Tutorial 4
Fourier Analysis Made easy – Part 1

2013/12

Fourier Analysis

- The process of breaking down any arbitrary wave into its harmonic components and identifying their contents, that is their amplitudes.
- Any arbitrary wave such as this that is periodic can be represented by a sum of sine and cosine waves.

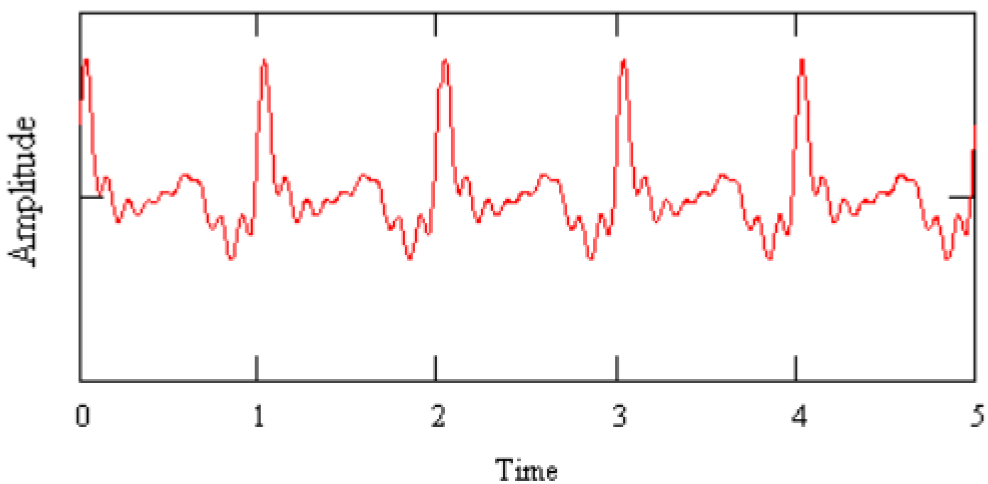


Figure 5 – An arbitrary signal of interest

Fourier Analysis

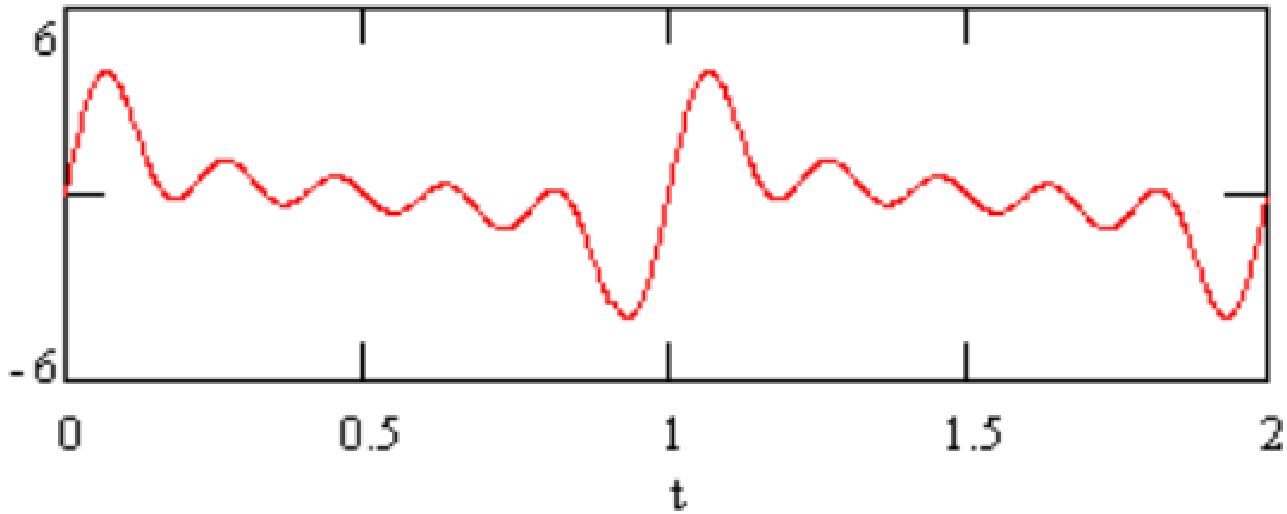


Figure 7 - The sum of four sine waves.

- Any two frequencies if their ratio is an integer are harmonic to each other, hence these waves are harmonics of each other. We write the sum of K such harmonic sine waves as

