Efficient Bases for the Galerkin Solution of Multiple-Scattering Problems

Ömer Aktepe

Boğaziçi University

August 11, 2016

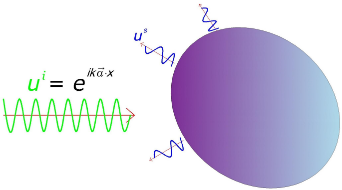
Overview

Main problem

α : Direction of incidence
k : Wave number
Finding the scattered field, u, for a given incident field

$$u^{inc}(x) = e^{ik\alpha\cdot x}, \qquad x \in \mathbb{R}^2,$$

for a smooth compact obstacle K.



Our aim is to solve this when K consisting of 2 convex sub obstacles K_1 and K_2.

Problem conditions

The scattered field has to satisfy

The Helmholtz equation: $$\Delta u(x) + k^2 u(x) = 0, \quad x \in \mathbb{R}^2 \backslash K,$$

The Sommerfeld condition: $$\lim_{|x|\to\infty} |x|^{1/2} \left[\left(\frac{x}{|x|}, \nabla u(x) \right) - iku(x) \right] = 0,$$

and the Dirichlet condition: $$u(x) = -u^{inc}(x) = -e^{ik\alpha\cdot x}, \quad x \in \partial K.$$

Single Layer Representation and Density Function

η: Density function (unknown)

$$u(x) = -\int_{\partial K} \Phi(x,y)\eta(y)ds(y), \quad x \in \mathbb{R}^2 \backslash K$$

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \neq y$$

$H_0^{(1)}$: The Hankel function of the first kind and order zero

Single Layer Representation and Density Function

η: Density function (unknown)

$$u(x) = -\int_{\partial K} \Phi(x,y)\eta(y)ds(y), \quad x \in \mathbb{R}^2 \backslash K$$

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \neq y$$

$H_0^{(1)}$: The Hankel function of the first kind and order zero

$$\eta(x) = \frac{\partial \left(u(x) + u^{inc}(x)\right)}{\partial \nu(x)}, \quad x \in \partial K.$$

ν: The exterior unit normal vector to ∂K

So, instead of u, we can find η.

Combined field integral equation

$$\eta(x) - \int_{\partial K} \left\{ \frac{\partial G(x,y)}{\partial \nu(x)} - ikG(x,y) \right\} \eta(y) ds(y)$$
$$= 2 \left\{ \frac{\partial u^{inc}(x)}{\partial \nu(x)} - iku^{inc}(x) \right\}, \quad x \in \partial K$$

$G(x,y) = -2\Phi(x,y)$

Combined field integral equation

$$\eta(x) - \int_{\partial K} \left\{ \frac{\partial G(x,y)}{\partial \nu(x)} - ikG(x,y) \right\} \eta(y) ds(y)$$
$$= 2 \left\{ \frac{\partial u^{inc}(x)}{\partial \nu(x)} - iku^{inc}(x) \right\}, \quad x \in \partial K$$

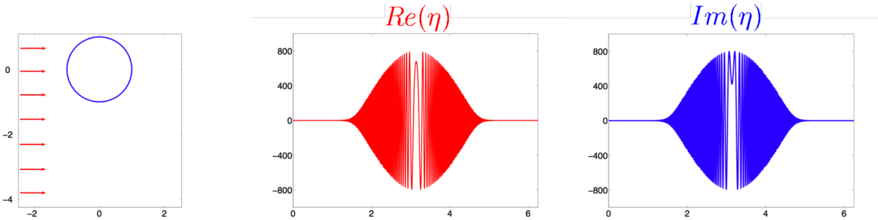
$G(x,y) = -2\Phi(x,y)$
Write shortly as

$$\mathcal{R}\eta = (I - R)\eta = f$$

This equation is uniquely solvable.

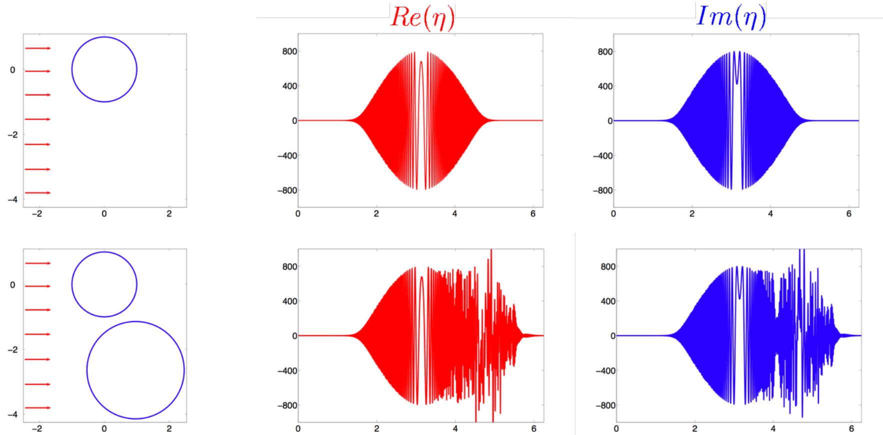
Single scattering vs Multiple scattering η values

$$k = 400$$



Single scattering vs Multiple scattering η values

$$k = 400$$

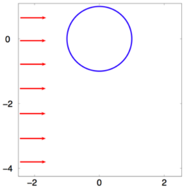


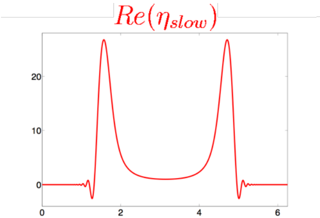
Single scattering vs Multiple scattering η^{slow} values

