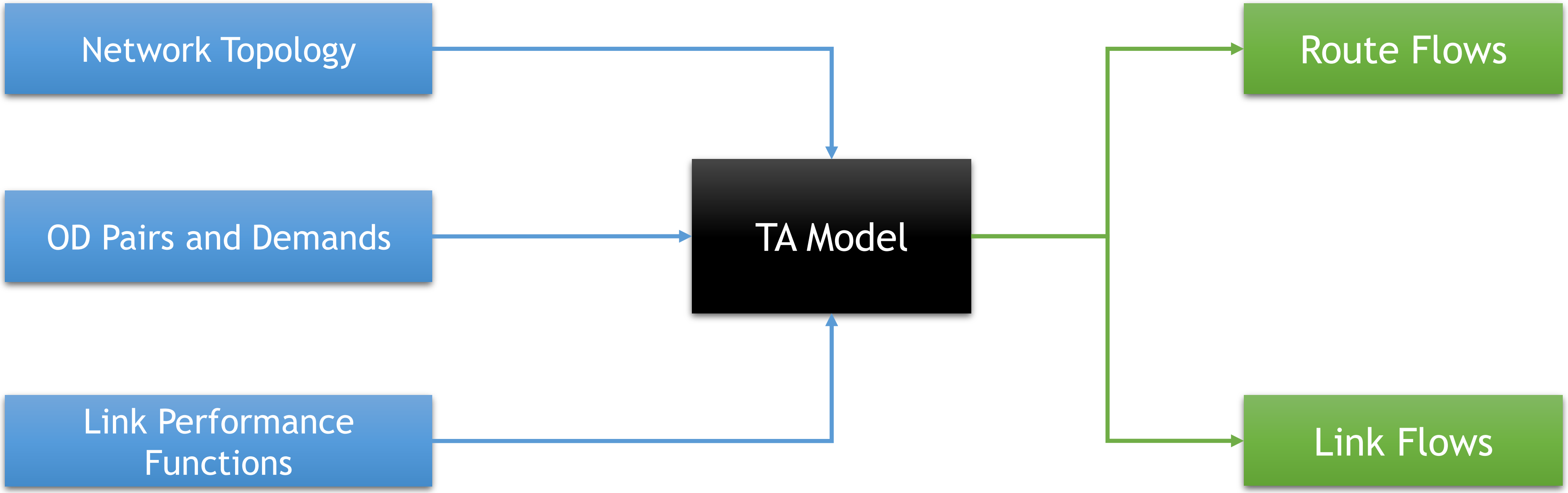
Traffic Assignment

Route choice

Traffic Assignment

Traffic assignment models are fundamental to the analysis of transportation systems.



Link Performance Functions

Link performance functions,

which capture the relationship between travel time per unit distance and traffic volume per unit time on the links of a network,

constitute an essential element in the equilibrium assignment of traffic flows to congested transportation networks *(Kim and Mahmassani, 1987)*.

a : link index

f_a : flow on link a

$\tau_a(f_a)$: generalized cost on link a

Link Performance Functions

There exists several important cost-flow functions (Smock, 1962; Overgaard, 1967; Akçelik function, etc.).

