{-t} \, t^{1/2 - 3/4} \, dt$$

$$= \frac{1}{4} \int_0^{\infty} e^{-t} \, t^{-1/4} \, dt$$

$$= \frac{1}{4} \int_0^{\infty} e^{-t} \, t^{3/4 - 1} \, dt$$

$$I_2 = \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$$

$$I = I_1 \times I_2 \Rightarrow \quad I = \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \times \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$$

$$= \frac{1}{8} \frac{\pi}{\sin \frac{\pi}{4}}$$

$$= \frac{1}{8} \frac{\pi}{\frac{1}{\sqrt{2}}} \Rightarrow \frac{\sqrt{2}\,\pi}{4 \times 2}$$

$$\boxed{I = \frac{\pi}{4\sqrt{2}}}$$

(4) (b)

$u = 2uv$

$$\int^{\infty} \frac{x^{m-1}}{(1+x^n)^p}$$

$$I = \int_0^{\infty} \frac{x^{m-1}}{(1+x^n)^p}\, dx$$

$x = 0 \Rightarrow t = 1$
$x = \infty \; t = 0$

Put $\frac{1}{1+x^n} = t$

$x^n = \frac{1}{t} - 1$

$\frac{1}{t} = 1 + x^n$

$n x^{n-1} \frac{dx}{dt} = \frac{-1}{t^2}$

$\frac{1}{t} - 1 = x^n$

$$\boxed{\left(\frac{1-t}{t}\right)^{1/n} = x}$$

$dx = \frac{-1}{n x^{n-1} t^2}\, dt$

$$\boxed{dx = \frac{-1}{n(1-t)^{\frac{n-1}{n}} t^{\frac{n+1}{n}}}\, dt}$$

$$\Rightarrow I = -\int_1^{\infty} \left(\left(\frac{1-t}{t}\right)^{1/n}\right)^{m-1} t^p \frac{dt}{n(1-t)^{\frac{n-1}{n}} t^{\frac{n+1}{n}}}$$

$$= \frac{1}{n}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{m-1}{n}} t^p (1-t)^{\frac{1-n}{n}} t^{\frac{-(n+1)}{n}}\, dt$$

$$= \frac{1}{n}\int_0^1 \frac{(1-t)^{\frac{m-1}{n}}}{t^{\frac{m-1}{n}}} t^p (1-t)^{\frac{1-n}{n}} t^{\frac{-(n+1)}{n}}\, dt \quad 2v\frac{\partial z}{\partial y}$$

$$= \frac{1}{n}\int_0^1 (1-t)^{\frac{m-1}{n}} t^{\frac{-(m-1)}{n}} t^p (1-t)^{\frac{1-n}{n}} t^{\frac{-(n+1)}{n}}\, dt \quad -(1)$$

$$= \frac{1}{n}\int^1 (1-t)^{\frac{m-\cancel{1}+\cancel{1}-n}{n}} t^{\frac{-m+\cancel{1}+np-n-\cancel{1}}{n}}\, dt$$

$$= \frac{1}{n}\int_0^1 (1-t)^{\frac{m-n}{n}} \; t^{p-\frac{m}{n}-1}\, dt$$

$$= \frac{1}{n}\int_0^1 t^{p-\frac{m}{n}-1}(1-t)^{\frac{m}{n}-1}\, dt$$

2.

$$= \frac{1}{n}\, B\left(p-\frac{m}{n}, \frac{m}{n}\right)$$

$$= \frac{\frac{1}{n}\Gamma\left(p-\frac{m}{n}\right)\Gamma\left(\frac{m}{n}\right)}{\Gamma\left(p-\frac{m}{n}+\frac{m}{n}\right)} = \frac{\frac{1}{n}\Gamma\left(p-\frac{m}{n}\right)\Gamma\left(\frac{m}{n}\right)}{\Gamma p}$$

$$\boxed{\int_0^\infty \frac{x^{m-1}}{(1+x^n)^p}\,dx = \frac{\frac{1}{n}\Gamma\left(p-\frac{m}{n}\right)\Gamma\left(\frac{m}{n}\right)}{\Gamma p}}$$

let $p = 1$

$$\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\frac{1}{n}\Gamma\left(1-\frac{m}{n}\right)\Gamma\left(\frac{m}{n}\right)}{\Gamma(1)}$$

$$= \frac{1}{n}\Gamma\left(\frac{m}{n}\right)\Gamma\left(1-\frac{m}{n}\right)$$

$$= \frac{1}{n}\frac{\pi}{\sin\left(\frac{m\pi}{n}\right)}$$

$$\boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad 0<n<1}$$

$$\boxed{\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx = \frac{\pi}{n\sin\left(\frac{m\pi}{n}\right)}}$$

let, $m=1$, $n=4$

$$\int_0^\infty \frac{x^{1-1}}{1+x^4}\,dx = \frac{\pi}{4\sin(\pi/4)}$$

$$\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{\pi}{4\cdot 1/\sqrt{2}} = \frac{\pi}{2\times\frac{2}{\sqrt{2}}\times\frac{1}{\sqrt{2}}}$$

$$\boxed{\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}}}$$ //

(5)

a)

$z = f(x, y)$ $\qquad x = u^2 - v^2 \qquad y = 2uv$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}(2u) + \frac{\partial z}{\partial y}(2v)$$

$$= 2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = \left(2u\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}\right)z$$

$$\frac{\partial}{\partial u} = 2u\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial y}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$$

$$= \left(2u\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}\right)\left(2u\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}\right)$$

$$= 2u\frac{\partial}{\partial x}\left(2u\frac{\partial z}{\partial x}\right) + 2u\frac{\partial}{\partial x}\left(2v\frac{\partial z}{\partial y}\right).$$

$$+ 2u\frac{\partial}{\partial y}\left(2u\frac{\partial z}{\partial x}\right)$$

$$+ 2v\frac{\partial}{\partial y}\left(2v\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial u^2} = 4u^2\frac{\partial^2 z}{\partial x^2} + 4uv\frac{\partial^2 z}{\partial x \partial y} + 4uv\frac{\partial^2 z}{\partial y \partial x} + 4v^2\frac{\partial^2 z}{\partial y^2} \quad (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}(2u)$$

$$\frac{\partial z}{\partial v} = -2v\frac{\partial z}{\partial x} + 2u\frac{\partial x}{\partial y}$$

$$= \left(-2u\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}\right) z$$

$$\frac{\partial \cdot}{\partial v} = -2u\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 x}{\partial v^2} = \left(-2v\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}\right)\left(-2v\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}\right)$$

$$= -2v\frac{\partial}{\partial x}\left(-2v\frac{\partial z}{\partial x}\right) - 2v\frac{\partial}{\partial x}\left(2u\frac{\partial z}{\partial y}\right)$$

$$+ 2u\frac{\partial}{\partial y}\left(-2v\frac{\partial z}{\partial x}\right) + 2u\frac{\partial}{\partial y}\left(2u\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial v^2} = 4v^2\frac{\partial^2 z}{\partial x^2} - 4uv\frac{\partial^2 z}{\partial x \partial y} - 4uv\frac{\partial^2 z}{\partial y \partial x} + 4u^2\frac{\partial^2 z}{\partial y^2} \quad (2)$$

(1) + (2) =>

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4u^2\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right] + 4v^2\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right]$$

$$= (4u^2 + 4v^2)\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right]$$

$$= 4(u^2 + v^2)\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right]$$