EFFICIENT BASES FOR THE GALERKIN SOLUTION OF MULTIPLE-SCATTERING PROBLEMS

by

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ABSTRACT

EFFICIENT BASES FOR THE GALERKIN SOLUTION OF MULTIPLE-SCATTERING PROBLEMS

In this thesis we consider high-frequency multiple scattering problems in the exterior of two-dimensional smooth compact scatterers consisting of two disjoint strictly convex obstacles. The motivation in considering this problem is the lack of fast yet rigorous numerical algorithms designed for its solution. Indeed, the only algorithm designed for the solution of this multiple scattering problem is the integral equation method developed by Bruno *et al.* in 2005 that uses a combination of geometrical optics to determine the phases of multiple scattering iterations, Nyström discretization to spectrally represent the unknowns, extensions of the stationary phase method to evaluate the arising integrals independent of the frequency, and a matrix free linear algebra solver. Unfortunately this algorithm is not supported with a rigorous convergence analysis.

In this thesis, we take an alternative approach and develop two classes of highly efficient Galerkin boundary element methods extending the recent single scattering algorithms, namely the *frequency-adapted Galerkin boundary element methods* and *change of variables Galerkin boundary element methods* recently developed by Ecevit *et al.*, to multiple scattering problems. In connection with each multiple scattering iterate, in both cases, we prove that the number of degrees of freedom necessary to obtain prescribed error tolerances independent of frequency needs increase as $\mathcal{O}(k^\epsilon)$ (for any $\epsilon > 0$) with increasing wavenumber k. Consequently, the theoretical developments supported with the numerical tests in this thesis fill an important gap in the literature.

ÖZET

ÇOKLU SAÇILIM PROBLEMLERİNİN GALERKİN ÇÖZÜMÜ İÇİN ETKİN BAZLAR

Bu tezde iki ayrık, düzgün, dışbükey, tıkız engel dışında oluşan iki boyutlu yüksek frekanslı çoklu saçılım problemlerini inceleyeceğiz. Bu problemi ele almamızın nedeni, literatürde problemle ilgili hızlı ve sayısal analizi yapılmış herhangi bir yöntemin bulunmamasıdır. Nitekim, bu çoklu saçılma probleminin çözümü için tasarlanmış tek algoritma 2005 yılında Bruno ve diğerleri tarafından geliştirilen, geometrik optik yöntemi ile fazların belirlendiği, Nyström ayrıklaştırma yöntemi ile bilinmeyenlerin spektral olarak temsil edildiği, durağan faz metodu ile numerik integrallerin hesaplandığı integral denklem metodudur. Ne yazık ki bu algoritmanın yakınsama analizi yapılmamıştır.

Bu tezde, alternatif bir yaklaşımla, yukarıda bahsi geçen çoklu saçılım problemlerinin nümerik çözümü için yüksek verimli Galerkin sınır elemanları yöntemleri geliştireceğiz. Bu yöntemleri geliştirirken yakın zamanda Ecevit ve diğerleri tarafından ortaya atılan tekil saçılım algoritmalarını, çoklu saçılım problemlerine genelleyeceğiz. Ayrıca her çoklu saçılım iterasyonuyla bağlantılı olarak, frekanstan bağımsız hata oranı elde edilmesi için problemin bilinmeyelerinin, her pozitif ϵ için, k dalga sayısı olmak üzere, k^{ϵ} sayısıyla doğru orantılı olarak artan serbestlik derecesiyle ifade edilmesinin yeterli olduğunu kanıtlayacağız. Sonuç olarak, bu tezde sayısal testler ile desteklenen teorik gelişmeler literatürde çoklu saçılım problemleri ile ilgili önemli bir boşluğu dolduracaktır.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

C^n	n times continuously differentiable functions
$\mathcal{C}_n$	Polynomials composed with the change of variable function
D_s^n	n^{th} derivative operator with respect to s
H_n^1	Hankel function of the first kind of order n
i	Imaginary unit
k	Wave number
K	Obstacle
L^2	Square integrable functions
$\mathbb{N}$	Natural numbers
$\mathcal{O}$	Big-O notation
$\mathcal{P}_n$	Polynomials of degree at most n
$\mathcal{R}$	Combined field integral operator
$\mathbb{R}$	Real numbers
$\mathfrak{Re}$	Real part of a number
S	Single layer operator
S'	Normal derivative of single layer operator
$S^{\mu}_{\varrho,\delta}$	Symbol class of Höremainder of order μ and type ϱ, δ
u	Scattered field
u^{inc}	Incident wave
$\mathcal{X}_j^m(x)$	$(j+1)$-th component of the broken $(m+1)$-rays terminating at $x \in \partial K$
$\mathbb{Z}$	Integers
α	Direction of incident wave
γ	Parametrization of an obstacle of ∂K
Δ	The laplacian
η	Normal derivative of the total field
η^{slow}	Slowly oscillating part of η
ν	Unit normal vector

Φ	Fundamental solution of the Helmholtz equation
ϕ	Change of variables functions
φ	Phase function
∂K	Boundary of K
∇	The gradient
$\bigoplus$	Direct sum
$\|\| \cdot \|\|_2$	L^2 norm
$\langle \cdot, \cdot \rangle$	Sesquilinear form
$\langle \cdot, \cdot \rangle_{\mathcal{R}}$	Adapted sesquilinear form

LIST OF ACRONYMS/ABBREVIATIONS

2D	Two Dimensional
DS	Deep shadow region
IL	Illimunated region
IT	Illuminated transition region
SB	Shadow boundary region
ST	Shadow transition region

1. INTRODUCTION

The last twenty years have witnessed important developments in the fields of computational acoustics and electromagnetism from both theoretical and practical perspectives. Indeed, algorithms relating to scattering simulations have mostly concentrated on finite elements [1–3] and integral equations [4–8]. The approaches based on the former are in general applicable in the low frequency regime since utilization of finite element methods in exterior scattering problems necessitate the construction of artificial interfaces to truncate the infinite computational domain as well as design of absorbing boundary conditions on this interface to represent the behaviour of solutions at infinity [9–13]. These requirements naturally limit the applicability of finite element methods to low frequency simulations since they easily result in loss of accuracy and related increase in computational cost in high frequency applications. In connection with the multiple scattering problems considered in this thesis, these difficulties are further coupled with the need to increase the size of aforementioned artificial interfaces due to the separation distance between the scatterers. Methods based on integral equation formulations are therefore better adopted to exterior scattering problems since they remove the need for an artificial interface simply by choosing an *outgoing* fundamental solution and they automatically restrict the computational domain to the region filled by the scatterers only. Moreover, in surface scattering problems considered in this thesis, they provide a dimensional reduction in the computational domain since the solutions can be computed based on a knowledge of the densities confined to the surface of the scattering obstacles [14]. Nonetheless, they give rise to dense linear systems whose sizes increases as $\mathcal{O}(k^p)$ with increasing wavenumber k where p is the dimension of the computational manifold.

The success of integral equation methods in high frequency problems is the ease of integrating the knowledge of asymptotic behaviour of the unknown into the problem formulation. This is naturally the path taken in this thesis since it converts the problem into the computation of a slowly oscillating envelope rather than the full solution whose oscillations are in harmony with those of the incoming field of radiation. The

first algorithm, in this direction, can be attributed to Nedélec *et al.* [15, 16] where they considered two-dimensional exterior surface scattering problems related with the impedance boundary condition, and developed a Galerkin method which reduces the number of degrees of freedom needed to represent the unknown surface densities to $\mathcal{O}(k^{1/3})$ through the use of stationary phase method [17]. This has clearly provided a significant improvement over the classical approaches since the number of degrees of freedom they require is $\mathcal{O}(k)$.

What has had a greater impact in the computational scattering community is the single scattering algorithm developed by Bruno *et al.* [18], which has successfully combined the Nyström method with suitable extensions of the the method of stationary phase and a change of variables around the shadow boundaries, since it has demonstrated the possibility of reducing the number of degrees of freedom needed to represent the unknown surface densities to $\mathcal{O}(1)$, and thus the possibility of obtaining prescribed error tolerances within fixed computational times for scattering problems of arbitrarily high frequency (see [19] for an extension of this algorithm to three-dimensional single scattering configurations). Alternative implementations of this method based on an appropriate combination of the collocation method and geometrical theory of diffraction [20], the collocation method and steepest descent method [21], and a p-version Galerkin method [22] have since been developed. Indeed, [22] has rigorously shown that an upper bound necessary to represent the unknown surface densities in these approaches is $\mathcal{O}(k^{1/9})$.

Among the aforementioned single scattering algorithms, [18,21,22] are asymptotic in the sense that the numerical solutions based on these algorithms do not converge to the actual solutions for any fixed wavenumber k as the number of degrees of freedom goes to infinity since these algorithms approximate the solutions beyond the $\mathcal{O}(k^{-1/3})$ shadow boundaries simply by zero.

On the other hand, the methods in [18, 20, 21] are not supported with a rigorous convergence analysis. Motivated with these observations, Ecevit and Özen [23] have recently proposed and rigorously analyzed *frequency-adapted Galerkin boundary ele-*

ment methods for single scattering problems which demand, for any convex scatterer, an increase of $\mathcal{O}(k^\epsilon)$ (for any $\epsilon > 0$) in the number of degrees of freedom to maintain a prescribed accuracy independent of frequency. More recently, Ecevit and Eruslu [24] have developed a class of *change of variables Galerkin boundary element methods* for the solution of single scattering problems that display similar characteristics when compared with that in [23] from both theoretical and practical points of view but that provide significantly improved numerical accuracies in the shadow regions. The aim of this thesis is to extend and rigorously analyze the single scattering algorithms [23, 24] to encompass the multiple scattering problems related to two disjoint strictly convex scatterers.

Indeed, as of today the only algorithm that displays the capability of reproducing multiple scattering returns in frequency independent computational times in the exterior of two smooth and convex structures is that of Bruno *et al.* [25] which is based on three main principles: 1) Representation of the solution as an infinite superposition of single scattering returns through a Neumann series, 2) Determination of the phases of these effects using a geometrical optics solver, and 3) Use of the single scattering algorithm they developed in [18] for the frequency independent solution of the associated integral equations. While the algorithms in [18] and [25] are not supported with rigorous numerical analysis, the numerical implementations in [25] have shown that the Neumann series converges spectrally for two convex obstacles. In this connection, considering a finite collection of smooth strictly convex obstacles Ecevit *et al.* have later shown for two-dimensional [26] and three-dimensional [27] configurations that the Neumann series can be rearranged into *primitive periodic orbits* and derived rigorous rate of convergence formulas on these orbits. The theoretical results obtained in [26] are central to the developments in this thesis as they provide the necessary tools (namely the Hörmander classes and asymptotic expansions of multiple scattering iterations) for the extension and rigorous numerical analysis of the single scattering algorithms in [23, 24] to the case of multiple scattering problems. Indeed, we rigorously prove for the numerical solution of each multiple scattering iteration that the required number of degrees of freedom to represent each one of these iterates needs to increase only as $\mathcal{O}(k^\epsilon)$ (for any $\epsilon > 0$) with increasing wavenumber k to obtain

prescribed error tolerances independent of the underlying frequency. Accordingly, this thesis fills an important gap in the literature by providing the first rigorous numerical algorithms capable of predicting multiple scattering returns in essentially frequency independent computational times when combined with methods for the evaluation of highly oscillatory integrals.

Parallel algorithms related to scattering by convex polygons have also been developed. These algorithms display similar performances as compared to those associated with smooth convex scatterers (see the survey article [28]). Recently non-convex polygons have also been considered [29] but the algorithm there applies only to *orthogonal non-convexities* which clearly allows only for finitely many reflections as opposed to the multiple scattering problems considered in this thesis.

This thesis is organized as follows. In Chapter 2, we introduce the scattering problem along with an equivalent integral equation formulation, and provide the necessary modifications needed to express the solution as a superposition of multiple scattering iterations when the scatterer consists of two disjoint components. In Chapter 3, we resort to [26] and present the Hörmander classes and asymptotic expansions of multiple scattering iterations along with the relevant wavenumber dependent derivative estimates of these iterates. In Chapter 4, we present the general ideas underlying the classical Galerkin method in connection with the scattering problems. In Chapter 5, we discuss the extension of the single scattering algorithm in [26] to multiple scattering problems, and present a rigorous convergence analysis. Chapter 6 is reserved for an alternative approach in forming the Galerkin equations which allows for the generation of orthogonal Galerkin basis functions and thus removes the need to solve the Galerkin equations. This is importance since the condition numbers of related matrices easily grow beyond computational framework with increasing polynomial degrees. In Chapter 7, we consider the extension of the single scattering algorithm in [24] to multiple scattering problems considered in this thesis, and present a full convergence analysis. Finally, in Chapter 8 we present numerical tests validating the theoretical findings in this thesis.

2. THE MULTIPLE SCATTERING PROBLEM

In this chapter, we introduce the two-dimensional multiple scattering problem along with its equivalent integral equation formulations. Specifically, we consider the solution of the multiple scattering problem in the exterior of a smooth compact obstacle K consisting of two disjoint strictly convex structures K_1 and K_2. For definiteness, we assume sound-soft boundary conditions and a plane-wave incidence $u^{inc}(x) = e^{ik\alpha\cdot x}$ with direction α ($|\alpha| = 1$). In this case, the *scattered field* u is sought to satisfy the Helmholtz equation in the exterior domain,

$$\Delta u(x) + k^2 u(x) = 0, \quad x \in \mathbb{R}^2 \backslash K$$

the Sommerfeld radiation condition at infinity,

$$\lim_{|x|\to\infty} |x|^{1/2} \left[\left(\frac{x}{|x|}, \nabla u(x) \right) - iku(x) \right] = 0$$

and the Dirichlet boundary condition

$$u(x) = -u^{inc}(x) = -e^{ik\alpha\cdot x}, \quad x \in \partial K.$$

By Green's identities, this problem can be converted into finding the solution of one of the following boundary integral equations

$$(I + S' - ikS)\eta = 2\left\{ \frac{\partial u^{inc}}{\partial \nu} - iku^{inc} \right\} \tag{2.1}$$

$$(I + S')\eta = 2\left\{ \frac{\partial u^{inc}}{\partial \nu} \right\} \tag{2.2}$$

$$(S)\eta = 2\left\{ u^{inc} \right\} \tag{2.3}$$

for the unknown *normal derivative of the total field*

$$\eta(x) = \frac{\partial \left(u(x) + u^{inc}(x)\right)}{\partial \nu(x)}, \quad x \in \partial K$$

where I is the identity operator, and S' and S are the boundary integral operators defined as

$$(S\eta)(x) := 2 \int_{\partial K} \Phi(x,y)\eta(y)ds(y), \quad x \in \partial K$$
$$(S'\eta)(x) := 2 \int_{\partial K} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\eta(y)ds(y), \quad x \in \partial K.$$

