Ozyegin University
CS 321 Programming Languages
Sample Problems on Lambda Calculus

1. Apply **one** step of beta reduction on the following lambda terms, if possible. If there is no beta reduction possibility, write NORMAL FORM.

   - $(\lambda x\,.\,x\,y\,z)\,a\,b\,c \;\Rightarrow\; \underline{\quad (a\,y\,z)\,b\,c \quad}$
   - $(\lambda w\,.\,\lambda p\,.\,p\,w) \;\Rightarrow\;$ **NORMAL FORM**
   - $(\lambda x\,.\,\lambda y\,.\,x\,(y\,x))\,(\lambda w\,.\,\lambda v\,.\,w)\,(\lambda z\,.\,z) \;\Rightarrow\; \underline{\quad (\lambda y\,.\,(\lambda w\,.\,\lambda v\,.\,w)\,(y\,(\lambda w\,.\,\lambda v\,.\,w)))\,(\lambda z\,.\,z) \quad}$
   - $(\lambda z\,.\,z\,(\lambda k\,.\,k)\,m) \;\Rightarrow\;$ **NORMAL FORM**
   - $(\lambda u\,.\,(\lambda x\,.\,x\,u)\,(\lambda y\,.\,u)) \;\Rightarrow\; \underline{\quad (\lambda u\,.\,(\lambda y\,.\,u)\,u) \quad}$
   - $(\lambda n\,.\,n\,n)\,(\lambda n\,.\,n\,n) \;\Rightarrow\; \underline{\quad (\lambda n\,.\,n\,n)\,(\lambda n\,.\,n\,n) \quad}$
   - $(\lambda g\,.\,(\lambda h\,.\,h)\,g) \;\Rightarrow\; \underline{\quad (\lambda g\,.\,g) \quad}$

   Solution: See `https://vimeo.com/250076161`

2. Reduce the following lambda term to its normal form. If a normal form does not exist, show that the term does not converge.

$$(\lambda x.\ \lambda y.\ x\ (y\ x))\ (\lambda w.\ \lambda v.\ w)\ (\lambda z.\ z)$$

   Solution: See `https://vimeo.com/250078355`

$$
\begin{aligned}
&(\lambda x.\ \lambda y.\ x\ (y\ x))\ (\lambda w.\ \lambda v.\ w)\ (\lambda z.\ z)\\
&\Rightarrow (\lambda y.\ (\lambda w.\ \lambda v.\ w)\ (y\ (\lambda w.\ \lambda v.\ w)))\ (\lambda z.\ z)\\
&\Rightarrow (\lambda w.\ \lambda v.\ w)\ ((\lambda z.\ z)\ (\lambda w.\ \lambda v.\ w))\\
&\Rightarrow (\lambda w.\ \lambda v.\ w)\ (\lambda w.\ \lambda v.\ w)\\
&\Rightarrow (\lambda v.\ (\lambda w.\ \lambda v.\ w))
\end{aligned}
$$

3. Reduce the following lambda expression to its normal form.

$$(\lambda f.\lambda m.\lambda p.m\,(f\,m\,p))\;(\lambda y.\lambda z.y\,z)\;(\lambda w.w)$$

Solution: See `https://vimeo.com/250079704`

$$\begin{aligned}
&(\lambda f.\lambda m.\lambda p.m\,(f\,m\,p))\;(\lambda y.\lambda z.y\,z)\;(\lambda w.w)\\
&\Rightarrow (\lambda m.\lambda p.m\,((\lambda y.\lambda z.y\,z)\,m\,p))\;(\lambda w.w)\\
&\Rightarrow \lambda p.(\lambda w.w)\,((\lambda y.\lambda z.y\,z)\,(\lambda w.w)\,p)\\
&\Rightarrow \lambda p.((\lambda y.\lambda z.y\,z)\,(\lambda w.w)\,p)\\
&\Rightarrow \lambda p.((\lambda z.(\lambda w.w)\,z)\,p)\\
&\Rightarrow \lambda p.((\lambda w.w)\,p)\\
&\Rightarrow \lambda p.p
\end{aligned}$$