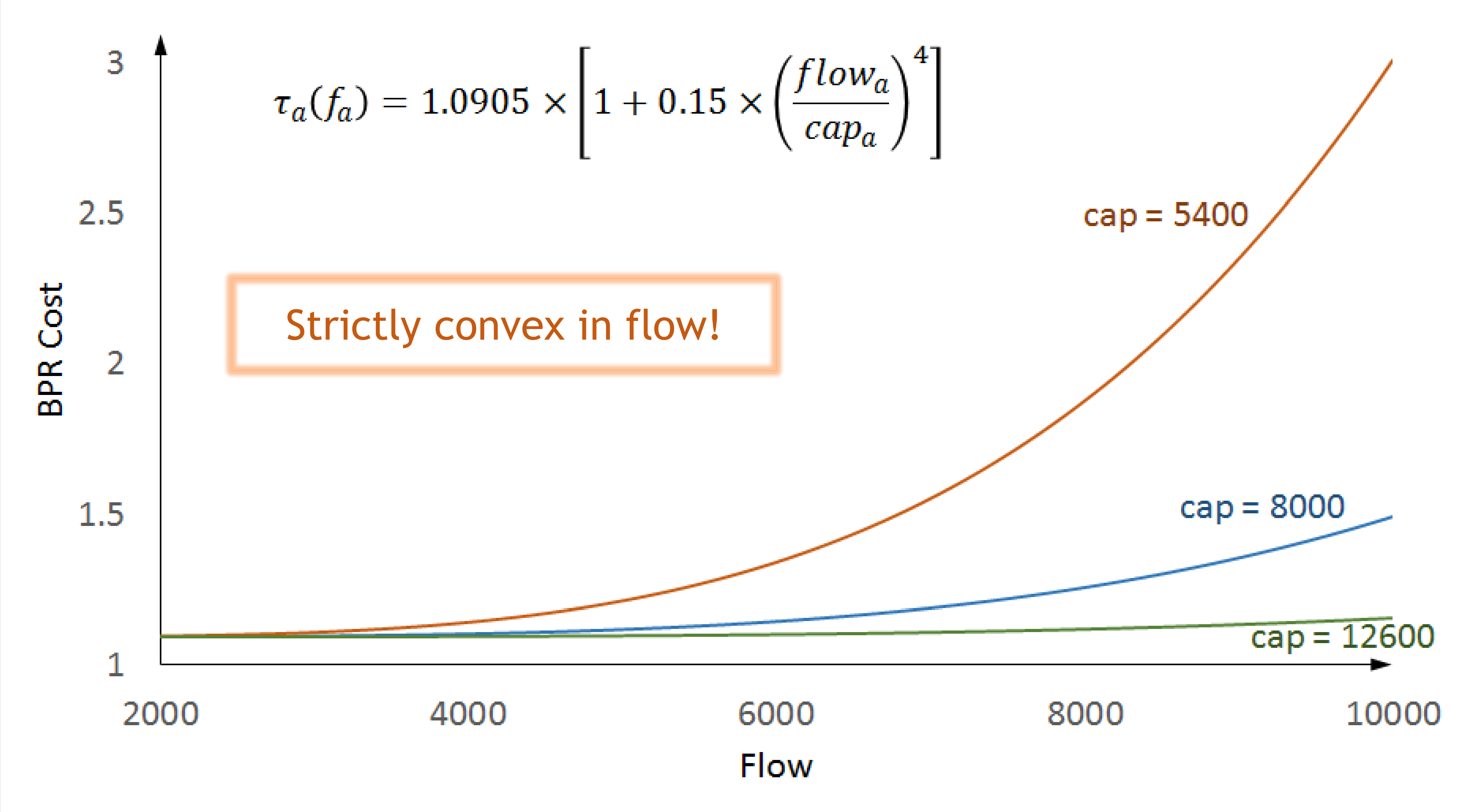
One of the most commonly used one is the Bureau of Public Roads (BPR) Function:

$$\tau_a(f_a) = FFTT_a \times \left[1 + B_a \times \left(\frac{f_a}{capacity_a}\right)^{power_a}\right]$$

On uncongested networks, link cost is basically the free-flow-travel-time:

$$\tau_a(0) = FFTT_a$$

Link Performance Functions



Traffic Assignment Models

		Congestion	
		No	Yes
Stochasticity	No	**All-or-Nothing** AoN	**User Equilibrium** (D)UE, Wardrop's principles
	Yes	**Pure Stochastic** Dial's, Burrell's	**Stochastic User Equilibrium** SUE

1. All-or-Nothing (AoN)

Congestion: No

The effect of link flows on travel times is ignored.

		Congestion	
		No	Yes
Stochasticity	No	All-or-Nothing AoN	User Equilibrium (D)UE, Wardrop's principles
	Yes	Pure Stochastic Dial's, Burrell's	Stochastic User Equilibrium SUE

Stochasticity: No

All drivers consider the same attributes for route choice; perceive the same route costs.

