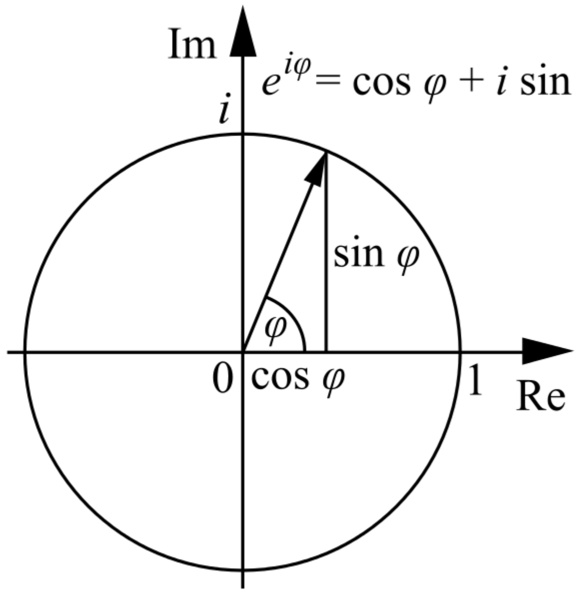
Tutorial 5
Fourier Analysis Made easy – Part 2

2013/12/25 Merry Christmas

Complex Representation of Fourier Series

- Euler's formula $e^{i\varphi} = \cos\varphi + i\sin\varphi$



- Complex Exponential $e^{j\omega t} = \cos\omega t + j\sin\omega t$, also called a Cisoid

Complex Representation of Fourier Series

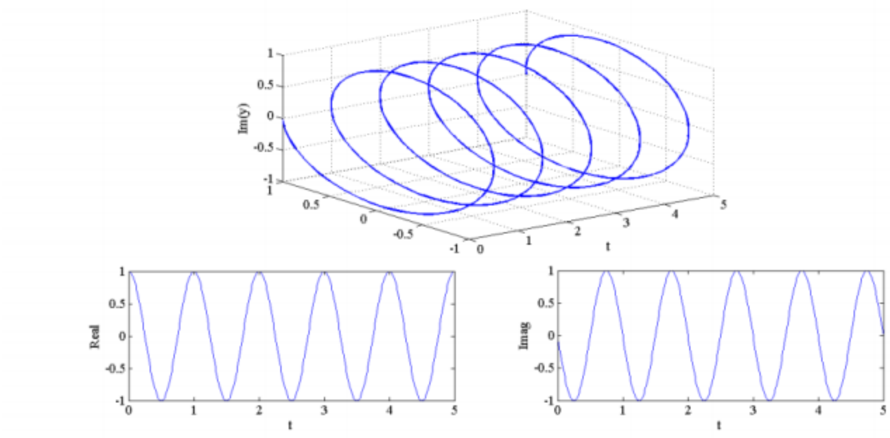
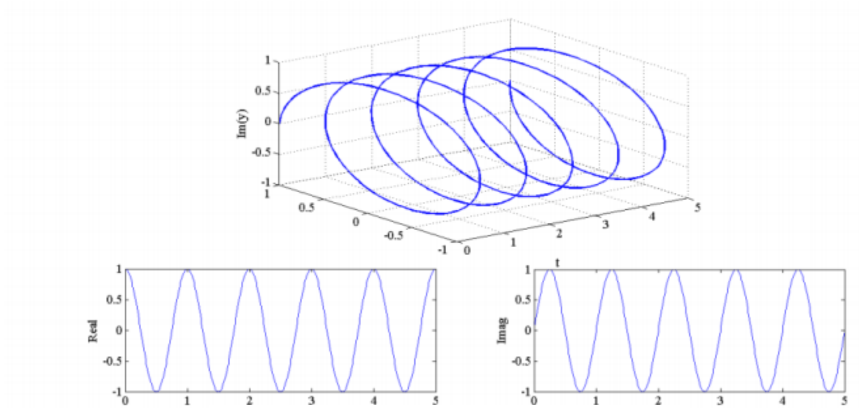


Figure 1 - $e^{j\omega t}$ is a helix. Its projection on the real axis is a cosine and on the imaginary axis is sine.



$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

Figure 2 – $e^{-j\omega t}$ is a helix rotating in the opposite direction. The projections on the real axis is a cosine and on imaginary axis is a negative sine.

