The Wave Equation (Chapter 7)

In this chapter we will look at hyperbolic equations. We will start with the wave equation

$$\frac{\partial^2 A(x,t)}{\partial t^2} = c^2 \frac{\partial^2 A(x,t)}{\partial x^2}$$

One standard way to solve this equation is to break it up into 2 first order equations by defining 2 variables P and Q

$$P = \frac{\partial A}{\partial t} \qquad\qquad Q = c\,\frac{\partial A}{\partial x}$$

$$\frac{\partial P}{\partial t} = c\,\frac{\partial Q}{\partial x} \qquad\qquad \frac{\partial Q}{\partial t} = c\,\frac{\partial P}{\partial x}$$

The wave equation is then written as

$$\frac{\partial \boldsymbol{a}}{\partial t} = c\boldsymbol{B}\,\frac{\partial \boldsymbol{a}}{\partial x}$$

Where $a = \begin{bmatrix} P \\ Q \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The last equation then implies an even simpler hyperbolic equation. In other words, a simpler version of

$$\frac{\partial \boldsymbol{a}}{\partial t} = c\boldsymbol{B}\,\frac{\partial \boldsymbol{a}}{\partial x}$$

is the so-called advection equation

$$\frac{\partial a}{\partial t} = -c\,\frac{\partial a}{\partial x}$$

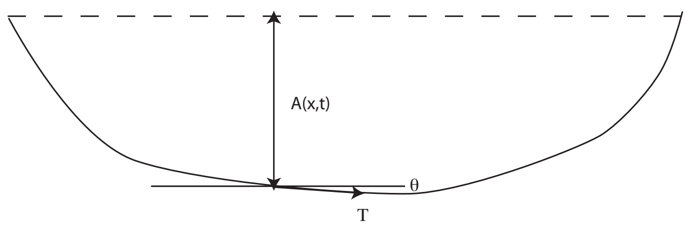
That describes the motion of a 1-D scalar field *a(x,t)* moving at constant velocity c (the negative sign makes it move the the right). This advection equation is the simplest example of a conservation-type equation such as the continuity equation

$$\frac{\partial \rho}{\partial t} = -\,\frac{\partial F}{\partial x}$$

Where F is the flux. Note, if $F = -\frac{\partial \rho}{\partial x}$, then the above equation becomes the diffusion equation.

Derivation of the wave equation

Consider a string under tension (T) attached to a wall at each end.



The equation of motion for an element of length Δx and mass density ρ is then

$$\rho\Delta x\,\frac{\partial^2 A(x,t)}{\partial t^2} = \sum Forces$$

$$= T(x+\Delta x)sin\theta(x+\Delta x) - T(x)sin\theta(x)$$

$$= T(x+\Delta x)sin\theta\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - T(x)sin\theta\left.\frac{\partial y}{\partial x}\right|_{x}$$

$$\approx \frac{\partial}{\partial x}\left(T(x)\,\frac{\partial y}{\partial x}\right)\Delta x$$

Derivation of the wave equation - 2

If we assume T is constant, then we have

$$\frac{\partial^2 A(x,t)}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 A(x,t)}{\partial t^2}$$

Where $c = \sqrt{\frac{T}{\rho}}$ is the speed of the wave

The Wave Equation – Numerical solution

If we then take the wave equation

$$\frac{\partial^2 A(x,t)}{\partial t^2} = c^2 \frac{\partial^2 A(x,t)}{\partial x^2}$$

In a finite difference approximation, it can be written as

$$\frac{A_j^{n+1} - 2A_j^n + A_j^{n-1}}{\tau^2} = \mathrm{c}^2 \frac{A_{j+1}^n - 2A_j^n + A_{j-1}^n}{h^2}$$

Where $x_j = -\frac{L}{2} + (j-1)h$ and $t_n = (n-1)\tau$ rewriting, we get

$$\boxed{A_j^{n+1} = 2A_j^n - A_j^{n-1} + \frac{c^2\tau^2}{h^2}\left(A_{j+1}^n - 2A_j^n + A_{j-1}^n\right)}$$

Wave Equation – Numerical solution

One advantage of this approach is that it allows solutions with wave speeds of both signs

$$\boxed{A_j^{n+1} = 2A_j^n - A_j^{n-1} + \frac{c^2\tau^2}{h^2}\left(A_{j+1}^n - 2A_j^n + A_{j-1}^n\right)}$$

One one the homework problems is to write a program to solve this equation.

