STATS 2

WEEK 1

→ Joint PMF of two discrete random variables
↳ Two events can either be independent; or
↳ one could be influencing the other.

→ Types of PMF of Multiple Random Variables:
1. Joint PMF
2. Marginal PMF
3. Conditional PMF

1. Joint PMF

→ X and Y → discrete random variables defined in the same probability space

Range of X and Y → T_X and T_Y

↳ Joint PMF of X and Y → f_{XY} → function from $T_X \times T_Y$ to $[0, 1]$

$$f_{XY}(t_1, t_2) = P(X = t_1 \text{ and } Y = t_2),\ t_2 \in T_X,\ t_2 \in T_Y$$

→ Usually written as a table or a matrix
↳

variable 1

variable 2

$P(x_1,x_2)$	$P(x_1,x_2)$	$P(x_1,x_2)$	...	..	..
...	...	...	...	...	...
...	...	...	..	...	...
...	...	...	...	..	..
...	...	...	...	...	...

2. Marginal PMF

→ To obtain the individual PMF of a random discrete variable from a joint PMF.
↳ Suppose X and Y random discrete variables, and their joint PMF → f_{XY}, then:

$$f_X(t) = P(X = t) = \sum_{t' \in T_Y} f_{XY}(t, t')$$

$$f_Y(t) = P(Y = t) = \sum_{t' \in T_X} f_{XY}(t', t)$$

3. Conditional PMF

→ $f_{X|A}(t)$ → PMF of a random variable X conditional on an event A.

$P(X = t \mid A)$, $t \in T_X$

$$f_{X|A}(t) = \frac{P((X = t) \cap A)}{P(A)}$$

→ $f_{Y|X}(t)$ → conditional PMF of Y given $X = t$

$$f_{Y|X=t}(t') = P(Y = t' \mid X = t) = \frac{P(Y = t', X = t)}{P(X = t)} = \frac{f_{XY}(t, t')}{f_X(t)}$$

$$f_{XY}(t, t') = f_{Y|X=t}(t')\, f_X(t)$$

→ $f_{XY}(t_1, t_2) = f_{Y|X=t_1}(t_2)\, f_X(t_1) = f_{X|Y=t_2}(t_1)\, f_Y(t_2)$

Q. Let $N \sim \text{Poisson}(\lambda)$. Given $N=n$, toss a fair coin n times and denote the number of heads obtained by X. What is the distribution of X?

A. $f_N(n) = \dfrac{e^{-\lambda}\lambda^n}{n!}$; $n = 0, 1, 2, \dots$ $\quad (X|N=n) \sim \text{Bin}(n, \tfrac{1}{2}) \quad f_{X|N=n}(k) = \binom{n}{k}\left(\tfrac{1}{2}\right)^n$

$$f_{XN}(k,n) = \frac{e^{-\lambda}\lambda^n}{n!} \cdot \frac{n!}{k!\,(n-k)!} \cdot \left(\frac{1}{2}\right)^n = \frac{e^{-\lambda}\lambda^n}{k!\,(n-k)!}\left(\frac{1}{2}\right)^n$$

$$\boxed{f_X(k) = \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{k!\,(n-k)!}\left(\frac{1}{2}\right)^n} \Rightarrow \frac{e^{-\lambda}\lambda^k}{k!\;2^k} \sum_{n=k}^{\infty} \frac{(\lambda)^{n-k}}{(n-k)!\,(2)^{n-k}}$$

$$X \sim \text{Poisson}(\lambda/2) \Rightarrow \boxed{f_X(k) = \frac{e^{-\lambda/2}(\lambda/2)^k}{k!}}$$

→ Joint PMF of multiple random variables

↳ $X_1, X_2, X_3, \dots X_n$ are discrete random variables defined in the same probability space.

$$f_{X_1X_2\dots X_n}(t_1, t_2, \dots, t_n) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and} \cdots \text{ and } X_n = t_n),\; t_i \in T_{X_i}$$

→ Marginal PMF with multiple random variables

↳ individual random variable ⇒ $f_{X_1}(t) = P(X_1 = t) = \displaystyle\sum_{t_2' \in T_2,\, t_3' \in T_3, \dots, t_n' \in T_n} f_{X_1,X_2,X_3\dots X_n}(t, t_2', t_3', \dots, t_n)$

↳ multiple random variables ⇒ $f_{X_1X_2}(t_1, t_2) = P(X_1 = t_1 \text{ and } X_2 = t_2) = \displaystyle\sum_{t_3' \in T_3} f_{X_1,X_2,X_3}(t_1, t_2, t')$

→ Conditional PMF with multiple random variables

↳ $(X_1, X_2 | X_3 = t_3) \sim f_{X_1,X_2|X_3=t_3}(t_1, t_2) = \dfrac{f_{X_1,X_2,X_3}(t_1, t_2, t_3)}{f_{X_3}(t_3)}$

↳ $(X_1 | X_2 = t_2, X_3 = t_3) \sim f_{X_1|X_2=t_2, X_3=t_3}(t_1) = \dfrac{f_{X_1,X_2,X_3}(t_1, t_2, t_3)}{f_{X_2X_3}(t_2, t_3)}$

↳ factors of a Joint PMF: $f_{X_1\dots X_4}(t_1, \dots, t_4) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$

$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$

$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 | X_3 = t_3, X_4 = t_4) \cdot P(X_3 = t_3 | X_4 = t_4) \cdot P(X_4 = t_4)$

$$f_{X_1\dots X_4}(t_1, \dots, t_4) = f_{X_1|X_2=t_2, X_3=t_3, X_4=t_4}(t_1) \cdot f_{X_2=t_2|X_3=t_3, X_4=t_4}(t_2) \cdot f_{X_3|X_4=t_4}(t_3) \cdot f_{X_4}(t_4)$$

WEEK 2

→ Independence of two random variables

↳ X and Y are independent if :

$$f_{XY}(t_1, t_2) = f_X(t_1) \cdot f_Y(t_2)$$

→ Joint PMF of f_{XY} is the product of the marginals PMF of X and Y.
→ Conditional PMF equals marginal PMF

→ Independence of multiple random variables

↳ $X_1, X_2, \ldots, X_n$ are independent iff

$$f_{X_1, X_2, \ldots X_n}(t_1, t_2, \ldots, t_n) = f_{X_1}(t_1) \cdot f_{X_2}(t_2) \cdot \ldots \cdot f_{X_n}(t_n)$$

→ Independent and Identically distributed (i.i.d.)

↳ Random variables $X_1, X_2, \ldots X_n$ are i.i.d. if → ① they are independent,
② Marginal PMFs f_{X_i} are identical.

→ $X_1, X_2 \ldots X_n \sim$ i.i.d. f_X

→ Memoryless property of Geometric distribution

→ Let $X \sim \text{Geometric}(p)$

① $P(X > n) = (1-p)^n$

② $P(X > m+n \mid X > m) = \dfrac{P(X > m+n \cap X > m)}{P(X > m)} = \dfrac{P(X > m+n)}{P(X>m)} = \dfrac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n$

→ Functions of random variable

↳ One-to-one function → $P(Y = f(x)) = P(X = x)$; e.g. → $y = x-5$, $y = 2^x$

↳ Many-to-one function → e.g. → $y = (x-5)^2$, $y = x(1-x)$ → $y_0 = f(x_1) = f(x_2) = \ldots = f(x_m)$ all the places f(x) takes the value y_0

⇓

↳ sum over all the probabilities of X for wherever the function takes the 'y' value. ⟶ $P(Y = y_0) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_m)$

→ Visualising function of 2 random variable

↳ $g(x,y)$: function

① Contours → values of (x,y) that result in $g(x,y) = c$
→ make a plot of those (x,y) for different c

② Regions → values of (x,y) that result in $g(x,y) \leq c$
→ make a plot of those (x,y) for different c

→ Function of 2 random variables

→ Let $X, Y \sim f_{XY}$; let $Z = g(X, Y)$ be a function of X and Y

· What is the PMF of Z?

→ Step 1 : Find the range of Z
→ Step 2 : Add over all the contours

↳ Suppose z is a possible value taken by Z :

$$P(Z = z) = \sum_{(x,y) : g(x,y) = z} f_{XY}(x,y)$$

→ Sum of 2 uniform random variables

$X, Y \sim$ iid Unif$\{1, 2, \ldots, n\}$, $W = X + Y$

range of $W = \{2, 3, \ldots, 2n\}$ $\quad W = w : (1, w-1), (2, w-2), \ldots, (w-1, 1)$

$W \in \{2, 3, \ldots, 2n\}$

$$P(W = w) = \begin{cases} \dfrac{w-1}{n^2}, & 2 \leq w \leq n+1 \\ \dfrac{2n-w+1}{n^2}, & n+2 \leq w \leq 2n \end{cases}$$

→ Max of 2 uniform random variables $X, Y \sim \text{iid Unif}\{1,2,\dots,n\}$, $Z = \max(X,Y)$

$$Z \in \{1,2,\dots,n\}$$

$$P(Z=z) = \frac{2z-1}{n^2}$$

→ PMF of $g(X_1, X_2, \dots X_n)$ → The PMF $X = g(X_1, X_2, \dots, X_n)$ is given by

$$f_X(t) = P(g(X_1,X_2,\dots,X_n)=t) = \sum_{(t_1,t_2,\dots,t_n):\, g(t_1,t_2,\dots,t_n)=t} f_{X_1,X_2,\dots,X_n}(t_1,t_2,\dots,t_n)$$

e.g. →

$t_2 \cdot t_3$	$t_1+t_2+t_3$	t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1,t_2,t_3)$
0	0	0	0	0	1/9
0	1	0	0	1	1/9
0	2	0	0	2	1/9
1	2	0	1	1	1/9
2	3	0	1	2	1/9
0	1	1	0	0	1/9
0	3	1	0	2	1/9
0	2	1	1	0	1/9
1	3	1	1	1	1/9

① $X = g(X_1,X_2,X_3) = X_1+X_2+X_3$ $\quad g \in \{\overset{1/9}{0}, \overset{2/9}{1}, \overset{3/9}{2}, \overset{3/9}{3}\}$

② $Y = h(X_1,X_2,X_3) = X_2 \cdot X_3$ $\quad h \in \{\overset{6/9}{0}, \overset{2/9}{1}, \overset{1/9}{2}\}$

f_{XY}	0	1	2	3
0	1/9	2/9	2/9	1/9
1	0	0	1/9	1/9
2	0	0	0	1/9

→ sum of n independent Bernoulli(p) = Binomial(n,p)

→ suppose X and Y take integer values and their joint PMF = f_{XY}. Let $Z = X+Y$

Convolution

$$P(Z=z) = \sum_{x=-\infty}^{\infty} P(X=x, Y=z-x) = \sum_{x=-\infty}^{\infty} f_{XY}(x, z-x) = \sum_{y=-\infty}^{\infty} f_{XY}(z-y, y)$$

$$= \sum_{x=-\infty}^{\infty} f_X(x) f_Y(z-x) = \sum_{y=-\infty}^{\infty} f_Y(y) f_X(z-y)$$

→ Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent.

→ $Z = X + Y$

$$f_Z(z) = \underbrace{\sum_{x=0}^{\infty} f_X(x) \cdot f_Y(z-x)}_{\text{can be replaced with } x \text{ going to } z \text{ because after } z \text{ this term will go to } 0.} = \sum_{x=0}^{z} \frac{e^{-\lambda_1} \cdot \lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2} \lambda_2^{(z-x)}}{(z-x)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{z!} \underbrace{\sum_{x=0}^{z} \frac{\lambda_1^x \lambda_2^{z-x} \cdot z!}{x!(z-x)!}}_{(\lambda_1+\lambda_2)^z}$$

$$f_Z(z) = \frac{e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1+\lambda_2)^z}{z!} \Rightarrow Z \sim \text{Poisson}(\lambda_1+\lambda_2)$$

→ conditional of $X|Z$

$$P(X=k|Z=n) = \frac{P(X=k, Z=n)}{P(Z=n)} = \frac{P(X=k) \cdot P(Z=n|X=k)}{P(Z=n)} = \frac{P(X=k) \cdot P(Y=n-k)}{P(Z=n)}$$

$$= \frac{\frac{\cancel{e^{-\lambda_1}} \cdot \lambda_1^k}{k!} \cdot \frac{\cancel{e^{-\lambda_2}} \cdot \lambda_2^{n-k}}{(n-k)!}}{\frac{\cancel{e^{-\lambda_1-\lambda_2}} \cdot (\lambda_1+\lambda_2)^n}{n!}} = \frac{n!}{k!(n-k)!} \frac{\lambda_1^k \cdot \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}$$

$$\longrightarrow (\lambda_1+\lambda_2)^n = (\lambda_1+\lambda_2)^k \cdot (\lambda_1+\lambda_2)^{n-k}$$

$$\longrightarrow \frac{n!}{k!(n-k)!} = {}^nC_k$$

$$= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$X|Z = \text{Bin}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$Y|Z = \text{Bin}\left(n, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$

→ Functions of non-overlapping indepent random variables are also independent

→ If X and Y are independent, $g(X)$ and $h(Y)$ are independent for any two functions g and h

→ Min/Max of two random variables

↳ $X, Y \sim f_{XY}$ $\quad Z = \min(X, Y)$

$f_Z(z) = P(\min(X,Y) = z) = P((X=z) \text{ and } (Y=z) \text{ or } (X=z \text{ and } Y>z) \text{ or } (X>z \text{ and } Y=z))$

$= f_{XY}(z,z) + \sum_{t_2>z} f_{XY}(z,t_2) + \sum_{t_1>z} f_{XY}(t_1,z)$

↳ CDF of a random variable

↳ CDF of a random variable X is a function $F_X : \mathbb{R} \to [0,1]$ defined as:

↳ $F_X(x) = P(X \le x)$

e.g. → X and Y are independent. $Z = \max(X,Y)$

$F_Z(z) = P(\max(X,Y) \le z)$

$= F_X(z) \cdot F_Y(z)$

↳ Let $X_1, X_2, \ldots, X_n \sim$ i.i.d.