All OD demand is assigned to a single route,

probably to the shortest route with respect to free-flow-travel-time.

2. User Equilibrium (UE)

Congestion: Yes

The link travel costs are assumed to be strictly increasing and convex in the link flows, $\tau_a(f_a)$.

		Congestion	
		No	Yes
Stochasticity	No	All-or-Nothing AoN	User Equilibrium (D)UE, Wardrop's principles
	Yes	Pure Stochastic Dial's, Burrell's	Stochastic User Equilibrium SUE

Stochasticity: No

All drivers consider the same attributes for route choice; perceive the same route costs.

Wardrop's Principles:

the journey costs on all the routes actually used are equal; and less than those which would be experienced by a traveler on any unused route (Wardrop, 1952).

$$x_{kw} > 0 \quad \Rightarrow \quad c_{kw} = \min_{l \in K_w} c_{lw}$$

User Equilibrium (UE)

$$f_a = \sum_{w} \sum_{k \in K_w} x_{kw}\, \delta_{ak}$$

$$c_{kw} = \sum_{a \in k} \tau_a(f_a)$$

$$x_{kw} > 0 \implies c_{kw} = \min_{l \in K_w}\{c_{lw}\}$$

$$\sum_{k \in K_w} x_{kw} = d_w$$

Exercise 1

Exercise1.xlsx contains a network with a single OD pair and 4 alternative routes.

The OD demand is 100 units.

In worksheets **a** and **b**, try to find the UE by trial-and-error.

In other words, assign the demand to the routes so that:

- Some of the route flows is equal to 100 units:

$$\sum_{k \in K_w} x_{kw} = d_w = 100$$

- The route flows satisfy Wardrop's principles:

$$x_{kw} > 0 \quad \Rightarrow \quad c_{kw} = \min_{l \in K_w} c_{lw}$$

User Equilibrium (UE)

UE flows can be obtained by solving the following optimization problem.

$$\min \sum_{a \in A} \int_0^{f_a} \tau_a(t) dt$$

$s.t$

$$f_a = \sum_{w \in W} \sum_{k \in K_w} \delta_{ak} \, x_{kw} \qquad \forall a \in A$$

$$\sum_{k \in K_w} x_{kw} = d_w \qquad \forall w \in W$$

$$x_{kw} \geq 0 \qquad \forall k \in K_w, w \in W$$

User Equilibrium (UE)

Karush-Kuhn-Tucker conditions:

$$L = \sum_a \int_0^{f_a} \tau_a(t)dt - \sum_a \alpha_a f_a + \sum_w \sum_k \sum_a \delta_{ak} \alpha_a x_{kw} + \sum_w \sum_k \beta_w d_w - \sum_w \sum_k \beta_w x_{kw} - \sum_w \sum_k \mu_{kw} x_{kw}$$

User Equilibrium (UE)

Karush-Kuhn-Tucker conditions:

$$L = \sum_a \int_0^{f_a} \tau_a(t)dt - \sum_a \alpha_a f_a + \sum_w \sum_k \sum_a \delta_{ak} \alpha_a x_{kw} + \sum_w \sum_k \beta_w d_w - \sum_w \sum_k \beta_w x_{kw} - \sum_w \sum_k \mu_{kw} x_{kw}$$

$$\frac{\partial L}{\partial f_a} = 0 \quad \Longrightarrow \quad \tau_a(f_a) - \alpha_a = 0$$

$$\frac{\partial L}{\partial x_{kw}} = 0 \quad \Longrightarrow \quad \sum_a \delta_{ak} \alpha_a - \beta_w - \mu_{kw} = 0$$

C-S $\Longrightarrow$ $x_{kw} \times \mu_{kw} = 0$

Feasibility $\Longrightarrow$ $x_{kw} \geq 0; \;\; \mu_{kw} \geq 0$

User Equilibrium (UE)