Here ν is the exterior unit normal vector to ∂K,

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \neq y$$

is the fundamental solution to the Helmholtz equation in $2D$, and $H_0^{(1)}$ is the Hankel function of the first kind and order zero. Indeed, once η is available, the scattered field can be constructed by means of the single-layer representation

$$u(x) = -\int_{\partial K} \Phi(x,y)\eta(y)ds(y), \quad x \in \mathbb{R}^2/K.$$

While (2.1) has a unique solution, (2.2) and (2.3) lack this property and therefore, in what follows, we will work with (2.1) which we re-write as

$$\eta(x) - \int_{\partial K} \left\{ \frac{\partial G(x,y)}{\partial \nu(x)} - ikG(x,y) \right\} \eta(y)ds(y)$$
$$= 2 \left\{ \frac{\partial u^{inc}(x)}{\partial \nu(x)} - iku^{inc}(x) \right\}, \quad x \in \partial K \tag{2.4}$$

where $G = -2\Phi$. For the sake of simplicity let us introduce

$$R := (-S' + ikS) \quad \text{and} \quad f = 2\left\{\frac{\partial u^{inc}}{\partial \nu} - iku^{inc}\right\} \quad \text{on } \partial K \tag{2.5}$$

in which case (2.4) becomes

$$(I - R)\eta = f \quad \text{on } \partial K. \tag{2.6}$$

Since we have $K = K_1 \cup K_2$, we can re-write (2.6) as

$$\begin{bmatrix} I - R_{11} & -R_{12} \\ -R_{21} & I - R_{21} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{2.7}$$

where we take $\eta = (\eta_1, \eta_2)^t$ and $f = (f_1, f_2)^t$ with η_j and f_j defined on ∂K_j and $f_j(x) = 2ik(\alpha \cdot \nu(x) - 1)e^{ik\alpha \cdot x}$ for $j \in \{1, 2\}$. Also, the integral operators $R_{jj'}$ are defined as

$$(R_{jj'}\eta_{j'})(x) = \int\limits_{\partial K_{j'}} \left\{\frac{\partial G(x,y)}{\partial \nu(x)} - ikG(x,y)\right\} \eta_{j'}(y)\, ds(y), \quad x \in \partial K_j$$

for $j, j' \in \{1, 2\}$. Now, let us define $D = \text{diag}(I - R)$ and multiply (2.7) with D^{-1} to get

$$(I - T)\eta = g \tag{2.8}$$

where

$$g = \begin{bmatrix} (I - R_{11})^{-1} f_1 \\ (I - R_{22})^{-1} f_2 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 0 & (I - R_{11})^{-1} R_{12} \\ (I - R_{22})^{-1} R_{21} & 0 \end{bmatrix}.$$

The Neumann series solution of (2.8) reads

$$\eta = \sum_{m=0}^{\infty} \eta^m = \sum_{m=0}^{\infty} T^m g$$

where we observe that

$$\eta^{m+1} = T\eta^m$$

which gives rise to

$$\eta^{m+1} = \begin{bmatrix} \eta_1^{m+1} \\ \eta_2^{m+1} \end{bmatrix} = \begin{bmatrix} (I - R_{11})^{-1} R_{12} \eta_2^m \\ (I - R_{22})^{-1} R_{21} \eta_1^m \end{bmatrix}.$$

Therefore, solving the multiple scattering problem (2.6) turns into recursively solving single scattering problems

$$\begin{cases} (I - R_{11})\eta_1^0 = f_1 & \text{on } \partial K_1 \\ (I - R_{22})\eta_2^0 = f_2 & \text{on } \partial K_2 \\ (I - R_{11})\eta_1^{m+1} = R_{12}\eta_2^m & \text{on } \partial K_1 \\ (I - R_{22})\eta_2^{m+1} = R_{21}\eta_1^m & \text{on } \partial K_2 \end{cases} \tag{2.9}$$

for any $m \geq 0$ and then adding η^m's to find η.

3. MULTIPLE SCATTERING ASYMPTOTIC EXPANSIONS

Equations in (2.9) motivate that the multiple scattering problem can be dismantled into contributions associated with the two paths $(K_{\tau_j})_{j\geq 0}$ distinguished with the requirements that $\tau_j \in \{1,2\}$ and $\tau_j \neq \tau_{j+1}$ for all $j \geq 0$. Precisely, in this case, $\tau_0 = 1$ determines the first path as $(K_1, K_2, K_1, K_2, \dots)$, and $\tau_0 = 2$ the second one as $(K_2, K_1, K_2, K_1, \dots)$. Equations in (2.9) can then be adapted to these paths to yield

$$(I - R_{\tau_0\tau_0})\eta_{\tau_0} = f_{\tau_0} \quad \text{on } \partial K_{\tau_0} \tag{3.1}$$

and for $j \geq 1$

$$(I - R_{\tau_j\tau_j})\eta_{\tau_j} = R_{\tau_j\tau_{j-1}}\eta_{\tau_{j-1}} \quad \text{on } \partial K_{\tau_j} \tag{3.2}$$

for $\tau_0 = 1$ and $\tau_0 = 2$. Furthermore, provided the obstacles K_1 and K_2 are strictly convex and satisfy the "no-occlusion condition" in the sense that at least one ray with direction α passes between K_1 and K_2 without touching them, the developments in [26] entail that the densities η_{τ_j} can be expressed as

$$\eta_{\tau_j}(x) = e^{ik\varphi_{\tau_j}(x)}\eta_{\tau_j}^{slow}(x)$$

where φ_{τ_j} is the *geometrical optics phase* which we shall shortly describe. Broadly speaking, this representation combined with the estimates on the derivatives of $\eta_{\tau_j}^{slow}$ allows the generation of Galerkin approximation spaces whose dimensions must increase only very mildly with increasing frequency to obtain solutions of equivalent accuracy independent of frequency. This will be further elaborated later in the thesis. Note explicitly that the geometrical optics phase φ_{τ_m} is defined at any point x on the

boundary of $K_{\tau_m} \in (K_{\tau_j})_{j\geq 0}$ as

$$\varphi_{\tau_m}(x) = \begin{cases} \alpha \cdot x & \text{if } m = 0 \\ \alpha \cdot \mathcal{X}_0^m(x) + \sum_{j=0}^{m-1} |\mathcal{X}_{j+1}^m(x) - \mathcal{X}_j^m(x)| & \text{if } m \geq 1 \end{cases}$$

where the points

$$(\mathcal{X}_0^m(x), \ldots, \mathcal{X}_m^m(x)) \in \partial K_{\tau_0} \times \cdots \times \partial K_{\tau_m}$$

called the "broken $(m+1)$-rays terminating at $x \in \partial K_{\tau_m}$," are uniquely determined by the requirements

$$\begin{cases} \text{(a) } \mathcal{X}_m^m(x) = x \\ \text{(b) } \alpha \cdot \nu(\mathcal{X}_0^m(x)) < 0 \\ \text{(c) } (\mathcal{X}_{j+1}^m(x) - \mathcal{X}_j^m(x)) \cdot \nu(\mathcal{X}_j^m(x)) > 0 \\ \text{(d) } \dfrac{\mathcal{X}_1^m(x) - \mathcal{X}_0^m(x)}{|\mathcal{X}_1^m(x) - \mathcal{X}_0^m(x)|} = \alpha - 2\alpha \cdot \nu(\mathcal{X}_0^m(x))\, \nu(\mathcal{X}_0^m(x)) \\ \text{(e) } \dfrac{\mathcal{X}_{j+1}^m(x) - \mathcal{X}_j^m(x)}{|\mathcal{X}_{j+1}^m(x) - \mathcal{X}_j^m(x)|} = \dfrac{\mathcal{X}_j^m(x) - \mathcal{X}_{j-1}^m(x)}{|\mathcal{X}_j^m(x) - \mathcal{X}_{j-1}^m(x)|} \\ \qquad -2\dfrac{\mathcal{X}_j^m(x) - \mathcal{X}_{j-1}^m(x)}{|\mathcal{X}_j^m(x) - \mathcal{X}_{j-1}^m(x)|} \cdot \nu(\mathcal{X}_j^m(x))\, \nu(\mathcal{X}_j^m(x)) \end{cases} \tag{3.3}$$

for $0 \leq j \leq m$. Equations in (3.3) simply state that the broken $(m+1)$-rays satisfy the law of reflection for each ray $\left[\mathcal{X}_j^m(x), \mathcal{X}_{j+1}^m(x)\right]$ when $j \in \{0, \ldots, m-2\}$ on the illuminated regions, however, the ray $\left[\mathcal{X}_{m-1}^m(x), \mathcal{X}_m^m(x)\right]$ is allowed to terminate in the shadow region. In fact, the broken rays allow us to define the *illuminated regions*, the *shadow regions* and the *shadow boundaries* as

$$\partial K_{\tau_m}^{IL} = \begin{cases} \left\{x \in \partial K_{\tau_0} : \alpha \cdot \nu(x) < 0\right\}, & m = 0 \\ \left\{x \in \partial K_{\tau_m} : (\mathcal{X}_m^m(x) - \mathcal{X}_{m-1}^m(x)) \cdot \nu(x) < 0\right\}, & m \geq 1 \end{cases}$$

and

$$\partial K^{SR}_{\tau_m} = \begin{cases} \{x \in \partial K_{\tau_0} : \alpha \cdot \nu(x) > 0\}, & m = 0 \\ \\ \left\{x \in \partial K_{\tau_m} : (\mathcal{X}^m_m(x) - \mathcal{X}^m_{m-1}(x)) \cdot \nu(x) > 0\right\}, & m \geq 1 \end{cases}$$

as well as

$$\partial K^{SB}_{\tau_m} = \begin{cases} \{x \in \partial K_{\tau_0} : \alpha \cdot \nu(x) = 0\}, & m = 0 \\ \\ \left\{x \in \partial K_{\tau_m} : (\mathcal{X}^m_m(x) - \mathcal{X}^m_{m-1}(x)) \cdot \nu(x) = 0\right\}, & m \geq 1. \end{cases}$$

Indeed, convexity and (3.3) ensure that the broken rays are uniquely determined and this, in turn, implies that the illuminated regions, shadow regions, shadow boundaries and the phase functions are all well-defined.

In what follows, for simplicity we are going to denote η_{τ_m}, $\eta^{slow}_{\tau_m}$, ∂K_{τ_m}, $\partial K^{IL}_{\tau_m}$ etc. as η_m, η^{slow}_m, ∂K_m, ∂K^{IL}_m etc. since we are going to stick to just one path. Our approach in designing the Galerkin approximation spaces is based on a detailed understanding of the derivative estimates of η^{slow}_m. This naturally calls for a characterization of Hörmander classes and asymptotic expansions of η^{slow}_m. Following [26] we introduce the definition of these concepts next.

Definition 3.1 (Symbol classes of Hörmander)**.** *Let $\mathcal{M}$ be an open subset of $\mathbb{R}^p$, and let Γ be an open conic subset of $\mathcal{M} \times \mathbb{R}^q$ (i.e. $(x, \xi) \in \Gamma$ implies $(x, t\xi) \in \Gamma$ when $t > 0$). The* symbol class of order $\mu \in \mathbb{R}$ and type $\varrho, \delta \in [0, 1]$ of Hörmander*, denoted as $S^\mu_{\varrho,\delta}(\Gamma)$, is defined to be the collection of all complex-valued functions $a \in C^\infty(\Gamma)$ such that, for any compact set $W \subset \Gamma$ and all multi-indices β, γ, the estimate*

$$\left|D^\beta_x D^\gamma_\xi a(x, \xi)\right| \leq C_{\beta,\gamma,W}(1 + |\xi|)^{\mu - \varrho|\gamma| + \delta|\beta|}, \qquad (x, \xi) \in W^c \tag{3.4}$$

holds for some constant $C_{\beta,\gamma,W}$ *where* $W^c = \{(x,t\xi) : (x,\xi) \in W,\, t \geq 1\}$. *Note that, if* $a \in S^{\mu}_{\varrho,\delta}(\Gamma)$, *then*

$$D_x^\beta D_\xi^\gamma a(x,\xi) \in S^{\mu-\varrho|\gamma|+\delta|\beta|}_{\varrho,\delta}(\Gamma)\,.$$

The upper bound in (3.4) is independent of the variable x on the left hand side. This permits us to extend this definition to the case where $\mathcal{M}$ is a p-dimensional C^∞ manifold for which atlas of charts are defined as

$$\{(U_\alpha,\chi_\alpha)|\chi_\alpha \,:\, U_\alpha \subset M \to \mathbb{R}^p\}$$

on $\mathcal{M}$, and where $D_x^\beta D_\xi^\gamma a(x,\xi)$ is defined by

$$D_x^\beta D_\xi^\gamma a(x,\xi) = \left\{D_y^\beta D_\xi^\gamma a(\chi_\alpha^{-1}(y),\xi) \,:\, (U_\alpha,\chi_\alpha) \text{ is a coordinate chart s.t. } \chi_\alpha(x) = y\right\}$$

such that

$$\left|D_x^\beta D_\xi^\gamma a(x,\xi)\right| = \max\left\{\left|D_y^\beta D_\xi^\gamma a(\chi_\alpha^{-1}(y),\xi)\right| \,:\, D_y^\beta D_\xi^\gamma a(\chi_\alpha^{-1}(y),\xi) \in D_x^\beta D_\xi^\gamma a(x,\xi)\right\}.$$

Definition 3.2 (Asymptotic expansions in the sense of Hörmander)**.** *Let* $\mathcal{M}$ *be a* p*-dimensional* C^∞ *manifold,* Γ *an open conic subset of* $\mathcal{M}\times\mathbb{R}^q$, *and* $a_j \in S^{\nu_j}_{\varrho,\delta}(\Gamma)$ *where* $\nu_j \to -\infty$ *as* $j \to \infty$. *We say that* $a \in S^{\mu}_{\varrho,\delta}(\Gamma)$ *admits the asymptotic expansion*

$$a \sim \sum_{j=0}^{\infty} a_j$$

if

$$a - \sum_{i<j} a_i \in S^{\mu_j}_{\varrho,\delta}(\Gamma)$$

for every $j = 0, 1, 2, \ldots$ *where* $\mu_j = \max_{i \geq j} \nu_i$ *and* $\mu = \mu_0$. *Moreover, the function* a *is uniquely determined modulo* $S^{-\infty}_{\varrho,\delta}(\Gamma) = \bigcap_\mu S^\mu_{\varrho,\delta}(\Gamma)$ *and has the same property relative to every rearrangement of the series* $\sum a_j$.

Next we present the Hörmander classes and asymptotic expansions of η_m^{slow} as were classified in [26].

Theorem 3.3 (Hörmander classes and uniform asymptotic expansions of η_m^{slow})**.** *The asymptotic characteristics of the slow densities* η_m^{slow} *are as follows:*

(i) *On the illuminated region* ∂K_m^{IL}, $\eta_m^{\text{slow}}(x) = \eta_m^{\text{slow}}(x, k)$ *belongs to* $S^1_{1,0}(\partial K_m^{IL} \times (0, \infty))$ *and admits the asymptotic expansion*

$$\eta_m^{\text{slow}}(x, k) \sim \sum_{j \geq 0} k^{1-j} a_{m,j}(x)$$

where $a_{m,j}(x)$ *are complex-valued* C^∞ *functions. Accordingly, for any* $N \in \mathbb{N} \cup \{0\}$, *the difference*

$$r_{m,N}(x, k) = \eta_m^{\text{slow}}(x, k) - \sum_{j=0}^{N} k^{1-j} a_{m,j}(x)$$

belongs to $S^{-N}_{1,0}(\partial K_m^{IL} \times (0, \infty))$ *and thus satisfies the estimates*

$$\left| D_x^\beta D_k^n r_{m,N}(x, k) \right| \leq C_{m,\beta,n,S} (1 + k)^{-N-n}$$

on any compact subset S *of* ∂K_m^{IL} *for any multi-index* β *and* $n \in \mathbb{N} \cup \{0\}$.

(ii) *Over the entire boundary* ∂K_m, $\eta^{\text{slow}}(x, k)$ *belongs to* $S^1_{2/3,1/3}(\partial K_m \times (0, \infty))$ *and admits the asymptotic expansion*

$$\eta_m^{\text{slow}}(x, k) \sim \sum_{p,q \geq 0} k^{2/3 - 2p/3 - q} \, b_{m,p,q}(x) \Psi^{(p)}(k^{1/3} Z_m(x))$$

where $b_{m,p,q}(x)$ are complex-valued C^∞ functions, $Z_m(x)$ is a real-valued C^∞ function that is positive on the illuminated region ∂K_m^{IL}, negative on the shadow region ∂K_m^{SR}, and vanishes precisely to first order on the shadow boundary ∂K_m^{SB}. Here, the function Ψ admits the asymptotic expansion

$$\Psi \sim \sum_{j=0}^{\infty} c_j \tau^{1-3j} \quad as \ \tau \to \infty$$

and Ψ is rapidly decreasing in the sense of Schwartz as $\tau \to -\infty$. In other words, for all $N, n \in \mathbb{N} \cup \{0\}$,

$$D_\tau^n \left\{ \Psi(\tau) - \sum_{j=0}^{N-1} c_j \tau^{1-3j} \right\} = \mathcal{O}(\tau^{1-3N-n}) \quad as \ \tau \to \infty$$

and

$$D_\tau^n \Psi(\tau) = \mathcal{O}(\tau^{-N}) \quad as \ \tau \to -\infty.$$

Note specifically then, for any $P, Q \in \mathbb{N} \cup \{0\}$, the difference

$$R_{m,P,Q}(x,k) = \eta_m^{\text{slow}}(x,k) - \sum_{p,q=0}^{P,Q} k^{2/3-2p/3-q} \, b_{m,p,q}(x) \Psi^{(p)}(k^{1/3} Z_m(x))$$

belongs to $S^{-\mu}_{2/3,1/3}(\partial K_m \times (0,\infty))$, $\mu = \min\{2P/3, Q\}$. Thus it satisfies the estimates

$$\left| D_x^\beta D_k^n R_{m,P,Q}(x,k) \right| \leq C_{m,\beta,n} (1+k)^{-\mu - 2n/3 + |\beta|/3}$$

for any multi-index β and $n \in \mathbb{N} \cup \{0\}$.

