High Resolution Methods for Conservation Equations

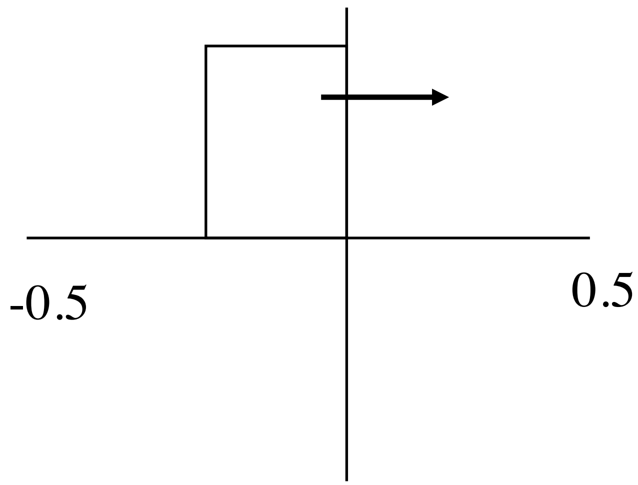
You may have noticed that in some cases the traffic program produced spurious oscillations that in come cases were negative. To understand this, lets go back to the advection equation that we looked at earlier, i.e.,

$$\frac{\partial u}{\partial t} = -c\,\frac{\partial u}{\partial x}$$

[Ref: Finite Volume methods for Hyperbolic problems, R. LeVeque, Cambridge Press, 2002]

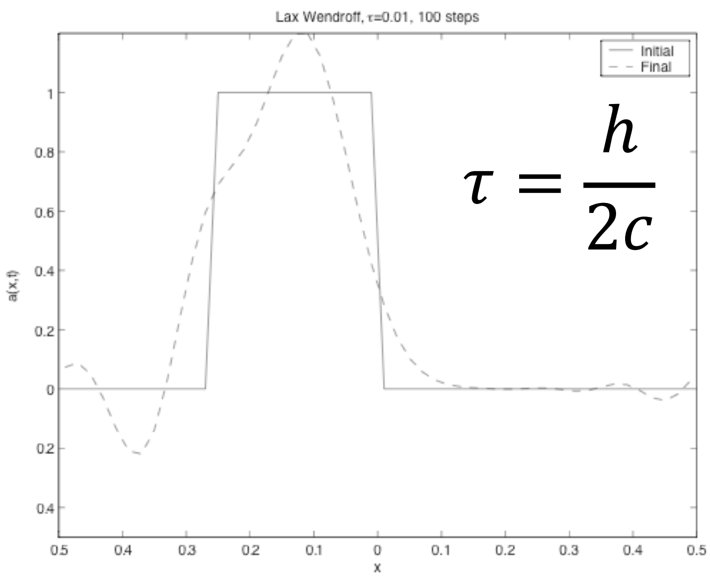
Square Wave Example

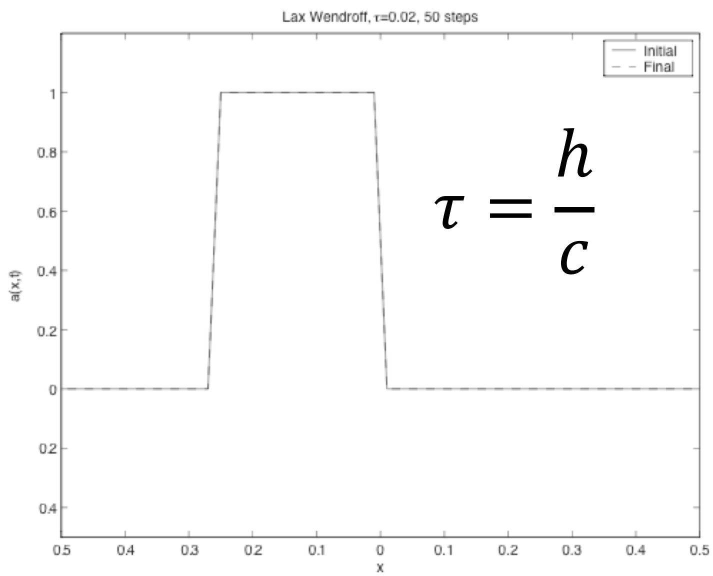
Lets look at what various numerical methods do in the case of a simple square wave moving to the right at speed 1 in a periodic system of length 1. We will run the code until the waveform returns to its starting place.