Karush-Kuhn-Tucker conditions:

$$L = \sum_a \int_0^{f_a} \tau_a(t)dt - \sum_a \alpha_a f_a + \sum_w \sum_k \sum_a \delta_{ak}\alpha_a x_{kw} + \sum_w \sum_k \beta_w d_w - \sum_w \sum_k \beta_w x_{kw} - \sum_w \sum_k \mu_{kw} x_{kw}$$

$$\frac{\partial L}{\partial f_a} = 0 \quad \Longrightarrow \quad \alpha_a = \tau_a(f_a)$$

$$\frac{\partial L}{\partial x_{kw}} = 0 \quad \Longrightarrow \quad \beta_w = \sum_a \delta_{ak}\tau_a(f_a) - \mu_{kw}$$

C-S $\quad \Longrightarrow \quad x_{kw} \times \mu_{kw} = 0$

Feasibility $\quad \Longrightarrow \quad x_{kw} \geq 0;\ \ \mu_{kw} \geq 0$

User Equilibrium (UE)

Karush-Kuhn-Tucker conditions:

$$\frac{\partial L}{\partial x_{kw}} = 0 \quad \Longrightarrow \quad \beta_w = c_{kw} - \mu_{kw}$$

C-S $\quad \Longrightarrow \quad x_{kw} \times \mu_{kw} = 0$

Feasibility $\quad \Longrightarrow \quad x_{kw} \geq 0;\ \ \mu_{kw} \geq 0$

Assume route k is used: $x_{kw} > 0$

$\mu_{kw} = 0$

$\beta_w = c_{kw}$

cost of all used routes are equal to β_w.

Assume route l is not used: $x_{lw} = 0$

$\mu_{lw} \geq 0$

$\beta_w = c_{lw} - \mu_{lw}$

Compare route k and route l

$c_{lw} = c_{kw} + \mu_{lw}$

$c_{lw} \geq c_{kw}$

cost of unused routes are ≥ cost of used routes.

User Equilibrium (UE)

Wardrop's Principles:

the journey costs on all the routes actually used are equal;

and less than those which would be experienced by a traveler on any unused route. (Wardrop, 1952).

Assume route k is used: $x_{kw} > 0$

$\mu_{kw} = 0$

$\beta_w = c_{kw}$

cost of all used routes are equal to β_w.

Assume route l is not used: $x_{lw} = 0$

$\mu_{lw} \geq 0$

$\beta_w = c_{lw} - \mu_{lw}$

Compare route k and route l

$c_{lw} = c_{kw} + \mu_{lw}$

$c_{lw} \geq c_{kw}$

cost of unused routes are $\geq$ cost of used routes.

User Equilibrium (UE)

Second-order conditions:

$$\frac{\partial^2 \left(\sum_a \int_0^{f_a} \tau_a(t)dt\right)}{\partial f_a^2} = \frac{d\big(\tau_a(f_a)\big)}{df_a} > 0$$

The objective function is strictly convex and the constraints are linear.

There exists a unique user equilibrium.

Method of Successive Averages (MSA) Algorithm – UE

Set $f_a^1 = 0$

Calculate link costs, $\tau_a(f_a^1)$

Set step number $n = 1$

Until link flows converges, do

  For each OD pair w

    Calculate route costs, $c_{kw} = \sum_a \delta_{ak}\, \tau_a(f_a^n)$

    Obtain route flows x_{kw} by assigning demand d_w to the minimum cost routes

  Calculate resulting link flows: $g_a = \sum_w \sum_k \delta_{ak} x_{kw}$

  Calculate new link flows: $f_a^{n+1} = \left(\frac{1}{n}\right) g_a + \left(1 - \frac{1}{n}\right) f_a^n$

