

MATH 2

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# WEEK 1

## → Vectors

- ↳ can be thought of as a list.
- ↳ can be in a row or a column form.
- ↳ Addition of vectors:

Add the corresponding entries. For e.g.  $\rightarrow (3, 5) + (2, 4) = (5, 9)$

## ↳ Scalar multiplication:

$$\text{e.g. } \rightarrow 2(8, 8, 10, 5) = (16, 16, 20, 10)$$

## ↳ Visualisation of a vector

Point  $(a, b) \equiv$  Vector  $(a, b) \equiv a\hat{i} + b\hat{j}$  where,  $\begin{matrix} \uparrow \\ \hat{i} \end{matrix} \rightarrow \text{one unit in } x\text{-axis}$   $\begin{matrix} \leftrightarrow \\ \hat{j} \end{matrix} \rightarrow \text{one unit in } y\text{-axis}$

$$\text{e.g.: Point } (-1, -1) = -\hat{i} - \hat{j}$$

- ↳ Vectors in  $\mathbb{R}^n$  are lists with  $n$  real entries.

## → Matrices

- ↳ rectangular array of numbers
- ↳ (rows × columns); e.g.  $\begin{bmatrix} 5 & 7 & 10 \\ 3 & 5 & 2 \end{bmatrix}$  is a  $2 \times 3$  matrix.

↳ (1, 2)ith entry  $\rightarrow 7$

- ↳ square matrix  $\rightarrow N \times N$

- ↳ Diagonal matrix  $\rightarrow$  all entries are 0 except the diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

- ↳ Scalar matrix  $\rightarrow$  all entries have the same value

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- ↳ Identity matrix  $\rightarrow$  denoted by 'I'; scalar matrix with values = 1

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ↳ Addition of matrices  $\rightarrow$  must be of the same size:

$$\begin{bmatrix} 1 & 0 \\ 5 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 8 & 3 \\ 7 & 4 \end{bmatrix}$$

- ↳ Scalar multiplication  $\rightarrow$  multiply each number with the scalar.

- ↳ Matrix multiplication  $\rightarrow A \times B = C$ ;  $C[i, j] = \sum_{k=1}^n A[i, k] \times B[k, j]$

↳ no. of columns in first matrix must = no. of rows in 2nd matrix

$$A_{m \times n} \times B_{n \times p} = (AB)_{m \times p}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

- ↳ Scalar multiplication is the same as multiplication by the scalar matrix

$$\text{ex. } \rightarrow \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

- ↳ Properties:
  - ①  $(A + B) + C = A + (B + C)$
  - ②  $(AB)L = A(BL)$
  - ③  $A + B = B + A$
  - ④  $AB \neq BA$

- ⑤  $\lambda(A + B) = \lambda A + \lambda B$
- ⑥  $\lambda(AB) = (\lambda A)B = A(\lambda B)$
- ⑦  $A(B + C) = AB + AC$
- ⑧  $(A + B)C = AC + BC$

## → System of Linear Equations

↪ collection of one or more linear equations involving the same set of variables.

↪ system of  $m$  linear equation with  $n$  variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

⋮

⋮

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

↪ system of lin. eq. in matrix form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

↑ coefficient matrix  
↑ column of variables  
↑ column of resulting values

$A = m \times n$  matrix

$x = \text{column vector with } n \text{ entries}$

$b = \text{column vector with } m \text{ entries}$

↪ Solutions to a system of lin. eq.:

- ① Infinite solution
- ② Single unique solution
- ③ No solution

## → Determinant

$$\hookrightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(A) = ad - bc$$

$$\text{e.g. } A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix} \quad \det(A) = 20 - 18 = 2$$

$$\hookrightarrow A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \det(A) = a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

↪ Determinant of Identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rightarrow \det(I_2) = 1$$

$$\rightarrow \det(I_3) = 1$$

↪ Determinant of a product of matrices

$$\hookrightarrow \det(AB) = \det(A) \cdot \det(B) ; \quad \det(ABC) = \det(A) \cdot \det(B) \cdot \det(C)$$

$$\hookrightarrow \det(A^n) = \det(A)^n \quad \hookrightarrow \det(A^{-1}) = \det(A)^{-1}$$

↪ Determinant of the inverse of a matrix

$$AA^{-1} = I \Rightarrow \det(AA^{-1}) = \det(I)$$

$$\hookrightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

↪ Switching rows

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\det(\tilde{A}) = cb - ad = -(ad - bc) = -\det(A)$$

$$\det(\tilde{A}) = -\det(A)$$

↪ Add multiple of a row/column to another row/column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$$

$$\det(\tilde{A}) = (a+tc)d - (b+td)c = ad + tcd - bc - tcd$$

$$\det(\tilde{A}) = \det(A)$$

↳ Scalar multiplication of a row / column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} ta & b \\ tc & d \end{bmatrix}$$

$$\det(\tilde{A}) = t \cdot \det(A)$$

↳ Upper / Lower triangle matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{bmatrix} \quad \text{upper triangle matrix}$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 0 \\ 3 & 4 & 9 \end{bmatrix} \quad \text{lower triangle matrix}$$

→ determinant is the product of diagonal elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \det(A) = a_{11} \cdot a_{22} \cdot a_{33}$$

↳ Transpose of a matrix and its determinants

→ Transpose of  $A_{m \times n} = A^T_{n \times m}$  with  $(i,j)$ -th entry  $A_{ji}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3 \times 2} \quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2 \times 3}$$

$$\det(A) = \det(A^T)$$

↳ Minors and cofactors

→ Minor of the entry in  $i$ -th row and  $j$ -th column is the determinant of the submatrix formed by deleting  $i$ -th row and  $j$ -th column.

e.g. →  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$(1,1)$ -th minor; denoted by  $M_{11}$

$M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

→ Cofactor  $(i,j)$ -th cofactor;  $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$C_{11} = M_{11}; \quad C_{23} = -M_{23}$$

→ For  $A_{3 \times 3}$

$$\det(A) = (a_{11} \times C_{11}) + (a_{12} \times C_{12}) + (a_{13} \times C_{13})$$

For  $A_{4 \times 4}$

$$\det(A) = \sum_{j=1}^4 a_{1j} C_{1j}$$

$$\text{For } A_{n \times n}; \quad \det(A) = \sum_{j=1}^n a_{1j} C_{1j}$$



# WEEK 2

## → Determinant (continued)

↳ Expansion along any row / column

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } i$$

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } j \quad (\text{determinant along a column})$$

↳ Properties :

① Determinant of a product is product of the determinants.

$$\hookrightarrow \det(AB) = \det(BA) = \det(A) \cdot \det(B)$$

② Switching two rows or columns changes the sign.

③ Adding multiples of a row to another row leaves the determinant unchanged.

④ Scalar multiplication of a row/column by a constant  $t$  multiplies the determinant by  $t$ .

$$⑤ \det(tA_{nn}) = (t)^n \det(A)$$

↳ Useful computational tips:

① The determinant of a matrix with a row or a column of zeros is 0.

② The determinant of a matrix in which one row (or column) is a linear combination of other rows (resp. columns) is 0.

③ Scalar multiplication of a row/column by a constant  $t$  multiplies the determinant by  $t$ .

④ While computing the determinant, you can choose to compute it using expansion along a suitable row or a column.

## → Cramer's rule

$$\rightarrow A_n = b, \text{ where } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\text{Cramer's rule} \Rightarrow A_{u_1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{21} \end{bmatrix}, A_{u_2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

$$\hookrightarrow \text{solutions for } u_1 \text{ & } u_2 \Rightarrow u_1 = \frac{\det(A_{u_1})}{\det(A)} ; \quad u_2 = \frac{\det(A_{u_2})}{\det(A)}$$

→ For a  $3 \times 3$  matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A_{u_1} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$A_{u_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$

$$A_{u_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

$$u_1 = \frac{\det(A_{u_1})}{\det(A)}$$

$$u_2 = \frac{\det(A_{u_2})}{\det(A)}$$

$$u_3 = \frac{\det(A_{u_3})}{\det(A)}$$

## → Inverse Matrix

↳ The inverse of a matrix  $A \rightarrow (A^{-1}) \Rightarrow A \cdot (A^{-1}) = I$

↳ if  $\det(A) \neq 0$ , then the matrix is invertible

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## ↳ Adjugate of a matrix

$M_{ij}$  = determinant of submatrix after deleting i-th row and j-th column.

$$C_{ij} = (-1)^{i+j} M_{ij}$$

→ Adjugate of a matrix  $\rightarrow$  Transpose of cofactor matrix

$$\text{Adj}(A) = C^T$$

$$\rightarrow A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

$\rightarrow A_{n \times n}$ , then  $\det(\text{adj}(A)) = (\det(A))^{n-1}$

## ↳ Solution to system of linear equations using inverse matrix.

$$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$$

## → Homogeneous System of Linear Equations

↳  $Ax = 0$ , where  $b = 0 \Rightarrow Ax = 0$

→ has unique solution  $0$  if  $A$  is invertible

→ has infinite solutions if  $A$  is not invertible

## → The Row Echelon Form (reduced form)

↳ Matrix is in row echelon form if:

- first non-zero element in each row (leading entry) is 1
- each leading entry is in a column to the right of the leading entry in the previous row.
- rows with all zero elements, if any, are below rows having a non-zero element.

e.g.  $\rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

→ reduced row echelon form  $\rightarrow$  For a non-zero row, the leading entry in the row is the only non-zero entry in its column

e.g.  $\rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

→ e.g. solution for  $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow Ax = b$

$$\begin{aligned} x_1 + 2x_2 &= b_1 \Rightarrow x_1 = b_1 - 2x_2 \\ x_3 &= b_2 \\ x_4 &= b_3 \end{aligned}$$

solution  $\Rightarrow x = \begin{bmatrix} b_1 - 2b_3 \\ b_2 \\ b_2 \\ b_3 \end{bmatrix}$

→ If the i-th column has the leading entry of some row, we call  $x_i$  a dependent variable.  
If the i-th column does not have the leading entry of any row,  $x_i \rightarrow$  independent variable.

