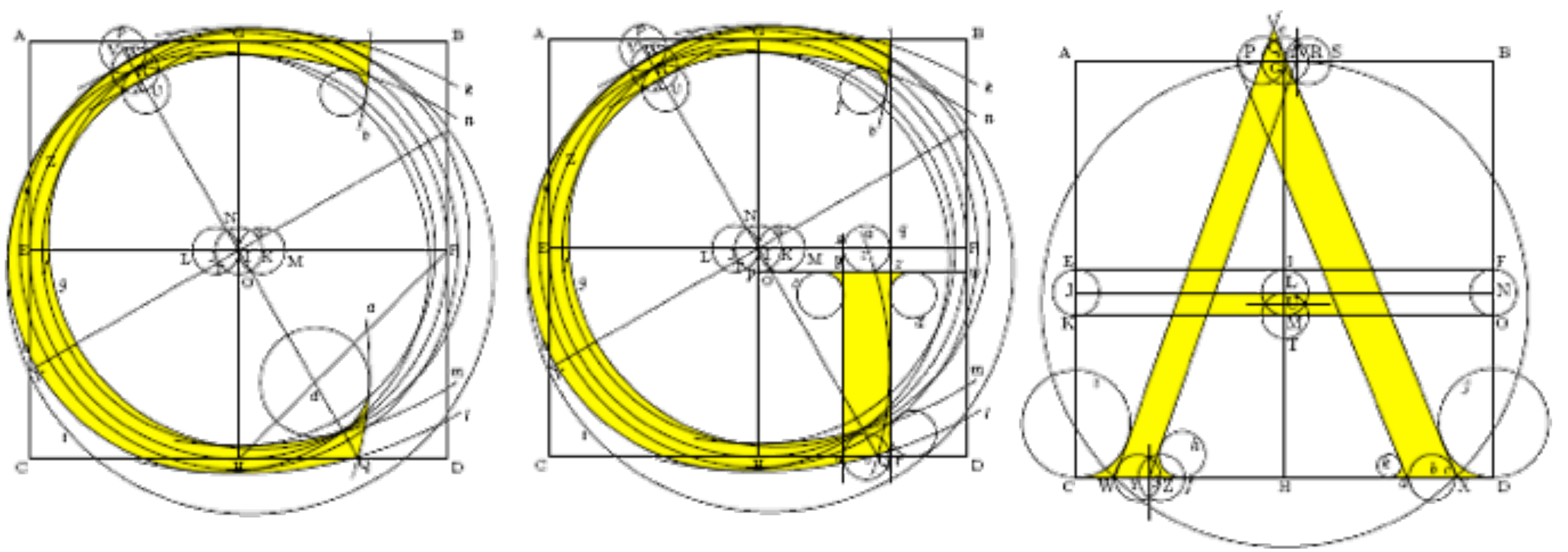
A VERY SHORT INTRODUCTION TO

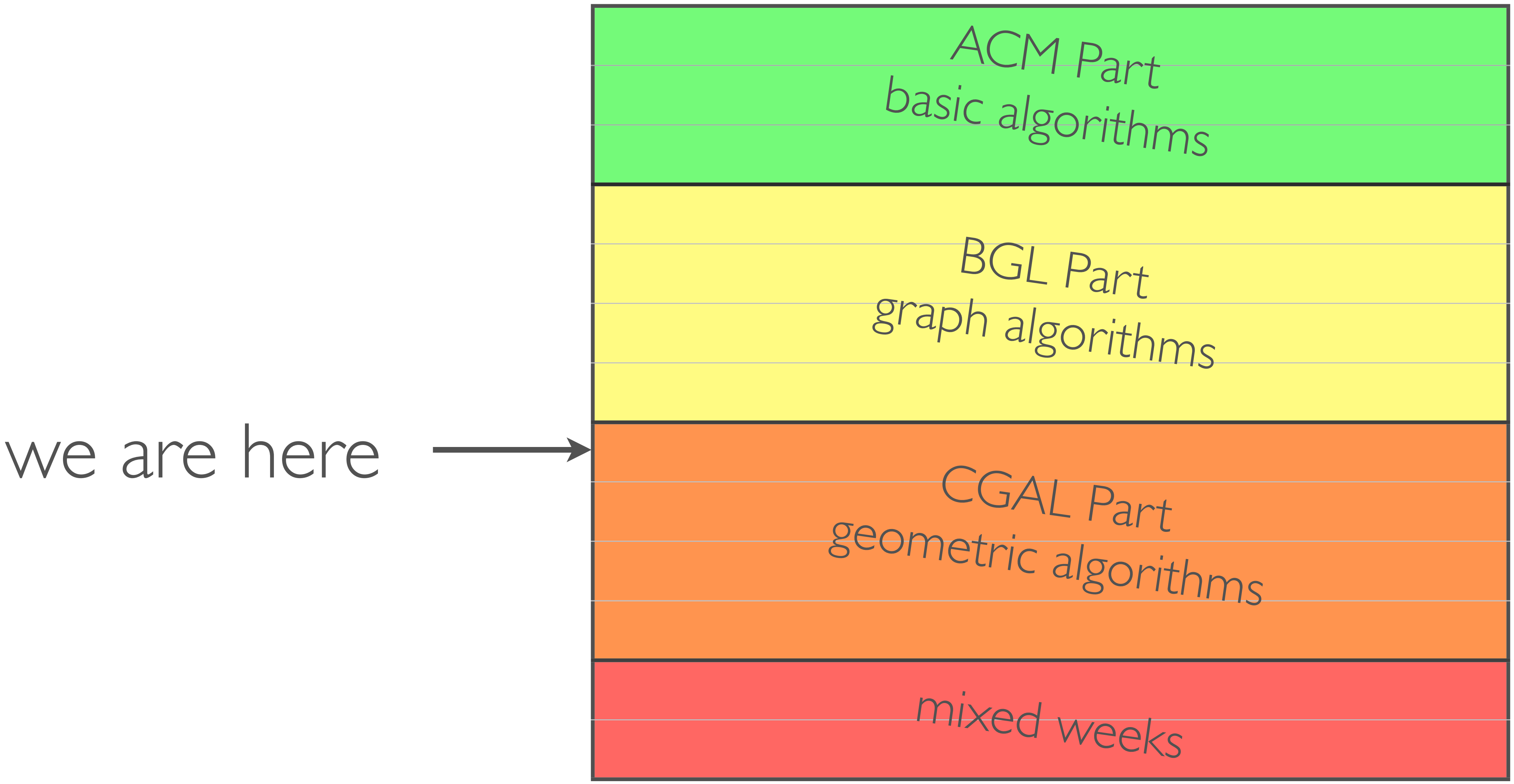


The Computational Geometry Algorithms Library

Michael Hoffmann <hoffmann@inf.ethz.ch>

(Based on work by Pierre Alliez, Andreas Fabri, Efi Fogel, Lutz Kettner, Sylvain Pion, Monique Teillaud, Mariette Yvinec, and probably many others.)

ALGOLAB TIMELINE



I: The CGAL Project

II: Exact Geometric Computing

III: Basic Programming using a CGAL Kernel

IV: Practical Information

PART I:

The CGAL Project:
History and Philosophy

THE MISSION

"Make the large body of geometric algorithms developed in the field of computational geometry available for industrial applications"

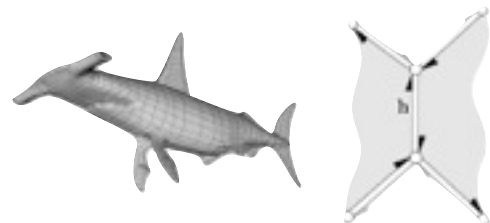
CGAL Project Proposal, 1996

Design goals: Reliability, efficiency, and flexibility.
Achieved through

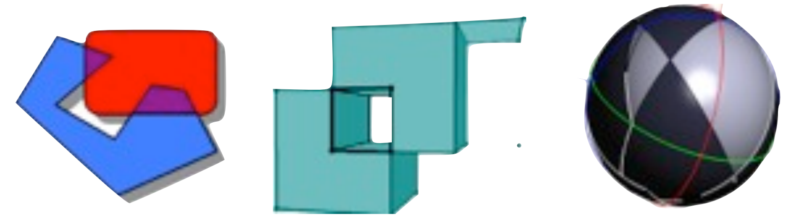
- Exact geometric computing
- Generic Programming
- ISO C++

