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Quantum Trajectory Simulations of Qubit Systems

by

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Abstract

The aim of this thesis is to explore quantum trajectory theory and how it relates to the conventional methods of studying open quantum systems, namely the Lindblad master equation. Quantum trajectories lead to stochastic evolution of systems consisting of discontinuous *jumps*. Quantum trajectories make use of state vectors rather than density matrices, making them more computationally efficient at simulating open quantum systems than master equation solvers. The thesis will compute these trajectories for a particular scheme of qubit systems for two types of interactions: CNOT and SWAP.

Acknowledgements

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Introduction

Open quantum systems is used to describe systems that have a coupling between the system state of interest and its environment(bath/reservoir/etc). In practice, systems of interest are rarely perfectly isolated, thus most systems are open. Studies of these systems have been powerful in fields of quantum optics, quantum statistical mechanics, quantum information theory, and many more. The study of these open systems also give important insights into quantum measurements as[1],

- All real systems are open to an extent, in a macroscopic system this coupling can lead to rapid decoherence. This is the conversion of a quantum superposition to classical mixture. Studying this will help in understanding the emergence of classical behaviour (the *quantum measurement problem*).
- The second reason they are important is due to the coupling between the system and the environemnt, a measurement of the environment can yield information about system.

The dynamics of open quantum systems are described using density matrices and their corresponding master equations. Particularly when the environment is weak, the description which is used, by invoking Markovian property and assuming an uncorrelated initial states, is the Gorini–Kossakowski–Sudarshan–Lindblad equation or GKSL equation or Lindblad equation.

However, these equations require the open quantum state to be described by density matrices. This is done because it allows one to trace over the environment and recover the evolution of the system state alone. Although useful, these calculations are computationally intensive. A Hilbert space of dimensions N would require N^2 density matrix elements if one were to use the master equation approach.

Quantum trajectory theory is a formulation of quantum mechanics useful in modelling these open systems. This approach works with state vectors and relies on a stochastic Schrodinger equation for its evolution. The evolution of the system state is mapped by continuous environment measurement. The interaction between the system and environment is exploited to gain information of the system without directly measuring it. Unlike master equations, the trajectories are not continuous and deterministic, rather they have jumps and randomness inherent to them due to the requirement for continuous environment measurement. Also unlike the master equation which requires N^2 elements, quantum trajectories require only N since they require state vectors only.

Although, the main draw of quantum trajectories is the computational efficiency, it also offers insights into the quantum measurement problem. As a way of exploring these concepts and understanding the properties of quantum trajectories, I will adapt a two level scheme [2] and provide simulations of the possible *unravelings* of the stochastic Schrodinger equation. I will also show how we can reach the same effective evolution of the Lindblad master equation.

Chapter 2

Quantum Trajectories

2.1 Introduction

In this chapter I will give a brief introduction to measurement theory and how different forms of it are related. I also show how to arrive at the master equation using an ensemble average of POVMs. Then Lindblad jump operators are introducted to lead into quantum trajectories and the algorithm by which we can construct them. I finally end with an example of spontaneous emission highlighting the concepts discussed.

2.2 Projective Measurements

Let us begin by understanding a simple model of measuring a quantum system and discuss its properties. Consider a quantum system of interest coupled to a measurement device[3]. The measurement device can be considered to be the "environment" of the system. The system is in state $|\psi\rangle_S$ and let us assume the measurement meter to be in pure state $|\psi\rangle_M$ having unitary coupling $\hat{U}_{SM}$ between them. The meter measures some observable $\hat{A}$ of the system and so the system state can be decomposed as

$$|\psi_S\rangle = \sum c_a |a\rangle \tag{2.1}$$

where $|a\rangle$ are eigenstates of $\hat{A}$.

The Hamiltonian of this system-meter apparatus:

$$\hat{H}_{SM} = \hbar\kappa \hat{A}_S \otimes \hat{P}_M \tag{2.2}$$

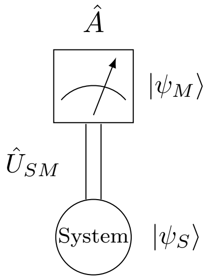

FIGURE 2.1: Schematic of System and Meter setup

where κ is energy scale of the apparatus and hence $\hbar\kappa$ is the energy. $\hat{A}_S$ is the system observable and $\hat{P}_M$ is the momentum of the meter's needle, this generates displacements of the meter position X_M

The coupling is given by:

$$\hat{U}_{SM} = e^{-i\kappa\Delta t \hat{A}_S \otimes \hat{P}_M} \tag{2.3}$$

where Δt is the time for which the meter is acting on the system.

Now let us "calibrate" the meter such that its initial state is $|0\rangle_M$ and this will give us the joint state of system and meter as:

$$\begin{aligned} |\Psi\rangle_{SM} &= \hat{U}_{SM} |\psi\rangle_S \otimes |0\rangle_M & (2.4)\\ &= e^{-i\kappa\Delta t \hat{A}_S \otimes \hat{P}_M} \sum_a c_a |a\rangle_S \otimes |0\rangle_M & (2.5)\\ &= \sum_a c_a |a\rangle_S \otimes e^{-ia\kappa\Delta t \hat{P}_M} |0\rangle_M & (2.6) \end{aligned}$$

The last step is done using the decomposition $|\psi_S\rangle = \sum c_a |a\rangle$. The second part of the tensored product $e^{-ia\kappa\Delta t \hat{P}_M} |0\rangle_M$ can be rewritten as

$$e^{-iX_a \hat{P}_M} |0\rangle_M$$

where $X_a = a\kappa\Delta t$ is the displacement of the needle based on the eigenvalue a that is obtained from the measurement of the observable $\hat{A}_S$. Hence the joint entangled state becomes:

$$\begin{aligned} |\Psi\rangle_{SM} &= \sum_a c_a |a\rangle_S \otimes e^{-iX_a \hat{P}_M} |0\rangle_M & (2.7)\\ &= \sum_a c_a |a\rangle_S \otimes |X_a = a\kappa\Delta t\rangle_M & (2.8) \end{aligned}$$

When the meter is observed and we find the displacement of the needle to be X_a (given by $X_a = a\kappa\Delta t$). So, the amount of displacement on the needle is a function of the eigenvalue of the operator $\hat{A}_S$.

Now, if we impose a condition that $\langle X_a|X_{a'}\rangle = \delta_{aa'}$ (von Neumann condition[4]), then on observation of the needle at any $|X_a\rangle$ will yield $|a\rangle$ as the state of the system. The von Neumann condition can be enforced by tuning κ and Δt.

More formally, measuring the meter in the $|X\rangle$ basis will give result $X_a = a\kappa\Delta t$ with probability $|c_a|^2$ and so the state of the system after measurement would be, $|a\rangle$.

Since the decomposition of states (2.1) and their probabilities are do not change due to the measurement, it is known as a *Quantum Non Demolition* (QND) measurement[5]. This idealized form of measurement is known as *Projective measurements.*

Projective measurements use a complete set of orthagonal projections $\hat{P}_n$ where n are the possible outcomes. These operators satisfy orthogonality and completeness relation:

$$\hat{P}_n\hat{P}_{n'} = \hat{P}_n\delta_{nn'} \quad \text{and} \quad \sum_n \hat{P}_n = \hat{1}$$

Projective measurements consist of two parts:

1. Probability of measurement outcome: In our case, the probability of outcome a

   $$p_a = |\langle a|\psi_s\rangle|^2 = \langle\psi|a\rangle\langle a|\psi\rangle = \langle\psi|\hat{P}_a|\psi\rangle \tag{2.9}$$

   where $\hat{P}_a = |a\rangle\langle a|$

2. Post measurement state: Given that the measurement gives outcome a, the post measurement state becomes (again, in our case),

   $$|\psi_{new}\rangle = \frac{\hat{P}_a|\psi_S\rangle}{||\hat{P}_a|\psi_S\rangle||} = \frac{\hat{P}_a|\psi_S\rangle}{p_a} = |a\rangle \tag{2.10}$$

   Here the initial state gets *projected* onto the eigenstate of the outcome (a) and is then normalized (denominator).

2.3 Generalized measurements (POVM)

In practice, it is rare to have a quantum system connected to the measurement device directly. This is because such apparatus destroys the system state after measurement(QND measurements are rare). Usually, there is an intermediary quantum probe which probes the system and is then measured by the meter. The system-probe interaction allows us to describe a more general set of measurements.

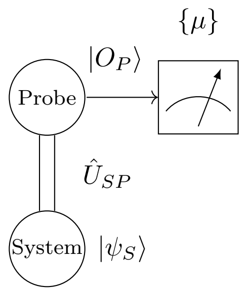

FIGURE 2.2: Schematic of system connected to the probe which is then measured

The system is in state $|\psi\rangle_S$ and the probe is $|O\rangle_P$ and the meter is being measured in the $\{\mu\}$ basis. The system and probe can be considered as a bipartite system.

From open quantum system dynamics, we get the interaction of the system and probe to be coupling $\hat{U}_{SP}$. The State of the system and probe is then given by:

$$\begin{aligned} |\Psi\rangle_{SP} &= \hat{U}_{SP} |\psi\rangle_S \otimes |O\rangle_P && (2.11) \\ \rho_{SP} &= \hat{U}_{SP} |\psi\rangle_S \otimes |O_P\rangle \langle O_P| \otimes \langle\psi|_S \hat{U}_{SP} && (2.12) \end{aligned}$$

First, let us trace over the probe, giving a mixed state:

$$\begin{aligned} tr_p(\rho)_{SP} &= tr_p(\hat{U}_{SP} |\psi\rangle_S \otimes |O_P\rangle \langle O_P| \otimes \langle\psi|_S \hat{U}_{SP}) && (2.13) \\ \rho_S &= \sum_\mu \hat{M}_\mu |\psi_S\rangle \langle\psi_S| \hat{M}^\dagger{}_\mu && (2.14) \\ &= \sum p_\mu |\psi_\mu\rangle \langle\psi_\mu| && (2.15) \end{aligned}$$

Where $\hat{M}_\mu$ are the Kraus operators in basis $|\mu_p\rangle$ given as:

$$\hat{M}_\mu = \langle\mu_p| \hat{U}_{SP} |O_P\rangle \tag{2.16}$$

The last line is an ensemble decomposition with states $|\psi_\mu\rangle$ as and probabilities p_μ:

$$\begin{aligned} |\psi_\mu\rangle &= \frac{M_\mu |\psi_S\rangle}{||M_\mu |\psi_S\rangle ||} && (2.17) \\ p_\mu &= \langle\psi_S| \hat{M}^\dagger{}_\mu \hat{M}_\mu |\psi_S\rangle && (2.18) \\ &= ||M_\mu |\psi_S\rangle ||^2 && (2.19) \end{aligned}$$

This describes the dynamics of the system after tracing over the probe. No measurement has taken place, this is simply open system dynamics.