Complex Representation of Fourier Series

- $$\sin \omega t = \frac{1}{2j}\left((\cos \omega t + j\sin \omega t) - (\cos \omega t - j\sin \omega t)\right)$$
$$= \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

- $$\cos \omega t = \frac{1}{2}\left((\cos \omega t + j\sin \omega t) + (\cos \omega t - j\sin \omega t)\right)$$
$$= \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

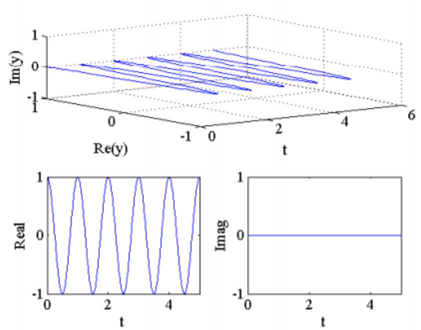


Figure 3 – Plotting $\left(e^{j\omega t} + e^{-j\omega t}\right)/2$ gives a a cosine wave with no projection on the imaginary axis.

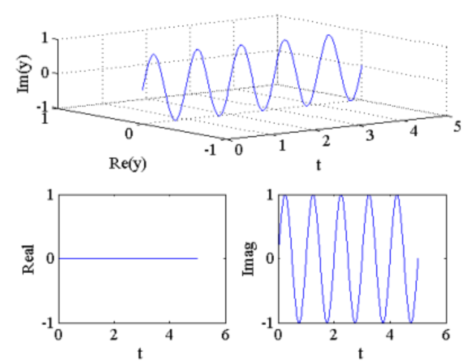


Figure 4 – Plotting $\left(e^{j\omega t} - e^{-j\omega t}\right)/2$ gives us a sine wave with no projection on the real axis.

Complex Representation of Fourier Series

- Taylor Series Representation of exponential e^x

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

- $$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots$$

$$= 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{3!} - \frac{\theta^2}{6!} - \frac{j\theta^7}{7!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^2}{6!} + \cdots$$ This is a cosine.

$$+ j\theta - \frac{j\theta^3}{3!} + \frac{j\theta^5}{3!} - \frac{j\theta^7}{7!} + \cdots$$ This is j times sine.

Complex Representation of Fourier Series

- $e^{j\pi/2} = \cancel{\cos(\pi/2)} + j\sin(\pi/2)$ $\quad$ $e^{j3\pi/2} = \cancel{\cos(3\pi/2)} + j\sin(3\pi/2)$

  $= 0 + j(1)$ $\quad$ $= 0 + j(-1)$

  $= j$ $\quad$ $= -j$

  $e^{j\pi} = \cos(\pi) + j\sin(\pi) = -1$

  $e^{j\pi} + 1 = 0$

- $x(t) = A\cos(\omega t + \theta)$

  $= \frac{A}{2} e^{j(\omega t+\theta)} + \frac{A}{2} e^{-j(\omega t+\theta)}$

  $= \frac{A}{2} e^{j\theta} e^{j\omega t} + \frac{A}{2} e^{-j\theta} e^{-j\omega t} = Q_+ e^{j\omega t} + Q_- e^{-j\omega t}$ $\qquad$ $Q_+ = \frac{A}{2} e^{j\theta}$:complex number

  $= A/2\left(\cos(\omega t + \theta) + \cancel{j\sin(\omega t + \theta)} + \cos(\omega t + \theta) - \cancel{j\sin(\omega t + \theta)}\right)$

  $= A\cos(\omega t + \theta)$

Complex Representation of Fourier Series

- Back to the Fourier Series…

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \cos(\omega_n t) + \sum_{n=1}^{N} b_n \sin(\omega_n t)$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \quad , \quad a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t \; dt \quad , \quad b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t \; dt$$

$$\Rightarrow \quad f(t) = a_0 + \sum_{n=1}^{N} \frac{a_n}{2}\left(e^{jn\omega t} + e^{-jn\omega t}\right) + \sum_{n=1}^{N} \frac{b_n}{2j}\left(e^{jn\omega t} - e^{-jn\omega t}\right)$$