$$f(t) = \sum_{n=1}^{K} \sin(n\omega t)$$

Fourier Analysis

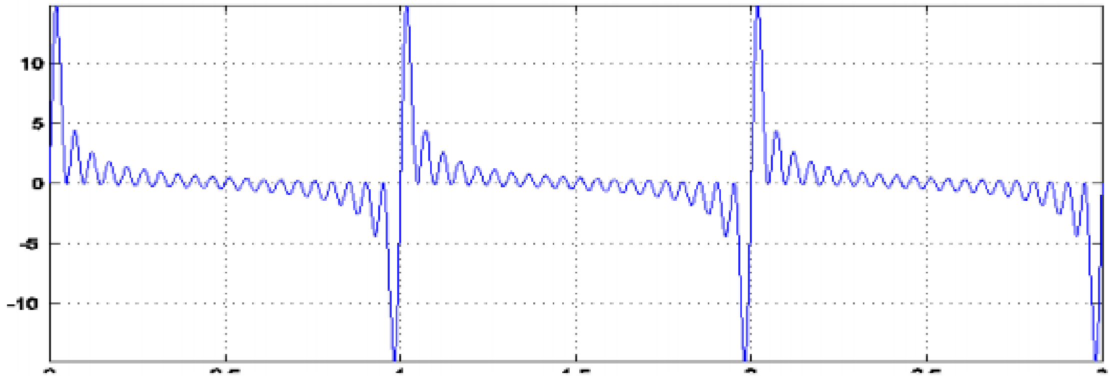


Figure 8 - The sum of 20 harmonic sine waves.

- the peak value is not the sum of the number of harmonics because all harmonic sine waves cross the x-axis at the same time but never peak at the same time - opposed to cosine
- The closed form of the summation of sine waves

$$\sum_{n=0}^{N} \sin(n\omega) = \frac{\sin\left(\frac{1}{2}N\omega\right)\sin\left(\frac{1}{2}(N+1)\omega\right)}{\sin\left(\frac{1}{2}N\omega\right)}$$

Fourier Analysis

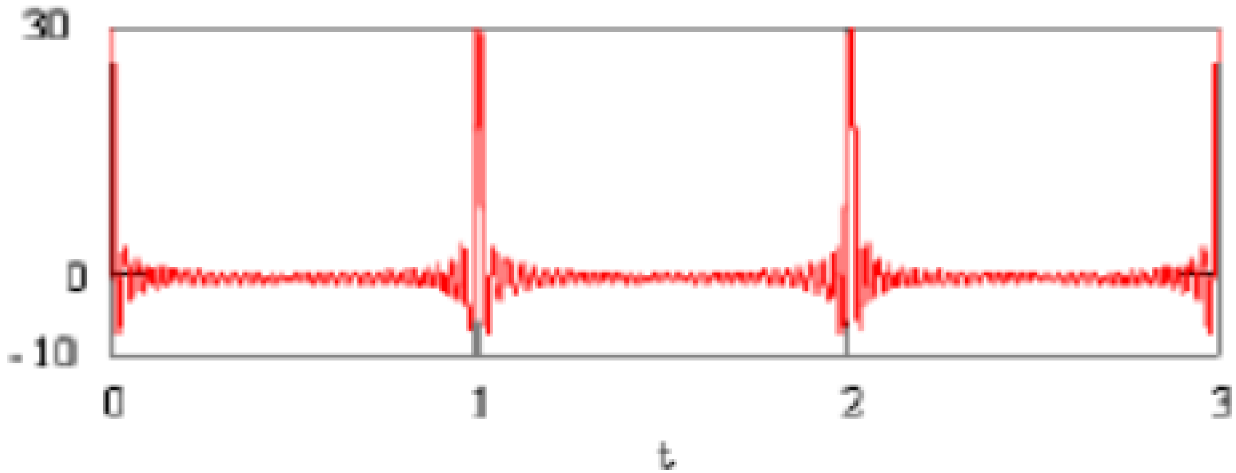


Figure 13 - Sum of 30 cosine waves of equal amplitude

- The closed from of the summation of cosine waves

$$\sum_{n=0}^{N}\cos(n\omega)=\frac{\cos\left(\frac{1}{2}N\omega\right)\sin\left(\frac{1}{2}(N+1)\omega\right)}{\sin\left(\frac{1}{2}N\omega\right)}$$

Special Signals

- We can create a **square wave** by the summing **odd** harmonics of sine waves (odd functions f(x)=-f(-x))