Now, suppose we measure the meter that is attached to the probe (rather than tracing the probe) and find result $|\mu_p\rangle$ then that would mean the state of the system after measurement is:

$$|\psi_S\rangle_{\text{After}} = \frac{\langle\mu_p|\,\hat{U}_{SP}\,|O_P\rangle\,|\psi_S\rangle}{||M_\mu\,|\psi_S\rangle\,||} \tag{2.20}$$

$$= \frac{M_\mu\,|\psi_S\rangle}{||M_\mu\,|\psi_S\rangle\,||} \tag{2.21}$$

This is a pure state and the probability of acquiring this state of the system (or finding the meter to be in state $|\mu_p\rangle$) is given by 2.18. Note the meter measures the probe and not the system directly. The meter has its own observables $\{\mu\}$ in the eigenbasis $|\mu\rangle$. This type of measurement is more generalised than the projective type we had discussed in the previous section. These measurements are called *Positive Operator-Valued Measure (POVMs)*. In these generalized measurements; the probability of the outcome μ is given by[6]:

$$p_\mu = \langle\psi_S|\,\hat{E}_\mu\,|\psi_S\rangle \tag{2.22}$$

where $\hat{E}_\mu$ are called the POVM elements positive and complete set, satisfying:

$$\hat{E}_\mu \geq 0 \text{ and } \sum_\mu \hat{E}_\mu = 1$$

In our case, the POVM elements can be related to the Kraus operators as:

$$\hat{E}_\mu = \hat{M^\dagger}_\mu \hat{M}_\mu \tag{2.23}$$

And the state after measurement is given by(2.20):

$$|\psi_S\rangle_{After} = \frac{M_\mu\,|\psi_S\rangle}{||M_\mu\,|\psi_S\rangle\,||} \tag{2.24}$$

Notice that if the Kraus operators were projectors, then $\hat{M}_\mu = \hat{M}_\mu^\dagger$ and $\hat{M}_n\hat{M}_{n'} = \hat{M}_n\delta_{nn'}$ and the generalized measurement would reduce to Projective measurements. So we see that all projective measurements are POVMs but all POVMs are not Projectors. The conditions on the POVM elements are looser than those on the projectors. Namely, the $\hat{E}_\mu$ are not required to be orthogonal.

Now what if we did not *know* the outcome of the measurement of the meter. In other words, we let the environment do the generalized measurement on the system, defined by the set of Kraus operators $\hat{M}_\mu$ and POVM elements $\hat{E}_\mu = \hat{M}_\mu^\dagger\hat{M}_\mu$. But we do not know the outcome of the measurement and so we must averaged over all possiblities of

μ giving us a mixed state

$$\rho = \sum_\mu p_\mu |\psi_\mu\rangle \langle\psi_\mu| \quad (2.25)$$

$$= \sum_\mu \hat{M}_\mu |\psi_S\rangle \langle\psi_S| \hat{M}^\dagger{}_\mu \quad (2.26)$$

The last line resembles (2.15) which we got by writing out the density matrix of the system/probe apparatus and tracing over the probe. The inference here is that open quantum system dynamics can be modelled as POVM measurements done by the *environment.*

2.4 Lindblad Dynamics

The Lindblad master equation is the most general generator of Markovian dyanamics in quantum systems. It is extensively used in the study of open quantum systems and is influencial in many fields such as quantum optics, quantum information, decoherence, etc.It is a completely positive and trace-preserving map, it takes density matrices to density matrices. The general form of the master equation is given by [7],

$$\dot{\rho} = -i[\hat{H} + \hat{H}_L s, \rho(t)] + \sum_i (\hat{L}_i \rho(t) \hat{L}_i^\dagger - \frac{1}{2}\{\hat{L}_i^\dagger \hat{L}_i, \rho(t)\}) \equiv \mathcal{L}\rho(t)$$

Here, the L_i are known as *jump operators.* Performing the Kraus decompostition of this master equation we get[8]:

$$\hat{\rho}(t+dt) = \hat{\rho}(t) + dt\mathcal{L}[\rho] \quad (2.27)$$

$$= \hat{M}_o(dt)\rho\hat{M}^\dagger{}_o(dt) + \sum_\mu \hat{M}_\mu(dt)\rho\hat{M}^\dagger{}_\mu(dt) \quad (2.28)$$

where the Kraus operators $\hat{M}$ are given as:

$$\hat{M}_\mu(dt) = \hat{L}_\mu\sqrt{dt} \quad \forall \quad \mu \neq o \quad (2.29)$$

$$\text{and} \qquad \hat{M}_o = \hat{1} - \frac{1}{\hbar}\hat{H}_{eff}dt \quad (2.30)$$

where, for coupling strength k

$$\hat{H}_{eff} = \hat{H} - i\frac{k}{2}\sum_\mu \hat{L}_i^\dagger \hat{L}_i \quad (2.31)$$

Now, suppose that at time t, $\hat{\rho} = |\psi(t)\rangle \langle\psi(t)|$ then from the equation(2.28) we get:

$$\hat{\rho}(t+dt) = \hat{M}_o(dt) |\psi(t)\rangle \langle\psi(t)| \hat{M}^\dagger{}_o(dt) + \sum_\mu \hat{M}_\mu(dt) |\psi(t)\rangle \langle\psi(t)| \hat{M}^\dagger{}_\mu(dt) \quad (2.32)$$

$$= \sum_\mu p_\mu |\psi_\mu(t)\rangle \langle\psi_\mu(t)| + p_o |\psi_o(t)\rangle \langle\psi_o(t)| \quad (2.33)$$

Where the last line comes from our discussion in the previous section where we showed that the mixed state obtained by doing a trace over the system probe apparatus can be expressed as a combination (ensemble average) of pure state evolutions (2.25). The final mixed state obtained by the Lindblad master equation resembles the same mixed state obtained by doing a POVM on the system! As shown in the previous section(2.18), the state $|\psi_\mu\rangle$ and the probability p_μ is given by:

$$|\psi_\mu\rangle = \frac{M_\mu |\psi(t)\rangle}{||M_\mu |\psi(t)\rangle ||} \quad (2.34)$$

$$= \frac{\hat{L}_\mu |\psi(t)\rangle}{||\hat{L}_\mu |\psi(t)\rangle ||} \quad (2.35)$$

$$\text{and} \quad p_\mu = \langle\psi(t)| \hat{M}^\dagger{}_\mu \hat{M}_\mu |\psi(t)\rangle \quad (2.36)$$

$$= \langle\psi(t)| \hat{L}^\dagger{}_\mu \hat{L}_\mu |\psi(t)\rangle \, dt \quad (2.37)$$

And for $\mu = o$ we get the state $|\psi_o\rangle$ with probability:

$$p_o = \langle\psi(t)| \hat{M}^\dagger{}_o \hat{M}_o |\psi(t)\rangle = 1 - \sum_\mu p_\mu \quad (2.38)$$

$$(2.39)$$

Where for small dt, the state $|\psi(t)\rangle$ goes to,

$$|\psi_o\rangle = \frac{M_o |\psi(t)\rangle}{||M_o |\psi(t)\rangle ||} \quad (2.40)$$

$$= \frac{e^{-\frac{i}{\hbar}\hat{H}_{eff}dt} |\psi(t)\rangle}{||e^{-\frac{i}{\hbar}\hat{H}_{eff}dt} |\psi(t)\rangle ||} \quad (2.41)$$

Now, let us interpret the above equations. We started with a pure state at time t, after interacting with the environment for a short time dt and the state became mixed(2.32). We interpret that mixed state as an averaging or a complex combination of pure processes (2.33). We say each process is pure because every state $|\psi_\mu\rangle$ is just a Lindblad jump operator applied to the original state $|\psi(t)\rangle$ and then normalized(2.35).

This allows for an algorithm to determine the state of a system after time dt. We start with initial state $|\psi(t)\rangle$ and choose a random number r between 0 and 1. If $r > p_\mu$ we say a jump happens and apply the jump operator $\hat{L}_\mu$ to the initial state (2.35). If $r < p_\mu$ then the jump has not happened and we evolve the state with $\hat{M}_o$ (2.41), which is a continuous non-Unitary evolution. Then we go to the next time instance, $t + 2dt$, and repeat the process of choosing r and evolving the state. Such a run over all t (at least the time period of interest) is known as a *Quantum Trajectory*. Using this algorithm and averaging over many such runs of it will yield the mixed state from Lindblad dynamics.

2.5 An example: Spontaneous Emission Two Level Atom

To understand the jump processes and the algorithm being used to generate these trajectories I will illustrate using the example of spontaneous emission of a two level atom. The master equation for this apparatus is given by[9][10],

$$\begin{aligned} \frac{d}{dt}\hat{\rho} &= \frac{-i}{\hbar}[\hat{H}, \hat{\rho}] = \frac{\Gamma}{2}\{\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{\sigma}_+\hat{\sigma}_- - 2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+\} && (2.42) \\ &= \frac{-i}{\hbar}[\hat{H}_{eff}, \hat{\rho}] + \Gamma\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- && (2.43) \end{aligned}$$

where

$$\begin{aligned} \hat{H}_{eff} &= \hat{H} - \frac{i\hbar}{2}\hat{L}^\dagger\hat{L} && (2.44) \\ \hat{L} &= \sqrt{\Gamma}\hat{\sigma}_- = \sqrt{\Gamma}\,|g\rangle\langle e| && (2.45) \end{aligned}$$

In this case the above described $\hat{L}$ is the only Lindblad operator and so,

$$\hat{H}_{eff} = \hat{H} - i\hbar\frac{\Gamma}{2}\,|e\rangle\langle e| \tag{2.46}$$

Let the state at time t be a pure state

$$|\psi(t)\rangle = c_g(t)\,|g\rangle + c_e(t)\,|e\rangle \tag{2.47}$$

In the next time instance, $t + dt$ there are two possible outcomes:

1. A photon is observed (a jump happens)

2. No photon is observed (no jump has happened)

2.5.1 Jump happens: A photon is observed

The probability to observe emission of photon (jump) is given by(2.36):

$$
\begin{aligned}
p_{jump} &= \langle\psi(t)|\,\hat{L}^{\dagger}\hat{L}\,|\psi(t)\rangle\,dt && (2.48)\\
&= \langle\psi(t)|\,(\sqrt{\Gamma}\,\langle g|e\rangle\,\sqrt{\Gamma}\,|g\rangle\,\langle e|)\,|\psi(t)\rangle\,dt && (2.49)\\
&= \langle\psi|e\rangle\,\langle e|\psi\rangle\,\Gamma dt && (2.50)\\
&= |c_e(t)|^2\,\Gamma dt && (2.51)\\
&= p_e(t)\Gamma dt && (2.52)
\end{aligned}
$$

where $p_e(t)$ is the probability of being in the excited state at time t. Which makes sense, given that the state is in a superposition of being in excited and ground state. The probability of emission would be the probability of it being in excited state (p_e) times the spontaneous emission rate (Γ) and the time interval (dt).