$$a_n = \frac{2}{T}\int_0^T f(t)\,\frac{1}{2}\left(e^{jn\omega t} + e^{jn\omega t}\right)dt \quad , \quad b_n = \frac{2}{T}\int_0^T f(t)\,\frac{1}{2j}\left(e^{jn\omega t} - e^{-jn\omega t}\right)dt$$

$$\Rightarrow \quad f(t) = a_0 + \sum_{n=1}^{\infty} \overset{An}{\boxed{\frac{1}{2}(a_n - jb_n)}}\, e^{jn\omega t} + \sum_{n=1}^{\infty} \overset{Bn}{\boxed{\frac{1}{2}(a_n + jb_n)}}\, e^{-jn\omega t}$$

$$a_n = \frac{1}{T}\int_0^T f(t)e^{jn\omega t}dt + \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt$$

$$\Rightarrow \quad f(t) = \cancelto{0}{a_0} + \sum_{n=1}^{\infty} A_n e^{jn\omega t} + \sum_{n=1}^{\infty} B_n e^{-jn\omega t}$$

$$A_n = \frac{1}{T}\int_0^T f(t)e^{jn\omega t}dt \quad , \quad B_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt$$

Complex Representation of Fourier Series

- $$f(t) = \sum_{n=0}^{\infty} A_n e^{jn\omega t} + \sum_{n=0}^{\infty} B_n e^{-jn\omega t}$$ (without a dc offset)
- $$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$C_n = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-jn\omega t} dt$$, where $C_n = A_n + jB_n$

$$C_n = \sqrt{A_n^{\ 2} + B_n^{\ 2}}$$

$$\phi_n = \tan^{-1}\left(\frac{B_n}{A_n}\right)$$

Magnitude And Power Spectrum

- Amplitude Spectrum – the plot of coefficients we draw from the Fourier series coefficients.
- Power Spectrum – converted from the amplitude spectrum by the Parseval's relationship

$$P = \frac{1}{T}\int_0^T |x(t)|^2\, dt \xrightarrow{\text{In time domain}} P = \sum_{k=-\infty}^{\infty} |x_k|^2$$

- For a discrete sinusoid of amplitude A, its average power is equal to $P_{avg} = A^2/2$
- For a constant signal, then its average power is just A^2. In frequency domain, we can calculate it by

$$P_{avg} = \mathrm{a}_0{}^2 + \sum_{n=1}^{\infty}\left(\frac{1}{2}a_n^2 + \frac{1}{2}b_n^2\right) \xrightarrow{\text{In complex domain}} = \sum_{n=-\infty}^{\infty} C_n C_{-n}$$

$$|C_0|^2 = \frac{1}{2}|a_0|^2,\ \ |C_n|^2 = |C_{-n}|^2 = \frac{1}{4}\left(a_n^2 + b_n^2\right)$$

Examples of Computing Fourier Complex Coefficients

- Example 1 – Computer complex coefficients of a cosine wave $f(t) = A\cos\omega t \;= \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t}$