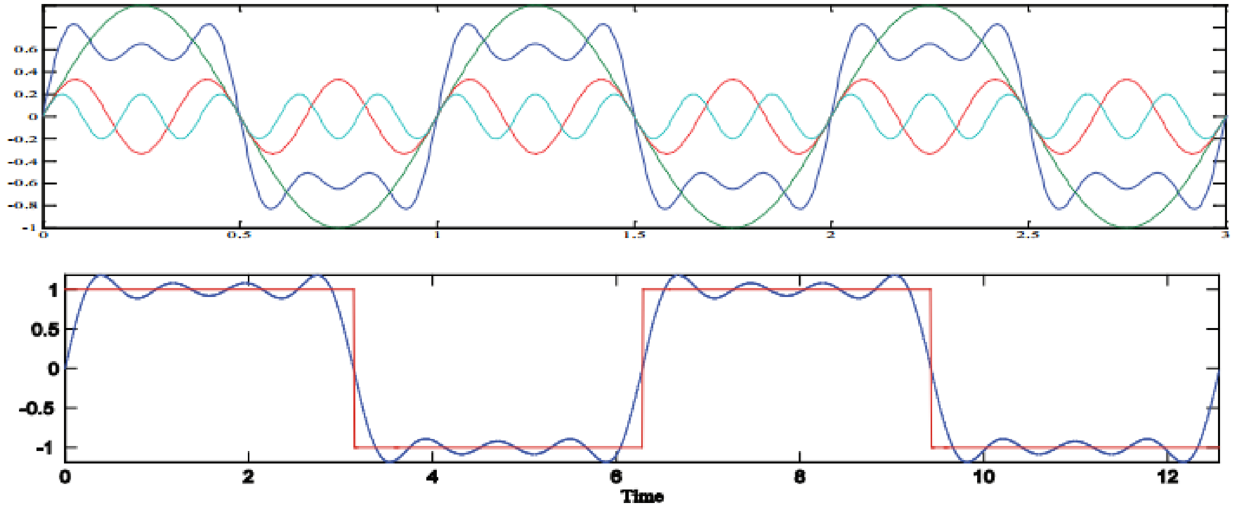


Figure 15 – Creating a square wave from odd harmonics of sines, (a) 3 sines waves, (b) 5 sine waves

$$square(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\left(\sin 2\pi\left(2k-1\right)ft\right)}{\left(2k-1\right)}$$

$$= \frac{4}{\pi}\left(\sin(2\pi f t) + \frac{1}{3}\sin(6\pi f t) + \frac{1}{5}\sin(10\pi f t) + \cdots\right)$$

Special Signals

- Triangular wave

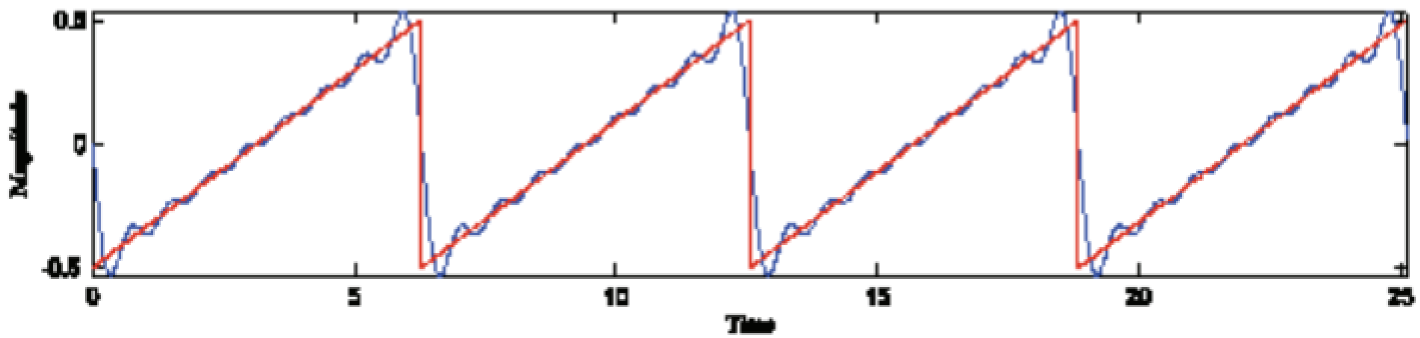


Figure 16 – Triangular wave represented by sines

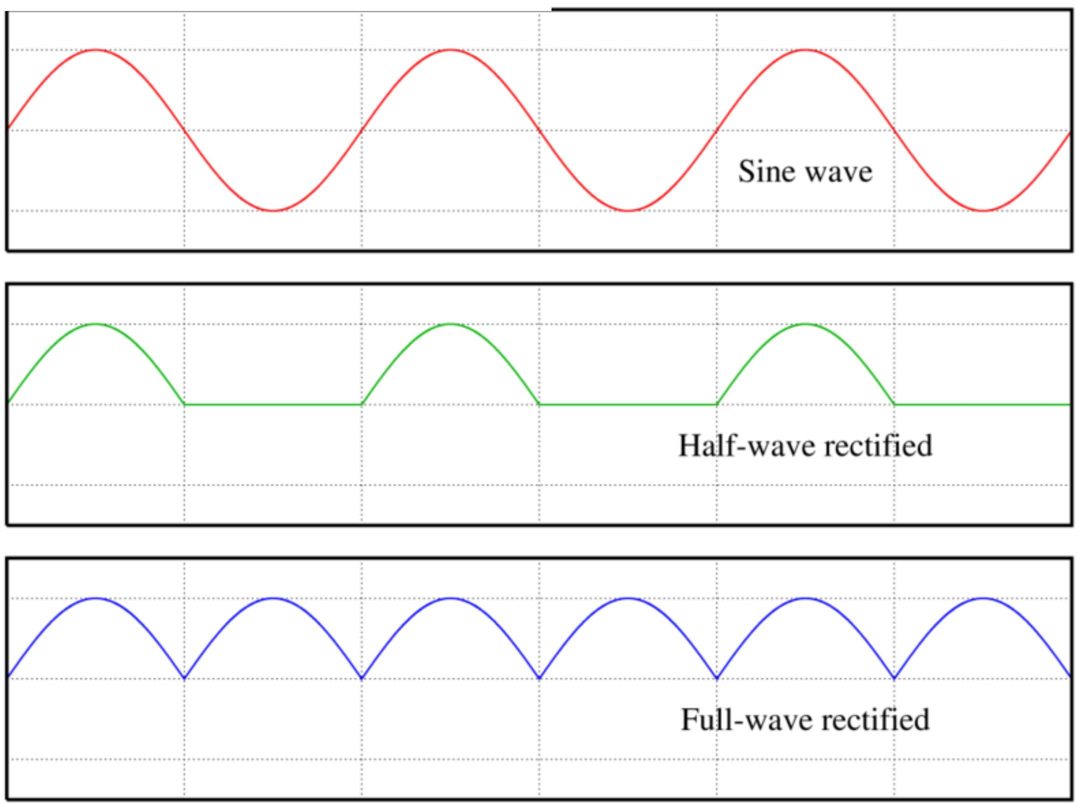
$$traingle(t) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}(-1)^k \frac{\left(\sin 2\pi\left(2k+1\right)ft\right)}{\left(2k+1\right)^2}$$