  Set $n = n + 1$

Exercise 2

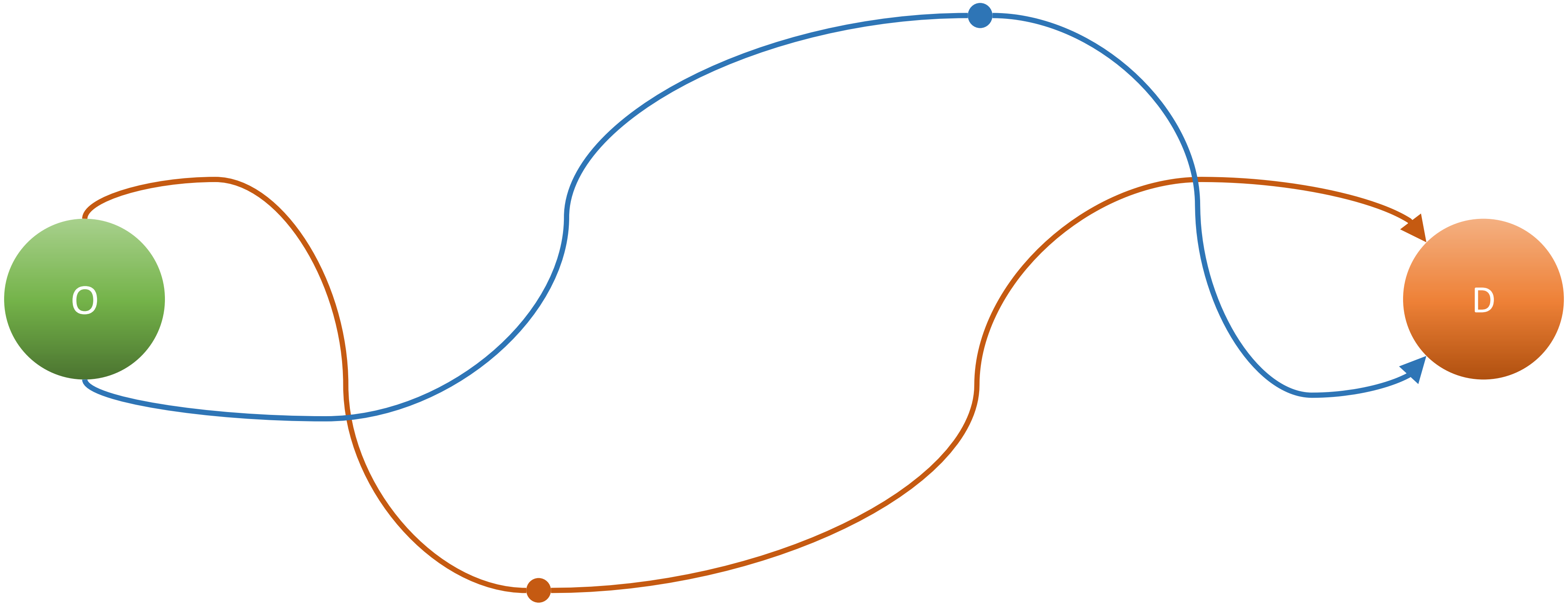
Exercise2.xlsx contains 1000 iterations of the MSA algorithm for UE problem.

- Check the calculations in a single iteration of the algorithm.
- From the link flows of the 1000^{th} iteration, try to obtain the route flows.
- Enter the route flows in **Exercise1.xlsx**, sheet **b**; and check whether the obtained solution is the UE.

Traffic Assignment Models

		Congestion: No	Congestion: Yes
Stochasticity	No	**All-or-Nothing** AoN	**User Equilibrium** (D)UE, Wardrop's principles
	Yes	**Pure Stochastic** Dial's, Burrell's	**Stochastic User Equilibrium** SUE

3. Pure Stochastic



3. Pure Stochastic

Congestion: No

The effect of link flows on travel times is ignored.

		Congestion	
		No	Yes
Stochasticity	No	All-or-Nothing AoN	User Equilibrium (D)UE, Wardrop's principles
	Yes	Pure Stochastic Dial's, Burrell's	Stochastic User Equilibrium SUE

Stochasticity: Yes

Drivers consider the heterogenic attributes for route choice; perceive different route costs.

p_{kw}: probability that a user in OD pair w chooses route k.

$$\sum_{k \in K_w} p_{kw} = 1$$

OD demand d_w is split to the routes with respect to the route choice probabilities p_{kw}.

$$x_{kw} = d_w \times p_{kw}$$

3. Pure Stochastic

Route choice probabilities are usually calculated using discrete choice models, such as:

- MNL: multinomial logit model
- MNP: multinomial probit model

$\tilde{U}_{kw}$ is the random perceived utility and $\tilde{\epsilon}_{kw}$ is the perception error term:

$$\tilde{U}_{kw} = -c_{kw}^0 + \tilde{\epsilon}_{kw}$$

where c_{kw}^0 is the free-flow-travel-cost.