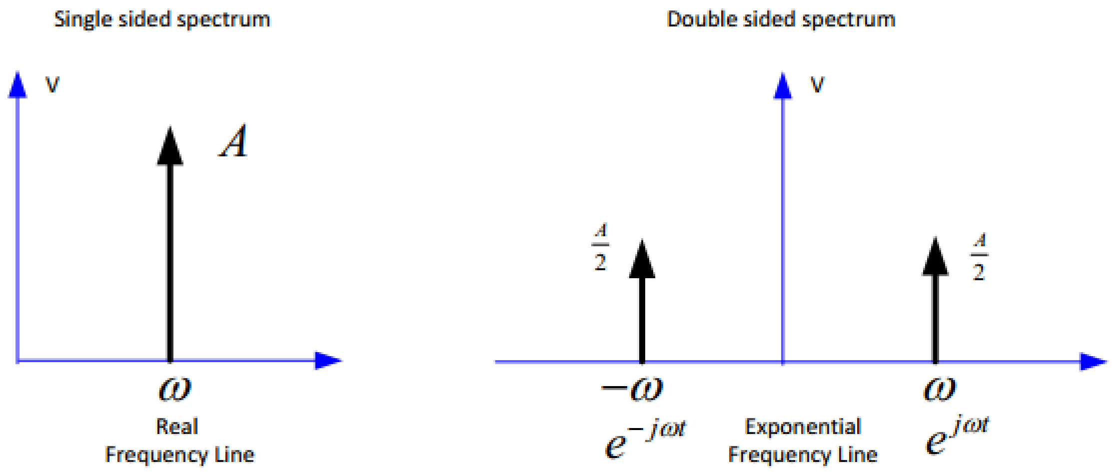


Figure 5 – Amplitude spectrum of $A\cos\omega t$

Examples of Computing Fourier Complex Coefficients

- Example 2 – Computer complex coefficients of a sine wave

$$f(t) = A\sin\omega t = \frac{A}{2j}e^{j\omega t} - \frac{A}{2j}e^{-j\omega t}$$

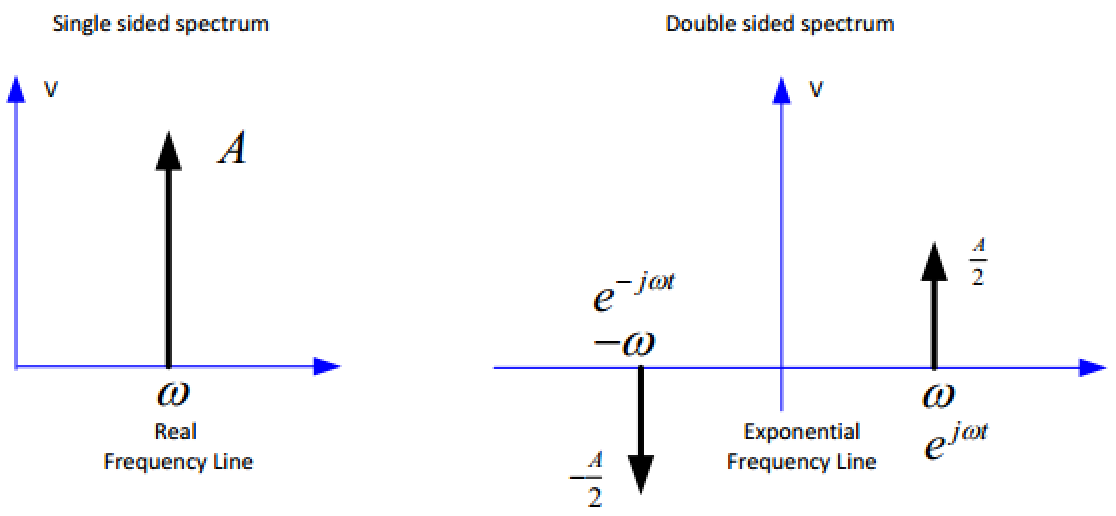
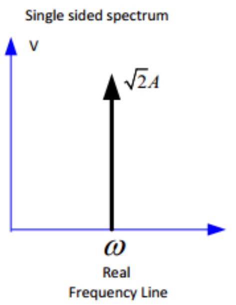


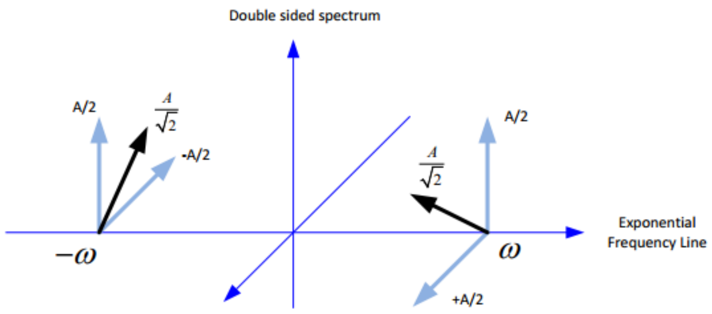
Figure 6 – Amplitude spectrum of $A\sin\omega t$

Examples of Computing Fourier Complex Coefficients

- Example 3 – Computer coefficients of

$$f(t) = A(\cos\omega t + \sin\omega t) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} + \frac{A}{2j}e^{j\omega t} - \frac{A}{2j}e^{-j\omega t}$$