4. The following are the Church encodings for boolean values and conditional expression. Write down an encoding for logical *and*.

$$\begin{aligned}
\textbf{true} &= \lambda a.\lambda b.\ a\\
\textbf{false} &= \lambda a.\lambda b.\ b\\
\textbf{if} &= \lambda c.\lambda t.\lambda e.\ c\ t\ e\\
\textbf{and} &= \underline{\quad \lambda p.\lambda q.\ p\ q\ \textbf{false}\quad}
\end{aligned}$$

Solution: See `https://vimeo.com/250079905`

5. Using the encodings below, show that **mult 3 2** is **6**.

$$
\begin{aligned}
\mathbf{0} &= (\lambda f.\lambda x.x)\\
\mathbf{1} &= (\lambda f.\lambda x.fx)\\
\mathbf{2} &= (\lambda f.\lambda x.f(fx))\\
\mathbf{3} &= (\lambda f.\lambda x.f(f(fx)))\\
\mathbf{mult} &= \lambda m.\lambda n.\lambda f.\lambda x.m(nf)x
\end{aligned}
$$

Solution:

$$
\begin{aligned}
\mathbf{mult\ 3\ 2} &= (\lambda m.\lambda n.\lambda f.\lambda x.\ m\ (nf)\ x)\ \mathbf{3}\ \mathbf{2}\\
&\Rightarrow (\lambda n.\lambda f.\lambda x.\ \mathbf{3}\ (nf)\ x)\ \mathbf{2}\\
&\Rightarrow \lambda f.\lambda x.\ \mathbf{3}\ (\mathbf{2}\ f)\ x\\
&= \lambda f.\lambda x.\ \mathbf{3}\ ((\lambda f.\lambda x.f(fx))\ f)\ x && \text{expanded } \mathbf{2}\\
&= \lambda f.\lambda x.\ \mathbf{3}\ (\underline{(\lambda f.\lambda x.f(fx))\ f})\ x && \beta\text{-reducing underlined term}\\
&\Rightarrow \lambda f.\lambda x.\ \mathbf{3}\ (\lambda x.f(fx))\ x\\
&= \lambda f.\lambda x.\ (\lambda f.\lambda x.f(f(fx)))\ (\lambda x.f(fx))\ x && \text{expanded } \mathbf{3}\\
&= \lambda f.\lambda x.\ \underline{(\lambda f.\lambda x.f(f(fx)))\ (\lambda x.f(fx))}\ x && \beta\text{-reducing underlined term}\\
&\Rightarrow \lambda f.\lambda x.\ (\lambda x.(\lambda x.f(fx))\ ((\lambda x.f(fx))\ ((\lambda x.f(fx))\ x)))\ x\\
&= \lambda f.\lambda x.\ (\lambda x.(\lambda x.f(fx))\ ((\lambda x.f(fx))\ (\underline{(\lambda x.f(fx))\ x})))\ x && \beta\text{-reducing underlined term}\\
&\Rightarrow \lambda f.\lambda x.\ (\lambda x.(\lambda x.f(fx))\ ((\lambda x.f(fx))\ (f(fx))))\ x\\
&= \lambda f.\lambda x.\ (\lambda x.(\lambda x.f(fx))\ (\underline{(\lambda x.f(fx))\ (f(fx))}))\ x && \beta\text{-reducing underlined term}\\
&\Rightarrow \lambda f.\lambda x.\ (\lambda x.(\lambda x.f(fx))\ (f(f(f(fx)))))\ x\\
&= \lambda f.\lambda x.\ (\lambda x.\underline{(\lambda x.f(fx))\ (f(f(f(fx))))})\ x && \beta\text{-reducing underlined term}\\
&\Rightarrow \lambda f.\lambda x.\ (\lambda x.(f(f(f(f(f(fx)))))))\ x\\
&= \lambda f.\lambda x.\ \underline{(\lambda x.(f(f(f(f(f(fx)))))))\ x} && \beta\text{-reducing underlined term}\\
&\Rightarrow \lambda f.\lambda x.(f(f(f(f(f(fx))))))\\
&= \mathbf{6}
\end{aligned}
$$