Lax Wendroff Solution

This shows the computed numerical solution (dashed) plotted against the exact solution for 1 period using the `advect` program





Lax Wendroff Scheme

When the timestep is half the courant number, the solution is inaccurate.

Let Q_j^n represent the numerical solution on the grid.

We can write out the Lax Wendroff scheme as

$$Q_j^{n+1} = Q_j^n - \frac{c\tau}{2h}\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\tau}{h}\right)^2 \left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right)$$

When $\tau = \frac{h}{c}$ the above reduces to

$$Q_j^{n+1} = Q_{j-1}^n$$

Which simply says that each cell is replaced with the value on the cell on its left.

Upwind Scheme

An even simpler scheme is the upwind scheme

$$Q_j^{n+1} = Q_j^n - \frac{c\tau}{h}\left(Q_j^n - Q_{j-1}^n\right)$$

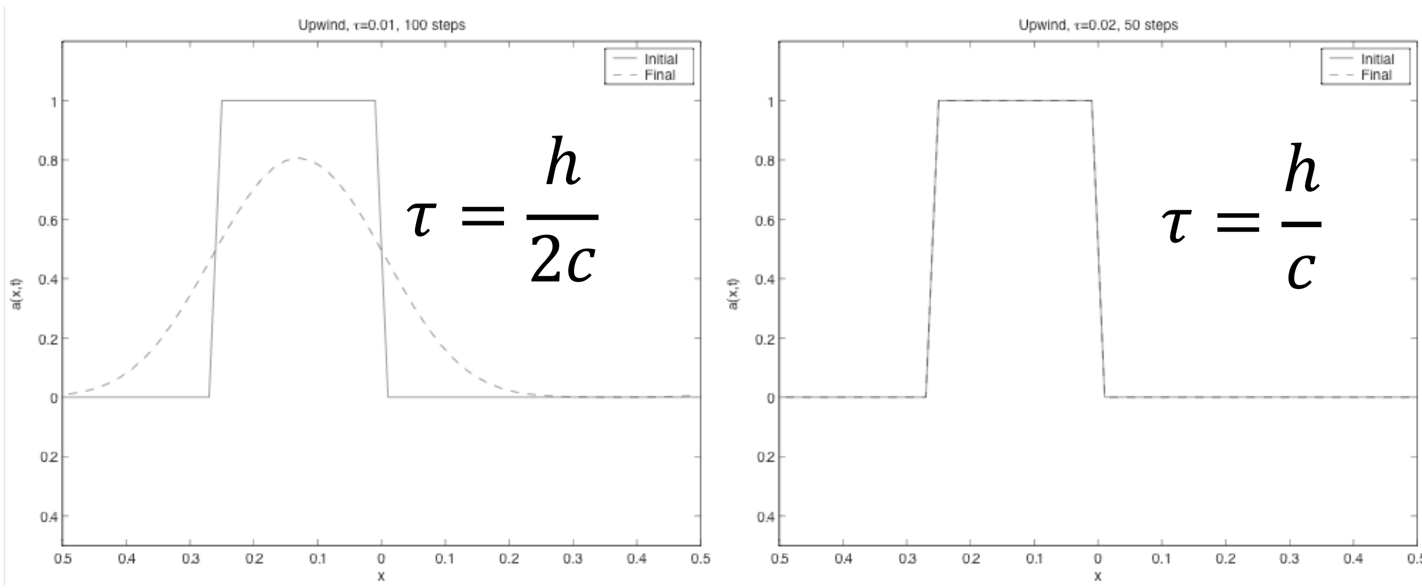
When $\tau = \frac{h}{c}$ the above reduces to

$$Q_j^{n+1} = Q_{j-1}^n$$

But when $\tau = \frac{h}{2c}$ we get that $Q_j^{n+1} = \frac{1}{2}(Q_j^n + Q_{j-1}^n)$