The Advection Equation

We will look now at the simplest hyperbolic equation, the Advection Equation. We shall see that the numerical solution of this simple equation is the cause of endless (numerical) grief. The basic equation is

$$\frac{\partial a(x,t)}{\partial t} = -\frac{c\partial a(x,t)}{\partial x}$$

with an initial condition $a(x,0)=f_0(x)$ where $f_0(x)$ is an arbitrary function. The analytical solution to the above equation is easy to obtain as $a(x,t)=f_0(x-ct)$. For example, if we have the initial condition function

$$a(x,0) = \cos[k(x - x_0)]\, exp\left[-\frac{(x - x_0)^2}{2\sigma^2}\right]$$

Then later in time it moves a distance ct:

$$a(x,t) = \cos[k(x - ct - x_0)]\, exp\left[-\frac{(x - ct - x_0)^2}{2\sigma^2}\right]$$

FTCS for the Advection Equation

The simplest approach to numerically solving the advection equation is to apply the FTCS as we did for the diffusion equation, so that the equation then becomes

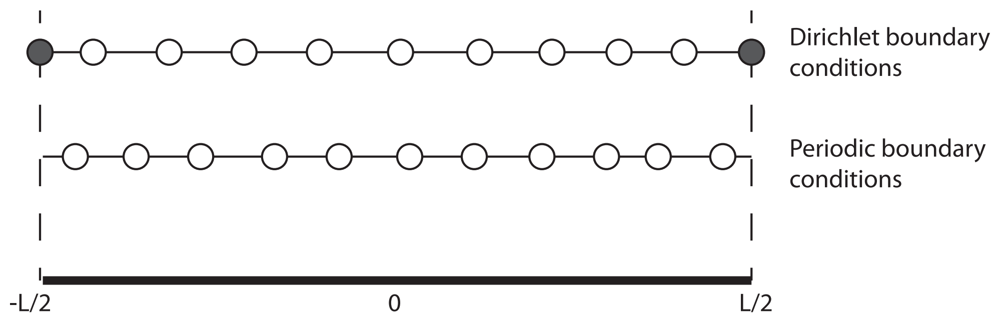
$$\frac{a_j^{n+1} - a_j^n}{\tau} = -c\,\frac{a_{j+1}^n - a_{j-1}^n}{2h}$$

or

$$\boxed{a_j^{n+1} = a_j^n - \frac{c\tau}{2h}\left(a_{j+1}^n - a_{j-1}^n\right)}$$

Unfortunately, this method is *unconditionally unstable* for all values of τ.

Boundary Conditions and Grid Spacing



Note the grid spacing changes. For periodic boundary conditions, the boundary lies between the first and last grid point.

Von Neumann Stability analysis for FTCS

Assume the solution takes the form

$$a_j^n = T^n e^{ikjh}$$

So that the FTCS method becomes

$$T^{n+1}e^{ikjh} = T^n e^{ikjh} - \frac{c\tau}{2h} T^n \left(e^{ik(j+1)h} - e^{ik(j-1)h}\right)$$

$$A \equiv \frac{T^{n+1}}{T^n} = 1 - \frac{c\tau}{2h}\left(e^{ikh} - e^{-ikh}\right)$$

Or

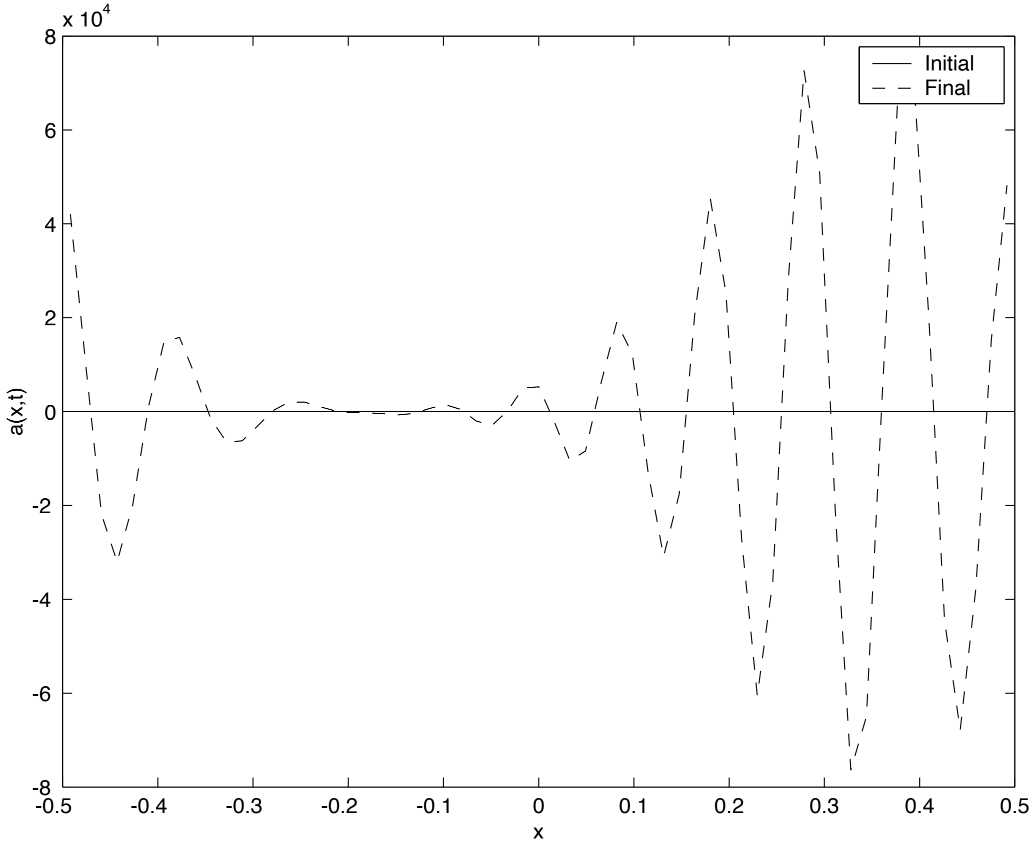
$$A^2 = 1 + \left(\frac{c\tau}{h}\right)^2 \sin^2 kh$$