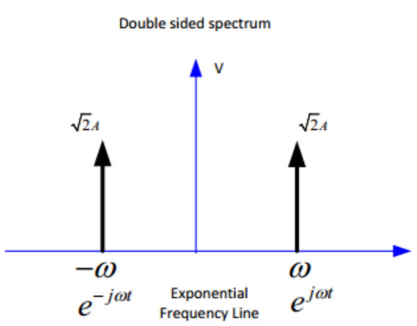


Figure 7 – Amplitude spectrum of $A\sin\omega t + A\cos\omega t$

Examples of Computing Fourier Complex Coefficients

- Example 4 – Computer coefficients of

$$f(t) = A(\cos w + j\sin wt) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} + \frac{A}{2}e^{j\omega t} - \frac{A}{2}e^{-j\omega t} = Ae^{j\omega t}$$

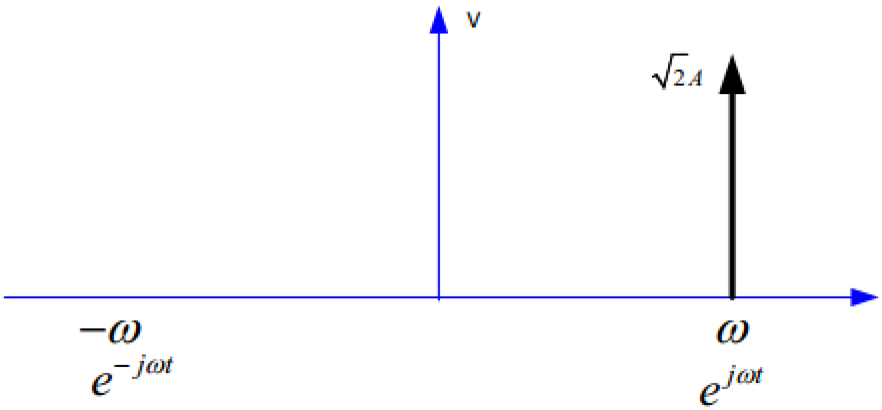


Figure 8 – Double-sided spectrum of $A\cos\omega t + jA\sin\omega t$

Examples of Computing Fourier Complex Coefficients

- Example 5 – Computer coefficients of

$$f(t) = A\cos(\omega = 0)t \ = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} = A$$

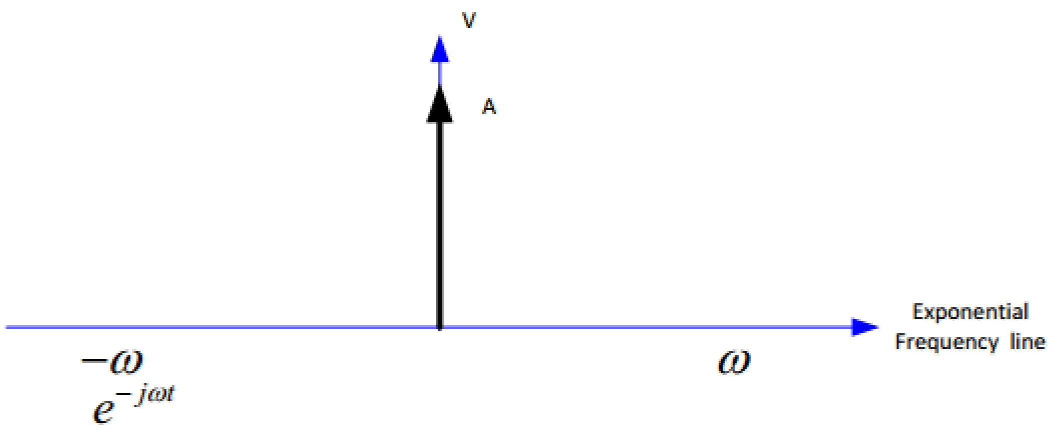


Figure 9 – Double-sided spectrum of A

Examples of Computing Fourier Complex Coefficients

- Example 6 – Computer coefficients of

$$x(t) = 2\cos^2(\omega t) \;\; = 2\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)^2 = 1 + \frac{1}{2}e^{j2\omega t} + \frac{1}{2}e^{-j2\omega t}$$