CONTENTS

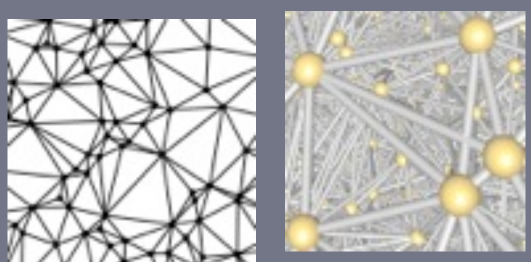
Bounding Volumes



Polyhedral Surfaces

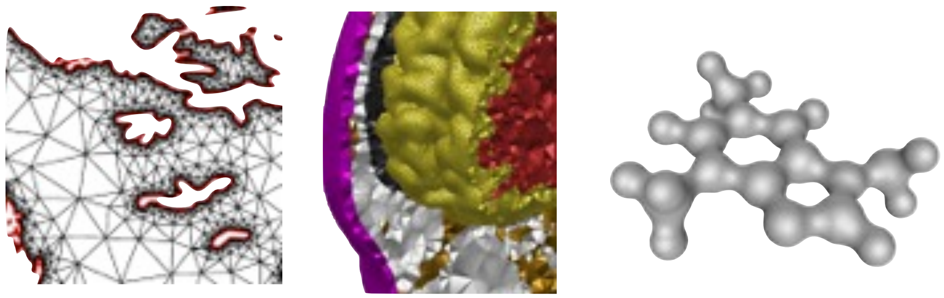


BooleanOperations



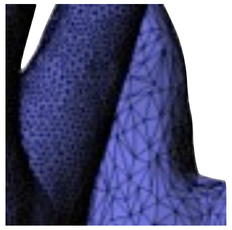
Triangulations

Voronoi Diagrams



Mesh Generation

Subdivision

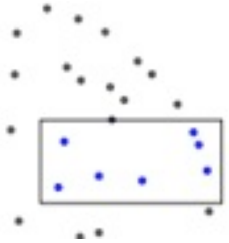


Simplification

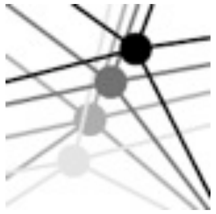
Parametrisation

Streamlines

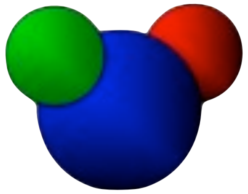
Ridge Detection



Neighbor Search



Kinetic Datastructures

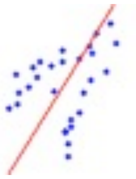


Lower Envelope

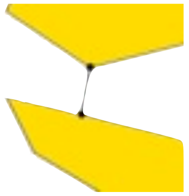
Arrangement

Intersection Detection

Minkowski Sum



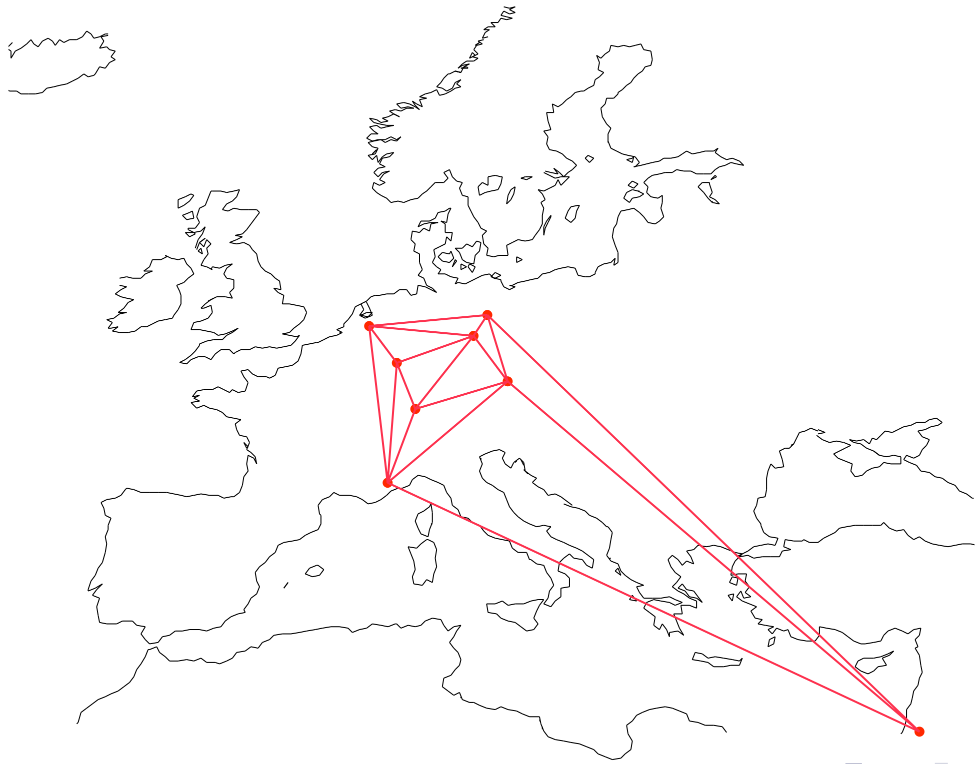
PCA



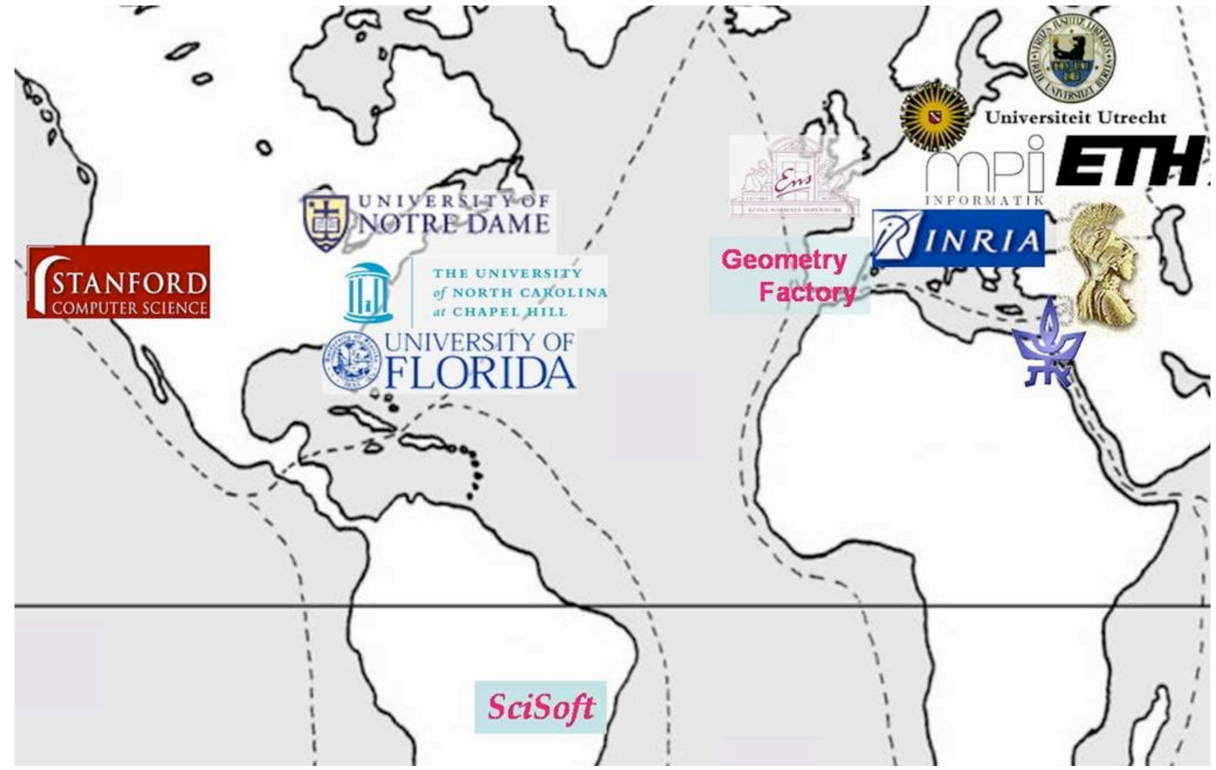
Polytope distance

QP Solver

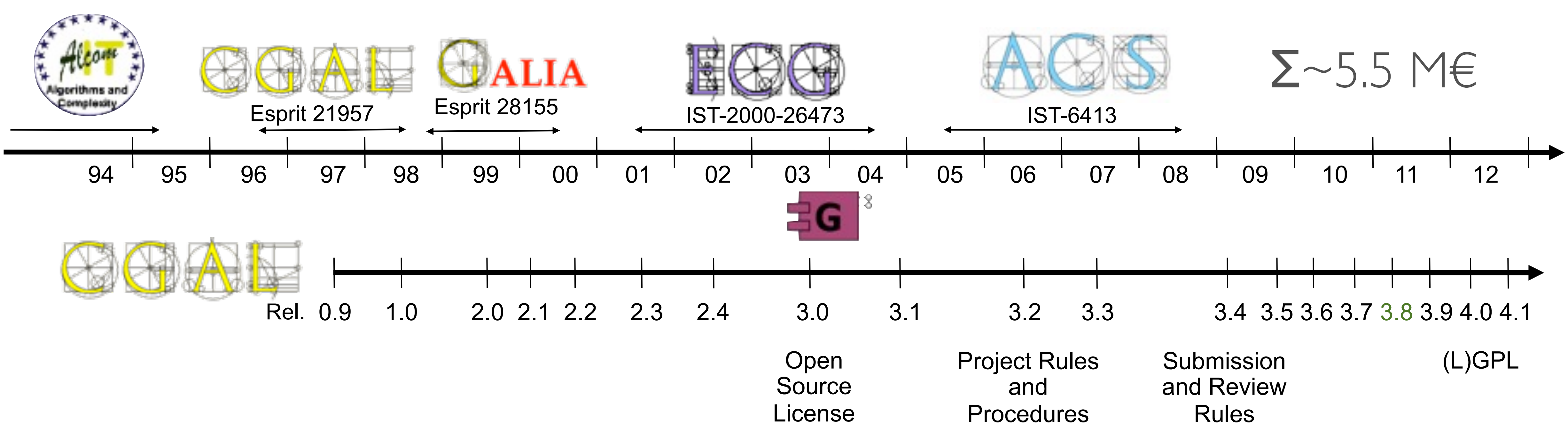
HISTORY



1996



2006



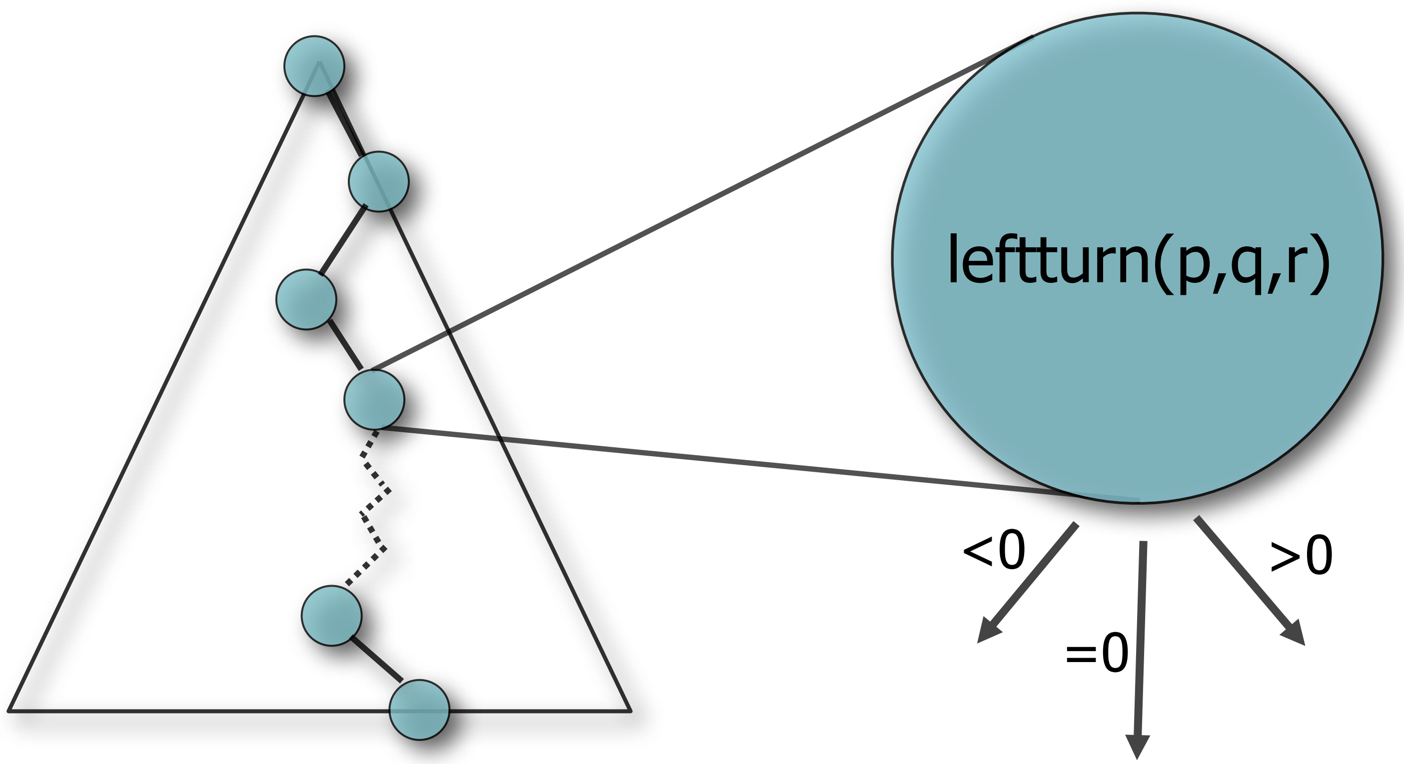
CGAL IN NUMBERS

- 500'000 lines of code (40 man years)
- 10'000 downloads per year
- 3'500 manual pages
- 4'000 subscribers to cgal-announce (7'000 for gcc)
- 1'400 subscribers to cgal-discuss (600 in gcc-help)