Which is <u>unconditionally</u> unstable.

Note that the fastest growing unstable mode is for when $kh = \pi/2$

Which corresponds to a wave of wavelength (since $k=2\pi/\lambda$) of $\lambda=4h$

FTCS N=61, 62.5 steps, τ=0.016, CFL=1



Lax Method for the Advection Equation

If we replace the term a_j^n with an average of the values around it, i.e.:

$$a_j^n \to \frac{a_{j+1}^n + a_{j-1}^n}{2}$$

we then get

$$\boxed{a_j^{n+1} = \frac{1}{2}\left(a_{j+1}^n + a_{j-1}^n\right) - \frac{c\tau}{2h}\left(a_{j+1}^n - a_{j-1}^n\right)}$$

This is the *LAX* method and is stable for values

$$\tau < \frac{h}{c}$$

which is known as the *Courant-Friedrichs-Lewy (CFL)* condition.

– It says that the velocity that information propagates (c) in the system should be faster than the velocity of the solution

Lax Method for the Advection Equation

– The averaging term in the Lax method succeeds in stabilizing the scheme by adding diffusion, we can see that if we rewrite the equation for the LAX method in the form

$$a_j^{n+1} = a_j^n + \frac{1}{2}\left(\mathrm{a}_{\mathrm{j+1}}^{\mathrm{n}} - 2\mathrm{a}_{\mathrm{j}}^{\mathrm{n}} + \mathrm{a}_{\mathrm{j-1}}^{\mathrm{n}}\right) - -\frac{c\tau}{2h}(a_{j+1}^n - a_{j-1}^n)$$

Which becomes a finite difference approximation to

$$a_j^{n+1} \approx a_j^n + \frac{h^2}{2}\frac{\partial^2 a}{\partial x^2} - \frac{c\tau}{2h}(a_{j+1}^n - a_{j-1}^n)$$

– Which effectively adds a diffusion term to the scheme, the amount of the diffusion is inversely proportional to the timestep since the lower the step adds more times the solution is updated (for a given time).

– So that if τ is too small and we end up taking many timesteps, then the diffusion ends up dominating the solution

– If τ is too large, the diffusion is too weak to stabilize the solution

Von Neumann Stability analysis for the Lax method

Again assume

$$a_j^n = T^n e^{ikjh}$$

So that the Lax method becomes

$$T^{n+1}\, e^{ikjh} = \frac{T^n e^{ik(j+1)h} + T^n e^{ik(j-1)h}}{2} - \frac{c\tau}{2h} T^n \left(e^{ik(j+1)h} - e^{ik(j-1)h}\right)$$

or

$$A \equiv \frac{T^{n+1}}{T^n} = \frac{\left(e^{ikh} + e^{-ikh}\right)}{2} - \frac{c\tau}{2h}\left(e^{ikh} - e^{-ikh}\right)$$