## → Row reduction

### Elementary Row Operations

Type	Action	Example and notation	Effect on determinant
1 Interchange two rows		$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = -\det(B)$
2 Scalar multiplication of a row by a constant $t$		$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = t \cdot \det(B)$
3 Adding multiples of a row to another row		$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_3} \begin{bmatrix} 3 & -19 & -2 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = \det(B)$

## ↳ Row echelon form

Steps : ① Find the left most non-zero column.  
 ② Use operations to get 1 in the top of that column.  
 ③ Make entries below that 1 to 0.  
 ④ Repeat the steps for the next row and onwards.

## ↳ Reduced row echelon form

↳ start from the right most column and make all the non-leading terms 0s.

↳ e.g.  $A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 2 & 1/2 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 2 & 1/2 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_3 - 5R_1} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 2 & 1/2 \\ 0 & -4 & 13/2 \end{bmatrix} \xrightarrow{R_2/2} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & -4 & 13/2 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 35 \end{bmatrix} \xrightarrow{R_3/35} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - \frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 35 \times 2 = 70$$

## → Gaussian Elimination Method

↳ Augmented matrix → matrix of size  $(m \times n+1)$

↳ First  $n$  columns are from  $A$  and the last column is  $b$ .

↳ denoted by  $[A|b]$

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

↳ Bring the matrix  $A$  to reduced row echelon form.

→ let  $R$  be the submatrix of obtained matrix of the first  $n$  columns and  $c$  be the submatrix of the obtained matrix of the last column.

$[R|c]$ , where  $R \rightarrow$  reduced row echelon form of  $A$

Solutions of  $Au=b$  are precisely the solutions of  $Ru=c$ .

## → Homogeneous system of linear equations

↳ 0 is always a solution a.k.a. trivial solution

↳ If there are more variables than equations, (i.e., more columns than rows) there will be infinite solutions because there must be some independent variables.



# WEEK 3

→ Properties of vectors Let  $v, w$  and  $v'$  be vectors in  $\mathbb{R}^n$  and  $a, b \in \mathbb{R}$

- i)  $v + w = w + v$
- ii)  $(v + w) + v' = v + (w + v')$
- iii) The 0 vector satisfies:  $v + 0 = 0 + v = v$
- iv) The vector  $-v$  satisfies:  $v + (-v) = 0$
- v)  $1v = v$
- vi)  $(ab)v = a(bv)$
- vii)  $a(v+w) = av + aw$
- viii)  $(a+b)v = av + bv$

↳ Vector space → a set with two operations (addition and multiplication)

$$\begin{array}{ccc} + : V \times V \rightarrow V & \text{and} & \cdot : \mathbb{R} \times V \rightarrow V \\ \text{for each pair of elements } v_1, v_2 \text{ in } V, \text{ there is} & & \text{for each } c \in \mathbb{R} \text{ and } v \in V \text{ there is a unique} \\ \text{a unique element } v_1 + v_2. & & \text{element } cv \text{ in } V. \end{array}$$

↳ Example of vector space → solutions of a homogeneous system  $Ax=0$

↳ non-examples:

① Define addition and scalar multiplication in  $\mathbb{R}^2$  as:

$$\begin{aligned} \rightarrow (u_1, u_2) + (y_1, y_2) &= (u_1 + y_1, u_2 - y_2) \\ \rightarrow c(u_1, u_2) &= (cu_1, cu_2) \end{aligned}$$

$$v + w \neq w + v \rightarrow \text{i), iii) and viii) fail}$$

$$\textcircled{2} \rightarrow (u_1, u_2) + (y_1, y_2) = (u_1 + y_1, 0)$$

$$\rightarrow c(u_1, u_2) = (cu_1, 0) \rightarrow \text{iii), iv) and v) fail}$$

→ Cancellation law of vector addition

↳ If  $v_1, v_2, v_3 \in V$  such that  $v_1 + v_3 = v_2 + v_3$ , then  $v_1 = v_2$

↳ Corollaries: ① The vector 0 described in iii) is unique.  $\rightarrow v + w = v + 0 \Rightarrow w = 0$   
 ② The vector  $v'$  " " iv) " " and referred to as  $-v$ .  
 $\rightarrow v + v' = 0 = v + v'' \Rightarrow v' = v''$

→ Some more properties:

$$\textcircled{1} 0v = 0 \text{ for each } v \in V \rightarrow (0+0)v = 0v + 0v \Rightarrow 0v = 0v + 0v \Rightarrow 0v + 0 = 0v + 0v$$

$$\textcircled{2} (-c)v = -cv = c(-v) \text{ for each } c \in \mathbb{R} \text{ and for each } v \in V$$

$$\rightarrow (c + (-c))v = cv + (-c)v \Rightarrow 0 = cv + (-c)v \Rightarrow -cv = (-c)v$$

$$\textcircled{3} c0 = 0 \text{ for each } c \in \mathbb{R}$$

→ Linear combinations:

↳ Let  $V$  be a vector space and  $v_1, v_2, \dots, v_n \in V$ .

Linear combinations of  $v_1, v_2, \dots, v_n$  with coefficients  $a_1, a_2, \dots, a_n \in \mathbb{R}$  is the vector  $\sum_{i=1}^n a_i v_i \in V$ .

↳ example:  $(2, 9)$  is a linear combination of vectors  $(2, 1)$  and  $(-2, 3)$ :

$$3(2, 1) + 2(-2, 3) = (6, 3) + (-4, 6) = (2, 9)$$

$$\rightarrow 3(2, 1) + 2(-2, 3) - (2, 9) = 0$$

↳ 0 vector is a linear combination of  $(2, 1), (-2, 3)$ , and  $(2, 9)$  with non-zero coefficients

↳ The plane of the two vectors  $(0, 2, 1)$  and  $(2, 2, 0)$  can be expressed by the equation  $2x - 2y + 4z = 0$

↳  $(1, 2, 0)$  does not lie on the plane

↳  $(3, 7, 2)$  does lie on this plane

## → Linear Dependence

↪ a set of vectors  $v_1, v_2, v_3, \dots, v_n \in V$  is said to be linearly dependent if there exists scalars  $a_1, a_2, a_3, \dots, a_n$ , not all zeros, such that  $\rightarrow a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n = 0$

↪ If a set of vectors are linearly dependent, then every superset of it is linearly dependent.  
↪ If  $v_1, v_2, v_3$  are " ", then  $v_1, v_2, v_3, v_n$  are also linearly dependent.

## → Linear Independence

↪ a set of vectors if they are not linearly dependent

↪  $v_1, v_2, v_3, \dots, v_n$  are linearly independent if :  $a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n = 0$   
can only be satisfied when  $a_i = 0$  for all  $i = 1, 2, 3, \dots, n$

↪ If a set of vectors  $v_1, v_2, \dots, v_n$  contain a vector  $v_i = 0$ , then the set is always linearly dependent.

↪ If two non-zero vectors are multiples of each other, then are linearly dependent.

$$a_1v_1 + a_2v_2 = 0 \Rightarrow c = -a_2/a_1, \text{ then } v_1 = cv_2$$

↪ Two vectors are linearly independent when they are not multiples of each other.

↪ for three vectors  $v_1, v_2, v_3$

$$\text{Suppose } a_1v_1 + a_2v_2 + a_3v_3 = 0 \Rightarrow v_1 = -a_2/a_1 v_2 - a_3/a_1 v_3$$

↪  $v_1$  is a linear combination of  $v_2$  and  $v_3$

↪ If three vectors are linearly independent then none of these vectors  
is a linear combination of the other two.

→ If you have more than  $n$  vectors in  $\mathbb{R}^n$ , then the homogeneous system  $Vu=0$  has infinitely many solutions. → They are linear dependent

↪ This corresponds to having more variables than there are equations.

→  $V$  is an  $n \times n$  matrix obtained by arranging a set of vectors in columns.

