Physics of Traffic Flow - Discrete Model

Another approach to modeling traffic flow is with a discrete model, in this case we model the system as a group of cars whose speed depends on how close the car in front is, the speed of the car is then given by

$$\frac{dx_c}{dt} = v(\rho(x_c(t), t))$$

in this model $\rho(x_c(t), t) = \frac{L_c}{x_{c+1} - x_c}$

where c refers to the index of an individual car and $c+1$ is the car in front of it, and L_c is the length of the car. In this model the speed is then

$$v\big(\rho(x_c(t), t)\big) = v_m(1 - \rho(x_c(x, t)))$$

so if there is no car in front, the speed of the car is v_m, and if the car in front is less than or equal to a car length, then the speed is zero (bump-to-bumper traffic)

Discrete MATLAB/python traffic programs

'trafd.m/.py' is designed to model a system of cars in a discrete fashion

- The initial condition is a set of cars, set bumper to bumper, at a stoplight, just like the traffic program that solves Burgers equation

'trafp.m/.py' is the same program but uses periodic boundary conditions

Of course one could get fancy and make changes such as:

- Make v_m different for each car
- Add a reaction time for each car

Physics of Traffic Flow – Fluid Model

In fluid mechanics, the equations of motion are of the form (1D)

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial F(x,t)}{\partial x}$$

where p is some conserved quantity (such as density, momentum, energy density) and F is a flux function.

One example is the continuity equation reads as

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial (\rho(x,t)v(x,t))}{\partial x}$$

One of the simplest flows involves a velocity (v) that is a function of the density (ρ), i.e., $v(x,t) = v(\rho)$.

Aside: Derivation of the Continuity Equation

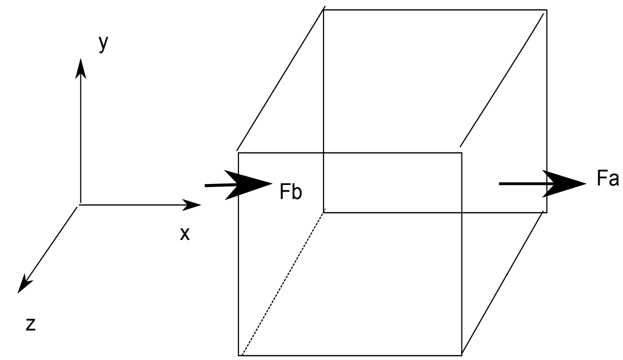
The continuity equation is a statement of conservation of Mass. In Cartesian coordinates, Consider a box of dimensions that has a fluid of density and in a flow field . In order to track the total mass in the box when there is flow into an out of the box, we have to consider the flux at each of the faces of the box

The flux out of the box, Fa is

$$Fa = \left[\rho\left(x + \frac{\delta x}{2}\right) v_x \left(1 + \frac{\delta x}{2}\right)\right] \delta y \, \delta z$$

and into the box is Fb

$$Fb = \left[\rho\left(x - \frac{\delta x}{2}\right) v_x \left(1 - \frac{\delta x}{2}\right)\right] \delta y \, \delta z$$



Expanding in a Taylor series

$$Fa \approx \left[\rho(x)v_x(x) + \frac{\delta x}{2}\left(\frac{\partial\rho}{\partial x}v_x + \frac{\rho\partial v_x}{\partial x}\right)\right]\delta y\ \delta z$$

$$Fb \approx \left[\rho(x)v_x(x) - \frac{\delta x}{2}\left(\frac{\partial\rho}{\partial x}v_x + \frac{\rho\partial v_x}{\partial x}\right)\right]\delta y\ \delta z$$

Taking the difference, we get

$$Fa - Fb \approx \frac{\partial}{\partial x}(\rho v_x)\delta x\delta y\delta \mathrm{z}$$