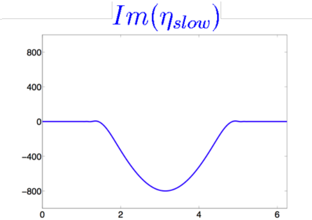
$$\eta = \eta^{slow} e^{ik\alpha \cdot x}$$

Single scattering vs Multiple scattering η^{slow} values

$$\eta = \eta^{slow} e^{ik\alpha \cdot x}$$
$$k = 400$$

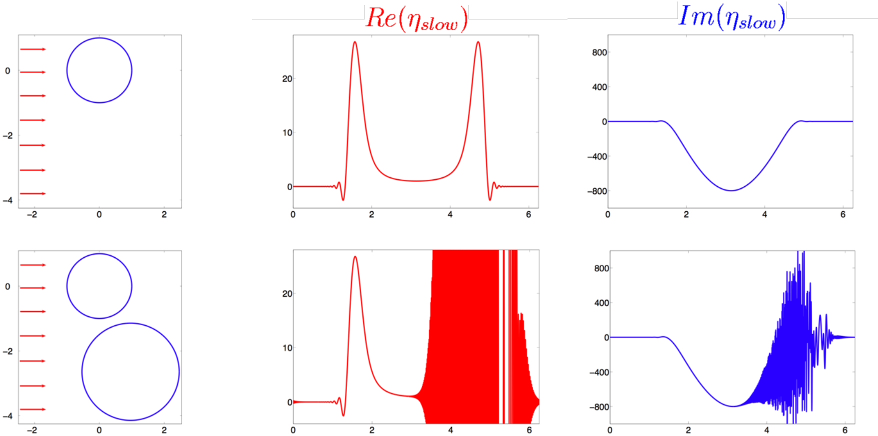






Single scattering vs Multiple scattering η^{slow} values

$$\eta = \eta^{slow} e^{ik\alpha \cdot x}$$
$$k = 400$$



Algorithms For Single Scattering Problem

Non-rigorous

- $\mathcal{O}(k^{1/3})$ - Abboud, Nédélec and Zhou (1994) [Galerkin method and stationary phase.]
- $\mathcal{O}(1)$ - Bruno et all. (2004)[Nystörm and stationary phase].
- $\mathcal{O}(1)$ - Giladi and Keller (2004) [Collocation and geometric theory of diffraction].
- $\mathcal{O}(1)$ - Huybrechs and Vandewalle (2007) [Collocation and steepest decent].

Algorithms For Single Scattering Problem

Non-rigorous

- $\mathcal{O}(k^{1/3})$ - Abboud, Nédélec and Zhou (1994) [Galerkin method and stationary phase.]
- $\mathcal{O}(1)$ - Bruno et all. (2004)[Nystörm and stationary phase].
- $\mathcal{O}(1)$ - Giladi and Keller (2004) [Collocation and geometric theory of diffraction].
- $\mathcal{O}(1)$ - Huybrechs and Vandewalle (2007) [Collocation and steepest decent].

Rigorous but approximation by zero in deep shadow

- $\mathcal{O}(k^{1/9})$ - Domínguez, Graham and Smyshlyaev (2007) [Galerkin]

Algorithms For Single Scattering Problem

Non-rigorous

- $\mathcal{O}(k^{1/3})$ - Abboud, Nédélec and Zhou (1994) [Galerkin method and stationary phase.]
- $\mathcal{O}(1)$ - Bruno et all. (2004)[Nystörm and stationary phase].
- $\mathcal{O}(1)$ - Giladi and Keller (2004) [Collocation and geometric theory of diffraction].
- $\mathcal{O}(1)$ - Huybrechs and Vandewalle (2007) [Collocation and steepest decent].

Rigorous but approximation by zero in deep shadow

- $\mathcal{O}(k^{1/9})$ - Domínguez, Graham and Smyshlyaev (2007) [Galerkin]

Rigorous and convergent

- $\mathcal{O}(k^{\epsilon})$ - Ecevit and Özen
- $\mathcal{O}(k^{\epsilon})$ - Ecevit and Eruslu [Better approximation in shadow region]

Algorithm For Multiple Scattering Problem

Non-rigorous

- $\mathcal{O}(1)$ Bruno et all. (2005) [Nystörm and stationary phase].

The Neumann series solution

Matrix representation of $(I - R)\eta = f$:

$$\begin{bmatrix} I_{11} - R_{11} & -R_{12} \\ -R_{21} & I_{22} - R_{21} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

where η_j and f_j are defined on the K_j

The Neumann series solution

Matrix representation of $(I - R)\eta = f$:

$$\begin{bmatrix} I_{11} - R_{11} & -R_{12} \\ -R_{21} & I_{22} - R_{21} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

where η_j and f_j are defined on the K_j
Multiply this by

$$\begin{bmatrix} I_{11} - R_{11} & 0 \\ 0 & I_{22} - R_{21} \end{bmatrix}^{-1}$$

We get

$$\begin{bmatrix} I_{11} & (I - R_{11})^{-1} R_{12} \\ (I - R_{22})^{-1} R_{21} & I_{22} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} (I - R_{11})^{-1} f_1 \\ (I - R_{22})^{-1} f_2 \end{bmatrix}$$

We get

$$\begin{bmatrix} I_{11} & (I - R_{11})^{-1} R_{12} \\ (I - R_{22})^{-1} R_{21} & I_{22} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} (I - R_{11})^{-1} f_1 \\ (I - R_{22})^{-1} f_2 \end{bmatrix}$$