If the jump happens, or a photon is emitted, the state is given by (2.35):

$$
\begin{aligned}
|\psi(t+dt)\rangle &= \frac{\hat{L}_\mu\,|\psi(t)\rangle}{||\hat{L}_\mu\,|\psi(t)\rangle\,||} && (2.53)\\
&= |g\rangle && (2.54)
\end{aligned}
$$

Which is also reasonable as after a jump we expect the atom to be in the ground state $|g\rangle$. An important note to make here is that after a jump is observed, subsequent measurements will not give different results. The system state has "collapsed" and will not be altered by further measurements.

2.5.2 No Jump: Non unitary evolution

The probability of no jump is given by the equation 2.38 and the state evolves as(2.41):

$$
|\psi(t+dt)\rangle = \frac{e^{-i\frac{\hat{H}}{\hbar}dt}e^{-\frac{\Gamma}{2}|e\rangle\langle e|dt}\,|\psi(t)\rangle}{p_o} \tag{2.55}
$$

which is a continuous, non-Unitary evolution. Notice that we do not just evolve the no jump state according to the Hamiltonian, we use the effective Hamiltonian. The imaginary part of the effective Hamiltonian is anti-Hermitian. This part of the evolution is at the crux of the concept of quantum trajectories, we will see how so below. Expanding the state $|\psi(t)\rangle$ for small dt will give:[9]

$$|\psi(t+dt)\rangle = \frac{c_g(dt)e^{-i\frac{\hat{H}}{\hbar}dt}e^{-\frac{\Gamma}{2}|e\rangle\langle e|dt}|g\rangle + c_e(dt)e^{-i\frac{\hat{H}}{\hbar}dt}e^{-\frac{\Gamma}{2}|e\rangle\langle e|dt}|e\rangle}{\sqrt{|c_g|^2 + |c_e|^2 e^{-\Gamma dt}}} \tag{2.56}$$

$$|\psi(t+dt)\rangle = \frac{c_g e^{-i\omega_g dt}|g\rangle + c_e e^{-i\omega_e dt}(1-\frac{\Gamma}{2}dt)|e\rangle}{\sqrt{|c_g|^2 + |c_e|^2(1-\Gamma dt)}} \tag{2.57}$$

$$= \frac{c_g e^{-i\omega_g dt}|g\rangle + c_e(1-\frac{\Gamma}{2}dt)e^{-i\omega_e dt}|e\rangle}{\sqrt{1-|c_e|^2\Gamma dt)}} \tag{2.58}$$

The last line follows because: $|c_e|^2 + |c_g|^2 = 1$

$$|\psi(t+dt)\rangle = c_g e^{-i\omega_g dt}(1+|c_e|^2\frac{\Gamma}{2}dt)|g\rangle + c_e e^{-i\omega_e dt}(1-|c_g|^2\frac{\Gamma}{2}dt)|e\rangle \tag{2.59}$$

In the non-jump case, the fact that we do not see a photon should update our probabilities of the state. As more time passes and we do not see a jump the probability of the electron being in the ground state should increase while the probability of it being in the excited state should decrease. This is what we see in equation 2.59: the coefficient of $|g\rangle$ increases while the coefficient of $|e\rangle$ decreases. No jump also gives us information!

Chapter 3

Qubit Trajectory Theory

3.1 Introduction

In this chapter I will introduce qubits and the types of gates used in the simulation. I will be discussing the model in which we are working as well as the equations that will be relevant. How the Lindblad picture applies to qubits and weak interactions between them. Finally I will walk through an example of CNOT interaction between environment and system bit.

3.2 Qubits and Gates

Qubits are quantum bits which are the basic units of quantum computing. They are two-state quantum systems for example: Spin of an electron or polarization of a photon. Unlike classical bits which can be in only one state at a given time, qubits can be in a superpostion of basis states.

The basis states of a qubit are defined as eigenstates of the Pauli spin matrix $\hat{\sigma}_z$ with eigenvalues ± 1 and are represented as follows:

$$|0\rangle \text{ and } |1\rangle \tag{3.1}$$

The most general pure state of a qubit is given by:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \quad |\alpha|^2 + |\beta|^2 = 1 \tag{3.2}$$

α and β are complex numbers. This can be parametriazed in terms of two angular variables θ and ϕ,

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)e^{-i\phi/2}|0\rangle + \sin(\theta/2)e^{i\psi/2}|1\rangle \tag{3.3}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. These two angles define a point on the *Bloch sphere* shown below.

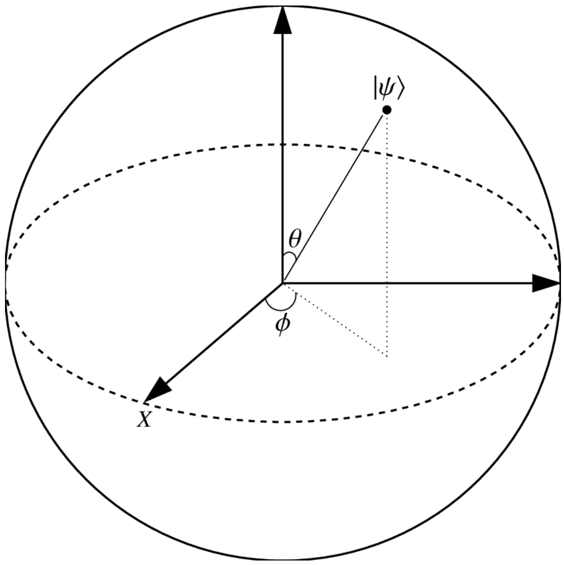

FIGURE 3.1: Bloch Sphere representation of a Qubit
(figure taken from Samson Abramsky.)

Each point on the Bloch sphere is a possible state of the qubit. Changing one state to another thus requires a *rotation* of the state vector which is facilitated by *unitary operators*. There are many specific rotations possible, which we will discuss. In general, the unitary operator for arbitrary rotation by angle(φ) about axis $\boldsymbol{n} = n_x, n_y, n_z$ is:

$$\hat{U} = \hat{1}\cos\varphi + i\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}\sin\varphi = \exp(i\varphi\boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}) \tag{3.4}$$

To specify the effect of a unitary transformation on qubits, it suffices to specify the effect on the basis states[6]. For two qubits A and B, the combined Hilbert space has a basis:

$$|0\rangle_A \otimes |0\rangle_B = |00\rangle_{AB} \tag{3.5}$$

$$|0\rangle_A \otimes |1\rangle_B = |01\rangle_{AB} \tag{3.6}$$

$$|1\rangle_A \otimes |0\rangle_B = |10\rangle_{AB} \tag{3.7}$$

$$|1\rangle_A \otimes |1\rangle_B = |11\rangle_{AB} \tag{3.8}$$

A popular example of a two bit transformation is the controlled-NOT gate or CNOT. The effect on the two-bit basis is as follows:

$$\hat{U}_{CNOT}\left|00\right\rangle_{AB} = \left|00\right\rangle_{AB} \tag{3.9}$$
$$\hat{U}_{CNOT}\left|01\right\rangle_{AB} = \left|01\right\rangle_{AB} \tag{3.10}$$
$$\hat{U}_{CNOT}\left|10\right\rangle_{AB} = \left|11\right\rangle_{AB} \tag{3.11}$$
$$\hat{U}_{CNOT}\left|11\right\rangle_{AB} = \left|10\right\rangle_{AB} \tag{3.12}$$

U_{CNOT} in terms of single bit operators is,

$$\hat{U}_{CNOT} = \left|0\right\rangle\left\langle 0\right| \otimes \hat{1} + \left|1\right\rangle\left\langle 1\right| \otimes \hat{\sigma}_x \tag{3.13}$$

and in matrix form:

$$\hat{U}_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Another two-bit gate is the $\hat{U}_{SWAP}$, it has the following effect on the basis states of A and B:

$$\hat{U}_{SWAP}\left|00\right\rangle_{AB} = \left|00\right\rangle_{AB} \tag{3.14}$$
$$\hat{U}_{SWAP}\left|01\right\rangle_{AB} = \left|10\right\rangle_{AB} \tag{3.15}$$
$$\hat{U}_{SWAP}\left|10\right\rangle_{AB} = \left|01\right\rangle_{AB} \tag{3.16}$$
$$\hat{U}_{SWAP}\left|11\right\rangle_{AB} = \left|11\right\rangle_{AB} \tag{3.17}$$

U_{SWAP} in matrix form:

$$\hat{U}_{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Both CNOT and SWAP are two-bit *quantum gates* that are both unitary and Hermitian, that is, $\hat{U}_{CNOT} = \hat{U}^{\dagger}_{CNOT}$. Because of this property we can define one-parameter families of the two-bit unitary transformations[2]:

$$\hat{U}_{CNOT}(\theta) = exp(-i\theta\hat{U}_{CNOT}) = \hat{1}\cos\theta - i\hat{U}_{CNOT}\sin\theta \tag{3.18}$$
$$\hat{U}_{SWAP}(\theta) = exp(-i\theta\hat{U}_{SWAP}) = \hat{1}\cos\theta - i\hat{U}_{SWAP}\sin\theta \tag{3.19}$$

where $0 \leq \theta \leq \pi$. We can see that when $\theta = 0$ we get identity and when $\theta = \pi/2$ we get the full CNOT (or SWAP gate). This way we can adjust the "strength" of the gate. Gates where $\theta \ll 1$ are said to be "weak" meaning the interaction between the two qubits is nearly identity, this leaves the state only slightly altered. These weak interactions are similar to those discussed in the POVM section 2.3 between the system and the probe.

Now, let us apply a CNOT gate to a two qubit state $(\alpha|0\rangle + \beta|1\rangle) \otimes |\psi\rangle$,

$$\hat{U}_{CNOT}(\theta)(\alpha|0\rangle + \beta|1\rangle) \otimes |\psi\rangle = (\hat{1}\cos\theta - i\hat{U}_{CNOT}\sin\theta)((\alpha|0\rangle + \beta|1\rangle) \otimes |\psi\rangle) \quad (3.20)$$

$$= (\alpha e^{-i\theta}|0\rangle + \beta|1\rangle) \otimes |\psi\rangle - i\beta\sin\theta|1\rangle \otimes (\hat{\sigma}_x|\psi\rangle) \quad (3.21)$$

To remove the relative phase $e^{-i\theta}$ between $|0\rangle$ and $|1\rangle$ we introduce $\hat{Z}(\theta) = (exp(+i\sigma_z\theta/2))\otimes \hat{1}$, such that $\hat{Z}(\theta)\hat{U}_{CNOT}$ undoes the extra relative phase and similarly for $\hat{U}_{SWAP}$. These two gates will be the primary interaction between the system and environment qubit. In the later sections I will setup the measurement apparatus and reexamine POVMs and Lindblad dynamics in the qubit picture.