DC

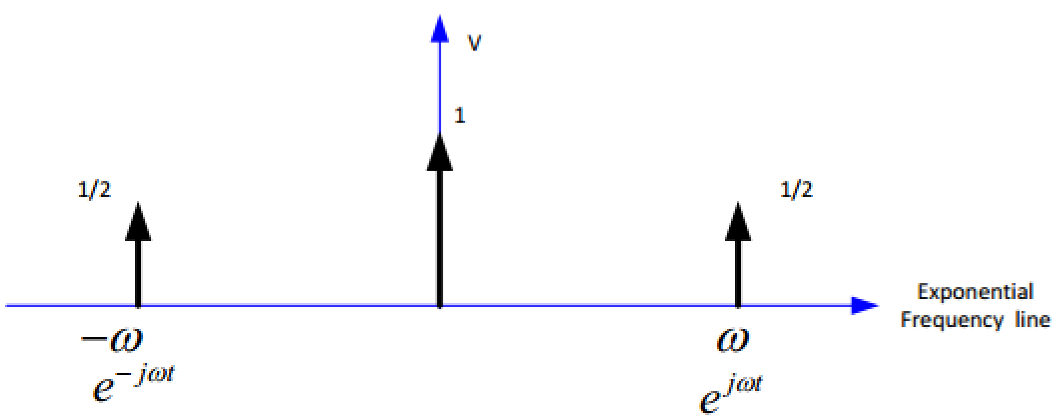


Figure 10 – Double-sided amplitude spectrum of $2\cos^2(\omega t)$

Examples of Computing Fourier Complex Coefficients

- Example 7 – Computer coefficients of

$$x(t) = 2\cos(\omega t)\cos(2\omega t) = \cos(\omega t) + \cos(3\omega t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} + \frac{1}{2}e^{j3\omega t} + \frac{1}{2}e^{-j3\omega t}$$

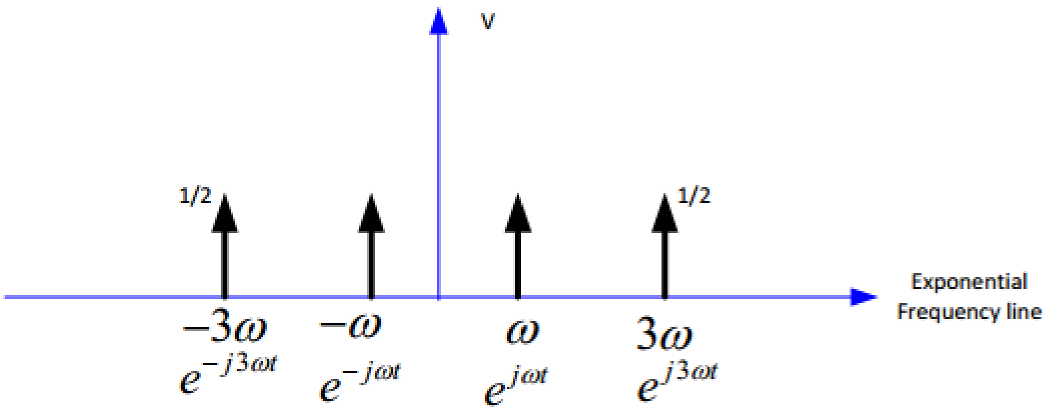


Figure 11 – Double-sided amplitude spectrum of $2\cos(\omega t)\cos(2\omega t)$

Examples of Computing Fourier Complex Coefficients

- Example 8 – Computer complex coefficients and power spectrum of a real signal

$$f(t) = .8\cos 2\pi t - .6\sin 2\pi t + .8\cos 4\pi t + .3\sin 7\pi t$$

$$= \frac{0.8}{2}e^{j2\pi t} + \frac{0.8}{2}e^{-j2\pi t} - \frac{0.6}{2j}e^{j2\pi t} + \frac{0.6}{2j}e^{-j2\pi t} + \frac{0.8}{2}e^{j4\pi t} + \frac{0.8}{2}e^{-j4\pi t} + \frac{0.3}{2j}e^{j7\pi t} - \frac{0.3}{2j}e^{-j7\pi t}$$