Matrix representation:

$$(I - T)\eta = g$$

$$T = \begin{bmatrix} 0 & (I - R_{11})^{-1} R_{12} \\ (I - R_{22})^{-1} R_{21} & 0 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} (I - R_{11})^{-1} f_1 \\ (I - R_{22})^{-1} f_2 \end{bmatrix}$$

The Neumann expansion of $(I - T)\eta = g$:

$$\eta = \sum_{m=0}^{\infty} \eta^m = \sum_{m=0}^{\infty} T^m g \quad \text{and} \quad \eta^{m+1} = T\eta^m$$

or

$$\eta^{m+1} = \begin{bmatrix} \eta_1^{m+1} \\ \eta_2^{m+1} \end{bmatrix} = \begin{bmatrix} (I - R_{11})^{-1} R_{12} \eta_2^m \\ (I - R_{22})^{-1} R_{21} \eta_1^m \end{bmatrix}$$

The Neumann expansion of $(I - T)\eta = g$:

$$\eta = \sum_{m=0}^{\infty} \eta^m = \sum_{m=0}^{\infty} T^m g \quad \text{and} \quad \eta^{m+1} = T\eta^m$$

or

$$\eta^{m+1} = \begin{bmatrix} \eta_1^{m+1} \\ \eta_2^{m+1} \end{bmatrix} = \begin{bmatrix} (I - R_{11})^{-1} R_{12} \eta_2^m \\ (I - R_{22})^{-1} R_{21} \eta_1^m \end{bmatrix}$$

We need to solve integral equation of the following form:

$$\begin{cases} (I - R_{jj})\eta_j^0 = f_j & \text{on } \partial K_j, \\ (I - R_{jj})\eta_j^{m+1} = R_{jj'}\eta_{j'}^m & \text{on } \partial K_j. \end{cases}$$

Hence, the problem is reduced to iteratively solving single scattering problems.

Phase-extraction

Recursive equations

$$\begin{cases} (I - R_{jj})\eta_j^0 = f_j & \text{on } \partial K_j \\ (I - R_{jj})\eta_j^{m+1} = R_{jj'}\eta_{j'}^m & \text{on } \partial K_j \end{cases}$$

suggests using

$$\eta^m(x) = e^{ik\varphi_m(x)}\eta^{m,slow}(x)$$

Phase-extraction

Recursive equations

$$\begin{cases} (I - R_{jj})\eta_j^0 = f_j & \text{on } \partial K_j \\ (I - R_{jj})\eta_j^{m+1} = R_{jj'}\eta_{j'}^m & \text{on } \partial K_j \end{cases}$$

suggests using

$$\eta^m(x) = e^{ik\varphi_m(x)}\eta^{m,slow}(x)$$

Geometrical optics phase

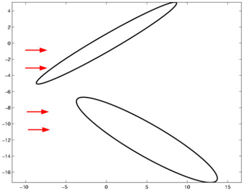
$$\varphi_m(x) = \begin{cases} \alpha \cdot x & \text{if } m = 0 \\ \alpha \cdot \mathcal{X}_0^m(x) + \sum_{j=0}^{m-1} |\mathcal{X}_{j+1}^m(x) - \mathcal{X}_j^m(x)| & \text{if } m \geq 1, \end{cases}$$

$(\mathcal{X}_0^m(x), \ldots, \mathcal{X}_m^m(x)) \in \partial K_{\tau_0} \times \cdots \times \partial K_{\tau_m}$: broken rays

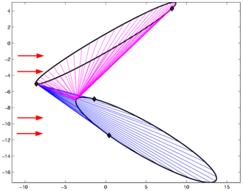
Broken rays are uniquely determined by the relations

$$\begin{cases} \text{(a) } \mathcal{X}_m^m(x) = x, \\ \text{(b) } \alpha \cdot \nu(\mathcal{X}_0^m(x)) < 0, \\ \text{(c) } (\mathcal{X}_{j+1}^m(x) - \mathcal{X}_j^m(x)) \cdot \nu(\mathcal{X}_j^m(x)) > 0, \\ \text{(d) } \dfrac{\mathcal{X}_1^m(x) - \mathcal{X}_0^m(x)}{|\mathcal{X}_1^m(x) - \mathcal{X}_0^m(x)|} = \alpha - 2\alpha \cdot \nu(\mathcal{X}_0^m(x))\, \nu(\mathcal{X}_0^m(x)), \\ \text{(e) } \dfrac{\mathcal{X}_{j+1}^m(x) - \mathcal{X}_j^m(x)}{|\mathcal{X}_{j+1}^m(x) - \mathcal{X}_j^m(x)|} = \dfrac{\mathcal{X}_j^m(x) - \mathcal{X}_{j-1}^m(x)}{|\mathcal{X}_j^m(x) - \mathcal{X}_{j-1}^m(x)|} \\ \qquad -2\dfrac{\mathcal{X}_j^m(x) - \mathcal{X}_{j-1}^m(x)}{|\mathcal{X}_j^m(x) - \mathcal{X}_{j-1}^m(x)|} \cdot \nu(\mathcal{X}_j^m(x))\, \nu(\mathcal{X}_j^m(x)) \end{cases}$$

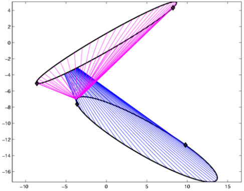
Reflection: 0



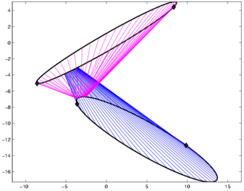
Reflection: 1



Reflection: 2



Reflection: 3



$$\eta^m(x) = \underbrace{e^{ik\varphi_m(x)}}_{\text{KNOWN}} \eta^{m,slow}(x)$$

$$\eta^m(x) = \underbrace{e^{ik\varphi_m(x)}}_{\text{KNOWN}} \eta^{m,slow}(x)$$

Theorem (Derivative estimates (Ecevit-Reitich))