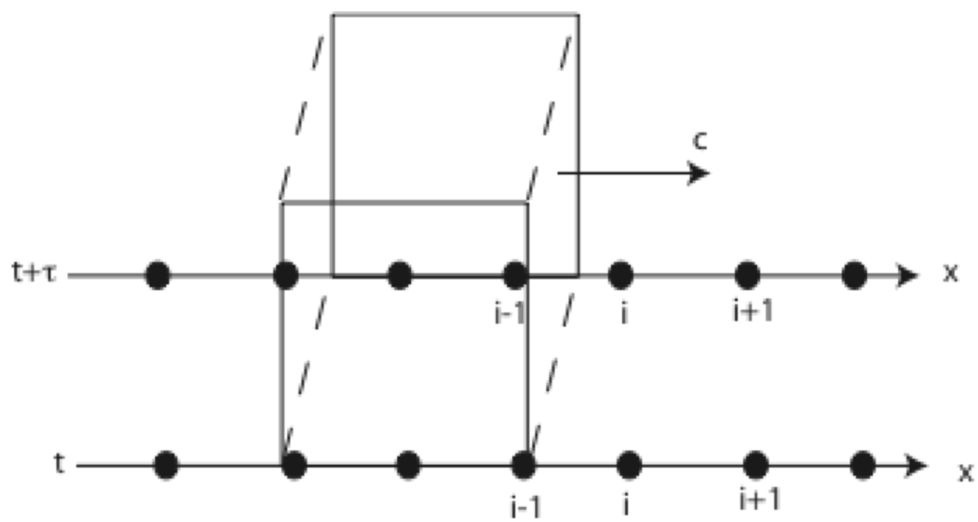
The results of which are illustrated on the next page

Upwind Scheme



Alternative Interpretation

Another way of looking at these schemes is in terms of characteristics



Lax Wendroff - again

An alternative way to write the Lax Wendroff scheme (assume c>0)

$$Q_j^{n+1} = Q_j^n - \frac{c\tau}{2h}\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{1}{2}\left(\frac{c\tau}{h}\right)^2 \left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right)$$

By rearranging some terms is as

$$Q_j^{n+1} = Q_j^n - \frac{c\tau}{h}\underbrace{\boxed{\left(Q_j^n - Q_{j-1}^n\right)}}_{\text{upwind}} \underbrace{-}_{\text{'anti diffusion'}} \frac{1}{2}\left(\frac{c\tau}{h}\right)\left[1 - \left(\frac{c\tau}{h}\right)\right]\underbrace{\boxed{\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right)}}_{\boxed{\text{diffusion}}}$$

Lax Wendroff - interpretation

In this view, the Lax Wendroff scheme adds a certain amount of 'anti diffusion' to the upwind scheme (due to the negative sign in front of the diffusion term on the RHS of the previous equation.) The basic idea of limiter schemes is to control the amount of anti-diffusion so as to not produce undesirable effects in regions of sharp gradients.

Reconstruct, Evolve and Average (REA)

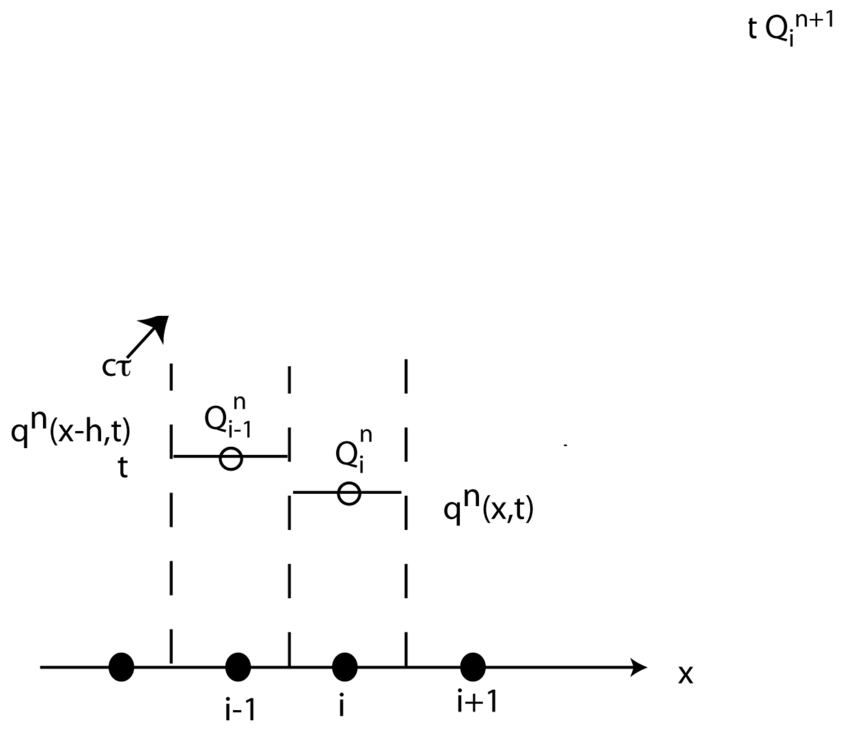
The upwind method for the advection equation is a special case of the so-called REA algorithm, which stands for Reconstruct, Evolve and Average.