- 120 components
- 80 commercialization licenses sold
- 24 Master Theses and 22 PhD Theses
- 20 active developers

PART II:

Exact Geometric Computing

LAYERS OF GEOMETRIC ALGORITHMS



$$\begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix}$$

Combinatorial Algorithm

Geometric Predicate

Algebraic Computation

Control flow depends on non-trivial algebraic computations.

How to do these efficiently and consistently?

(Tough, no universally applicable solution…)

ARITHMETIC

All operations beyond + and - are computed using limited precision floating point arithmetic.

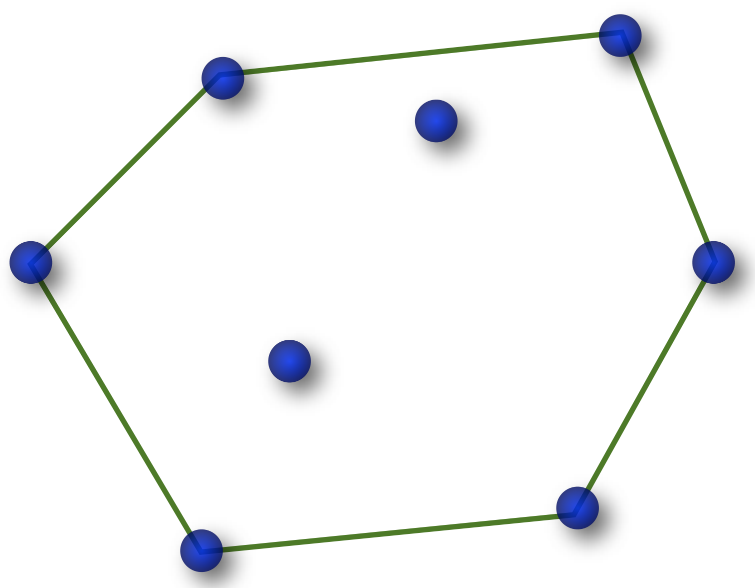
Integer multiplication and division are usually slower, often considerably.
And the precision is limited regardless...

Results may be incorrect due to roundoff.

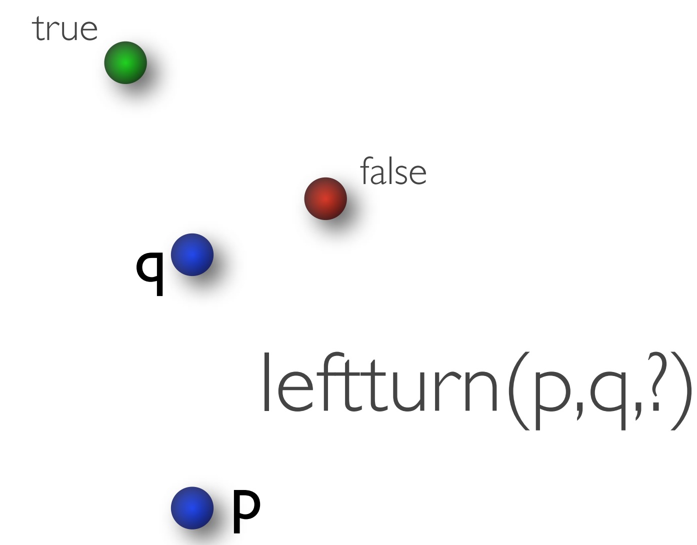
Difference to numeric computing:
Results are interpreted combinatorially: yes or no.

Incorrect results often lead to a complete failure rather than to a reasonable approximation.

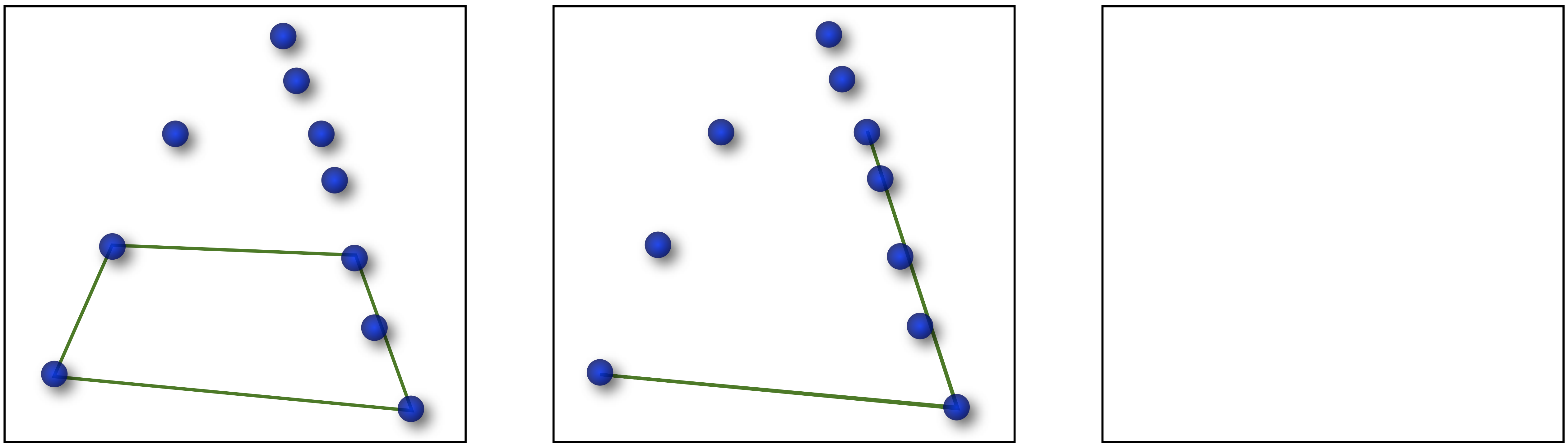
CONVEX HULL



Based on orientation test.