Let $m \geq 0$, and denote by $y(s) = (y^1(s), y^2(s))$ the arc-length parametrization of ∂K_m. Then, for all $n \in \mathbb{N} \cup \{0\}$, there exist constants $C_n > 0$ independent of k and s such that for all k sufficiently large,

$$\left| D_s^n \eta^{m,slow}(y(s)) \right| \leq k \begin{cases} C_n, & n = 0, 1, \\ C_n \left[1 + \sum_{j=2}^n k^{(j-1)/3} (1 + k^{1/3} |w(s)|)^{-(j+2)} \right], & n \geq 2, \end{cases}$$

where $w(s) = (s - t_1)(b - t_2)$ and $\partial K_m^{SB} = \{y(t_1), y(t_2)\}$.

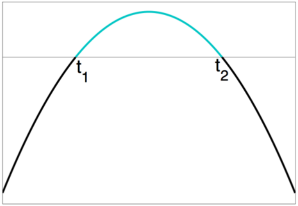


Figure: $w(s) = (s - t_1)(b - t_2)$

When $s = t_1$ or $s = t_2$:

As $k \to \infty$, $1 + \sum_{j=2}^{n} k^{(j-1)/3}(1 + k^{1/3}|w(s)|)^{-(j+2)} = \mathcal{O}(k^{(n-1)/3})$

We need to control this

Galerkin Method

$$\begin{cases} (I - R_{jj})\eta_j^0 = f_j & \text{on } \partial K_j \\ (I - R_{jj})\eta_j^{m+1} = R_{jj'}\eta_{j'}^m & \text{on } \partial K_j \end{cases}$$

Every equation has the same form. Fix m, and denote the related equation as:

$$\mathcal{R}\eta = f, \qquad \eta \in L^2(\partial\mathcal{K})$$

Galerkin Method

$$\begin{cases} (I - R_{jj})\eta_j^0 = f_j & \text{on } \partial K_j \\ (I - R_{jj})\eta_j^{m+1} = R_{jj'}\eta_{j'}^m & \text{on } \partial K_j \end{cases}$$

Every equation has the same form. Fix m, and denote the related equation as:

$$\mathcal{R}\eta = f, \qquad \eta \in L^2(\partial\mathcal{K})$$

Galerkin Approximation Spaces

Galerkin approximation space, $\mathcal{G}$: A finitely spanned subset of $L^2(\partial\mathcal{K})$.

Best approximation error estimate

By Céa's lemma, there exist unique $\hat{\eta} \in \mathcal{G}$

$$\langle \hat{\mu}, \mathcal{R}\hat{\eta} \rangle = \langle \hat{\mu}, f \rangle, \qquad \forall \hat{\mu} \in \mathcal{G}$$

and for the continuity and coercivity constants, C and c we have

$$\|\eta - \hat{\eta}\| \leq \frac{C}{c} \inf_{\hat{\mu} \in \mathcal{G}} \|\eta - \hat{\mu}\|$$

Best approximation error estimate

By Céa's lemma, there exist unique $\hat{\eta} \in \mathcal{G}$

$$\langle \hat{\mu}, \mathcal{R}\hat{\eta} \rangle = \langle \hat{\mu}, f \rangle, \qquad \forall \hat{\mu} \in \mathcal{G}$$

and for the continuity and coercivity constants, C and c we have

$$\|\eta - \hat{\eta}\| \leq \frac{C}{c} \inf_{\hat{\mu} \in \mathcal{G}} \|\eta - \hat{\mu}\|$$

Matrix representation

When $\mathcal{G} = \text{span}\{\hat{\mu}_1, \cdots, \hat{\mu}_n\}$ and $\hat{\eta} = \sum_{i=1}^{n} c_i \hat{\mu}_i$

$$\begin{bmatrix} \langle \mathcal{R}\hat{\mu}_1, \hat{\mu}_1 \rangle & \dots & \langle \mathcal{R}\hat{\mu}_n, \hat{\mu}_1 \rangle \\ \vdots & & \vdots \\ \langle \mathcal{R}\hat{\mu}_1, \hat{\mu}_n \rangle & \dots & \langle \mathcal{R}\hat{\mu}_n, \hat{\mu}_n \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, \hat{\mu}_1 \rangle \\ \vdots \\ \langle f, \hat{\mu}_n \rangle \end{bmatrix}.$$

Recent Achievements

$\mathcal{O}(k^{\epsilon})$ - Ecevit and Özen (2016)

(i) Number of degrees of freedom: $\mathcal{O}(k^{\epsilon})$, ϵ can be chosen arbitrarily small.

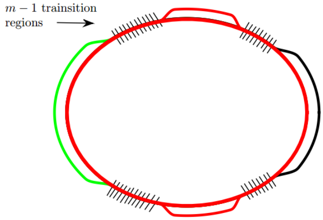
(ii) Mimicking the behavior of solution between illuminated region and shadow boundaries: $4m-4$ transition regions. $4m$ in total.