① Distribution of $\underline{\min(X_1, X_2, \ldots, X_n)}$ → $P(\min(X_1, X_2, \ldots, X_n) \ge z) = (P(X \ge z))^n$

② Distribution of $\underline{\max(X_1, X_2, \ldots, X_n)}$ → $P(\max(X_1, X_2, \ldots, X_n) \le z) = (P(X \le z))^n = (F_X(z))^n$

↳ Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(p)$ be independent. Find the distr. of $\underline{\min(X,Y)}$

$Z = \min(X,Y)$ $\quad P(Z \ge z) = P(X \ge z, Y \ge z) = (1-p)^{z-1} \cdot (1-p)^{z-1}$

$= ((1-p)^2)^{z-1}$

$P(Z \ge z+1) = ((1-p)^2)^z$

$P(Z = z) = P(Z \ge z) - P(Z \ge z+1)$

$= ((1-p)^2)^{z-1} - ((1-p)^2)^z = ((1-p)^2)^{z-1}(1-(1-p)^2)$ ← Geometric probability

$\min(X,Y) \sim \text{Geometric}(1-(1-p)^2)$

Let $X_1 \sim \text{Geometric}(p_1)$ and $X_2 \sim \text{Geometric}(p_2)$

↳ $\min(X_1, X_2) \sim \text{Geometric}(1-(1-p_1)(1-p_2))$

$Z = \max(X_1, X_2, \ldots, X_{10})$ $\quad X \sim \text{Bin}(6, \tfrac{1}{2})$

$P(Z \le 2) = P(X_1 \le 2, X_2 \le 2, \ldots, X_{10} \le 2)$

$= \left(22 \cdot \left(\tfrac{1}{2}\right)^6\right)^{10}$

X	0	1	2	3
	$\left(\tfrac{1}{2}\right)^6$	$6\left(\tfrac{1}{2}\right)^6$	$15\left(\tfrac{1}{2}\right)^6$	

$\binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6 \times 5}{2} = 15$

$X \le 2 = \left(\tfrac{1}{2}\right)^6 \cdot 22$

$Z = \min(X_1, X_2, \ldots, X_{10})$ $\quad X \sim \text{Bin}(6, \tfrac{1}{2})$

$F_Z(2) = 1 - P(X_1 > 2, X_2 > 2, \ldots X_{10} > 2)$ $\quad X > 2 = 42\left(\tfrac{1}{2}\right)^6$

$= 1 - \left(\frac{42}{2^6}\right)^{10} = 1 - \left(\frac{21}{2^5}\right)^{10}$

— ✗ — ✗ — ✗ —

WEEK 3

→ Expected value of a discrete random variable:

↳ $X \to$ discrete rand. variable ; range $\to T_X$; PMF $\to f_X$

$$E[X] = \sum_{t \in T_X} t \cdot f_X(t) = \sum_{t \in T_X} t \cdot P(X=t)$$

↳ example ⇒ $X \sim \text{Bernoulli}(p) \Rightarrow E[X] = 0(1-p) + p = p$

$X \sim \text{Uniform}\{1,2,3,4,5,6\} \Rightarrow E[X] = \sum_{t \in T_X} t \cdot 1/6 = 3.5$

$X \sim \text{Uniform}\{a, a+1, \ldots, b\} \Rightarrow E[X] = \frac{a}{b-a+1} + \frac{a+1}{b-a+1} + \ldots + \frac{b}{b-a+1} = \frac{a+b}{2}$

Identity ⇒ $a + (a+1) + (a+2) + \ldots + b = (b-a+1)\left(\frac{a+b}{2}\right)$

Simplify summations

① Difference Equation (DE): $a_{t+1} - r a_t = b_t \; (r \neq 1)$ $\quad \sum_{t=1}^{n} a_t = \frac{a_1 - r a_n}{1-r} + \frac{1}{1-r} \sum_{t=1}^{n-1} b_t$

② Geometric Progression (GP): $a_{t+1} - r a_t = 0 \; (r \neq 1)$ $\quad \sum_{t=1}^{n} a_t = \frac{a_1 - r a_n}{1-r} \xrightarrow[n \to \infty]{|r|<1} \frac{a_1}{1-r}$

③ Exponential Function : $\sum_{t=0}^{\infty} e^{-\lambda} \frac{\lambda^t}{t!} = 1$ $\quad e^{\lambda} = \sum_{t=0}^{\infty} \frac{\lambda^t}{t!}$

④ Binomial Formula : $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$

↳ $X \sim \text{Geometric}(p)$ $\quad E[X] = \sum_{t=1}^{\infty} t \cdot (1-p)^{t-1} \cdot p$

↳ GP : $a_1 = p$, $r = 1-p$, $b_t = r^t p$ $\quad E[X] = 1/p$

↳ $X \sim \text{Poisson}(\lambda)$ $\quad E[X] = \sum_{t=0}^{\infty} t \cdot e^{-\lambda} \frac{\lambda^t}{t!}$

↳ using exponent function → $E[X] = \lambda$

↳ $X \sim \text{Binomial}(n,p)$ $\quad E[X] = \sum_{t=0}^{n} t \cdot \binom{n}{t} p^t \cdot (1-p)^{n-t}$

↳ $E[X] = np$

→ Expected value of a function of random variables

↳ $X_1, X_2, \ldots, X_n$ have joint PMF $f_{X_1, X_2, \ldots X_n}$ with range X_i denoted T_{X_i}

Let $g : T_{X_1} \times T_{X_2} \times \ldots \times T_{X_n} \to \mathbb{R}$ be a function, $Y = g(X_1, X_2, \ldots, X_n)$; T_Y and f_Y

$$E[Y] = \sum_{t \in T_Y} t \cdot f_Y(t) = \sum_{t_i \in T_{X_i}} g(t_1, t_2, \ldots, t_n) \cdot f_{X_1, X_2, \ldots X_n}(t_1, t_2, \ldots, t_n)$$

→ Linearity of Expected value :

① $E[cX] = cE[X]$ for a random variable X and a constant c

② $E[X+Y] = E[X] + E[Y]$ for any two random variables X and Y

③ $E[aX + bY] = aE[X] + bE[Y]$

→ Centering operation

↳ $Y = X - E[X]$ is a translated version of X and $E[Y] = 0$

$X - E[X]$ is a zero-mean random variable.

→ Variance of a random variable

↳ $Var(X) = E[(X - E[X])^2]$

$Var(X) = E[X^2] - (E[X])^2$

↳ If X and Y are independent:

- $E[XY] = E[X]E[Y]$
- $Var(X+Y) = Var(X) + Var(Y)$

Distribution	Expected Value	Variance
Bernoulli(p)	p	$p(1-p)$
Binomial(n,p)	$n \cdot p$	$np(1-p)$
Geometric(p)	$1/p$	$(1-p)/p^2$
Poisson(λ)	λ	λ
Uniform $\{1,2,\dots,n\}$	$(n+1)/2$	$(n^2-1)/12$

→ A random variable is standardised if $E(X)=0$ and $Var(X)=1$

↳ $Y = \dfrac{X - E[X]}{SD(X)}$, then Y is a standardised rand. var. obtained from X.

→ Covariance of two random variables

↳ X and Y are rand. var. on the same prob. space

$$Cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]\cdot E[Y]$$

↳ $Cov(X, aY+bZ) = a\,Cov(X,Y) + b\,Cov(X,Z)$

↳ $Cov(aX+bY, Z) = a\,Cov(X,Z) + b\,Cov(Y,Z)$

↳ If X and Y are independent, then $Cov(X,Y) = 0$

If X and Y are independent, then they are uncorrelated.
If X and Y are uncorrelated, they might still be dependent.

→ Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{SD(X)\,SD(Y)}$$

→ Markov's Inequality $P(X \geq c) \leq \frac{\mu}{c}$, where $\mu = E(X)$

→ Chebyshev's Inequality

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

WEEK 4

→ Cumulative Distribution Function (CDF): $F_X(x) = P(X \le x)$

Properties:

→ $F_X(b) - F_X(a) = P(a < X \le b)$
→ As $x \to -\infty$, F_X goes to 0
→ As $x \to \infty$, F_X goes to 1

↳ if F_X jumps at a point x, then $P(X=x) > 0$.
if F_X is continuous at a point x, then $P(X=x) = 0$.

→ Integral

Indefinite : $F(x) = \int f(x)\,dx$

Definite : $$\int_a^b f(x)\,dx = F(b) - F(a)$$

→ Probability Density Function

↳ $F_X(x_0) = \int_{-\infty}^{x_0} f_X(x)\,dx$

↳ Properties: ① $f(x) \ge 0$ ② $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ③ $f(x)$ is a piecewise function

→ $supp(X) = \{x : f_X(x) > 0\}$ → interval in which X can fall with positive probability

→ Common Distributions:

↳ $X \sim Exp(\lambda)$

PDF $f_X(x) \begin{cases} \lambda \exp(-\lambda x) & x > 0 \\ 0 & \text{otherwise} \end{cases}$

CDF $F_X(x) \begin{cases} 0 & x \le 0 \\ 1 - \exp(-\lambda x) & x > 0 \end{cases}$

↳ $X \sim Normal(\mu, \sigma^2)$

PDF $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$

CDF $F_X(x) = \int_{-\infty}^{x} f_X(v)\,dv$

Standardization: if $X \sim Normal(\mu, \sigma^2)$, then $(X-\mu)/\sigma \sim Normal(0,1)$

↳ $Z \sim Normal(0,1)$ PDF : $f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$

CDF : $F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)\,du$

WEEK 5

→ Function of Continuous Random Variable:

↳ Example: Suppose $X \sim \text{Uniform}[0,1]$, and $Y = 2X \in [0,2]$

For $y \in [0,2]$, $CDF_Y(y) = P(Y \le y) = P(2X \le y) = P(X \le y/2)$

$$P(X \le y/2) = \int_0^{y/2} f_X(x)\,dx = x\Big|_0^{y/2} = \frac{y}{2}$$

So, $CDF_Y(y) = \frac{y}{2}$ PDF of Y → derivative of CDF_Y

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2}$$; So, $Y \sim \text{Uniform}[0,2]$

↳ Always find CDF first

↳ General case: CDF of $g(X)$

→ Suppose $g: \mathbb{R} \to \mathbb{R}$ is a function

$Y = g(X)$

$$F_Y(y) = P(Y \le y) = P(g(x) \le y) = P(X \in \{x : g(x) \le y\})$$

↳ Monotonic, differentiable function → Let X be a continuous rand. var.

↳ Let $g(x)$ be monotonic (increasing or decreasing) for $x \in supp(X)$

PDF of $Y = g(X)$ is

$$f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))$$

example:

Translation ① $Y = X + a$

→ $g(x) = x + a$ $g^{-1}(y) = y - a$ $g'(x) = 1$ $g'(g^{-1}(y)) = 1$

⇒ $f_Y(y) = f_X(g^{-1}(y)) = f_X(y-a)$

scaling ② $Y = aX$

→ $g(x) = ax$ $g^{-1}(y) = y/a$ $g'(x) = a$

⇒ $f_Y(y) = \frac{1}{|a|} f_X(y/a)$

Affine ③ $Y = aX + b$

→ $g(x) = ax + b$ $g^{-1}(y) = (y-b)/a$ $g'(x) = a$

⇒ $f_Y(y) = \frac{1}{|a|} f_X((y-b)/a)$

↳ Normal distribution

↳ affine transformation of a normal dist. is normal.

↳ Let $X \sim \text{Exp}(\lambda)$. Find PDF of X^2 $Y = X^2$

$f_X(x) = \lambda e^{-\lambda x}$ $g(x) = x^2$ $g^{-1}(y) = \sqrt{y}$ $g'(x) = 2x$ $g'(g^{-1}(y)) = 2\sqrt{y}$

$$f_Y(y) = \frac{1}{|2\sqrt{y}|} f_X(\sqrt{y}) = \frac{1}{2\sqrt{y}} \lambda e^{-\lambda\sqrt{y}}$$

→ Expectation of a function of Continuous Random Variable

↳ Let X → cont. rand. var., and $g: \mathbb{R} \to \mathbb{R}$ be a function

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$ → may diverge to $\pm\infty$

→ Expectation of a Continuous Random Variable

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

Variance → $Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$

→ Joint Distribution

↳ (X, Y) are jointly distributed. For each $x \in T_X$, we have continuous Y_x with PDF $f_{Y_x}(y)$

Y_x : Y given $X = x$, $(Y \mid X = x)$

Marginal density of $Y \to f_Y(y) = \sum_{x \in T_X} f_X(x) \cdot f_{Y|X=x}(y)$

↳ Example: Let $X \sim \text{Uniform}\{0,1,2\}$. $Y|X=0 \sim \text{Normal}(5, 0.4)$
$Y|X=1 \sim \text{Normal}(6, 0.5)$ and $Y|X=2 \sim \text{Normal}(7, 0.6)$

Marginal of Y?

$$f_Y(y) = \frac{1}{3} \cdot \frac{1}{\sqrt{2\pi}} \left(\frac{1}{0.4} \cdot \exp\left(\frac{-(y-5)^2}{2 \cdot (0.4)^2}\right) + \frac{1}{0.5} \cdot \exp\left(\frac{-(y-6)^2}{2 \cdot (0.5)^2}\right) + \frac{1}{0.6} \cdot \exp\left(\frac{-(y-7)^2}{2 \cdot (0.6)^2}\right) \right)$$

↳ Conditional probability of discrete given continuous

$$P(X = x \mid Y = y_0) = \frac{f_X(x) \cdot f_{Y|X=x}(y_0)}{f_Y(y_0)}$$

, where $f_Y \to$ marginal density of Y, and $Y = y_0 \in \text{supp}(Y)$

↳ Example: Let $X \sim \text{Uniform}\{-1, 1\}$. $Y|X=-1 \sim \text{Uniform}(-2, 2)$ and $Y|X=1 \sim \text{Exp}(5)$

→ Find $f_{X|Y=-1}$, $f_{X|Y=1}$ and $f_{X|Y=3}$

$$f_Y(y) = \frac{1}{2} f_{Y|X=-1}(y) + \frac{1}{2} f_{Y|X=1}(y) = \begin{cases} 0 & , y < -2 \\ \frac{1}{2} \cdot \frac{1}{4} & , -2 \le y < 0 \\ \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} 5e^{-5y} & , 0 \le y \le 2 \\ \frac{1}{2} 5e^{-5y} & , y > 2 \end{cases}$$

$$P(X=-1 \mid Y=-1) = \frac{f_X(-1) \cdot f_{Y|X=-1}(-1)}{f_Y(-1)} = \frac{(\frac{1}{2}) \cdot (\frac{1}{4})}{(\frac{1}{2}) \cdot (\frac{1}{4})} = 1$$

$$P(X=1 \mid Y=-1) = \frac{f_X(1) \cdot f_{Y|X=1}(-1)}{f_Y(-1)} = 0$$

$$P(X=-1 \mid Y=1) = \frac{f_X(-1) \cdot f_{Y|X=-1}(1)}{f_Y(1)} = \frac{(\frac{1}{2}) \cdot (\frac{1}{4})}{(\frac{1}{2}) \cdot (\frac{1}{4}) + (\frac{1}{2}) 5e^{-5}}$$

WEEK 6

→ Joint density

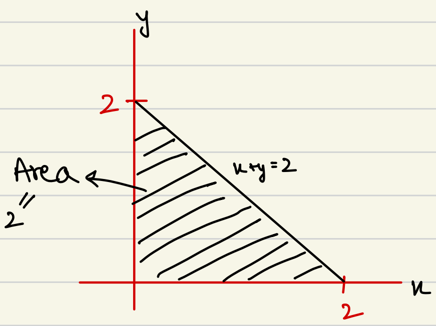
↳ A function $f(x,y)$ is a joint density function if:
 → $f(x,y) \geq 0$
 → $\iint_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

↳ For any two dimensional region A,
$$P((X,Y) \in A) = \iint_A f(x,y)\, dx\, dy$$

↳ $\text{supp}(X,Y) = \{(x,y) : f_{XY}(x,y) > 0\}$

↳ example:

Let $(X,Y) \sim \text{Uniform}(D)$, where $D = \{(x,y) : x+y < 2,\ x>0,\ y>0\}$
Sketch the support and find $P(X+Y<1)$, $P(X+2Y>1)$



Area of support $= 2$. uniform means $P(X,Y) = \frac{1}{2}$

$$f_{XY}(x,y) = \begin{cases} 1/2, & (x,y) \in D \\ 0, & \text{otherwise} \end{cases}$$

$$P(X+Y<1) = \frac{\text{Area of } x+y<1}{\text{Area of } x+y<2} = \frac{1/2}{2} = \frac{1}{4}$$

only works for uniform

$$P(X+2Y>1) = 1 - P(X+2Y<1) = 1 - \frac{1/4}{2} = 1 - \frac{1}{8} = \frac{7}{8}$$

↳ example:

Let (X,Y) have joint density:

$$f_{XY}(x,y) = \begin{cases} x+y, & 0 < x,y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Show it is a valid density. Find $P(X<1/2, Y<1/2)$, and $P(X+Y<1)$

$$\int_{y=0}^{1}\int_{x=0}^{1} (x+y)\, dx\, dy = \int_{y=0}^{1} \tfrac{1}{2} + y\, dy = 1 \quad \checkmark \text{ valid density}$$

$$P(X<1/2, Y<1/2) = \int_{y=0}^{1/2}\int_{x=0}^{1/2} x+y\, dx\, dy = \int_{y=0}^{1/2} (1/8 + y/2)\, dy = y/8 + y^2/4 \Big|_0^{1/2} = \frac{1}{8}$$

$P(X+Y<1) \rightarrow$ when $Y=y$, $x < 1-y$

$$= \int_{y=0}^{1}\int_{x=0}^{1-y} (x+y)\, dx\, dy = \int_{y=0}^{1} \left(\frac{x^2}{2} + yx \Big|_0^{1-y} \right) dy = \int_{y=0}^{1} \frac{(1-y)^2}{2} + y - y^2\, dy = \int_0^1 \frac{1}{2} - \frac{y^2}{2}\, dy = \frac{1}{3}$$

$\frac{1}{2} + \frac{y^2}{2} - y + y - y^2$

→ Marginal Density of jointly distributed continuous random variables

↳ $f_{XY}(x,y)$ is joint continuous density

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\, dy \qquad f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\, dx$$

↳ Joint density determines the marginal density but marginal density can be the same for different joint densities so you cannot determine the joint density from marginal densities alone.