As was shown in [26], Theorem 3.3 yields the following estimates on the derivatives of η_m^{slow}.

Theorem 3.4 (Derivative estimates)**.** *Let $m \geq 0$, and denote by $y(s) = (y^1(s), y^2(s))$ the arc-length parametrization of ∂K_m. Then, for all $n \in \mathbb{N} \cup \{0\}$, there exist constants $C_n > 0$ independent of k and s such that for all k sufficiently large,*

$$\left|D_s^n \eta_m^{\text{slow}}(y(s))\right| \leq k \begin{cases} C_n, & n = 0, 1 \\ C_n \left[1 + \sum_{j=2}^{n} k^{(j-1)/3}(1 + k^{1/3}|w(s)|)^{-(j+2)}\right], & n \geq 2 \end{cases}$$

where $w(s) = (s-a)(b-s)$ and $\partial K_m^{SB} = \{y(a), y(b)\}$.

4. THE GALERKIN METHOD

As is well known, the Galerkin method provides a convenient mechanism for the solution of operator equations in Hilbert spaces [30]. In this chapter, we describe how the multiple scattering integral equations (3.1)-(3.2) can be solved through the Galerkin method. Indeed, while each of (3.1)-(3.2) is an operator equation of the form

$$\mathcal{R}\eta = f \tag{4.1}$$

in $L^2(\partial\mathcal{K})$ with either $\mathcal{K} = K_1$ or $\mathcal{K} = K_2$, clearly the unique solution of (4.1) is also the unique solution of its weak formulation

$$B(\eta, \mu) = F(\mu) \quad \text{for all } \mu \in L^2(\partial\mathcal{K}) \tag{4.2}$$

where the sesquilinear form $B : L^2(\partial\mathcal{K}) \times L^2(\partial\mathcal{K}) \to \mathbb{C}$ and the bounded linear functional $F : L^2(\partial\mathcal{K}) \to \mathbb{C}$ are defined as

$$B(\mu, \eta) = \langle \mu, \mathcal{R}\eta \rangle \quad \text{and} \quad F(\mu) = \langle \mu, f \rangle \tag{4.3}$$

for the L^2 inner product given by $\langle g, h \rangle = \int_{\partial\mathcal{K}} g(s)\overline{h(s)}ds$ and the operator $\mathcal{R} = I - R$ (see (2.5) for the definition of R). In order to use the Galerkin method, we need to work on a finite dimensional subspace $\hat{X}$ of $L^2(\partial\mathcal{K})$ and find an approximate solution $\hat{\eta} \in \hat{X}$ to (4.2). Hence, we are going to restrict (4.2) on a "nice" finite dimensional subspace of $L^2(\partial\mathcal{K})$, namely $\hat{X}$, and work on the equation

$$B(\hat{\mu}, \hat{\eta}) = F(\hat{\mu}) \quad \text{for all } \hat{\mu} \in \hat{X}. \tag{4.4}$$

By Céa's lemma, under certain conditions we have a unique solution to (4.4) and a bound for the error term.

Lemma 4.1. *(Céa's Lemma) Suppose $B : X \times X \to \mathbb{C}$ is a bounded strictly coercive sesquilinear form on a Hilbert space X, that is, there exist positive constants C and c such that*

$$|B(\mu, \eta)| \leq C\|\mu\|\|\eta\|, \quad \textit{for all } \eta, \mu \in X$$
$$\mathcal{R}eB(\eta, \eta) \geq c\|\eta\|^2, \quad \textit{for all } \eta \in X$$

and $F : X \to \mathbb{C}$ is a a bounded linear functional. Given a finite dimensional subspace $\hat{X}$ (called a Galerkin approximation space) of X, there exists a unique $\hat{\eta} \in \hat{X}$ (called the Galerkin solution) such that

$$B(\hat{\mu}, \hat{\eta}) = F(\hat{\mu}), \quad \textit{for all } \hat{\mu} \in \hat{X}.$$

Furthermore, we have the error estimate

$$\|\eta - \hat{\eta}\| \leq \frac{C}{c} \inf_{\hat{\mu} \in \hat{X}} \|\eta - \hat{\mu}\|$$

where η is the solution of

$$B(\mu, \eta) = F(\mu), \quad \textit{for all } \mu \in X.$$

The constants C and c in the lemma are called the continuity and coercivity constants respectively. For the operator $\mathcal{R} = I - R$ in (2.6) and the sesquilinear form B in (4.2) we have $C/c = \mathcal{O}(k^{1/3})$ as $k \to \infty$.

Incidentally, if $\mathcal{B} = \{\hat{\mu}_1, \ldots, \hat{\mu}_m\}$ is a basis for a finite dimensional Galerkin approximation space $\hat{X}$ of dimension m, then the Galerkin solution $\hat{\eta} \in \hat{X}$ can be

expressed as $\hat{\eta} = \sum_{j=1}^{m} c_j \hat{\mu}_j$ and (4.4) becomes

$$\langle \hat{\mu}_i, f \rangle = F(\hat{\mu}_i) = B(\hat{\mu}_i, \hat{\eta}) = \sum_{j=1}^{m} \overline{c}_j B(\hat{\mu}_i, \hat{\mu}_j) = \sum_{j=1}^{m} \overline{c}_j \langle \hat{\mu}_i, \mathcal{R}\hat{\mu}_j \rangle$$

for $i = 1, \ldots, m$. Upon taking conjugates this gives rise to the linear system

$$\begin{bmatrix} \langle \mathcal{R}\hat{\mu}_1, \hat{\mu}_1 \rangle & \ldots & \langle \mathcal{R}\hat{\mu}_m, \hat{\mu}_1 \rangle \\ \vdots & & \vdots \\ \langle \mathcal{R}\hat{\mu}_1, \hat{\mu}_m \rangle & \ldots & \langle \mathcal{R}\hat{\mu}_m, \hat{\mu}_m \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} \langle f, \hat{\mu}_1 \rangle \\ \vdots \\ \langle f, \hat{\mu}_m \rangle \end{bmatrix}. \tag{4.5}$$

We see that for any given finite dimensional subspace $\hat{X}$, solving (4.4) is equivalent to solving (4.5) for the vector $\hat{c} = (c_1, \ldots, c_{\dim(\hat{X})})$. Therefore, in what follows we focus on constructing Galerkin approximation spaces $\hat{X} \subset L^2(\partial\mathcal{K})$ whose dimensions increase only very slowly with increasing frequency to fix the error independent of frequency. The main idea is to construct Galerkin approximation spaces by mimicking the asymptotic behaviour of the density η_{τ_m} locally for each reflection.

Bearing these in mind, in the next three chapters, we fix the obstacle path $(K_{\tau_j})_{j\geq 0}$ (i.e. we choose either $\tau_0 = 1$ or $\tau_0 = 2$), and we also fix the reflection number m and set $\mathcal{K} = K_{\tau_m}$. Further, we write $\eta = \eta_{\tau_m}$, $\eta^{slow} = \eta^{slow}_{\tau_m}$, and $\varphi = \varphi_{\tau_m}$ so that

$$\eta(x) = e^{ik\varphi(x)}\eta^{slow}(x), \quad x \in \partial\mathcal{K}. \tag{4.6}$$

Finally, we write $R = R_{\tau_m\tau_m}$ so that the corresponding integral equation in (3.1) or (3.2) takes on the form

$$(I - R)\eta = f. \tag{4.7}$$

5. FREQUENCY-ADAPTED GALERKIN BOUNDARY ELEMENT METHODS FOR MULTIPLE SCATTERING PROBLEMS

In this chapter, we describe how the *frequency-adapted Galerkin boundary element methods* developed in [23] can be extended to encompass the multiple scattering problems considered in this thesis for the efficient solution of integral equations (3.1)-(3.2).

As mentioned in the end of Chapter 4, we fix the obstacle path $(K_{\tau_j})_{j\geq 0}$ and the reflection number m, and set $\mathcal{K} = K_{\tau_m}$. Further, we write $\eta = \eta_{\tau_m}$, $\eta^{slow} = \eta^{slow}_{\tau_m}$, and $\varphi = \varphi_{\tau_m}$ so that we have the phase extraction (4.6), and we consider the solution of the integral equation (4.7) by the Galerkin method.

To this end, we use γ to denote an arc-length parametrization, with period $2L$, of the boundary of the obstacle $\mathcal{K}$. Further, we assume that γ is chosen so that $\gamma(0)$ is in the shadow region, $\gamma(t_1)$ and $\gamma(t_2)$ are the shadow boundary points with $0 < t_1 < t_2 < 2L$, and (t_1, t_2) is mapped onto the illuminated region whereas (t_2, t_1+2L) is to the shadow region of this obstacle.

Given $m \in \mathbb{N}$, $0 \leq \epsilon_m < \epsilon_{m-1} < \ldots < \epsilon_1 < 1/3$, and constants $\xi_1, \xi_2 > 0$, we now divide the interval $[0, 2L]$ into $4m$ subregions as follows:

(i) *Illuminated region*:

$$\Lambda^{IL} = [t_1 + \xi_1 k^{-1/3} k^{\epsilon_1}, t_2 - \xi_2 k^{-1/3} k^{\epsilon_1}]$$

(ii) *Deep shadow region*:

$$\Lambda^{DS} = [0, t_1 - \xi_1 k^{-1/3} k^{\epsilon_1}] \cup [t_2 + \xi_2 k^{-1/3} k^{\epsilon_1}, 2L]$$

(iii) *Shadow boundaries* ($\ell = 1, 2$):

$$\Lambda^{SB_\ell} = [t_\ell - \xi_\ell k^{-1/3} k^{\epsilon_m}, t_\ell + \xi_\ell k^{-1/3} k^{\epsilon_m}]$$

(iv) *Illuminated transitions*: For $j = 1, 2, \ldots, m-1$,

$$\Lambda_j^{IT_1} = [t_1 + \xi_1 k^{-1/3} k^{\epsilon_{j+1}}, t_1 + \xi_1 k^{-1/3} k^{\epsilon_j}]$$

$$\Lambda_j^{IT_2} = [t_2 - \xi_2 k^{-1/3} k^{\epsilon_j}, t_2 - \xi_2 k^{-1/3} k^{\epsilon_{j+1}}]$$

(v) *Shadow transitions*: For $j = 1, 2, \ldots, m-1$,

$$\Lambda_j^{ST_1} = [t_1 - \xi_1 k^{-1/3} k^{\epsilon_j}, t_1 - \xi_1 k^{-1/3} k^{\epsilon_{j+1}}]$$

$$\Lambda_j^{ST_2} = [t_2 + \xi_2 k^{-1/3} k^{\epsilon_{j+1}}, t_2 + \xi_2 k^{-1/3} k^{\epsilon_j}].$$

We define the Galerkin approximation spaces $\mathcal{P}_j$ for $j \in \{1, \ldots, 4m\}$ on these subregions as

$$\mathcal{P}_j = \chi_j \, e^{ik\varphi} \, \mathbb{P}_{d_j}$$

where χ_j is the characteristic function of Λ_j ($j = 1, \ldots, 4m$) and $\mathbb{P}_{d_j}$ is the space of polynomials of degree at most d_j which is a d_j+1 dimensional polynomial space defined on each subregion. Hence, our $4m+\sum_{j=1}^{4m} d_j$ dimensional Galerkin approximation space is the direct sum of these spaces

$$\mathcal{P}_{\mathbf{d}} = \bigoplus_{j=1}^{4m} \mathcal{P}_j$$

where $\mathbf{d}$ stands for $\mathbf{d} = (d_1, \ldots, d_{4m})$. Exactly as demonstrated in [23], these choices give rise to the following result.

Theorem 5.1. *For all $n_j \in \{0, \ldots, d_j + 1\}$ ($j = 1, \ldots, 4m$) and all sufficiently large $k \geq 1$, we have*

$$\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})} \lesssim_{n_1,\ldots,n_{4m}} \frac{C}{c} \, k \sum_{j=1}^{4m} \frac{E(k,j)}{(d_j)^{n_j}} \tag{5.1}$$

for the Galerkin solution $\hat{\eta}$ to (4.4), where C and c are the continuity and coercivity constants of sesquilinear form B in Lemma 4.1, and for

$$E(k,j) = 1 + \begin{cases} k^{-(1+3\epsilon_1)/2}\left(k^{(1/3-\epsilon_1)/2}\right)^{n_j}, & \text{if } \Lambda_j = \Lambda^{IL} \text{ or } \Lambda_j = \Lambda^{DS} \\ k^{-1/2}\left(k^{\epsilon_m}\right)^{n_j}, & \text{if } \Lambda_j = \Lambda^{SB_1} \text{ or } \Lambda_j = \Lambda^{SB_2} \\ k^{-(1+3\epsilon_{r+1})/2}\left(k^{(\epsilon_r-\epsilon_{r+1})/2}\right)^{n_j}, & \text{if } \Lambda_j = \Lambda_r^{IT_\ell} \text{ or } \Lambda_j = \Lambda_r^{ST_\ell}. \end{cases}$$

The error bound in (5.1) can be brought into a more simple form. Indeed, if $d_{j^*} = \min(d_1, \ldots, d_{4m})$, we can have a better upper bound simply by increasing d_{j^*} as the inequality depends only on n_j values. Hence, this suggest us setting $d = d_1 = \cdots = d_{4m}$. In addition, for simplicity, let us take $n = n_1 = \cdots = n_{4m}$ too. Hence, for all $n \in \{0, \ldots, d+1\}$ and all sufficiently large $k \geq 1$, inequality in (5.1) becomes

$$\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})} \lesssim_n \frac{C}{c}\, k \sum_{j=1}^{4m} \frac{\hat{E}(k,j)}{d^n}$$

where and $\hat{E}(k,j)$ is defined as

$$E(k,j) = 1 + \begin{cases} k^{-(1+3\epsilon_1)/2}\left(k^{(1/3-\epsilon_1)/2}\right)^{n}, & \text{if } \Lambda_j = \Lambda^{IL} \text{ or } \Lambda_j = \Lambda^{DS} \\ k^{-1/2}\left(k^{\epsilon_m}\right)^{n}, & \text{if } \Lambda_j = \Lambda^{SB_1} \text{ or } \Lambda_j = \Lambda^{SB_2} \\ k^{-(1+3\epsilon_{r+1})/2}\left(k^{(\epsilon_r-\epsilon_{r+1})/2}\right)^{n}, & \text{if } \Lambda_j = \Lambda_r^{IT_\ell} \text{ or } \Lambda_j = \Lambda_r^{ST_\ell}. \end{cases}$$

Corollary 5.2. *If $d = d_1 = \cdots = d_{4m}$ and $n = n_1 = \cdots = n_{4m}$ such that $n \in \{0, \ldots, d+1\}$, and if*

$$\epsilon_j = \frac{1}{3}\frac{2m-2j+1}{2m+1}, \qquad j = 1, \ldots, m \tag{5.2}$$

then, for all sufficiently large $k \geq 1$, we have

$$\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})} \lesssim_n \frac{C}{c} km \frac{1 + k^{-\frac{1}{2}}\left(k^{\frac{1}{6m+3}}\right)^n}{d^n}.$$

