N-body Choreography

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In 1888 King Oscar II of Sweden established a prize for whoever could solve the n-body problem. Newton had solved the 2-body problem, but a solution to the n-body problem remained (and still remains) elusive. Euler in 1767 and Lagrange in 1772 found periodic solutions to the 3-body problem that were ellipses, however these solutions were unstable. The prize was eventually awarded to Henri Poincaré, while he did not solve the problem he introduced the idea of surfaces of section, phase space that now form the roots of deterministic chaos. In this unit we will look at Lagrange's periodic solution to the 3 body problem as well as recently discovered novel solutions. The other solution by Euler assumes that the 3 masses are collinear, and requires the solution of a 5th order polynomial. Until recently it was thought that there were no other solutions to the 3 body problem, but in 2000 Alain Chenciner and Richard Montgomery proved the existence of a figure 8 orbit (Annals of Mathematics, 152 (2000), 881–901. "A remarkable periodic solution of the three-body problem in the case of equal masses".) that was discovered numerically by Cristopher Moore in 1993 (http://dx.doi.org/10.1103/PhysRevLett.70.3675) An example of this solution and some others can be found in the program called 'nbody.m/py')

Circle of planets

v 1.01

Lets setup a problem where we have N planets symmetrically distributed and of equal mass (unit 1) moving anticlockwise in a circle centered at the origin of radius r. If the system is symmetric, then they all should move in a circle provided the initial conditions are setup correctly. From the figure 1 below and by symmetry of the setup at *t*=0, the angle θ_n of the location relative to the x-axis of the *n*-th planet is given by

$$\theta_n = 2\pi \frac{n-1}{N}$$

Without loss of generality, we can look at the mass labeled 1 in the figure. At *t*=0 we set the position and velocity at

$$\vec{r}_{01} = r\hat{x}$$
$$\vec{v}_{01} = v_0\hat{y}$$

the key to getting the right orbits is to determine v_0 so that all the masses continue to move in a circle. If we compute the acceleration on mass 1, due to all the other masses, by symmetry the y-component of acceleration cancels out to zero while the x-component is given by