→ Conditional density

↳ $f_{X|Y=b}(u) = \dfrac{f_{XY}(u,b)}{f_X(a)}$

$f_{XY}(x,y) = f_X(u) \cdot f_{Y|X=u}(y) = f_Y(y) \cdot f_{X|Y=y}(u)$

WEEK 7

→ Empirical distribution

↳ Let $X_1, X_2, \ldots, X_n \sim X$ be i.i.d. samples. $\#(X_i = t) \to$ no. of time t occurs.
The empirical distribution is the discrete distribution with PMF: $p(t) = \dfrac{\#(X_i = t)}{n}$

↳ Sample mean denoted $\bar{X}$

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

↳ Expected value and variance of the sample mean
Let $X_1, X_2, X_3, \ldots, X_n \sim X$ be iid with a finite mean μ and variance σ^2. Sample mean $\to \bar{X} = \dfrac{X_1 + X_2 + X_3 + \ldots X_n}{n}$

$$E[\bar{X}] = \mu \qquad Var(\bar{X}) = \frac{\sigma^2}{n}$$

↳ Sample variance denoted S^2

$$S^2 = \frac{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \ldots + (X_n - \bar{X})^2}{(n-1)}$$

$$E[S^2] = \sigma^2$$

↳ Sample proportion
Let A be an event and $P(A)$ be the probability of A.

$$S(A) = \frac{\#(X_i \text{ for which } A \text{ is true})}{n} \qquad E[S(A)] = P(A) \qquad Var(S(A)) = \frac{P(A)(1-P(A))}{n}$$

↳ Sum of random variables
Let $X_1, X_2, \ldots, X_n$ be random variables. Let $S = X_1 + X_2 + \ldots + X_n$ be their sum.

$$E[S] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

→ If $X_1, X_2, \ldots, X_n$ are pairwise uncorrelated, then $E[X_i X_j] = E[X_i] \cdot E[X_j]$ for all $i, j, i \neq j$

$$Var(S) = Var(X_1) + Var(X_2) + \ldots + Var(X_n)$$

↳ Weak law of large numbers
Let $X_1, X_2, \ldots, X_n \sim$ iid X, where $E[X] = \mu$ and $Var(X) = \sigma^2$
Let $\bar{X} = (X_1 + X_2 + \ldots + X_n)/n$, then $E[\bar{X}] = \mu$ and $Var(\bar{X}) = \sigma^2/n$

$$P(|\bar{X} - \mu| > \delta) \leq \frac{\sigma^2}{n\delta^2}$$ → similar to Markov's inequality

WEEK 8

Moment Generating Function

- Let X be a zero-mean random variable. MGF of X, denoted $M_X(\lambda)$ is $f: \mathbb{R} \to \mathbb{R}$

$$M_X(\lambda) = E[e^{\lambda X}]$$

  - $X \to$ discrete $\Rightarrow$ $M_X(\lambda) = f_X(x_1)e^{\lambda x_1} + f_X(x_2)e^{\lambda x_2} + \ldots + f_X(x_n)e^{\lambda x_n}$, where $supp(X) = \{x_1, x_2, \ldots, x_n\}$

    $X \to$ continuous $\Rightarrow$ $M_X(\lambda) = \int_{x \in T_X} f_X(x) e^{\lambda x} dx$

    $X \sim Normal(0, \sigma^2)$

    - $M_X(\lambda) = e^{\lambda^2 \sigma^2 / 2}$

- $E[e^{\lambda X}] = E[1 + \lambda X + \lambda^2 X^2/2! + \lambda^3 X^3/3! + \ldots]$

$$= 1 + \lambda \underbrace{E[X]}_{1^{st}\text{ moment}} + \frac{\lambda^2}{2!}\underbrace{E[X^2]}_{2^{nd}\text{ moment}} + \frac{\lambda^3}{3!}\underbrace{E[X^3]}_{3^{rd}\text{ moment}} + \ldots$$

- Sum of i.i.d. random variables

  Let $Y = X_1 + X_2$, where $X_1, X_2 \sim$ iid X

  example: $X \sim$ iid centralised Bernoulli (p)

  → $M_X(\lambda) = (1-p)e^{-p\lambda} + pe^{(1-p)\lambda}$

  → $M_Y(\lambda) = (M_X(\lambda))^2 = (1-p)^2 e^{-2p\lambda} + 2p(1-p)e^{(1-2p)\lambda} + p^2 e^{2(1-p)\lambda}$

Y	$-2p$	$1-2p$	$2(1-p)$
$f_Y(y)$	$(1-p)^2$	$2p(1-p)$	p^2

Central Limit Theorem

- Let $X_1, X_2, \ldots, X_n \sim$ iid X with $E[X] = 0$ and $Var(X) = \sigma^2$

  Let $Y = \frac{X_1 + X_2 + \ldots X_n}{\sqrt{n}}$, Then,

$$M_Y(\lambda) \to e^{\lambda^2 \sigma^2 / 2}$$

  - Y is said to converge to $Normal(0, \sigma^2)$

- Application:

  $X_1, X_2, \ldots X_n \sim$ iid X Let $\mu = E[X]$ and $Var(X) = \sigma^2$

  Let $Y = X_1 + X_2 + \ldots + X_n$ What is $P(Y - \mu n > \delta \mu n)$?

  → $E[Y] = \mu n$

  $(Y - \mu n)/\sqrt{n} \approx Normal(0, \sigma^2)$

  $(Y - \mu n)/(\sqrt{n}\sigma) \approx Normal(0, 1)$

$$P(Y - \mu n > \delta \mu n) = 1 - F\left(\frac{\delta \mu \sqrt{n}}{\sigma}\right)$$

Linear Combination of independent Normals

- Let $X_1, X_2, \ldots, X_n \sim Normal$. Let $X_i \sim Normal(\mu_i, \sigma_i^2)$

  Suppose $Y = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$, then

  $Y \sim Normal(\mu, \sigma^2)$, where $\mu = a_1\mu_1 + a_2\mu_2 + \ldots + a_n\mu_n$

  $\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots + a_n^2\sigma_n^2$

  Linear combinations of normals is normally distributed

Gamma distribution

- $X \sim Gamma(\alpha, \beta)$. PDF $f_X(x) \propto (x^{\alpha - 1} e^{-\beta x})$, $x > 0$

  → Sum of n iid $Exp(\beta) = Gamma(\alpha, \beta)$

  → Square of $Normal(0, \sigma^2) = Gamma(1/2, 1/2\sigma^2)$

- $\alpha > 0$: shape parameter

  $\beta > 0$: rate parameter

  $\theta = 1/\beta$: scale parameter

→ Cauchy distribution

↳ $X \sim \text{Cauchy}(\theta, \alpha^2)$, where θ : location parameter
$\alpha > 0$: scale parameter

Suppose $X, Y \sim$ iid Normal$(0, \sigma^2)$. Then,

$$\frac{X}{Y} \sim \text{Cauchy}(0, 1)$$

→ Beta distribution

↳ $X \sim \text{Beta}(\alpha, \beta)$ if PDF $f_X(x) \propto (x)^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$

↳ has finite support $\alpha > 0, \beta > 0$: shape parameters

→ Sum of n independent Gamma(α, β) is Gamma$(n\alpha, \beta)$

Sum of squared Normal$(0,1)$ iid. $\sim$ Gamma$(n/2, 1/2)$

↳ a.k.a. chi-square distribution with n degrees of freedom

→ Sample mean and variance of normal samples

↳ Suppose $X_1, X_2, \dots, X_n \sim$ Normal(μ, σ^2). Then,

① $\bar{X} \sim$ Normal$(\mu, \sigma^2/n)$

② $\frac{(n-1)(S^2)}{\sigma^2} \sim \chi^2_{n-1}$, Chi-square with n-1 deg. freedom

③ $\bar{X}$ and S^2 are indepent.

WEEK 9

→ Estimation of a parameter θ

↳ function of the samples : $\hat{\theta}(X_1, X_2, \dots, X_n)$

↳ θ → real constant number

$\hat{\theta}$ → function of n random variables; will have a PDF/PMF

→ Estimation Error :

↳ Error : $\hat{\theta}(X_1, X_2, \dots, X_n) - \theta$ → is a random variable

→ If $P(|\text{Error}| > \delta)$ is small , it's a good estimator

↳ Chebyshev bound : $P(|\text{Error} - E[\text{Error}]| > \delta) \leq \frac{\text{Var}(\text{Error})}{\delta^2}$

↳ Good estimator : $P(|\text{Error}| > \delta)$ will fall with n

→ Second moment : $E[X^2]$

Variance$(X) = E[X^2] - \mu^2$ $E[X^2] = \text{Var}(X) + \mu^2$

→ Bias , Risk , Variance

↳ $X_1, X_2, \dots, X_n \sim$ i.i.d. X , parameter θ

$\hat{\theta}$ → estimator for θ

$\text{Bias}(\hat{\theta}, \theta) = E[\hat{\theta} - \theta] = E[\hat{\theta}] - \theta$

$\text{Risk}(\hat{\theta}, \theta) = E[(\hat{\theta} - \theta)^2]$ ← 2^{nd} moment of Error ; Mean-squared error

$\text{Var}(\hat{\theta}) = E[(\hat{\theta} - E[\hat{\theta}])^2]$

$$\text{Risk}(\hat{\theta}, \theta) = \text{Var}(\hat{\theta}) + \text{Bias}(\hat{\theta}, \theta)^2$$

↳ example:

$X_1, X_2, \dots, X_n \sim \text{Bernoulli}(p)$. Consider estimator $\hat{p} = \dfrac{X_1 + X_2 + \dots X_n + \sqrt{n}/2}{n + \sqrt{n}}$

$$\text{Bias}(\hat{p}, p) = E[\hat{p}] - p = \frac{n \cdot p + \sqrt{n}/2}{n + \sqrt{n}} - p = \frac{p + \frac{1}{2\sqrt{n}}}{1 + \frac{1}{\sqrt{n}}} - p = \frac{\sqrt{n}\,(\frac{1}{2} - p)}{n + \sqrt{n}}$$

$$\text{Var}(\hat{p}) = \left(\frac{1}{n + \sqrt{n}}\right)^2 (np(1-p))$$

$$\text{Risk} = \frac{(\sqrt{n}(\frac{1}{2}-p))^2}{(n+\sqrt{n})^2} + \frac{np(1-p)}{(n+\sqrt{n})^2} = \frac{n}{4(n+\sqrt{n})^2}$$

→ Method of moments

↳ $X \sim f_X(x)$, parameters $\theta_1, \theta_2, \dots$

Moments $E[X], E[X^2]$, etc. can be expressed as functions of parameters

Bernoulli(p): $E[X] = p$

Exponential(λ): $E[X] = 1/\lambda$

Normal(μ, σ^2): $E[X] = \mu$, $E[X^2] = \mu^2 + \sigma^2$

↳ Sample moments

$X_1, X_2, \dots, X_n \sim$ iid X

$$M_k(X_1, X_2, \dots, X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$ ← random variable

We expect M_k to revolve $E[X^k]$

sample moments: m_1, m_2

Distribution ": $E[X] = f(\theta_1, \theta_2)$, $E[X^2] = g(\theta_1, \theta_2)$

Solve for θ_1, θ_2 from $f(\theta_1, \theta_2) = m_1$, $g(\theta_1, \theta_2) = m_2$ in terms of m_1, m_2

$\hat{\theta}_1, \hat{\theta}_2$: replace m_1 by M_1 and m_2 by M_2

example:

→ $X_1, X_2, \dots, X_n \sim$ iid Exponential(λ)

→ $E[X] = 1/\lambda$, $m_1 = 1/\lambda \Rightarrow \lambda = 1/m_1$

$$\hat{\lambda} = \frac{1}{M_1} = \frac{n}{\sum_{i=1}^{n} X_i}$$

→ $X_1, X_2, \dots, X_n \sim$ iid Normal(μ, σ^2)

→ $E[X] = \mu$, $E[X^2] = \sigma^2 + \mu^2$

$m_1 = \mu$, $m_2 = \sigma^2 + \mu^2 \Rightarrow \sigma^2 = m_2 - (m_1)^2$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$$

→ $X_1, X_2, \dots, X_n \sim$ iid Bin(N, p)

→ $E[X] = Np$, $E[X^2] = Np(1-p) + N^2p^2$

$m_1 = Np \Rightarrow N = m_1/p$

$m_2 = Np(1-p) + N^2p^2 \Rightarrow m_2 = \dfrac{m_1}{p} p(1-p) + \dfrac{m_1^2}{p^2} p^2$

$$m_2 = m_1(1-p) + m_1^2 \Rightarrow \frac{m_2 - m_1^2}{m_1} = 1 - p$$

$$p = 1 - \frac{m_2 - m_1^2}{m_1} = \frac{m_1^2 + m_1 - m_2}{m_1}$$

$$N = \frac{m_1^2}{m_1^2 + m_1 - m_2}$$

→ A die. $P(x \in [1,5]) = \theta$. Sample of 10 rolls: 1, 6, 5, 5, 4, 6, 6, 2, 3, 4. Estimate θ.