Proof. Here, we observe that the terms $(k^{(1/3-\epsilon_1)/2})^n$, $(k^{\epsilon_m})^n$ and $(k^{(\epsilon_r-\epsilon_{r+1})/2})^n$ in the definition of $E(k,j)$ go to infinity as k goes to infinity. Therefore, in order to make the error term less dependent on k we must assign ϵ values in such a way that the increase of the error term should not be mainly determined by one region. Hence, this suggests to have proportional increase in every region as k increases which is equivalent to having

$$\frac{1/3-\epsilon_1}{2} = \epsilon_m = \frac{\epsilon_j - \epsilon_{j+1}}{2}, \qquad j = 1, \dots, m-1. \tag{5.3}$$

Since $\frac{\epsilon_j-\epsilon_{j+1}}{2}$ for $j = 1, \dots, m-1$ are equal to each other, we conclude that the interval (ϵ_m, ϵ_1) must be divided into $m-1$ many subintervals with equal length in the form $(\epsilon_m, \epsilon_{m-1})$, $(\epsilon_{m-1}, \epsilon_{m-2})$, $\dots$, (ϵ_3, ϵ_2), (ϵ_2, ϵ_1) which means

$$\frac{\epsilon_j - \epsilon_{j+1}}{2} = \frac{\epsilon_1 - \epsilon_m}{m-1} = \frac{\epsilon_1 - \frac{1/3-\epsilon_1}{2}}{m-1}, \qquad j = 1, \dots, m-1 \tag{5.4}$$

and this must be equal to $\frac{1/3-\epsilon_1}{2}$ by (5.3). So, we have

$$\frac{\epsilon_1 - \frac{1/3-\epsilon_1}{2}}{m-1} = \frac{1/3-\epsilon_1}{2}.$$

By simple algebraic manipulations we find that

$$\epsilon_1 = \frac{1}{3}\frac{2m-1}{2m+1}$$

and we combine this with

$$\epsilon_j = \epsilon_1 - j\frac{\epsilon_1 - \frac{1/3-\epsilon_1}{2}}{m-1}, \qquad j = 1, \dots, m-1$$

which we can conclude from (5.4). Finally, simple algebraic manipulations yield (5.2). The choice of ϵ_j's as in (5.2) turns the error bound in (5.1) into

$$\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})} \lesssim_n \frac{C}{c}\, k\, \Gamma(k,m,n)$$

where

$$\Gamma(k,m,n) = \frac{1 + k^{-\frac{2m}{2m+1}} \left(k^{\frac{1}{6m+3}}\right)^n}{d^n} + \frac{1 + k^{-\frac{1}{2}} \left(k^{\frac{1}{6m+3}}\right)^n}{d^n} + \sum_{j=1}^{m-1} \frac{1 + k^{-\frac{2m-j}{2m+1}} \left(k^{\frac{1}{6m+3}}\right)^n}{d^n}$$

Since $-\frac{1}{2} \geq -\frac{2m-j}{2m+1} \geq -\frac{2m}{2m+1}$ for $j = 1, \ldots, m-1$ and $k \geq 1$, the result follows. □

We know that C/c is of order $\mathcal{O}(k^{1/2})$ by [31], and η is of order $\mathcal{O}(k)$ by [32]. Also, we can choose $d, n \in \mathbb{N}$ so that $n\epsilon \geq 1/2$ where $d \sim k^{\epsilon}$. Additionally, we observe that the equality $k^{\frac{1}{6m+3}} = e^{\frac{\log k}{6m+3}}$ suggests us taking m proportional to $\log k$ so that $k^{\frac{1}{6m+3}}$ and so $1 + k^{-\frac{1}{2}} k^{\frac{1}{6m+3}}$ are bounded by a constant independent of k. Here, we also know that for all $\delta > 0$ there exist k_0 such that, for all $k > k_0$, $\log k$ is less than k^{δ} ($\lim_{k\to\infty}(\log k)/k^{\delta} = 0$). So, $\frac{C}{c}\frac{\log k}{d^n}$ is proportional to $k^{1/2+\delta-\epsilon n}$ which is less than k^{δ} for all $\delta \geq 0$ when k is sufficiently large. Combining these, if ϵ_j $(j = 1, \ldots, m)$ are chosen as in (5.2), the preceding corollary entails for the relative error

$$\frac{\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})}}{\|\eta\|_{L^2(\partial\mathcal{K})}} \lesssim_n \frac{C}{c}\frac{\log k}{d^n}.$$

This verifies that our boundary element method can be tuned to demand only an $\mathcal{O}\left(k^{\varepsilon}\right)$ increase in the number of degrees of freedom, namely $4m\,(d+1)$, to maintain a fixed accuracy with increasing wavenumber k.

6. AN ALTERNATIVE APPROACH

In the previous chapter, in order to approximate the solution η of (4.7), we used the Galerkin method considering the L^2 inner product $\langle a, b\rangle_{L^2(\partial\mathcal{K})} = \int_{\partial\mathcal{K}} a(s)\overline{b(s)}ds$ and the Galerkin approximation spaces $\bigoplus_{j=1}^{4m} \mathcal{P}_j$ that are spanned by the basis functions of the form $\chi_j\, e^{ik\,\varphi_m} x^i$ for $i \in \{1, \ldots, d_j\}$ and $j \in \{1, \ldots, 4m\}$. In general, these basis functions fail to be orthogonal, and therefore the corresponding Galerkin solutions $\hat{\eta}$ must be computed as the solution of Galerkin equations (4.5) through use of a linear algebra solver. Unfortunately, however, the condition numbers of the Galerkin matrices appearing in (4.5) quickly grow beyond computational range with increasing polynomial degrees, and this renders the Galerkin equations (4.5) numerically ill-conditioned.

In this chapter, we introduce a new inner product

$$\langle a, b\rangle_{\mathcal{R}} = \langle \mathcal{R}a, \mathcal{R}b\rangle_{L^2(\partial\mathcal{K})}$$

on $L^2(\mathcal{K})$ where $\mathcal{R} = I - R$ is the integral operator defined as in (2.6) and (2.9), and we apply the Gram-Schmidt process to the aforementioned basis functions to obtain a new basis $\{\hat{\mu_1}, \ldots, \hat{\mu_n}\}$ for the Galerkin approximation space $\bigoplus_{j=1}^{4m} \mathcal{P}_j$ that is orthonormal with respect to this new inner product. While applying Gram-Schmidt process, we use a numerically stable algorithm [33] which starts by setting $w_j^{(1)} = \mu_j$, for $i = 1, \ldots, n$, and recursively computes

$$\begin{aligned} \hat{\mu}_j &= \frac{w_j^{(j)}}{\|w_j^{(j)}\|_{\mathcal{R}}}, && i = 1, \ldots, n \\ w^{(j+1)} &= w^{(j)} - \hat{\mu}_j \langle w^{(j+1)}, \hat{\mu}_j\rangle_R, && k = j+1, \ldots, n \end{aligned}$$

where $\|\cdot, \cdot\|_{\mathcal{R}}$ is the norm obtained by the inner product $\langle \cdot, \cdot\rangle_{\mathcal{R}}$.

Theorem 6.1. *$L^2(\partial\mathcal{K})$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{R}}$.*

Proof. It is obvious that $\langle\cdot,\cdot\rangle_{\mathcal{R}}$ is conjugate symmetric and linear in the first variable since $\langle\cdot,\cdot\rangle_{L^2(\partial\mathcal{K})}$ is an inner product and $\mathcal{R}$ is a linear operator on $L^2(\partial\mathcal{K})$. In addition, for all $\eta \in L^2(\partial\mathcal{K})$, we have $\langle\eta,\eta\rangle_{\mathcal{R}} = \langle\mathcal{R}\eta,\mathcal{R}\eta\rangle_{L^2(\partial\mathcal{K})} \geq 0$. Further, $\langle\eta,\eta\rangle_{\mathcal{R}} = 0$ if and only if $\mathcal{R}\eta = 0$ and this is equivalent to $\eta = 0$ since $\mathcal{R}$ being invertible is known by [34]. As a result, $\langle\cdot,\cdot\rangle_{\mathcal{R}}$ is an inner product on $L^2(\partial\mathcal{K})$.

Now, assuming $\{\eta_n\}$ is a Cauchy sequence with respect to $\langle\cdot,\cdot\rangle_{\mathcal{R}}$, $\{\mathcal{R}\eta_n\}$ becomes a Cauchy sequence with respect to $\langle\cdot,\cdot\rangle_{L^2(\partial\mathcal{K})}$. Therefore, there exist a unique $\mu \in L^2(\partial\mathcal{K})$ such that $\mathcal{R}\eta_n \to \mu$ as $n \to \infty$, and the invertibility of $\mathcal{R}$ guarantees the existence of a unique $\eta \in L^2(\partial\mathcal{K})$ such that $\mathcal{R}\eta = \mu$. Hence, $\{\mathcal{R}\eta_n\}$ converges to $\mathcal{R}\eta$ with respect to $\langle\cdot,\cdot\rangle_{L^2(\partial\mathcal{K})}$, or equivalently $\{\eta_n\}$ converges with respect to $\langle\cdot,\cdot\rangle_{\mathcal{R}}$. □

Theorem 6.2. *$\langle\cdot,\cdot\rangle_{\mathcal{R}}$ is a bounded and strictly coercive sesquilinear form on $L^2(\partial\mathcal{K})$, that is, there exist positive constants C^* and c^* such that*

$$|\langle\eta,\mu\rangle_{\mathcal{R}}| \leq C^*\|\eta\|\|\mu\|, \quad \textit{for all } \eta,\mu \in L^2(\partial\mathcal{K})$$
$$\mathcal{R}e\langle\eta,\eta\rangle_{\mathcal{R}} \geq c^*\|\eta\|^2, \quad \textit{for all } \eta \in L^2(\partial\mathcal{K})$$

where $\|\cdot\|$ is the norm on $L^2(\partial\mathcal{K})$ generated by the inner product $\langle\cdot,\cdot\rangle_{L^2(\partial\mathcal{K})}$.

Proof. Theorem 6.1 shows that $\langle\cdot,\cdot\rangle_{\mathcal{R}}$ is a sesquilinear form on $L^2(\partial\mathcal{K})$. By Theorem 4.1 B is a bounded sesquilinear form on $L^2(\partial\mathcal{K})$ so that, for any $\eta,\mu \in L^2(\partial\mathcal{K})$, we have

$$|\langle\eta,\mu\rangle_{\mathcal{R}}| = |\langle\mathcal{R}\eta,\mathcal{R}\mu\rangle_{L^2(\partial\mathcal{K})}| = |B(\eta,\mathcal{R}\mu)| \leq C\|\eta\|\|\mathcal{R}\mu\| \leq C^2\|\eta\|\|\mu\|.$$

For strict coercivity we first observe the following identity

$$\mathcal{R}e\langle\eta,\eta\rangle_{\mathcal{R}} = \mathcal{R}e\langle\mathcal{R}\eta,\mathcal{R}\eta\rangle_{L^2(\partial\mathcal{K})} = \|\mathcal{R}\eta\|^2.$$

Now if we assume that $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ is not strictly coercive, then for all $n \in \mathbb{N}$ we can choose $\eta_n \in L^2(\partial\mathcal{K})$ such that $\|\eta_n\| = 1$ and $\|\mathcal{R}\eta_n\|^2 \leq \frac{1}{n}$. This implies that $\|\mathcal{R}\eta_n\| \to 0$ as $n \to \infty$. Since $\mathcal{R}$ is a uniquely solvable integral operator and $L^2(\partial\mathcal{K})$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2(\partial\mathcal{K})}$, we conclude that $\|\eta_n\| \to 0$ too. However, this contradicts with $\|\eta_n\| = 1$ for all $n \in \mathbb{N}$. Hence, $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ must be strictly coercive. □

As a result of Theorem 6.1 and Theorem 6.2, this new inner product meets the assumptions of Céa's Lemma. Accordingly, all the results in the previous chapter hold exactly when we modify Céa's Lemma 4.1 with $B(\eta, \mu) := \langle \eta, \mu \rangle_{\mathcal{R}}$ and $F(\mu) := \langle \mathcal{R}f, \mathcal{R}\mu \rangle$ for all $\eta, \mu \in \mathrm{L}^2(\partial\mathcal{K})$. Hence, identity (4.4) turns into $\langle \mathcal{R}\hat{\mu}, \mathcal{R}\hat{\eta} \rangle = \langle \mathcal{R}\hat{\mu}, \mathcal{R}f \rangle$ and the Galerkin equations in (4.5) takes on the form

$$\begin{bmatrix} \langle \mathcal{R}\hat{\mu}_1, \mathcal{R}\hat{\mu}_1 \rangle & \dots & \langle \mathcal{R}\hat{\mu}_n, \mathcal{R}\hat{\mu}_1 \rangle \\ \vdots & & \vdots \\ \langle \mathcal{R}\hat{\mu}_1, \mathcal{R}\hat{\mu}_n \rangle & \dots & \langle \mathcal{R}\hat{\mu}_n, \mathcal{R}\hat{\mu}_n \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle \mathcal{R}f, \mathcal{R}\hat{\mu}_1 \rangle \\ \vdots \\ \langle \mathcal{R}f, \mathcal{R}\hat{\mu}_n \rangle \end{bmatrix} \tag{6.1}$$

where $\hat{\eta} = \sum_{i=1}^{n} c_i \hat{\mu}_i$ is the approximation of η in $\operatorname{span}\{\hat{\mu_1}, \dots, \hat{\mu_n}\}$. Because of orthonormality of the basis functions with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{R}}$, the Galerkin matrix in (6.1) is simply the identity matrix, and therefore $\hat{\eta} = \sum_{i=1}^{n} \langle f, \hat{\mu}_i \rangle_{\mathcal{R}} \hat{\mu}_i$. This shows that we do not need to solve the Galerkin equations (6.1) and thus we do not have to deal with high condition numbers during the numerical computations.

7. CHANGE OF VARIABLES GALERKIN BOUNDARY ELEMENT METHODS FOR MULTIPLE SCATTERING PROBLEMS

In Chapter 5, we designed Galerkin approximation spaces for the efficient computation of multiple scattering iterates as the direct sum of approximation spaces confined to the asymptotic behaviour of the solutions. Yet, the number of direct summands forming the Galerkin approximation spaces had to increase in proportion to $\log k$ as k increases. This naturally complicates the implementation of the frequency-adapted Galerkin boundary element methods for multiple scattering problems developed in Chapter 5. In this chapter we take a different approach that removes the need to increase the number of direct summands defining the Galerkin approximation spaces with increasing wavenumber k. To this end, following the prescriptions in [24], we introduce novel changes of variables in the transition regions adopted to the asymptotic behavior of the solutions therein. Consequently we generate new Galerkin approximation spaces that perform comparable to the approach taken in Chapter 5 but with reduced numbers of degrees of freedom and significantly better approximations in the shadow regions.

As in Chapter 5, we choose a $2L$-periodic smooth arclength parametrization γ of $\partial\mathcal{K}$ in the counterclockwise orientation so that $\varphi(\gamma(0)) \cdot \nu(\gamma(0)) = 1$ where ν is the unit outward normal vector to $\partial\mathcal{K}$. So, given that $\gamma(\{t_1, t_2\})$ are shadow boundaries with $0 < t_1 < t_2 < 2L$, the illuminated and shadow regions are given, respectively, by $\gamma((t_1, t_2))$ and $\gamma((t_2, t_1 + 2L))$.

Following the developments in [24] when $k > 1$, we construct the Galerkin spaces in two different ways.