Reconstruct: Each numerical quantity Q_j^n on the grid can be interpreted as the average of a function $q_j^n(x)$ over the grid cell.

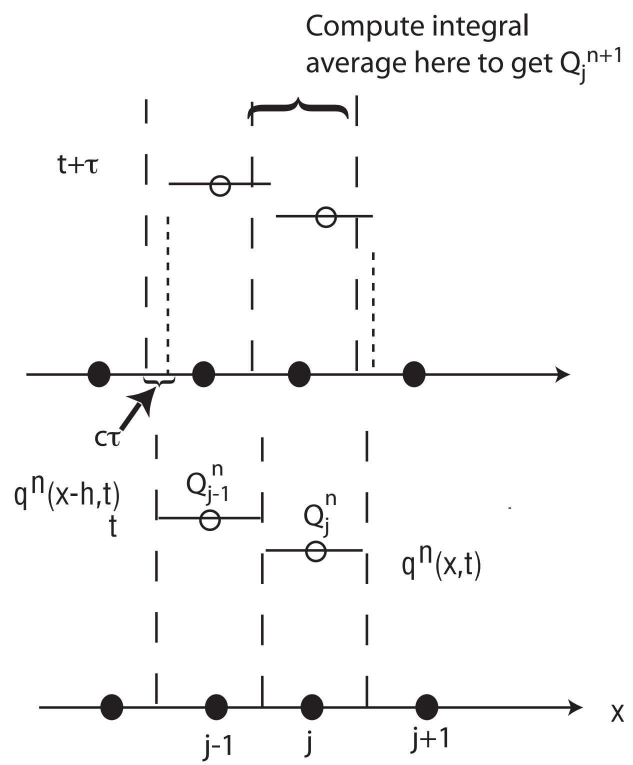
Evolve: Move the function $q_j^n(x)$ a distance $c\tau$ to obtain the function at the new timestep $q_i^{n+1}(x)$

Average: Compute the average of $q_j^{n+1}(x)$ to compute the new estimate of Q_j^{n+1}

REA - Upwind method



REA - Upwind method



The Upwind method-Piecewise constant

Reconstruct: We interpret the values stored at each grid point as some characteristic average value of the quantity over the cell and if we assume that the values are stored as step functions $q_j^n(x,t)$ (piecewise constant)

Evolve: Move the functions $q_j^n(x,t)$ a distance $c\tau$

Average: Compute the new value Q_j^{n+1} by computing the average value over the cell from $j-\frac{1}{2}\ to\ j+\frac{1}{2}$

$$Q_j^{n+1} = \frac{1}{h}\left((h-c\tau)Q_j^n + c\tau Q_{j-1}^n\right)$$
$$= Q_j^n - \frac{c\tau}{h}(Q_j^n - Q_{j-1}^n)$$