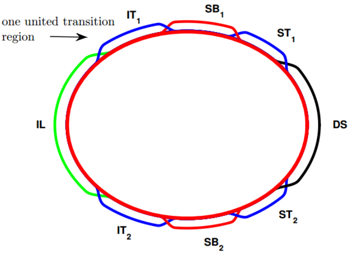
$\mathcal{O}(k^{\epsilon})$ - Ecevit and Eruslu (2016)

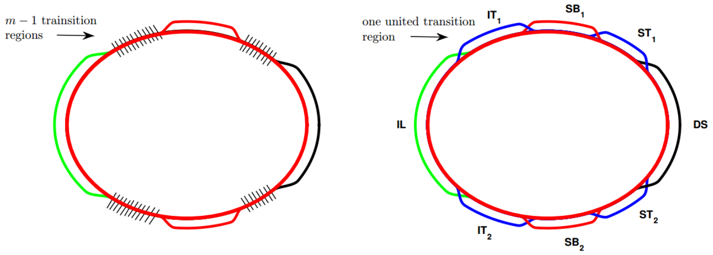
(i) Number of degrees of freedom: $\mathcal{O}(k^{\epsilon})$, ϵ can be chosen arbitrarily small.

(ii) Mimicking the behavior of solution between illuminated region and shadow boundaries: 4 transition regions. 6 in total.

We transfer their ideas in multiple scattering case.


$m-1$ trainsition
regions

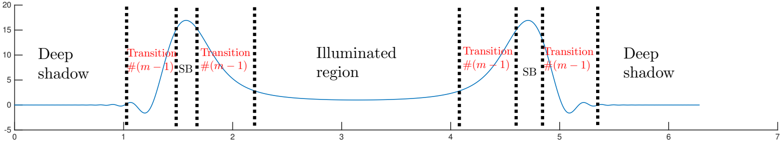

one united transition
region
IT1
SB1
ST1
IL
DS
IT2
SB2
ST2

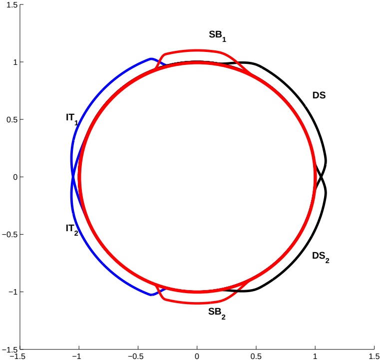


Ecevit-Özen: $\mathcal{G} = \bigoplus_{j=1}^{4m} \chi_j \, e^{ik\varphi} \, \mathbb{P}_d$ where $\mathbb{P}_d$ is polynomials of degree at most d defined on each region. For all $n \in \{0, \cdots, d+1\}$ and sufficiently large $k \geq 1$

$$\frac{\|\eta - \hat{\eta}\|_{L^2(\partial K)}}{\|\eta\|_{L^2(\partial K)}} \lesssim_n \frac{C}{c} m \frac{1 + k^{-\frac{1}{2}} \left(k^{\frac{1}{6m+3}}\right)^n}{d^n} = \mathcal{O}(k^\epsilon)$$

by a clever choice of d and m.


Deep
shadow
Transition
$\#(m-1)$
SB
Transition
$\#(m-1)$
Illuminated
region
Transition
$\#(m-1)$
SB
Transition
$\#(m-1)$
Deep
shadow


SB_1
DS
IT_1
IT_2
DS_2
SB_2

Ecevit-Eruslu: $\mathcal{G} = \bigoplus_{j=1}^{J} \chi_j \ e^{ik\varphi} \ \mathcal{C}_d$ where

$$\mathcal{C}_d = \begin{cases} \mathbb{P}_d \circ \phi^{-1}, & \text{transition region,} \\ \mathbb{P}_d, & \text{otherwise,} \end{cases}$$

for ϕ is the change of variable function. For all $n \in \{0, \cdots, d+1\}$ and sufficiently large $k \geq 1$

$$\frac{\|\eta - \hat{\eta}\|_{L^2(\partial K)}}{\|\eta\|_{L^2(\partial K)}} \lesssim_n \frac{C}{c} \frac{(\log k)^{n+1/2}}{d^n} = \mathcal{O}(k^{\epsilon})$$

by a clever choice of d.

$$\phi\left(s\right)=\begin{cases} t_1+\varphi\left(s\right)k^{\psi\left(s\right)}, & s\in I_{IT_1},\\ t_2-\varphi\left(s\right)k^{\psi\left(s\right)}, & s\in I_{IT_2},\\ t_1-\varphi\left(s\right)k^{\psi\left(s\right)}, & s\in I_{ST_1},\\ t_2+\varphi\left(s\right)k^{\psi\left(s\right)}, & s\in I_{ST_2}.\end{cases}$$

$$\varphi(s)=\begin{cases} \xi_1+\left(\xi_1'-\xi_1\right)\dfrac{s-a_1}{b_1-a_1}, & s\in I_{IT_1},\\ \xi_2'+\left(\xi_2-\xi_2'\right)\dfrac{s-a_2}{b_2-a_2}, & s\in I_{IT_2},\\ \zeta_1'+\left(\zeta_1-\zeta_1'\right)\dfrac{s-a_3}{b_3-a_3}, & s\in I_{ST_1},\\ \zeta_2+\left(\zeta_2'-\zeta_2\right)\dfrac{s-a_4}{b_4-a_4}, & s\in I_{ST_2},\end{cases}$$

$$\psi(s) = -\frac{1}{3}\begin{cases} \dfrac{b_1 - s}{b_1 - a_1}, & s \in I_{IT_1}, \\ \dfrac{s - a_2}{b_2 - a_2}, & s \in I_{IT_2}, \\ \dfrac{s - a_3}{b_3 - a_3}, & s \in I_{ST_1}, \\ \dfrac{b_4 - s}{b_4 - a_4}, & s \in I_{ST_2}. \end{cases}$$

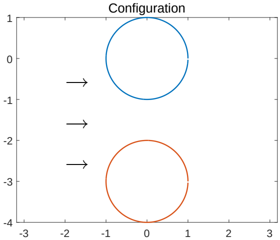
Multiple Scattering Algorithms Developed In This Thesis