First, we define illuminated transition intervals, shadow transition intervals and shadow boundary intervals as

$$\begin{aligned}
I_{IT_1} &:= [t_1 + \xi_1 k^{-1/3}, t_1 + \xi_1'] = [a_1, b_1] \\
I_{IT_2} &:= [t_2 - \xi_2', t_2 - \xi_2 k^{-1/3}] = [a_2, b_2] \\
I_{ST_1} &:= [t_1 - \zeta_1', t_1 - \zeta_1 k^{-1/3}] = [a_3, b_3] \\
I_{ST_2} &:= [t_2 + \zeta_2 k^{-1/3}, t_2 + \zeta_2'] = [a_4, b_4] \\
I_{SB_1} &:= [t_1 - \zeta_1 k^{-1/3}, t_1 + \xi_1 k^{-1/3}] = [a_5, b_5] \\
I_{SB_2} &:= [t_2 - \xi_2 k^{-1/3}, t_2 + \zeta_2 k^{-1/3}] = [a_6, b_6]
\end{aligned}$$

where we chose $\xi_j, \xi_j', \zeta_j, \zeta_j' > 0$ for $j \in \{1, 2\}$ so that

$$t_1 + \xi_1 \leq t_1 + \xi_1' \leq t_2 - \xi_2' \leq t_2 - \xi_2$$

and

$$t_2 + \zeta_2 \leq t_2 + \zeta_2' \leq L + t_1 - \zeta_1' \leq L + t_1 - \zeta_1.$$

In the second approach, we complement these intervals with the illuminated and deep shadow intervals defined as

$$\begin{aligned}
I_{IL} &:= [t_1 + \xi_1', t_2 - \xi_2'] = [a_7, b_7] \\
I_{DS} &:= [t_2 + \zeta_2', L + t_1 - \zeta_1'] = [a_8, b_8].
\end{aligned}$$

Similar to the previous cases, given $\mathbf{d} = (d_1, \ldots, d_J) \in \mathbb{Z}_+^J$ for $J \in \{6, 8\}$, $J + \sum_{j=1}^{4m} d_j$ dimensional change of variable Galerkin approximation space is defined as

$$\mathcal{C}_{\mathbf{d}} = \bigoplus_{j=1}^{J} \chi_{\mathcal{I}_j} \; e^{ik\,\alpha\cdot\gamma} \; \mathcal{C}_{d_j}$$

where $\chi_{\mathcal{I}_j}$ is the characteristic function of $\mathcal{I}_j = [a_j, b_j]$, and

$$\mathcal{C}_{d_j} := \begin{cases} \mathbb{P}_{d_j} \circ \phi^{-1}, & \text{if } \mathcal{I}_j \text{ is a transition region} \\ \mathbb{P}_{d_j}, & \text{otherwise.} \end{cases}$$

As before $\mathbb{P}_{d_j}$ is the space of polynomials of degree at most d_j, and ϕ is the frequency dependent change of variables on the transition intervals defined by

$$\phi(s) = \begin{cases} t_1 + \varphi(s)\, k^{\psi(s)}, & s \in I_{IT_1} \\ t_2 - \varphi(s)\, k^{\psi(s)}, & s \in I_{IT_2} \\ t_1 - \varphi(s)\, k^{\psi(s)}, & s \in I_{ST_1} \\ t_2 + \varphi(s)\, k^{\psi(s)}, & s \in I_{ST_2}. \end{cases}$$

Here φ is given by

$$\varphi(s) = \begin{cases} \xi_1 + (\xi_1' - \xi_1)\dfrac{s - a_1}{b_1 - a_1}, & s \in I_{IT_1} \\ \xi_2' + (\xi_2 - \xi_2')\dfrac{s - a_2}{b_2 - a_2}, & s \in I_{IT_2} \\ \zeta_1' + (\zeta_1 - \zeta_1')\dfrac{s - a_3}{b_3 - a_3}, & s \in I_{ST_1} \\ \zeta_2 + (\zeta_2' - \zeta_2)\dfrac{s - a_4}{b_4 - a_4}, & s \in I_{ST_2} \end{cases}$$

which maps every boundary point of the transition regions onto itself, and ψ is given by

$$\psi(s) = -\frac{1}{3} \begin{cases} \dfrac{b_1 - s}{b_1 - a_1}, & s \in I_{IT_1} \\ \dfrac{s - a_2}{b_2 - a_2}, & s \in I_{IT_2} \\ \dfrac{s - a_3}{b_3 - a_3}, & s \in I_{ST_1} \\ \dfrac{b_4 - s}{b_4 - a_4}, & s \in I_{ST_2}. \end{cases}$$

Note that when constructing these intervals, we choose $\mathcal{I}_j$ so that if any two of them have a nonempty intersection, then either they are the same intervals or they intersect each other at one boundary point. Furthermore, we have $\gamma\left(\cup_{j=1}^{J}\mathcal{I}_j\right) = \partial\mathcal{K}$. Therefore we can clearly identify $L^2(\partial\mathcal{K})$ and $L^2(\cup_{j=1}^{J}\mathcal{I}_j)$ through the parametrization γ.

So (4.4) needs to be modified according to this framework. In this respect, observe that the change of variables in the Galerkin formulation is equivalent to finding the unique $\hat{\eta} \in \mathcal{C}_{\mathbf{d}}$ such that

$$B(\hat{\mu}, \hat{\eta}) = F(\hat{\mu}), \quad \text{for all } \hat{\mu} \in \mathcal{C}_{\mathbf{d}}. \tag{7.1}$$

Hence most of the results in Chapter 4 hold with this new approximation space too. The analogues of Theorem 5.1 and Corollary 5.2 are as follows.

Theorem 7.1. *Given $k_0 > 1$ and $k \geq k_0$, suppose that the sesquilinear form B in (4.3) associated with the integral operator $\mathcal{R} = I - R$ in (2.6) is continuous with a continuity constant C and coercive with a coercivity constant c. Then, for all $n_j \in \{0, \ldots, d_j + 1\}$ $(j = 1, \ldots, J)$, we have*

$$\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})} \lesssim_{n_1,\ldots,n_J,k_0} \frac{C}{c}\, k \sum_{j=1}^{J} \frac{E(k,j)}{(d_j)^{n_j}}$$

for the Galerkin solution $\hat{\eta}$ to (7.1) where

$$E(k,j) = \begin{cases} (\log k)^{n_j+1/2}, & j = 1,2,3,4 \ (\textit{transition regions}) \\ k^{-1/6}, & j = 5,6 \qquad (\textit{shadow boundaries}). \end{cases}$$

If $J = 8$, then

$$E(k,j) = 1, \qquad j = 7,8, \quad (\textit{illuminated and shadow regions}).$$

Since η has the same asymptotic behaviour as k, Theorem 5.2 entails the following corollary.

Corollary 7.2. *Under the assumptions of Theorem 7.1, if the same polynomial degree $d = d_1 = \ldots = d_J$ is used on each interval, then for all $n \in \{0, \ldots, d+1\}$, there holds*

$$\frac{\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})}}{\|\eta\|_{L^2(\partial\mathcal{K})}} \lesssim_{n,k_0} \frac{C}{c} \frac{(\log k)^{n+1/2}}{d^n}$$

for the Galerkin solution $\hat{\eta}$ to (7.1).

According to Theorem 7.1 and Corollary 7.2 theoretically the change of variables Galerkin approximation spaces have improved convergence behaviour compared with the method discussed in Chapter 5. In addition, numerically this new method has better accuracy in shadow region. For the proof of Theorem 7.1, we need the following results.

Theorem 7.3 (Best approximation by algebraic polynomials [35])**.** *Given an interval $\mathcal{I} = (a, b)$ and $n \in \mathbb{Z}_+$, introduce the semi–norms (for suitable f) by*

$$|f|_{n,\mathcal{I}} = \left[\int_a^b |D^n f(s)|^2 (s-a)^n (b-s)^n \, ds\right]^{1/2}.$$

Then, for all $n \in \{0, \ldots, d+1\}$, following holds:

$$\inf_{p \in \mathbb{P}_d} \|f - p\| \lesssim_n |f|_{n,\mathcal{I}} \, d^{-n}.$$

Theorem 7.4. *For given $k_0 > 0$, we have*

$$\left|D_s^n \eta^{\text{slow}}(s,k)\right| \lesssim_{n,k_0} k + \sum_{m=4}^{n+2} \left(k^{-1/3} + |w(s)|\right)^{-m}$$

for all $n \in \mathbb{Z}_+$ and all $k \geq k_0$. Here $w(s) = (s - t_1)(t_2 - s)$.

Proof. The same estimate is shown to hold for all sufficiently large k in [22]. Thus the result follows from the continuous dependence of $D_s^n \eta^{\text{slow}}(s,k)$ on s and k. □

Proposition 7.5. *For given $k_0 > 1$, we have*

$$|D_s^n \phi| \lesssim_{n,k_0} (\log k)^n \, k^\psi \quad \text{on } \mathcal{I}_j \quad (j = 1,2,3,4)$$

for all $n \in \mathbb{N}$ and all $k \geq k_0$.

Proof. Since the argument is the same for $j = 1,2,3,4$, we prove the result for $j = 1$. Using $\varphi'' = \psi'' = 0$, we obtain

$$D_s^n \phi = \sum_{l=0}^{n} \binom{n}{l} D_s^{n-l} \varphi \; D_s^l k^\psi = \varphi \; D_s^n k^\psi + D_s^1 \varphi \; D_s^{n-1} k^\psi, \qquad n \geq 1 \tag{7.2}$$

and

$$D_s^n k^\psi = \left(D_s^1 \psi\right)^n (\log k)^n \, k^\psi, \qquad n \geq 0. \tag{7.3}$$

Next, plugging (7.3) in (7.2), we get

$$D_s^n \phi = \left(\varphi \; D_s^1 \psi \; \log k + D_s^1 \varphi\right) \left(D_s^1 \psi\right)^{n-1} (\log k)^{n-1} \, k^\psi, \qquad n \geq 1. \tag{7.4}$$

Since, for $k \geq k_0 > 1$, we have

$$\frac{1}{|b_1 - a_1|} = \frac{1}{\xi_1' - \xi_1 k^{-1/3}} \leq \frac{1}{\xi_1' - \xi_1 k_0^{-1/3}} \lesssim_{k_0} 1$$

for $s \in I_1 = I_{IT_1}$, we obtain

$$\left|D_s^1 \psi\right| = \frac{1}{3} \frac{1}{b_1 - a_1} \lesssim_{k_0} 1 \qquad \text{and} \qquad \left|D_s^1 \varphi\right| = \frac{\xi_1' - \xi_1}{b_1 - a_1} \lesssim_{k_0} 1.$$

Note further that $\xi_1 \le \varphi \le \xi_1'$. Use of these in (7.4) completes the proof. □

Proposition 7.6. *For given $k_0 > 0$, the inequality*

$$\left|D_s^n(\eta^{slow} \circ \phi)\right| \lesssim_{n,k_0} k\,(\log k)^n \quad \text{on } \mathcal{I}_j \quad (j = 1,2,3,4)$$

holds for all $n \in \mathbb{N}$ and all $k \ge k_0$.

Proof. Let us fix $k \ge k_0 > 1$ and also $j = 1, \dots, 4$. Faá Di Bruno's formula for the derivatives of a composition reads

$$D^n\left(f \circ g\right)(t) = \sum_{\{m_\ell\}} (D^m f)(g(t)) \prod_{\ell=1}^{n} \frac{\ell}{m_\ell!} \left(\frac{D^\ell g(t)}{\ell!}\right)^{m_\ell}.$$

Here the summation is over all non-negative integers m_ℓ satisfying $n = \sum_{\ell=1}^n \ell m_\ell$, and $m = \sum_{\ell=1}^n m_\ell$. Therefore

$$\left|D_s^n\left(\eta^{\text{slow}} \circ \phi\right)\right| \lesssim_n \sum_{\{m_\ell\}} \left|(D_s^m \eta^{\text{slow}})(\phi)\right| \prod_{\ell=1}^{n} \left|D_s^\ell \phi\right|^{m_\ell}$$

so Proposition 7.5 yields

$$\left|D_s^n\left(\eta^{\text{slow}} \circ \phi\right)\right| \lesssim_{n,k_0} \sum_{\{m_\ell\}} \left|(D_s^m \eta^{\text{slow}})(\phi)\right| \prod_{\ell=1}^{n} \left((\log k)^\ell\, k^\psi\right)^{m_\ell}.$$

Accordingly, we have

$$\begin{aligned}\left|D_s^n\left(\eta^{\text{slow}} \circ \phi\right)\right| &\lesssim_{n,k_0} (\log k)^n \sum_{\{m_\ell\}} \left|(D_s^m \eta^{\text{slow}})(\phi)\right| k^{m\psi} \\ &\lesssim_{n,k_0} (\log k)^n \sum_{m=0}^{n} \left|(D_s^m \eta^{\text{slow}})(\phi)\right| k^{m\psi}.\end{aligned}$$

Thus it suffices to show that

$$\left|(D_s^m \eta^{\text{slow}})(\phi)\right| k^{m\psi} \lesssim_{m,k_0} k$$

holds for all non-negative integers m. Indeed, if $0 \leq \ell \leq m$, then

$$\begin{aligned}
\left(k^{-1/3} + |\omega(\phi)|\right)^{-\ell} &= \left(k^{-1/3} + |\omega(\phi)|\right)^{-m} \left(k^{-1/3} + |\omega(\phi)|\right)^{m-\ell} \\
&\leq \left(k^{-1/3} + |\omega(\phi)|\right)^{-m} \left(k_0^{-1/3} + L^2\right)^{m-\ell} \\
&\lesssim_{m,k_0} \left(k^{-1/3} + |\omega(\phi)|\right)^{-m}
\end{aligned}$$

so Theorem 7.4 combined with the inequality $\psi \leq 0$ implies

$$\begin{aligned}
\left|D_s^m \eta^{\text{slow}}(\phi)\right| k^{m\psi} &\lesssim_{m,k_0} \left[k + \sum_{\ell=4}^{m+2} \left(k^{-1/3} + |\omega(\phi)|\right)^{-\ell}\right] k^{m\psi} \\
&\lesssim_{m,k_0} \left[k + \left(k^{-1/3} + |\omega(\phi)|\right)^{-(m+2)}\right] k^{m\psi} \\
&\lesssim_{m,k_0} k + \left(\frac{k^\psi}{k^{-1/3} + |\omega(\phi)|}\right)^m \left(k^{-1/3} + |\omega(\phi)|\right)^{-2} \\
&\lesssim_{m,k_0} k + \left(\frac{k^\psi}{|\omega(\phi)|}\right)^m k^{2/3}.
\end{aligned}$$

It remains to show that $k^\psi / |\omega(\phi)|$ is bounded independent of k. As this requires the same argument for $j = 1, 2, 3, 4$, we concentrate on $j = 1$. In this case, we have $\phi - t_1 = \varphi k^\psi$, $\varphi \geq \xi_1 > 0$ and $t_2 - \phi \geq t_2 - (t_1 + \xi_1') \geq \xi_2' > 0$ and therefore

$$\frac{k^\psi}{|\omega(\phi)|} = \frac{k^\psi}{(\phi - t_1)(t_2 - \phi)} = \frac{k^\psi}{\varphi k^\psi (t_2 - \phi)} \leq \frac{1}{\xi_1 \xi_2'}$$

completing the proof. □

Corollary 7.7. *On the transition intervals ($j = 1, 2, 3, 4$), for a given $k_0 > 1$, we have*

$$\left|\eta^{\text{slow}} \circ \phi\right|_{n,\mathcal{I}_j} \lesssim_{n,k_0} k (\log k)^n$$

for all $n \in \mathbb{N}$ and all $k \geq k_0$.