Possible results with an unreliable orientation test:



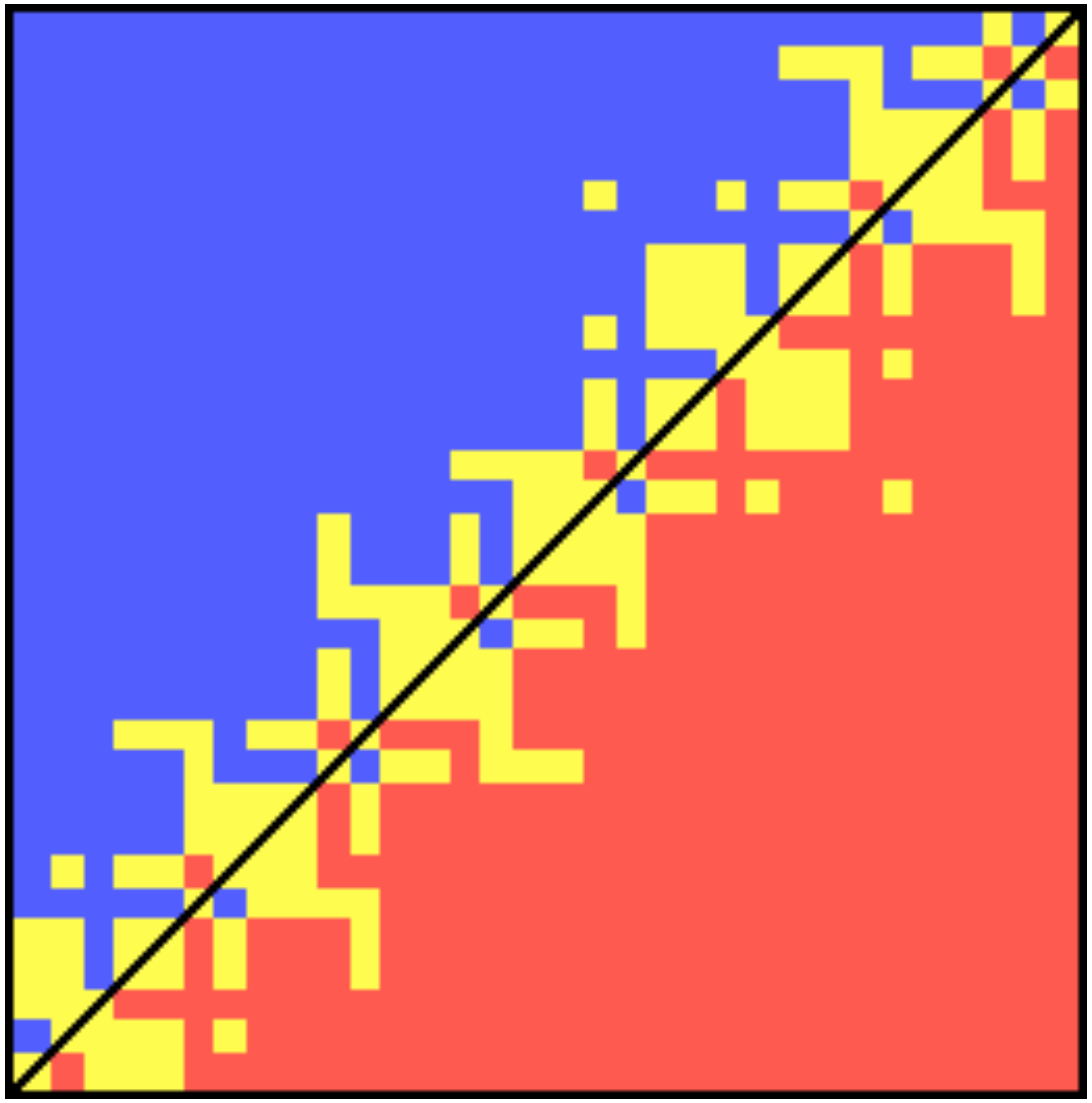
STRAIGHT LINES ?

$$\text{Orientation}(p, q, r) = \begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix} = (q.x-p.x)(r.y-p.y)-(q.y-p.y)(r.x-p.x)$$

$p = (0.5+x \cdot u, 0.5+y \cdot u)$
$q = (12, 12)$
$r = (24, 24)$

$0 \leq x, y < 256, \; u = 2^{-53}$

256x256 pixel image
red: <0, yellow: =0, blue: >0
evaluated with `double`



STRAIGHT LINES ?

$$\text{Orientation}(p, q, r) = \begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix} = (q.x-p.x)(r.y-p.y)-(q.y-p.y)(r.x-p.x)$$

$p = (0.5+x\cdot u, 0.5+y\cdot u)$

$q = (8.800000000000007,$
$\quad 8.800000000000007)$

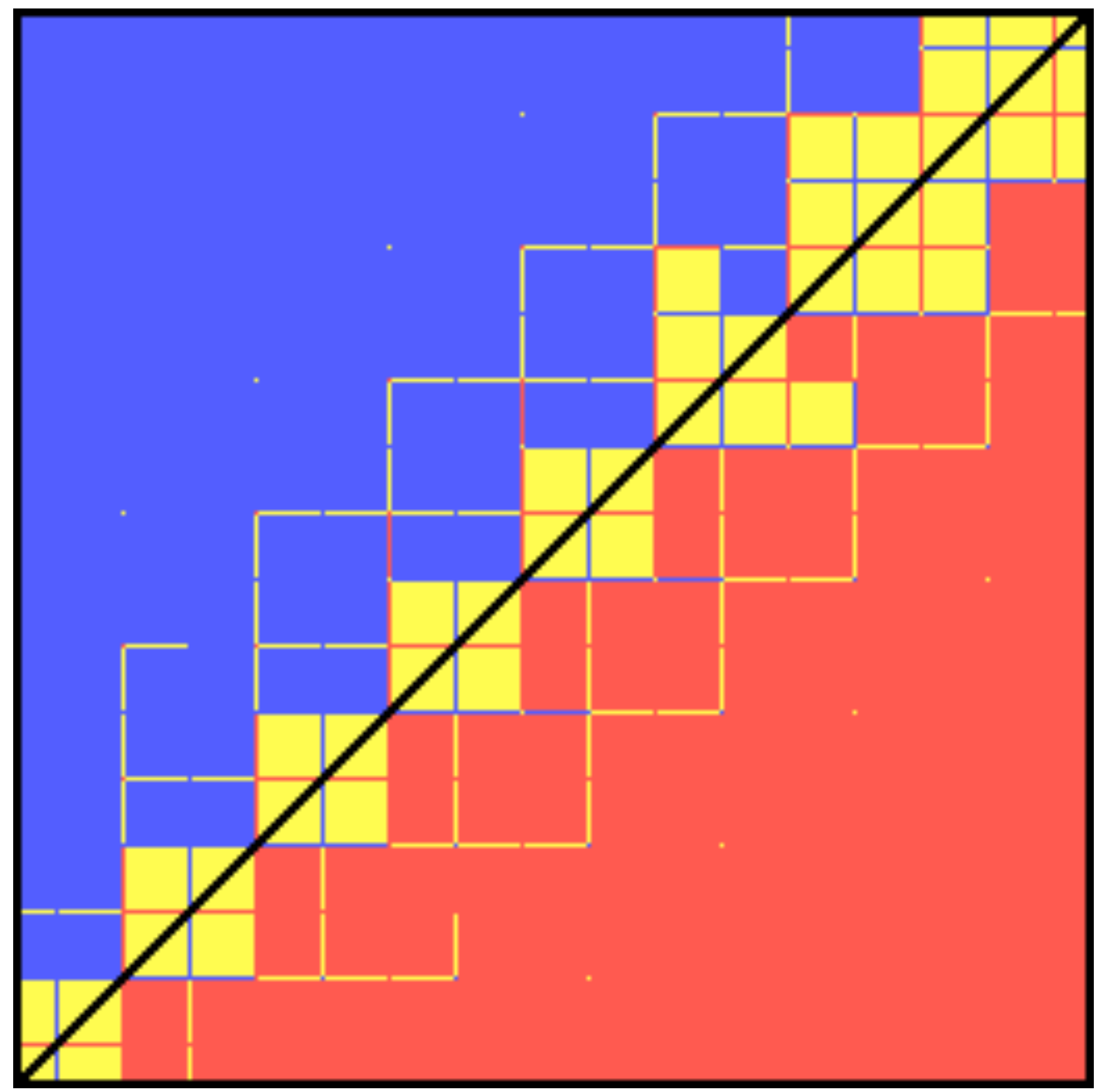
$r = (12.1, 12.1)$

$0 \leq x, y < 256, \;\; u = 2^{-53}$

256x256 pixel image

red: <0, yellow: =0, blue: >0

evaluated with **`double`**



STRAIGHT LINES ?

$$\text{Orientation}(p, q, r) = \begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix} = (q.x-p.x)(r.y-p.y)-(q.y-p.y)(r.x-p.x)$$

$p = (0.50000000000002531 + x \cdot u,$
$\quad 0.5000000000000171 0 + y \cdot u)$