that becomes

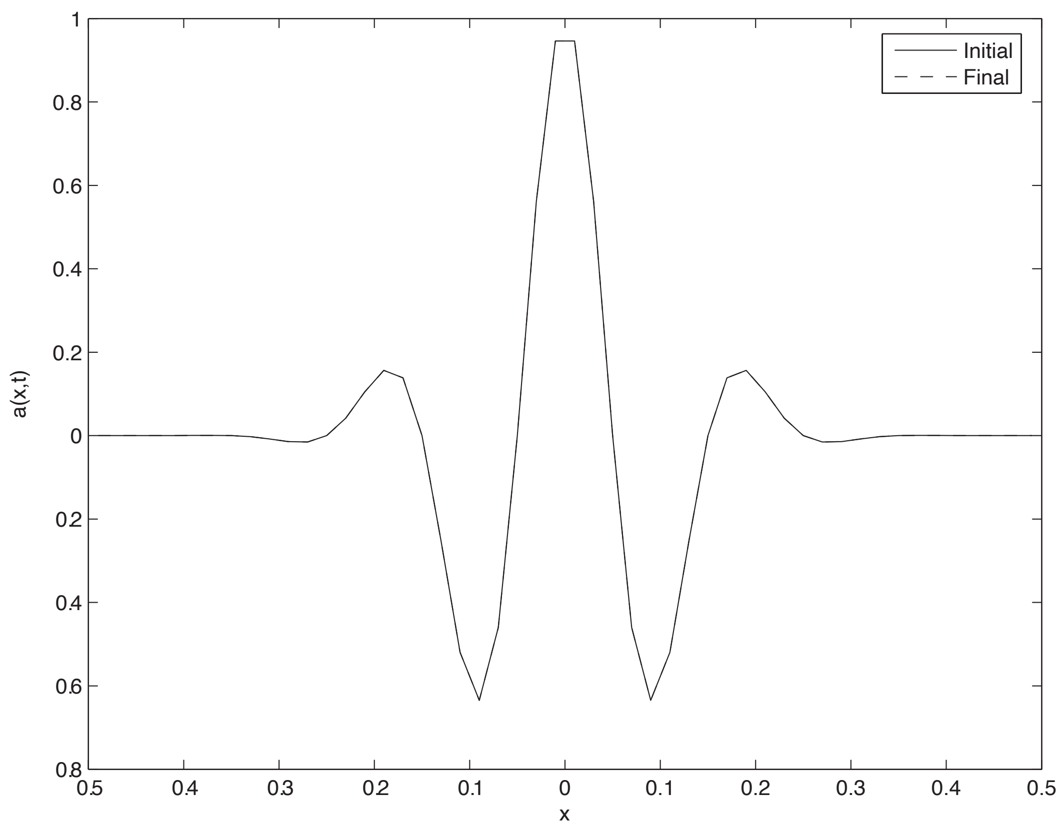
$$A^2 = \cos^2 kh + \left(\frac{c\tau}{h}\right)^2 \sin^2 kh$$