$$= \frac{8}{\pi^2}\left(\sin(2\pi f t) - \frac{1}{9}\sin(6\pi f t) + \frac{1}{25}\sin(10\pi f t) + \cdots\right)$$

- Sawtooth wave $sawtooth(t) = \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin(2\pi kft)}{k}$

Special Signals

- Rectified Wave
  - Even fuction f(x)=f(-x), created by cosine waves
  - $f(t)=\frac{2A}{\pi}-\frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2\pi f_0 t)}{(4n^2-1)}$



Fourier Series

- $f_1(t) = \sum_{n=1}^{\infty} \sin(2n\pi f t)$, $f_2(t) = \sum_{n=1}^{\infty} \cos(2n\pi f t)$, where f is the fundamental frequency.
- $f_1(t) = \sum_{n=1}^{\infty} \sin(2\pi f_n t)$, $f_2(t) = \sum_{n=1}^{\infty} \cos(2\pi f_n t)$, where f is the nth harmonic of the fundamental.
- $$f(t) = \sum_{n=1}^{N} a_n \sin(2\pi f_n t) + \sum_{n=1}^{N} b_n \cos(2\pi f_n t)$$
  - The coefficients an represent the coefficient of the nth sine wave and bn of the nth cosine wave.
  - The sum of sine and cosines is always symmetrical about the x-axis so there is no possibility of representing a wave with a dc offset.

provides us with the needed dc offset

- $$\boxed{f(t) = a_0 + \sum_{n=1}^{N} a_n \sin(2\pi f_n) + \sum_{n=1}^{N} b_n \cos(2\pi f_n t)} \Rightarrow$$ Fourier series equation

Fourier series with many faces

- The representation is by radial frequency w

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \sin(\omega_n t) + \sum_{n=1}^{N} b_n \cos(\omega_n t)$$

- For discrete representation

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \sin 2\pi \frac{n}{T} t + b_n \cos 2\pi \frac{n}{T} t$$

  - T: the period of the fundamental frequency
  - T/n: the period of the nth harmonic
  - Fn = n/T
- The cosine representation

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(w_n t + \phi_n) \xrightarrow[\text{amplitude term}]{\text{Pull out the constant}} f(t) = a_0 + \frac{1}{2\pi} \sum_{n=1}^{\infty} (a_n \sin f_n t + b_n \cos f_n t)$$

  - written by incorporating a variable for the phase
- The equation starting at zero frequency

$$f(t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} (a_n \sin f_n t + b_n \cos f_n t)$$

- The exponent representation

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\pi t/T}$$

Computing the Fourier Coefficients

- Computing a_0 the dc coefficient

  $$f(t) = a_0 + \frac{1}{2\pi}\sum_{n=1}^{\infty}(a_n \sin f_n t + b_n \cos f_n t)$$