If we repeat for all 6 faces we find that the total mass loss of the box is

$$\left(\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right)\delta x\delta y\delta \mathrm{z}$$

$$\frac{\partial\rho}{\partial t}\delta x\delta y\delta \mathrm{z} = -\left(\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right)\delta x\delta y\delta \mathrm{z}$$

Which simplifies to the continuity equation

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

Physics of Traffic Flow -2

For a fluid version, consider for example the following functional form for v

$$v(\rho) = v_m(1 - \frac{\rho}{\rho_m})$$

where $v_m > 0$ is the maximum velocity and $\rho_m > 0$ is the maximum density. This is a rough approximation to car traffic. The maximum velocity is the speed limit and when the maximum density occurs the flow grinds to a halt. The continuity equation then becomes

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(\rho v_m \left(1 - \frac{\rho}{\rho_m}\right)\ \right)$$

$$= -\frac{\partial}{\partial \rho}\left(\rho v_m (1 - \frac{\rho}{\rho_m})\right)\ \frac{\partial \rho}{\partial x}$$

$$= -v_m\left(1 - \frac{2\rho}{\rho_m}\right)\frac{\partial \rho}{\partial x} = -\mathrm{c}(\rho)\,\frac{\partial \rho}{\partial x}$$

Physics of Traffic Flow -3

So we have

$$\frac{\partial \rho}{\partial t} = -\mathrm{c}(\rho)\,\frac{\partial \rho}{\partial x}$$

and

$$c(\rho) = v_m\left(1 - \frac{2\rho}{\rho_m}\right)$$

note:

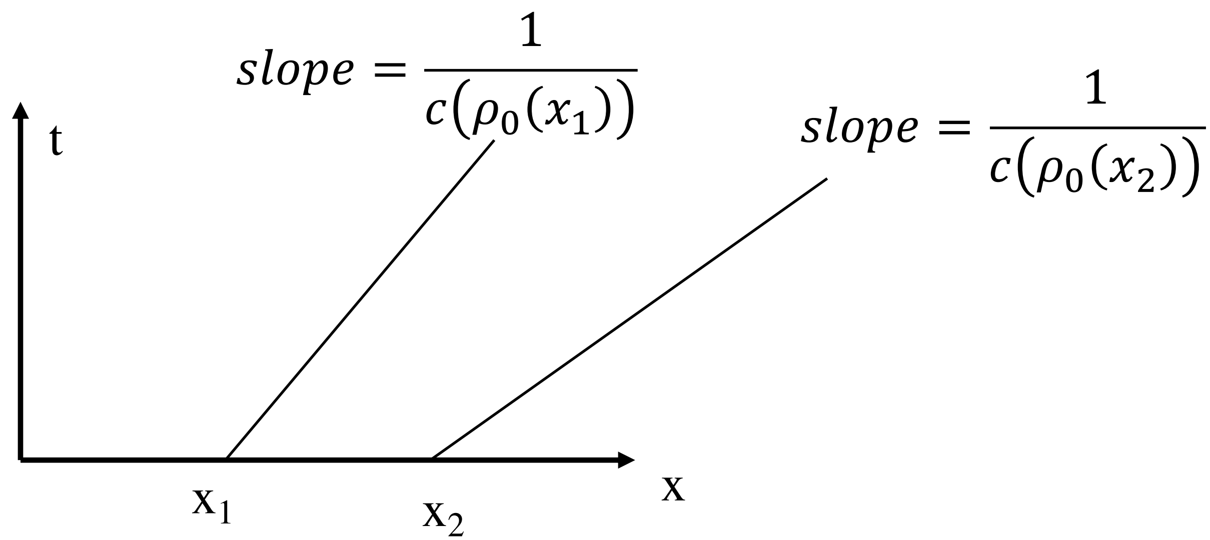
- $c(\rho)$ can be both positive and negative
- $c(\rho)$ is not the speed of the traffic
- $c(\rho)$ is linear in ρ
- $v(\rho) \geq 0$
- $c(\rho) \leq v(\rho)$