Which is stable when

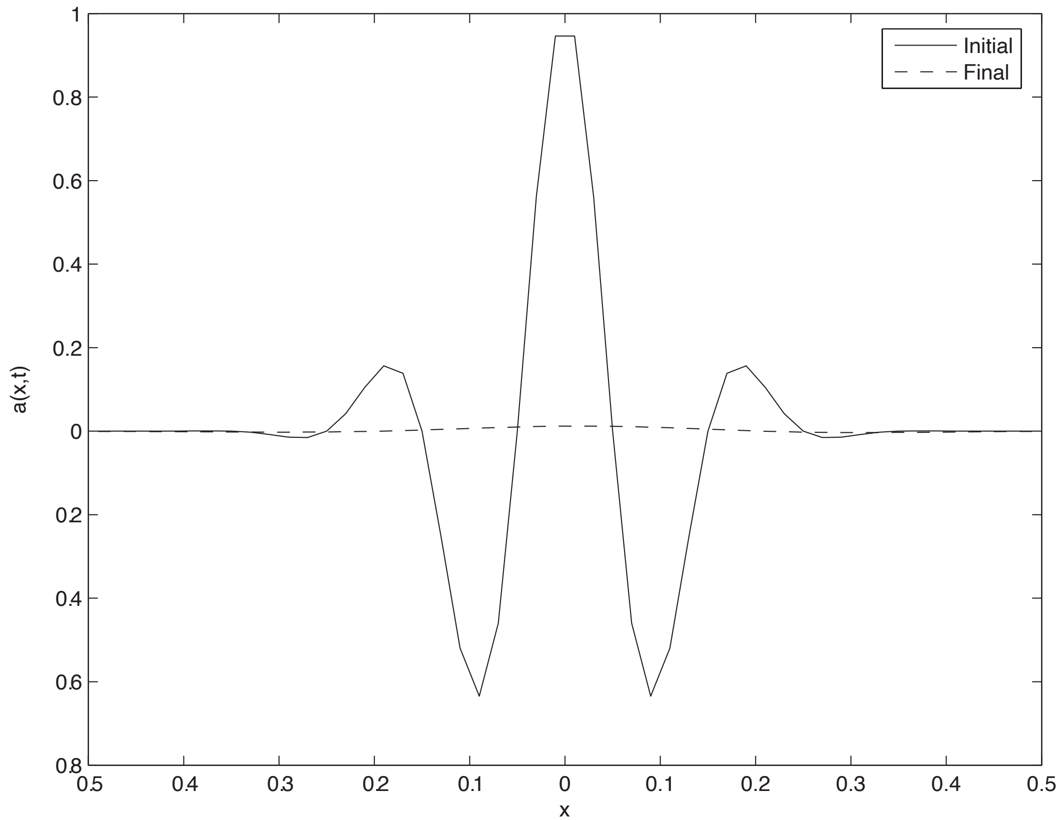
$$\tau \leq \frac{h}{c}$$

Note that when $\tau = \frac{h}{c}$, then $|A| = 1$

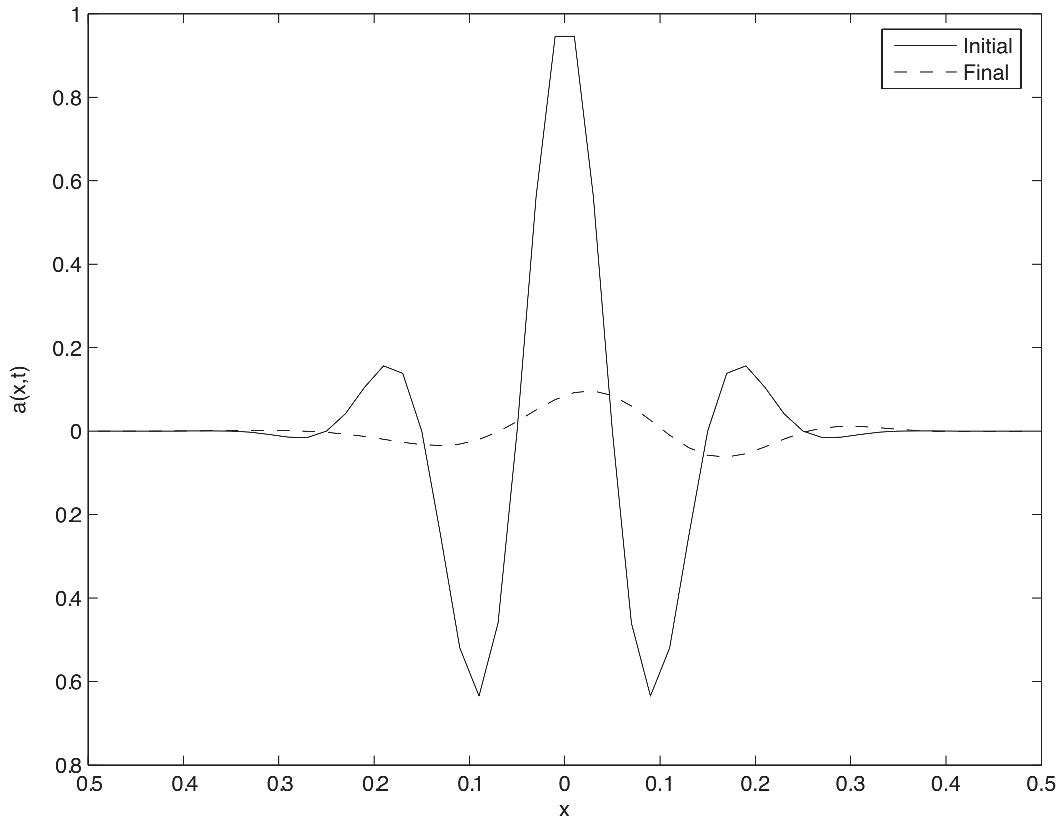
Lax N=50, 50 steps, τ=0.02, CFL=1



Lax N=50, 100 steps, τ=0.01, CFL=0.5



Lax N=50, 60 steps, τ=0.016, CFL=0.83



Phase Error

Another error that arises from the numerical approximation is a phase error. For the simple wave equation, the exact solution is a wave that moves to the right at speed c. Numerical solutions will typically change the amplitude and the phasing of the wave, the amount will vary depending on the wavenumber (k) of the wave. The amplification factor A can be written as

Where

$$A = |A|e^{i\theta}$$

And

$$|A| = \sqrt{Re(A)^2 + Im(A)^2}$$

$$\theta = -\tan^{-1}\left(\frac{Im(A)}{Re(A)}\right)$$

Where Re and Im represent the real and imaginary parts of A

Phase Error

If we had an initial condition

$$\psi(x,0) = \sum_k a_k e^{ikx}$$

Where *k* is the wavenumber and a_k is the Fourier amplitude, then at some later time *t*, the exact solution takes the form

$$\psi(x,t) = \sum_k a_k e^{i(kx-ct)}$$
$$= \mathrm{e}^{-\mathrm{ict}}\psi(x,0)$$

That simply states that the wave at time t has moved a distance *ct*. To be exact, a numerical solution should have the same phase shift. So a measure of the phase shift error would then be how close to 1 is ratio

$$\varepsilon \equiv 1 - \frac{\theta}{kct}$$

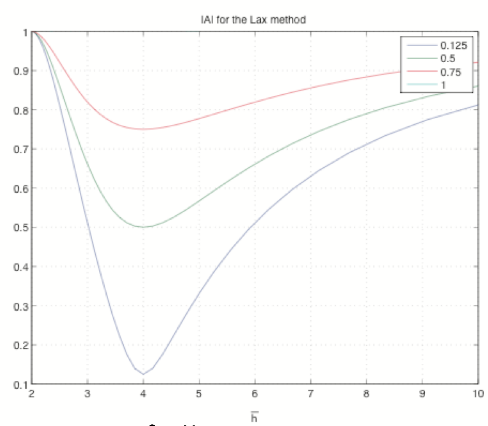
Errors for the LAX method

For the LAX method, the phase error is

$$\theta = -\tan^{-1}\left(\frac{Im(A)}{Re(A)}\right) = \tan^{-1}\left(\frac{\frac{c\tau}{h}\sin(kh)}{\cos(kh)}\right)$$

