The Definition of The Derivative

Given that you are reading this note, I assume you are already familiar with the concepts of limits and functions. In Limits we know that the computation of the slope of a tangent line, the instantaneous rate of change of a function, and the instantaneous velocity of an object at $x = a$ all required us to compute the following limit.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We also saw that with a small change of notation this limit could also be written as,

$$\lim_{x \to a} \frac{f(a + h) - f(a)}{h} \qquad (1)$$

This is such an important limit and it arises in so many places that we give it a name. We call it a derivative. Here is the official definition of the derivative.Even though familiarity with the concepts of functions and limits is necessary, if you are wondering why (h) is added to the function, please refer to page number 2 for a detailed explanation.

Definition of the Derivative

The derivative of $f(x)$ with respect to x is the function $f'(x)$ and is defined as,

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \qquad (2)$$

We replaced all the a's in (1) with x's to acknowledge the fact that the derivative is really a function as well.

Practice Problems

1) $f(x) = 2x^2 - 16x + 35$

2) $f(x) = \sqrt{x}$

3) $f(x) = \dfrac{1}{x}$

4) $f(x) = \dfrac{x^2 + 3x}{x + 1}$

5) A car's position along a straight road is given by the function $s(t) = t^2 - 4t$, where s is the position in meters and t is the time in seconds. Use the definition of the derivative to find the car's velocity at $t = 3$ seconds.

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Meaning of Adding h

1) Incremental Change:
   - The variable h represents a small increment of change in x from the point a.
   - Instead of considering x approaching a, we consider x as $a+h$, where h is a small distance from a.
2) Simplifying the Approach:
   - By letting $h \to 0$, we are examining what happens to the function f as the point $x = a + h$ gets infinitesimally close to a.
   - This simplification makes the algebra more manageable, especially when differentiating functions.
3) Instantaneous Rate of Change:
   - The expression $\lim_{x \to a} \frac{f(a+h) - f(a)}{h}$ is the average rate of change of the function over the interval from a to $a+h$.
   - Taking the limit as $h \to 0$ gives the instantaneous rate of change, which is the derivative of the function at $x = a$.
4) Understanding Tangent Lines:
   - Geometrically, the slope of the tangent line at $x = a$ is the limit of the slopes of the secant lines between $(a, f(a))$ and $(a+h, f(a+h))$ as h approaches 0.
   - This approach provides a more intuitive understanding of how the tangent line just "touches" the curve at a single point without crossing it.

Summary

The change in notation to s allows for a clearer and more practical way to compute limits, which are fundamental in calculus for understanding derivatives and the behavior of functions. By using h, we can directly analyze the change in the function's value over an infinitesimally small interval, leading to a precise calculation of instantaneous rates of change.