$q = (17.300000000000001,$
$\quad 17.300000000000001)$

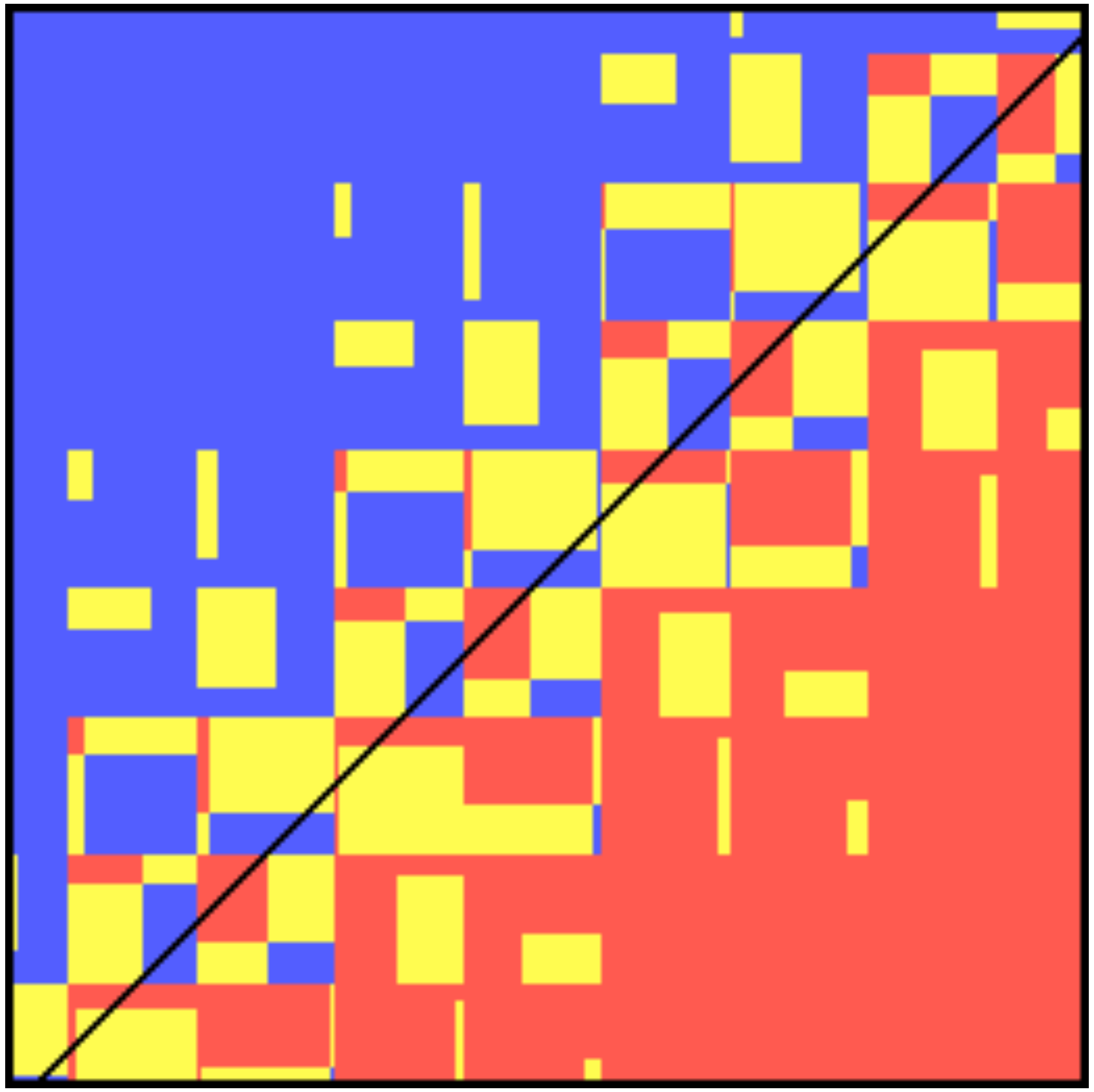
$r = (24.00000000000000500000,$
$\quad 24.00000000000005 17765)$

$0 \leq x, y < 256, \ u = 2^{-53}$

256x256 pixel image

red: <0, yellow: =0, blue: >0

evaluated with `ext double`



HOW TO OBTAIN CORRECTNESS?

Several options:

- Hope things go fine and fiddle around if not

Sometimes possible, often hard, always messy. Very problem-specific, no general machinery.

- Adapt algorithm to cope with imprecisions

- Restrict input

Good in special cases, hard to impossible for general purpose implementations . Document and check properly!

- Use exact algebra

General approach. Easy to use. Can be very slow...

- Filtering: Check whether things go fine and use exact algebra only when needed.

General approach. Easy to use. Often quite efficient...

FLOATING POINT NUMBERS

IEEE 754 double precision

+/-	exponent	mantissa
1 bit	11 bits	53 bits

0.1 is not exactly representable

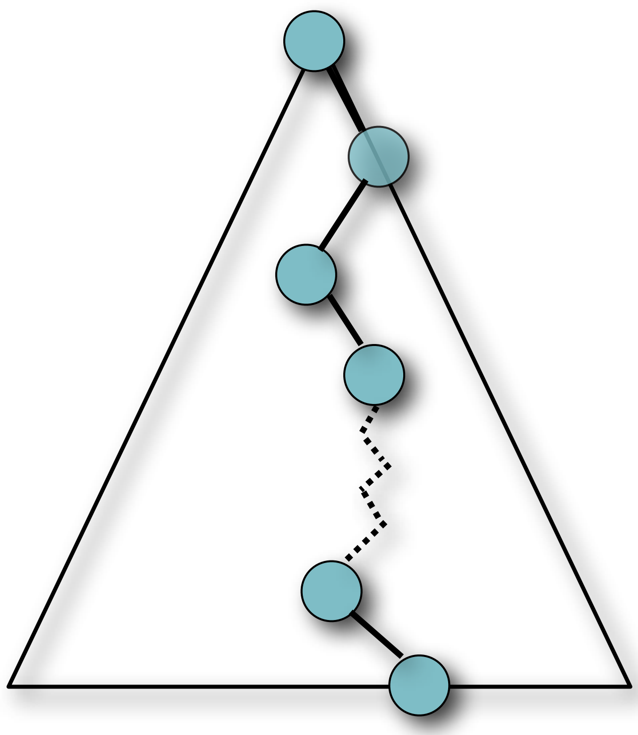
Numbers $\pm m \cdot 2^x$, $0 \leq m < 2^{53}$, $-1022 \leq x \leq 1023$.

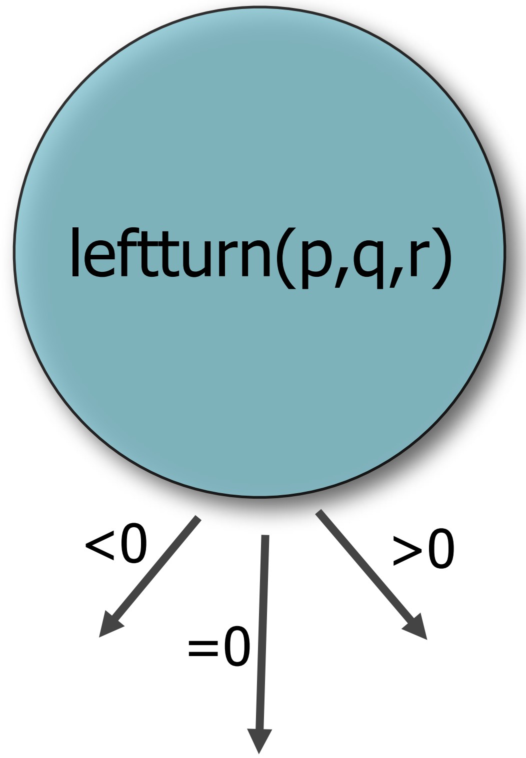
b bits	$\pm$	b bits	$\approx$	b+1 bits
b bits	$\cdot$	b bits	$\approx$	2b bits

(q.x-p.x)(r.y-p.y)-(q.y-p.y)(r.x-p.x)

→ orientation test ≈ 2b+3 bits, can be done exactly for 25-bit integer coordinates.

EXACT COMPUTATION





$$\begin{vmatrix} p.x & p.y & 1 \\ q.x & q.y & 1 \\ r.x & r.y & 1 \end{vmatrix}$$

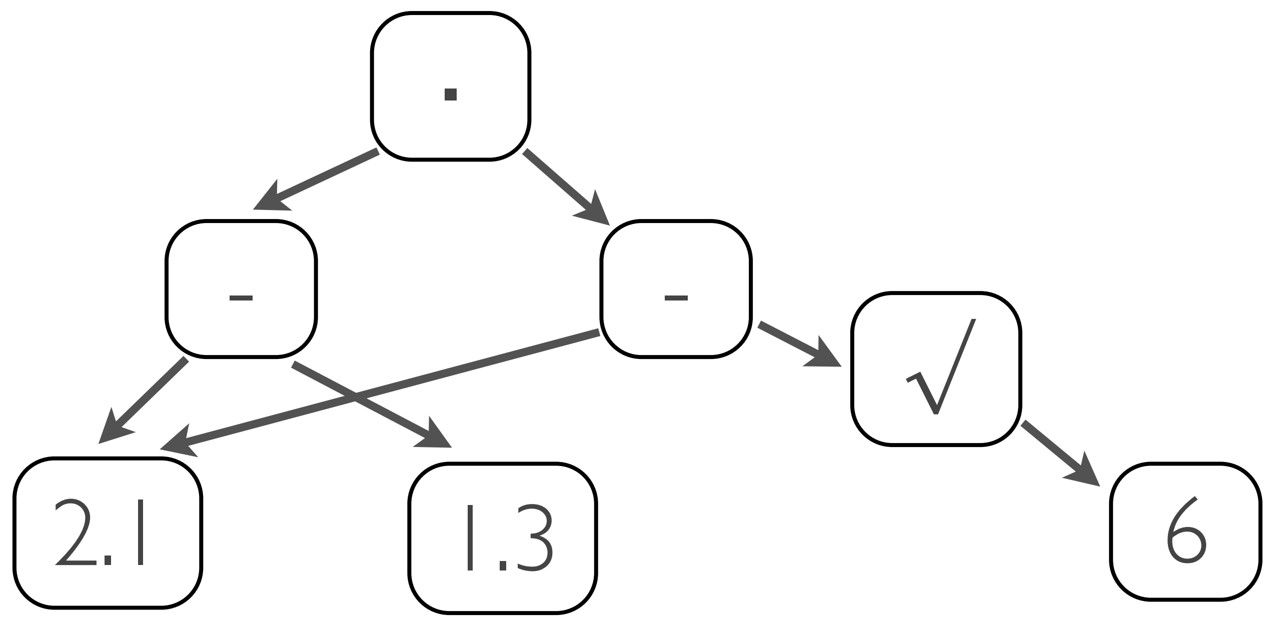
Combinatorial Algorithm	Geometric Predicate	Algebraic Computation

Ensure that the control flow in the algorithm is the same, as if all algebraic computations were made exactly.

→ **Correctness**

EXACT ALGEBRAIC COMPUTATION

$(2.1-1.3)(2.1-\sqrt{6})$



- numbers represented as expression-dags
- arbitrary precision floating point data types (array of digits) to compute approximations
- sign(x): compute finer and finer approximations for x, until it becomes clear that $x>0$ or $x<0$;
- for any algebraic expression there is a *separation bound* that tells where to stop and conclude $x=0$.

FLOATING POINT FILTERS

Exact algebraic computation is expensive.