$k = 50, 100, 200, 400, 800$
$d = 4, 8, 12, 16, 20$

$$\begin{cases} (I - R_{jj})\eta_j^0 = f_j & \text{on } \partial K_j \\ (I - R_{jj})\eta_j^{m+1} = R_{jj'}\eta_{j'}^m & \text{on } \partial K_j \end{cases}$$

Relative Error:

$$\log_{10}\left(\frac{\|\eta^m - \hat{\eta}^m\|_{L^2(\partial K)}}{\|\eta^m\|_{L^2(\partial K)}}\right)$$



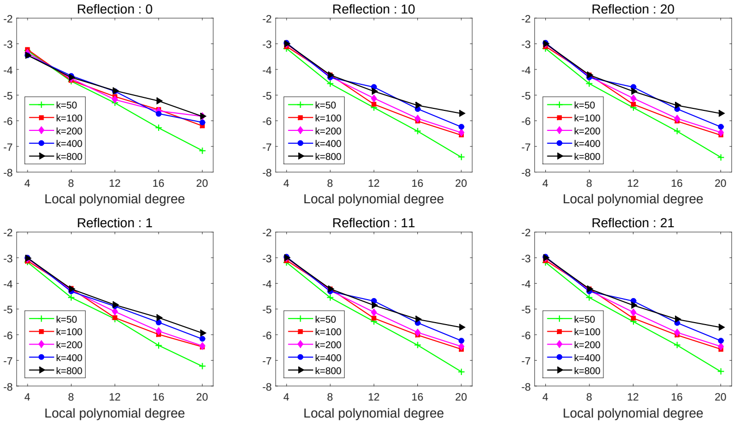


Figure: Relative errors using Nyström solutions as a right hand side for two circles: First path.

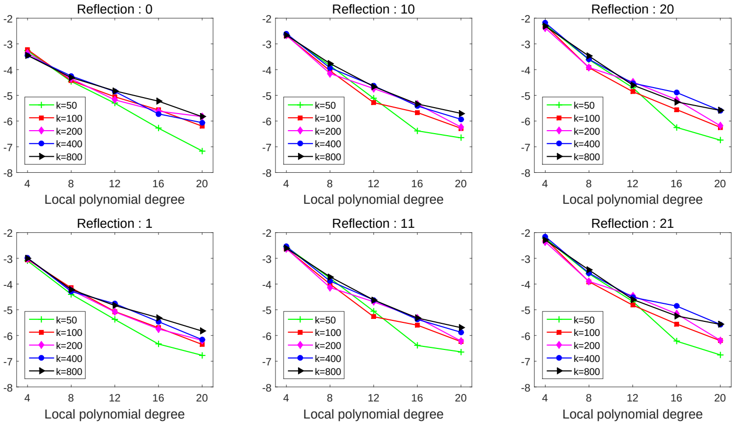


Figure: Relative errors using Galerkin solutions as a right hand side for two circles: First path.

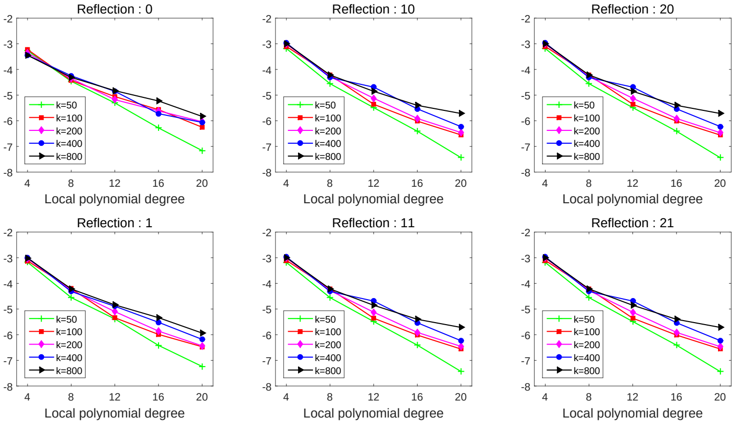


Figure: Relative errors using Nyström solutions as a right hand side for two circles: Second path.

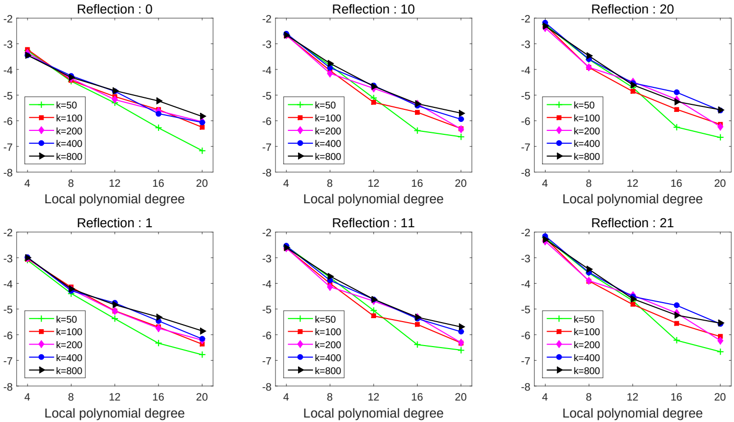


Figure: Relative errors using Galerkin solutions as a right hand side for two circles: Second path.