Proof. We use Proposition 7.6 combined with the inequality $0 < b_j - a_j < L$ to obtain

$$\begin{aligned}\left|\eta^{\text{slow}} \circ \phi\right|^2_{n,\mathcal{I}_j} &= \int_{a_j}^{b_j} \left|D_s^n \left(\eta^{\text{slow}} \circ \phi\right)(s)\right|^2 (s-a_j)^n (b_j - s)^n \, ds \\ &\lesssim_{n,k_0} k^2 (\log k)^{2n} \int_{a_j}^{b_j} (s-a_j)^n (b_j - s)^n \, ds \\ &\lesssim_{n,k_0} k^2 (\log k)^{2n}.\end{aligned}$$

Taking square roots we obtain the desired result. □

Proof. (of Theorem 7.1): Céa's lemma implies that

$$\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})} \leq \frac{C_k}{c_k} \inf_{\hat{\mu} \in \mathcal{C}_{\mathbf{d}}} \|\eta - \hat{\mu}\|_{L^2(\partial\mathcal{K})} \tag{7.5}$$

where $\hat{\eta}$ is the solution of (7.1). Since we identify $L^2(\partial\mathcal{K})$ with $L^2\left(\cup_{j=1}^J \mathcal{I}_j\right)$, we have

$$\|\eta - \hat{\mu}\|_{L^2(\partial\mathcal{K})} = \|\eta - \hat{\mu}\|_{L^2\left(\cup_{j=1}^J \mathcal{I}_j\right)} \leq \sum_{j=1}^J \|\eta - \hat{\mu}\|_{L^2(\mathcal{I}_j)}$$

for any $\hat{\mu} \in \mathcal{C}_{\mathbf{d}}$. Therefore the particular change of variables employed in the Galerkin approximation spaces $\mathcal{C}_{\mathbf{d}}$ implies

$$\inf_{\hat{\mu} \in \mathcal{C}_{\mathbf{d}}} \|\eta - \hat{\mu}\|_{L^2(\partial\mathcal{K})} \leq \sum_{j=1}^4 \inf_{p \in \mathbb{P}_{d_j}} \|\eta^{\text{slow}} - p \circ \phi^{-1}\|_{L^2(\mathcal{I}_j)} + \sum_{j=5}^J \inf_{p \in \mathbb{P}_{d_j}} \|\eta^{\text{slow}} - p\|_{L^2(\mathcal{I}_j)}. \tag{7.6}$$

Further, for any $p \in \mathbb{P}_{d_j}$, using Proposition 7.5 and the inequality $k^\psi < 1$, we obtain

$$\begin{aligned}\|\eta^{\text{slow}} - p \circ \phi^{-1}\|^2_{L^2(\mathcal{I}_j)} &= \int_{a_j}^{b_j} \left|\left(\eta^{\text{slow}} - p \circ \phi^{-1}\right)(s)\right|^2 ds \\ &= \int_{a_j}^{b_j} \left|\left(\eta^{\text{slow}} \circ \phi - p\right)(s)\right|^2 D_s^1 \phi(s) \, ds \\ &\lesssim_{k_0} \log k \, \|\eta^{\text{slow}} \circ \phi - p\|^2_{L^2(\mathcal{I}_j)}.\end{aligned}$$

This inequality when combined with the inequalities (7.5) and (7.6) implies

$$\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})} \lesssim_{k_0} \frac{C_k}{c_k} \left\{ \sum_{j=1}^{4} (\log k)^{1/2} \inf_{p\in\mathbb{P}_{d_j}} \|\eta^{\text{slow}} \circ \phi - p\|_{L^2(\mathcal{I}_j)} + \sum_{j=5}^{J} \inf_{p\in\mathbb{P}_{d_j}} \|\eta^{\text{slow}} - p\|_{L^2(\mathcal{I}_j)} \right\}$$

so that Theorem 7.3 entails

$$\|\eta - \hat{\eta}\|_{L^2(\partial\mathcal{K})} \lesssim_{n_1,\ldots,n_J,k_0} \frac{C_k}{c_k} \left\{ \sum_{j=1}^{4} (\log k)^{1/2} \left|\eta^{\text{slow}} \circ \phi\right|_{n_j,\mathcal{I}_j} d_j^{-n_j} + \sum_{j=5}^{J} \left|\eta^{\text{slow}}\right|_{n_j,\mathcal{I}_j} d_j^{-n_j} \right\}.$$

Accordingly, we have to show that

$$\left|\eta^{\text{slow}} \circ \phi\right|_{n_j,\mathcal{I}_j} \lesssim_{n_j,k_0} k\,(\log k)^{n_j}\,, \qquad j = 1,2,3,4 \tag{7.7}$$

and

$$\left|\eta^{\text{slow}}\right|_{n_j,\mathcal{I}_j} \lesssim_{n_j,k_0} k\,k^{-1/6}, \qquad j = 5,6 \tag{7.8}$$

and, in case $J = 8$,

$$\left|\eta^{\text{slow}}\right|_{n_j,\mathcal{I}_j} \lesssim_{n_j,k_0} k, \qquad j = 7,8.$$

Inequalities in (7.7) are exactly the content of Corollary 7.7. For $j = 5,6$, Theorem 7.4 implies

$$\left|D_s^{n_j}\eta^{\text{slow}}(s,k)\right| \lesssim_{n_j,k_0} k + \sum_{m=4}^{n_j+2} \left(k^{-1/3} + |w(s)|\right)^{-m} \lesssim_{n_j,k_0} k + k^{(n_j+2)/3}$$

so that

$$\begin{aligned}\left|\eta^{\text{slow}}\right|_{n_j,\mathcal{I}_j}^2 &= \int_{a_j}^{b_j} \left|D_s^{n_j}\eta^{\text{slow}}(s)\right|^2 (s-a_j)^{n_j}\,(b_j-s)^{n_j}\,ds \\ &\lesssim_{n_j,k_0} \left(k + k^{(n_j+2)/3}\right)^2 (b_j-a_j)^{2n_j+1} \\ &\lesssim_{n_j,k_0} \left(k + k^{(n_j+2)/3}\right)^2 \left(k^{-1/3}\right)^{2n_j+1} \\ &\lesssim_{n_j,k_0} \left(k\,k^{-1/6}\right)^2.\end{aligned}$$

Taking square roots delivers inequalities in (7.8). Therefore the proof is complete when $J = 6$.

For $J = 8$, we appeal to Theorem 7.4 for $j = 7, 8$, and thereby obtain

$$\left|D_s^{n_j}\eta^{\text{slow}}(s,k)\right| \lesssim_{n_j,k_0} k + \sum_{m=4}^{n_j+2} \left(k^{-1/3} + |w(s)|\right)^{-m} \lesssim_{n_j,k_0} k + \sum_{m=4}^{n_j+2} |w(s)|^{-m} \lesssim_{n_j,k_0} k.$$

Thus it follows that

$$\begin{aligned}\left|\eta^{\text{slow}}\right|_{n_j,\mathcal{I}_j}^2 &= \int_{a_j}^{b_j} \left|D_s^{n_j}\eta^{\text{slow}}(s)\right|^2 (s-a_j)^{n_j}\,(b_j-s)^{n_j}\,ds \\ &\lesssim_{n_j,k_0} k^2\,(b_j-a_j)^{2n_j+1} \\ &\lesssim_{n_j,k_0} k^2\end{aligned}$$

which completes the proof when $J = 8$. □

8. NUMERICAL RESULTS

In this chapter we present numerical experiments to validate the theoretical developments in this thesis. Indeed, all our tests have shown that the numerical results based on implementations of the three different algorithms developed in Chapters 5,6 and 7 display only very minor differences. Therefore here we present only the numerical results that relate to the multiple-scattering change of variables Galerkin boundary element methods discussed in Chapter 5.

We consider two different multiple scattering configurations. The first one consists of two unit circles centered on the y-axis and separated by a distance of 1, and the direction of incidence is $\alpha = (1, 0)$. We first present the logarithmic relative errors

$$\log_{10}\left(\frac{\|\eta_{\tau_m} - \hat{\eta}_{\tau_m}\|_{L^2(\partial K_{\tau_m})}}{\|\eta_{\tau_m}\|_{L^2(\partial K_{\tau_m})}}\right)$$

for the wave-numbers $k = 50, 100, 200, 400, 800$ versus the local polynomial degrees $p = 4, 8, 12, 16, 20$ on the two paths

$$(K_{\tau_m})_{m\geq 0} = (K_1, K_2, K_1, K_2, \ldots)$$

and

$$(K_{\tau_m})_{m\geq 0} = (K_2, K_1, K_2, K_1, \ldots)$$

for the reflection numbers

$$m = 0, 1, 10, 11, 20, 21.$$

Figures 8.1 and 8.2 correspond to the numerical results associated with the first path.

Specifically, Figure 8.1 we presents the results when $\eta_{\tau_0}, \ldots, \eta_{\tau_{m-1}}$ and the right-hand sides in the integral equations (3.1)-(3.2) are computed through use of a combination of Nyström and trapezoidal quadrature rules, and η_{τ_m} is constructed via the numerical solution of (3.1)-(3.2) by the multiple scattering change of variables Galerkin boundary element methods developed in Chapter 7. On the other hand, the results in Figure 8.2 correspond to the solutions obtained by an implementation of the multiple scattering change of variables Galerkin boundary element methods at each reflection. Note that a simple comparison of these figures depict that the second approach results in a minor loss of accuracy only even though the right-hand sides in the second approach are significantly restricted in accuracy as compared to those of the first one. This clearly displays that the multiple scattering change of variables Galerkin boundary element methods are immune to the loss of accuracy in the right-hand sides of the integral equations (3.1)-(3.2), and this is clearly related with the smoothing characters of the integral operators $R_{jj'}$ with $j \neq j'$.

Similarly, Figures 8.3 and 8.4 present the same comparison as in Figures 8.1 and 8.2 but for the second path. As expected due to the symmetrical structure of the configuration with respect to the direction of incidence, the numerical results are in harmony with those corresponding to the first path.

Finally, in Figures 8.5 and 8.6, we present the logarithmic relative errors

$$\log_{10}\left(\frac{\|\eta - \hat{\eta}^{(m)}\|_{L^2(\partial K)}}{\|\eta\|_{L^2(\partial K)}}\right)$$

for the wave-numbers $k = 50, 100, 200, 400, 800$ versus the number of terms m used in the Neumann series solution of the multiple scattering problem for the local polynomial degrees $p = 4, 8, 12, 16, 20$. Note specifically that here while η is a highly accurate reference solution obtained by a direct solution of the multiple scattering problem using a combination of Nystöm and trapezoidal quadrature rules [30], $\hat{\eta}^{(m)}$ is exactly the sum of the first m-terms in the Neumann series solution where each term is associated with

the corresponding iterates in the two paths and numerically computed in Figure 8.5 using the Nyström solutions as in Figures 8.1 and 8.3, and in Figure 8.6 the Galerkin solutions as in Figures 8.2 and 8.4.

The second configuration we consider consists of the ellipses with parametrizations

$$(\frac{3}{2}\cos t, \frac{1}{2}\sin t)$$

and

$$(\frac{3}{2}\cos t, \frac{1}{3}\sin t)$$

and rotated in the counter clockwise direction by an angle of $\pi/6$ and $\pi/12$ radians respectively, and the second one shifted with the vector $s = (0.5, -1.6)$ (see Figure 8.11). The direction of incidence is taken to be $\alpha = (1, 0)$. Figures 8.7–8.12 are arranged in exactly the same way as Figures 8.1–8.6 for the configuration consisting of two circles considered above, and they clearly depict that the performance of the Galerkin boundary element methods developed in this thesis is not influenced by geometrical changes.

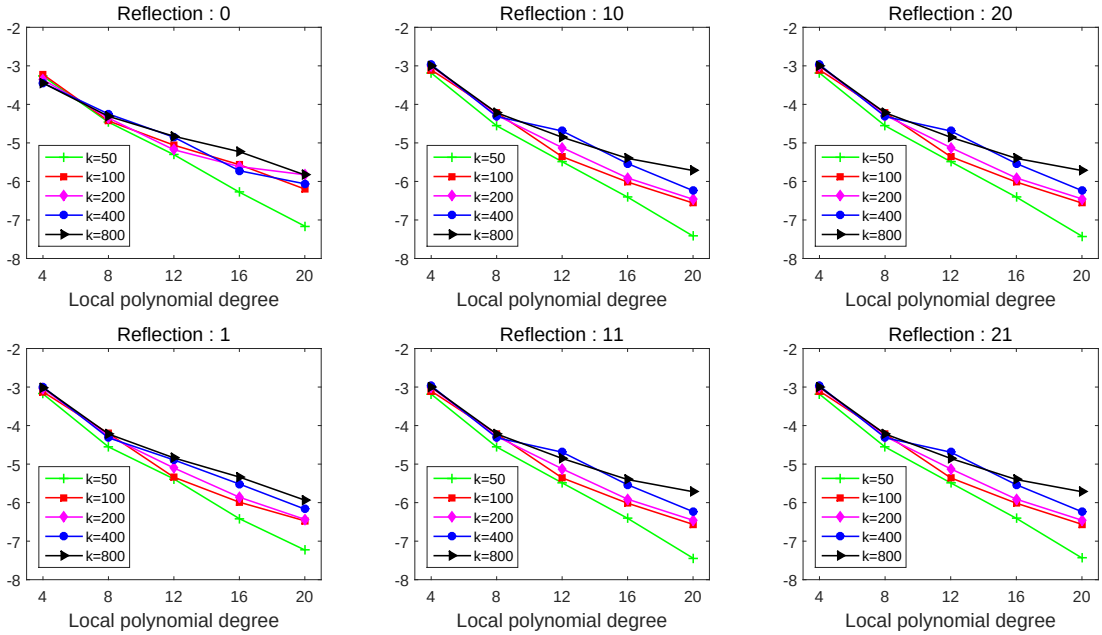

Figure 8.1. Relative errors using Nyström solutions as a right hand side for two circles: First path.

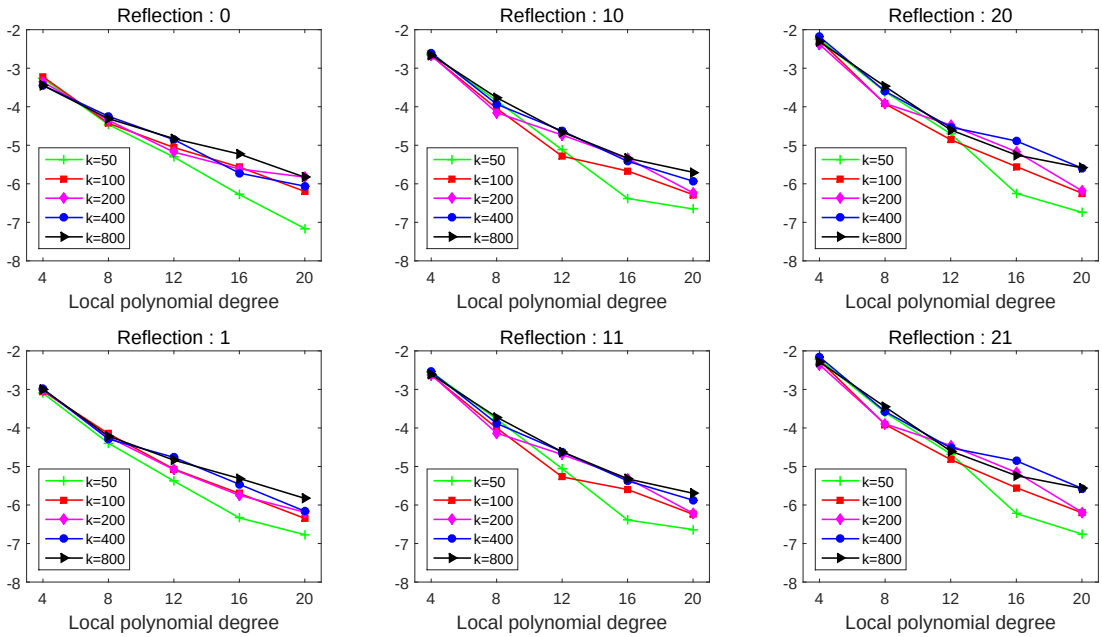

Figure 8.2. Relative errors using Galerkin solutions as a right hand side for two circles: First path.