Under random utility maximization, route choice probabilities are calculated as follows:

$$p_{kw} = \Pr\left\{\tilde{U}_{kw} = \max_{l \in K_w}\{\widetilde{U}_{lw}\}\right\} = \Pr\left\{-c_{kw}^0 + \tilde{\epsilon}_{kw} = \max_{l \in K_w}\{-c_{lw}^0 + \tilde{\epsilon}_{lw}\}\right\}$$

3. Pure Stochastic

In general, calculating the route choice probabilities is challenging.

$$p_{kw} = \Pr\left\{\tilde{U}_{kw} = \max_{l \in K_w}\{\tilde{U}_{lw}\}\right\} = \Pr\left\{-c_{kw}^0 + \tilde{\epsilon}_{kw} = \max_{l \in K_w}\{-c_{lw}^0 + \tilde{\epsilon}_{lw}\}\right\}$$

Assuming that $\tilde{\epsilon}_{kw}$ are independently and identically distributed Gumbel random variables, we obtain the MNL model.

Assuming that $\tilde{\epsilon}_{kw}$ are normal random variables, we obtain the MNP model.

3. Pure Stochastic

Assuming that $\tilde{\epsilon}_{kw}$ are independently and identically distributed Gumbel random variables, we obtain the MNL model.

With logit assumption, we may obtain a simplified route choice probability expression.

$$p_{kw} = \frac{\exp\left[-\theta_w c_{kw}^0\right]}{\sum_{l \in K_w} \exp\left[-\theta_w c_{lw}^0\right]}$$

where θ_w is the dispersion parameter.

Assuming that $\tilde{\epsilon}_{kw}$ are normal random variables, we obtain the MNP model.

Unfortunately, MNP model does not lead to a closed-form choice probability expression.

MNP choice probabilities may be calculated using Monte Carlo simulations.

It is computationally expensive.

Exercise 3

Exercise3.xlsx demonstrates pure stochastic assignment using MNL choice model with a dispersion parameter of 0.1 ($\theta_w = 0.1$).

- Check route choice probability calculation.
- Compare:
  - MNL route choice probabilities calculated using the free-flow-travel-times (c_{kw}^0), with
  - MNL route choice probabilities calculated using the route costs c_{kw} obtained by the pure stochastic traffic assignment,

  and comment.
- You may try to find equilibrium by assigning choice probabilities in range M15:M18 by trial-and-error.

4. Stochastic User Equilibrium (SUE)

Congestion: Yes

The link travel costs are assumed to be strictly increasing and convex in the link flows, $\tau_a\,(f_a)$.

Stochasticity: Yes

Drivers consider the heterogenic attributes for route choice; perceive different route costs.

		Congestion: No	Congestion: Yes
Stochasticity	No	All-or-Nothing AoN	User Equilibrium (D)UE, Wardrop's principles
Stochasticity	Yes	Pure Stochastic Dial's, Burrell's	Stochastic User Equilibrium SUE

Daganzo and Sheffi (1977) defined the SUE as

the traffic state at which no user can improve his/her perceived travel time by unilaterally changing routes.