$$a_x = -\sum_{n=2}^{N} GM \frac{\cos\alpha}{R^2}$$

which points towards the origin, also from the geometry then

$$\alpha = \frac{\pi}{2} - \theta_n$$

and *R* can be computed thanks to Pythagoras as

$$R^2 = (r - r\cos\theta_n)^2 + r^2\sin^2\theta_n = 2r^2(1-\cos\theta_n)$$

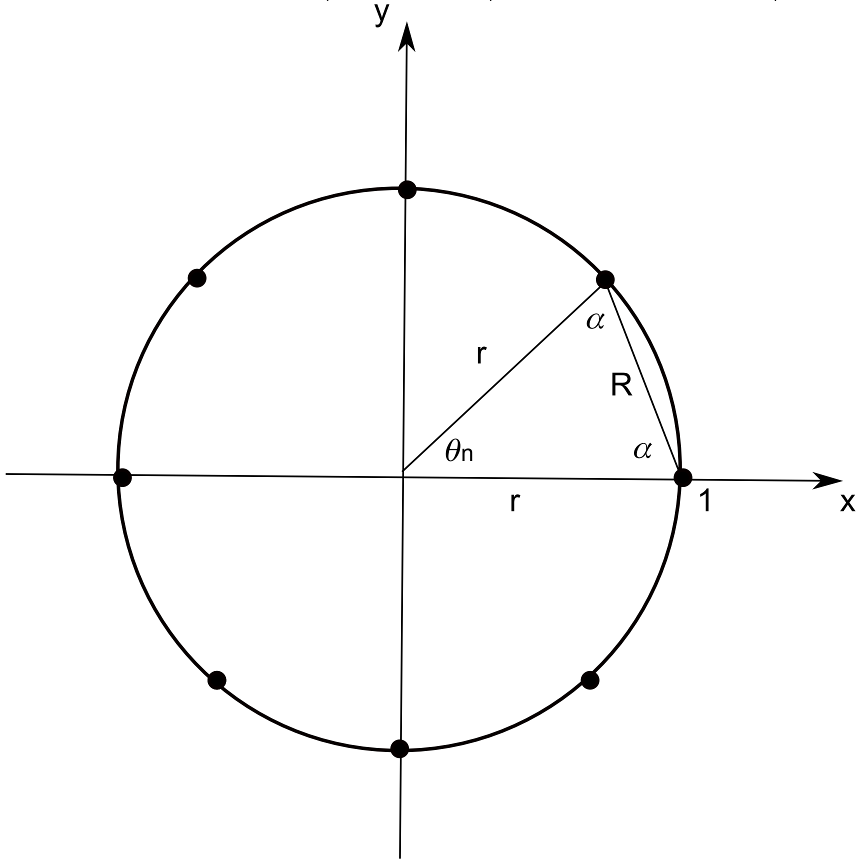

Figure 1

Putting this into the above acceleration equation we get

$$a_x = -\sum_{n=2}^{N} GM \frac{\sin(\theta_n/2)}{2r^2(1-\cos\theta_n)}$$

this can be simplified even further to

$$a_x = -\sum_{n=2}^{N} GM \frac{1}{4r^2\sin(\theta_n/2)}$$

However, for coding reasons it turns out the first form is more convenient as $\sin\left(\frac{1}{2}\theta_n\right)$ goes to zero when one of the masses crosses the x-axis, which can cause a divide by zero problem. To get the velocity that imposes circular motion, we equate the acceleration to v_0^2/r and solve for v_0 to get an expression for the initial speed:

$$v_0 = \sqrt{\sum_{n=2}^{N} GM \frac{\sin(\theta_n/2)}{2r(1-\cos\theta_n)}}$$

All the masses are then initialized with this speed. Their initial position and velocities for all N masses is then given by

$$\vec{r}_{0n} = \hat{x} r \cos(\theta_n) + \hat{y} r \sin(\theta_n)$$
$$\vec{v}_{0n} = -\hat{x} v_0 \sin(\theta_n) + \hat{y} v_0 \cos(\theta_n)$$

You will find that besides the binary system (N=2) this system is unstable. That can be seen if you run the code long enough the planets will start to deviate from circular motion.

Lagrange's solution to the 3-body problem

The simplest n-body problem is one with 3 bodies and here we will explore a specific special case solution that was originally derived by Lagrange in 1772. We start with the equations of motion of 3 bodies written as

$$\begin{aligned}
\ddot{\vec{x}}_1 &= -GMm_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} - GMm_3 \frac{\vec{x}_1 - \vec{x}_3}{|\vec{x}_1 - \vec{x}_3|^3} \\
\ddot{\vec{x}}_2 &= -GMm_1 \frac{\vec{x}_2 - \vec{x}_1}{|\vec{x}_2 - \vec{x}_1|^3} - GMm_3 \frac{\vec{x}_2 - \vec{x}_3}{|\vec{x}_2 - \vec{x}_3|^3} \\
\ddot{\vec{x}}_3 &= -GMm_1 \frac{\vec{x}_3 - \vec{x}_1}{|\vec{x}_3 - \vec{x}_1|^3} - GMm_2 \frac{\vec{x}_3 - \vec{x}_2}{|\vec{x}_3 - \vec{x}_2|^3}
\end{aligned} \tag{1}$$

Where m_i is the mass of each object in solar masses, and *GM* in our units is $4\pi^2$. If we put center or mass (CM) at the origin, then

$$\sum_{i=1}^{3} m_i \vec{x}_i = 0 \tag{2}$$

It is also useful to introduce another set of variables (see Figure 2) called relative position vectors $\vec{s}_i$ defined by

$$\begin{aligned}
\vec{s}_1 &\equiv \vec{x}_3 - \vec{x}_2 \\
\vec{s}_2 &\equiv \vec{x}_1 - \vec{x}_3 \\
\vec{s}_3 &\equiv \vec{x}_2 - \vec{x}_1
\end{aligned} \tag{3}$$

These variables are related so that

$$\sum_{i=1}^{3} \vec{s}_i = 0 \tag{4}$$

Solving (2) and (3) for the position vectors

$$\begin{aligned}
m\vec{x}_1 &= m_3\vec{s}_2 - m_2\vec{s}_3 \\
m\vec{x}_2 &= m_1\vec{s}_3 - m_3\vec{s}_1 \\
m\vec{x}_3 &= m_2\vec{s}_1 - m_1\vec{s}_2
\end{aligned} \tag{5}$$