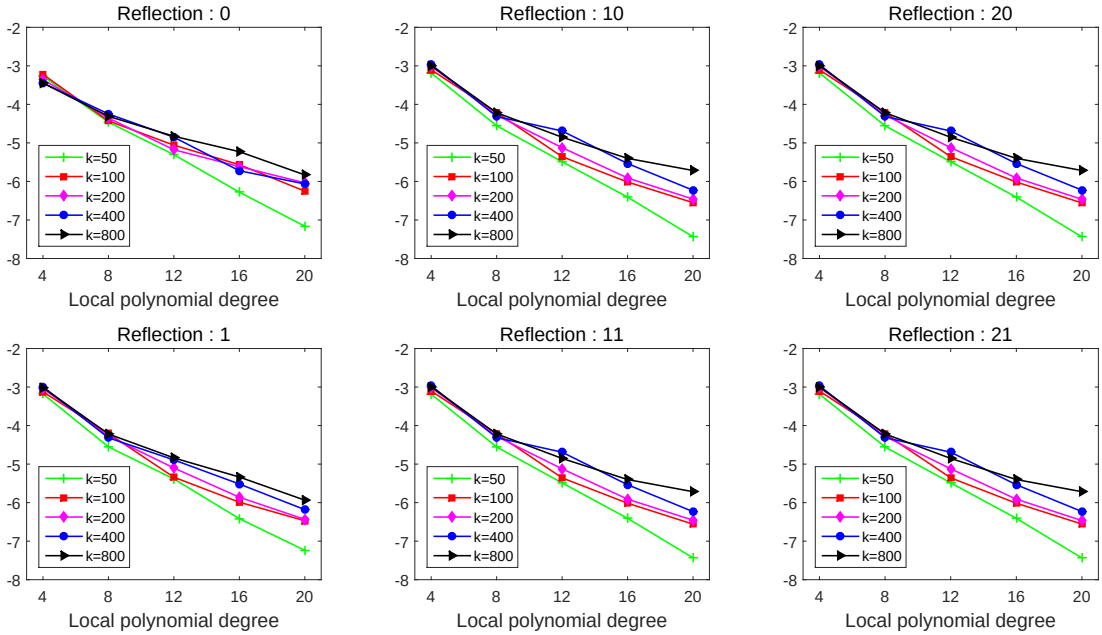

Figure 8.3. Relative errors using Nyström solutions as a right hand side for two circles: Second path.

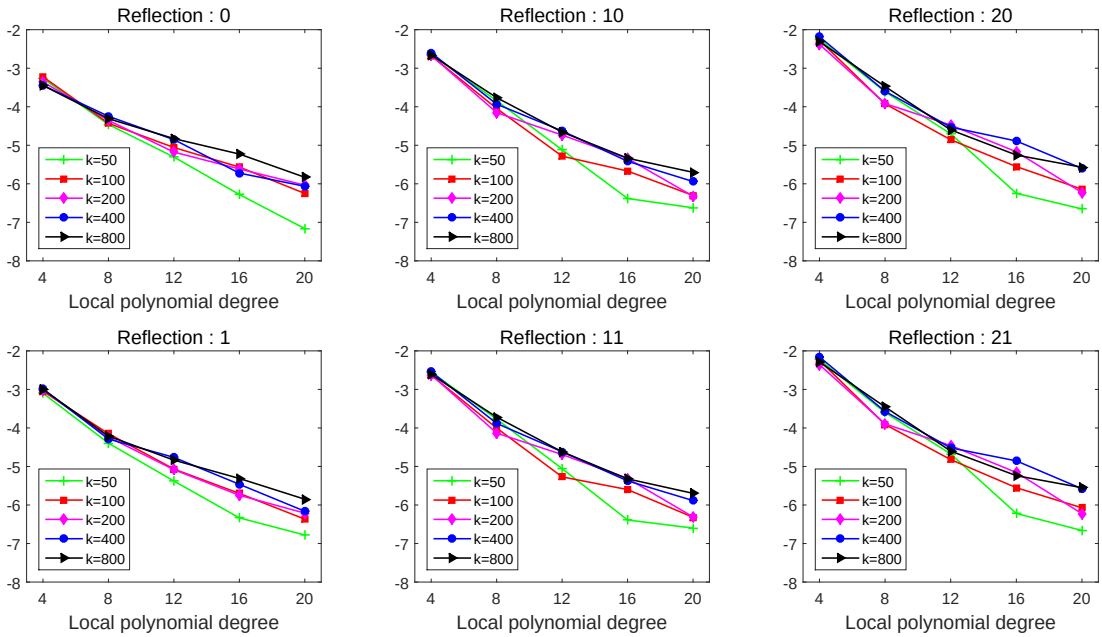

Figure 8.4. Relative errors using Galerkin solutions as a right hand side for two circles: Second path.

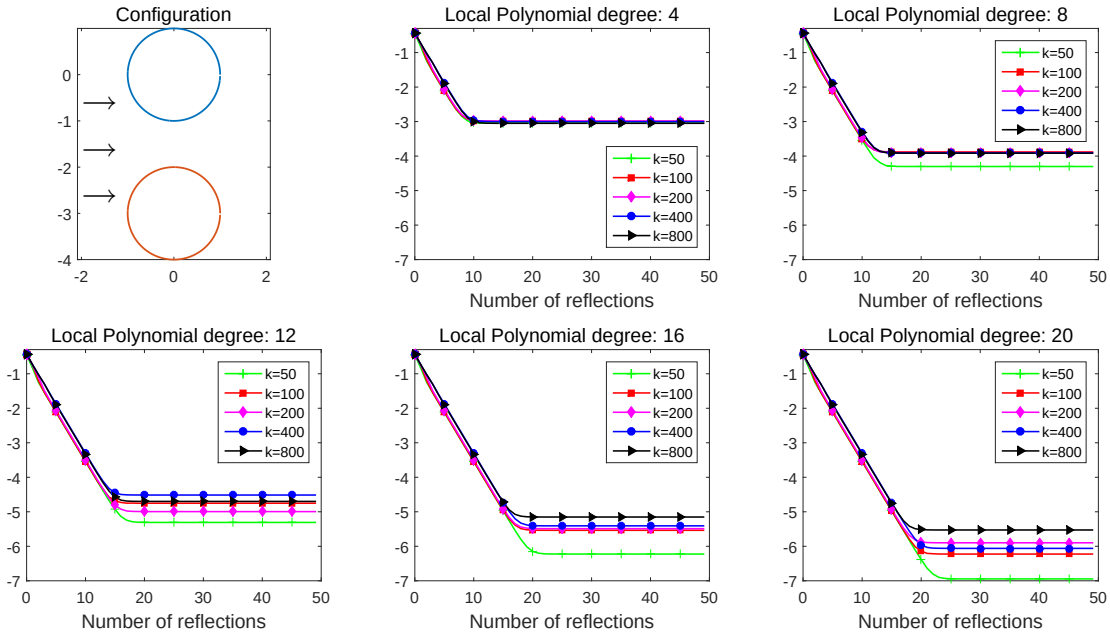

Figure 8.5. Sum of Galerkin solutions obtained by using Nyström solutions as a right hand side for two circles.

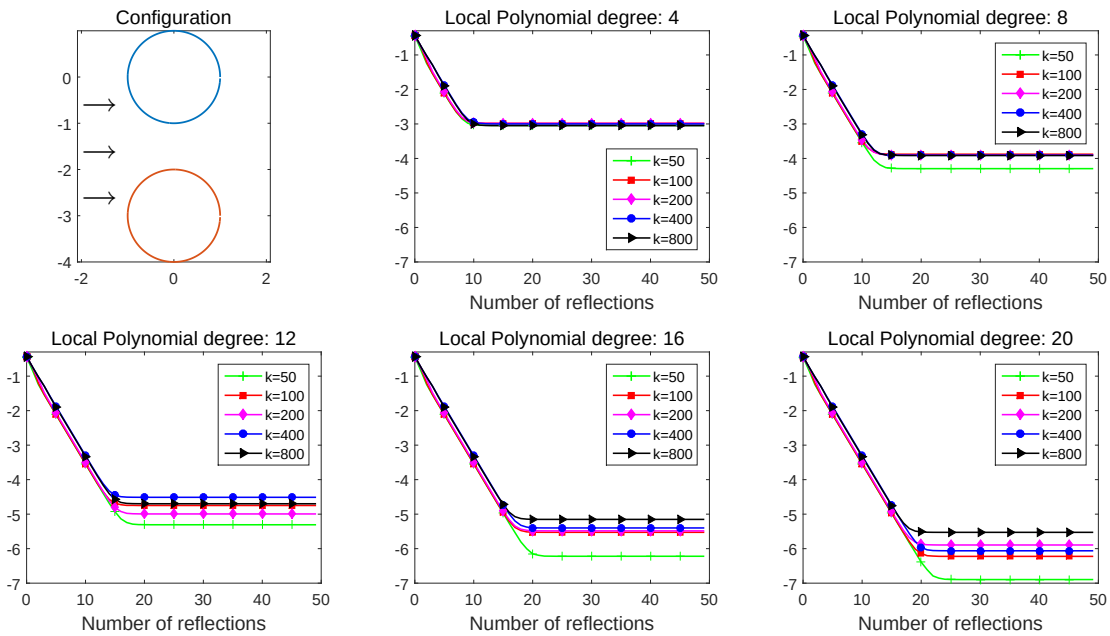

Figure 8.6. Sum of Galerkin solutions obtained by using Galerkin solutions as a right hand side for two circles.

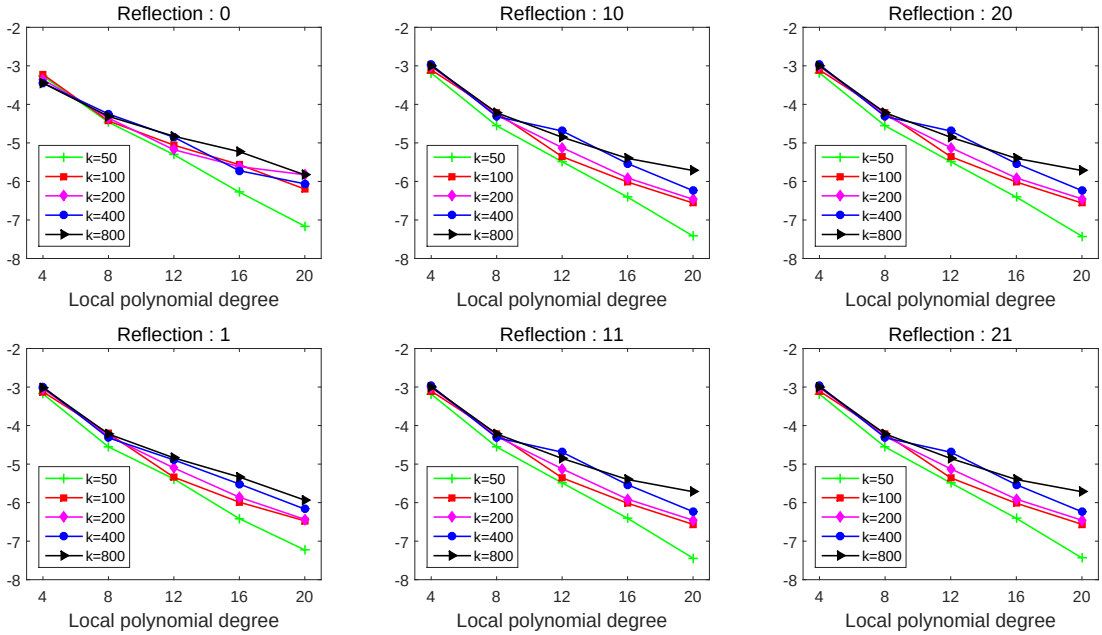

Figure 8.7. Relative errors using Nyström solutions as a right hand side for two ellipses: First path.

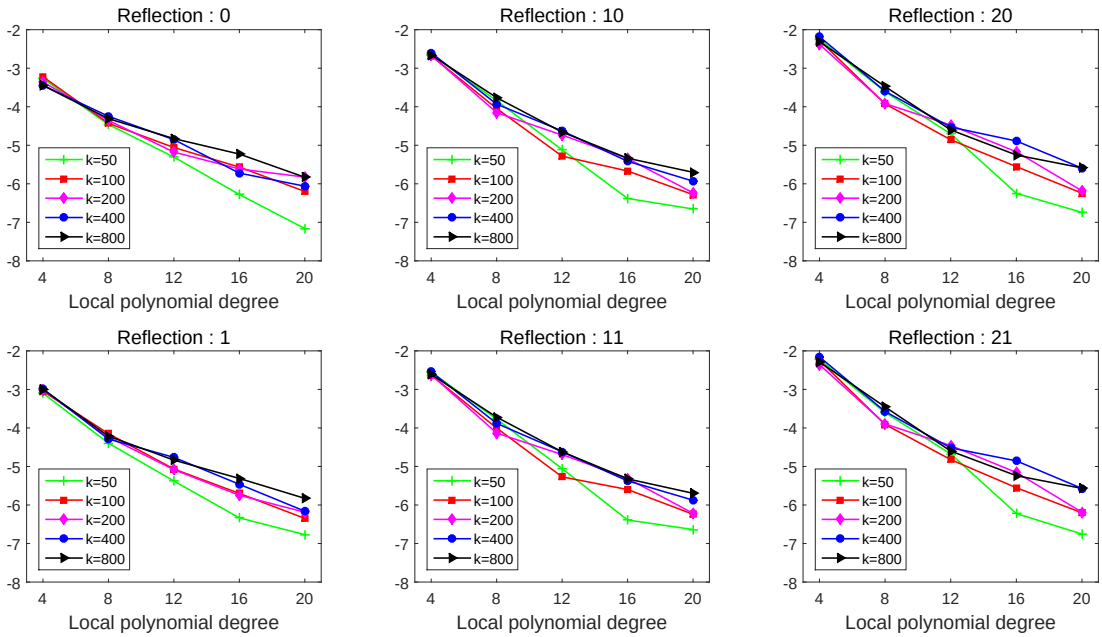

Figure 8.8. Relative errors using Galerkin solutions as a right hand side for two ellipses: First path.

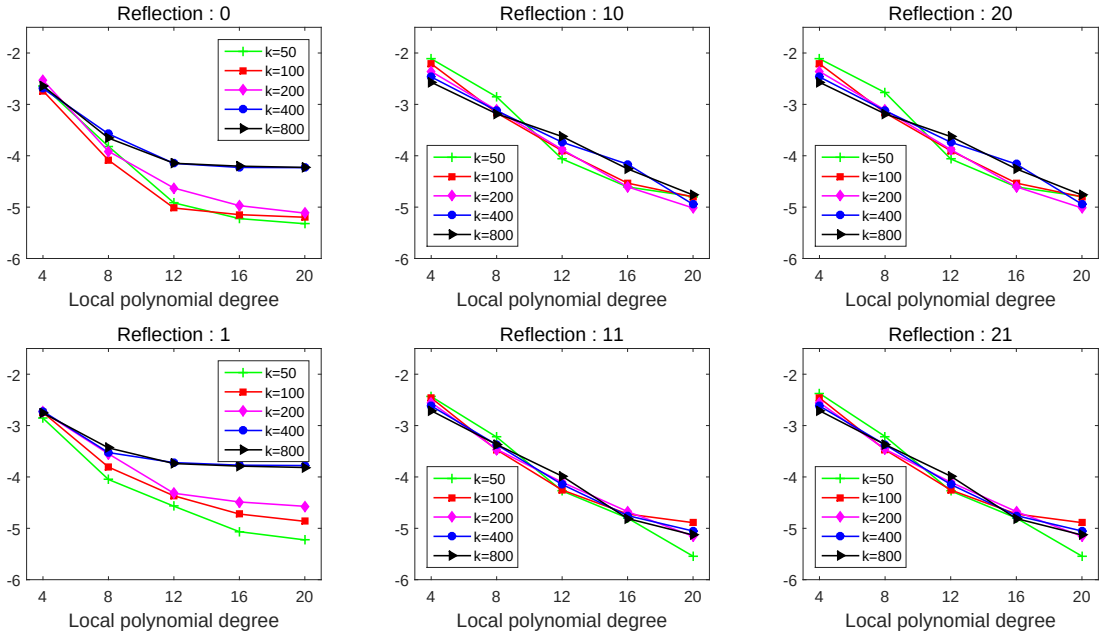

Figure 8.9. Relative errors using Nyström solutions as a right hand side for two ellipses: Second path.