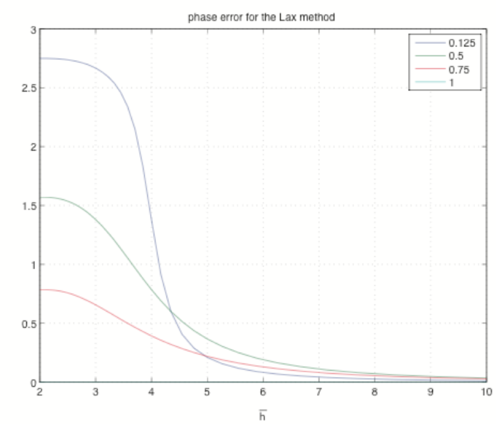
Note that when $\frac{c\tau}{h} = 1$ $\varepsilon = 0$ and $|A| = 1$

|A|



λ/h
Amplitude Plot

ε



λ/h
Phase error plot

Upwind Method for the Advection Equation

Since information is coming from the upstream direction, a physically meaningful approach is to use the upwind information only, in other words replace the spatial gradient with a one sided upwind difference

$$\boxed{a_j^{n+1} = a_j^n - \frac{c\tau}{h}\left(a_j^n - a_{j-1}^n\right)}$$

Note if you reverse this by using the so-called downwind method, the method is unstable.

$$a_j^{n+1} = a_j^n - \frac{c\tau}{h}\left(a_{j+1}^n - a_j^n\right)$$

Von Neumann Stability analysis for the Upwind method

Again assume

$$a_j^n = T^n e^{ikjh}$$

So that the upwind method becomes

$$T^{n+1}\, e^{ikjh} = T^n e^{ikjh} - \frac{c\tau}{h} T^n \left(e^{ikjh} - e^{ik(j-1)h}\right)$$

So that

$$A \equiv \frac{T^{n+1}}{T^n} = 1 - \frac{c\tau}{h}\left(1 - e^{-ikh}\right) = 1 - \frac{c\tau}{h}(1 - coskh + isinkh)$$

that becomes

$$A^2 = \left[1 - \frac{c\tau}{h} + \frac{c\tau}{h} coskh\right]^2 + \left[\frac{c\tau}{h} sinkh\right]^2$$

and is stable for values

$$\tau < \frac{h}{c}$$

Again, note that when $\tau = \frac{h}{c}$, then $|A| = 1$

Proof for Upwind stability requirement

Setting $\eta = \frac{c\tau}{h}$ A becomes

$$A^2 = [1 - \eta + \eta coskh]^2 + [\eta sinkh]^2$$

That simplifies to

$$A^2 = 1 - 2\eta + 2\eta^2 + 2\eta(1 - \eta)coskh$$

Since we want $|A|^2 \leq 1$ it simplifies to

$$-2\eta + 2\eta^2 + 2\eta(1 - \eta)coskh \leq 0$$

Or

$$2\eta(1 - \eta)(-1 + coskh\,) \leq 0$$

Since the term involving cosine is always negative, that leaves the requirement that

$$\eta(1 - \eta) \geq 0$$

Which can only be true if $\eta = \frac{c\tau}{h} \leq 1$

Von Neumann Stability analysis for the Downwind method

If you replace c with $-c$, you have the downwind method, and the Von Neumann Stability analysis becomes

$$A^2 = \left[1 + \frac{c\tau}{h} - \frac{c\tau}{h} coskh\right]^2 + \left[\frac{c\tau}{h} sinkh\right]^2$$

Using the notation from the previous page, this reduces to

$$2\eta(1+\eta)(1 - coskh\,) \leq 0$$

Since the cosine term is always be greater than 0, as is the second term, this says that you cannot satisfy the requirement, and therefore the method is unconditionally unstable.

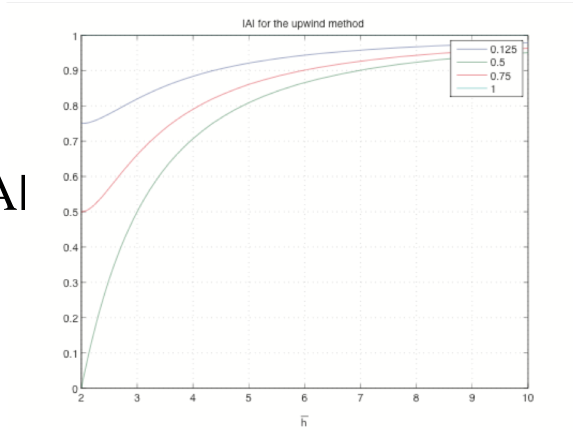
Errors for the Upwind method

For the Upwind method, the phase error is

$$\theta = -\tan^{-1}\left(\frac{Im(A)}{Re(A)}\right) = \tan^{-1}\left(\frac{\frac{c\tau}{h}\sin(kh)}{1-\frac{c\tau}{h}+\frac{c\tau}{h}\cos(kh)}\right)$$