where *m* is the total mass of the system

$$m \equiv \sum_{i=1}^{3} m_i$$

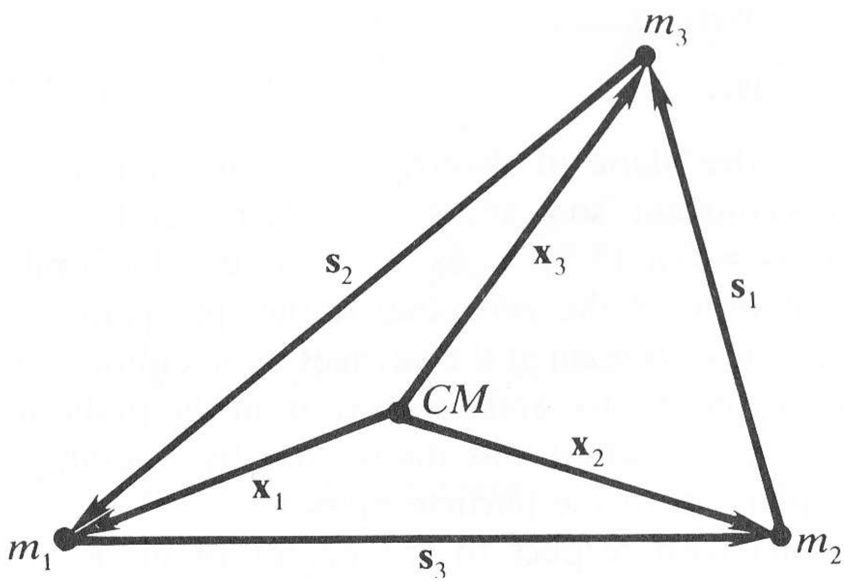

Figure 2: Coordinate System for the 3 masses

If we now substitute (3) into (1) we get

$$\begin{aligned}\ddot{\vec{s}}_1 &= -GMm\frac{\vec{s}_1}{|\vec{s}_1|^3} + GMm_1\vec{G} \\ \ddot{\vec{s}}_2 &= -GMm\frac{\vec{s}_2}{|\vec{s}_2|^3} + GMm_2\vec{G} \\ \ddot{\vec{s}}_3 &= -GMm\frac{\vec{s}_3}{|\vec{s}_3|^3} + GMm_3\vec{G}\end{aligned} \tag{6}$$

where

$$\vec{G} \equiv \sum_{i=1}^{3} \frac{\vec{s}_i}{|\vec{s}_i|^3} \tag{7}$$

One nice feature is that equations (7) are decoupled into a set of 3 similar equations if $\vec{G} = 0$ that looks like a set of 2 body equations. One way for this to be true is to set

$$|\vec{s}_1| = |\vec{s}_2| = |\vec{s}_3| \equiv s \tag{8}$$

In other words the masses are at the edge of an equilateral triangle as shown in Figure 3. One can then express the sides of the triangle $\vec{s}_{1,2}$ in terms of one side $\vec{s}_3$ as a rotation

$$\begin{aligned}\vec{s}_1 &= \mathbf{R}\left(\frac{\mathbf{2}\pi}{\mathbf{3}}\right)\tilde{\mathbf{s}}_\mathbf{3} \\ \vec{s}_2 &= \mathbf{R}\left(-\frac{\mathbf{2}\pi}{\mathbf{3}}\right)\tilde{\mathbf{s}}_\mathbf{3}\end{aligned} \tag{9}$$

where $\mathbf{R}(\theta)$ is the rotation matrix defined as

$$\mathbf{R}(\theta) \equiv \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{10}$$

If we substitute (9) into (5) we get

$$
\begin{aligned}
m\vec{x}_1 &= \left[m_3\mathbf{R}\left(-\frac{\mathbf{2}\pi}{\mathbf{3}}\right) - \mathbf{m_2 1}\right]\vec{s}_3 \\
m\vec{x}_2 &= \left[m_1\mathbf{1} - \mathbf{m_3 R}\left(\frac{\mathbf{2}\pi}{\mathbf{3}}\right)\right]\vec{s}_3 \\
m\vec{x}_3 &= \left[m_2\mathbf{R}\left(\frac{\mathbf{2}\pi}{\mathbf{3}}\right) - \mathbf{m_1 R}\left(-\frac{\mathbf{2}\pi}{\mathbf{3}}\right)\right]\vec{s}_3
\end{aligned} \tag{11}
$$

where $\mathbf{1}$ is the 2x2 identity matrix. Equation (11) can be used to set the initial positions of the masses, by specify a side vector $\vec{s}_3$ all the position vectors can be determined.

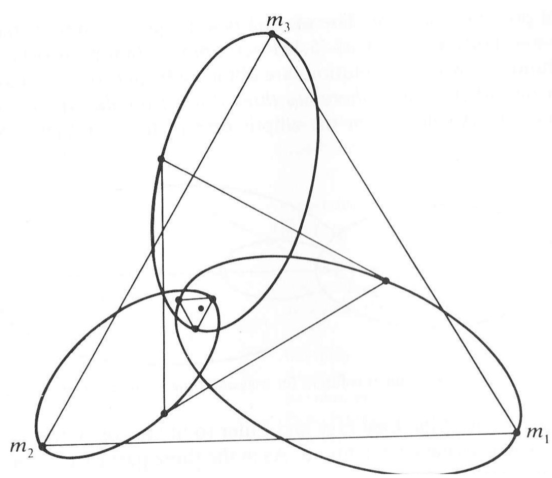

Figure 3: An example orbit configuration.