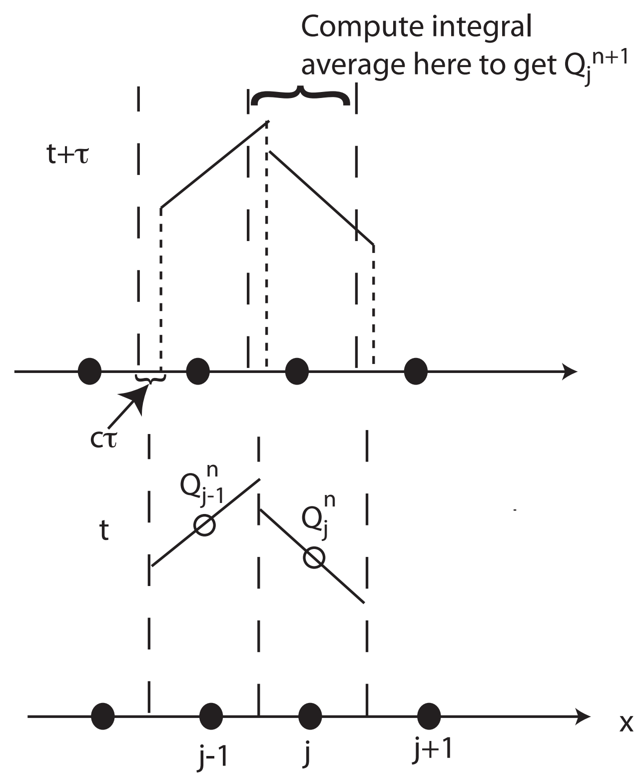
Piecewise Linear

If we generalize the approximation and assume that the values vary linearly between grid points with a function of slope σ_j^n, then in the range

$j - \frac{1}{2}, j + \frac{1}{2}$

The function takes on the form

$$q_j^n(x) = Q_j^n + \sigma_j^n(x - x_j)$$



Piecewise Linear (cont.)

$$Q_j^{n+1} = \frac{1}{h}\left[\int_{x_j-\frac{h}{2}-c\tau}^{x_j-\frac{h}{2}} q_{j-1}^n(x)dx + \int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}-c\tau} q_j^n(x)dx\right]$$

That becomes

$$Q_j^{n+1} = \frac{1}{h}\left[\int_{x_j-\frac{h}{2}-c\tau}^{x_j-\frac{h}{2}} Q_{j-1}^n + \sigma_{j-1}^n\left(x - x_{j-1}\right)dx + \int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}-c\tau} Q_j^n + \sigma_j^n\left(x - x_j\right)dx\right]$$

Piecewise Linear (cont.)

$$Q_j^{n+1} = \frac{c\tau}{2h}\left[2Q_{j-1}^n + \sigma_{j-1}^n(x_j - \frac{h}{2} - x_{j-1} + x_j - \frac{h}{2} - c\tau - x_{j-1}\right] + \frac{h - c\tau}{2h}\left[2Q_j^n + \sigma_j^n(\frac{h}{2} - c\tau - \frac{h}{2})\right]$$
$$= \frac{c\tau}{h}\left[Q_{j-1}^n + \frac{1}{2}\sigma_{j-1}^n(h - c\tau)\right] + \frac{h-c\tau}{2h}\left[Q_j^n - \frac{1}{2}\sigma_j^n c\tau\right]$$
$$= Q_j^n - \frac{c\tau}{h}\left(Q_j^n - Q_{j-1}^n\right) - \frac{c\tau}{2h}(h - c\tau)\left(\sigma_j^n - \sigma_{j-1}^n\right)$$

This is basically an upwind method with a correction term that depends on the slopes.

One has a choice of slopes σ

Upwind: $\sigma_j^n = 0$

Lax-Wendroff: $\sigma_j^n = \frac{Q_{j+1}^n - Q_j^n}{h}$ (Downwind)

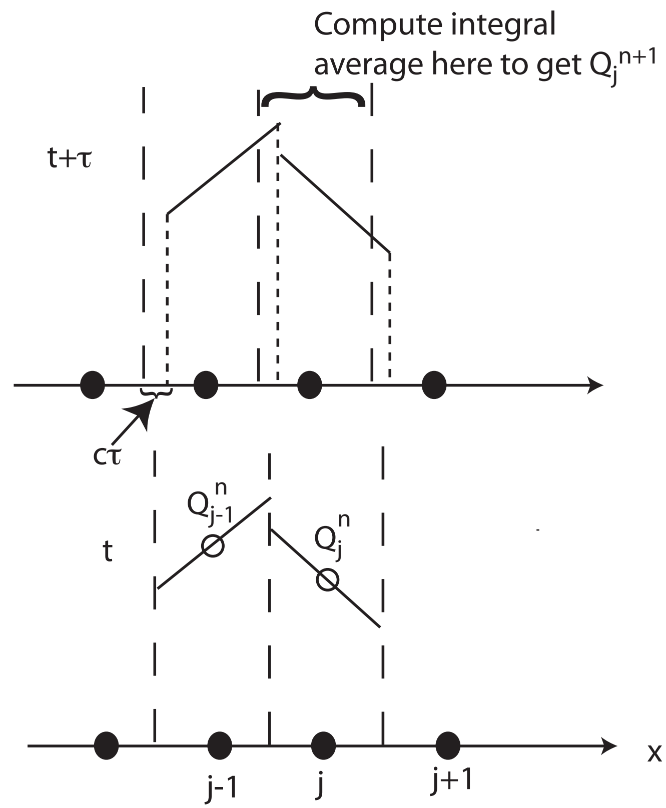
Beam-Warming: $\sigma_j^n = \frac{Q_j^n - Q_{j-1}^n}{h}$ (Upwind)