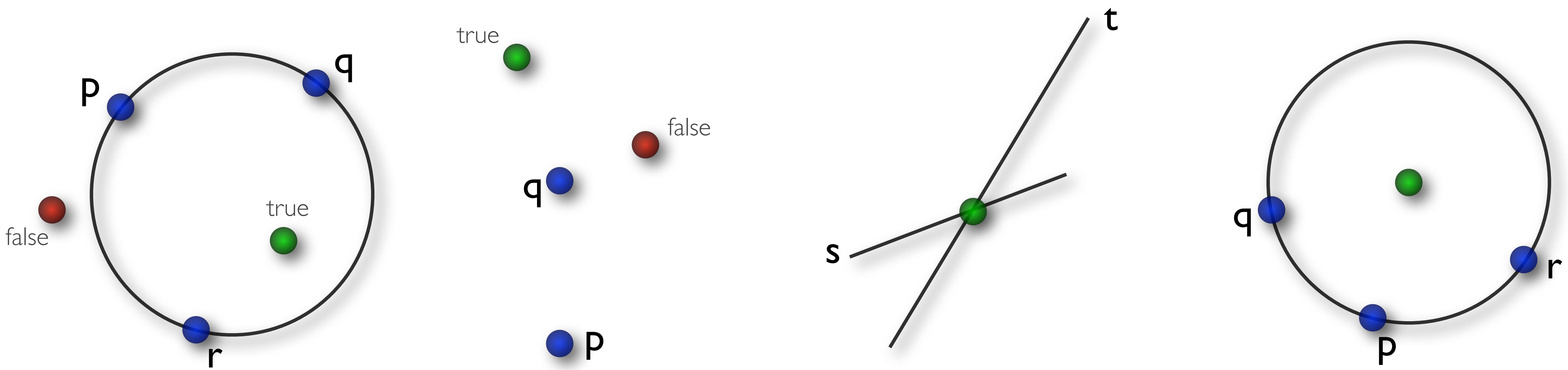
→ use when absolutely necessary only.

- maintain double approximation [l,h] using interval arithmetic (hardware support => fast)
- if $0 \notin [l,h]$, this is good enough to decide about sign.
- use exact machinery only if $0 \in [l,h]$.

Minimal overhead as long as filter works.

In particular, if only predicates are used and no constructions.

GEOMETRIC OPERATIONS



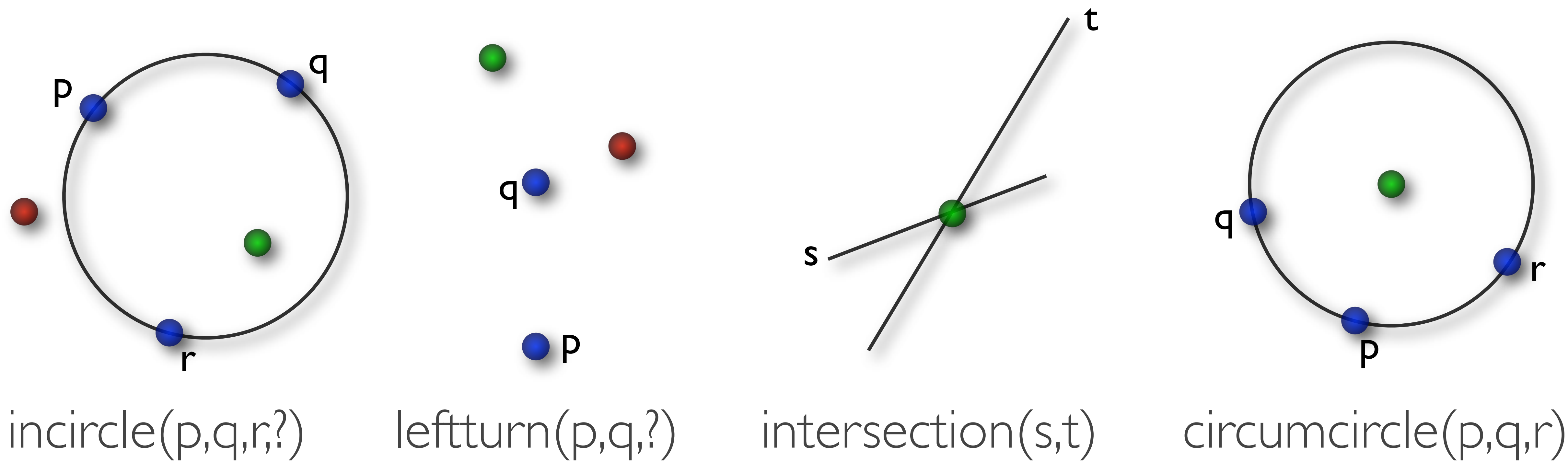
incircle(p,q,r,?) leftturn(p,q,?) intersection(s,t) circumcircle(p,q,r)

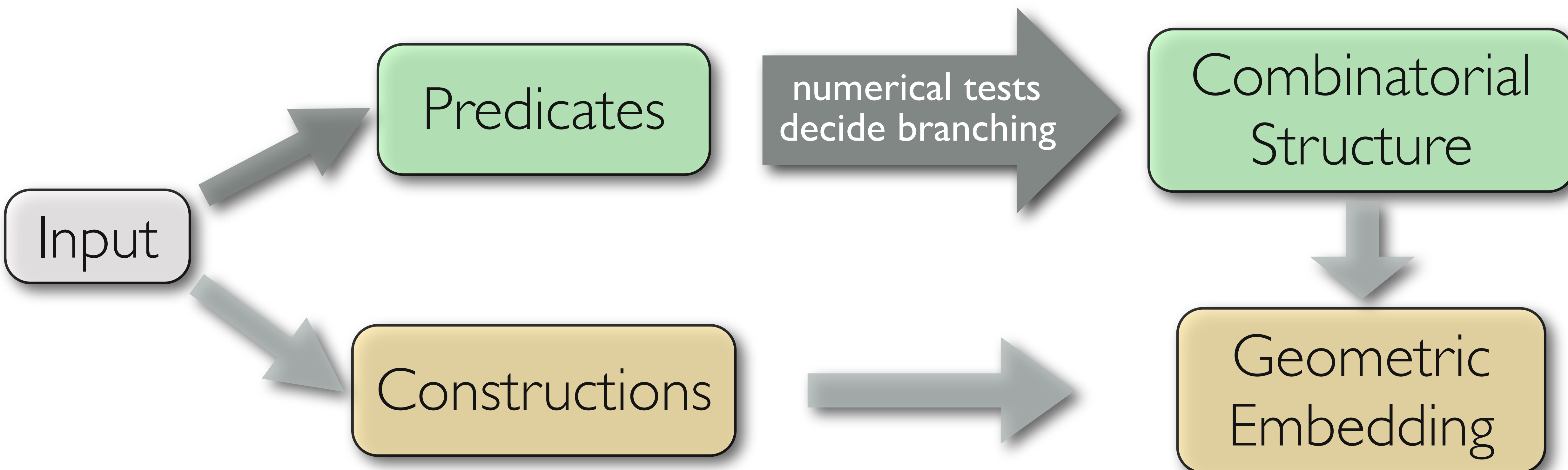
Predicates

Constructions

Do you need (exact) constructions?

GEOMETRIC OPERATIONS





FLEXIBILITY

Collection of geometric data types and operations.

There is no single true way to do geometric computing.

CGAL offers different kernels to serve various needs

You have to choose the right one for your particular case.

Predefined defaults:

All three compute predicates exactly using filters for efficiency.

- `CGAL::Exact_predicates_inexact_constructions_kernel`
  Constructions use `double`.
- `CGAL::Exact_predicates_exact_constructions_kernel`
  Constructions use an exact number type supporting +,-,*,/.
- `CGAL::Exact_predicates_exact_constructions_kernel_with_sqrt`
  Constructions use an exact number type supporting +,-,*,/, and roots.

fast

slow

PART III:

Basic Programming using a CGAL Kernel

HELLO POINT

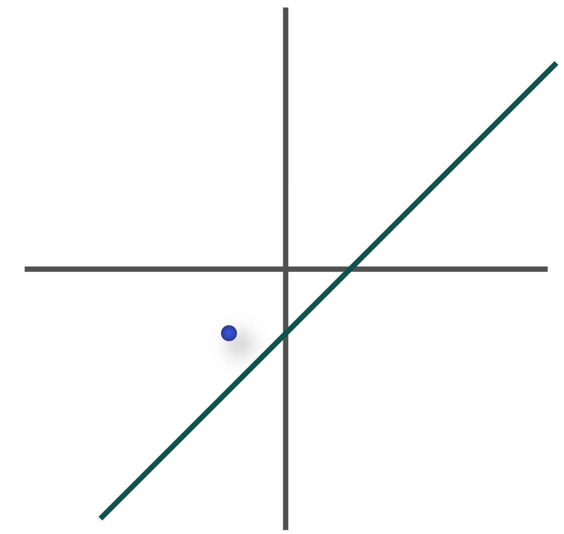
```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <iostream>

typedef CGAL::Exact_predicates_inexact_constructions_kernel K;

int main()
{
  K::Point_2 p(2,1), q(1,0), r(-1,-1);
  K::Line_2 l(p,q);
  K::FT d = CGAL::squared_distance(r,l);
  std::cout << d << std::endl;
}
```

There is a bunch of hyperlinks here.
Click me to get to the CGAL manual.

Does this code use constructions?
YES!



Output: 0.5

FT = field type

The number type used for the underlying algebra. Supports all field operations, i.e., +-*/.

Some (few) field types also support exact roots.

avoids square root computation

To obtain an approximation of the real distance, use

```
std::sqrt(CGAL::to_double(CGAL::squared_distance(r,l)))
```

This function must be defined for any field type.