Physics of Traffic Flow - Method of Characteristics

It is simple to show that along a characteristic of slope

$$\frac{dt}{dx} = \frac{1}{c\big(\rho_0(x_1)\big)}$$

the solution is unchanged

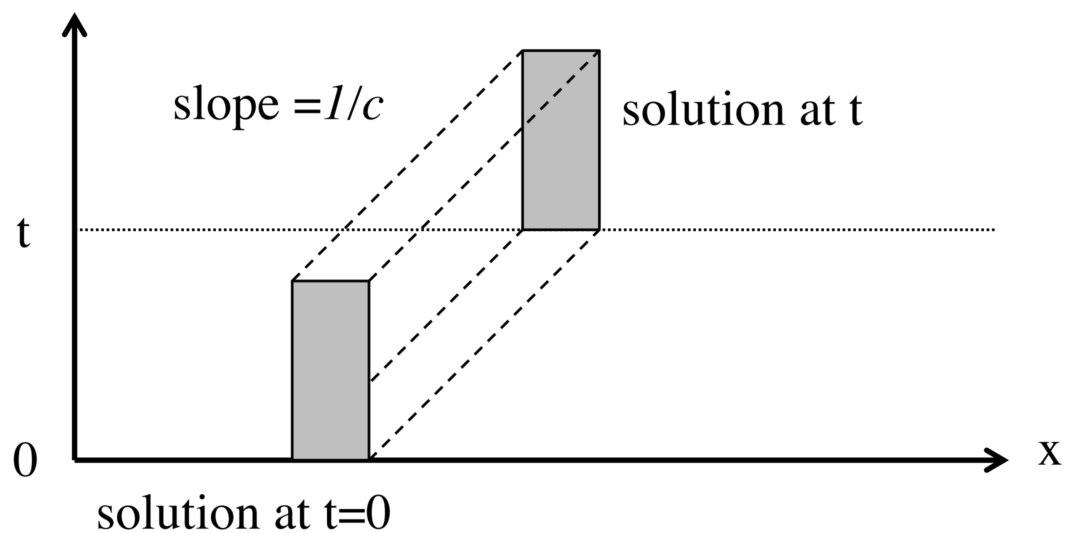


Method of Characteristics

For the advection equation

$$\frac{\partial a(x,t)}{\partial t} = -c\,\frac{\partial a(x,t)}{\partial x}$$

the solution was analytical and is easy to obtain as $a(x,t)=f_0(x-ct)$, where $f_0(x)$ is the initial condition. This can be also interpreted geometrically:

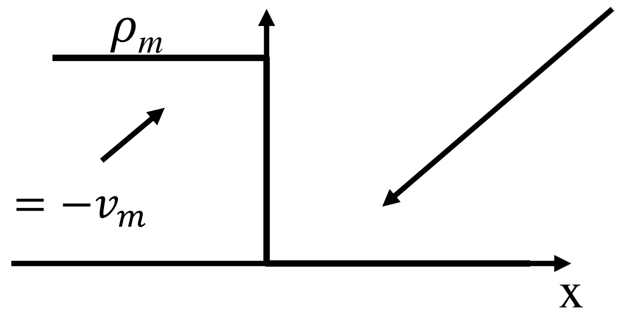


Physics of Traffic Flow -Traffic Light

Consider the initial condition

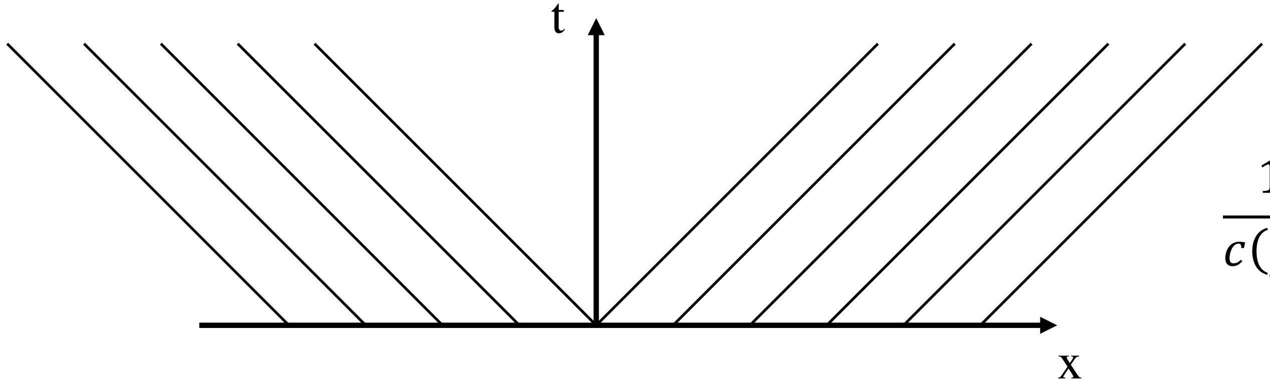
$$\rho(x,0) = \rho_0(x) = \begin{cases} \rho_m & x < 0 \\ 0 & x \geq 0 \end{cases}$$