Fromm: $\sigma_j^n = \frac{Q_{j+1}^n - Q_{j-1}^n}{2h}$ (Centered)

The last 3 choices of slopes determine the formal order of accuracy, but only the downwind version results in a symmetric formula.

Back to Lax Wendroff

Lets look at the Lax Wendroff method when it encounters a discontinuity - note the overshoot in the code runs



Limiters

Another way to write equation (1) is in the form

$$Q_j^{n+1} = Q_j^n - \frac{\tau}{h}(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}})$$

Where $F^n_{j\pm\frac{1}{2}}$ corresponds to the flux values that move across cell interfaces.

The idea behind the limiters is to limit the amount of flux to prevent undershoots and overshoots.

Slope Limiters

This approach limits the slope σ_j^n to reduce artifacts such as overshoots while at the same time minimizing diffusion. To proceed, we need to generalize our expressions for the flux to take into account all signs of the velocity c (so far we have implicitly assumed that c >0). We can then write the flux as

$$\mathrm{F}^{\mathrm{n}}_{\mathrm{j}\pm\frac{1}{2}} = \begin{cases} cQ_{j-1}^n + \dfrac{1}{2}c(h - c\tau)\sigma_{j-1}^n & c \geq 0 \\ cQ_j^n - \dfrac{1}{2}c(h + c\tau)\sigma_j^n, & c \leq 0 \end{cases}$$

minmod Slope Limiters

Slope limiters take various function forms, each having advantages and disadvantages. One popular one is the so called minmod limiter, where

$$\sigma_j^n = minmod\left[\frac{Q_j^n - Q_{j-1}^n}{h}, \frac{Q_{j+1}^n - Q_j^n}{h}\right]$$

Where $minmod(a,b) \equiv \begin{cases} a & \text{if } |a| < |b| \quad \text{and } ab > 0 \\ b & \text{if } |a| > |b| \quad \text{and } ab > 0 \\ 0 & \text{if } ab < 0 \end{cases}$

In words, the minmod method does the following: If a and b have the same sign, it chooses the one with the smallest absolute value. If the signs are opposite, then we have a local maxima or minima, in which case the slope is set to zero and diffusion is turned on to ‘clean’ it out.

Total Variation Diminishing (TVD)

So what does it mean to 'clean' up a solution? Essentially we need a measure of the amount of oscillations in the system, a measure of which is known as the total variation of a function

$$TV(Q) = \sum_j \left| Q_j - Q_{j-1} \right|$$

What is want is that TV(Q) be constant in time. The following schemes can be shown to have this property.

Flux Limiters

Rather than using the slope we will look at the jumps across cell faces where we will have the jump

$$\Delta Q^{n}_{j-\frac{1}{2}} = \mathrm{Q}^{\mathrm{n}}_{\mathrm{j}} - \mathrm{Q}^{\mathrm{n}}_{\mathrm{j}-1}$$

So a more general way to write the flux is

$$F^{n}_{j-\frac{1}{2}} = \mathrm{c}^{+}\mathrm{Q}^{\mathrm{n}}_{\mathrm{j}-1} + \mathrm{c}^{-}\mathrm{Q}^{\mathrm{n}}_{\mathrm{j}} + \frac{1}{2}|c|\left(1 - \left|\frac{c\tau}{h}\right|\right)\delta^{n}_{j-\frac{1}{2}}$$

Where $\mathrm{c}^{+} = \max(c, 0)$ and $\mathrm{c}^{-} = \mathrm{m}in(c, 0)$ and $\delta^{n}_{j-\frac{1}{2}} =$ a limited version of $\Delta \mathrm{Q}^{\mathrm{n}}_{\mathrm{j}-\frac{1}{2}}$ where *c* is the average velocity in cell *j* and *j-1*

Flux Limiters (cont.)

