PHYS 416 – Chapter 6 – The Diffusion Equation

Due by 11:59 PM, Thursday, April 2, 2020

6. Suppose we replace our Dirichlet boundary conditions with the following Neumann boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=-\frac{L}{2}} = \left.\frac{\partial T}{\partial x}\right|_{x=\frac{L}{2}} = 0$$

(a) Using the method of images find the solution *T(x,t)* for the initial condition, $T(x,0)=\delta(x)$. [Pencil]

Note: the Gaussian solution for the diffusion equation is

$$T_G(x,t) = \frac{1}{\sigma(t)\sqrt{2\pi}} exp\left[-\frac{(x-x_0)^2}{2\sigma^2(t)}\right]$$

where x_0 is the location of the maximum and $\sigma(t)$ is the standard deviation, defined by

$$\sigma(t) = \sqrt{2\kappa t}$$

(c) Modify `dftcs` to implement these boundary conditions by setting $T_1^n = T_2^n$ and $T_N^n = T_{N-1}^n$. Explain why the spatial discretization is $x_i = \left(i - \frac{3}{2}\right)h - \frac{L}{2}$ with $h = \frac{L}{N-2}$ for these boundary conditions. [Computer, turn in program, plots and answers to questions.]

8. The Richardson scheme for solving the diffusion equation uses the following discretization

$$\frac{T_i^{n+1} - T_i^{n-1}}{2\tau} = \kappa\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{h^2}$$

(a) Modify `dftcs` to implement this method. Use the FTCS method to get it started. Try a variety of timesteps (τ) and show that the scheme is <u>always</u> unstable.

(b) Perform a Von Neumann stability analysis of this scheme. Assume that

$$T_j^n = T^n e^{ikhj}$$

And that

$$\frac{T^{n+1}}{T^n} = \frac{T^n}{T^{n-1}} \equiv A$$

And solve for A. Modify the program `vn_demo` to plot the amplification factors as a function of phase angle *kh*. Show that the scheme is <u>always</u> unstable. [Computer, Turn in your programs, and plots along with answers to questions.]

12. Consider the Neumann boundary conditions for the 1D neutron decay problem:

$$\frac{\partial}{\partial t}n(x,t) = D\frac{\partial^2}{\partial x^2}n(x,t) + Cn(x,t)$$

Where *D* is the diffusion rate, and *C* is the creation rate for neutrons. The boundary condition is:

$$\left.\frac{\partial n}{\partial x}\right|_{x=-\frac{L}{2}} = \left.\frac{\partial n}{\partial x}\right|_{x=\frac{L}{2}} = 0$$

(a) Using the separation of variables approach, show that this system is always supercritical. [Pencil]

(b) Modify the `neutrn2` program to implement these boundary conditions, by setting $n_1^n = n_2^n$ and $n_N^n = n_{N-1}^n$. In this case the spatial discretization is $x_i = \left(i - \frac{3}{2}\right) h - \frac{L}{2}$ with $h = \frac{L}{N-2}$ for these boundary conditions. Compare the programs output to the results predicted in part (a). (i.e., compare the growth of the average n with the prediction.) [Computer, Turn in your program, and plots comparing the theoretical versus the numerical result.]