$Vu=0 \rightarrow$  homogeneous system

↪ square matrix, therefore  $u=0$  if and only if  $V$  is invertible  
if and only if  $\det(V) \neq 0$



# WEEK 4

→ Span of a set of vectors

↪ Span of a vector space  $V \rightarrow \text{Span}(V) = \left\{ \sum_{i=1}^n a_i v_i \in V \mid a_1, a_2, a_3, \dots, a_n \in \mathbb{R} \right\}$

e.g. → Let  $V = \{(1, 0)\} \subset \mathbb{R}^2$ . Then,  $\text{Span}(V) = \{a(1, 0) | a \in \mathbb{R}\} = \{(a, 0) | a \in \mathbb{R}\}$

$$\rightarrow \text{Let } V = \{(1,0,0), (0,1,0)\} \subset \mathbb{R}^3. \quad \text{Span}(V) = \{a(1,0,0) + b(0,1,0) \mid a, b \in \mathbb{R}\} = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$$

## → Spanning set

→ A set  $S \subseteq V$  is a spanning set for  $V$  if  $\text{span}(S) = V$ .

↪ If a vector  $T \subseteq V$ , then  $\text{Span}(T) \subseteq \text{Span}(V)$

$T \subseteq \text{Span}(V) \rightarrow \text{then } \text{Span}(T) \subseteq \text{Span}(V)$

→ Basis

→ Basis  $B$  of a vector space  $V$  is a linearly independent subset of  $V$  that spans  $V$ .

↳ e.g. Standard Basis  $\rightarrow$  let  $e_i \in \mathbb{R}^n$  be the vector with  $i^{\text{th}}$  coordinate 1 and all other coordinates 0.  $\Rightarrow e_i = (1, 0, 0, \dots, 0)$

The set  $E = \{e_1, e_2, e_3, \dots, e_n\} \subseteq \mathbb{R}^n$  is a basis for  $\mathbb{R}^n$

→ How to find a basis:

① Start with  $\emptyset$  and keep appending vectors which are not a linear combination of the set thus far obtained, until we obtain a spanning set.

② Take a spanning set and keep deleting vectors which are linear combinations of other vectors, until the remaining vectors satisfy that they are not a linear combination of the remaining ones.

→ Rank / dimension of a vector space

↳ size of (or cardinality) of a basis of the vector space.

6 Rank of a matrix Let A be an  $m \times n$  matrix.

→ column space → subspace of  $\mathbb{R}^m$  spanned by column vectors of A.

$$\text{Column rank} = \text{Row rank} = \text{Rank of A}$$

→ number of non-zero rows when reduced to row echelon

e.g.  $\rightarrow$  vector space  $W$  is spanned by  $\{(1,0,1), (-2,-3,1), (3,3,0)\}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \quad \text{RREF of } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank / dimension of  $W$  is 2 and a basis is given by  $(1, 0, 1), (0, 1, -1)$ .

6 Row method  $\rightarrow$  put the spanning vectors in rows and reduce the matrix. No. of non-zero rows is the rank and the rows themselves form a basis.

Column method  $\rightarrow$  put the spanning vectors in columns and reduce the matrix. No. non-zero rows is the rank. The columns of original matrix corresponding to the columns of reduced matrix containing the pivots (i.e., dependent variables) form a basis.

# WEEK 5

## → Null space

↪ Let  $A$  be an  $m \times n$  matrix.

Subspace  $W = \{u \in \mathbb{R}^n \mid Au = 0\} \Rightarrow$  solution space of homogeneous system  $Au = 0$

↪ a.k.a. null space of  $A$ .

↪ dimension of null space  $\Rightarrow$  nullity of  $A$ .

↪ Finding nullity and basis of an  $m \times n$  matrix  $A$ .

↪ nullity( $A$ ) = no. of independent variables

↪ by letting the independent variables vary in  $\mathbb{R}$  and calculating the values of dependent variables, you get the solution space of  $Au = 0$ .

↪ basis  $\Rightarrow$  substitute one ind. variable as 1 and the others as 0.

↪ all the vectors you get by fixing 1 for each independent variable form the basis.

## → Rank-nullity theorem

↪ For a matrix  $R$  in RREF, no. of non-zero rows (i.e., rank of  $R$ ) is the number of dependent variables.

↪ For an  $m \times n$  matrix  $A$ ,  $\text{rank}(A) + \text{nullity}(A) = n$

↪ To check whether  $n$  vectors form a basis for  $\mathbb{R}^n$ :

↪ put  $n$  vectors as columns to form an  $n \times n$  matrix

↪ if determinant  $\neq 0$ , then this set of vectors form a basis.

## → Linear Mapping

↪ Linear mapping of  $f$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  can be defined as:

$$f(u_1, u_2, u_3, \dots, u_n) = \left( \sum_{j=1}^n a_{1j} u_j, \sum_{j=1}^n a_{2j} u_j, \dots, \sum_{j=1}^n a_{mj} u_j \right)$$

↪ can be rewritten as  $Au$ , where  $A =$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

## → Linear Transformation

↪ in any linear transformation,  $f(0) = 0$  because  $\Rightarrow f(0+0) = f(0) + f(0)$

$$f(0) = f(0) + f(0) \Rightarrow 0 = f(0)$$

↪ a function  $f: V \rightarrow W$  is one-to-one  $\Rightarrow$  if  $f(v_1) = f(v_2)$ , then  $v_1 = v_2$ .

" " "  $f: V \rightarrow W$  " onto  $\Rightarrow$  if for every  $w \in W$ , there exists  $v \in V$  such that  $f(v) = w$ .

↪ Isomorphism  $\Rightarrow$  if a linear transformation is a bijection (both one-to-one and onto).

Rough example

$$\begin{aligned} f(u, y) &= u \cdot y (2, 0) + y (0, 1) \\ &= (2u - 2y, 0) + (0, y) \Rightarrow (2u - 2y, y) \end{aligned}$$

↪ for a function to be linear transformation, it has to satisfy:

$$\textcircled{1} \quad f(u+y) = f(u) + f(y); \text{ and}$$

$$\textcircled{2} \quad f(\alpha u) = \alpha \cdot f(u)$$



# WEEK 6

→ Example:  $W = \{(u, y, z) \mid u+y+z=0\}$

basis  $\rightarrow (-1, 1, 0)$  and  $(-1, 0, 1)$

$(u, y, z) \in W$  can be expressed as:

$$(u, y, z) = y(-1, 1, 0) + z(-1, 0, 1)$$

$$\text{Hence } f: W \rightarrow \mathbb{R}^2 \text{ is } f(u, y, z) = y(1, 0) + z(0, 1) \\ = (y, z)$$

→ Matrix corresponding to a linear transformation with respect to ordered bases

↳ Let  $f: V \rightarrow W$  be a linear transformation

Let  $\beta = v_1, v_2, \dots, v_n$  be an ordered basis of  $V$  and  $\gamma = w_1, w_2, \dots, w_m$  ordered basis of  $W$ .

Each  $f(v_i)$  can be written as a linear combination of  $w_j$ :

$$f(v_1) = a_{11}w_1 + a_{21}w_2 + a_{31}w_3 + \dots + a_{m1}w_m \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ f(v_n) = a_{1n}w_1 + a_{2n}w_2 + a_{3n}w_3 + \dots + a_{mn}w_m$$

The matrix corresponding to the linear transformation  $f$  is given by

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & \dots & a_{2m} \\ a_{31} & \dots & a_{3m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{bmatrix}$$

↳ example: Let  $V = W = \mathbb{R}^2$ ,  $\beta = \gamma = (1, 0), (1, 1)$  and  $f(u, y) = f(2u, y)$

$$f(1, 0) = (2, 0) = 2(1, 0) + 0(1, 1) \\ f(1, 1) = (2, 1) = 1(1, 0) + 1(1, 1)$$

↳ the matrix corresponding to  $f$  w.r.t. the ordered bases  
 $\{(1, 0), (1, 1)\}$  is  $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

↳ Changing the ordered basis gives us different matrices corresponding to the same linear transformation.

Q. Let  $W = \{(u, y, z) \mid u=2y+z\}$ . Let  $\beta = \{(2, 1, 0), (1, 0, 1)\}$  be a basis of  $W$ . Let  $T: W \rightarrow \mathbb{R}^2$  such that  $T(2, 1, 0) = (1, 0)$  and  $T(1, 0, 1) = (0, 1)$ .

↳ Definition of  $T \Rightarrow T(u, y, z) = (y, z)$  or  $T(u, y, z) = (u-y-z, u-2y)$

↳ What will be the matrix representation of  $T$  w.r.t.  $\beta$  for  $W$  and  $\gamma = \{(1, 1), (1, -1)\}$  for  $\mathbb{R}^2$ ?

$$T(2, 1, 0) = (1, 0) = a(1, 1) + b(1, -1) \rightarrow a+b=1 \\ T(1, 0, 1) = (0, 1) = c(1, 1) + d(1, -1) \rightarrow a-b=0 \Rightarrow a=b=\frac{1}{2} \\ c+d=0 \Rightarrow c=-d \\ c-d=1 \Rightarrow -2d=1 \Rightarrow d=-\frac{1}{2}, c=\frac{1}{2}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

→ Image and kernel of linear transformations

Let  $f: V \rightarrow W$  be a linear transformation

↳ Kernel

$$\ker(f) = \{v \in V \mid f(v) = 0\} \xrightarrow{\text{zeros of function}}$$

↳ Image

$$\text{Im}(f) = \{w \in W \mid \exists v \in V \text{ for which } f(v) = w\} \xrightarrow{\text{range of function}}$$

↳ A linear transformation  $f$  is 1-1 (injective) if and only if  $\ker(f) = 0$

↳ A linear transformation  $f$  is onto (surjective) if  $\text{Im}(f) = W$  for  $f: V \rightarrow W$

↳ Kernel of a function is in the null space of matrix  $A$ , where  $A$  is the matrix corresponding to the transformation.