$E[X] = 15\theta + 6(1 - 5\theta) = 15\theta + 6 - 30\theta = 6 - 15\theta$

$\theta = \dfrac{6 - m_1}{15}$ $\qquad \hat{\theta} = \dfrac{6 - M_1}{15}$

M_1 of samples $= 4.2$

$\hat{\theta} = \dfrac{6 - 4.2}{15} = 0.12$

→ Maximum Likelihood → easy and very useful

↳ $X_1, X_2, \dots, X_n \sim \text{iid } X$, prameters: $\theta_1, \theta_2, \dots$

Likelihood of a sampling $x_1, x_2, \dots, x_n$ denoted $L(x_1, x_2, \dots, x_n)$

$$L(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f_X(x_i : \theta_1, \theta_2, \dots)$$

product of PDF evaluated at each sample

function $\theta_1, \theta_2, \dots$

example:

→ Bernoulli(p): 1, 0, 0, 1, 0, 1, 1, 1, 0, 1

$$L = (p)^6 (1-p)^4$$

↳ Maximum Likelihood estimator

↳ which set of parameters gave you max likelihood for that sample ?

$$\theta_1, \theta_2, \dots = \arg\max_{\theta_1, \theta_2} \prod_{i=1}^{n} f_X(x_i : \theta_1, \theta_2, \dots)$$

gives the arguments (the inputs) of the func. at max.

Maximising a function : Maximising L ⟷ Maximising $\log L$

example

→ $X_1, X_2, \dots X_n \sim \text{iid Bernoulli}(p)$

$$L(x_1, x_2, \dots, x_n) = p^{w}(1-p)^{n-w}$$, where $w = \#$ of $x_i = 1$

$$\log L = w \cdot \log(p) + (n-w)\log(1-p)$$

$$\frac{d \log L}{dp} = \frac{w}{p} + \frac{(-1)(n-w)}{1-p}$$

$$0 = \frac{w}{p} - \frac{(n-w)}{1-p} \Rightarrow 0 = \frac{w(1-p) - (n-w)p}{p(1-p)} \Rightarrow p(n-w) = w(1-p)$$

$$np - \cancel{wp} = w - \cancel{wp} \quad = \boxed{p = \frac{w}{n}}$$

→ $X_1, X_2, \dots, X_n \sim \text{iid Exp}(\lambda)$

$$L(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n}(\lambda e^{-\lambda x_i}) = \lambda^n \exp(-(x_1 + x_2 + \dots + x_n)\lambda)$$

$$\text{MLE}(\lambda) \Rightarrow \log L = n\log(\lambda) - \lambda\left(\sum_{i=1}^{n} x_i\right)$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i = \frac{n}{\lambda} \Rightarrow \boxed{\lambda = \frac{n}{\sum_{i=1}^{n} x_i}}$$

→ $X_1, X_2, \dots, X_n \sim \text{iid Normal}(\mu, \sigma^2)$

$$L(x_1, x_2, \dots, x_n) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^{n}(x_i - \mu)^2\right)$$

$$\log(L) = -n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i^2 + \mu^2 - 2x_i\mu)$$

$$\frac{\partial \log(L)}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$$

$$0 = -n + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$$

$$n\sigma^2 = \sum_{i=1}^{n}(x_i - \hat{\mu})^2 \Rightarrow \boxed{\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})^2}{n}}$$

$$\frac{\partial \log(L)}{\partial \mu} = \text{derivative of } \sum_{i=1}^{n}(x_i^2 + \mu^2 - 2x_i\mu)$$

$$0 = \sum_{i=1}^{n}(2\mu - 2x_i) \Rightarrow 0 = n\mu - \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i = n\mu$$

$$\boxed{\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

WEEK 10

→ Schools of thoughts for design of parameter estimation:

① Frequentist : treat θ as an unknown constant

↳ Method of moments , Maximum likelihood

② Bayesian : θ is random variable with a known distribution

↳ Bayesian estimation

→ Bayesian estimation

↳ example

→ $X_1, X_2, \dots X_n \sim$ iid Bernoulli(p) . Suppose $p \sim$ Uniform$\{0.25, 0.75\}$

Samples : $S = 1,0,1,1,0$

$$P(p=0.25|S) = \frac{P(S|p=0.25)\,P(p=0.25)}{P(S)} = \frac{((0.25)^3 \times (0.75)^2) \times (0.5)}{(0.5)((0.25^3 \times 0.75^2)+(0.25^2 \times 0.75^3))} = 0.25$$

$$P(p=0.75|S) = 0.75$$

Posterior mode Estimator 1 : $p = \text{mode}\{P(p=0.25|S), P(p=0.75|S)\}$

Posterior mean Estimator 2 : Posterior mean :

$$(0.25)(P(p=0.25|S)) + (0.75)(P(p=0.75|S)) = 0.625$$

↳ Posterior density $\propto$ Maximum likelihood × prior density

↳ example with continuous prior

$X_1, \dots, X_n \sim$ Bernoulli(p) Prior → $p \sim$ Uniform$[0,1]$

Posterior : $P(X_1 = x_1, \dots, X_n = x_n | p = p) \cdot f_p(p)$ ← always 1

$= (p)^w (1-p)^{n-w}$, $0 \le p \le 1$, where $w = \#$ of 1 in sample

Posterior $\sim$ Beta$(\alpha = w+1, \beta = n-w+1)$

posterior mean $= \frac{\alpha}{\alpha+\beta} = \frac{w+1}{n+2} \Rightarrow \hat{p} = \frac{(\sum x_i)+1}{n+2}$

→ $X_1, \dots, X_n \sim$ Bernoulli(p) Prior : $p \sim$ Beta(α, β)

Posterior : $(p)^{w+\alpha-1}(1-p)^{n-w+\beta-1}$, $0 \le p \le 1$, where $w = \#$ of 1 in sample

Posterior $\sim$ Beta$(w+\alpha, n-w+\beta)$

posterior mean $= \frac{w+\alpha}{n+\alpha+\beta} = \frac{(\sum x_i)+\alpha}{n+\alpha+\beta}$

→ Examples:

↳ Consider n i.i.d. samples $\sim$ Geometric(p)

MME : $E[X] = \frac{1}{p}$ $\quad \hat{p} = \frac{1}{M_1} = \frac{n}{\sum_{i=1}^{n} x_i}$

ML : $L = (p)^n (1-p)^{\sum x_i - n}$ $\Rightarrow \log(L) = n\log(p) - n\log(1-p) + \sum_i^n x_i \log(1-p)$

$$\frac{\partial \log(L)}{\partial p} = \frac{n}{p} + \frac{n}{1-p} - \frac{\sum x_i}{1-p} \Rightarrow 0 = \frac{n}{p} - \frac{\sum x_i - n}{1-p} \Rightarrow p(\sum x_i - n) = n - np$$

$$p = \frac{n}{\sum x_i}$$

Bayes:

post. distr : $(p)^n (1-p)^{x_1+\dots x_n - n} \sim$ Beta$(n+1, x_1+x_2+\dots+x_n-n+1)$

post mean : $\frac{n+1}{x_1+x_2+\dots+x_n+2}$

$(p)^{w}(1-p)^{n-w}\,(p)^{\alpha-1}(1-p)^{\beta-1}$

$(p)^{w+\alpha-1}(1-p)^{n-w+\beta-1} \sim \text{Beta}(w+\alpha,\ n-w+\beta)$

$\hat{p} = \dfrac{w+\alpha}{n+\alpha+\beta}$ if $\alpha=\beta=1$, $\hat{p} = 1/12$

$\dfrac{\alpha}{\alpha+\beta} = \dfrac{1}{10} \Rightarrow 10\alpha = \alpha+\beta$

$9\alpha = \beta$

from Activity 4

① $P(\text{picking 15 marked turtles}) = \dfrac{\binom{30}{15}\binom{N-30}{35}}{\binom{N}{50}}$

$E[X] = \dfrac{50\times 30}{N}$ $\qquad \hat{N} = \dfrac{50\times 30}{15} = \boxed{100}$

② $n = 68$ $\quad$ Sample $p = \dfrac{21}{68}$ $\quad X \sim \text{Bernoulli}(p)$

Posterior density: $(p)^{w}(1-p)^{n-w}(p)^{0}(1-p)^{0}$, where $w = \#$ of people who support construction

$= (p)^{w}(1-p)^{n-w} \sim \text{Beta}(w+1,\ n-w+1)$

Posterior mean: $\dfrac{\alpha}{\alpha+\beta} = \dfrac{w+1}{n+2}$, where $w = 21$, $n = 68$ $\qquad \hat{p} = \boxed{0.33}$

③ $\dfrac{\alpha}{\alpha+\beta} = 0.6 \Rightarrow \alpha = 0.6\alpha + 0.6\beta \Rightarrow \dfrac{2}{3}\alpha = \beta$ — ①

$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = (0.15)^2 \Rightarrow \dfrac{2/3\,\alpha^2}{\left(\frac{5}{3}\alpha\right)^2\left(\frac{5\alpha+3}{3}\right)} = (0.15)^2 \Rightarrow$

$\dfrac{25\alpha^2(5\alpha+3)}{27} = \dfrac{125\alpha^3+75\alpha^2}{27}$

$\dfrac{(2/3)\,\cancel{\alpha^2}}{\cancel{\alpha^2}\left(\frac{125\alpha+75}{27}\right)} = (0.15)^2$

$27 \times \dfrac{2}{3} \times \dfrac{1}{(0.15)^2} = 125\alpha + 75$ $\qquad \alpha = 5.8,\ \beta = 3.87$

Prior $\sim \text{Beta}(5.8, 3.87)$. Posterior $\propto (p)^{w+4.8}(1-p)^{n-w+2.87}$

Posterior density $\sim \text{Beta}(w+5.8,\ n-w+3.87)$

$\hat{p} = \dfrac{w+5.8}{n+9.67} = \boxed{0.345}$

④ $X \sim \text{Bernoulli}(p)$

Sample $= \{1,1,1,0,0,0,0,0,0,0,0\}$

Posterior density $\propto (p)^{w}(1-p)^{n-w}(p)^{-0.5}(1-p)^{-0.5}$

$= (p)^{w-0.5}(1-p)^{n-w-0.5} \sim \text{Beta}(w+0.5,\ n-w+0.5)$

$\hat{p} = \dfrac{w+0.5}{n+1} = \dfrac{3.5}{11} = \boxed{0.318}$

WEEK 11

→ Metrics for Hypothesis Testing

$$X_1, X_2, \ldots, X_n \sim \text{iid } X$$

↳ H_0 : null ; H_A : alternative

Test : defined by set A

↳ if a sample $\in A$, then accept H_0 ; otherwise reject H_0

↳ ① Significance level denoted α

→ Type I error : Reject H_0 when H_0 is true

$\alpha = P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ is true})$

② Power of test $1-\beta$

→ Type II error : Accept H_0 when H_A is true

$\beta = P(\text{Type II error}) = P(\text{Accept } H_0 \mid H_A \text{ is true})$

$1-\beta = P(\text{Reject } H_0 \mid H_A \text{ is true})$

→ example:

→ Consider 100 samples $X_1, X_2, \ldots, X_{100} \sim \text{iid Normal}(\mu, 25)$. $H_0 : \mu = 0$; $H_A : \mu > 0$ at significance $\alpha = 0.05$.

$T = \frac{X_1 + X_2 + \ldots + X_{100}}{100}$ reject H_0 if $T > c$. Find c.

$E[T] = \mu$ $\quad SD(T) = 1/2$ $\quad$ z-score$(c) = \frac{c-\mu}{1/2}$

assuming $H_0 : \mu = 0$,

$\alpha = 1 - F_Z(2c) \Rightarrow 0.05 = 1 - F_Z(2c)$

$F_Z(2c) = 0.95 \Rightarrow c = (1/2) F_Z^{-1}(0.95)$

→ P-value

→ Suppose $T = t$ in one sampling. The lowest significance level α at which the null will be rejected for $T = t$ is said to be the P-value of the sampling.

→ Calculation : put t value as critical value and compute α.

→ example :

→ Check whether a coin is fair or biased. $H_0 : p = 0.5$; $H_A : p > 0.5$.
You toss the coin 100 times and observe 55 heads. What is the p-value?

$E[X] = 50$; $SD(X) = 10\sqrt{0.25}$

$$\text{p-value} = 1 - F_Z\left(\frac{55-50}{10\sqrt{0.25}}\right) = 1 - F_Z(1) = 0.16$$

→ $H_0 : \mu = 65$; $H_A : \mu > 65$. $\alpha = 0.05$. $\sigma = 6$. Find n . $1-\beta = 0.95$ and $H_A : \mu = 67$

$$\alpha = P(\mu > c \mid \mu = 65) \Rightarrow 0.05 = 1 - F_Z\left(\frac{c-65}{6/\sqrt{n}}\right) \Rightarrow c = \frac{6}{\sqrt{n}} \cdot F_Z^{-1}(0.95) + 65 \quad \text{— ①}$$

$$1-\beta = P(\mu > c \mid \mu = 67) \Rightarrow 0.95 = 1 - F_Z\left(\frac{c-67}{6/\sqrt{n}}\right) \Rightarrow c = \frac{6}{\sqrt{n}} \cdot F_Z^{-1}(0.05) + 67 \quad \text{— ②}$$

$$F_Z^{-1}(0.95) = -F_Z^{-1}(0.05)$$

$$\frac{6}{\sqrt{n}} \cdot F_Z^{-1}(0.05) + 67 = -\frac{6}{\sqrt{n}} \cdot F_Z^{-1}(0.05) + 65$$

$$2 = -\frac{12}{\sqrt{n}} \cdot F_Z^{-1}(0.05)$$

$$n = (-6 \cdot F_Z^{-1}(0.05))^2 \approx 98$$

WEEK 12

→ Two-sample Z-test

↳ $X_1, X_2, \ldots, X_n \sim \text{iid Normal}(\mu_1, \sigma_1^2)$
$Y_1, Y_2, \ldots, Y_n \sim \text{iid Normal}(\mu_2, \sigma_2^2)$

$H_0: \mu_1 = \mu_2$; $H_A: \mu_1 \neq \mu_2$
$T = \bar{X} - \bar{Y}$; Reject H_0 if $|T| > c$

Given H_0, $T \sim \text{Normal}(0, \sigma_T^2)$, where $\sigma_T^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

$\alpha = P(|T| > c \mid H_0) = 2F_Z\left(\frac{-c}{\sigma_T}\right)$

→ Two-sample F-test

↳ $X_1, X_2, \ldots, X_n \sim \text{iid Normal}(\mu_1, \sigma_1^2)$
$Y_1, Y_2, \ldots, Y_n \sim \text{iid Normal}(\mu_2, \sigma_2^2)$

$H_0: \sigma_1^2 = \sigma_2^2$; $H_A: \sigma_1^2 \neq \sigma_2^2$
$T = \frac{S_X^2}{S_Y^2}$; Reject H_0 if $T > 1 + c_R$ or $T < 1 - c_L$

Given H_0, $T \sim F(n_1 - 1, n_2 - 1)$

$\alpha/2 = P(T > 1 + c_R \mid H_0) = P(T < 1 - c_L \mid H_0)$

WEEK 1 GA

① $T_x = \{0,1,2,3\}$ $T_y = \{-1,1,2,3\}$ $X \sim Bin(3, 1/2)$ $P(X<3) = 1 - P(X=3) = 7/8$ $P(X=3) = \binom{3}{3}\cdot(1/2)^3 = 1/8$

$P(Y\leq 1) = P(Y=1) + P(Y=-1)$ $P(Y=1) = 1/2$ $P(Y=-1) = 1/8$ $P(Y\leq 1) = 1/2 + 1/8 = 5/8$

$f_{XY}(t_x<3, t_y \leq 1) = \frac{5}{8} \times \frac{7}{8} = \frac{35}{64} \approx \boxed{0.5469}$

