# PHYS 416 – Chapter 7

*Due Tuesday, April 14, 2020 at 11:59 PM*

## Wave Equation

**A.** Modify `advect` to use Dirichlet boundary conditions. What happens when the wave hits the boundary? Physically, is this what would you expect to happen? Explain why or why not. Note: be careful about grid spacing when you go from periodic to Dirichlet boundary conditions.

**B.** a. Create a program called `wave.m/.py`, that is used to solve the wave equation,

$$\frac{\partial^2 A}{\partial t^2} = c^2 \frac{\partial^2 A}{\partial x^2}$$

and uses the method:

$$A_i^{n+1} = 2A_i^n - A_i^{n-1} + \frac{c^2\tau^2}{h^2}\left(A_{i+1}^n - 2A_i^n + A_{i-1}^n\right) \tag{1}$$

assuming Dirichlet Boundary conditions at both ends, i.e.

$$A\left(x = -\frac{L}{2}, t\right) = 0 \text{ and } A\left(x = \frac{L}{2}, t\right) = 0$$

and initial conditions are:

$$A(x,0) = \cos[k(x - x_o)]\, exp\left[-\frac{(x-x_o)^2}{2\sigma^2}\right] \text{ and } \frac{\partial}{\partial t}A(x,0) = 0$$

Assume $c$=1. [Hint: Start with `advect.m/.py`]

b. You will notice in part (a) that once the wave starts moving, it splits it to 2 parts, one part moving left and the other moving right. Make a simple change to the initial condition so that the wave moves to the right instead.
[*Turn in the programs and sample plots. If you like, you can combine the program for parts a and b into one with an option for the initial condition.]*

## Physics of Traffic Flow

7.11 Modify the traffic program so that it uses the initial conditions

$$\rho(x, t = 0) = \frac{\rho_m}{2}[1 + \cos\left(\frac{4\pi x}{L}\right)]$$

Plot the density versus position for a variety of times and show that the cosine wave turns into a sawtooth wave. In nonlinear acoustics, this is referred to as an N-wave. If you have ever been to a very loud rock concert, you may have heard one of these.
[*Turn in the program and the sample plots*]

7.12 Suppose that we have a uniform density of traffic with a small congested area. Modify the traffic program so that the initial conditions are

$$\rho(x, t = 0) = \rho_0[1 + \alpha \exp\left(-\frac{x^2}{2\sigma^2}\right)]$$

where $\alpha = \frac{1}{5}$, $\sigma = \frac{L}{8}$ and $\rho_0$ are constants.

(a) Show that for light traffic, (e.g., $\rho_0 = \frac{\rho_m}{4}$) the perturbation moves forward.

(b) Show that for heavy traffic, (e.g., $\rho_0 = \frac{3\rho_m}{4}$) the perturbation moves backward. Interpret this result physically.
(c) Show that for $\rho_0 \sim \frac{\rho_m}{2}$)the perturbation is almost stationary; it drifts and distorts slightly.
[*Turn in the program and the sample plots*]

## All of the following are 1 credit optional exercises

7.8. Call $x_c(t)$ the position of a given car; then

$$\frac{dx_c(t)}{dt} = v(\rho(x_c(t), t))$$

(a) Show *that*

$$x_c(t) = \begin{cases} x_c(0) & t < -\frac{x_c(0)}{v_m} \\ v_m t - 2\sqrt{-x_c(0)v_m t} & t > -\frac{x_c(0)}{v_m} \end{cases}$$

by using the solution to the stoplight problem, i.e.,

$$\rho(x,t) = \begin{cases} \rho_m & \text{for } x \leq -v_m t \\ \frac{1}{2}\left(1 - \frac{x}{v_m t}\right)\rho_m & \text{for } -v_m t < x < v_m t \\ 0 & \text{for } x \geq v_m t \end{cases} \qquad (*)$$

And $v(\rho) = v_m(1 - \frac{\rho}{\rho_m})$ [Note that initially the car's location is < 0.] [*Pencil*]
(b) Modify the `trafd.m/.py` program to also plot the analytic solution from (a) along with the numerical solution. [Turn in the program and sample plots.]

## High resolution methods for Hyperbolic Equations

C. Test the high-resolution methods (`advecth`, that also needs `hires` and `limiter`) on how they deal with the original Gaussian wave form. Use N=50, and $\tau$=0.01. How do they compare to the methods we used earlier such as the Lax and Lax-Wendroff methods?

Modify the limiter program to use the so-called *superbee* method

$$\phi(\theta) = \max(0, \min(1, 2\theta), \min(2\theta))$$

compare the results to the other limiter methods for both the 2square waves and the Gaussian initial waveform.
[Note: Be careful how you implement MATLAB's max and min functions],

D. Modify your traffic program to use the limiter methods. In this case, the velocity (u) is defined as the average velocity in the cell, a good approximation is to use

$$u_i = \frac{1}{2}(u_{i-1} - u_{i+1})$$

Compare the results you get with the various methods.