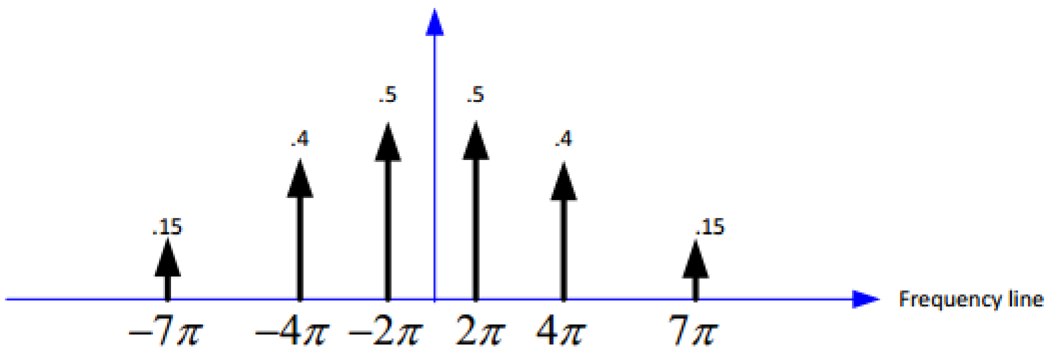


Figure 12 - Two-sided spectrum

Examples of Computing Fourier Complex Coefficients

- Example 9 – Computer complex coefficients of a real signal with variable phase

$$x(t) = 3 + 6\cos(4\pi t + 2) + j4\sin(4\pi t + 3) - j6\sin(10\pi t + 1.5)$$
$$= 3 + \left(3e^{4\pi t}e^{2} + 3e^{-4\pi t}e^{-2}\right) + \left(2e^{4\pi t}e^{3} - 2e^{-4\pi t}e^{-3}\right) + \left(3e^{10\pi t}e^{1.5} - 3e^{-10\pi t}e^{-1.5}\right)$$
$$= 3 + e^{j4\pi t}\left(3e^{2j} + 2e^{3j}\right) + e^{-j4\pi t}\left(3e^{-2j} - 2e^{-3j}\right) + 3e^{j10\pi t}\left(e^{j1.5}\right) + 3e^{-j10\pi t}\left(e^{-j1.5}\right)$$

$$e^{j4\pi t}\left(3e^{2j} + 2e^{3j}\right)$$
$$= \sqrt{\left(3\cos(2) + 2\cos(3)\right)^2 + \left(3\sin(2) + 2\sin(3)\right)^2}$$
$$= 6.16e^{j4\pi t}$$

$$e^{-j4\pi t}\left(3e^{-2j} - 2e^{-3j}\right)$$
$$= \sqrt{\left(3\cos(2) - 2\cos(3)\right)^2 + \left(3\sin(2) - 2\sin(3)\right)^2}$$
$$= 3.46e^{-j4\pi t}$$

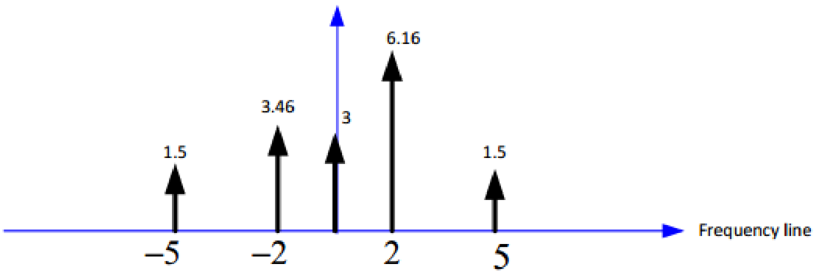


Figure 13 - Two-sided spectrum of a complex signal

A Little Bit about Complex Numbers

- Converting forms
  - Rule
    - 1. A rectangular form z = x +jy then its polar form is equal to

$$M\angle\theta = \sqrt{x^2+y^2}\angle\tan^{-1}(y/x)$$
$$= \sqrt{x^2+y^2}\angle\left(\tan^{-1}(y/x)+\pi\right) \quad \text{if } x < 0$$

    - 2. Given a polar form z $= M\angle\theta$ then its rectangular form is given by

$$x + jy = M\cos\theta + jM\sin\theta$$

  - Example 1: Convert z = ∠. 927 to rectangular form
    - Real part = 5cos(.927) = 3
    - Imaginary part = 5sin(.927)=4
    - Z = 3+j4