② $T_x = T_y = T_z = \{0,1,2\}$

X	Y	Z
0	0	2
2	0	0
1	0	1

$= \boxed{1/3}$

$f_{XY}(2,0) = \sum_{t_z=0}^{2} f_{XYZ}(2,0,t_z) = P(2,0,0) = 1/9$

$f_Y(0) = \sum_{t_x \in T_x, t_z \in T_z} f_{XYZ}(t_x, 0, t_z) = P(0,0,2) + P(1,0,1) + P(2,0,0) = 3/9$

$f_{X|Y=0}(2) = \frac{f_{XY}(2,0)}{f_Y(0)}$

$f_{X|Y=0}(2) = \frac{1/9}{3/9} = \boxed{1/3}$

③ $1/4 + 1/4 + 1/8 + 1/8 + k = 1$ $k = 1 - (2/4 + 1/4) = 1 - 3/4 = 1/4$

$f_{Y|X=1}(2) = \frac{f_{YX}(2,1)}{f_X(1)} = \frac{1/4}{1/8 + 1/8 + 1/4} = \frac{1/4}{1/2} = \frac{1}{4} \times 2 = \boxed{\frac{1}{2}}$

⑤ $f_Y(1) = \sum_{t' \in T_x} f_{XY}(t', 1)$ $f_X(1) = 12/13$ $f_X(0) = 1/13$

$Y \sim Bin(n,p)$

$f_{Y|X=1}(1) =$ $n=3$, $p = 7/20$

$= \binom{3}{1}\left(\frac{7}{20}\right)\left(\frac{13}{20}\right)^2$

$= \frac{3 \times 7 \times (13)^2}{(20)^3}$

$= 0.443625$

$f_{Y|X=0}(1) =$ $n=2$, $p = 4/15$

$= \binom{2}{1}\left(\frac{4}{15}\right)\left(\frac{11}{15}\right)$

$= \frac{2 \times 4 \times 11}{(15)^2}$

$= 0.39111$

$f_{XY}(0,1) = f_{Y|X=0}(1) \cdot f_X(0)$
$= 0.03008547$

$f_{XY}(1,1) = f_{Y|X=1}(1) \cdot f_X(1)$
$= 0.4095$

$f_Y(1) = f_{XY}(0,1) + f_{XY}(1,1) \approx \boxed{0.4396}$

⑦ $f_{XY}(x,y) = a(bx+y)$

x \ y	0	1	2	3
0	0	a	2a	3a
1	ab	ab+a	ab+2a	ab+3a
2	2ab	2ab+a	2ab+2a	2ab+3a
	3ab	3ab+2a	3ab+4a	

$3\left(12a + 3ab = \frac{3}{7}\right) = 36a + 9ab = \frac{9}{7}$

$2\left(6a + 9ab = \frac{4}{7}\right) = 12a + 18ab = \frac{8}{7}$

$-12a - 3ab = -\frac{3}{7}$

$15ab = \frac{5}{7}$

$36a + 9ab = \frac{9}{7}$

$-6a - 9ab = -\frac{4}{7}$

$30a = \frac{5}{7}$

$\boxed{a = \frac{1}{42}}$

$ab = \frac{1}{21}$

$\frac{b}{42} = \frac{1}{21} \Rightarrow \boxed{b=2}$

$f_{XY}(1,1) = \frac{1}{42}(2+1) = \boxed{\frac{1}{14}}$

⑧ $f_{Y|X=2}(0) = \frac{f_{YX}(0,2)}{f_X(2)}$

$f_X(2) = \binom{6}{2}(1/2)^6$

$f_{X|Y=0}(2) = \binom{5}{2}(1/2)^5$; $f_Y(0) = 1/2$; $f_{YX}(0,2) = \binom{5}{2}(1/2)^6$

$f_{Y|X=2}(0) = \frac{\binom{5}{2}\cdot(1/2)^6}{\binom{6}{2}\cdot(1/2)^6} = \frac{\binom{5}{2}}{\binom{6}{2}}$

$= \frac{5!}{2!\,3!} \times \frac{2!\,4!}{6!} = \frac{5!}{6!} \times \frac{4!}{3!} = \frac{4}{6} = \boxed{0.666}$

(9) $\dfrac{\binom{5}{1}\times\binom{4}{1}}{\binom{12}{2}} = \dfrac{5\times 4}{\frac{12!}{10!\,2!}} = \dfrac{5\times\cancel{4}}{\cancel{12}_{3}\times 11}\times 2 = \dfrac{10}{33} = \boxed{0.30}$

(10) $N \sim Bin(7, 1/2)$ $\quad X \sim Bin(n, 1/2)$

$x \backslash N$	0	1	2	3	4
0	0	$^{7}C_1(1/2)^8$			$\underline{\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11}}$
1	0	$^{7}C_1(1/2)^8$		$\underline{\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10}}$	
2	0	0	$\underline{\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9}}$		
3	0	0	0		
4	0	0	0	0	

$\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11} = \dfrac{7!}{4!\,3!}\times\dfrac{1}{2048} = \dfrac{7\times6\times5}{3\times2}\times\dfrac{1}{2048} = \dfrac{35}{2048}$

$\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10} = \dfrac{7!}{3!\,4!}\times\dfrac{3}{1024} = \dfrac{210}{2048}$

$\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9} = \dfrac{7!}{2!\,5!}\times\dfrac{1}{512} = \dfrac{7\times6}{2}\times\dfrac{1}{512} = \dfrac{84}{2048}$

$\text{Total} = \dfrac{35+210+84}{2048} = \dfrac{329}{2048} = \boxed{0.161}$

WEEK 2 GA

(2)

	0	1	2	
0	0.06	0.18	0.12	0.36
1	0.04	0.12	0.48	$\underline{0.64}$
	0.1	0.3	0.6	

	0	1	2	
0	2/24	1/24	1/24	4/24
1	4/24	2/24	3/24	9/24
2	6/24	3/24	2/24	11/24
	12/24	6/24	6/24	

	0	1	2	
0	1/10	2/10	2/10	5/10
1	1/10	1/10	3/10	5/10
	2/10	3/10	5/10	

	0	1	
0	1/10	1.5/10	2.5/10
1	2/10	3/10	5/10
2	1/10	1.5/10	2.5/10
	4/10	6/10	

(3) $X \sim Bernoulli(0.2)$ $\quad Y \sim Bernoulli(0.4)$ $\quad Z = X+Y$

$f_{X|Z=1}(1) = \dfrac{P(X=1, Z=1)}{P(Z=1)} = \dfrac{P(X=1, Y=0)}{P(X=1, Y=0 \text{ or } X=0, Y=1)} = \dfrac{(0.2)(0.6)}{(0.2)(0.6)+(0.8)(0.4)} = \dfrac{0.12}{0.12+0.32} = \boxed{0.2727}$

(4) $Z = X+Y$ $\qquad f_{XY}(k, y) = \dfrac{9}{16\cdot(4)^{k+y}}$

$f_Z(k) = P(X=k, Y=k-k) = \displaystyle\sum_{k=0}^{k} f_{XY}(k, k-k) = \boxed{\dfrac{(k+1)\cdot 9}{16\cdot(4)^k}}$

(5) $Z = \max(X, Y)$

$f_Z(k) = P(X=k, Y\le k \text{ or } X\le k, Y=k) = \displaystyle\sum_{y=0}^{k} f_{XY}(k, y) + \sum_{k=0}^{k} f_{XY}(k, k)$

$= 2\cdot\displaystyle\sum_{k=0}^{k}\dfrac{9}{16\cdot(4)^{k+k}} =$

(6)

$y \backslash k$	1	2	3	4	5	6
1				*		
2					*	
3						*
4						
5						
6						

$f_X(4)\cdot f_Y(1) + f_X(5)\cdot f_Y(2) + f_X(6)\cdot f_Y(3)$

$= \left(\dfrac{1}{36}\right)\times 3 = \dfrac{3}{36} = \dfrac{1}{12}$

(7) $X \sim \text{Geometric}(p)$ $Y \sim \text{Geometric}(p)$ iid $Z = X + Y$

$f_Y(k) = p(1-p)^{k-1}$

$f_Z(k) = P(X=u, Y=k-u) = \sum_{u=1}^{k-1} p(1-p)^{u} \cdot p(1-p)^{k-u}$

$= p^2(1-p)^k (k-1)$

$f_Z(18) = p^2(1-p)^{18}(17)$ $f_Z(19) = p^2(1-p)^{19}(18)$

$f_Z(18) < f_Z(19) \Rightarrow \cancel{p^2}\cancel{(1-p)^{18}}(17) < \cancel{p^2}(1-p)^{19}(18)$

$\frac{17}{18} < 1-p$

$\boxed{p < 0.056}$

(8) $X \sim \text{Poisson}(2)$ $f_X(1) = e^{-2} 2$ $f_X(2) = e^{-2} 2$ $f_X(\neq 1 \text{ or } 2) = 1 - 4e^{-2}$

X_1	X_2	X_3
1	2	
1		2
2	1	
2		1
	2	1
	1	2

$6 \cdot (2e^{-2})^2 (1-4e^{-2}) = 0.439575(1-4e^{-2}) = \boxed{0.2016}$

(9) $X \sim \text{Bernoulli}(0.8)$ $Y \sim \text{Bernoulli}(0.3)$ $Z = X + Y - XY$

$f_Z(1) = P(X=1, Y=0 \text{ or } X=0, Y=1 \text{ or } X=1, Y=1) = 1 - P(X=0, Y=0) = 1-(0.2)(0.7) = 0.86$

$Z \sim \text{Bernoulli}(0.86)$

(10) $X, Y \sim \text{Geometric}(0.8)$ $P(X=1 \mid X+Y=2) = \frac{P(X=1, X+Y=2)}{P(X+Y=2)} = \frac{P(X=1, Y=1)}{P(X=1, Y=1)}$

(11) $X \sim \text{Poisson}(5)$ $Y \sim \text{Poisson}(1)$ $Z = X + Y$

$Z \sim \text{Poisson}(6)$

$Y|Z=4 \sim \text{Binomial}(4, 1/6)$

$Y|Z=4\,(3) = \binom{4}{3}(1/6)^3(5/6) = \frac{4 \cdot 5}{(6)^4} = \boxed{0.0154}$

WEEK 3 GA

(2) $X \sim \text{Bin}(2, 6/14)$ $Y \sim \text{Bin}(2, 8/14)$ $\rho(X,Y) = \frac{\text{Cov}(X,Y)}{SD(X)SD(Y)}$

Y \ X	0	1	2
0	0	0	30/182
1	0	96/182	0
2	56/182	0	0

$E(X) = 156/182$ $E(X^2) = 216/182$ $\text{Var}(X) = 14976/33124$

$E(Y) = 208/182$ $E(Y^2) = 320/182$ $\text{Var}(Y) = 14976/33124$

$SD(X)SD(Y) = 14976/33124$ $E(XY) = 96/182$

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 96/182 - \frac{156 \cdot 208}{(182)^2} = \frac{-14976}{33124}$

$\rho(X,Y) = \frac{-14976}{33124} \times \frac{33124}{14976} = \boxed{-1}$

(3) $8^{th} \to P(G) = 0.4$ $G_8 \sim \text{Bin}(2, 0.4)$

$9^{th} \to 0.4$ $G_9 \sim \text{Bin}(2, 0.4)$

$10^{th} \to 0.6$ $G_{10} \sim \text{Bin}(2, 0.6)$

$E(G_{total}) = E(G_8) + E(G_9) + E(G_{10})$

$E(G_8) = E(G_9) = 0.8$ $E(G_{10}) = 1.2$

$E(G_{total}) = 0.8 + 0.8 + 1.2 = \boxed{2.8}$

(5) $\sigma = \sqrt{2}$ $\quad 7 = \mu - k\sigma \Rightarrow \left|\frac{7-15}{\sqrt{2}}\right| = k \Rightarrow k \simeq 5.66$ $\quad P(7 < X < 23) = 1 - \frac{1}{(5.66)^2} \simeq 0.969$

(6) $X \sim \text{Geometric}(0.04)$ $\quad Var(X) = (1-p)/p^2 = 600$

(8) $Var(XY) = Var(X) \cdot Var(Y) + Var(Y)(E(X))^2 + Var(X)(E(Y))^2$

$= 15 + 3969 + 605 = 4589$

(9)

X	Y	Z	P(X,Y,Z)	R
0	1	2	0.4	4
0	2	3	0.3	6
1	0	-2	0.3	-4

$R = XY + 2Z$

$E(R) = 2.2$ $\quad E(R^2) = 12.4$

$Var(R) = 12.4 - (2.2)^2 = 7.56$

(10) both losing $= \left(\frac{498 \times 497}{500 \times 499}\right) \times ₹0$

one wins 500 $= \left(\frac{1}{500} \times \frac{498}{499} \times 2\right) \times ₹500 = 1.99$

one wins 2000 $= \left(\frac{1}{500} \times \frac{498}{499} \times 2\right) \times ₹2000 = 7.98$

both tickets win $= \left(\frac{2}{500} \times \frac{1}{499}\right) \times ₹2500 = 0.02$

total = 10

(11)

$P(UU) = (0.4)(0.6)$	Gain(UU) = 2400	$\rightarrow$ 576		1050
$P(UD) = (0.4)(0.4)$	" (UD) = 150	$\rightarrow$ 24		300
$P(DU) = (0.6)(0.6)$	" (DU) = -1200	$\rightarrow$ -432		-450
$P(DD) = (0.6)(0.4)$	" (DD) = -2550	$\rightarrow$ -612		-1200

+

-36
48
-162

MOCK WEEK 1 - 2

(1)

X	Y	S	Z_1	Z_2
1	1	2	1	1
1	2	3	2	1
2	1	3	2	1
2	2	4	2	2
3	1	4	3	1
3	2	5	3	2

S	2	3	4	5
P(S)	0.1	0.4	0.4	0.1

Z_2	1	2
$P(Z_2)$	0.6	0.4

Z_1	1	2	3
$P(Z_1)$	0.1	0.7	0.2

(4)

y\x	0	1	2	
0	2/24	3/24	1/24	1/4
1	6/24	9/24	3/24	3/4
	1/3	1/2	1/6	

$P(Y=1 \mid X=2) = \frac{3}{24} \times 6 = \frac{3}{4}$

$P(X \leq 1, Y=0) = 5/24$ $\quad P(X=2, Y=1) = 3/24$

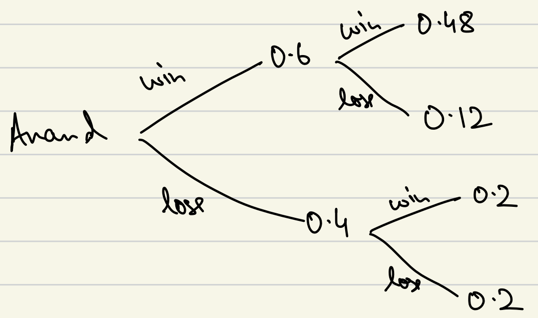
sum $= 8/24 = 1/3$ $\quad \frac{1}{\text{sum}} = \boxed{3}$

⑥ $X \sim \text{Poisson}(2)$ $\quad Y \sim \text{Poisson}(3)$ $\quad Z = X+Y$ $\quad Z \sim \text{Poisson}(5)$ $\quad X|Z=5 \sim \text{Bin}(5, 2/5)$

$Y|Z=5 \sim \text{Bin}(5, 3/5)$

$$P(X=1|Z=5) = \binom{5}{1}\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^4 = 0.2592$$