We will write $\delta^n_{i-\frac{1}{2}} = \phi\left(\theta^n_{j-\frac{1}{2}}\right)\Delta \mathrm{Q}^n_{\mathrm{j}-\frac{1}{2}}$

where $\theta^n_{j-\frac{1}{2}} = \dfrac{\Delta Q^n_{J-\frac{1}{2}}}{\Delta Q^n_{j-\frac{1}{2}}}$

and J= $\begin{cases} j-1 \text{ if } c > 0 \\ j+1 \text{ if } c \leq 0 \end{cases}$

A flux limiter does the following:

- For smooth parts of a solution it will do second order accurate flux-conserved advection.
- For regions near a jump or a very sharp gradient it will switch to first order (i.e. upwind) flux-conserved advection.

Flux limiter method

In general then, the flux limiter method takes the form

$$Q_j^{n+1} = Q_j^n - \frac{c\tau}{h}\left(Q_j^n - Q_{j-1}^n\right) - \frac{c\tau}{2h}\left(1 - \frac{c\tau}{h}\right)\left[\phi\left(\theta_{j+\frac{1}{2}}^n\right)\left(Q_{j+1}^n - Q_j^n\right) - \phi\left(\theta_{j-\frac{1}{2}}^n\right)\left(Q_j^n - Q_{j-1}^n\right)\right] \; if\ c > 0$$

$$Q_j^{n+1} = Q_j^n - \frac{c\tau}{h}\left(Q_{j+1}^n - Q_j^n\right) - \frac{c\tau}{2h}\left(1 + \frac{c\tau}{h}\right)\left[\phi\left(\theta_{j+\frac{1}{2}}^n\right)\left(Q_{j+1}^n - Q_j^n\right) - \phi\left(\theta_{j-\frac{1}{2}}^n\right)\left(Q_j^n - Q_{j-1}^n\right)\right] \; if\ c < 0$$

Linear Methods

upwind: $\phi(\theta) = 0$

Lax-Wendroff: $\phi(\theta) = 1$

Beam-Warming: $\phi(\theta) = \theta$

Fromm: $\phi(\theta) = \frac{1}{2}(1 + \theta)$

Example- Linear Methods- Lax Wendroff

In the special case of the Lax Wendroff limiter, for c>0, the equation on page 21 becomes

$$Q_j^{n+1} = Q_j^n - \frac{c\tau}{h}\left(Q_j^n - Q_{j-1}^n\right) - \frac{c\tau}{2h}(1\text{-}\ \frac{c\tau}{2h})\left[\left(\mathrm{Q}_{\mathrm{j+1}}^{\mathrm{n}} - \mathrm{Q}_{\mathrm{j}}^{\mathrm{n}}\right) - \left(\mathrm{Q}_{\mathrm{j}}^{\mathrm{n}} - \mathrm{Q}_{\mathrm{j-1}}^{\mathrm{n}}\right)\right]$$

$$= Q_j^n - \frac{c\tau}{h}\left(Q_j^n - Q_{j-1}^n\right) - \frac{c\tau}{2h}(1\text{-}\ \frac{c\tau}{2h})\left[\mathrm{Q}_{\mathrm{j+1}}^{\mathrm{n}} - 2\mathrm{Q}_{\mathrm{j}}^{\mathrm{n}} - \mathrm{Q}_{\mathrm{j-1}}^{\mathrm{n}}\right]$$

Which is the Lax Wendroff method (slide 8)

High Resolution limiters

minmod: $\phi(\theta) = minmod(1, \theta)$

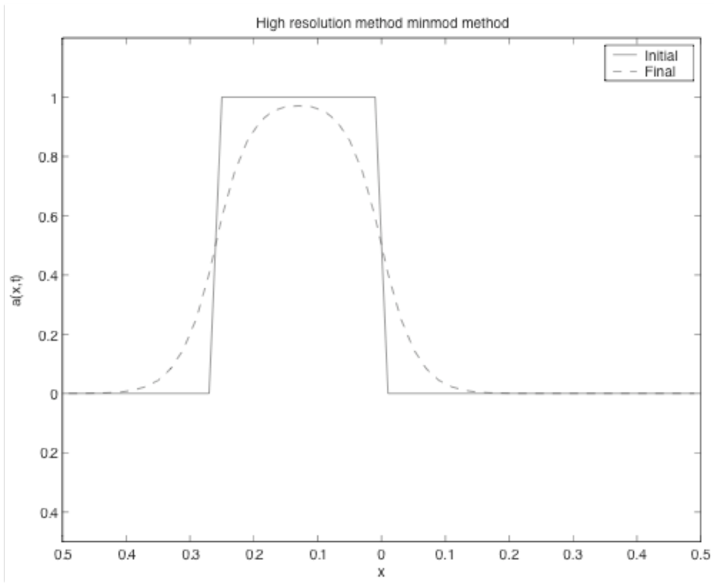
$$\text{where } \min mod(a, b) \equiv \begin{cases} a & \text{if } |a| < |b| \quad \text{and } ab > 0 \\ b & \text{if } |a| > |b| \quad \text{and } ab > 0 \\ 0 & \text{if } ab < 0 \end{cases}$$