↳ A vector  $w \in \text{Im}(f)$  iff  $d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$  is in the column space of  $A$ , where  $w = \sum_{i=1}^m d_i w_i$

→ Bases for the kernel and image of a linear transformation

$$T: V \rightarrow W$$

↪  $B = \text{RREF}(A)$ , where  $A$  is the matrix corresponding to  $T$ .

→ Basis for  $\ker(T) =$  Basis for the null space of  $A$  if the basis for  $V$  and  $W$  were standard ordered basis.

→ Basis for  $\text{Im}(T) =$  columns of  $A$  corresponding to the pivot columns in  $B$ .

↪ Example: Let  $V = \mathbb{R}^2$ ,  $W = \{(u, y, z) \mid u+y+z=0\}$ . Respective ordered basis  $\beta = \{(1, 1), (1, -1)\}$   
Let  $T: V \rightarrow W$ , where  $T(u, y) = (0, u+2y, -u-2y)$   $\gamma = \{(-1, 1, 0), (-1, 0, 1)\}$

$$\begin{aligned} T(1, 1) &= (0, 3, -3) = a(-1, 1, 0) + b(-1, 0, 1) \\ T(1, -1) &= (0, -1, 1) = c(-1, 1, 0) + d(-1, 0, 1) \end{aligned}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \quad \text{Rank}(A) = 1$$

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$$

$$\begin{aligned} \text{Basis of null space} &= \{(-1/3, 1)\} \\ \text{Basis of } \ker(T) &= (-1/3)(1, 1) + 1(1, -1) \\ &= (4/3, -2/3) \end{aligned}$$

$$\begin{aligned} \text{Basis of } \text{Im}(T) &= 3(-1, 1, 0) - 3(-1, 0, 1) \\ &= (0, 3, -3) \end{aligned}$$

↪ Rank-nullity theorem for linear transformation

Let  $T: V \rightarrow W$  be a linear transformation

$$\rightarrow \text{Rank}(T) = \dim(\text{Im}(T))$$

$$\rightarrow \text{Nullity}(T) = \dim(\ker(T))$$

$$\rightarrow \text{rank}(T) + \text{nullity}(T) = \dim(V)$$



# WEEK 7

## → Equivalence of matrices

Let  $A$  and  $B$  be two  $m \times n$  matrices.  $A$  is equivalent to  $B$  if  $B = QAP$  for some invertible  $P_{m \times m}$  and  $Q_{n \times n}$ .

Characteristics:

- ①  $A$  can be transformed into  $B$  by a combination of elementary row and column operations.
- ②  $\text{rank}(A) = \text{rank}(B)$

Consider a linear transformation  $T: V \rightarrow W$ , two ordered bases  $\beta_1$  and  $\beta_2$  for  $V$  and  $\gamma_1$  and  $\gamma_2$  for  $W$ .

Let  $A \rightarrow$  matrix corresponding to  $T$  w.r.t.  $\beta_1$  and  $\gamma_1$

$B \rightarrow$  " " " " "  $\beta_2$  and  $\gamma_2$

$A$  is equivalent to  $B$ , where  $P \rightarrow$  express  $\beta_2$  in terms of  $\beta_1$

$Q \rightarrow$  "  $\gamma_2$  " " "  $\gamma_1$

## → Similar matrices

An  $n \times n$  matrix  $A$  is similar to an  $n \times n$  matrix  $B$  if there exists an  $n \times n$  invertible matrix  $P$  such that  $B = P^{-1}AP$

Properties of similar matrices

- ①  $A$  and  $B$  are equivalent
- ②  $\text{rank}(A) = \text{rank}(B)$
- ③  $\det(B) = \det(A)$
- ④  $\text{Trace}(A) = \text{Trace}(B)$

Consider a linear transformation  $T: V \rightarrow V$  and two ordered bases  $\beta$  and  $\gamma$  for  $V$

let  $A \rightarrow$  matrix corresponding to  $T$  w.r.t.  $\beta$

$B \rightarrow$  " " " " "  $\gamma$

$A$  is similar to  $B$ , where  $P \rightarrow$  Express  $\gamma$  in terms of  $\beta$ .

$P^{-1} \rightarrow$  "  $\beta$  " " "  $\gamma$ .

## → Affine Subspace

Let  $V$  be a vector space. An affine subspace of  $V$  is a subset  $L$  such that there exists  $v \in V$  and a vector subspace  $U \subseteq V$  such that

$$L = v + U := \{v + u \mid u \in U\}$$

$U$  is unique;  $v$  is not unique (can be any vector  $v \in L$ )

Affine subspaces are translates of a vector subspace of  $V$ .

Solution set of system of linear equations

$$Ax = b$$

If  $b \in$  column space of  $A$ . Solution set  $L$  is an affine subspace of  $\mathbb{R}^n$ . It can be described as  $\rightarrow L = v + N(A)$  where  $v$  is any solution of  $Ax = b$  and  $N(A) \rightarrow$  null space of  $A$

example: let  $T(x, y, z) = (2x+3y+2, 4x-5y+3)$ .

This is not a linear transformation because  $T(0, 0, 0) = (2, 3) \neq (0, 0)$

But this is an affine transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^2$

Let  $T: V \rightarrow W$  be a lin. trans. and  $w \in W$ , then the mapping:

$$T': V \rightarrow W$$

$T'(v) = w + T(v)$ , where  $T(x, y, z) = (2x+3y, 4x-5y)$  and  $w = (2, 3)$

## → Length of a vector

↳ for any vector  $v$ , length =  $\sqrt{\text{dot product of } v}$

example in  $\mathbb{R}^2$

$$\hookrightarrow \text{len}(x, y) = \sqrt{(x, y)(x, y)} = \sqrt{x^2 + y^2}$$

in  $\mathbb{R}^3$

$$\hookrightarrow \text{len}(x, y, z) = \sqrt{(x, y, z)(x, y, z)} = \sqrt{x^2 + y^2 + z^2}$$

## → Angle between two vectors

$$\hookrightarrow \text{for any two vectors } v \text{ and } u, \theta = \cos^{-1} \left( \frac{u \cdot v}{\sqrt{(v \cdot v) \times (u \cdot u)}} \right)$$

$$\theta = \cos^{-1} \frac{u \cdot v}{\|u\| \|v\|}, \text{ where } \|u\| = \sqrt{u \cdot u}$$

## → Inner product

↳ inner product on a vector space  $V$  is a function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$

↳  $\langle v, v \rangle > 0$  for all  $v \in V \setminus \{0\}$

$$\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$$

$$\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$$

$$\langle \langle v_1, v_2 \rangle, v_3 \rangle = \langle v_1, \langle v_2, v_3 \rangle \rangle = \langle v_1, v_2 \rangle$$

↳ example on  $\mathbb{R}^2$ :

$$\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\langle u, v \rangle = u_1 v_1 - (u_1 v_2 - u_2 v_1) + 2u_2 v_2, \text{ where } u = (u_1, u_2) \text{ and } v = (v_1, v_2)$$

$$\text{matrix form} \rightarrow \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

↗ if this symmetric, then the function is an inner product.

## → Norm

↳ A norm on a vector space is a function

$$\|\cdot\| : V \rightarrow \mathbb{R}$$

satisfies these conditions:

$$\textcircled{1} \quad \|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in V$$

$$\textcircled{2} \quad \|cu\| = |c| \|u\| \quad \forall c \in \mathbb{R} \text{ and } \forall u \in V$$

$$\textcircled{3} \quad \|u\| \geq 0 \quad \forall u \in V; \quad \|u\| = 0 \text{ iff } u = 0$$

$$\hookrightarrow \|v_i\| = \sqrt{u_1^2 + u_2^2 + u_3^2 + \dots + u_n^2}, \text{ where } v_i = (u_1, u_2, u_3, \dots, u_n)$$



# WEEK 8

## → Orthogonality and linear independence

↳ If the dot product of two vectors = 0, then they are orthogonal.

↳ Orthogonal set  $S = \{v_1, v_2, \dots, v_n\} \subseteq V$ ,  $S$  is an orthogonal set of vectors if:

$$\langle v_i, v_j \rangle = 0 \quad \forall i, j \in \{1, 2, \dots, n\} \text{ and } i \neq j$$

→ If a set of vectors is orthogonal, then they are linearly independent

## ↳ Orthogonal basis

↳ A basis consisting of mutually orthogonal vectors.

if  $\dim(V) = n$ , then

orthogonal basis = orthogonal set of  $n$  vectors

## → Orthonormal basis

↳ orthogonal set of vectors such that the norm of each vector of the set = 1.