Note that when $\frac{c\tau}{h} = 1$ $\varepsilon = 0$ and $|A| = 1$

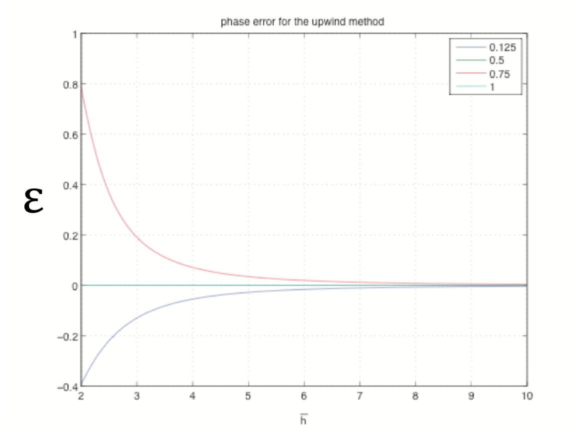
|A|



λ/h

Amplitude Plot

ε



λ/h

Phase error plot

Leap Frog Method for the Advection Equation

To get second order in time accuracy we can use the following scheme

$$\frac{a_j^{n+1} - a_j^{n-1}}{2\tau} = -c\,\frac{a_{j+1}^n - a_{j-1}^n}{2h}$$

we then get

$$\boxed{a_j^{n+1} = a_j^{n-1} - \frac{c\tau}{h}\left(a_{j+1}^n - a_{j-1}^n\right)}$$

With the timestep restriction $\tau \leq \frac{h}{c}$

The difficulty here with this method is that 2 previous time levels have to be kept, requiring more memory and the code is not self-starting.

The scheme has the advantage that $|A|=1$ for all timesteps.

Lax Wendroff Method for the Advection Equation

Another popular method is the Lax Wendroff scheme. It can be derived from the Taylor series expansion for $a(x,t+\tau)$

$$a(x,t+\tau) = a(x,t) + \tau\frac{\partial a}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2 a}{\partial t^2} + O(\tau^3)$$

using a general form of the advection equation

$$\frac{\partial a(x,t)}{\partial t} = -\frac{\partial F(a)}{\partial x}$$

Where $F(a)$ is a *FLUX* function. Differentiating the above wrt t

$$\frac{\partial^2 a(x,t)}{\partial t^2} = -\frac{\partial}{\partial t}\frac{\partial F(a)}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial F(a)}{\partial t}$$

But

$$\frac{\partial F(a)}{\partial t} = \frac{dF(a)}{da}\frac{\partial a}{\partial t} = F'(a)\frac{\partial a}{\partial t} = -F'(a)\frac{\partial F}{\partial x}$$

The last step uses the advection equation

Lax Wendroff Method

Going back to

$$a(x, t+\tau) = a(x,t) + \tau \frac{\partial a}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2 a}{\partial t^2} + O(\tau^3)$$

and using

$$\frac{\partial a(x,t)}{\partial t} = -\frac{\partial F(a)}{\partial x}$$

and

$$\frac{\partial^2 a(x,t)}{\partial t^2} = -\frac{\partial}{\partial x}\frac{\partial F(a)}{\partial t} = \frac{\partial}{\partial x}(F'(a)\frac{\partial F}{\partial x})$$

we get

$$a(x, t+\tau) = a(x,t) - \tau \frac{\partial F}{\partial x} + \frac{\tau^2}{2}\frac{\partial}{\partial x}\left(F'(a)\frac{\partial F}{\partial x}\right) + O(\tau^3)$$

Which is the general form of the Lax Wendroff scheme

Lax Wendroff Method - Finite difference form

In finite difference form, we have

$$a_j^{n+1} = a_j^n - \tau \frac{F_{j+1}(a) - F_{j-1}(a)}{2h} + \frac{\tau^2}{2h}\left(\left[F'(a)\frac{\partial F}{\partial x}\right]_{j+\frac{1}{2}} - \left[F'(a^n)\frac{\partial F}{\partial x}\right]_{j-\frac{1}{2}}\right)$$

$$= a_j^n - \tau \frac{F_{j+1}(a) - F_{j-1}(a)}{2h} + \frac{\tau^2}{2h^2}\left(F'_{j+\frac{1}{2}}(a)(F_{j+1} - F_i) - F'_{j-\frac{1}{2}}(a)(F_j - F_{j-1})\right)$$