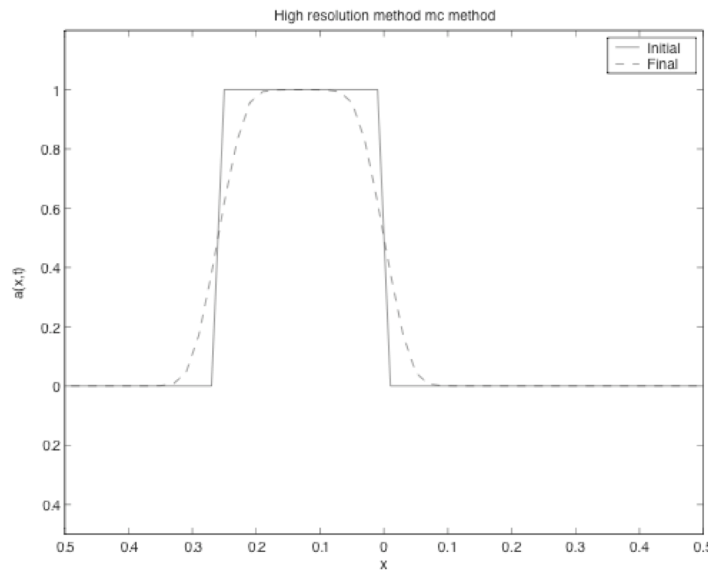
superbee: $\phi(\theta) = \max(0, \min(1,2\theta), \min(2, \theta))$

MC: $\phi(\theta) = \max\left(0, \frac{\min(1+\theta)}{2}\right), 2,2\theta)$

van Leer: $\phi(\theta) = \frac{\theta+|\theta|}{1+|\theta|}$

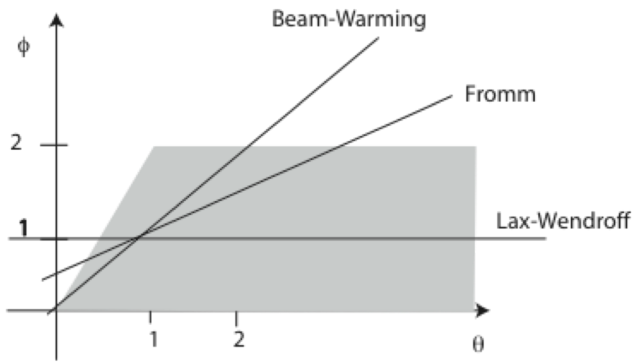
Two high resolution methods for τ=0.01 (=h/2c)



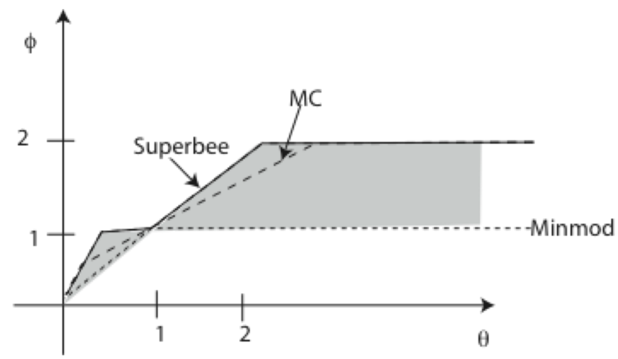


Methods give exact answer for τ=0.02 (=h/c)

One can show that for the scheme to be TVD, it must line inside the shaded region



The shaded region shows where the schemes must lie to be TVD



The shaded region shows where the schemes must lie to be TVD