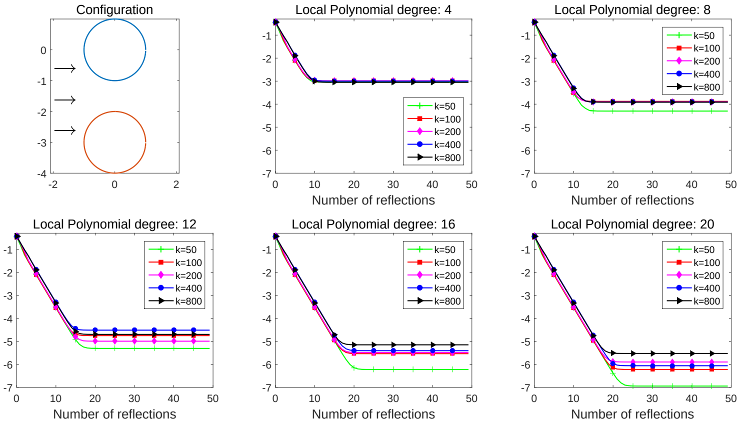


Figure: Sum of Galerkin solutions obtained by using Nyström solutions as a right hand side for two circles.

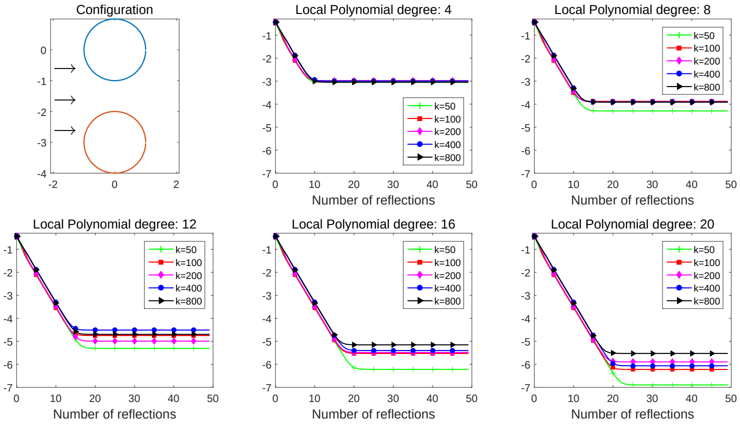
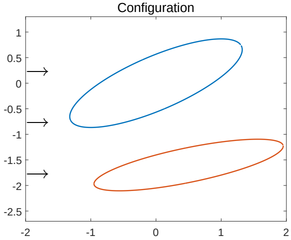


Figure: Sum of Galerkin solutions obtained by using Galerkin solutions as a right hand side for two circles.

$k = 50, 100, 200, 400, 800$
$d = 4, 8, 12, 16, 20$



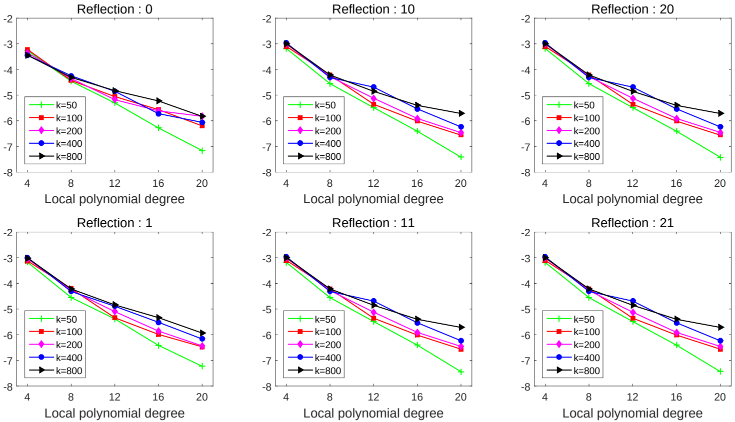


Figure: Relative errors using Nyström solutions as a right hand side for two ellipses: First path.

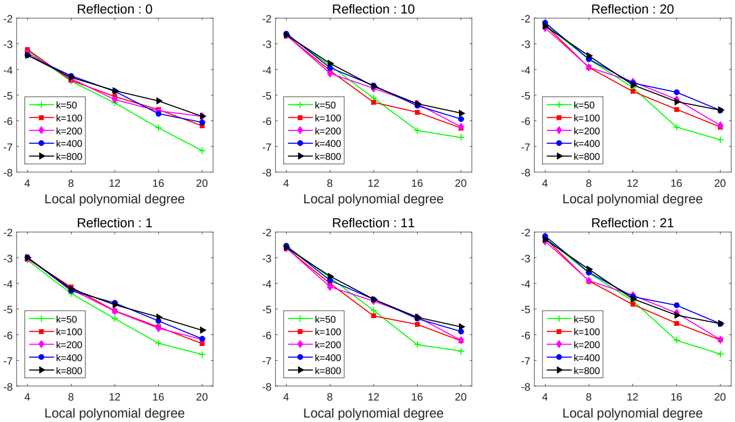


Figure: Relative errors using Galerkin solutions as a right hand side for two ellipses: First path.

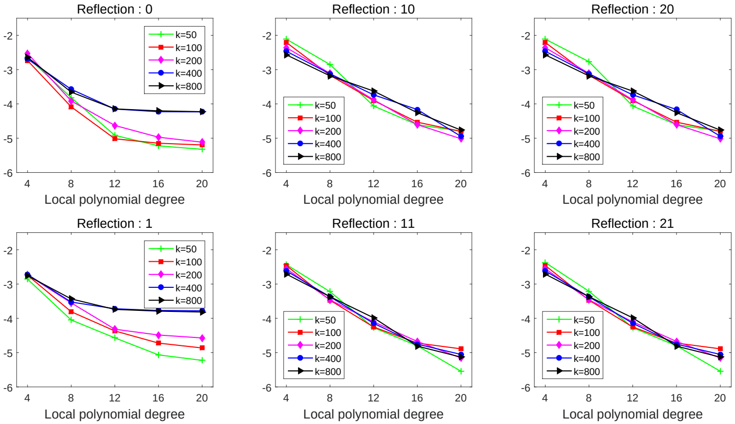


Figure: Relative errors using Nyström solutions as a right hand side for two ellipses: Second path.