If  $S = \{v_1, v_2, \dots, v_k\} \subseteq V$ ,  $S$  is orthonormal set if:

$$\langle v_i, v_j \rangle = 0 \quad \forall i, j \in \{1, 2, \dots, k\} \text{ and } i \neq j$$

and  $\|v_i\| = 1 \quad \forall i \in \{1, 2, \dots, k\}$

## ↳ Orthonormal basis

↳ it is an orthogonal basis where the norm of each vector = 1.  
↳ is a maximal orthonormal set.

↳ to obtain an orthonormal set from an orthogonal set:  
divide each vector by its norm.

→ Let  $\Gamma = \{v_1, v_2, \dots, v_k\}$  be an orthogonal set.

Let  $B$  be an orthonormal set.

$$B = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots, \frac{v_k}{\|v_k\|} \right\}$$

example: orthogonal basis =  $\{(1, 3), (-3, 1)\}$

$$\text{orthonormal basis} = \left\{ \frac{1}{\sqrt{10}}(1, 3), \frac{1}{\sqrt{10}}(-3, 1) \right\}$$

↳ orthonormal basis is a very useful way to find the coefficients that make any vector  $v$ .

Explanation → Suppose  $\Gamma = \{v_1, v_2, \dots, v_n\}$  is an orthonormal basis of an inner product space  $V$  and let  $v \in V$ .

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

To find  $c_1, c_2, \dots, c_n$ , we need to write a system of linear equations and solve. But since  $\Gamma$  is an orthonormal basis, we can use the inner product to find  $c_i = \langle v, v_i \rangle$

$$\text{Proof} \rightarrow \langle v, v_i \rangle = \langle c_1 v_1 + c_2 v_2 + \dots + c_n v_n, v_i \rangle$$

$$\text{because they're orthogonal} = c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle$$

$$\text{because they're orthonormal} = c_i \langle v_i, v_i \rangle$$

$$= c_i$$

## → Projection of a vector to a subspace

orthonormal

↪  $V \rightarrow$  inner product space.  $W \subseteq V \rightarrow$  a subspace. Projection of  $v \in V$  onto  $W$  is the vector in  $W$ , denoted by  $\text{proj}_W(v)$ , computed as:

Find an orthonormal basis  $\{v_1, v_2, \dots, v_n\}$  for  $W$

$$\text{Define } \text{proj}_W(v) = \sum_{i=1}^n \langle v, v_i \rangle v_i$$

example:  $V = \mathbb{R}^2$ ,  $W = \langle (3,1) \rangle$ ,  $v = (1,3)$

$\left\{ \frac{1}{\sqrt{10}}(3,1) \right\}$  is the orthonormal basis for  $W$

$$\begin{aligned} \text{proj}_W(v) &= \left\langle v, \frac{1}{\sqrt{10}}(3,1) \right\rangle \frac{1}{\sqrt{10}}(3,1) = \langle (1,3), (3,1) \rangle \frac{1}{\sqrt{10}}(3,1) \\ &= \frac{6}{\sqrt{10}}(3,1) = (1.8, 0.6) \end{aligned}$$

orthogonal

↪ if  $\{v_1, v_2, \dots, v_n\}$  is an orthogonal basis for a subspace  $W$ , then  $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots, \frac{v_n}{\|v_n\|} \right\}$

is an orthonormal basis for  $W$  and hence:

$$\text{proj}_W(v) = \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} v_i = \sum_{i=1}^n \text{proj}_{v_i}(v)$$

example:  $W$  is 2-dim subspace of  $V = \mathbb{R}^3$  spanned by orthogonal vectors  $v_1 = (1, 2, 1)$  and  $v_2 = (1, -1, 1)$ . What is projection of  $v = (-2, 2, 2)$  on  $W$ ?

$$\text{proj}_{v_1}(v) = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \frac{4}{6} (1, 2, 1) = \frac{2}{3} (1, 2, 1)$$

$$\text{proj}_{v_2}(v) = \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = -\frac{2}{3} (1, -1, 1)$$

$$\text{proj}_W(v) = \text{proj}_{v_1}(v) + \text{proj}_{v_2}(v)$$

$$= \frac{2}{3} (1, 2, 1) - \frac{2}{3} (1, -1, 1)$$

$$= (0, 2, 0)$$

## ↪ Properties of the projection $P_W$

$$\textcircled{1} \quad P_W(v) = v \quad \forall v \in W$$

$$\textcircled{2} \quad \text{Range}(P_W) = W$$

$$\textcircled{3} \quad W^\perp = \{v \mid v \in V, \text{ such that } \langle v, w \rangle = 0 \quad \forall w \in W\} \text{ is the null space of } P_W.$$

$$\textcircled{4} \quad P_W^2 = P_W$$

$$\textcircled{5} \quad \|P_W(v)\| \leq \|v\|$$

→ Projection matrix iff:  $A^T = A$  and  $A^2 = A$

→ Gram-Schmidt process

↪ Let  $V$  be an inner product space with a basis  $\{u_1, u_2, \dots, u_n\}$ .

Define orthogonal basis  $\{v_1, v_2, \dots, v_n\}$  and orthonormal basis  $\{w_1, w_2, \dots, w_n\}$  as:

$$v_1 = u_1 ; w_1 = v_1 / \|v_1\|$$

$$v_2 = u_2 - \langle u_2, w_1 \rangle w_1 ; w_2 = v_2 / \|v_2\|$$

$$v_3 = u_3 - \langle u_3, w_1 \rangle w_1 - \langle u_3, w_2 \rangle w_2 ; w_3 = v_3 / \|v_3\|$$

$$\vdots \quad \vdots \quad \vdots$$

$$v_n = u_n - \langle u_n, w_1 \rangle w_1 - \langle u_n, w_2 \rangle w_2 - \dots - \langle u_n, w_{n-1} \rangle w_{n-1} ; w_n = v_n / \|v_n\|$$

## → Orthogonal Transformations

↪ Let  $T: V \rightarrow V$ .  $T$  is said to be orthogonal transformation if:

$$\langle T_v, T_w \rangle = \langle v, w \rangle \quad \forall v, w \in V$$

↪ Preserves the lengths and angles.

→ Orthogonal matrix iff:  $AA^T = A^TA = I$

→ Rotation → preserves the lengths and angles

↳ Rotation in  $\mathbb{R}^2$

$$T_\theta(1, 0) = (\cos \theta, \sin \theta)$$

$$T_\theta(0, 1) = (-\sin \theta, \cos \theta)$$

$$\text{matrix} \rightarrow R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

↳ Rotation in  $\mathbb{R}^3$

along the Z-axis  $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

along the Y-axis  $R_\theta = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$

along the X-axis  $R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$



## WEEK 9

→ Scalar-valued multivariable functions

↳ a function  $f: D \rightarrow \mathbb{R}$ , where  $D$  is a domain in  $\mathbb{R}^n$  where  $n > 1$ .  
example ⇒ linear transformations, polynomial function

→ Vector-valued multivariable functions

↳  $f: D \rightarrow \mathbb{R}^m$ , where  $D$  is a domain in  $\mathbb{R}^n$  where  $m, n > 1$ .  
example of  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ :  $f(u, v, w) = (u^2 + v^2, w)$

→ Partial Derivatives

↳ If  $a$  is a point in  $\mathbb{R}^n$ , then an open ball of radius  $r$  around  $a$  is the set defined as:

$$\{u \in \mathbb{R}^n \mid \|u-a\| < r\}$$

↳  $e_1, e_2, \dots, e_n$  is the standard ordered basis of  $\mathbb{R}^n$ .

↳ Let  $f(u_1, u_2, \dots, u_n)$  be a function on a domain  $D$  in  $\mathbb{R}^n$  containing a point  $a$  and an open ball around it.