Even if the field type supports exact square roots, in order to output it numerically you have to resort to an approximation...

https://elabs.inf.ethz.ch/file.php/29/CGALWeek1/Sample_Programs/hello.cpp

HELLO POINT

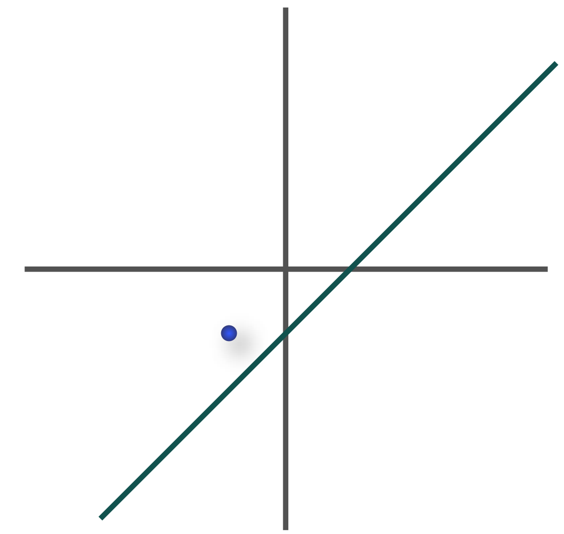
```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <iostream>

typedef CGAL::Exact_predicates_inexact_constructions_kernel K;

int main()
{
  K::Point_2 p(2,1), q(1,0), r(-1,-1);
  K::Line_2 l(p,q);
  K::FT d = CGAL::squared_distance(r,l);
  std::cout << d << std::endl;
}
```

For the small coordinates used here, things are probably fine. But in general...

this code is not safe!



Constructing a line from two points. Trivial?

Depends on representation of lines... equation => non-trivial construction

Constructing a point from Cartesian `double` coordinates. All default kernels can do this exactly, by just storing the coordinates.
=> trivial construction, no problem

Also a non-trivial construction. (Squared distance may be considerably larger than input coordinates, which may lead to overflow.)

CGAL::Line_2<Kernel>

Class

Definition

An object *l* of the data type *Line_2*<*Kernel*> is a directed straight line in the two-dimensional Euclidean plane $\mathbb{E}^2$. It is defined by the set of points with Cartesian coordinates (x,y) that satisfy the equation l : ax + by + c = 0

The line splits $\mathbb{E}^2$ in a *positive* and a *negative* side. A point p with Cartesian coordinates (px, py) is on the positive side of *l*, iff $a\,px + b\,py + c > 0$, it is on the negative side of *l*, iff $a\,px + b\,py + c < 0$. The positive side is to the left of *l*.

https://elabs.inf.ethz.ch/file.php/29/CGALWeek1/Sample_Programs/hello.cpp

HELLO POINT (EXACTLY)

```
#include <CGAL/Exact_predicates_exact_constructions_kernel_with_sqrt.h>
#include <iostream>
#include <iomanip>

typedef CGAL::Exact_predicates_exact_constructions_kernel_with_sqrt K;

int main()
{
  K::Point_2 p(2,1), q(1,0), r(-1,-1);
  K::Line_2 l(p,q);
  K::FT d = sqrt(CGAL::squared_distance(r,l));
  std::cout << CGAL::to_double(d) << std::endl;
  std::cout << std::setiosflags(std::ios::fixed) << std::setprecision(2)
            << CGAL::to_double(d) << std::endl;
}
```

Set precision (number of digits after the decimal point) for floating point number output. Round to nearest, but tie-breaking is not well defined!

Output:

```
0.707107
0.71
```

Round to some **double** nearby. (There is no easy way to output the exact internal representation.)

Output floating point numbers in fixed point notation from now on. `std::resetiosflags(std::ios::fixed)` switches back to default behaviour.

Compute squareroot (here: exactly).

Problem: No guarantee on precision and rounding.

https://elabs.inf.ethz.ch/file.php/29/CGALWeek1/Sample_Programs/hello-exact.cpp

HELLO POINT (EVEN MORE EXACTLY)

```
#include <CGAL/Exact_predicates_exact_constructions_kernel_with_sqrt.h>
#include <iostream>
#include <cmath>

typedef CGAL::Exact_predicates_exact_constructions_kernel_with_sqrt K;

double floor_to_double(const K::FT& x)
{
  double a = std::floor(CGAL::to_double(x));
  while (a > x) a -= 1;
  while (a+1 <= x) a += 1;
  return a;
}

int main()
{
  K::Point_2 p(2,1), q(1,0), r(-1,-1);
  K::Line_2 l(p,q);
  K::FT d = sqrt(CGAL::squared_distance(r,l));
  std::cout << floor_to_double(d) << std::endl;
}
```

for `std::floor(...)`

Compute approximation of the closest integer ≤ x.
(Usually, this is pretty good. But we cannot be sure that it is always...)

Compare to the exact value to be sure.

Compute squareroot exactly.

Output:
0

We need a precise specification for all output, in order to compare on the judge.

This is the recommended way to round down to an integer.
(The symmetric function `ceil_to_double(...)` to round up should be an easy exercise...)

https://elabs.inf.ethz.ch/file.php/29/CGALWeek1/Sample_Programs/hello-really-exact.cpp

TWO KERNELS IN ONE PROGRAM

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <iostream>
#include <stdexcept>

typedef CGAL::Exact_predicates_inexact_constructions_kernel IK;
typedef CGAL::Exact_predicates_exact_constructions_kernel   EK;

int main()
{
  IK::Point_2 p(2,1), q(1,0), r(-1,-1);
  // do something that needs predicates only, e.g., ...
  std::cout << (CGAL::left_turn(p, q, r) ? "y" : "n") << "\n";

  // now we use non-trivial constructions...
  EK::Point_2 ep(p.x(), p.y()), eq(q.x(), q.y()), er(r.x(), r.y());
  EK::Circle_2 c(ep, eq, er);
  if (!c.has_on_boundary(ep))
    throw std::runtime_error("ep not on c");
}
```

This works because the coordinates of `IK::Point_2` are actually `double`.

It would not work the other way round, because the coordinates of `EK::Point_2` are of some elaborate number type.

We cannot just write `c(p, q, r)` because these are `IK::Point_2` and there is no general conversion between points from different kernels.

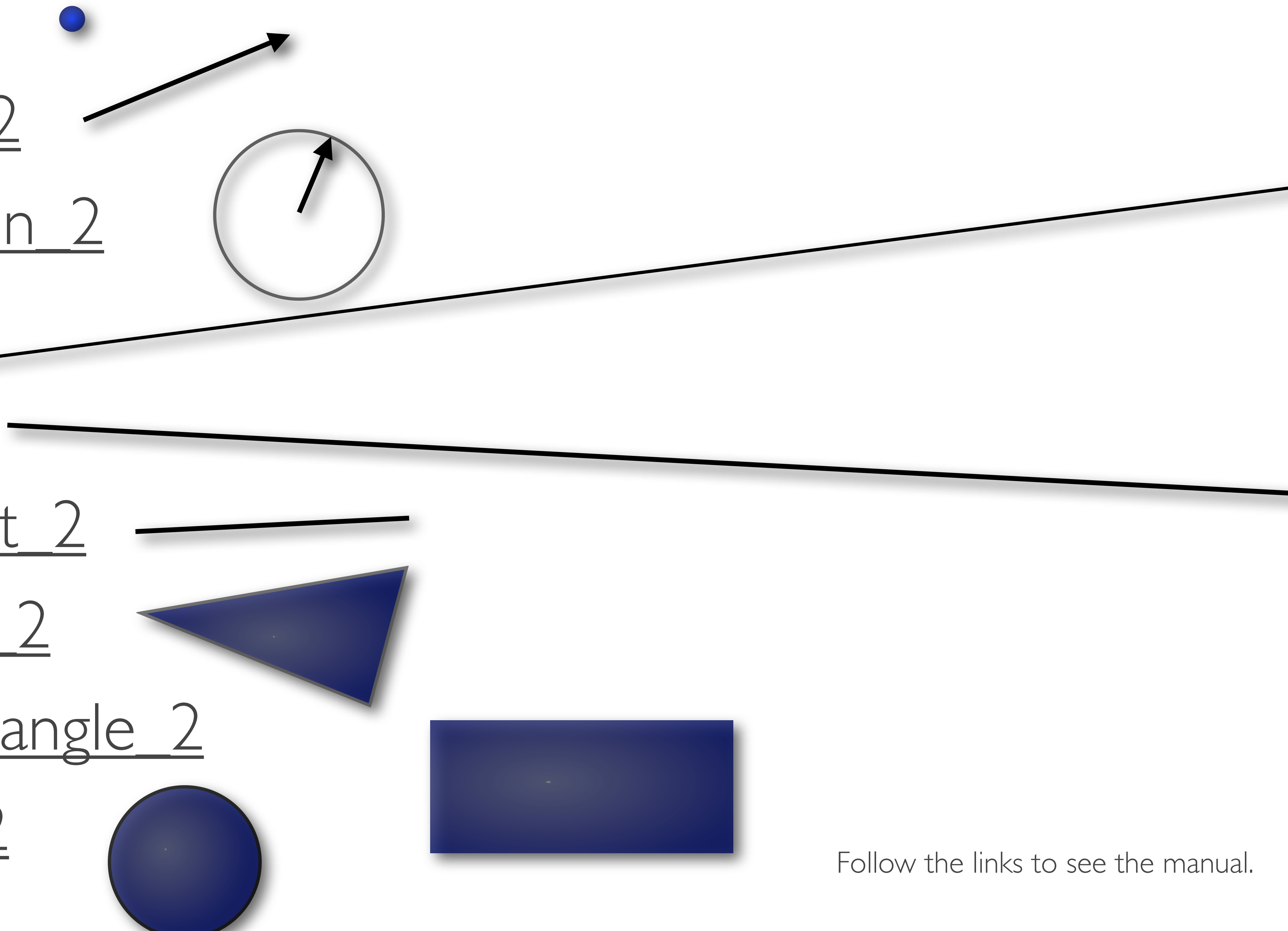
Output:

n

https://elabs.inf.ethz.ch/file.php/29/CGALWeek1/Sample_Programs/two-kernels.cpp

2D (LINEAR) KERNEL

- Point_2
- Vector_2
- Direction_2
- Line_2
- Ray_2
- Segment_2
- Triangle_2
- Iso_rectangle_2
- Circle_2



Follow the links to see the manual.

2D KERNEL FUNCTIONALITY

See the CGAL Manual: http://www.cgal.org

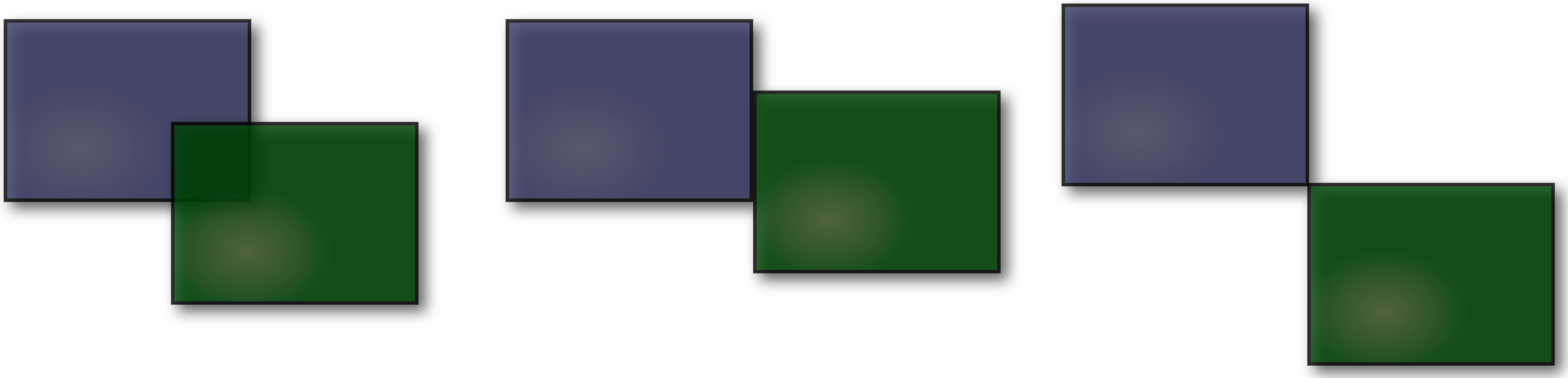
Most manual chapters have two parts:

- User Manual: general introduction and examples.
- Reference Manual: complete list of functionality.

Often one deals with several different interacting types and has to jump back and forth.
=> html is very convenient

http://www.cgal.org/Manual/3.8/doc_html/cgal_manual/Kernel_23_ref/Chapter_intro.html

INTERSECTIONS



Problem: We do not know the return type.