Stochastic User Equilibrium (SUE)

$$f_a = \sum_{w} \sum_{k \in K_w} x_{kw}\, \delta_{ak}$$

$$c_{kw} = \sum_{a \in k} \tau_a(f_a)$$

$$p_{kw} = \Pr\{-c_{kw} + \tilde{\epsilon}_{kw} \geq -c_{lw} + \tilde{\epsilon}_{lw}\, ; \forall l \in K_w\}$$

$$x_{kw} = d_w p_{kw}$$

Stochastic User Equilibrium (SUE)

UE flows can be obtained by solving the following optimization problem.

$$\min \mathrm{Z} = -\sum_{w \in W} d_w \, E\left[\max_{k \in K_w}\{-c_{kw} + \tilde{\epsilon}_{kw}\}\right] + \sum_{a \in A} f_a \tau_a(f_a) - \sum_{a \in A} \int_0^{f_a} \tau_a(t) dt$$

KKT conditions of the optimization problem reduces to SUE conditions.

$$\text{hint:} \qquad \frac{\partial E\left[\max_{k \in K_w}\{-c_{kw} + \tilde{\epsilon}_{kw}\}\right]}{\partial c_{kw}} = p_{kw}$$

Sheffi (1985) shows that the objective function is strictly convex; and the SUE is unique.

Method of Successive Averages (MSA) Algorithm – SUE

Set $f_a^1 = 0$

Calculate link costs, $\tau_a(f_a^1)$

Set step number $n = 1$

Until link flows converges, do

For each OD pair w

Calculate route costs, $c_{kw} = \sum_a \delta_{ak}\, \tau_a(f_a^n)$

Calculate route choice probabilities p_{kw}

Set route flows $x_{kw} = d_w p_{kw}$

Calculate resulting link flows: $g_a = \sum_w \sum_k \delta_{ak} x_{kw}$

Calculate new link flows: $f_a^{n+1} = \left(\frac{1}{n}\right) g_a + \left(1 - \frac{1}{n}\right) f_a^n$

Set $n = n + 1$

Exercise 4

Exercise4.xlsx contains 1000 iterations of the MSA algorithm for SUE problem.

- Check the calculations in a single iteration of the algorithm in **MSA-SUE** sheet.
- The route choice probabilities in range L15:L18 in **SUE** sheet are obtained from the route choice probabilities of the 1000^{th} iteration of the MSA algorithm.
- Note that the equilibrium is obtained.

seSue

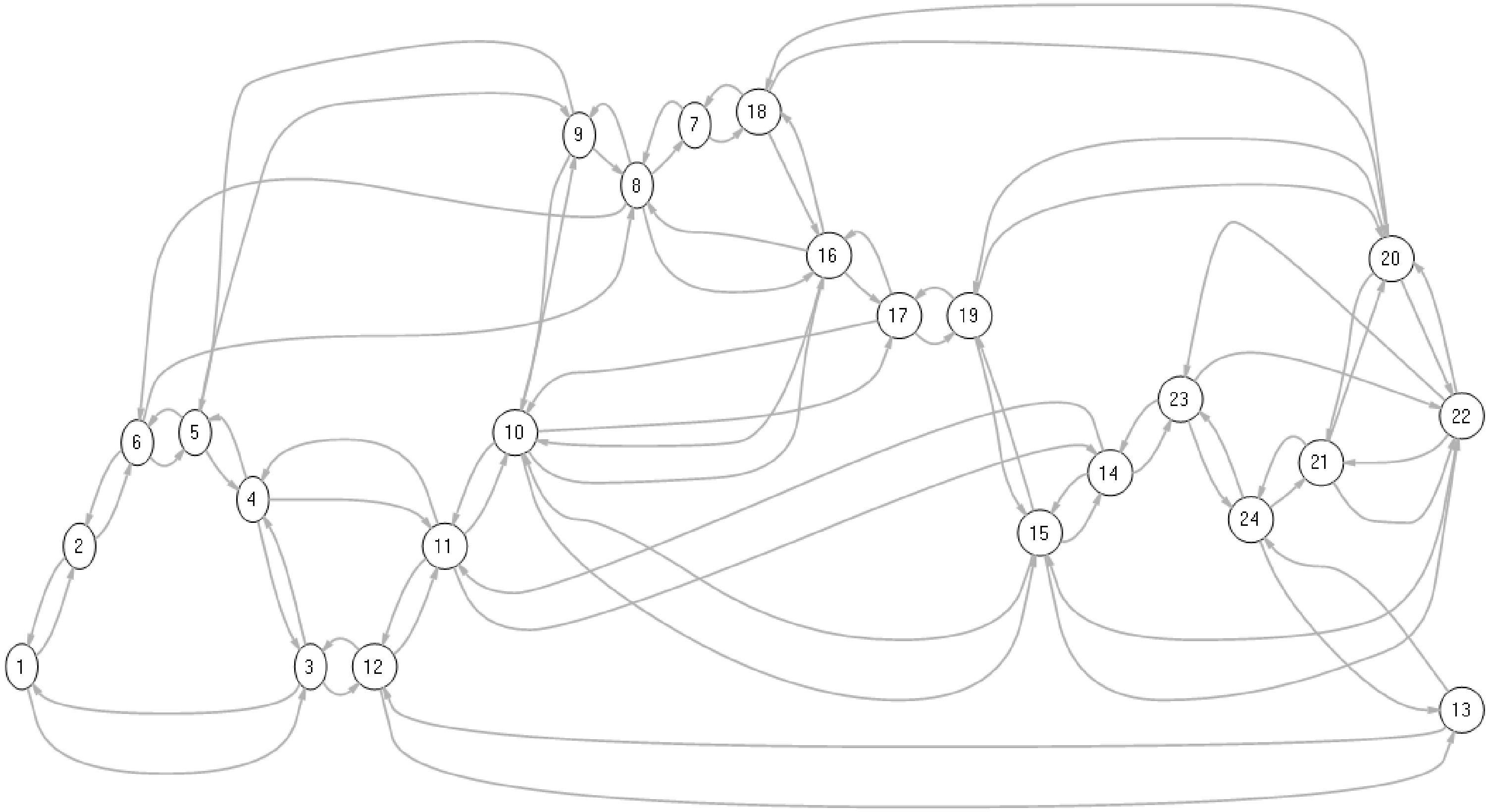
seSue is an open source tool to aid research on static path-based Stochastic User Equilibrium models developed in SUTD.