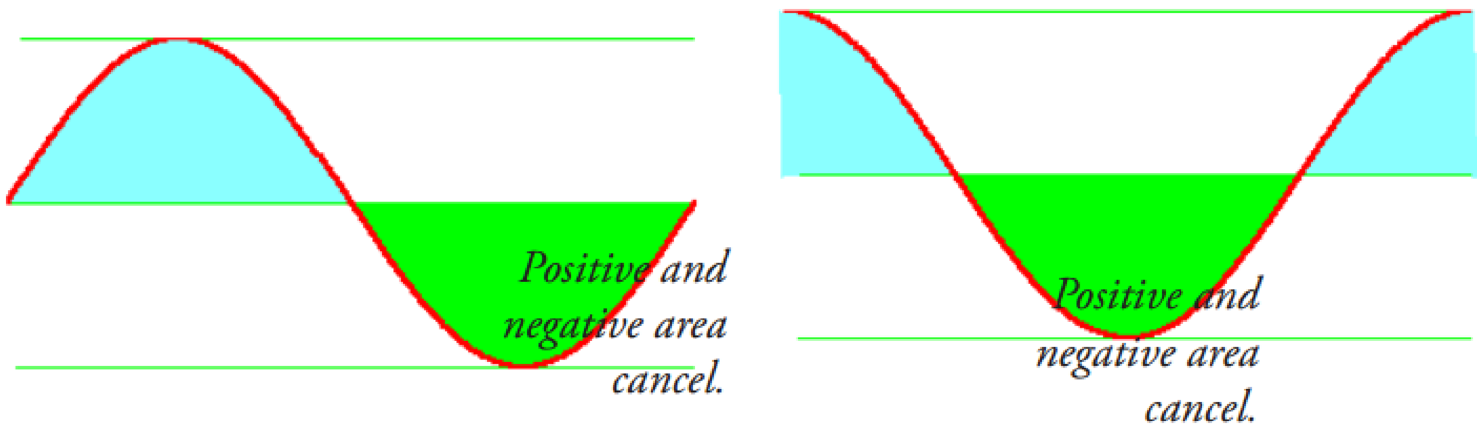
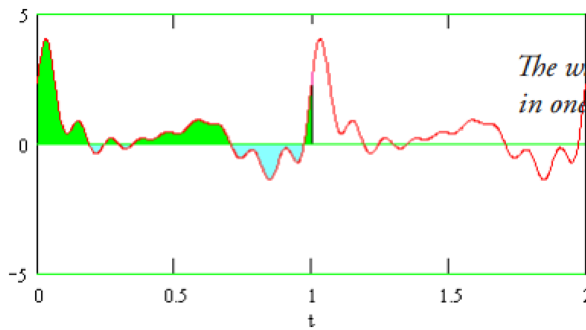
  - 

Figure 19 - The area under a sine and a cosine wave over one period is always zero.

  - $$\int_0^T f(t)dt = \int_0^T a_o dt + \int_o^T \left(\cancel{\sum_{n=1}^{\infty} a_n \sin nwt + b_n \cos nwt} \right) dt = \int_0^T a_o dt = a_0 T$$

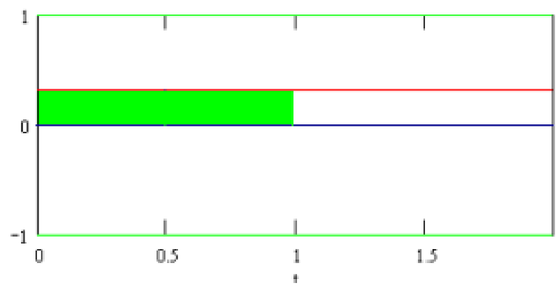
  - $$\boxed{a_0 = \frac{1}{T}\int_0^T f(t)dt}$$

Computing the Fourier Coefficients



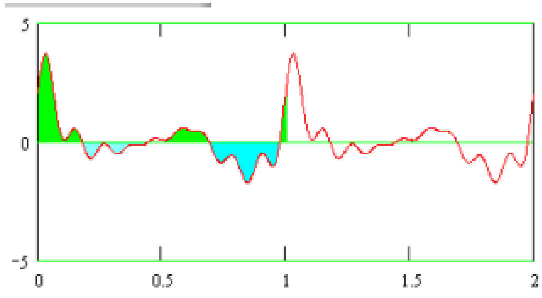
The wave has non-zero area in one period, which means it has a DC offset.

Figure 20 - Signal to be analyzed



All area comes from the a_0 coefficient.

Figure 21a - The dc offset of the signal

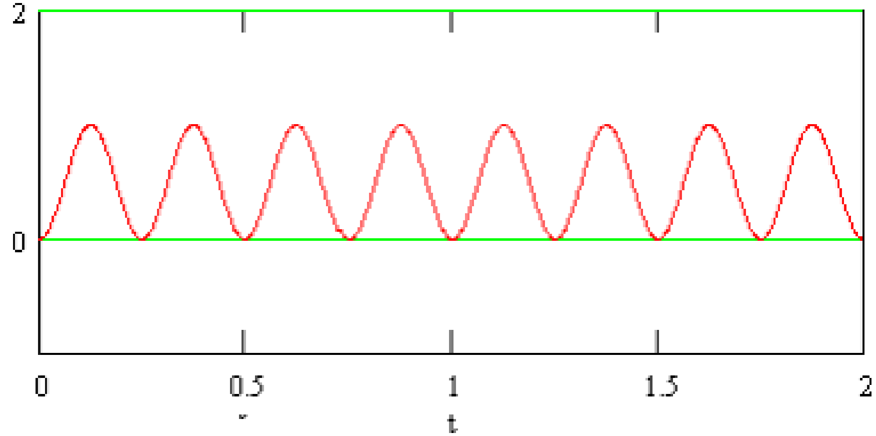


Area under the wave when shifted down is zero.

Figure 21b - Signal without the dc component

Computing the Fourier Coefficients

- Computing the coefficients of sine waves a_n
  - $f(t) = \sin n\omega t \times \sin n\omega t$



Multiplying one sine wave by any other causes the area under the new wave to become zero.