⑧



$X_2 \backslash X_1$	0	1	
0	0.2	0.12	0.32
1	0.2	0.48	0.68
	0.4	0.6	

⑪ $P(X=3, Y=3) = P(N=6)$ and $P(X=3)$

$$P(N=6) = \frac{e^{-5}5^6}{6!} = 0.146223$$

$$P(X=3|N=6) = \binom{6}{3}(0.3)^3(0.7)^3 = 20(0.21)^3 = 0.18522$$

$$P(X=3, N=6) = \boxed{0.0271}$$

⑬ $$P(X_1=0|Y=8) = \frac{P(X_1=0, Y=8)}{P(Y=8)} \qquad P(Y=8) = \binom{10}{8}\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)^2$$

$$P(X_1=0, Y=8) = (1/3)\cdot\binom{9}{8}\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)$$

$$\frac{\cancel{(1/3)^2}\,\cancel{(2/3)^8}\;{}^9C_8}{\cancel{(1/3)^2}\,\cancel{(2/3)^8}\;{}^{10}C_8} = \frac{9!}{8!} \times \frac{8!\,2!}{10!} = \frac{2}{10}$$

⑭ $$f_{Z|Y=15}(50) = \frac{P(X=10, Y=5 \text{ or } X=5, Y=10)}{f_{xy}\,(10,5) \text{ or } (5,10) \text{ or } (7,8) \text{ or } (8,7) \text{ or } (6,9) \text{ or } (9,6)}$$

$$f_{xy}(10,5) = f_{xy}(5,10) = \binom{10}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5 \cdot \binom{10}{10}\left(\frac{1}{3}\right)^{10} = \binom{10}{5}\left(\frac{1}{3}\right)^{15}\left(\frac{2}{3}\right)^5$$

$$f_{xy}(6,9) = f_{xy}(9,6) = \binom{10}{6}\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^4 \cdot \binom{10}{9}\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right) = \binom{10}{6}\binom{10}{9}\left(\frac{1}{3}\right)^{15}\left(\frac{2}{3}\right)^5$$

$$f_{xy}(7,8) = f_{xy}(8,7) = \binom{10}{8}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^2 \cdot \binom{10}{7}\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^3 = \binom{10}{8}\binom{10}{7}\left(\frac{1}{3}\right)^{15}\left(\frac{2}{3}\right)^5$$

$$\frac{\binom{10}{5}\cancel{\left(\frac{1}{3}\right)^{15}}\cancel{\left(\frac{2}{3}\right)^5}}{\cancel{\left(\frac{1}{3}\right)^{15}}\cancel{\left(\frac{2}{3}\right)^5}\left[\binom{10}{5}+\binom{10}{6}+\binom{10}{8}\right]} = \frac{\frac{10!}{5!\,5!}}{\frac{10!}{5!\,5!} + \frac{10!}{6!\,4!} + \frac{10!}{8!\,2!}}$$

$$\frac{10!}{5!\,5!} = \frac{\cancel{10}\times9\times\cancel{8}\times7\times\cancel{6}}{\cancel{5\times4\times3\times2}} = 28\times9$$

$$\frac{10!}{6!\,4!} \times \frac{10!}{9!} = \frac{10\times\cancel{9}\times\cancel{8}\times7}{\cancel{4\times3\times2}} \times 10 = 100\times3\times7$$

$$\frac{10!}{8!\,2!} \times \frac{10!}{7!\,3!} = \frac{10\times9}{\cancel{2}} \times \frac{10\times\cancel{9}\times\cancel{8}}{\cancel{3\times2}} = 600\times9$$

⑮ $X \sim \text{Uniform}\{0,1,\dots,9\}$ $\quad Y \sim \text{Uniform}\{0,1,\dots,9\}$

$$P(Z \geq 2) = 1 - P(Z=1) - P(Z=0)$$

$P(Z=0) \rightarrow X=Y = 1/10$ $\qquad P(Z=1) \rightarrow |X-Y| = 1 = 18/100$

$$P(Z\geq 2) = 1 - \frac{28}{100} = \frac{72}{100}$$

01	43	78
10	45	87
12	54	89
21	56	98
23	65	
32	67	
34	76	

WEEK 4 GA

(2) $P(-19 < X < 16) = P(X < 16) = 1 - \exp(-832)$

(3) $X \sim \text{Exponential}(\lambda = 1)$ $\quad k = \lambda = 1$

(4) $P(45 < X < 48) = F_X(48) - F_X(45)$ $\quad F_X(48) = 1 - \exp(-48)$ $\quad F_X(45) = 1 - \exp(-45)$

$= 1 - \exp(-48) - 1 + \exp(-45)$

$= \exp(-45) - \exp(-48) = e^{-45} - e^{-48}$

(5) $X \sim \text{Exponential}(\lambda = 1/900)$ $\quad P(X > 700) = 1 - P(X \le 700) = \exp(-700/900)$

$= \exp(-0.78)$

(6) $f_X(x) = 5x^4$ $\quad F_X(x) = x^5$ $\quad P(X < 6/9 \mid X > 1/9) = \dfrac{P(1/9 < X < 6/9)}{P(X > 1/9)} = \dfrac{F_X(6/9) - F_X(1/9)}{1 - F_X(1/9)}$

$= \dfrac{(6/9)^5 - (1/9)^5}{1 - (1/9)^5} = \dfrac{(6)^5 - 1}{\cancel{(9)^5}} \times \dfrac{\cancel{(9)^5}}{(9)^5 - 1} = \dfrac{(6)^5 - 1}{(9)^5 - 1} = \dfrac{7775}{59048} \simeq 0.132$

(7) $Z = \dfrac{X - 206}{35} \Rightarrow -1 = \dfrac{X - 206}{35} \Rightarrow X = -35 + 206 = 171$

WEEK 5 GA

(1) $A = 7/37$ $\quad B = 30/37$

$A \sim \text{Exp}(1/7)$ $\quad B \sim \text{Exp}(1/15)$ $\quad f_Y(y) = \frac{7}{37}(1/7)\exp(-y/7) + \frac{30}{37}(1/15)\exp(-y/15)$

$P(X = A \mid Y = 5) = \dfrac{f_{Y|X=A}(y) \cdot f_X(A)}{f_Y(y)} = \dfrac{(1/37) \cdot \exp(-5/7)}{\frac{7}{37}(1/7)\exp(-5/7) + \frac{30}{37}(1/15)\exp(-5/15)}$

$= \dfrac{\exp(-5/7)}{\exp(-5/7) + 2\exp(-1/3)} = 0.2546$

$f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(\dfrac{-(x-\mu)^2}{2\sigma^2}\right)$

(2) $Y = XZ + X$ $\quad X \sim \text{Uniform}\{1,2,3\}$ $\quad Z \sim \text{Normal}(1,4)$

$P(X = 2 \mid Y = 2) = \dfrac{f_{Y|X=2}(2) \cdot f_X(2)}{f_Y(2)}$ $\quad f_X(2) = 1/3$ $\quad f_{Y|X=2}(2) = \dfrac{P(X=2, Z=0)}{P(X=2)} = P(Z=0) = \dfrac{1}{2\sqrt{2\pi}} \exp(-1/8)$

$= \dfrac{\exp(-1/8)}{1 + \exp(-1/8) + \exp(-2/9)}$

$f_Y(y) = \dfrac{1}{3} \dfrac{1}{2\sqrt{2\pi}} \left(\exp\left(-\dfrac{(y-2)^2}{8}\right) + \exp\left(-\dfrac{(y-4)^2}{32}\right) + \exp\left(-\dfrac{(y-6)^2}{72}\right) \right)$

$f_Y(2) = \dfrac{1}{6\sqrt{2\pi}} \cdot \left(\cancel{\exp\left(\dfrac{-(2-2)^2}{8}\right)} + \exp\left(\dfrac{-1}{8}\right) + \exp\left(\dfrac{-2}{9}\right) \right)$

$= 0.32889$

(3) $X = (6)(1-x)^5$ $\quad Y = (1-X)^9$ $\quad g^{-1}(y) = 1 - (y)^{1/9}$ $\quad g'(x) = 9(1-x)^8$ $\quad g'(g^{-1}(y)) = 9(y)^{8/9}$

$f_Y(y) = \dfrac{1}{|g'(g^{-1}(y))|} \cdot f_X(g^{-1}(y)) = \dfrac{1}{9(y)^{8/9}} (6)(1 - (1 - (y)^{1/9}))^5$

$= \dfrac{2}{3} \dfrac{(y)^{5/9}}{(y)^{8/9}} = \dfrac{2}{3}(y)^{-3/9}$

(4) $\text{supp}(y) = \dfrac{24}{3} < y < \dfrac{33}{3}$ $\quad g(x) = \dfrac{27 - x}{3}$ $\quad g^{-1}(y) = 27 - 3y$ $\quad g'(x) = \dfrac{1}{3}$ $\quad f_X(g^{-1}(y)) = \dfrac{(27-3y)^2}{81}$

$f_Y(y) = \dfrac{1}{|g'(g^{-1}(y))|} \cdot f_X(g^{-1}(y)) = \dfrac{3(27-3y)^2}{81}$

(5) $E[X] = \int_0^1 x \cdot x(4x^2 + 3x - 2)\,dx = \int_0^1 4x^4 + 3x^3 + 2x^2\,dx = \dfrac{4x^5}{5}\Big|_0^1 + \dfrac{3x^4}{4}\Big|_0^1 + \dfrac{2x^3}{3}\Big|_0^1$

$= \dfrac{4}{5} + \dfrac{3}{4} + \dfrac{2}{3} = \dfrac{48 + 45 + 40}{60} = \dfrac{133}{60}$

$9 \cdot E[X] = \dfrac{9 \cdot (133)}{60} = 19.95$

(6) $E[X] = \int_0^1 x^2\,dx + \int_1^2 2x - x^2\,dx = \frac{x^3}{3}\Big|_0^1 + x^2\Big|_1^2 - \frac{x^3}{3}\Big|_1^2 = \frac{1}{3} + 3 - \frac{7}{3} = \frac{-6+9}{3} = 1$

$Var(X) = \int_0^1 (x-1)^2 x\,dx + \int_1^2 (x-1)^2(2-x)\,dx$

$= \int_0^1 x^3 + x - 2x^2\,dx + \int_1^2 -x^3 + 4x^2 - 5x + 2\,dx = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\Big|_0^1 + -\frac{x^4}{4} + \frac{4x^3}{3} - \frac{5x^2}{2} + 2x\Big|_1^2$

$= \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) + \left(-4 + \frac{32}{3} - 10 + 4\right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{5}{2} + 2\right)$

$= \frac{1}{4} - \frac{2}{3} + \frac{2}{4} - \frac{16}{4} + \frac{32}{3} - \frac{40}{4} + \frac{16}{4} + \frac{1}{4} - \frac{4}{3} + \frac{10}{4} - \frac{8}{4}$

$= \frac{1+2-16-40+16+1+10-8}{4} + \frac{32-2-4}{3} = \frac{-34}{4} + \frac{26}{3} = \frac{104-102}{12} = \frac{1}{6}$

$Var(Y) = (48)^2(Var(X)) = 384$

(9) $g(x) = x^3 + 22 \quad g^{-1}(y) = (y-22)^{1/3} \quad g'(x) = 3x^2 \quad g'(g^{-1}(y)) = 3(y-22)^{2/3}$

$f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} \cdot f_X(g^{-1}(y)) = \frac{1}{3}(y-22)^{-2/3} \cdot \frac{1}{20} = \frac{(y-22)^{-2/3}}{60}$

$E[Y] = \int_{22}^{8022} \frac{(y-22)^{1/3}}{60}\,dx = \frac{3(y-22)^{4/3}}{(4)(60)}\Big|_{22}^{8022} = \frac{(8000)^{4/3}}{80} - \frac{1}{80} \approx 2000$

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$

(10) male ~ Normal (60, 25) $\quad$ X = Gender $\quad$ $Y|X=M \sim$ Normal (60, 25)

female ~ Normal (55, 36) $\quad$ Y = Age $\quad$ $Y|X=F \sim$ Normal (55, 36)

$P(X=M|Y=60) = \frac{f_{Y|X=M}(60) \cdot f_X(M)}{f_Y(60)}$

$= \frac{(3/5)\cdot(1/\sqrt{50\pi})}{f_Y(60)}$

$= \frac{3}{\cancel{5}} \cdot \frac{\cancel{15}\,3}{9 + 5\exp(-\frac{25}{72})}$

$= \frac{9}{9 + 5\exp(\frac{-25}{72})}$

$f_Y(y) = \frac{3}{5}\cdot\frac{1}{5\sqrt{2\pi}}\exp\left(\frac{-(y-60)^2}{50}\right) + \frac{2}{5}\cdot\frac{1}{6\sqrt{2\pi}}\exp\left(\frac{-(y-55)^2}{72}\right)$

$= \frac{1}{\sqrt{50\pi}}\left(\frac{3\exp}{5}\left(\frac{-(y-60)^2}{50}\right) + \frac{1}{3}\exp\left(\frac{-(y-55)^2}{72}\right)\right)$

$f_Y(60) = \frac{1}{\sqrt{50\pi}}\left(\frac{3}{5} + \frac{1}{3}\exp\left(\frac{-25}{72}\right)\right)$

$= \frac{1}{\sqrt{50\pi}}\left(\frac{9 + 5\exp\left(\frac{-25}{72}\right)}{15}\right)$

Mock for Quiz 1

(1)

$Y \backslash X$	1	2	
1	2/8	3/8	5/8
2	2/8	1/8	3/8
	4/8	4/8	

(2) $P(X+Y=3) = f_{XY}(1,2) + f_{XY}(2,1) = 2/8 + 3/8 = 5/8$

(3) $P(\max(X,Y) \le 2) = f_{XY}(x \le 2, y \le 2) = 1$

(4) $P(X<2) = P(X>2) = 0.1 \quad ; \quad P(-2<X<2) = 1-(P(X<2)+P(X>2)) = 0.8$

(5) Probability of getting same $= \frac{1}{6}$ ~ Geometric (1/6)

$E[X] = 1/p = 6$

(7) $X \sim Exp(1/20) \qquad F_X(15) = e^{-3/4} \approx 0.527633$

(8) $f_X(15 < X < 30) = F_X(30) - F_X(15) \approx 0.249236$

(9)

X	-10	20	30
P(X=x)	9/36	18/36	9/36

$\frac{9}{36}(30-10) + \frac{18}{36}(20) = 20\left(\frac{9+18}{36}\right) = 15$

11 13 15
33 31 35
55 51 53

12 14 16
21 23 25
32 34 36
41 43 45
52 54 56
61 63 65

(10) $P(Y>5 \mid X=2) \sim \text{Geometric}(0.5) = (1-0.5)^5 = 0.03125$

(11) $P(Y>5) = 0.5\left(\text{Geometric}(1/2) + \text{Geometric}(1/3)\right) \approx 0.1629$

(13) $P(X \geq 180) + P(X \leq 120) = 1 - P(120 \leq X \leq 180)$

$P(3\sigma \leq X \leq 3\sigma) \geq 1 - \frac{1}{9} \geq \frac{8}{9}$ $\qquad 1 - P(3\sigma \leq X \leq 3\sigma) \leq \frac{1}{9}$