The rate of change of  $f$  at  $a$  w.r.t. the variable  $u_i$  is

$$\lim_{h \rightarrow 0} \frac{f(a + he_i) - f(a)}{h}$$

↓  
Let  $g(h) = f(a + he_i)$ , then

$$\lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h}$$

example:  $f(u, y) = u + y$

$$\begin{aligned}\frac{\partial f}{\partial u}(u, y) &= \lim_{h \rightarrow 0} \frac{f(u, y) + h(1, 0) - f(u, y)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(u+h, y) - f(u, y)}{h} = \lim_{h \rightarrow 0} \frac{u+h+y - u-y}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} = 1\end{aligned}$$

↳ To calculate partial derivative w.r.t.  $u_i$ , think of  $f$  only as a function of  $u_i$  and treat all other variables as constants.

example :

$$\rightarrow f(u, y, z) = uy + yz + zu$$

$$\frac{\partial f}{\partial u}(u, y, z) = y + 0 + z = y + z$$

$$\frac{\partial f}{\partial y}(u, y, z) = u + z$$

$$\rightarrow f(u, y) = \sin(uy)$$

$$\frac{\partial f}{\partial u} = y \cdot \cos(uy)$$

$$\frac{\partial f}{\partial y} = u \cdot \cos(uy)$$

### → Directional derivative

↳ to calculate rate of change in a particular direction at a point

→ choose a unit vector  $u \in \mathbb{R}^n$  in the direction we want and compute:

$$\lim_{h \rightarrow 0} \frac{f(\underline{a} + hu) - f(\underline{a})}{h}$$

examples:

→ rate of change of  $f(u, y) = u + y$  at  $(0, 0)$  in the direction of line  $u=y$ .  
 $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$\lim_{h \rightarrow 0} \frac{f(0 + h\frac{1}{\sqrt{2}}, 0 + h\frac{1}{\sqrt{2}}) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{2h\frac{1}{\sqrt{2}}}{h} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

→ rate of change of  $f(u, y, z) = uy + yz + zu$  at  $(1, 2, 3)$  in the direction of vector  $(4, 3, 0)$ .

$$u = (\frac{4}{5}, \frac{3}{5}, 0)$$

$$\lim_{h \rightarrow 0} \frac{f(1 + \frac{4h}{5}, 2 + \frac{3h}{5}, 3) - f(1, 2, 3)}{h} = \frac{8 + 3 + 9 + 12}{5} = \frac{32}{5}$$

↳ directional derivative of  $f$  in the direction of the unit vector  $u$  is the function  $f_u(\underline{a})$  and defined as

$$f_u(\underline{a}) = \lim_{h \rightarrow 0} \frac{f(\underline{a} + hu) - f(\underline{a})}{h}$$

Properties:

$$\textcircled{1} \quad (cf + g)_u(\underline{a}) = cf_u(\underline{a}) + g_u(\underline{a})$$

$$\textcircled{2} \quad (fg)_u(\underline{a}) = f_u(\underline{a})g(\underline{a}) + f(\underline{a})g_u(\underline{a})$$

$$\textcircled{3} \quad (\frac{f}{g})_u(\underline{a}) = \frac{f_u(\underline{a})g(\underline{a}) - f(\underline{a})g_u(\underline{a})}{g(\underline{a})^2}$$

example:

$$\rightarrow f(u, y) = u + y \quad u = (u_1, u_2)$$

$$f_u(u, y) = \lim_{h \rightarrow 0} \frac{f(u + hu_1, y + hu_2) - f(u, y)}{h} = \lim_{h \rightarrow 0} \frac{u + hu_1 + y + hu_2 - (u + y)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{hu_1 + hu_2}{h} = u_1 + u_2$$

## → Limits of scalar-valued multivariable function at a point

Let  $f \rightarrow$  scalar-valued multi-variable function and  $\alpha$  be a point such that there exists a sequence in the domain that converges to  $\alpha$ . If there exists  $L, L \in \mathbb{R}$  such that  $f(\alpha_n) \rightarrow L$  for all sequences  $\alpha_n$  such that  $\alpha_n \rightarrow \alpha$ , then limit of  $f$  at  $\alpha$  exists and equals  $L$ .

$$\lim_{\alpha \rightarrow \alpha} f(\alpha) = L$$

Rules: If  $\lim_{\alpha \rightarrow \alpha} f(\alpha) = F$  and  $\lim_{\alpha \rightarrow \alpha} g(\alpha) = G$ , and  $c \in \mathbb{R}$

$$\rightarrow \lim_{\alpha \rightarrow \alpha} (cf + g)(\alpha) = cF + G$$

$$\rightarrow \lim_{\alpha \rightarrow \alpha} (fg)(\alpha) = FG$$

$$\rightarrow \lim_{\alpha \rightarrow \alpha} (f/g)(\alpha) = \frac{F}{G}$$

Composition → If  $\lim_{\alpha \rightarrow \alpha} f(\alpha) = F$ ,  $\lim_{\alpha \rightarrow F} g(\alpha) = L$

$$\lim_{\alpha \rightarrow \alpha} (g \circ f)(\alpha) = L$$

Sandwich → If  $\lim_{\alpha \rightarrow \alpha} f(\alpha) = L$ ,  $\lim_{\alpha \rightarrow \alpha} g(\alpha) = L$  and  $f(\alpha) \leq h(\alpha) \leq g(\alpha)$

$$\lim_{\alpha \rightarrow \alpha} h(\alpha) = L$$

example:

$$\rightarrow h(u, y, z) = u^2y^3 + y^3z^2 + xyz$$

$$\lim_{(u, y, z) \rightarrow (1, 2, 3)} h(u, y, z) = (1)^2(2)^3 + (2)^3(3)^2 + (1)(2)(3) = 86$$

## → Limit of a vector-valued function at a point

$f: D \rightarrow \mathbb{R}^m$ ,  $D \subset \mathbb{R}^n$

If  $f_i$  is  $i^{th}$  component of the function, then  $f_i$  is a scalar-valued function  $D \rightarrow \mathbb{R}$

example:

$$\rightarrow \lim_{\alpha \rightarrow (1, 2)} (u^2y + y^3, e^{uy}, \frac{u^2-1}{y^2-2}) = \left( \lim_{\alpha \rightarrow (1, 2)} u^2y + y^3, \lim_{\alpha \rightarrow (1, 2)} e^{uy}, \lim_{\alpha \rightarrow (1, 2)} \frac{u^2-1}{y^2-2} \right)$$

$$= (10, e^2, 0)$$

→ To find the limit of a function, find the limit of that function at the chosen point, along different curves and if they match, the limit exists.

Tip: check along the axes, check along  $u_1 = u_2 = u_3 = \dots = u_n$  in  $\mathbb{R}^n$

## → Directional derivative in terms of the gradient

→  $\nabla f$  at  $\alpha$  is the vector  $(f_{x_1}(\alpha), f_{x_2}(\alpha), \dots, f_{x_n}(\alpha))$  in  $\mathbb{R}^n$   $\nabla f(\alpha) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$  in  $\mathbb{R}^n$

example:

$$\rightarrow f(u, y) = \sin(uy) \quad \frac{\partial f}{\partial u} = y \cdot \cos(uy) \quad \frac{\partial f}{\partial y} = u \cdot \cos(uy)$$

$$\nabla f(u, y) = (y \cdot \cos(uy), u \cdot \cos(uy)) \quad ; \quad \nabla f(\pi, 1) = (\cos(\pi), \pi \cos(\pi)) \\ = (-1, -\pi)$$

Properties

$$\rightarrow \nabla(f+g)(\alpha) = (\nabla f(\alpha) + \nabla g(\alpha))$$

$$\rightarrow \nabla(fg)(\alpha) = \nabla f(\alpha)g(\alpha) + f(\alpha)\nabla g(\alpha)$$

$$\rightarrow \nabla(f/g)(\alpha) = \frac{\nabla f(\alpha)g(\alpha) - f(\alpha)\nabla g(\alpha)}{(g(\alpha))^2}$$

→ Direction derivative at point  $\alpha$  in the direction of  $u$ :  $f_u(\alpha) = \nabla f(\alpha) \cdot u$



# WEEK 10

→ Minimizing directional derivative

↳  $f(x_1, x_2, \dots, x_n)$  in  $\mathbb{R}^n$ .  $\nabla f$  is continuous around the point  $\mathbf{x}$ . Find the direction where  $f$  is the steepest negative, i.e.  $\nabla f$  is minimised.

Let  $f_u$  be the directional derivative in the direction of  $u$ .

$$f_u = \nabla f(\mathbf{x}) \cdot u \\ = \|\nabla f(\mathbf{x})\| \cdot \|u\| \cdot \cos(\theta), \text{ where } \theta = \text{angle b/w } \nabla f(\mathbf{x}) \text{ and } u \\ = \|\nabla f(\mathbf{x})\| \cdot \cos(\theta)$$

$$f_u \text{ is minimised when } u = -\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}$$

$$f_u = -\|\nabla f(\mathbf{x})\| \text{ is minimised}$$

Steepest ascent → when  $u = \nabla f / \|\nabla f\|$

Steepest descent → when  $u = -\nabla f / \|\nabla f\|$

No change → when  $u$  is orthogonal to  $\nabla f$

→ Tangent for scalar-valued multi-variable function

↳ Tangent for  $f(x, y)$

→  $f(x, y)$  on domain  $D$  in  $\mathbb{R}^2$ . Consider a line  $L$  in  $D$  passing through  $\mathbf{x}$  and restrict  $f$  to  $L$ . If the line  $L$  is in the direction of the unit vector  $u$ , then tangent will be the line with slope  $f_u(\mathbf{x})$  and passing through  $(\mathbf{x}, f(\mathbf{x}))$ .