The features of the software are:

- route generation;
- calculating UE and SUE with different logit-based, weibit-based, and marginal distribution models;
- making sensitivity analysis; such as analyzing what-if scenarios on changes in demands or in link cost parameters.

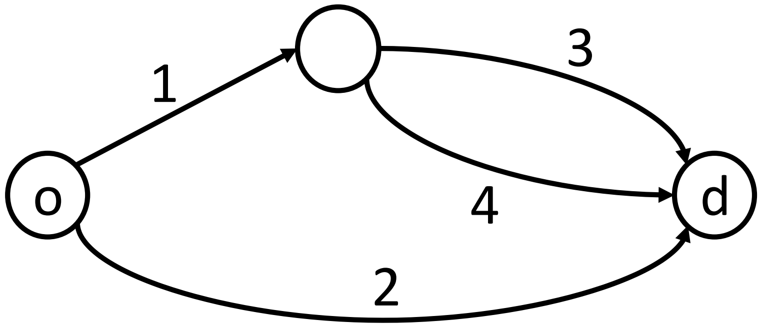
Exercise 5

Follow the steps in the file **seSueTrafficAssignment.pdf** to solve traffic assignment problem on a medium-sized real network.



Exercise 6 (exercise 12-27 from Garber & Hoel)

A flow of 10,000 vehicles in peak hour is to be distributed between three routes for the (o, d) pair.



BPR parameters of the links are given on the right.

Perform 2 MSA iterations using:

- UE conditions (Wardrop)

$$x_{kw} > 0 \quad \Rightarrow \quad c_{kw} = \min_{l \in K_w} c_{lw}$$

- SUE with MNL(0.1)

$$p_{kw} = \frac{\exp[-0.1\ c_{kw}]}{\sum_{l \in K_w} \exp[-0.1 c_{lw}]}$$

Link	FFTT	B	Capacity	Power
1	17	0.2	3.8	2.8
2	30	0.2	5.0	4.0
3	15	0.3	4.2	3.7
4	12	0.4	6.6	4.3

$$\tau_a(f_a) = FFTT_a \times \left[1 + B_a \times \left(\frac{f_a}{capacity_a}\right)^{power_a}\right]$$