(14) $k \cdot \int_0^1 (x)^{1/2}\, dx = 1 \Rightarrow \frac{2k(x)^{3/2}}{3}\Big|_0^1 = 1 \qquad \frac{2k}{3} = 1 \qquad k = 3/2$

(15) $P(X \geq 0.4 \mid X < 0.8) = \frac{P(0.4 \leq X < 0.8)}{P(X<0.8)} = \frac{\int_{0.4}^{0.8} 3/2 (x)^{1/2}\, dx}{\int_0^{0.8} 3/2(x)^{1/2}\, dx} = \frac{(0.8)^{3/2} - (0.4)^{3/2}}{(0.8)^{3/2}} \approx 0.64645$

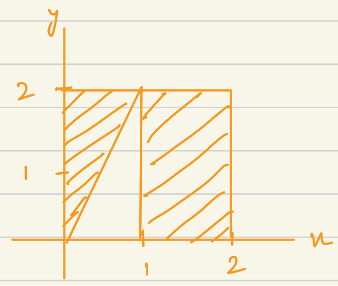
(18) $P(X_1 = 0 \mid Y=8) = \frac{P(X_1=0, Y=8)}{P(Y=8)} = \frac{\binom{9}{8}\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^2}{\binom{10}{8}\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^2} = \frac{9}{45} = 1/5 = 0.2$

WEEK 6 GA

(1) $f_X(x) = \int_0^5 \frac{4xy}{5^4}\, dy = \frac{4x}{5^4}\left(\frac{y^2}{2}\Big|_0^5\right) = \frac{2x}{5^2}$

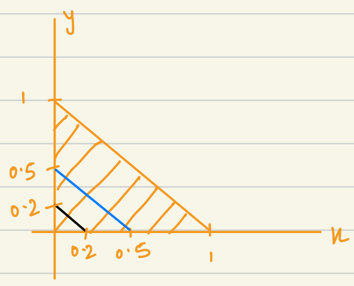
$f_Y(y) = \frac{2y}{5^2} \qquad f_X(x)\cdot f_Y(y) = \frac{4xy}{5^4} = f_{XY}(x,y)$

(3)



Total area $= \left(\frac{1}{2}\times 2 \times 1\right) + (1 \times 2) = 1 + 2 = 3$

(4)



$P(X+Y \leq 1/2) = \int_{y=0}^{1/2}\int_{x=0}^{1/2-y} 24xy\, dx\, dy = \int_{y=0}^{1/2} 24y\left(\frac{x^2}{2}\Big|_0^{1/2-y}\right) dy = \int_{y=0}^{1/2} 3y + 12y^3 - 12y^2\, dy = \frac{3y^2}{2} + 3y^4 - 4y^3\Big|_0^{1/2}$

$= \frac{3}{8} + \frac{3}{16} - \frac{4}{8} = \frac{3}{16} - \frac{2}{16} = \frac{1}{16}$

$P(X+Y \leq 1/5) = \int_{y=0}^{1/5}\int_{x=0}^{1/5-y} 24xy\, dx\, dy = \int_{y=0}^{1/5} 24y\left(\frac{x^2}{2}\Big|_0^{1/5-y}\right) dy = \int_{y=0}^{1/5} 24y\left(1/50 + y^2/2 - 1/5y\right) dy = \frac{6y^2}{25} + 3y^4 - \frac{8y^3}{5}\Big|_0^{1/5}$

$= \frac{6}{625} + \frac{3}{625} - \frac{8}{625} = \frac{1}{625}$

(5) $f_{XY}(x,y) = 3xy - 3x^2y \qquad f_X(x) = \int_{y=0}^{2} 3xy - 3x^2y\, dy = \frac{3xy^2}{2} - \frac{3x^2y^2}{2}\Big|_0^2 = 6x - 6x^2 = 6x(1-x)$

$f_Y(y) = \int_{x=0}^{1} 3xy - 3x^2y\, dx = \frac{3x^2y}{2} - x^3y\Big|_0^1 = \frac{3y}{2} - y = \frac{y}{2}$

$f_{X|Y=1}(>1/3) = \frac{f_{XY}(x>1/3, 1)}{f_Y(1)} \qquad f_{XY}(x>1/3, 1) = \int_{1/3}^{1} 3x - 3x^2\, dx = \frac{3x^2}{2} - x^3\Big|_{1/3}^{1} = \frac{3}{2} - 1 - \frac{3}{18} + \frac{1}{27} = \frac{27-18-3}{18} + \frac{1}{27} = \frac{1}{3} + \frac{1}{27}$

$= \frac{10}{27}$

$= \frac{10/27}{1/2} = 20/27 \approx 0.74$

(6) $P(x-y<3) = P(y>x-3) = \int_{x=0}^{10}\int_{y=x-3}^{x} 1/50\, dy\, dx = \int_0^{10} \frac{y}{50}\Big|_{x-3}^{x} dx = \frac{3x}{50}\Big|_0^{10} = \frac{3}{5}$

(7) $\int_{x=6}^{\infty}\int_{y=0}^{8} ke^{-x}(e)^{-y}\, dy\, dx = \int_6^{\infty} ke^{-x}(1-e^{-8})\, dx = ke^{-6}(1-e^{-8})$

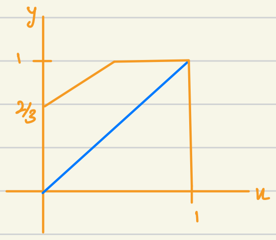
(8) $f_X(x) = \int_0^2 x/8 + y/8\, dy = \frac{xy}{8} + \frac{y^2}{16}\Big|_0^2 = \frac{x+1}{4} \qquad f_X(1/6) = \frac{7/6}{4} = 7/24 \qquad P(2/6 \leq Y \leq 2, X=1/6) = \int_{y=2/6}^{2} 1/48 + 6y/48\, dy = y/48 + 3y^2/48\Big|_{2/6}^{2}$

$= \frac{2}{48} + \frac{12}{48} - \frac{(1/3)}{48} - \frac{(1/3)}{48} = 5/18$

$P(2/6 \leq Y \leq 2 \mid X=1/6) = \frac{5/18}{7/24} = \frac{5}{\cancel{18}_{3}} \times \frac{\cancel{24}^{4}}{7} = \frac{20}{21} \approx 0.95$

(9) $f_X(x) = \int_0^1 \frac{2x}{3} + \frac{2}{3} \, dy = \frac{2xy}{3} + \frac{2y}{3} \Big|_0^1 = \frac{2x}{3} + \frac{2}{3}$

$f_Y(y) = \int_0^1 2x/3 + 2/3 \, dx = \frac{x^2}{3} + \frac{2x}{3} \Big|_0^1 = \frac{1}{3} + \frac{2}{3} = 1$



$P(X>Y) = \int_{x=0}^{1} \int_{y=0}^{x} 2x/3 + 2/3 \, dy \, dx = \int_0^1 2x^2/3 + 2x/3 \, dx = \frac{2x^3}{9} + \frac{x^2}{3} \Big|_0^1 = \frac{2}{9} + \frac{1}{3} = \frac{2+3}{9} = \frac{5}{9}$

WEEK 7 GA

(1) $Var(A) = \sigma^2/3$ $Var(B) = 0.46\,\sigma^2$ $Var(C) = 0.38\,\sigma^2$

(2) $\sigma^2 = 25$ $n = 30$ $Var(\text{sample mean}) = \sigma^2/n = 0.83$

(3) $p = 1/6$ $\sigma^2 = 5/36$ $\mu = 1/6$ $\delta = 0.6$ $P(|\bar{X} - \mu| < \delta) \geq 1 - \frac{\sigma^2}{n\delta^2}$

$\geq 1 - \frac{5}{36} \times \frac{1}{(300)(0.36)} \Rightarrow \geq 1 - \frac{5}{3888} \Rightarrow \frac{3883}{3888}$

(5) $Var(X_i) = \frac{1}{(i)^2}$, $i \in [1,9]$ $Var(i \cdot X_i) = 1$, $i \in [1,9]$ $Var(Y) = 9$

(6) $\sigma^2 = 12$ $P(|\bar{X} - \mu| < \delta) \geq 1 - \frac{\sigma^2}{n\delta^2}$, where $n = 50$, $\delta = 3$

(7) $Var(S(A)) = 0.0004$ $n = (0.45)(0.55)/0.0004 = 619$

(8) $\mu = 2$ $n = 50$ $\delta = 1$ $P(|\bar{X} - 2| < 1) \geq 1 - \frac{4}{(50)(1)}$
$\sigma^2 = 4$

≥ 0.92

WEEK 8 GA

(1) $E[Y] = 2$ $Var(Y) = 50(0.04) = 2$

$P(Y > 8) \rightarrow$ z-score $= \frac{6}{\sqrt{2}} =$

(3) $E[Y] = 600$ $Var(Y) = 1200(0.25) = 300$

z-score of $Y = 580 \rightarrow \frac{580 - 600}{\sqrt{300}} = -1.15$ $F_Z(-1.15) \approx 0.125$

(4) $E[Y] = 250/0.5 = 500$ $Var(Y) = 1000$

$P(Y > 565) = 1 - P(Y < 565) = 1 - F_Z\left(\frac{565 - 500}{\sqrt{1000}}\right) = 1 - 0.98030 = 0.0197$

(5) $E[X] = 2n$ $Var(X) = 2n$

X is a sum of i.i.d. Gamma(2, 1)

$E\left[\frac{X}{2n}\right] = 1$ $Var\left(\frac{X}{2n}\right) = \frac{2n}{(2n)^2} = \frac{1}{2n}$ $P\left(\frac{X - E[X]}{Var(X)} > 0.01\right) \leq \frac{1}{(0.01)^2}$

$\frac{Var(X/2n)}{(0.01)^2} = 0.01$

$\frac{1}{2n} = (0.01)^3$ $n = \frac{1}{2(0.01)^3} = 500000$

(7) $E[L_i] = 20 \quad Var(L_i) = 400 \quad SD(L_i) = 20$

$E[T] = 1000 \quad Var(T) = 20{,}000 \quad SD(T) = 141.4214$

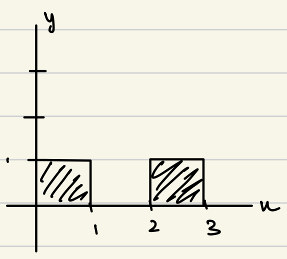
$$P(T > 1950) = 1 - P(T < 1950) = 1 - F_Z\left(\frac{1950-1000}{\sqrt{20000}}\right) = 1 - F_Z(6.7)$$

(8) $E[\bar{X}] = 36 \quad Var(\bar{X}) = 4/35 \quad SD(X) = 2/\sqrt{35}$

$$P(27 < \bar{X} < 40) = F_Z\left(\frac{40-36}{2/\sqrt{35}}\right) - F_Z\left(\frac{27-36}{2/\sqrt{35}}\right)$$

MOCK QUIZ 2

(2)



$\text{Area}(D) = 2(1\times 1) = 2$

$f_{XY}(x,y) = \frac{1}{2} \quad \text{for } x \in D$

(4) $E[X] = 0$

$(0.3e^{-4\lambda} + 0.2e^{\lambda} + 0.5e^{2\lambda})(0.3e^{-4\lambda} + 0.2e^{\lambda} + 0.5e^{2\lambda})$

$= 0.09e^{-8\lambda} + 0.06e^{-3\lambda} + 0.15e^{-2\lambda}$

$0.04e^{2\lambda}$

$0.25e^{4\lambda}$

(10) 60% → male 40% → female

$N(160, 25) \quad N(150, 25)$

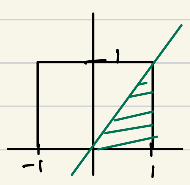
$$P(M \mid H=160) = \frac{P(H \mid M) \times P(M)}{P(H)} \qquad \frac{1}{5\sqrt{2\pi}} \quad 0.60$$

$$P(H=160) = \frac{1}{5\sqrt{2\pi}}(0.60) + \frac{1}{5\sqrt{2\pi}}\exp\left(\frac{-100}{50}\right)(0.40)$$

$$= \frac{1}{5\sqrt{2\pi}}\left(0.6 + 0.4\exp(-2)\right)$$

$$P(M \mid H=160) = \frac{0.6}{(0.6 + 0.4\exp(-2))} \approx 0.917$$

(11)



$$P(X > Y) = \frac{\int_{x=0}^{1}\int_{y=0}^{x} \frac{3}{2}x^2\,dy\,dx}{\int_{x=-1}^{1}\int_{y=0}^{1} \frac{3}{2}x^2\,dy\,dx} = \frac{\int_{x=0}^{1} \frac{3}{2}x^3\,dx}{\int_{x=-1}^{1} \frac{3}{2}x^2\,dx} = \frac{\frac{3}{8}x^4\Big|_0^1}{\frac{1}{2}x^3\Big|_{-1}^1} = \frac{\frac{3}{8}}{\frac{1}{2}-(-\frac{1}{2})} = \frac{3}{8}$$

$$P(|X| > Y) = \int_{x=0}^{1}\int_{y=0}^{x} \frac{3}{2}x^2\,dy\,dx + \int_{x=-1}^{0}\int_{y=0}^{-x} \frac{3}{2}x^2\,dy\,dx$$

$$= \frac{3}{8} + \left(-\frac{3}{8}x^4\Big|_{-1}^{0}\right) = \frac{3}{4}$$

(14) $X \sim Exp(1/2) \quad Y \sim Exp(1)$

$$P(Y > X \mid X = x) = \int_x^\infty e^{-y}\,dy = -e^{-y}\Big|_x^\infty = e^{-x}$$

$$P(Y>X \mid X=x) = \frac{P(Y>X)}{P(X=x)} \qquad P(Y>X) = P(Y>X \mid X=x)\cdot P(X=x)$$

$$= \int_0^\infty e^{-x}\cdot(1/2)e^{-x/2}\,dx = (1/2)\int_0^\infty e^{-3x/2}\,dx = \frac{1}{2}\left(\frac{-2}{3}e^{-3x/2}\Big|_0^\infty\right)$$

$$= \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$$

(15) $\mu = 240 \quad \sigma^2 = (2)^2 \times 100 = 400$

$\sigma = 20$

$$z\text{-score} = \frac{250-240}{20} = 0.5$$

(16) σ^2 of sample mean $= 2$

$$P(|\bar{X} - \mu| < 4) \geq 1 - \frac{2}{50(16)} = 0.9975$$

(17) $X, Y \sim \text{Exp}(1)$ $\qquad f_X(x) = e^{-x} \qquad f_Y(y) = e^{-y}$

$$P(X<2Y) = P(X<2Y \mid Y=y) \times P(Y=y)$$

$$P(X<2Y \mid Y=y) = \int_0^{2y} e^{-x}\,dx = -e^{-x}\Big|_0^{2y} = -e^{-2y}+1$$

$$P(X<2Y) = \int_0^{\infty} e^{-y}(-e^{-2y}+1)\,dy = \int_0^{\infty} -e^{-3y}+e^{-y}\,dy$$

$$= \frac{e^{-3y}}{3} - e^{-y}\Big|_0^{\infty} = 0 - (1/3 - 1) = 1 - 1/3 = \frac{2}{3}$$