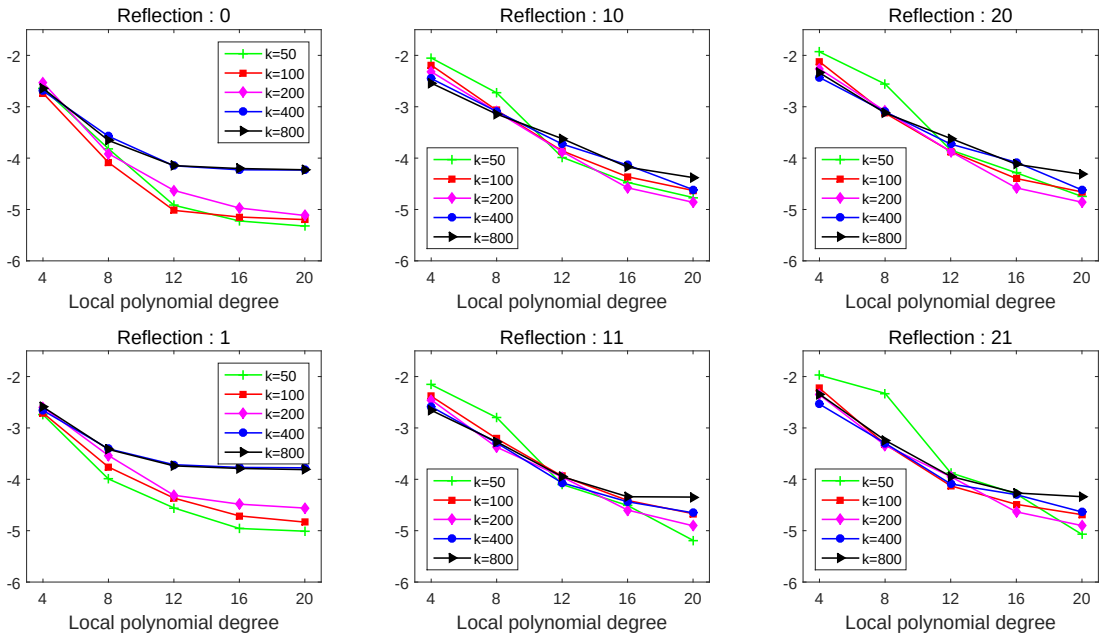

Figure 8.10. Relative errors using Galerkin solutions as a right hand side for two ellipses: Second path.

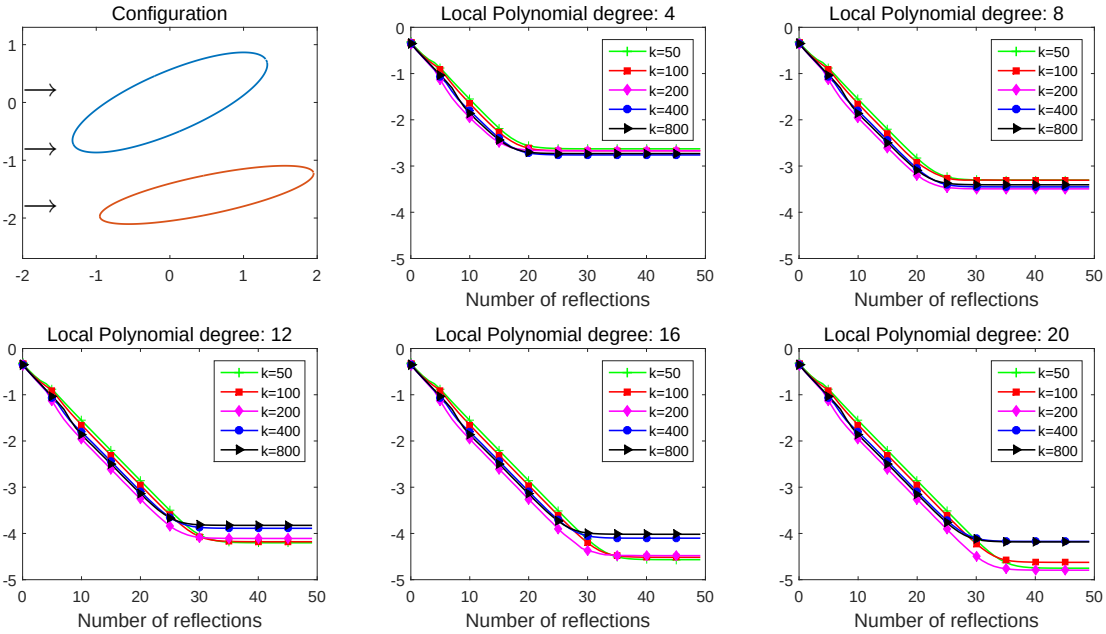

Figure 8.11. Sum of Galerkin solutions obtained by using Nyström solutions as a right hand side for two ellipses.

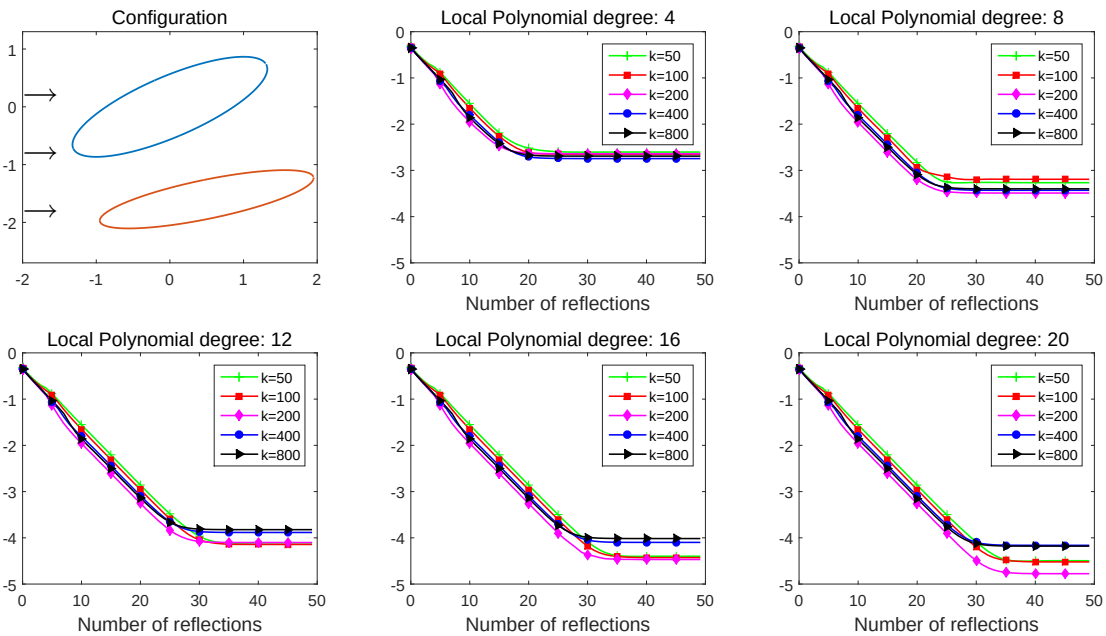

Figure 8.12. Sum of Galerkin solutions obtained by using Galerkin solutions as a right hand side for two ellipses.

9. CONCLUSION

In this thesis, we developed Galerkin boundary element methods for the efficient and rigorous numerical solution of multiple scattering problems in the exterior of two-dimensional smooth compact obstacles consisting of two disjoint strictly convex structures. From a theoretical point of view, the Galerkin approximation spaces developed herein have the property that the number of degrees of freedom to obtain a prescribed accuracy independent of frequency needs to increase as $\mathcal{O}(k^{\epsilon})$ (for any $\epsilon > 0$) with increasing wave-number k. Accordingly, the algorithms developed in this thesis fill an important gap in the literature as they provide the very first examples of rigorously error controllable algorithms based on extraction of the phases of associated multiple scattering iterates.

Possible future directions of research include the extension of the algorithms generated in this thesis to multiple scattering configurations consisting of several smooth convex obstacles, and the development of acceleration strategies for the efficient truncation of the Neumann series in this case.

REFERENCES

1. Boffi, D., "Finite Element Approximation of Eigenvalue Problems", *Acta Numerica*, Vol. 19, pp. 1–120, 2010.

2. Davies, R. W., K. Morgan, and O. Hassan, "A High Order Hybrid Finite Element Method Applied to the Solution of Electromagnetic Wave Scattering Problems in the Time Domain", *Computational Mechanics*, Vol. 44, No. 3, pp. 321–331, 2009.

3. Hesthaven, J. S., and T. Warburton, "High-Order Accurate Methods for Time-Domain Electromagnetics", *CMES Computer Modeling in Engineering and Sciences*, Vol. 5, pp. 395–408, 2004.

4. Banjai, L., and W. Hackbusch, "Hierarchical Matrix Techniques for Low- and High-Frequency Helmholtz Problems", *IMA Journal of Numerical Analysis*, Vol. 28, No. 1, pp. 46–79, 2008.

5. Bruno, O. P., V. Domínguez, and F.-J. Sayas, "Convergence Analysis of a High-Order Nyström Integral-Equation Method for Surface Scattering Problems", *Numerische Mathematik*, Vol. 124, No. 4, pp. 603–645, 2013.

6. Bruno, O. P., and L. A. Kunyansky, "A Fast, High-Order Algorithm for the Solution of Surface Scattering Problems: Basic Implementation, Tests and Applications", *Journal of Computational Physics*, Vol. 169, pp. 80–110, 2001.

7. Tong, M. S., and W. C. Chew, "Multilevel Fast Multipole Acceleration in the Nyström Discretization of Surface Electromagnetic Integral Equations for Composite Objects", *IEEE Transactions on Antennas and Propagation*, Vol. 58, No. 10, pp. 3411–3416, 2010.

8. Amini, S., and A. Profit, "Multi-Level Fast Multipole Solution of the Scattering

Problem", *Engineering Analysis with Boundary Elements*, Vol. 27, pp. 547–564, 2003.

9. Antoine, X., "Advances in the on-Surface Radiation Condition Method: Theory, Numerics and Applications", *Computational Methods for Acoustics Problems*, pp. 207–232, Saxe-Coburg Publications, 2008.

10. Grote, M. J., and C. Kirsch, "Nonreflecting Boundary Conditions for Time-Dependent Multiple Scattering", *Journal of Computational Physics*, Vol. 221, No. 1, pp. 41–62, 2007.

11. Engquist, B., and A. Majda, "Absorbing Boundary Conditions for the Numerical Simulation of Waves", *Mathematics of Computation*, Vol. 31, No. 139, pp. 629–651, 1977.

12. Givoli, D., "High-Order Local Non-Reflecting Boundary Conditions: A Review", *Wave Motion*, Vol. 39, No. 4, pp. 319–326, 2004.

13. Grote, M., and I. Sim, "Local Non-reflecting Boundary Condition for Time-Dependent Multiple Scattering", *Journal of Computational Physics*, Vol. 230, pp. 3135–3154, 2011.

14. Colton, D., and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, 1992.

15. Abboud, T., J.-C. Nédélec, and B. Zhou, "Méthode des Équations Intégrales pour les Hautes Fréquences", *Comptes Rendus de l'Académie des Sciences*, Vol. 318, pp. 165–170, 1994.

16. Abboud, T., J.-C. Nédélec, and B. Zhou, "Improvement of the Integral Equation Method for High Frequency Problems", *Proceedings of 3rd International Conference on Mathematical Aspects of Wave Propagation Problems*, 1995.

17. Fedoryuk, M. V., "The Stationary Phase Method and Pseudo Differential Operators", *Russian Mathematical Surveys*, Vol. 26, No. 1, pp. 65–115, 1971.

18. Bruno, O. P., C. A. Geuzaine, J. A. Monroe, and F. Reitich, "Prescribed Error Tolerances within Fixed Computational Times for Scattering Problems of Arbitrarily High Frequency: The Convex Case", *Philosophical Transactions of The Royal Society*, Vol. 362, pp. 629–645, 2004.

19. Bruno, O. P., and C. A. Geuzaine, "An $\mathcal{O}(1)$ Integration Scheme for Three-Dimensional Surface Scattering Problems", *Journal of Computational and Applied Mathematics*, Vol. 204, pp. 463–476, 2007.

20. Giladi, E., "An Asymptotically Derived Boundary Element Method for the Helmholtz Equation in High Frequencies", *Journal of Computational and Applied Mathematics*, Vol. 198, pp. 52–74, 2007.

21. Huybrechs, D., and S. Vandewalle, "A Sparse Discretization for Integral Equation Formulations of High Frequency Scattering Problems", *SIAM Journal on Scientific Computing*, Vol. 29, No. 6, pp. 2305–2328, 2007.

22. Domínguez, V., I. G. Graham, and V. P. Smyshlyaev, "A Hybrid Numerical-Asymptotic Boundary Integral Method for High-Frequency Acoustic Scattering", *Numerische Mathematik*, Vol. 106, No. 3, pp. 471–510, 2007.

23. Ecevit, F., and H. Ç. Özen., "Frequency-Adapted Galerkin Boundary Element Methods for Convex Scattering Problems", *Numerische Mathematik*, DOI: 10.1007/s00211-016-0800-7, 2016.

24. Ecevit, F., and H. H. Eruslu, *Change of Variables Galerkin Schemes for High-Frequency Single-Scattering Problems*, Submitted.

25. Bruno, O. P., C. A. Geuzaine, and F. Reitich, "On the $\mathcal{O}(1)$ Solution of Multiple-

Scattering Problems, *IEEE Transactions on Magnetics*, Vol. 41, pp. 1488–1491, 2005.

26. Ecevit, F., and F. Reitich, "Analysis of Multiple Scattering Iterations for High-Frequency Scattering Problems. I. The Two-Dimensional Case", *Numerische Mathematik*, Vol. 114, No. 2, pp. 271–354, 2009.

27. Anand, A., Y. Boubendir, F. Ecevit, and F. Reitich, "Analysis of Multiple Scattering Iterations for High-Frequency Scattering Problems. II. The Three-Dimensional Scalar Case", *Numerische Mathematik*, Vol. 114, No. 3, pp. 373–427, 2010.

28. Chandler-Wilde, S. N., I. G. Graham, S. Langdon, and E. A. Spence, "Numerical-Asymptotic Boundary Integral Methods in High-Frequency Acoustic Scattering", *Acta Numerica*, Vol. 21, pp. 89–305, 2012.

29. Chandler-Wilde, S. N., D. P. Hewett, S. Langdon, and A. Twigger, "A High Frequency Boundary Element Method for Scattering by a Class of Nonconvex Obstacles", *Numerische Mathematik*, Vol. 129, No.4, pp. 647–689, 2015.

30. Kress, R., *Linear Integral Equations*, Springer, New York, 1999.

31. Spence, E. A., S. N. Chandler-Wilde, I. G. Graham, and V. P. Smyshlyaev, "A New Frequency-Uniform Coercive Boundary Integral Equation for Acoustic Scattering", *Communications on Pure and Applied Mathematics*, Vol. 64, No. 1, pp. 1384–1415, 2011.

32. Melrose, R. B., and M. E. Taylor, "Near Peak Scattering and the Corrected Kirchhoff Approximation for a Convex Obstacle", *Advances in Mathematics.*, Vol. 55, pp. 242–315, 1985.

33. Olver, P., *Orthogonal Bases and the QR Algorithm.*, 2016, `http://www.math.umn.edu/~olver/aims_/qr.pdf`, accessed at May 2016.

34. Özen, H. Ç., *Robust High-Frequency Solvers*, M.S. Thesis, Boğaziçi University, 2012.

35. Schwab, C., *p– and hp–Finite Element Methods. Theory and Applications in Solid and Fluid Mechanics*, Oxford University Press, Oxford, 1998.