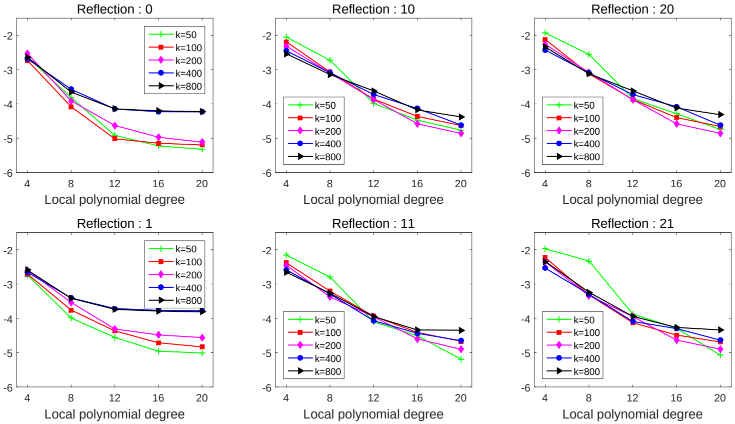


Figure: Relative errors using Galerkin solutions as a right hand side for two ellipses: Second path.

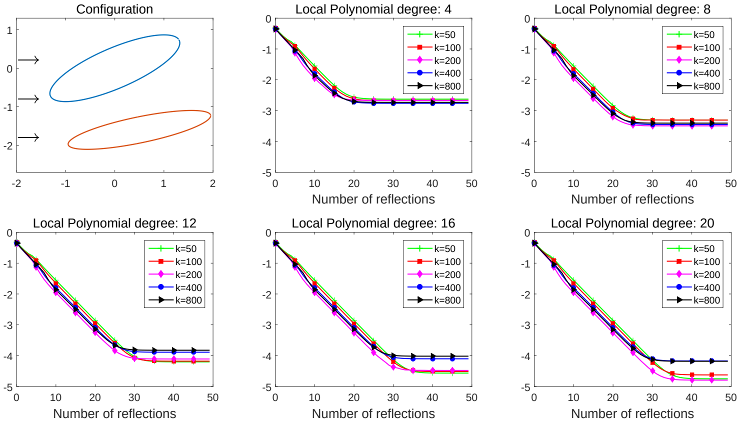


Figure: Sum of Galerkin solutions obtained by using Nyström solutions as a right hand side for two ellipses.

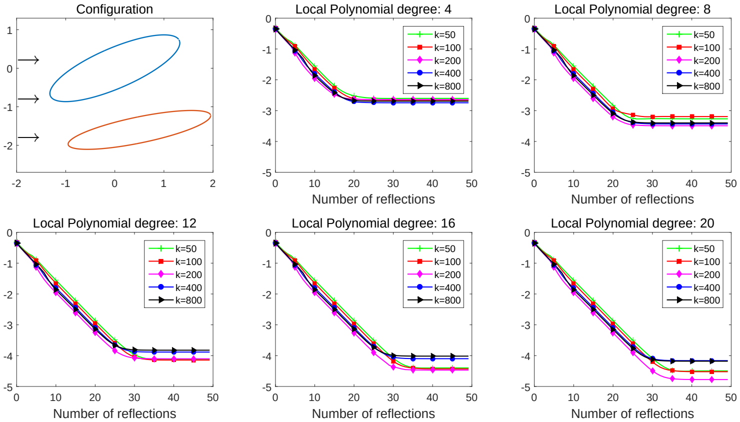


Figure: Sum of Galerkin solutions obtained by using Galerkin solutions as a right hand side for two ellipses.

Bruno, O. P., C. A. Geuzaine, J. A. Monroe, and F. Reitich, "Prescribed error tolerances within fixed computational times for scattering problems of arbitrarily high frequency: the convex case", *Phil. Trans. Roy. Soc. London*, Vol. 362, pp. 629–645, 2004.

Giladi, E., "An asymptotically derived boundary element method for the Helmholtz equation in high frequencies", *J. Comput. Appl. Math.*, Vol. 198, pp. 52–74, 2007.

Huybrechs, D., and S. Vandewalle, "A sparse discretization for integral equation formulations of high frequency scattering problems", *SIAM J. Sci. Comput.*, Vol. 29, No. 6, pp. 2305–2328, 2007.

Domínguez, V., I. G. Graham, and V. P. Smyshlyaev, "A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering", *Numer. Math.*, Vol. 106, No. 3, pp. 471–510, 2007.

Ecevit, F., and H. Ç. Özen., "Frequency-adapted Galerkin boundary element methods for convex scattering problems", *Numer. Math.*, DOI: 10.1007/s00211-016-0800-7, 2016.

Ecevit, F., and H. H. Eruslu, "Change of variables Galerkin schemes for high-frequency single-scattering problems", In press, 2016.

Bruno, O. P., C. A. Geuzaine, and F. Reitich, "On the $\mathcal{O}(1)$ solution of multiple-scattering problems, *IEEE Trans. Magn.*, Vol. 41, pp. 1488–1491, 2005.

Ecevit, F., and F. Reitich, "Analysis of multiple scattering iterations for high-frequency scattering problems. I. The two-dimensional case", *Numer. Math.*, Vol. 114, No. 2, pp. 271–354, 2009.

The End