CHAPTER 3

→ Appendix :

↳ Define the risk model in two parts:

Model returns as:

$$\underset{\text{vector of returns}}{r} = \underset{\text{N×K Factor exposures}}{X} \cdot \underset{\text{K vector Factor returns}}{b} + \underset{\text{N vector specific returns}}{u} \qquad (3A.1)$$

Assume : ① $u \perp b$ i.e., $Cov\{u_n, b_k\} = 0 \quad \forall n, k$

② $u_n \perp u_m$ i.e., $Cov\{u_n, u_m\} = 0 \quad \forall m, n$ for $m \neq n$

Model risk as :

$$\underset{\text{N×N cov. matrix of stocks}}{V} = \underset{\text{N×K Factor exposures}}{X} \cdot \underset{\text{K×K cov. matrix of Factors}}{F} \cdot X^T + \underset{\text{N×N diag. matrix specific variance}}{\Delta} \qquad (3A.2)$$

↳ Model Estimation

→ Estimate Factor returns via multiple regressions, Fama-Macbeth.

→ Regress stock returns against Factor exposures.

→ Run GLS regressions, weighting each observed return by the inverse of its specific variance.

→ These regressions can involve a lot of Factors and they won't suffer from multi collinearity.

↳ Factor Portfolios

→ estimated Factor returns:

$$b = (\underset{\text{exposure matrix}}{X^T} \cdot \underset{\text{added weightage term because GLS}}{\Delta^{-1}} \cdot X)^{-1} \cdot X^T \cdot \underset{\text{specific variance}}{\Delta^{-1}} \cdot r \qquad (3A.3)$$

$$b_k = \sum_{n=1}^{N} \underbrace{c_{k,n}}_{\text{portfolio weights}} \cdot r_n \qquad (3A.4)$$

can interpret Factor return as return to a portfolio.

→ Factor portfolios resemble characteristic portfolios, except they are multiple-Factor in nature.

Characteristic portfolios have unit exposure to their characteristic, not necessarily zero exposure to other Factors.

→ a.k.a. Factor-mimicking portfolios.

↳ not investable portfolios because they contain every single asset with some weight.

↳ Factor Covariance Matrix

→ Estimate the Factor covariance matrix using Factor returns of each period.

→ There are more advanced techniques of Forecasting Factor covariance matrix.

↳ Specific Risk

→ model specific risk as

$$u_n^2(t) = S(t)[1 + v_n(t)] \qquad (3A.5)$$

where,

$$S(t) = \frac{1}{N} \sum_{n=1}^{N} u_n^2(t) \Big\} \text{— average specific variance across the universe} \qquad (3A.6)$$

$$0 = \frac{1}{N} \sum_{n=1}^{N} v_n(t) \Big\} \text{ cross-sectional variation in specific variance} \qquad (3A.7)$$

↳ Risk Analysis

→ Portfolio P described by h_p that gives holdings.

Factor exposures of P:

$$x_p = X^T \cdot h_p \qquad (3A.8)$$

Variance of P:

$$\sigma_p^2 = x_p^T F \cdot x_p + h_p^T \Delta \cdot h_p \qquad (3A.9)$$
$$= h_p^T \cdot V \cdot h_p$$

→ Active risk (tracking error) ; $h_{PA} = h_p - h_B$

$$\psi_p^2 = h_{PA}^T \cdot V \cdot h_{PA} \qquad (3A.12)$$

→ To seperate market risk from residual risk,
N vector of stock betas relative to h_B

$$\beta = \frac{V \cdot h_B}{\sigma_B^2} = \frac{X \cdot F \cdot x_B + \Delta \cdot h_B}{\sigma_B^2} \qquad (3A \cdot 13)$$

$$b = \frac{F \cdot x_B}{\sigma_B^2} \quad \text{and} \quad d = \frac{\Delta \cdot h_B}{\sigma_B^2}, \text{ then}$$

$$\beta = X \cdot b + d \qquad (3A \cdot 16)$$

The portfolio beta:

$$\beta_P = h_P^T \cdot \beta = x_P^T \cdot b + h_P^T \cdot d \qquad (3A \cdot 17)$$

Systematic and residual risk:

$$\sigma_P^2 = \beta_P^2 \cdot \sigma_B^2 + \omega_P^2 \qquad (3A \cdot 18)$$

Residual covariance matrix:

$$VR = V - \beta \cdot \sigma_B^2 \cdot \beta^T \qquad (3A \cdot 19)$$

↳ Marginal Contribution

→ Marginal impact on risk is measured by the partial derivative of the risk w.r.t asset holding

→ Marginal Contribution to Total Risk:

$$MCTR = \frac{\partial \sigma_P}{\partial h_P^T} = \frac{V \cdot h_P}{\sigma_P} \qquad (3A \cdot 20)$$