3.3 System and Environment

In this section I will describe the system and environment interaction adapted from [2]. We assume that the system to be a qubit and the environment to be a series of qubits. For a simple picture the system qubit interacts with only one environment qubit at a time and after interacting never come in contact again. We also assume that $\hat{H}_E = 0$, that is, the environment qubits have no Hamiltonian of their own. The Hilbert space for the system is $\mathcal{H}_S = \mathcal{H}_2$ and the environment $\mathcal{H}_E = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2 \ldots$. We also assume the enviroment qubit to be in the pure state $|0\rangle$. Refer to the schematic[2] below.

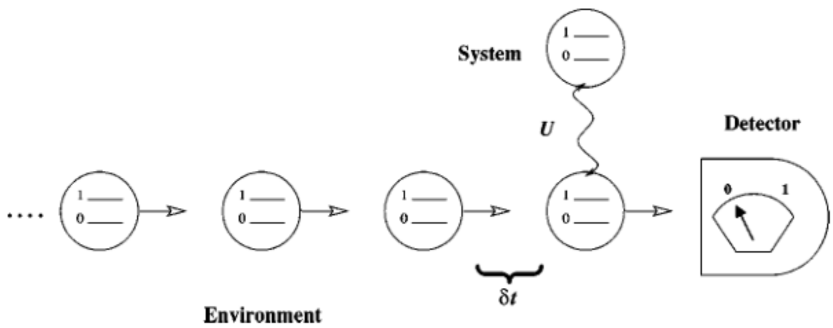

FIGURE 3.2: Schematic of the system-environment apparatus. Each environment bit interacts with the system only once and is then measured.

Now, let us suppose that the system qubit(3.2) interacts with the environment bits($|0\rangle$) in a way that a CNOT is performed from the system bit onto the environment bit. This would make the Hilbert space $\mathcal{H}_{SE} = \mathcal{H}_2 \otimes \mathcal{H}_2$ for each pairwise interaction. (Here we are talking about the full CNOT gate, i.e. $\theta = \pi/2$),

$$\begin{aligned} \left|\Psi'\right\rangle &= \hat{U}_{CNOT}(\alpha\left|0\right\rangle_S \otimes \left|0\right\rangle_E + \beta\left|1\right\rangle_S \otimes \left|0\right\rangle_E) && (3.22) \\ &= \alpha\left|0\right\rangle_S \otimes \left|0\right\rangle_E + \beta\left|1\right\rangle_S \otimes \left|1\right\rangle_E && (3.23) \end{aligned}$$

Tracing over environment bit:

$$\rho_S = |\alpha|^2 \left|0\right\rangle\left\langle 0\right| + |\beta|^2 \left|1\right\rangle\left\langle 1\right| \tag{3.24}$$

Now doing a projective measurement on the environment qubit of $|\Psi'\rangle$ in the z-basis will give us the $|0\rangle_E$ ($|1\rangle_E$) with probability $p_0 = \alpha(p_1 = \beta)$ and so system qubit will be projected onto state $|0\rangle_S$ ($|1\rangle_S$). This scheme acts as a projective measurement on the system itself. A similar setup with SWAP gate instead of a CNOT will give the post interaction state,

$$\left|\Psi'\right\rangle = \left|0\right\rangle_S (\alpha\left|0\right\rangle_E + \beta\left|1\right\rangle_E) \tag{3.25}$$

Measuring the environment qubit in z-basis will leave the system state in $|0\rangle$ regardless of the outcome of the measurement. Here the type of measurement makes no difference on the system state. In the case of CNOT the interaction yields an entangled state whereas SWAP does not (both gates give entangled states when $\theta \neq \pi/2$). The important thing here is that *after* measurement the state is in a product state and no longer entangled for either gate. This allows us to further manipulate or disregard the environment qubit after measurement without any consequence on the system bit. And so it becomes redundant to keep tabs on the environment qubit to describe the system state. Although this is a very simplified model, it allows us to discuss the generation of trajectories effectively.

3.4 Lindblad dyanamics in qubit picture.

Open quantum systems, or systems that interact with an environment usually cannot be described by a pure state. However, if the interaction between the system and environment has a simple form and the initial state of the environment has certain properties, it is possible to find an effective evolution equation for the system[11]. This evolution equation has to be completely positive, take density matrices to density matrices and be Markovian. The equation that satisfies this is the Lindblad Master equation we had discussed in 2.4. Here we assume $H_E = 0$ and so the equation becomes:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}_S, \rho] + \sum_k [\hat{L}_k \rho \hat{L}_k^\dagger - (1/2)\hat{L}_k \hat{L}_k^\dagger \rho - (1/2)\rho \hat{L}_k \hat{L}_k^\dagger] \tag{3.26}$$

In this section we will show the above master equation can be approximated from the discrete time evolution picture of experiment described above. We will start with a generic system state $|\psi\rangle$ and environment $|E\rangle$ undergoing some unitary transformation[2]:

$$|\Psi\rangle = |\psi\rangle \otimes |E\rangle \longrightarrow |\Psi'\rangle = \hat{U}(|\psi\rangle \otimes |E\rangle) \tag{3.27}$$

Since $\hat{U}$ is an operator on $\mathcal{H}_S \otimes \mathcal{H}_E$ it can be written as a sum of product operators:

$$\hat{U} = \sum_j \hat{A}_j \otimes \hat{B}_j \tag{3.28}$$

The state of the system after unitary transformation is

$$\begin{aligned} \rho'_S &= \mathrm{Tr}_{\mathrm{env}}\{\hat{U} |\Psi\rangle \langle\Psi| \hat{U}^\dagger\} && (3.29)\\ &= \sum_{j,j'} \mathrm{Tr}_{\mathrm{env}}\{(\hat{A}_j |\psi\rangle \langle\psi| \hat{A}_{j'}^\dagger) \otimes (\hat{B}_j |E\rangle \langle E| \hat{B}_{j'}^\dagger)\} && (3.30)\\ &= \sum_{j,j'} \{(\hat{A}_j |\psi\rangle \langle\psi| \hat{A}_{j'}^\dagger)(\langle E| \hat{B}_{j'}^\dagger \hat{B}_j |E\rangle)\} && (3.31) \end{aligned}$$

The matrix $M_{jj'} \equiv \langle E| \hat{B}_{j'}^\dagger \hat{B}_j |E\rangle$ has a set of orthonormal eigenvectors μ_k with real eigenvalues λ_k. Using this we can define a new operator;

$$\hat{O}_k \equiv \sqrt{\lambda_k} \sum_j \mu_{kj} \hat{A}_j \tag{3.32}$$

and so the expression (3.31) becomes,

$$\rho'_S = \mathrm{Tr}_{\mathrm{env}}\{\hat{U} |\Psi\rangle \langle\Psi| \hat{U}^\dagger\} = \sum_k \hat{O}_k |\psi\rangle \langle\psi| \hat{O}_k^\dagger \tag{3.33}$$

After interacting with successive n environment bits and tracing out the environment we are left with,

$$\rho_S^{(n)} = \sum_{k_1,\ldots,k_n} \hat{O}_{k_n} \cdots \hat{O}_{k_1} |\psi\rangle \langle\psi| \hat{O}_{k_1}^\dagger \cdots \hat{O}_{k_n}^\dagger \tag{3.34}$$

This is a type of discrete master equation, an intial pure state has, in general, evolved into a mixed state. Note this looks like the outcome of a POVM measurement but since we do not have any particular measurement result we get a mixture of the possible outcomes like we did in 2.25. Now, consider a unitary operator that is weak, meaning it is close to identity,

$$\hat{U} = exp\{-i\epsilon \sum_j \hat{A}_j \otimes \hat{B}_j\} \tag{3.35}$$

here "weak" means that $\epsilon \ll 1$, and $\mathrm{Tr}\{\hat{A}_j^\dagger \hat{A}_j\}$ $\mathrm{Tr}\{\hat{B}_j^\dagger \hat{B}_j\}$ $O(1)$. Expanding in second order in ϵ,

$$\begin{aligned}
\rho'_S &= \mathrm{Tr}_{\mathrm{env}}\{\hat{U} |\Psi\rangle \langle\Psi| \hat{U}^\dagger\} \qquad (3.36)\\
&\approx |\psi\rangle\langle\psi| - i\epsilon \sum_j [\hat{A}_j, |\psi\rangle\langle\psi|] \langle E| \hat{B}_j |E\rangle \\
&+ \frac{\epsilon^2}{2} \sum_{j,j'} \langle E| \hat{B}_{j'}^\dagger \hat{B}_j |E\rangle \\
&\times (2\hat{A}_j |\psi\rangle\langle\psi| \hat{A}_{j'}^\dagger - \hat{A}_{j'}^\dagger \hat{A}_j |\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| \hat{A}_{j'}^\dagger \hat{A}_j) \qquad (3.37)
\end{aligned}$$

To simplify we assume that the first-order term vanishes:

$$\sum_j \hat{A}_j \langle E| \hat{B}_j |E\rangle = 0 \tag{3.38}$$

Here too we define $M_{jj'} \equiv \langle E| \hat{B}_{j'}^\dagger \hat{B}_j |E\rangle$ with its orthonormal eigenvectors and eigenvalues as above. Taking δt as the interaction time, we can define operators:

$$\hat{L}_k = \sqrt{\frac{\epsilon^2 \lambda_k}{\delta t}} \sum_j \mu_{kj} \hat{A}_j \tag{3.39}$$

In terms of these *Lindblad operators* the equation 3.37 becomes

$$\frac{\rho' - \rho}{\delta t} = \sum_k [\hat{L}_k \rho \hat{L}_k^\dagger - (1/2)\hat{L}_k^\dagger \hat{L}_k \rho - (1/2)\rho \hat{L}_k^\dagger \hat{L}_k] \tag{3.40}$$

Which has the same form as 3.26 with a vanishing Hamiltonian. We can recover the continuous evolution equation if we limit the interaction to infinitesimal transformations ($\epsilon \ll 1$).