A Little Bit about Complex Numbers

- Example 2: Convert z = -1-j to rectangular form

$$M = \sqrt{(-1)^2 + (-j)^2} = \sqrt{2}$$

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{y}{x}\right) + \pi \quad \text{If } x < 0 \quad = \arctan\left(\frac{-1}{-1}\right) + \pi = 3\pi / 4$$

$$z = \sqrt{2}\angle 3\pi / 4$$

- Example 3: Convert z = 1+j to polar form

$$\text{Z} = \sqrt{1^2 + j^2} = \sqrt{2} \text{ and } \theta = \arctan(1) = \pi / 4 = .785$$

A Little Bit about Complex Numbers

- Adding and Multiplying
  - add in rectangular form, multiply in polar.
  - Rule
    - 1. Given Z1 = a + jb and Z2 = c + jd then Z1+Z2 = (a+c)+j(b+d)
    - 2. Given Z1=$M_1\angle\theta_1$ and Z2=$M_2\angle\theta_2$ then Z3=Z1*Z2=$M_1M_2\angle(\theta_1+\theta_2)$
  - Example 1: Add Z1=$\sqrt{2}\angle.785$ and Z2=$5\angle.927$
    - Convert both to rectangular form
    - Z1=1+j and Z2=3+4j
    - Z3=(1+3)+j(1+4)=4+j5
  - Example 2: Multiply Z1=1+j and Z2=3+4j
    - Z1 =$\sqrt{2}\angle.785$*Z2=$5\angle.927$=5 $\sqrt{2}\angle 1.71$
  - Example 3: Divide Z1=1+j and Z2=3+4j
    - $z_2 = 5\angle.927 \div z_1 = \sqrt{2}\angle.785$
    - $z_3 = \frac{z_2}{z_1} = \frac{5}{\sqrt{2}}\angle(.927 - .785)$

A Little Bit about Complex Numbers

- Conjugation
  - The conjugate for a complex number z, is given by $z^* = x - jy$
  - $e^{j\omega t}$ is the complex conjugate of $e^{-j\omega t}$
  - $z = M\angle\theta$, $z^* = M\angle -\theta \longrightarrow$
  - $|z|^2 = zz^*$ (used to compute the power of the signal)
  - $|z| = \frac{1}{2}\left(z + z^*\right)$ (used to computer the magnitude of the signal)

Matlab Programs

Matlab Program no. 1 for plotting the complex exponential

```
%comexp produces the complex exponential diagram in Chapter 1.
%Try changing target function to see effect on signal.
t = 0:0.01:5;
y=5*exp(-(j*(2*pi))*t); % change this equation for different cases
subplot(2,2,1);
plot3(t,real(y),imag(y));
grid
xlabel('t'),ylabel('Re(y)'),zlabel('Im(y)');
title('3-D plot of a Complex Exponential');

subplot(2,2,3),plot(t,real(y)),xlabel('t'),
ylabel('Magnitude'),title('Re(y(t))');

subplot(2,2,4),plot(t,imag(y)),xlabel('t'),
ylabel('Angle'),title('Im(y(t))');
```

Matlab Programs

Matlab Program no. 2 – Compute double sided DFT of a signal

```
clc;close all; clear all;
%Generate the signal
fs=128; % Sampling rate should be at least 16 times higher frequency to get a good picture.
N=1024; % FFT size.
t=0:1/fs:((N/fs)-(1/fs)); %Time it takes to creat N points.
x=3+  6*cos(8*pi*t+2) +j*8*sin(8*pi*t+3) - j*6*sin(30*pi*t+1); %The target signal.
figure(1) ;
plot(t, x); %Plot signal, all points.
%fplot('x', [0,1]) % want to plot the first 1 second only.
xlabel('Time (Hz)');ylabel('Signal in time domain ' ) ;

figure(2) ;
xf1=abs(fft(x) )/N;% Compute the Double sided amplitude spectrum
xf=fftshift(xf1);
P=xf.*xf; %compute the power spectrum
% Map the frequency bin to the frequency (hz)
f=LINSPACE(0, fs, N);
f3=[-fs/2:fs/2:1024];%fk=k fs/N where k=0,1,2,...N-1
f2 = linspace(-fs/2, fs/2, 1024);
% now we will plot the DFT spectrums
plot(f2,xf);grid
xlabel('Frequency (Hz)');ylabel('Double-sided Amplitude Spectrum (DFT) ' ) ;
```