(18) $f_{XY}(x,y) = e^{-x-y}$

$$P(X>Y \mid X<2Y) = \frac{P(X>Y, X<2Y)}{P(X<2Y)} = \frac{1/6}{2/3} = \frac{1}{6} \times \frac{3}{2} = \frac{1}{4}$$

$$\int_{y=0}^{\infty} e^{-y} \int_{x=y}^{2y} e^{-x}\,dx = e^{-y}\left(-e^{-x}\Big|_y^{2y}\right) = e^{-y}\left(-e^{-2y}+e^{-y}\right)$$

$$= -e^{-3y}+e^{-2y}$$

$$P(X>Y, X<2Y) = \int_0^{\infty} -e^{-3y}+e^{-2y}\,dy = \frac{e^{-3y}}{3} - \frac{e^{-2y}}{2}\Big|_0^{\infty} = -\left(\frac{1}{3}-\frac{1}{2}\right) = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$$

$$P(X<2Y) = \int_{y=0}^{\infty}\int_{x=0}^{2y} e^{-y}e^{-x}\,dx\,dy$$

$$= \int_0^{\infty} e^{-y}\left(-e^{-x}\Big|_0^{2y}\right)dy$$

$$= e^{-y}(1-e^{-2y}) = e^{-y}-e^{-3y}$$

$$\int_0^{\infty} e^{-y}-e^{-3y}\,dy = -e^{-y}+\frac{e^{-3y}}{3}\Big|_0^{\infty}$$

$$= -\left((-1)+\frac{1}{3}\right) = 1-\frac{1}{3} = \frac{2}{3}$$

WEEK 9 GA

(1) $$\text{Risk}(\hat{\mu},\mu) = \text{Var}(\hat{\mu}) + \text{Bias}(\hat{\mu},\mu)$$

$$\text{Var}(\hat{\mu}) = \text{Var}(7X_1) + \text{Var}(7X_2) + \dots + \text{Var}(7X_n)$$

$$= 49\,\text{Var}(X)$$

$$\text{Risk} = \frac{49\sigma^2}{n} + 36\mu^2$$

$$\text{Bias}(\hat{\mu},\mu) = E[\hat{\mu}] - \mu$$

$$= 7\mu - \mu$$

$$= 6\mu$$

$$\text{Bias}(\hat{\mu},\mu)^2 = 36\mu^2$$

(4) $X \sim \text{Normal}(\mu, 1600)$

$$L(x_1,\dots,x_{10}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(\frac{-(x_i-\mu)^2}{2\sigma^2}\right)$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(\frac{1}{2\sigma^2}\right) \exp\left(-\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

$$\hat{\mu} = 161$$

$$\log(L) = -n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$f_X(161) = \frac{\exp(-0.5)}{\sqrt{2\pi}} = \frac{\exp(-0.5)}{2.5066283} \approx 0.24$$

$$\frac{\partial \log(L)}{\partial \mu} = n\mu - \sum_{i=1}^{n} x_i$$

$$\frac{\sum_{i=1}^{n} x_i}{n} = \hat{\mu}$$

(5) $$f_X(x) = \frac{\theta}{(x)^{\theta+1}} \qquad E[X] = \frac{\theta}{\theta-1} \qquad m_1 = \frac{\theta}{\theta-1} \Rightarrow m_1(\theta-1) = \theta$$

$$\frac{\theta}{\theta} - \frac{1}{\theta} = \frac{1}{m_1} \Rightarrow 1-\frac{1}{\theta} = \frac{1}{m_1} \Rightarrow 1-\frac{1}{m_1} = \frac{1}{\theta} \Rightarrow \frac{m_1-1}{m_1} = \frac{1}{\theta}$$

$$\theta = \frac{m_1}{m_1-1}$$

$$\hat{\theta} = \frac{M_1}{M_1-1}$$

$$\hat{\theta} = \frac{x}{83} \times \frac{1}{x/83-1} = \frac{x}{\cancel{83}} \times \frac{\cancel{83}}{x-83}$$

$$\hat{\theta} = \frac{\sum x_i}{\sum x_i - 83}$$

(6) Bin(8, θ)

$$L = \binom{8}{x_i}(\theta)^{x_i}(1-\theta)^{8-x_i}$$
$$= \binom{8}{x_1}\cdots\binom{8}{x_n}(\theta)^{\Sigma x_i}(1-\theta)^{8n}(1-\theta)^{-\Sigma x_i}$$
$$\log(L) = \log\left(\binom{8}{x_1}\cdots\binom{8}{x_n}\right) + \log(\theta)\Sigma x_i + 8n\log(1-\theta) - \log(1-\theta)\Sigma x_i$$
$$\frac{\partial \log(L)}{\partial\theta} = \frac{\Sigma x_i}{\theta} - \frac{8n - \Sigma x_i}{1-\theta}$$
$$(1-\theta)\Sigma x_i = \theta(8n - \Sigma x_i)$$
$$\frac{1-\theta}{\theta} = \frac{8n - \Sigma x_i}{\Sigma x_i} \Rightarrow \frac{1}{\theta} = \frac{8n-\Sigma x_i}{\Sigma x_i} + 1 \Rightarrow \theta = \frac{\Sigma x_i}{8n}$$

(7) $E[X] = \theta + 1$ $\quad m_1 - 1 = \theta$

$$\hat{\theta} = \frac{\Sigma x_i - n}{n} \qquad \text{Bias}(\hat{\theta}, \theta) = E[\hat{\theta}] - \theta = 0$$

(9) $L = (\theta)^{-n}\exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} x_i\right)$ $\quad \log(L) = -n\log(\theta) - \frac{\Sigma x_i}{\theta}$

$$\frac{\partial \log(L)}{\partial\theta} = \frac{-n}{\theta} + \frac{\Sigma_i x_i}{\theta^2}$$
$$0 = -n\theta^2 + \theta\Sigma x_i$$
$$n\theta^2 = \theta\Sigma x_i \Rightarrow \hat{\theta} = \frac{\Sigma_i x_i}{n}$$

(10) $E[X] = \frac{1-p+2p+3-3p+4p}{2} = \frac{4+2p}{2} = 2+p$

$m_1 - 2 = p$ $\quad \hat{p} = 0.2$

WEEK 10 GA

(1) Prior ~ Uniform {8, 10} $\quad P(S) = \frac{0.5}{9!}\left(\exp(-8)(8)^9 + \exp(-10)(10)^9\right) = 0.1246$

$\exp(-8)(8)^9 \to 45025$, $\exp(-10)(10)^9 \to 45399$

Posterior mode = 10

(2) Posterior distribution : $(p)^w(1-p)^{n-w} \sim \text{Beta}(w+1, n-w+1)$

$$\hat{p} = \frac{w+1}{n+2} = \frac{7}{12} = 0.583$$

(3) Post. dist. : $(\lambda)^n \exp(-\lambda(x_1+x_2+\dots+x_n))\,(\mu)\,(\exp(-\mu\lambda))$ $\quad \lambda e^{-\lambda x}$

$\propto (\lambda)^n \exp(-\lambda(x_1+x_2+\dots+x_n+\mu)) \sim \text{Gamma}(n+1, \mu+\Sigma x_i)$ $\quad$ Post. mean $= \frac{\alpha}{\beta} = \frac{n+1}{\mu+\Sigma x_i}$

(4) Post. dist. $\propto$ Gamma$(11, 1/20 + \Sigma x_i)$ $\quad$ post. mean $= \frac{11}{1/20 + 237} = 0.046$

(5) $\hat{\mu} = \bar{X}\left(\frac{n\sigma_0^2}{n\sigma_0^2+\sigma^2}\right) + \mu_0\left(\frac{\sigma^2}{n\sigma_0^2+\sigma^2}\right)$ $\quad \sigma^2 = 16$ $\quad n = 10$ $\quad \mu_0 = 45$ $\quad \sigma_0^2 = 36$

$\bar{X} = 59.5$

$$\hat{\mu} = 59.5\left(\frac{360}{376}\right) + 45\left(\frac{16}{376}\right) = 58.882$$

(6) $\frac{2}{2+\beta} = 0.4 \Rightarrow \beta = 3$ $\quad$ Post. Dist. $\propto (p)^{w+2}(1-p)^{n-w+3} \sim \text{Beta}(w+3, n-w+4)$

post. mean $= \frac{w+3}{n+7} = \frac{7}{17} = 0.4117$

(7) Post. dist. $\propto (p)^{w+40}(1-p)^{n-w+60} \sim \text{Beta}(w+41, n-w+61)$

$\frac{40}{40+b} = 0.4 \Rightarrow b = 60$ $\quad \hat{p} = \frac{44}{112} = 0.393$

(8) $X \sim \text{Normal}(\mu, 256)$ $\quad \bar{X} = 536.8$

$\hat{\mu} = 533.04$

(9) $E[X] = \frac{1}{\lambda}$ $\quad \hat{\lambda} = \frac{1}{M_1} = \frac{n}{\Sigma x_i} = 0.039$

Post. Dist. $\propto (\lambda)^n \exp(-\lambda(\Sigma x_i)) \sim \text{Gamma}(n+1, \Sigma x_i)$

$\hat{\lambda}_B = \frac{n+1}{\Sigma x_i} = 0.043$

ML : $(\lambda)^n \exp(-\lambda(\Sigma x_i))$ $\quad \log(L) = n\log(\lambda) - \lambda(\Sigma x_i)$

$$\frac{\partial \log(L)}{\partial\lambda} = \frac{n}{\lambda} - \Sigma x_i$$
$$\hat{\lambda}_{ML} = \frac{n}{\Sigma x_i} = 0.039$$

WEEK 11 GA

① $H_0: \mu = 400$; $H_A: \mu = 390$ $SD(\bar{X}) = 5/\sqrt{n}$

$\alpha = 0.05$ $1-\beta = 0.95$

$\alpha = P(\mu < c \mid \mu = 400) \Rightarrow 0.05 = F_z\left(\frac{c-400}{5/\sqrt{n}}\right) \Rightarrow c = F_z^{-1}(0.05)\cdot 5/\sqrt{n} + 400$

$1-\beta = P(\mu < c \mid \mu = 390) \Rightarrow 0.95 = F_z\left(\frac{c-390}{5/\sqrt{n}}\right) \Rightarrow c = -F_z^{-1}(0.05)\cdot 5/\sqrt{n} + 390$

$0 = F_z^{-1}(0.05)\cdot 10/\sqrt{n} + 10 \Rightarrow n = (F_z^{-1}(0.05))^2 \approx 3$

③ $\alpha = P(|\bar{X}-\mu| > c \mid \mu = 10.5)$

$\alpha = P\left(\frac{|\bar{X}-10.5|}{3/10} > \frac{10c}{3}\right) \Rightarrow 0.05 = 2\cdot F_z\left(\frac{-10c}{3}\right) \Rightarrow -F_z^{-1}(0.025)\cdot\frac{10}{3} = c \Rightarrow c \approx 6.53$

$\Rightarrow 0.10 = 2\cdot F_z\left(\frac{-10c}{3}\right) \Rightarrow -F_z^{-1}(0.05)\cdot\frac{10}{3} = c \Rightarrow c \approx 5.48$

④ $H_0: \mu = 0$ $H_A: \mu = 1.5$ $\sigma = 1/2$

$\alpha = P(\bar{x} > c \mid \mu = 0) \Rightarrow 0.05 = 1 - F_z\left(\frac{c-0}{1/2}\right) \Rightarrow (1/2) F_z^{-1}(0.95) = c \Rightarrow c \approx 0.82$

⑤ $1-\beta = P(\bar{x} > c \mid \mu = 1.5) = 1 - F_z(2(0.82-1.5)) = 1 - F_z(-1.36) = 0.91$

⑥ $H_0: p = 0.9$ $H_A: p > 0.9$ $\sigma = 0.3$ Test $= 92$

p-value $= P(p > c \mid p = 0.9) = 1 - F_z\left(\frac{92-90}{3}\right) = 0.25$

⑧ $H: p = 0.25$ $H_A: p > 0.25$ $c = 0.36$ $\sigma = 0.0433$

$\alpha = P(p > 0.36 \mid p = 0.25) \Rightarrow \alpha = 1 - F_z\left(\frac{0.36-0.25}{0.0433}\right) = 1 - F_z(2.54) = 0.00554$

$\beta = P(p \le 0.36 \mid p = 0.5) \Rightarrow \beta = F_z\left(\frac{0.36-0.5}{0.05}\right) = F_z(-2.8) = 0.0026$

⑩ $H_0: p = 0.6$ $H_A: p < 0.6$ $\sigma = \frac{\sqrt{0.24}}{100}\cdot\sqrt{n}$

$\alpha = P(p < c \mid p = 0.6) \Rightarrow 0.10 = F_z\left(\frac{c-0.6}{\frac{\sqrt{0.24}}{100}\cdot\sqrt{n}}\right) \Rightarrow F_z^{-1}(0.10)\cdot\sqrt{n}\cdot\frac{\sqrt{0.24}}{100} + 0.6 = c$

$1-\beta = P(p < c \mid p = 0.4) \Rightarrow 0.90 = F_z\left(\frac{c-0.4}{\frac{\sqrt{0.24}}{100}\cdot\sqrt{n}}\right) \Rightarrow -F_z^{-1}(0.10)\cdot\sqrt{n}\cdot\frac{\sqrt{0.24}}{100} + 0.4 = c$

$F_z^{-1}(0.10)\cdot\sqrt{n}\cdot\frac{\sqrt{0.24}}{50} + 0.2 = 0 \Rightarrow n \approx 254$

⑫ $H_0: \mu = 150$; $H_A: \mu < 150$ $\bar{X} = 145$; $\sigma = 5/6$

$0.05 = P(\bar{x} < c \mid \mu = 150) = F_z\left(\frac{c-150}{5/6}\right)$

$F_z^{-1}(0.05)\cdot\frac{5}{6} + 150 = c \Rightarrow c = 148.6$, reject H_0

⑬ $H_0: p = 0.90$; $H_A: p \ne 0.90$ sample $p = 0.894$ $\sigma = 0.045$

$0.01 = P(|p - 0.90| > c) \Rightarrow 0.01 = 2 F_z\left(\frac{-c}{0.045}\right) \Rightarrow -F_z^{-1}(0.005)\cdot(0.045) = c$

$c = 0.1161$

⑮ $H_0: \mu = 2.5$ $H_A: \mu \ne 2.5$ $\sigma = 0.01$

$\alpha = P(|\bar{x}-\mu| > c \mid \mu = 2.5) \Rightarrow 0.05 = 2\cdot F_z\left(\frac{-c}{0.01}\right) \Rightarrow c = -F_z^{-1}(0.025)\cdot(0.01) = 0.0196$

⑯ $H_0: \mu = 800$ $H_A: \mu \ne 800$ $\sigma = 40/\sqrt{30}$

$\alpha = 2\cdot F_z\left(\frac{-c\sqrt{30}}{40}\right) = 0.00614$

$1-\beta = P(|\bar{X}-800| > 20 \mid \mu = 788)$

$= F_z\left(\frac{780-788}{40/\sqrt{30}}\right) + 1 - F_z\left(\frac{820-788}{40/\sqrt{30}}\right)$

$= 1 + F_z(-1.09) - F_z(1.09)$

WEEK 12 GA

1. $\bar{X} - \bar{Y} \sim \text{Normal}(0, 113/25)$

   $F_Z^{-1}(0.025) = \dfrac{-c}{\sqrt{113/25}} \Rightarrow c = 4.167$

4. $-1.646 = \dfrac{c - 40}{2/30} \Rightarrow c = 39.89$

5. $S^2 = \dfrac{30.14(6.25)}{19} \Rightarrow S = 3.1487$