$$c(0) = v_m\left(1 - 2\frac{0}{\rho_m}\right) = v_m$$



$$c(\rho_m) = v_m\left(1 - 2\frac{\rho_m}{\rho_m}\right) = -v_m$$

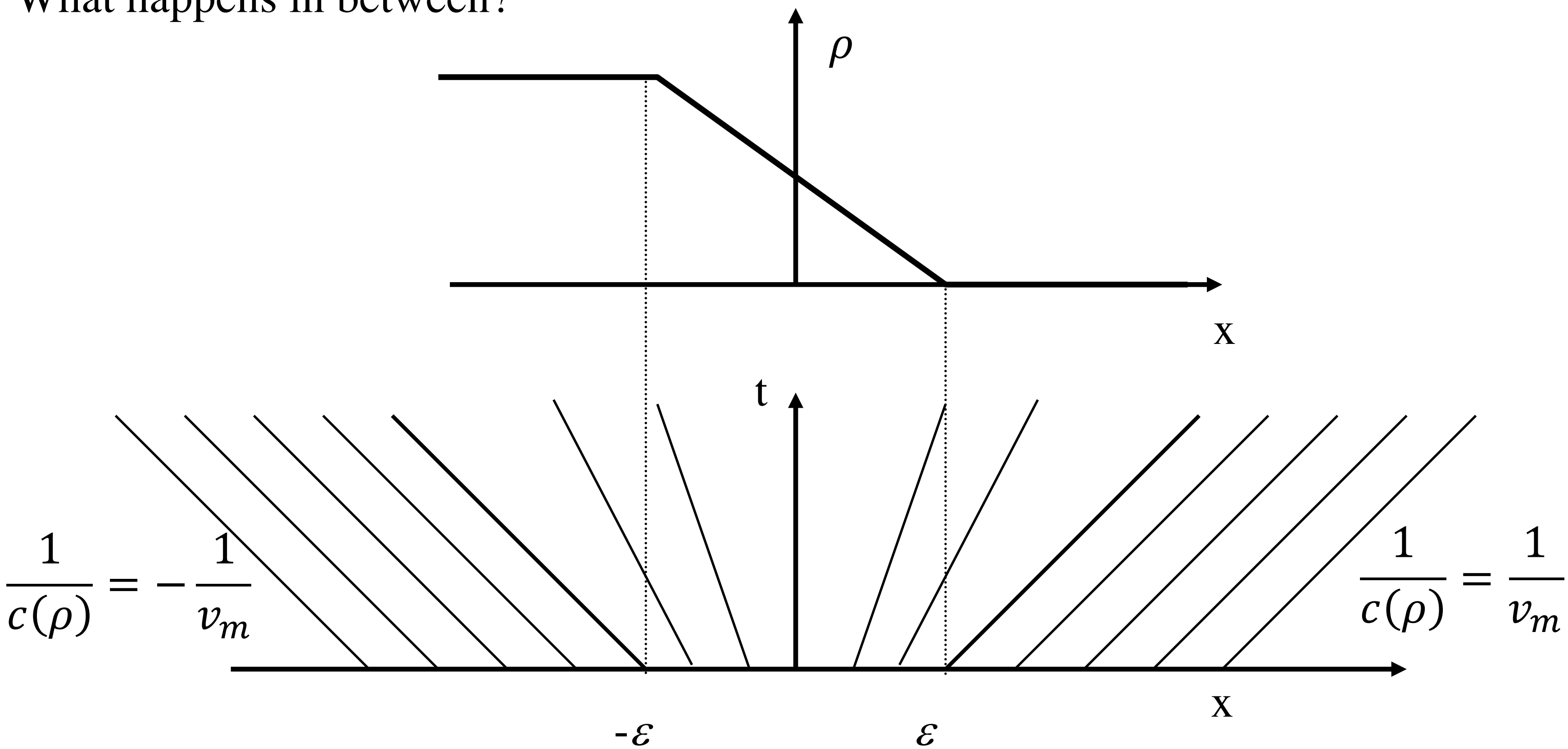
$$\frac{1}{c(\rho)} = -\frac{1}{v_m}$$



$$\frac{1}{c(\rho)} = \frac{1}{v_m}$$

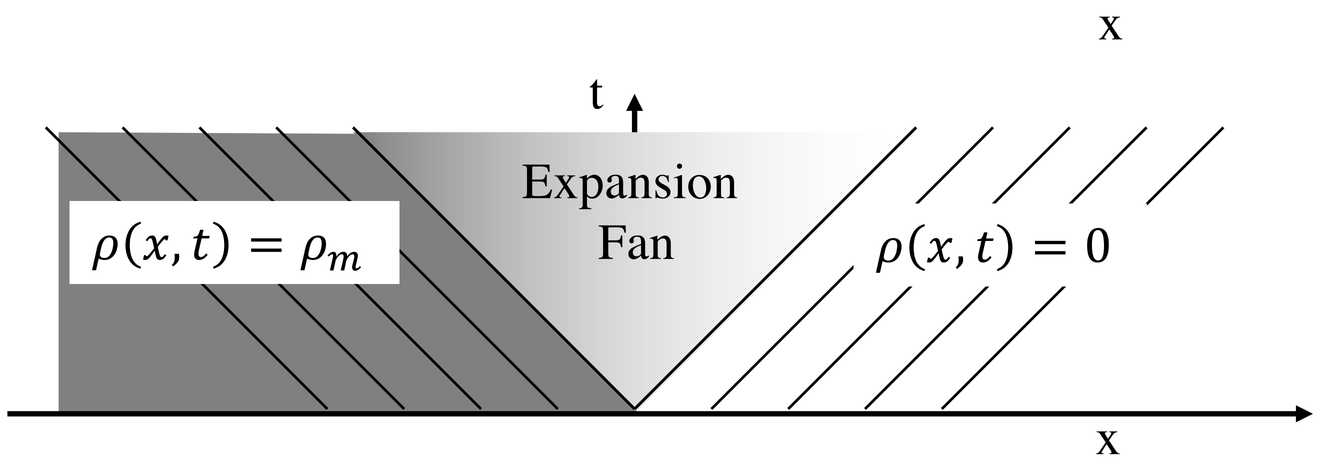
Physics of Traffic Flow -Traffic Light -2

What happens in between?