→ Marginal Contribution to Residual Risk:

$$MCRR = \frac{VR \cdot h_P}{\omega_P} = \frac{V \cdot h_{PR}}{\omega_P} \qquad (3A \cdot 22)$$

where $h_{PR} = h_P - \beta_P \cdot h_B$ → residual holdings vector

→ Marginal Contribution to Active Risk:

$$MCAR = \frac{V \cdot h_{PA}}{\psi_P} \qquad (3A \cdot 23)$$

can decompose active risk into market and residual

$$MCAR = \beta \cdot k_1 + MCRR \cdot k_2 \qquad (3A \cdot 24)$$

where

$$k_1 = \frac{\beta_{PA} \cdot \sigma_B^2}{\psi_P} \quad \text{and} \quad k_2 = \frac{\omega_P}{\psi_P}$$

↳ Factor Marginal Contribution

→ To add exposure to a factor, add a portfolio with exposure to factor k and zero exposure to other factors.

$$h_{PA} \longrightarrow h_{PA} + \underbrace{\left[(X^T \cdot \Delta^{-1} \cdot X)^{-1} \cdot X^T \cdot \Delta^{-1}\right]^T}_{\text{K×N vector of factor portfolios}} \cdot \delta_k \qquad (3A \cdot 27)$$

↳ k×1 vector containing zeros except in k-th row

where δ_k is a k×1 vector containing zeros in all but k-th row, where it contains δ_k.

→ Change in active risk by adding this portfolio:

$$\Delta \psi_P = \left[\left[(X^T \cdot \Delta^{-1} \cdot X)^{-1} \cdot X^T \cdot \Delta^{-1}\right]^T \cdot \delta_k\right]^T \cdot MCAR \qquad (3A \cdot 28)$$
$$= \delta_k^T \cdot (X^T \cdot \Delta^{-1} \cdot X)^{-1} \cdot X^T \cdot \Delta^{-1} \cdot \left(\frac{V \cdot h_{PA}}{\psi_P}\right)$$

Simplification using factor decomposition of cov. matrix

$$\frac{\Delta \psi_P}{\delta_k^T} = \underbrace{\left(\frac{\overbrace{F \cdot x_{PA}}^{\text{Resembles (3A·23)}}}{\psi_P}\right)}_{\text{Factor variance}} + \underbrace{\left[\frac{\overbrace{(X^T \cdot \Delta^{-1} \cdot X)^{-1}}^{\text{very small}} \cdot x_{PA}}{\psi_P}\right]}_{\text{Residual variance}} \qquad (3A \cdot 29)$$

$$\frac{\Delta \psi_P}{\delta_k^T} = \frac{(X^T \cdot \Delta^{-1} \cdot X)^{-1} \cdot X^T \cdot \Delta^{-1} \cdot V \cdot h_{PA}}{\psi_P} \qquad V = X \cdot F \cdot X^T + \Delta$$

$$= (X^T \cdot \Delta^{-1} \cdot X)^{-1} \cdot X^T \cdot \Delta^{-1} \cdot (X \cdot F \cdot X^T + \Delta) \cdot h_{PA}$$
$$= (X^T \cdot \Delta^{-1} \cdot X)^{-1} \; X^T \cdot \Delta^{-1} \cdot (X \cdot F \cdot X^T + \Delta) \cdot (X^T)^{-1} \cdot x_{PA}$$
$$= (X^T \cdot \Delta^{-1} \cdot X)^{-1} \cdot \Delta^{-1} \cdot (X \cdot F \cdot X^T + \Delta) \cdot x_{PA}$$
$$= \left[(X^T \cdot \Delta^{-1} \cdot X)^{-1} \cdot \Delta^{-1} \cdot X \cdot F \cdot X^T + (X^T \cdot \Delta^{-1} \cdot X)^{-1}\right] \cdot x_{PA}$$
$$= \left[F + (X^T \cdot \Delta^{-1} \cdot X)^{-1}\right] \cdot x_{PA} = \boxed{F \cdot x_{PA} + (X^T \cdot \Delta^{-1} \cdot X)^{-1} \cdot x_{PA}}$$

$$x_{PA} = X^T \cdot h_{PA}$$
$$(X^T)^{-1} \cdot x_{PA} = h_{PA}$$

↳ Attribution of Risk

- Example with decomposition of active risk, but idea applies to total or residual risk too.

$$h_{PA}^{T} \cdot MCAR = \psi_P \qquad (3A.31)$$

In percentage terms:

$$\frac{h_{PA}^{T} \cdot MCAR}{\psi_P} \qquad (3A.32)$$

- If we increase the holding in asset n.

$$\Delta \psi_P = \Delta h_{PA}(n) \cdot MCAR(n) \qquad (3A.33)$$

↳ Attribution to Factors

- Using the factor risk model:

$$\psi_P = x_{PA}^{T} \cdot F \cdot x_{PA} \quad h_{PA}^{T} \cdot \Delta \cdot h_{PA} \qquad (3A.37)$$

where $x_{PA} = X^{T} \cdot h_{PA}$

Therefore,

$$FMCAR = \frac{\partial \psi_P}{\partial x_{PA}^{T}} = \frac{F \, x_{PA}}{\psi_P} \qquad (3A.38)$$