Figure 22 - The area under a sine wave multiplied by itself is always non-zero.

  - $\int_0^T a_n \sin nwt \sin nwt \, dt = a_n T / 2 \quad for\ n = m$

Computing the Fourier Coefficients

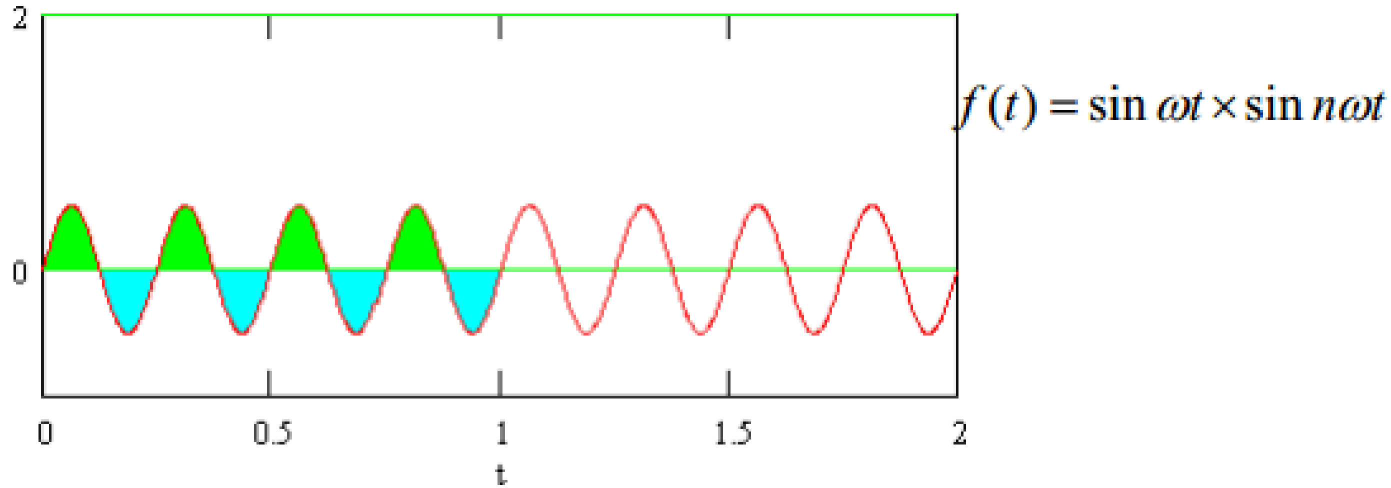


Figure 23 - The area under a sine wave multiplied by its own harmonic is always zero.

$$\int_0^T a_n \sin nwt \ \sin mwt \ dt = 0 \quad for\ n \neq m$$

$$\int_0^T a_n \sin nwt \sin nwt \ dt = a_n T / 2 \quad for\ n = m$$

Computing the Fourier Coefficients

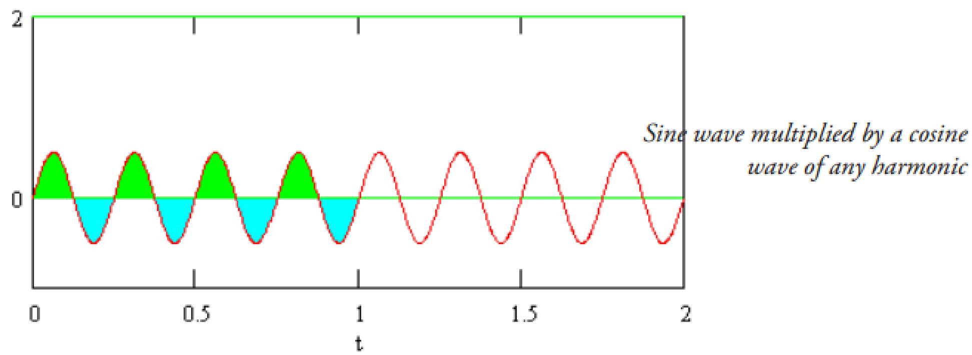
- 

Figure 24 - The area under a cosine wave multiplied by a sine wave is always zero.

$$\int_0^T a_n \sin nwt \ \sin mwt \ dt = 0 \quad \textit{for } n \neq m$$

$$\int_0^T a_n \sin nwt \sin nwt \ dt = a_n T/2 \quad \textit{for } n = m$$

$$\int_0^T a_n \cos nwt \sin mwt \ dt = 0$$

- $$\int_o^T \boxed{f(t)} \sin nwt \ dt = \int_0^T \boxed{a_0} \sin wt \ dt + \int_0^T \boxed{a_n \sin nwt} \sin nwt \ dt + \int_0^T \boxed{b_n \cos nwt} \sin nwt \ dt$$

$$\int_0^T a_n \sin nwt \times \sin nwt \ dt = \frac{a_n T}{2} \Rightarrow \boxed{a_n = \frac{2}{T}\int_0^T f(t) \sin nwt \ dt}$$