3.5 Stochastic Schrodinger Equation

Now let us repeat the above exercise for an environment bit in state $|0\rangle$ and then see what happens to the system state after we *measure* the environment bit. We start with a general state $|\Psi\rangle = |\psi\rangle_S \otimes |0\rangle_E$ and a weak unitary transformation as before,

$$\hat{U} = exp\{-i\theta \sum_j \hat{A}_j \otimes \hat{B}_j\} \tag{3.41}$$

where $\theta \ll 1$, and $\mathrm{Tr}\{\hat{A}_j^\dagger \hat{A}_j\}$ $\mathrm{Tr}\{\hat{B}_j^\dagger \hat{B}_j\}$ $O(1)$, allowing us to expand the exponential in powers of θ. The state of the system and environment after interaction becomes:

$$\begin{aligned} \hat{U}|\Psi\rangle &= |\psi\rangle \otimes |0\rangle - i\theta \sum_j \hat{A}_j |\psi\rangle_S \otimes \hat{B}_j |0\rangle_E \\ &- \theta^2 \sum_{j,j'} \hat{A}_{j'} \hat{A}_j |\psi\rangle_S \otimes \hat{B}_{j'} \hat{B}_j |0\rangle_E + O(\theta^3) \end{aligned} \tag{3.42}$$

Now defining matrix $M_{jj'} \equiv \langle 0| \hat{B}_{j'} \hat{B}_j |0\rangle$ and its corresponding Lindblad operator as $\hat{L}_k = \sqrt{\frac{\epsilon^2 \lambda_k}{\delta t}} \sum_j \mu_{kj} \hat{A}_j$ as above along with simplifying assumptions [2], we are left with the Lindblad operator:

$$\hat{L} = \sqrt{\frac{\theta^2}{\delta t}} \sum_j \hat{A}_j \langle 1| \hat{B}_j |0\rangle \tag{3.43}$$

and so the joint state of system and environment in terms of $\hat{L}$ is,

$$\hat{U}|\Psi\rangle = (1 - (1/2)\hat{L}^\dagger \hat{L}\delta t)|\psi\rangle_S \otimes |0\rangle_E - i\sqrt{\delta t}\hat{L}|\psi\rangle_S \otimes |0\rangle_E + O(\theta^3) \tag{3.44}$$

Here we see the dependence of $\hat{L}$ $\theta/\sqrt{\delta t}$ which can parametrize the strength of the interaction between the system and the environment. In the limit of continuous measurement ($\delta t \to 0$) the right hand side of equation 3.44 vanishes leaving only $|\Psi\rangle$, which means the state has not evolved. This is an example of the *quantum Zeno effect*[12]. Most master equations have time scales built in, such as characteristic collision time, decay time, etc. In our case of qubits the time scales are much larger such that the system bit has interacted with several environment bits.

Now that the interaction has taken place, we measure the environment qubit in the z-basis. The probabilities of the outcomes($|0\rangle$ or $|1\rangle$) are:

$$\begin{aligned} p_0 &= 1 - \theta^2 \sum_{j,j'} \langle\psi| \hat{A}_{j'} \hat{A}_j |\psi\rangle \langle 0| \hat{B}_{j'} |1\rangle \langle 0| \hat{B}_j |1\rangle \\ &= 1 - \langle\psi| \hat{L}^\dagger \hat{L} |\psi\rangle \delta t = 1 - p_1 \end{aligned} \tag{3.45}$$

And the change in the system state will be,

$$\delta|\psi\rangle = |\psi'\rangle - |\psi\rangle = -\frac{1}{2}(\hat{L}^\dagger \hat{L} - \langle \hat{L}^\dagger \hat{L}\rangle)|\psi\rangle \delta t \tag{3.46}$$

Since the transformation is weak, the result 0 is highly likely. Thus with each subsequent environment qubit $|0\rangle_E$ that give measurement result 0; the state will evolve according to the nonlinear equation 3.46. We saw this in the no-jump case of the spontaneous emission example discussed in the previous chapter(2.5). Whereas if we measure the environment bit and get 1, the system state will change to:

$$|\psi\rangle \longrightarrow |\psi'\rangle = \frac{\hat{L}\,|\psi\rangle}{\sqrt{\langle \hat{L}^\dagger \hat{L}\rangle}} \tag{3.47}$$

where, $\langle \hat{L}^\dagger \hat{L}\rangle$ is the quantum expectation value of $\hat{L}$ in state $|\psi\rangle$. This sudden evolution of the system is exactly the same as what we saw in the *jump* case for spontaneous emission.

Combining these two types of evolution into a single equation called the *stochastic Schrodinger equation*[13]

$$\begin{aligned} \delta\,|\psi\rangle &= |\psi'\rangle - |\psi\rangle && (3.48)\\ &= -\frac{1}{2}(\hat{L}^\dagger \hat{L} - \langle \hat{L}^\dagger \hat{L}\rangle)\,|\psi\rangle\,\delta t + (\frac{\hat{L}}{\sqrt{\langle \hat{L}^\dagger \hat{L}\rangle}} - \hat{1})\,|\psi\rangle\,\delta N && (3.49)\end{aligned}$$

Where δN is a stochastic variable which is 0 with probability p_0 given above and 1 with probability p_1. This means that when the measurement result is 0, $\delta N = 0$ and when the result is 1, $\delta N = 1$. When $\delta N = 0$, the state evolves in the continuous and deterministic way whereas when $\delta N = 1$ we see the sudden *jumps*. The ensemble average of the stochastic variable over all possible outcomes,

$$M[\delta N] = \langle \hat{L}^\dagger \hat{L}\rangle \delta t \tag{3.50}$$

where M[ ] denotes the average value over all possible outcomes. Notice that the right hand side has the same form as p_{jump} (2.48). Armed with these equations we can move onto the CNOT and SWAP gates we introduced at the beginning of the chapter.

3.6 Weak CNOT Gate Quantum Trajectory Equations

Now we will explicitly use the modified CNOT transformation $\hat{Z}(\theta)\hat{U}_{CNOT}(\theta)$ in the above derived Schrodinger's equation to derive the traditional trajectory equations.

We will start with observing the effect the transformation has on the starting system-environment state $|\psi\rangle \otimes |0\rangle = \alpha\,|00\rangle + \beta\,|10\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$

$$\begin{aligned}
\hat{Z}(\theta)\hat{U}_{CNOT}(\theta)\,|\psi\rangle \otimes |0\rangle &= \hat{Z}(\theta)(\hat{1}\cos\theta - i\hat{U}_{CNOT}(\theta)\sin\theta)(\alpha\,|00\rangle + \beta\,|10\rangle) \\
&= \hat{Z}(\theta)(\alpha\cos\theta\,|00\rangle + \beta\cos\theta\,|10\rangle - i\alpha\sigma\theta\,|00\rangle - i\beta\sigma\theta\,|11\rangle) \\
&= \hat{Z}(\theta)\{\alpha\exp\{-i\theta\}\,|00\rangle + \beta\cos\theta\,|10\rangle - i\beta\sin\theta\,|11\rangle\} \\
&= \alpha\,|00\rangle + \beta\cos\theta\,|10\rangle - i\beta\sin\theta\,|11\rangle = |\Psi'\rangle \qquad (3.51)
\end{aligned}$$

This is an entangled state, now if we measure the environment bit in the z-basis, we will observe the following states with probability,

$$|0\rangle \implies p_0 = |\alpha|^2 + |\beta|^2\cos^2\theta \qquad (3.52)$$

and,

$$|1\rangle \implies p_1 = |\beta|^2\theta^2 \qquad (3.53)$$

It is apparent that, regardless of α and β, that $p0 \gg p1$ for small values of $\theta \ll 1$. And the probability we will measure $|0\rangle_E$ is higher, giving us the system's new state as,

$$\begin{aligned}
|\psi'\rangle &= (\alpha\,|0\rangle + \beta\cos\theta\,|1\rangle)/\sqrt{p_0} \qquad (3.54) \\
&\approx \alpha(1 + |\beta|^2\theta^2/2)\,|0\rangle + \beta(1 - |\alpha|^2\theta^2/2)\,|1\rangle \qquad (3.55)
\end{aligned}$$

The coefficient of $|0\rangle$ increases while that of $|1\rangle$ decreases, which is what we expect. This is the case of "no-jump" and so, while not changing the state drastically, we still have the ability to update the probabilities of the state. Continuing this for many bits will allow us to plot out an *unraveling* of the quantum trajectory.

On the other hand, measuring environment bit as $|1\rangle$ changes the system drastically to,

$$|\psi'\rangle = |1\rangle \qquad (3.56)$$

This state remains unchanged by further environment bit interactions and measurements. These series of weak interactions followed by projective measurements are the same as doing a POVM on the state. Similar to having a probe interacting with the system and then measuring probe described in the second chapter. This use of an ancillary bit to expand the Hilbert space of the system and do measurements is known as the *Neumark's Theorem* [14].

Repeating the above procedure for successive environment qubits, measuring each to be $|0\rangle$, will yield a system state after n iterations as,

$$|\psi_n\rangle = \alpha_n |0\rangle + \beta_n |1\rangle \tag{3.57}$$

where,

$$\frac{\beta_n}{\alpha_n} = (1 - \frac{\theta^2}{2})^n \frac{\beta}{\alpha} \approx \exp\{-n\theta^2/2\}\frac{\beta}{\alpha} \tag{3.58}$$

and with $|\alpha_n|^2 + |\beta_n|^2 = 1$ gives the probabilities,

$$|\alpha_n|^2 = \frac{|\alpha|^2}{|\alpha_n|^2 + |\beta_n|^2 \exp\{-n\theta^2\}} \tag{3.59}$$

$$|\beta_n|^2 = \frac{|\beta|^2 \exp\{-n\theta^2\}}{|\alpha_n|^2 + |\beta_n|^2 \exp\{-n\theta^2\}} \tag{3.60}$$

As $n \to \infty$, $|\alpha_n|^2 \longrightarrow 1$ which is what we expect after not measuring a single 1 result. The component of $|1\rangle_S$ must be very small. Each step of this process evolves the system state toward $|0\rangle_S$ Which is to say, the probability of the system being in $|0\rangle$ (p_0) after many measurements is the same as $|\alpha|^2$, where α comes from the generic system state we started with 3.2. Mathematically,

$$\prod_{n=0}^{\infty} p_0(n) \approx |\alpha|^2 \tag{3.61}$$

And so the probability at some point(large n), to get result 1 leaving the system in $|1\rangle$ is $1 - |\alpha|^2 = |\beta|^2$. This is equivalent to doing a single strong (Projective) measurement on the system bit itself, with the exact same probabilities.

Going back to the Lindblad operator and Stochastic Schrodinger equation, we can simplify what we derived in the previous section as,

$$|\psi'\rangle - |\psi\rangle = -\frac{\theta^2}{2}(|1\rangle\langle 1| - |\langle 1|\psi\rangle|^2 |\psi\rangle) \tag{3.62}$$

$$+ \; (\frac{|1\rangle\langle 1|}{|\langle 1|\psi\rangle|} - \hat{1})|\psi\rangle\,\delta N \tag{3.63}$$

where the Lindblad operator used is,

$$\hat{L} = \sqrt{\theta^2/\delta t}\,|1\rangle\langle 1| \tag{3.64}$$

and the mean of the stochastic variable gives $M[\delta N] = \theta^2 |\langle 1|\psi\rangle|^2 = \theta^2|\beta|^2 = p_1$.