Physics of Traffic Flow -Traffic Light -3

In the limit as $\varepsilon \to 0$ then



the solution is then

$$\rho(x,t) = \begin{cases} \rho_m & for\, x \le -v_m t \\ \frac{1}{2}\left(1 - \frac{x}{v_m t}\right)\rho_m & for\, -v_m t < x < v_m t \\ 0 & for\, x \ge v_m t \end{cases}$$

Traffic Program

This program solves the equation

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial F(x,t)}{\partial x}$$

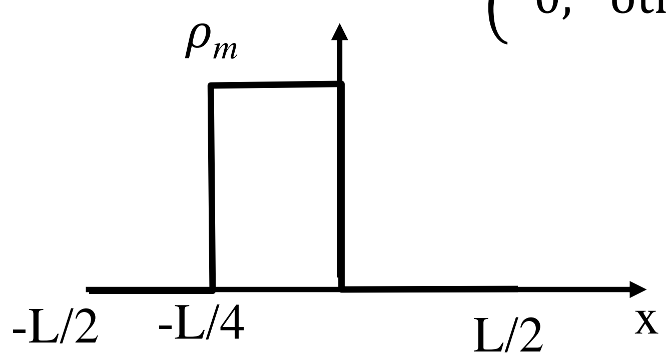
using 3 different methods

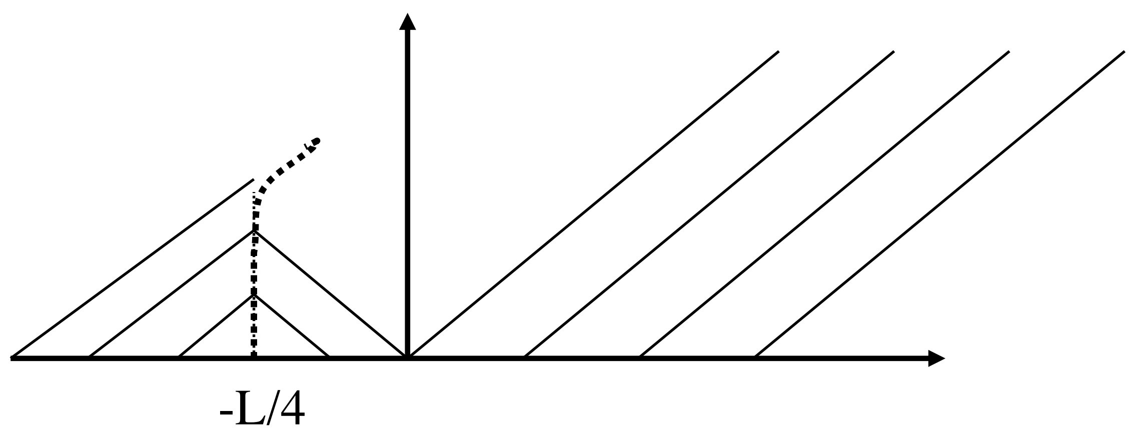
- FTCS: $\rho_i^{n+1} = \rho_i^n - \frac{\tau}{2h}(F_{i+1}^n - F_{i-1}^n)$
- Lax Scheme: $\rho_i^{n+1} = \frac{1}{2}(\rho_{i-1}^n + \rho_{i-1}^n) - \frac{\tau}{2h}(F_{i+1}^n - F_{i-1}^n$
- Lax Wendroff: $\rho_i^{n+1} = \rho_i^n - \frac{\tau}{2h}(F_{i+1}^n - F_{i-1}^n) + \frac{\tau^2}{2h}(c_{i+\frac{1}{2}} \frac{F_{i+1}^n - F_i^n}{h} - c_{i-\frac{1}{2}} \frac{F_i^n - F_{i-1}^n}{h})$

