Phys 416/517 – Chapter 9 Exercises

[Due Wednesday May 6, 11:59PM]

A. Modify the `advect2` program to use the Crank-Nicolson method and compare the output from this program for the original Gaussian wave [*Turn in your program and sample plots*].

9.15 Modify the `schro` program to include the delta function potential $V(x) = U\delta(x - L/2)$. Vary the amplitude U and do runs where it is less than, equal to, and greater than $E = \hbar^2 k_0^2 / 2m$, the energy of the particle. Show that some of the wave function penetrates the potential even when E < U. If memory allows, increase L, the system size, to distinctly separate the reflected and transmitted waves [*Turn in your program and sample plots*].

9.9 The Lax scheme for the advection equation with periodic boundary conditions may be written as

$$\mathbf{a}^{n+1} = \left(\frac{1}{2}\mathbf{C} - \frac{c\tau}{2h}\mathbf{B}\right)\mathbf{a}^n = \mathbf{A}\mathbf{a}^n$$

where **a**, **B** and **C** are defined as

$$\mathbf{a}^n = \begin{bmatrix} a_1^n \\ a_2^n \\ a_3^n \\ \vdots \\ a_N^n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Demonstrate that the matrix stability for the Lax scheme is given by the CFL condition. Specifically, find the spectral radius of **A** using:

(a) The power method (function power1.m/.py provided)

(b) MATLAB's eig function (or np.linalg.eigvals in numpy/python)

(c) $\|A\|_1$

(d) $\|A\|_\infty$

as estimates for the spectral radius. [Assume N=51 and c=1]
[*Turn in your program and plots of the spectral radius as a function of the CFL number.]*

Optional Extra Credit – 1 point

9.17 An important equation from the theory of solitons is the *Korteweg-de Vries* (KdV) equation,

$$\frac{\partial \rho}{\partial t} = -6\rho\frac{\partial \rho}{\partial x} - \frac{\partial^3 \rho}{\partial x^3}$$

Write a program that solves it using the explicit/implicit scheme

$$\frac{\rho_j^{n+1} - \rho_j^n}{\tau} = -6D_j\rho_j^n - \frac{1}{2}\left(\frac{\rho_{j+2}^n - 2\rho_{j+1}^n + 2\rho_{j-1}^n - \rho_{j-2}^n}{2h^3} + \frac{\rho_{j+2}^{n+1} - 2\rho_{j+1}^{n+1} + 2\rho_{j-1}^{n+1} - \rho_{j-2}^{n+1}}{2h^3}\right)$$

where $D_j = \dfrac{\rho_{j+1}^n - \rho_{j-1}^n}{2h}$

Use **Periodic** boundary conditions. Test your program for the solitary wave solution of the Kdv equations: $\rho(x,t) = 2\,\mathrm{sech}^2(x - 4t)$, but consider the periodic nature of the domain. [*Turn in your program and plots. Use L=20, and be careful of the timestep.]*