The process of measuring the environment bit is what generates the trajectories with jumps. Suppose now we do not measure it and allow subsequent environment bit interactions. In this case the system evolves into a mixed state which is the average of all possible measurement outcomes. Starting with state ρ and allowing it to interact with only one environment qubit and averaging over the possible outcomes 0 and 1 with appropriate probabilities will give the next state as:

$$\rho' = \sum_{k=0}^{1} \hat{A}_k \rho \hat{A}_k^\dagger \tag{3.65}$$

where,

$$\hat{A}_0 = |0\rangle\langle 0| + \cos\theta\, |1\rangle\langle 1| \tag{3.66}$$

$$\hat{A}_1 = \sin\theta\, |1\rangle\langle 1| \tag{3.67}$$

Repeating this for every next environment qubit will take us to the final state after n qubits.

In the limit of $\theta \ll 1$, this evolution yields the Lindblad master equation,

$$\frac{\rho' - \rho}{\delta t} = \hat{L}\rho\hat{L}^\dagger - (1/2)\hat{L}\hat{L}^\dagger\rho - (1/2)\rho\hat{L}\hat{L}^\dagger \tag{3.68}$$

Which is what we expect from our previous discussions relating the average of trajectory outcomes to the master equation. This evolution of density matrix is deterministic and continuous, contrary to the trajectories which are random and discontinuous. The randomness is due to the measurement of the environment bit and the discontinuities are due to the jumps.

We can do a similar exercise for the SWAP gate, the state after interacting with the environment bit would be,

$$|\Psi'\rangle = \alpha\, |00\rangle + \beta\cos\theta\, |10\rangle - i\beta\sin\theta\, |01\rangle \tag{3.69}$$

A measurement of 0 would leave the system in the same state as in the CNOT case, leaving it only slightly altered. The probabilities are the same in both cases. Unlike the CNOT case, a measurement of 1 would change the system drastically to $|0\rangle$ rather than $|1\rangle$. Similar to the case of spontaneous decay.

Chapter 4

Qubit Quantum Trajectory simulations

4.1 Introduction

In this chapter the results of the simulations are plotted. The purpose of these is to highlight the various possible unravelings of the quantum trajectories and the affect of the strength of the interaction between system and environment. To facilitate the following simulations, I will make use of the Python library, QuTiP. QuTiP is an open-source software for simulating the dynamics of open quantum systems. The QuTiP class Quantum Object class allows us to create operator and qubit states which will be useful in creating the system and environment bits as well as the weak quantum gates. We will make use of several functions that operate on these quantum objects that produce the complex conjugate of a matrix, normalize a state, etc.

QuTiP also has an inbuilt Monte Carlo Solver which I have used to plot the spontaneous emission example. This method is highly efficient over master equation solvers for Hilbert space sizes that are on the order of a few hundred states or larger. In our case, the Hilbert space is given by 3.3 but it can be approximated by $\mathcal{H}_{SE} = \mathcal{H}_2 \otimes \mathcal{H}_2$ for each interaction between qubit and environment. While, not extremely efficient in the qubit scheme we are discussing, it serves as a good illustration of the trajectory method being applied to qubits.

4.2 The algorithm

In this section I will outline the specific algorithm I have used to generate the trajectory plots using QuTiP[15]. As mentioned above, the method "mcsolve" is a useful tool in solving trajectory equations and provide the evolution of such systems. However, in our case I will write my own algorithm that will output the different unraveling of the trajectory equations. As an added benefit, I have the freedom of simulating qubit scheme(3.3) we have been discussing.

1. Initialize a generic system state $|\psi\rangle$ and environment in state $|0\rangle$. This makes up the apparatus $|\Psi\rangle = |\psi\rangle \otimes |0\rangle$.

2. Interaction between the system and environment state happens by applying the operator $\hat{H} = \hat{Z}(\theta)\hat{U}_{CNOT}$ (or SWAP).

3. Chose a random number r from a Uniform (0,1) distribution. This represents the probability of a quantum jump occurring. In our case, it is the probability of measuring the environment bit to be in state $|1\rangle$.

4. If $r < p0$, where $p0$ is given by 3.52, then we assume that the environment bit measurement was $|0\rangle$ and generate the new system state accordingly,
   $|\psi'\rangle = (\hat{1}_S \otimes |0\rangle_E \langle 0|_E) |\psi\rangle$ then it is normalized.

   - We then tensor a new incoming environment bit and repeat the loop, $|\Psi'\rangle = |\psi'\rangle \otimes |0\rangle$

5. If $r > p0$, then we say a jump has occurred. Here we assume that the environment bit measurement yielded $|1\rangle$ and we break the loop of interacting new environment bits because the system cannot be altered by further measurements.

Each curve constructed constitutes a different *unraveling* of the quantum trajectory. An average over the outcome of all such trajectories will yield the behaviour of a master equation (2.32). In the following sections, I will display the constructed plots for different values of θ, which is a proxy for the "strength" of interaction between system and environment bit.

4.3 Results and Discussion

4.3.1 Controlled NOT gate

The interaction considered here is the Controlled Not gate which we discussed in 3.6. Recall its effect on our total state as,

$$\begin{aligned} |\Psi'\rangle &= \hat{Z}(\theta)\hat{U}_{CNOT}(\theta)\,|\psi\rangle \otimes |0\rangle && (4.1)\\ &= \alpha\,|00\rangle + \beta\cos\theta\,|10\rangle - i\beta\sin\theta\,|11\rangle && (4.2)\end{aligned}$$

Measuring the environment state as 0 will change the system to,

$$|\psi'\rangle = (\alpha\,|0\rangle + \beta\cos\theta\,|1\rangle)/\sqrt{p_0} \quad (4.3)$$

Repeat this for several environment bits causes the system state slowly changes to $|0\rangle$ as $|\beta|^2 \to 0$ (assuming measurement result 0 at each iteration). The abrupt jumps to $|\beta|^2 \to 1$ is indication that a 'jump' has occurred leaving the system in $|1\rangle$ (assumes measurement result is 1). I have repeated these simulations for different starting system states and plotted them all in one figure. We will see that most of the time the β probability decays to zero non-linearly, however there are few cases (due to the low probability of jumps) in which we observe the jump. The different plots are for changing values of θ, highlighting the effect the interaction strength has on the rate of decay of probabilities.

4.3.1.1 'Strong' interaction ($\theta = 1$)

At $\theta \approx 1$ the interaction between the system and environment is strong. So instead of a POVM, we end up to something close to a projective measurement on the system itself. This is apparent through the almost instant jump to either the $|1\rangle$ ($\beta = 1$) or $|0\rangle$ ($\beta = 0$) states of the system (4.1). In 4.2 I have removed the trajectory that takes β to 1 to highlight that the change is takes place in a single environment measurement. Much like it would if we measured the system itself.

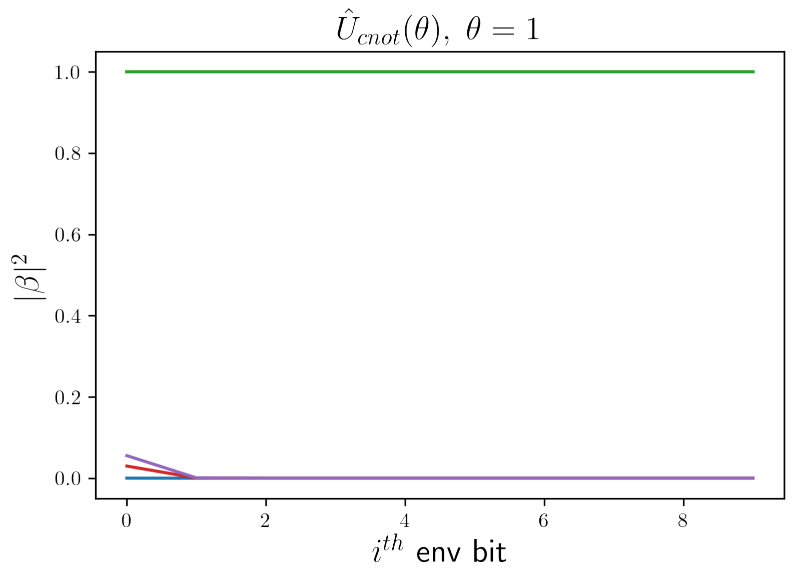

FIGURE 4.1: 5 runs starting with different starting states

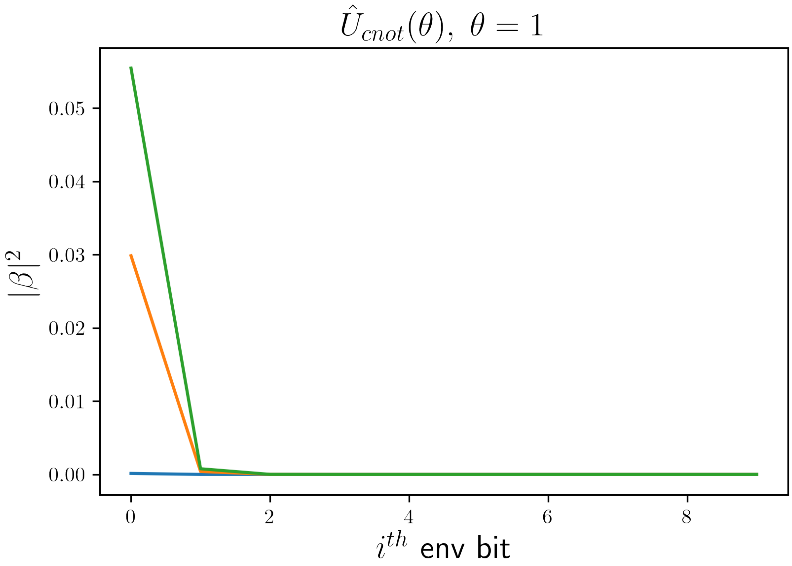

FIGURE 4.2: The same 5 runs excluding the "jumped" states

4.3.1.2 'Weak' interaction ($\theta = 0.1$)

In this regime, the weak interaction causes the each subsequent measurement of the environment bit to act as a POVM on the system state. In 4.3 I have done 5 such runs, all with different starting states, seeing just one (in orange) jump. Running this 100 times (4.4) has 8 such jumps, further highlighting the low probability of a jump occurring. All the 0 result states decay to zero after interacting with 50 bits. This number is proportional to the weakness of the interaction.