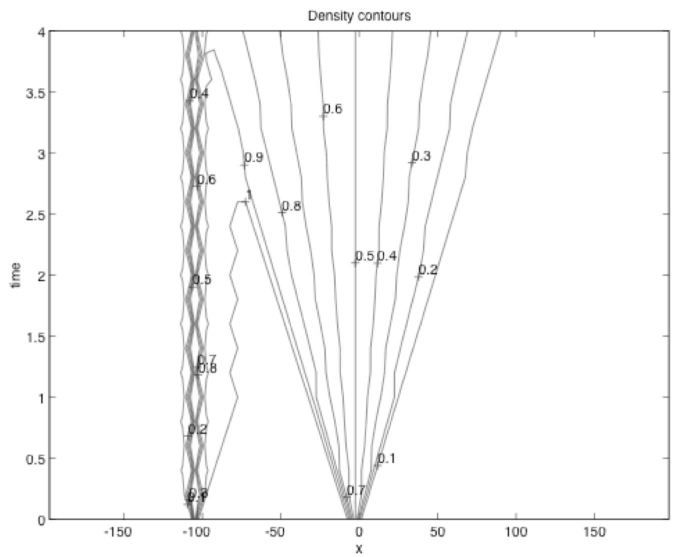
where $c_{i\pm\frac{1}{2}}^n = c\left(\frac{\rho_{i\pm 1}^n + \rho_i^n}{2}\right)$

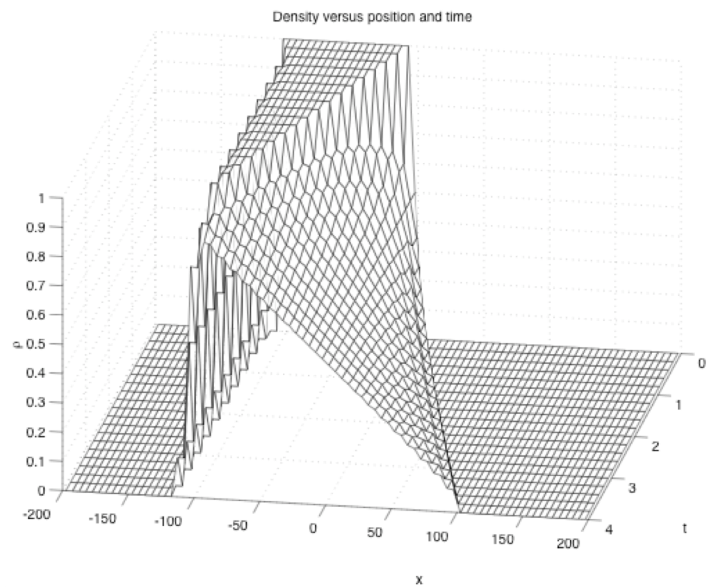
Traffic Problem - Example

Look at the initial condition $\rho(x,0) = \rho_0(x) = \begin{cases} \rho_m, -\frac{L}{4} < x < 0 \\ 0, \ \text{otherwise} \end{cases}$