Computing the Fourier Coefficients

- Computing the coefficients of cosine waves **$\underline{b_n}$**

$$\int_{o}^{T} f(t)\cos nwt\, dt = \overset{0}{\int_{0}^{T} a_0 \cos wt\, dt} + \overset{0}{\int_{0}^{T} a_n \sin nwt \times \cos nwt\, dt} + \int_{0}^{T} b_n \cos nwt \times \cos nwt\, dt$$

$$\int_{0}^{T} b_n \cos nwt \times \cos nwt\, dt = \frac{b_n T}{2}$$

$$\boxed{b_n = \frac{2}{T}\int_{0}^{T} f(t)\cos nwt\, dt}$$

- Summary

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \sin(2\pi f_n t) + \sum_{n=1}^{N} b_n \cos(2\pi f_n t)$$

$$a_0 = \frac{1}{T}\int_{0}^{T} f(t)dt\ , \quad a_n = \frac{2}{T}\int_{0}^{T} f(t)\sin nwt\, dt\ , \quad b_n = \frac{2}{T}\int_{0}^{T} f(t)\cos nwt\, dt$$

Computing the Fourier Coefficients

- Example 1: Show that this wave has only odd harmonics.

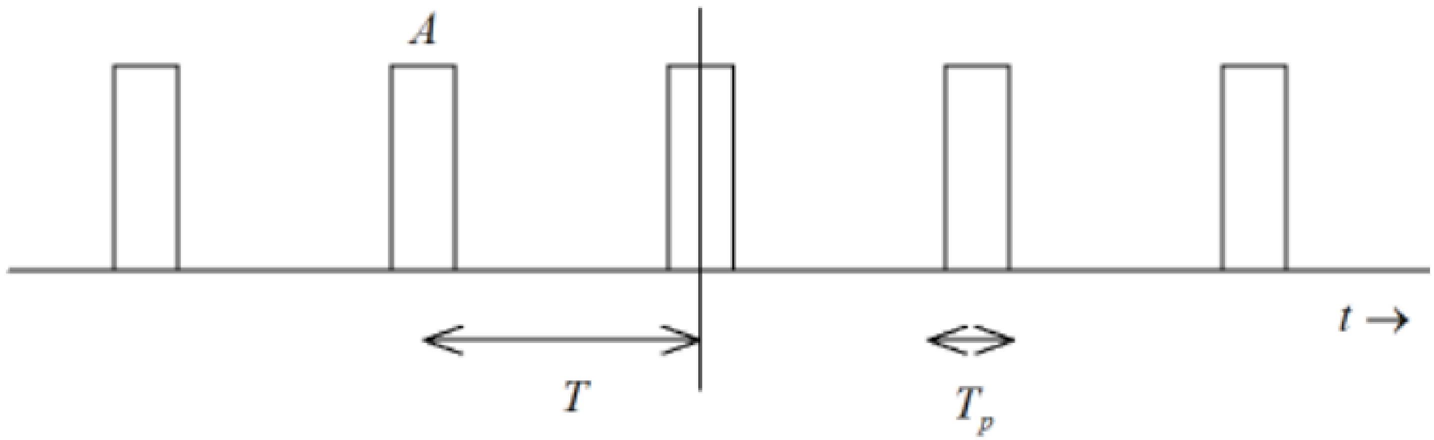


Figure 26 - Computing coefficients of a square pulse

$$b_n = \frac{2}{T}\int_{-T/8}^{T/8} A\cos\left(\frac{2\pi nx}{T}\right)dt$$

$$= \frac{A}{2\pi} 2\sin\left(\frac{2\pi n}{T}\cdot\frac{T}{8}\right)$$

$$= \frac{A}{2}\sin c\left(\frac{\pi n}{4}\right)$$

Computing the Fourier Coefficients

- Example 2: Find the coefficients of this wave.