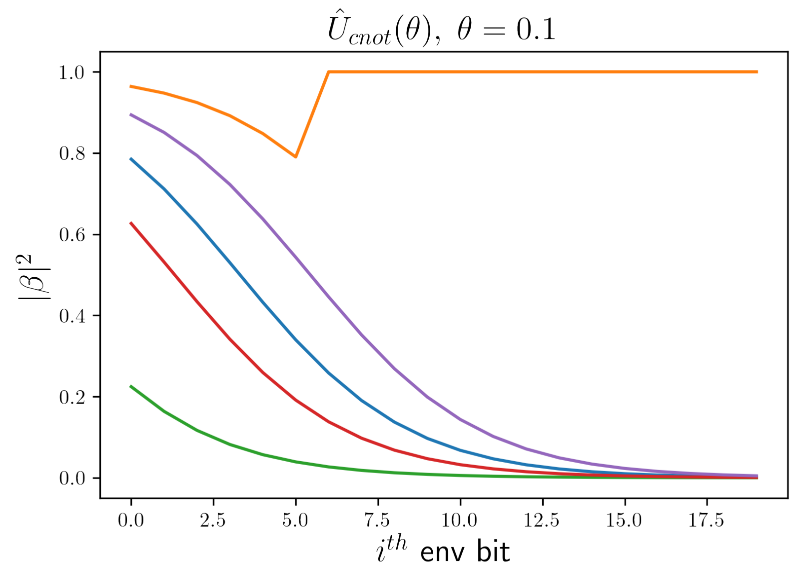

FIGURE 4.3: 5 runs with different initial states, we can see one jump.

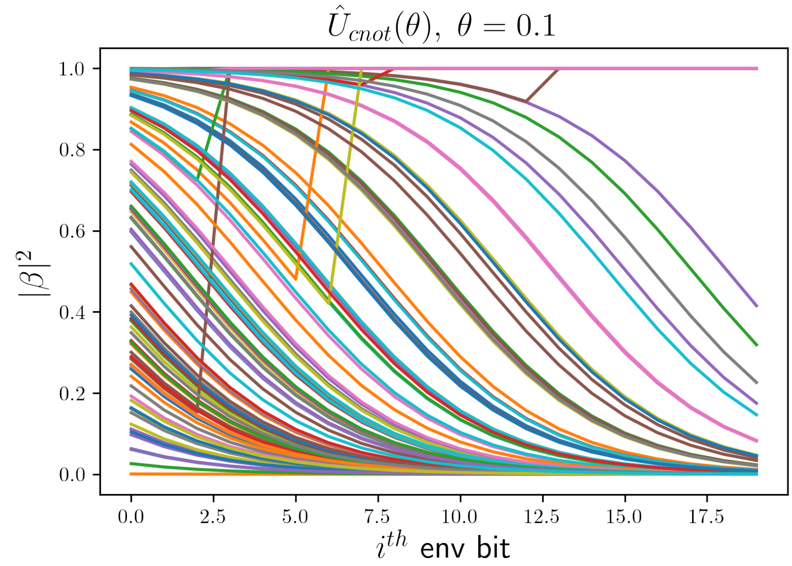

FIGURE 4.4: 100 runs with different initial states, there are 8 jumps in this particular plot.

4.3.1.3 $\theta = 0.01$

Here, none of the states jump, even with 100 different starting states. The reasoning for this may be that the interaction is *so* weak that the probability p_0 of getting the result 1, is really low. This causes all the system states to decay rather than jump.

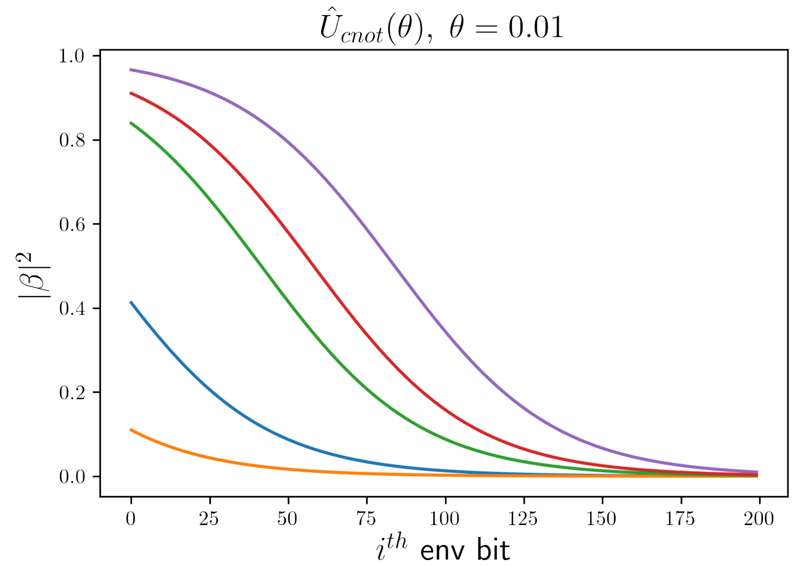

FIGURE 4.5: 5 runs starting with different starting states

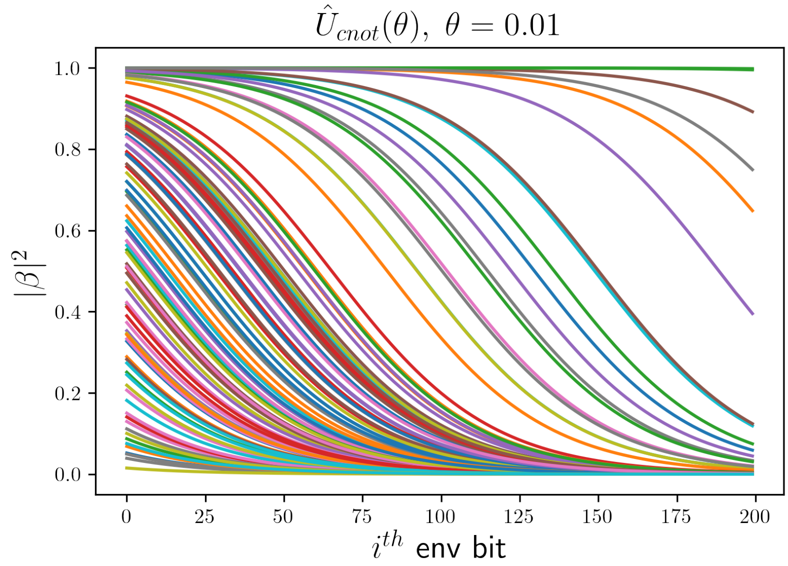

FIGURE 4.6: 100 runs, there are no jumps

4.3.2 SWAP Gate

As we discussed before, the SWAP gate interaction between system and environment bit gives,

$$\left|\Psi'\right\rangle = \hat{Z}(\theta)\hat{U}_{SWAP}(\theta)\left|\psi\right\rangle \otimes \left|0\right\rangle \tag{4.4}$$

$$= \alpha\left|00\right\rangle + \beta\cos\theta\left|10\right\rangle - i\beta\sin\theta\left|01\right\rangle \tag{4.5}$$

The entangled state has the same probabilities of measurement outcomes as the CNOT case. Measuring the environment state as 0 will change the system to,

$$\left|\psi'\right\rangle = (\alpha\left|0\right\rangle + \beta\cos\theta\left|1\right\rangle)/\sqrt{p_0} \tag{4.6}$$

The only difference between the two cases is that while in CNOT, the system state jumps to $|1\rangle$, here it jumps to $|0\rangle$. The decay of $|\beta|^2$ is the same in both cases as well. Here too I have repeated the simulation for multiple starting system states and for varying θ.