```
K::Iso_rectangle_2 r1 = … ;
K::Iso_rectangle_2 r2 = … ;
??? i = CGAL::intersection(r1, r2);
```

You might say that a point is nothing but a degenerate rectangle. Then think about a plane and a line in $\mathbb{R}^3$.

Solution: Use a generic class `CGAL::Object`.

Test whether it contains an object of type `T` using `object_cast<T>`.

Note: `CGAL::Object` is not a common base class but just a generic wrapper.

INTERSECTIONS

```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <iostream>
#include <stdexcept>

typedef CGAL::Exact_predicates_exact_constructions_kernel K;
typedef K::Point_2 P;
typedef K::Segment_2 S;

int main()
{
  P p[] = { P(0,0), P(2,0), P(1,0), P(3,0), P(.5,1), P(.5,-1) };
  S s[] = { S(p[0],p[1]), S(p[2],p[3]), S(p[4],p[5]) };
  for (int i = 0; i < 3; ++i)
    for (int j = i+1; j < 3; ++j)
      if (CGAL::do_intersect(s[i],s[j])) {
        CGAL::Object o = CGAL::intersection(s[i],s[j]);
        if (const P* op = CGAL::object_cast<P>(&o))
          std::cout << "point: " << *op << "\n";
        else if (const S* os = CGAL::object_cast<S>(&o))
          std::cout << "segment: " << os->source() << " "
                    << os->target() << "\n";
        else // how could this be? -> error
          throw std::runtime_error("strange segment intersection");
      } else
        std::cout << "no intersection\n";
}
```

Test for intersection (predicate)

Construct intersection (construction :-))

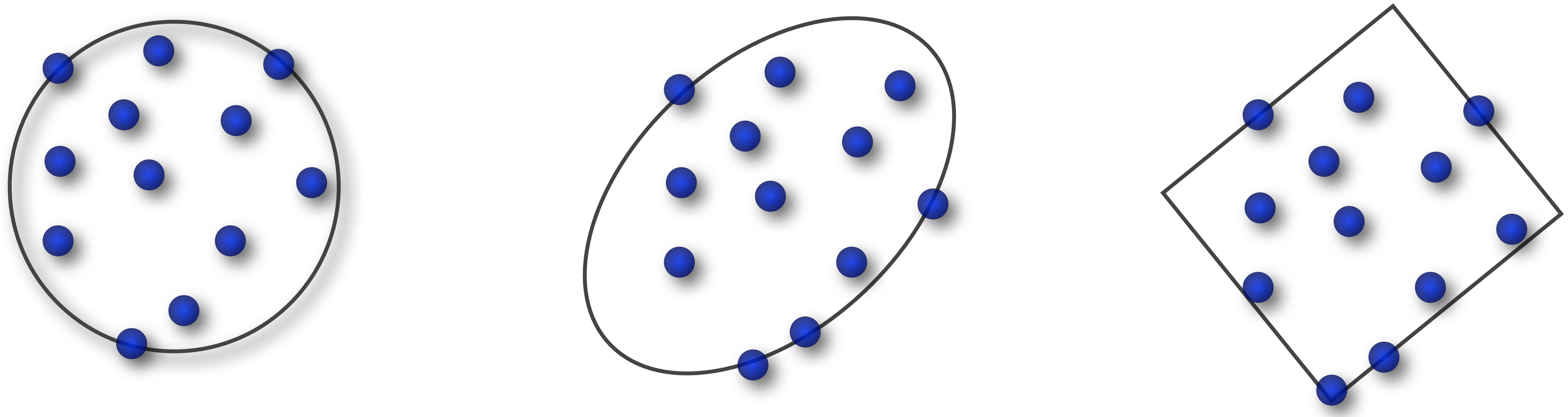
Cast fails (=0) if o is not of type P.

Output:

```
segment: 1 0 2 0
point: 0.5 0
no intersection
```

https://elabs.inf.ethz.ch/file.php/29/CGALWeek1/Sample_Programs/intersect.cpp

BOUNDING VOLUMES



Problem: Given n points in $\mathbb{R}^2$, what is their minimum enclosing ... ?

- Circle
- Ellipse
- (Circular) annulus

Can be computed in expected linear time.

- Rectangle
- Parallelogram
- Strip

Can be computed in linear time once the convex hull is known.

http://www.cgal.org/Manual/3.8/doc_html/cgal_manual/Bounding_volumes/Chapter_main.html

MINIMUM ENCLOSING CIRCLE

```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <CGAL/Min_circle_2.h>
#include <CGAL/Min_circle_2_traits_2.h>
#include <iostream>

// typedefs
typedef  CGAL::Exact_predicates_exact_constructions_kernel K;
typedef  CGAL::Min_circle_2_traits_2<K>  Traits;
typedef  CGAL::Min_circle_2<Traits>      Min_circle;

int main()
{
  const int n = 100;
  K::Point_2 P[n];

  for (int i = 0; i < n; ++i)
    P[i] = K::Point_2((i % 2 == 0 ? i : -i), 0);
  // (0,0), (-1,0), (2,0), (-3,0), ...

  Min_circle mc(P, P+n, true);
  Traits::Circle c = mc.circle();
  std::cout << c.center() << " " << c.squared_radius() << std::endl;
}
```

Many data structures and algorithms have their own traits concept. It defines the geometric primitives needed.

Separate: Combinatorial algorithm <=> geometry

Build from a range of points.

Randomize input order? Generally a good idea, unless input is known to be random, anyway.

This part needs the incircle predicate only...

Construct and return the circle.

Only here the construction is needed...

Output:
```
-0.5 0 9702.25
```

https://elabs.inf.ethz.ch/file.php/29/CGALWeek1/Sample_Programs/miniball.cpp

PART IV:

Practical Information

USING CGAL

Best start in a new directory, name source file s.t. it ends with `.cpp`.

Run **`cgal_create_cmake_script`** in this directory.

`cmake .`

Note the dot (current directory)!

These scripts should be in PATH on the lab PCs and the provided VirtualBox environment.
http://csa.inf.ethz.ch/~trast/vm_cgal_stuff.html

This creates a makefile with rules and targets for every `.cpp` file.
You can then build your program using **`make`**

You have to re-run `cgal_create_cmake_script` whenever you add a new application/`.cpp` file.
No need to re-run `cmake` because that's done by `make` automatically.
As a default, makefiles are created in release mode. If you want to debug, run `cmake -DCMAKE_BUILD_TYPE=Debug .`
To go back to release mode, run `cmake -DCMAKE_BUILD_TYPE=Release .`
If you want to see the actual compiler and linker calls, run `cmake -DCMAKE_VERBOSE_MAKEFILE=ON .`
Warning: Do not use valgrind with CGAL.

That's it!

If you want to install CGAL on your private computer:
- Check/install prerequisites first: compiler, cmake, boost, gmp, mpfr, (qt)
- Install cgal, c.f.
  `http://www.cgal.org/Manual/3.8/doc_html/installation_manual/contents.html`
- Or download CGAL packages of your distribution if they exist (don't forget cgal-devel).

For more, see...

HTTP://WWW.CGAL.ORG