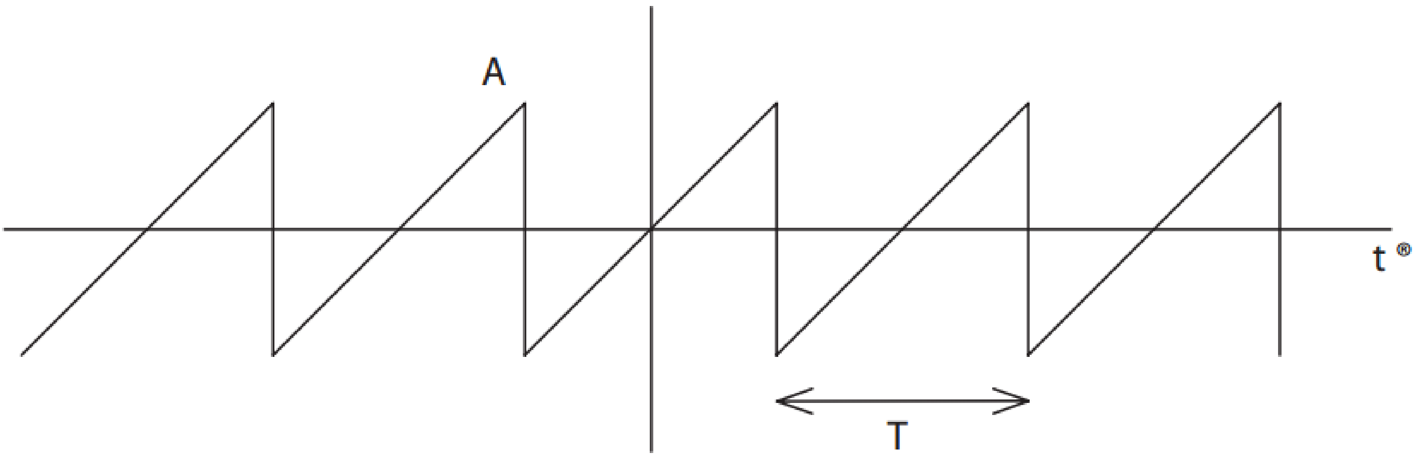


Figure 27 - A saw-tooth wave

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \frac{2tA}{T}\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$= 4\frac{A}{T^2}\left[-t\cos\left(\frac{2\pi nt}{T}\right)\frac{T}{2\pi n} + \frac{T^2}{4\pi^2 n^2}\sin\left(\frac{2\pi nt}{T}\right)\right]_{-T/2}^{T/2}$$

$$= -\frac{2A}{\pi n}\cos(\pi n)$$

$$a_0 = 0$$

$$a_n = (-1)^{n+1}\frac{2A}{\pi n}$$

Coefficients Become the Spectrum

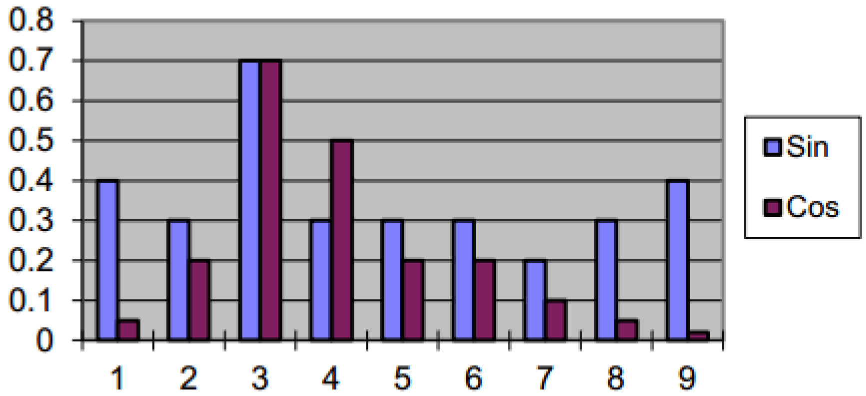


Figure 25 - The Fourier series coefficients for each harmonic

$$c_n = \sqrt{a_n^2 + b_n^2} \qquad \phi_n = \tan^{-1}(b_n / a_n)$$

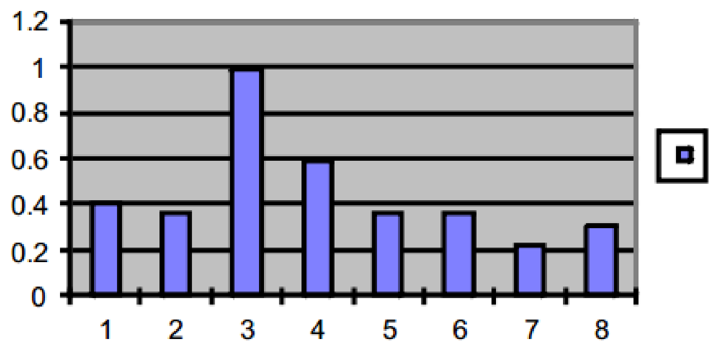


Figure 23 - A traditional looking spectrum created from the Fourier coefficients

Periodicity

- A real signal may not be periodic at all. The theory allows us to extend the "period" to infinity so we just pick any section of a signal or even the whole signal and call it "The Period", representing the whole signal.

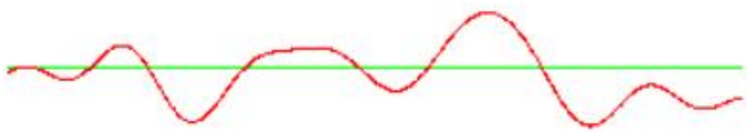


Figure 24 - We call the signal periodic, even though we don't know what lies at each end.

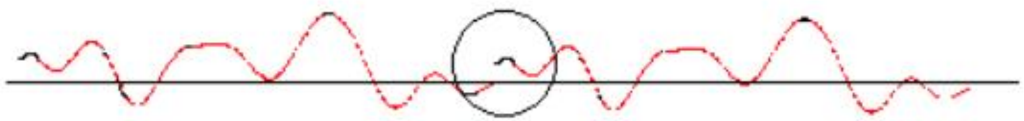


Figure 25 - Our signal repeated to make it mathematically periodic, but ends do not connect and have discontinuity