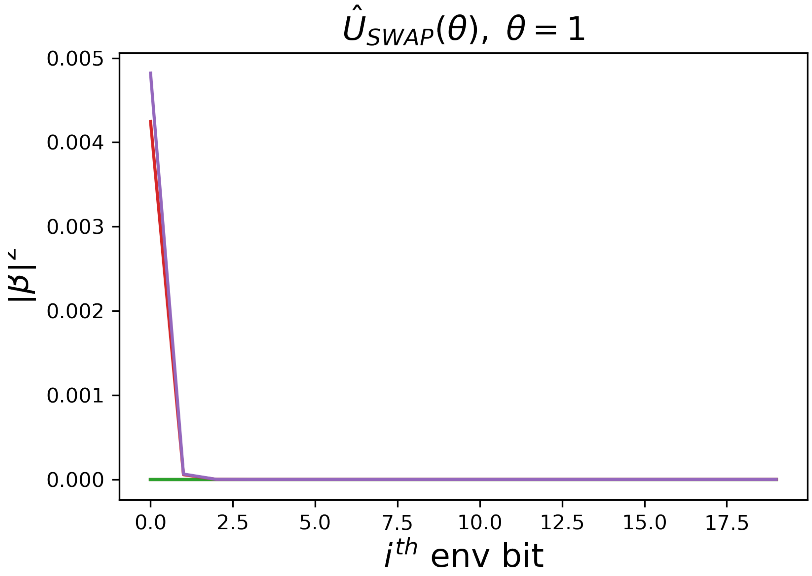

FIGURE 4.7

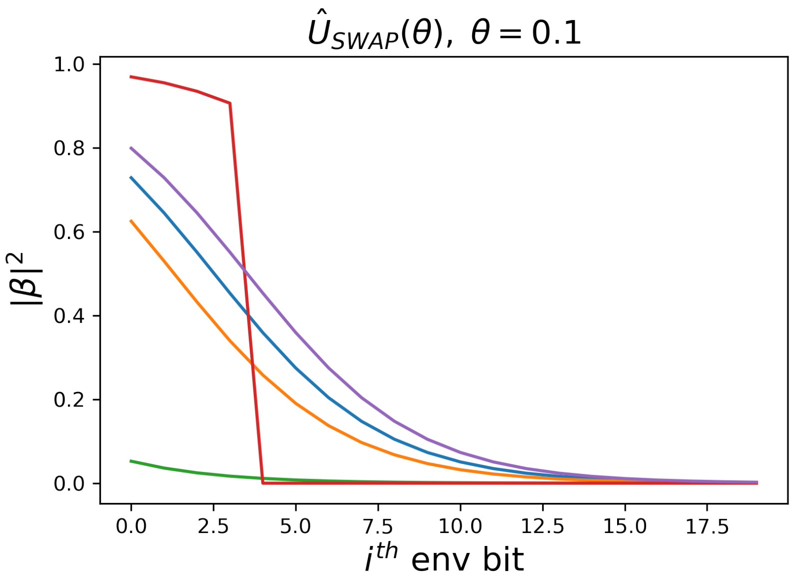

FIGURE 4.8

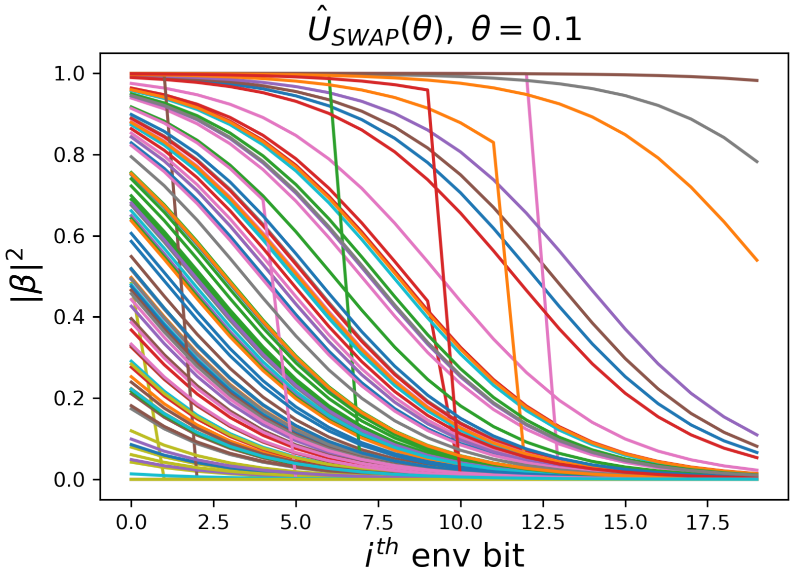

FIGURE 4.9

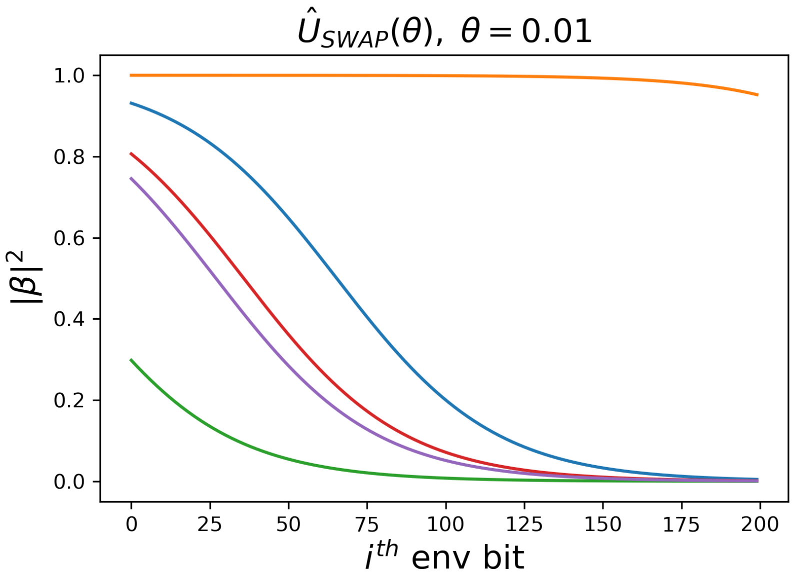

FIGURE 4.10

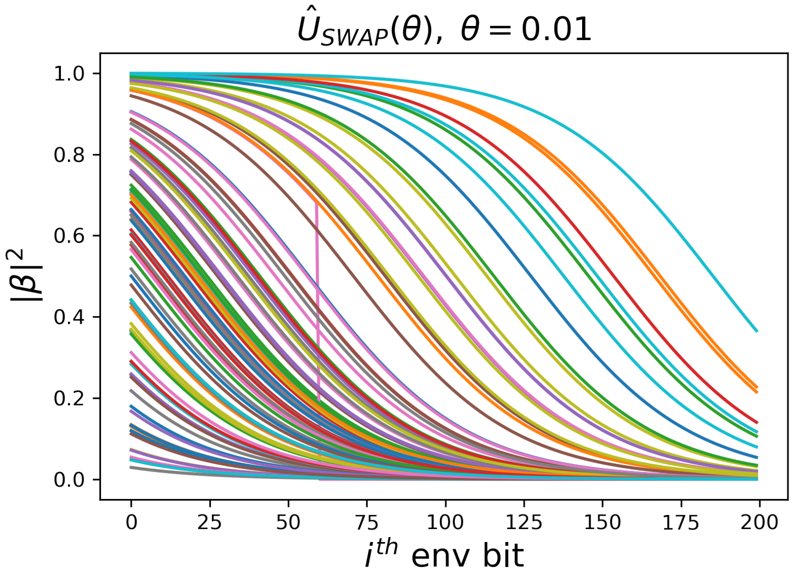

FIGURE 4.11

The behaviour of these curves is the similar to CNOT. The occurrence of jumps are rare compared to the decaying states. For weaker interactions, the probability of getting measurement result 1 low. This is the reason the number of jumps decrease as θ decreases. The number of interactions required to decay the state completely also increases, this can be seen in the x-axis of the plots. The lower the θ the more interactions that are required. This is because the "weakeness" causes the information learnt about the system to also reduce, requiring more measurements.

4.4 Conclusion

We discussed measurement theory and how POVM results relate to those from the Lindblad master equation. Lindblad dynamics in the context of qubits and the derivation of the Stochastic Schrodinger's equation was done. All of this set us up to formulate an algorithm to do stochastic evolution of system qubits interacting with environment bits through CNOT and SWAP gates. This was used to generate plots of quantum trajectories showcasing the non-linear, non-jump, evolution as well as the drastic, jump case.

Appendix A

A.1 Code

A.1.1 CNOT

To initialize an arbitrary initial state I have used a random number generator for α while using the total probability condition to fix β. I have also added arbitrary phase to β, but this has no physical significance since we are interested only in the probabilities.

```
def genstate():
    #every run of this function will generate new alpha beta
    alpha = np.random.random()
    beta = np.sqrt(1-alpha**2)*np.exp(np.random.uniform(0,2*np.pi))
    return(alpha, beta)
#Note that this does not generate uniform(0,1) for beta but it
#has no affect on the measurement outcomes
```

The interaction between the environment bit and system bit is done using:

```
#Strength of the interaction, theta:
theta = 0.01
#sigmaz operator that undoes the phase difference:
weak_sz = qeye(2) * np.cos(theta/2) + 1j*sigmaz() * np.sin(theta/2)
Z = tensor(weak_sz, qeye(2))
#Ucnot operator:
weak_cx = tensor(qeye(2),qeye(2)) * np.cos(theta) - 1j* cnot() * np.sin(
    ↪ theta)
#Complete interaction:
H = Z*weak_cx
```

The "measurement" operators used to both, find the probabilities of each outcome with the expect() method.

```
#Measurement Operators
e = basis(2,1)
g = basis(2,0)

#The following projectors are used to define the "collapse"
#or measurement operators
proj0 = g * g.dag()
proj1 = e * e.dag()

#operators used to change state to post #measurement state as well as
    ↪ the probability
#of measurement outcomes. These are applied to
#the environment bit only.
m0 = tensor(qeye(2), proj0)
m1 = tensor(qeye(2), proj1)

#operators used on the system to find values of
#alpha and beta
n0 = tensor(proj0, qeye(2))
n1 = tensor(proj1, qeye(2))
```

Finally, the simulation is run using the above setup. There are many arrays that are initialized to make the plots.

```
#arr_of_arr contains every run with different initial states

arr_of_arr = []

#100 runs will generate 100 curves on the plot

for j in range(0,100):

    #Generating the system and environment state
    alpha, beta = genstate()
    sys = (alpha*basis(2,0) + beta*basis(2,1)).unit()
    env = basis(2,0)
    psi = tensor(sys,g).unit()
```

```
########################################
arr_e0_s0 = []
arr_e0_s1 = []
arr_e1_s0 = []
arr_e1_s1 = []

tlist = []
psit0 = psi
########################################

n = 200
for i in range(0,n):
    #sets up x axis, number of env bits
    tlist = tlist + [i]
    #interact with ZUcnot
    psiH = (H*psit0).unit()
    #prob of no jump
    p0 = expect(m0,psiH)
    #prob of jump
    p1 = 1-p0
    #create RNG number r
    r = np.random.random()
    #no jump case
    if(r<p0):
        #measure env 0, new state with env 0:
        psit0 = (m0*psiH).unit()
        #measure beta
        e0_s1 = expect(n1,psit0)
        #add to array
        arr_e0_s1 = arr_e0_s1 + [e0_s1]
        #reinitialize the state
        psit0 = tensor((psit0[0][0][0]*g + psit0[2][0][0]*e), g)
        continue
    #jump case
    else:
        #measure env 1, new state w env 1:
        psit0 = (m1*psiH).unit()
        #measure beta
        e1_s1 = expect(n1,psit0)
```

```
            #add to array and set all subsequent values to 1.
            arr_e0_s1 = arr_e0_s1 + [e1_s1] + [1]*(n-i-1)
            break
    arr_of_arr = arr_of_arr + [arr_e0_s1]
```

A.1.2 SWAP

The only difference between SWAP and CNOT is that when the measurement outcome is 1, the system goes to $|0\rangle$ rather than $|1\rangle$. The new interaction:

```
#Strength of interaction
theta = 0.01
#sigmaz to remove phase difference.
weak_sz = qeye(2) * np.cos(theta/2) + 1j*sigmaz() * np.sin(theta/2)
Z = tensor(weak_sz, qeye(2))

#Ucnot operator
weak_sw = tensor(qeye(2),qeye(2)) * np.cos(theta) - 1j* swap() * np.sin(
    ↪ theta)

#Interaction
H = Z*weak_sw
```

The main loop:

```
#arr_of_arr contains every run with different initial states
arr_of_arr = []

#100 runs will generate 100 curves on the plot

for j in range(0,100):
    j = j+1
    #Generating the system and environment state
    alpha, beta = genstate()
    sys = (alpha*basis(2,0) + beta*basis(2,1)).unit()
    env = basis(2,0)
    psi = tensor(sys,g).unit()

    ########################################
    arr_e0_s0 = []
```

```
arr_e0_s1 = []
arr_e1_s0 = []
arr_e1_s1 = []

tlist = []
psit0 = psi
########################################

n = 200
for i in range(0,n):
    #sets up x axis, number of env bits
    tlist = tlist + [i]
    #interact with ZUcnot
    psiH = (H*psit0).unit()
    #prob of no jump
    p0 = expect(m0,psiH)
    #prob of jump
    p1 = 1-p0
    #create RNG number r
    r = np.random.random()
    #no jump case
    if(r<p0):
        #measure env 0, new state with env 0:
        psit0 = (m0*psiH).unit()
        #measure beta
        e0_s1 = expect(n1,psit0)
        arr_e0_s1 = arr_e0_s1 + [e0_s1]
        #reinitialize the state
        psit0 = tensor((psit0[0][0][0]*g + psit0[2][0][0]*e), g)
        continue
    #jump case
    else:
        count = count + 1
        #measure env 1, new state w env 1:
        psit0 = (m1*psiH).unit()
        #measure beta
        e1_s1 = expect(n1,psit0)
        arr_e0_s1 = arr_e0_s1 + [e1_s1] + [0]*(n-i-1)
        break
```

```
        #reinitialize, adds new env qubit
    arr_of_arr = arr_of_arr + [arr_e0_s1]
```

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