↳ Equation of tangent line

→  $u = (u_1, u_2) \rightarrow$  unit vector on  $L$ .  $f_u(a, b) \rightarrow$  directional derivative at  $(a, b)$

$$= \underbrace{(a, b, f(a, b)) + t(u_1, u_2, f_u(a, b))}_{\substack{\rightarrow \text{ passing through} \\ \rightarrow \text{ the point } f(a, b)}} + \underbrace{t(u_1, u_2, f_u(a, b))}_{\substack{\rightarrow \text{ rate of the same as } f(a, b) \\ \rightarrow \text{ in the direction of } u}}$$

$$\begin{aligned} \text{Parametric Form: } & \quad x(t) = a + tu_1, \\ & \quad y(t) = b + tu_2, \\ & \quad z(t) = f(a, b) + t f_u(a, b) \end{aligned}$$

→ examples

$$\rightarrow f(x, y) = xy; \text{ tangent at } (1, 1) \text{ in the direction of } (3, 4)$$

$$u = \left( \frac{3}{5}, \frac{4}{5} \right) \quad \nabla f(x, y) = (y, x) \quad \nabla f_u(1, 1) = \frac{7}{5}$$

$$\text{tangent: } (x(t), y(t), z(t)) = (1, 1, 1) + t \left( \frac{3}{5}, \frac{4}{5}, \frac{7}{5} \right)$$

$$= (1 + \frac{3t}{5}, 1 + \frac{4t}{5}, 1 + \frac{7t}{5})$$

$$\rightarrow f(x, y) = \sin(xy); \text{ tangent at } (\pi, 1) \text{ in direction } (1, 2)$$

$$u = \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) \quad \nabla f(x, y) = (y \cdot \cos(xy), x \cdot \cos(xy))$$

$$\nabla f_u(\pi, 1) = (-1, -\pi) \cdot \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) = -\frac{1+2\pi}{\sqrt{5}}$$

$$(x(t), y(t), z(t)) = (\pi, 1, 0) + t \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{1+2\pi}{\sqrt{5}} \right) = \left( \pi + \frac{t}{\sqrt{5}}, 1 + \frac{2t}{\sqrt{5}}, -\frac{1+2\pi}{\sqrt{5}} \right)$$

Equation of tangent line:  $f(x, y)$  at point  $(a, b)$  in the direction of  $(u_1, u_2)$

$$x(t) = a + tu_1, \quad y(t) = b + tu_2, \quad z(t) = f(a, b) + t \left( \frac{\partial f}{\partial x}(a, b)u_1 + \frac{\partial f}{\partial y}(a, b)u_2 \right)$$

$$z(t) - f(a, b) = \frac{\partial f}{\partial u}(a, b)(u(t) - a) + \frac{\partial f}{\partial y}(a, b)(y(t) - b)$$

↳ Tangent Plane:

$$z = f(a, b) + \frac{\partial f}{\partial u}(a, b)(u-a) + \frac{\partial f}{\partial y}(a, b)(y-b)$$

→ example:

$$\rightarrow f(u, y) = u + y ; \text{ tangent at } (1, 1)$$

$$\frac{\partial f}{\partial u}(1, 1) = \frac{\partial f}{\partial y} = 1 \quad f(1, 1) = 2$$

$$z = 2 + u - 1 + y - 1 \Rightarrow z = u + y$$

basis for tangent plane =  $\{(1, 0, 1), (0, 1, 1)\}$

$$\rightarrow f(u, y) = \sin(uy) ; \text{ tangent at } (1, 0)$$

$$f(1, 0) = 0 \quad \frac{\partial f}{\partial u}(1, 0) = 0 \quad \frac{\partial f}{\partial y}(1, 0) = 1$$

$$z = f(1, 0) + \frac{\partial f}{\partial u}(1, 0)(u-1) + \frac{\partial f}{\partial y}(1, 0)(y-0)$$

$$z = y$$

basis for plane =  $\{(0, 1, 1)\}$

→ let  $f(\underline{x})$  on domain  $D$  in  $\mathbb{R}^n$ , and  $\nabla f$  continuous on  $D$ .

equation of tangent hyperplane to  $f$  at  $(\underline{a})$ :

$$z = f(\underline{a}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\underline{a})(x_i - a_i)$$

$$z = f(\underline{a}) + \nabla f(\underline{a}) \cdot (\underline{x} - \underline{a})$$

→ Critical points for multivariable functions

↳ for all  $\underline{a}$  for which  $\nabla f(\underline{a}) = 0$

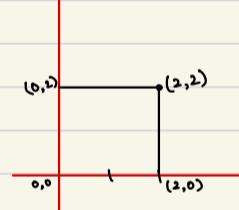
↳ Absolute maxima/minima of  $f(u, y) = u^3 + y^3 - 3u - 3y^2 + 1$  over the square with vertices  $(0, 0), (2, 0), (2, 2), (0, 2)$

$$\nabla f(u, y) = (3u^2 - 3, 3y^2 - 6y)$$

$$\text{critical points: } 3u^2 - 3 = 0, 3y^2 - 6y = 0$$

$$\rightarrow u = \pm 1; y = 0 \text{ or } 2$$

points to consider:  $(1, 0), (1, 2), (-1, 0), (-1, 2), (2, 0), (0, 2)$



$$\text{endpoints: } f(u, 0) = u^3 - 3u + 1$$

$$f'(u, 0) = 0 \Rightarrow 3u^2 - 3 = 0 \Rightarrow (1, 0)$$

$$f(u, 2) = u^3 - 3u - 3$$

$$f'(u, 2) = 0 \Rightarrow 3u^2 - 3 = 0$$

$$f(0, y) = y^3 - 3y^2 + 1$$

$$f'(0, y) = 0 \Rightarrow 3y^2 - 6y = 0 \Rightarrow 0 \text{ or } 2$$

$$f(2, y) = y^3 - 3y^2 + 3$$

$$f(2, y) = 0 \Rightarrow y = 0 \text{ or } 2$$

$$f(1, 0) = -1 \quad f(0, 2) = -3$$

$$f(1, 2) = -5 \quad f(2, 2) = -1$$

$$f(-1, 0) = 1 \quad f(2, 0) = 3$$

Global max at  $(2, 0)$

Global min at  $(1, 2)$



# WEEK 11

## → Second Order Partial Derivatives

↳ Second order partial derivatives of  $f$  are the partial derivatives of the partial derivatives.

→ If the second order mixed partial derivatives  $f_{xy}$  and  $f_{yx}$  are continuous in an open ball around  $\mathbf{z}$ , then  $f_{xy}(\mathbf{z}) = f_{yx}(\mathbf{z})$

→ example

$$\rightarrow f(u, y) = \sin(uy)$$

$$\frac{\partial f}{\partial u} = y \cos(uy) \quad \frac{\partial f}{\partial y} = u \cos(uy)$$

$$\frac{\partial^2 f}{\partial u^2} = -y^2 \sin(uy) \quad \frac{\partial^2 f}{\partial y^2} = -u \sin(uy)$$

$$\frac{\partial^2 f}{\partial y \partial u} = \cos(uy) - uy \sin(uy)$$

$$\frac{\partial^2 f}{\partial u \partial y} = \cos(uy) - uy \sin(uy)$$

## ↳ Higher order partial derivatives

Hessian matrix : Let  $f(u_1, u_2, \dots, u_n)$  be defined on  $D$  in  $\mathbb{R}^n$

The hessian matrix of  $f$  is defined as:

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial u_1^2} & \frac{\partial^2 f}{\partial u_1 \partial u_2} & \dots & \frac{\partial^2 f}{\partial u_1 \partial u_n} \\ \frac{\partial^2 f}{\partial u_2 \partial u_1} & \frac{\partial^2 f}{\partial u_2^2} & \dots & \frac{\partial^2 f}{\partial u_2 \partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial u_n \partial u_1} & \frac{\partial^2 f}{\partial u_n \partial u_2} & \dots & \frac{\partial^2 f}{\partial u_n^2} \end{bmatrix}$$

→ Using  $H_f$  to classify critical points.

If  $\det(H_f(\mathbf{z})) > 0$        $f_{xx}(\mathbf{z}) > 0 \rightarrow$  then  $\mathbf{z}$  — local minima  
 $f_{xx}(\mathbf{z}) < 0 \rightarrow$  then  $\mathbf{z}$  — local maxima

If  $\det(H_f(\mathbf{z})) < 0 \rightarrow$  then  $\mathbf{z}$  — saddle point

If  $\det(H_f(\mathbf{z})) = 0 \rightarrow$  then inconclusive

example :  $f(u, y) = \sin(uy)$

$$\nabla f(u, y) = (y \cos(uy), u \cos(uy))$$

critical points :  $(0, 0)$ , or any point where  $\cos(uy) = 0$

$$H_f(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \det(H_f(0, 0)) = -1 \quad \therefore (0, 0) \text{ is a saddle point}$$

## ↳ Classification of critical points of $f(u, y, z)$

→ If  $\det(H_f(\mathbf{z})) > 0$  AND  $(f_{xx} f_{yy} - f_{xy}^2)(\mathbf{z}) > 0$  AND  $f_{xx}(\mathbf{z}) > 0$   $\Rightarrow$  local minimum

→ If  $\det(H_f(\mathbf{z})) < 0$  AND  $(f_{xx} f_{yy} - f_{xy}^2)(\mathbf{z}) > 0$  AND  $f_{xx}(\mathbf{z}) < 0$   $\Rightarrow$  local maximum

→ If  $\det(H_f(\mathbf{z})) = 0$  AND cases 1 or 2 do not occur  $\Rightarrow$  saddle point