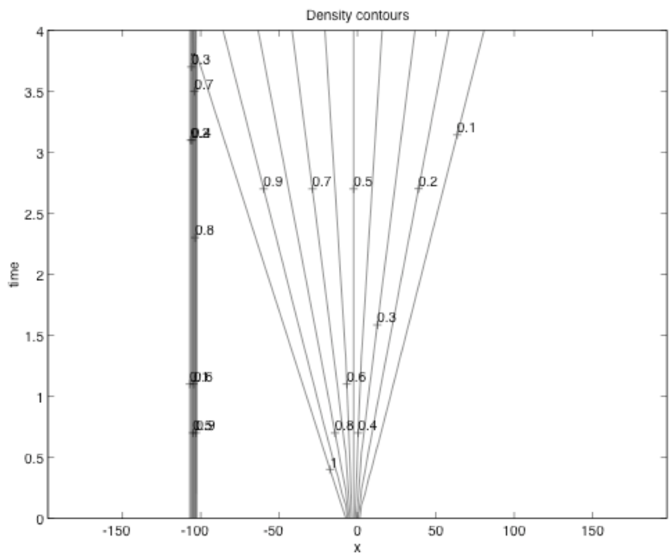


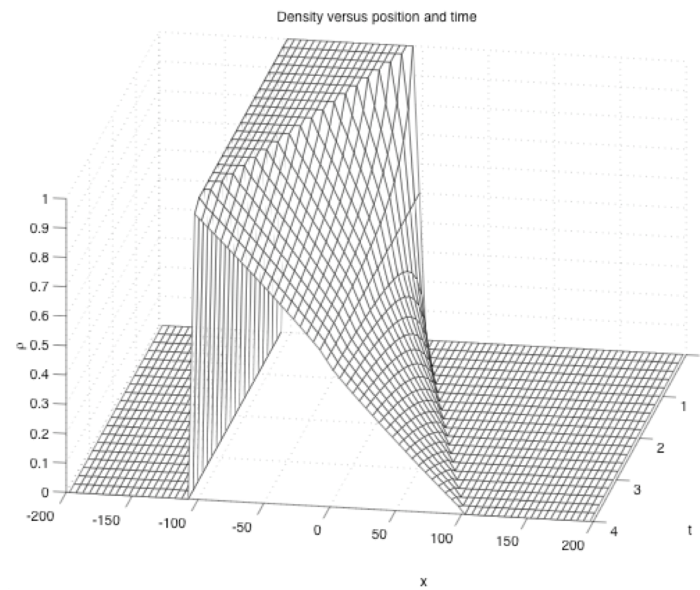




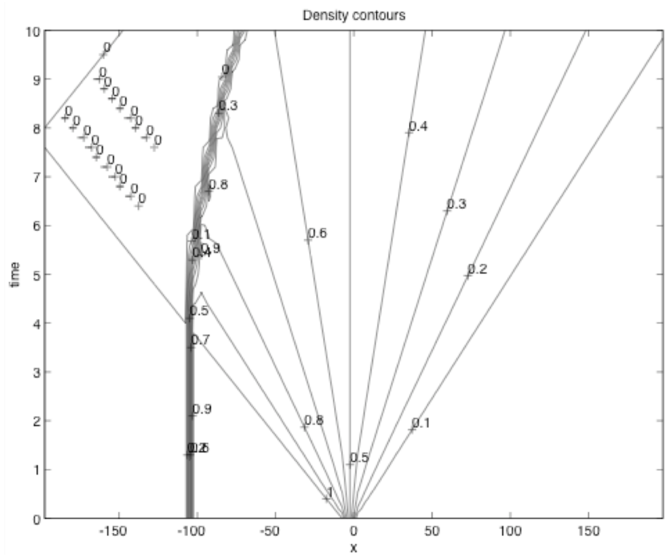


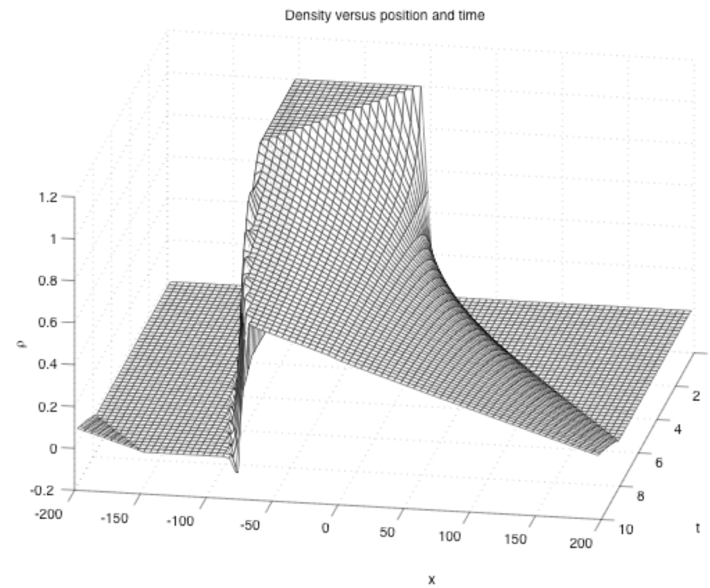
Lax Method τ=0.2s, N=80, 20 steps





Lax-Wendroff Method τ=0.2s, N=80, 20 steps





Lax-Wendroff Method τ=0.2s, N=80, 50 steps