If we take the second derivative w.r.t. time of equation (11) and then use equation (6) to eliminate $\vec{s}_i$ you end up (after some tedious matrix algebra) with and equation of motion that says the force on each particle is towards the center or mass of the system

$$
\begin{aligned}
\ddot{\vec{x}}_1 &= -GM\frac{(m_2^2 + m_2 m_3 + m_3^2)^{3/2}}{m^2}\frac{\vec{x}_1}{|\vec{x}_1|^3} \\
\ddot{\vec{x}}_2 &= -GM\frac{(m_1^2 + m_1 m_3 + m_3^2)^{3/2}}{m^2}\frac{\vec{x}_2}{|\vec{x}_2|^3} \\
\ddot{\vec{x}}_3 &= -GM\frac{(m_1^2 + m_1 m_2 + m_2^2)^{3/2}}{m^2}\frac{\vec{x}_3}{|\vec{x}_3|^3}
\end{aligned} \tag{12}
$$

Equation (12) can then be used to setup the initial condition for velocity. If we assume as in Figure 3 that all the masses are at their aphelion, then their velocity should be perpendicular to $\vec{x}_i$. If we want a circular orbit, then from (12)

$$
\frac{v_{10}^2}{|\vec{x}_1|} = GM\frac{(m_2^2 + m_2 m_3 + m_3^2)^{3/2}}{m^2}\frac{1}{|\vec{x}_1|^2} \tag{13}
$$

which is basically saying that the centripetal acceleration balances gravitational acceleration. Equation (13) then simplifies to

$$v_{10} = \sqrt{GM}\frac{(m_2^2 + m_2 m_3 + m_3^2)^{3/4}}{m}\frac{1}{\sqrt{|\vec{x}_1|}} \tag{14}$$

Where v_{10} is the initial speed of mass 1. Similarly, for the other 2 masses

$$v_{20} = \sqrt{GM}\frac{(m_1^2 + m_1 m_3 + m_3^2)^{3/4}}{m}\frac{1}{\sqrt{|\vec{x}_2|}}$$

$$v_{30} = \sqrt{GM}\frac{(m_1^2 + m_1 m_2 + m_2^2)^{3/4}}{m}\frac{1}{\sqrt{|\vec{x}_3|}} \tag{15}$$

The direction of motion should be perpendicular to $\vec{x}_i$, so that

$$\vec{v}_{i0} = -v_{i0}\sin(\alpha_i)\hat{x} + v_{i0}\cos(\alpha_i)\hat{y} \tag{16}$$

where

$$\tan(\alpha_i) \equiv= \frac{y_{i0}}{x_{i0}} \tag{17}$$

and y_{i0} and x_{i0} are the components of the initial location as determined by equation (11) for the *i*-th mass.

Euler's Collinear Solution

Euler found another exact solution to the three body equations. For this case, lets make mass 2 lie in between masses 1 and 3. We can satisfy equation (4) by assuming that

$$\vec{s}_1 = \lambda\vec{s}_3 \qquad\qquad \vec{s}_2 = -(1+\lambda)\vec{s}_3 \tag{18}$$

where λ is a positive scalar to be determined. We can eliminate $\vec{G}$ from equations (6) to get

$$\begin{aligned} \ddot{\vec{s}}_1 + m\frac{\vec{s}_1}{|\vec{s}_1|^3} &= \frac{m_1}{m_3}\left(\ddot{\vec{s}}_3 + m\frac{\vec{s}_3}{|\vec{s}_3|^3}\right) \\ \ddot{\vec{s}}_2 + m\frac{\vec{s}_2}{|\vec{s}_2|^3} &= \frac{m_2}{m_3}\left(\ddot{\vec{s}}_3 + m\frac{\vec{s}_3}{|\vec{s}_3|^3}\right) \end{aligned} \tag{19}$$

Inserting (18) in (19) to eliminate the vectors $\vec{s}_1$ and $\vec{s}_2$ we get

$$(m_2 + m_3(1+\lambda)\ddot{\vec{s}}_3 = -(m_2 + m_3(1+\lambda)^{-2})\frac{m\vec{s}_3}{|\vec{s}_3|^3}$$

$$(m_1 - m_3\lambda)\ddot{\vec{s}}_3 = -(m_1 - m_3\lambda^{-2})\frac{m\vec{s}_3}{|\vec{s}_3|^3}$$

Which means then

$$\frac{m_2 + m_3(1+\lambda)}{m_1 - m_3\lambda} = \frac{m_2 + m_3(1+\lambda)^{-2}}{m_1 - m_3\lambda^{-2}}$$

Rearranging we get

$$\begin{aligned} &(m_1+m_2)\lambda^5 + (3m_1 + 2m_2)\lambda^4 + (3m_1 + m_2)\lambda^3 - \\ &\qquad (m_2 + 3m_3)\lambda^2 - (2m_2 + 3m_3)\lambda - (m_2 + m_3) = 0 \end{aligned} \tag{20}$$

Equation (20) is a 5[th] order polynomial that can be solved for λ given the masses.

Reference

New Foundations for Classical Mechanics, by David Hestenes, Kluwer, 1990.