Where

$$F'_{j\pm\frac{1}{2}} \equiv F'\left(\frac{a_{j\pm1}^n + a_j^n}{2}\right)$$

Lax Wendroff Method for the Advection Equation

For the advection equation

$$\frac{\partial a(x,t)}{\partial t} = -c\,\frac{\partial a(x,t)}{\partial t}$$

Then $F_i = c a_i^n$ and $F_i' = c$

Then the scheme becomes

$$\boxed{a_j^{n+1} = a_j^n - \frac{c\tau}{2h}\left(a_{j+1}^n - a_{j-1}^n\right) + \frac{c^2\tau^2}{2h^2}\left(a_{j+1}^n - 2a_j^n + a_j^n\right)}$$

Note the last term on the RHS is effectively a diffusion term that stabilizes the scheme. The diffusion rate depends on the timestep τ so is less aggressive than the LAX scheme.

Again, we have the CFL condition states that $\tau \le \frac{h}{c}$

Note that when $\tau = \frac{h}{c}$ then for the simple advection problem, the scheme becomes

$$a_j^{n+1} = \mathrm{a}_\mathrm{j}^\mathrm{n}$$

which is the exact solution.

Von Neumann Stability analysis for the Lax Wendroff method

Again assume $a_j^n = T^n e^{ikjh}$ so that the Lax Wendroff method becomes

$$T^{n+1}\, e^{ikjh}$$
$$= T^n e^{ikjh} - \frac{c\tau}{2h} T^n\left(e^{ik(j+1)h} - e^{ik(j-1)h}\right) + \frac{1}{2}\left(\frac{c\tau}{h}\right)^2 T^n\left(e^{ik(j+1)h} - 2e^{ikjh} + e^{ik(j-1)h}\right)$$

Or

$$A \equiv \frac{T^{n+1}}{T^n} = 1 - \frac{c\tau}{2h}\left(e^{ikh} - e^{-ikh}\right) + \frac{1}{2}\left(\frac{c\tau}{h}\right)^2 \left(e^{ikh} - 2 + e^{-ikh}\right)$$
$$= 1 - \frac{ic\tau}{h} sinkh \; - 2\left(\frac{c\tau}{h}\right)^2 \sin^2\left(\frac{kh}{2}\right)$$

Which becomes

$$|A|^2 = 1 - 4\left(\frac{c\tau}{h}\right)^2\left[1 - \left(\frac{c\tau}{h}\right)^2\right]\sin^4\left(\frac{kh}{2}\right)$$

Which is stable when $\tau \leq \frac{h}{c}$

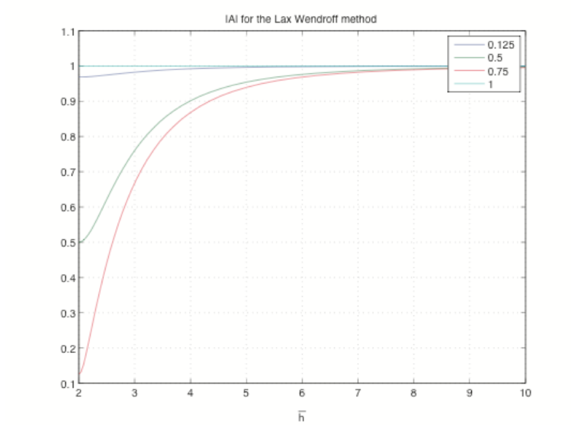
Errors for the Lax Wendroff method

For the Upwind method, the phase error is

$$\theta = -\tan^{-1}\left(\frac{Im(A)}{Re(A)}\right) = \tan^{-1}\left(\frac{\frac{c\tau}{h} sinkh}{1-\left(\frac{c\tau}{h}\right)^2 \sin^2\left(\frac{kh}{2}\right)}\right)$$

Again note that when $\frac{c\tau}{h} = 1$ $\varepsilon = 0$ and $|A| = 1$

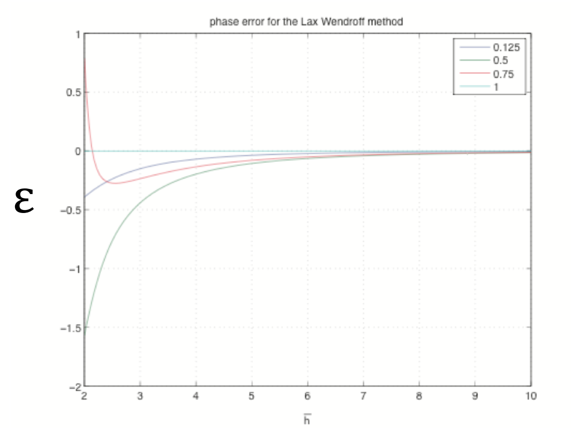
|A|



λ/h

Amplitude Plot

ε



λ/h

Phase error plot

Lax Wendroff N=50, 60 steps, τ=0.016, CFL=0.83