→ If  $\det(H_f(\mathbf{z})) = 0$   $\Rightarrow$  inconclusive





## WEEK 1 GA

①  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$   $A - xI = \begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix}$

$$\det(A - xI) = (a-x)(d-x) - bc$$

$$= ad - ax - dx + x^2 - bc$$

$$= \underbrace{ad - bc}_{\det(A)} - \underbrace{x(a+d)}_{\text{trace}(A)} + \underbrace{x^2}_{c^2}$$

②  $3A = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$   $\det(3A) = 9ad - 9bc = (3)^2 \det(A)$

$$3A = \begin{bmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$$

$$|3A| = 3^3 |A| = \boxed{27|A|}$$

③  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$   $I + A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$   $5A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$   $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$

$$5A + I = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$$

$$(I + A)^3 - (5A + I) = mA$$

$$\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \quad m = 8$$

④  $A = \begin{bmatrix} 20 & 30 & 40 \\ 8 & 16 & 24 \\ 8 & 10 & 12 \end{bmatrix}$   $\det(A) = 20 \det \begin{bmatrix} 16 & 24 \\ 10 & 12 \end{bmatrix} - 30 \det \begin{bmatrix} 8 & 24 \\ 8 & 12 \end{bmatrix} + 40 \det \begin{bmatrix} 8 & 16 \\ 8 & 10 \end{bmatrix}$   
 $= 20(-48) - 30(-96) + 40(-48) = -960 + 2880 - 1920 = \boxed{0}$

⑤  $\det(A) = 3 \quad \det(B) = 3 \quad \det(B^{-1}) = \frac{1}{3}$

$$\det(3A^2B^{-1}) = (3)^3 \cdot (\det(A))^2 \cdot \left(\frac{1}{3}\right) = 9(3)^2 = \boxed{81}$$

⑥  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$   $A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$   $A^3 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix}$   $A^6 = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 729 & 729 & 729 \\ 729 & 729 & 729 \\ 729 & 729 & 729 \end{bmatrix} \quad \text{sum of diagonals} = \boxed{2187}$$

⑦  $A = \begin{bmatrix} 12 & 19 & 26 \\ 17 & 24 & 31 \\ 22 & 29 & 36 \end{bmatrix}$   $\det(A) = 12 \det \begin{bmatrix} 24 & 31 \\ 29 & 36 \end{bmatrix} - 19 \det \begin{bmatrix} 17 & 31 \\ 22 & 36 \end{bmatrix} + 26 \det \begin{bmatrix} 17 & 24 \\ 22 & 29 \end{bmatrix}$   
 $= 12(-35) - 19(-70) + 26(-35) = \boxed{0}$

⑧  $30u_1 + 20u_2 + 25u_3 = 670 - \textcircled{1}$   
 $20u_1 + 35u_2 + 25u_3 = 730 - \textcircled{2}$   
 $20u_1 + 10u_2 + 15u_3 = 400 - \textcircled{3}$

$$\textcircled{1} - 2 \cdot \textcircled{3} \Rightarrow 30u_1 + 20u_2 + 25u_3 - 20u_1 - 20u_2 - 30u_3 = 670 - 400 - 800$$

$$= -10u_1 - 5u_3 = -130$$

$$\textcircled{2} - \textcircled{3} \Rightarrow 20u_1 + 15u_2 + 10u_3 = 330$$

$$-20u_1 - 10u_2 - 15u_3 = -400$$

$$= 25u_2 + 10u_3 = 330 - \textcircled{4}$$

$$10u_1 + 5u_3 = 130 - \textcircled{5}$$

$$2 \cdot \textcircled{5} = 20u_1 + 10u_3 = 260 \Rightarrow 20u_1 = 260 - 10u_3$$

$$u_1 + u_2 + u_3 = 9 + 10 + 8 = \boxed{27}$$

$$260 - 10u_3 + 10u_2 + 15u_3 = 400$$

$$\Rightarrow 10u_2 + 5u_3 = 140 - \textcircled{6}$$

⑩  $A = \begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix}$

$$\textcircled{4} - 2 \cdot \textcircled{6} \Rightarrow 0 \ 25 \ 10 \ 330$$

$$0 \ -20 \ -10 \ -280$$

$$= 5u_2 = 50 \quad \boxed{u_2 = 10}$$

$$250 + 10u_3 = 330 \Rightarrow \boxed{u_3 = 8}$$

$$10u_1 + 40 = 130 \Rightarrow \boxed{u_1 = 9}$$

## WEEK 2 GA

$$\textcircled{1} \quad P(1) = -45 \quad \begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 4 & 2 & 1 & -19 \\ 9 & 3 & 1 & 3 \end{array} \quad R_2 - 4R_1 \quad \begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 0 & -2 & -3 & 161 \\ 0 & -6 & -8 & 408 \end{array} \quad R_2 - \frac{1}{2}R_3 \quad \begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & -6 & -8 & 408 \end{array} \quad R_3 + 6R_2 \quad \begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{array}$$

$$\begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{array} \quad R_1 - R_2 \quad \begin{array}{r|rrr} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 1 & -75 \end{array} \quad R_2 - R_3 \quad \begin{array}{r|rrr} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 32 \\ 0 & 0 & 1 & -75 \end{array}$$

$$P(n) = -2n^2 + 32n - 75 \quad P'(n) = -4n + 32$$

$$\frac{-32}{-4} = n \quad n=8$$

$P(8)$  is peak  
 $P(8) = 53$

$$\textcircled{4} \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & c & d \end{array} \quad R_2 - 2R_1 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array} \quad R_3 + 3R_2 \quad \begin{array}{r|rrr} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \quad \text{comic book} = ?$$

$\frac{\text{if } c=7}{d=4000} \rightarrow$

$\text{if } c=2$   
 $d=4000$

$$\begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 4000 \end{array} \quad R_2 - 2R_1 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -2 & 0 \end{array}$$

$\text{if } c=7$   
 $d=3000$

$$\begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 7 & 3000 \end{array} \quad R_2 - 2R_1 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & -1000 \end{array}$$

$\text{if } c=2$   
 $d=3000$

$$\begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 3000 \end{array} \quad R_2 - 2R_1 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 1000 \end{array} \quad R_3 - 3R_2 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 1000 \end{array} \Rightarrow \begin{pmatrix} 400 \\ 200 \\ 200 \end{pmatrix}$$

$$\textcircled{6} \quad (-\frac{1}{2}) + (-\frac{1}{2}) + \alpha/6 = -1 \Rightarrow \alpha = 0$$

$$(-\frac{1}{3}) + b/6 = 0 \Rightarrow \frac{b}{6} = \frac{1}{3} \Rightarrow b = 2$$

$$(\frac{1}{2}) + (-\frac{1}{6}) + c/6 = 0 \Rightarrow \frac{c-1}{6} = -\frac{1}{2} \Rightarrow c-1 = -3 \Rightarrow c = -2$$

$$\alpha + b + c = 0$$

$$\textcircled{7} \quad \begin{array}{r|rrr} 1 & 3k & 3k+4 & 61 \\ 1 & k+4 & 4k+2 & 65 \\ 1 & 2k+2 & 3k+4 & 66 \end{array} \quad R_2 - R_1 \quad \begin{array}{r|rrr} 1 & 3k & 3k+4 & 61 \\ 0 & -2k+4 & k-2 & 4 \\ 0 & -k+2 & 0 & 5 \end{array} \quad R_3 \leftrightarrow R_2 \quad \begin{array}{r|rrr} 1 & 3 & 7 & 61 \\ 0 & 1 & 0 & 5 \\ 0 & 2 & -1 & 4 \end{array} \quad -R_3 + 2R_2 \quad \boxed{\begin{array}{r|rr} 1 & 3 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \left| \begin{array}{r} 61 \\ 5 \\ 6 \end{array} \right.}$$

$\frac{4k+2-3k-4}{k-2} = 1$   
 $k=1$

$$\textcircled{8} \quad \begin{array}{r|rrr} 2 & 3 & 5 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 44 \end{array} \quad R_1 - R_2 \quad \begin{array}{r|rrr} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 1 & 1 & 2 & 44 \end{array} \quad R_3 - R_1 \quad \begin{array}{r|rrr} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 0 & 44 \end{array} \rightarrow \text{shows no solution}$$

$$\textcircled{9} \quad \begin{array}{r|rrr} 1 & 3 & 0 & 0 \\ 4 & 1 & 5 & 34 \\ 2 & 2 & 7 & 32 \\ 3 & 9 & 0 & 0 \end{array} \quad R_2 - 4R_1 \quad \begin{array}{r|rrr} 1 & 3 & 0 & 0 \\ 0 & -11 & 5 & 34 \end{array}$$

$$R_3 - 2R_1 \quad \begin{array}{r|rrr} 1 & 3 & 0 & 0 \\ 0 & -4 & 7 & 32 \end{array}$$

$$R_4 - 3R_1 \quad \begin{array}{r|rrr} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \quad R_4 \text{ shows } \det(A) = 0$$

$$\textcircled{10} \quad \begin{array}{r|rrr} 7 & 2 & 1 & 8 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{array} \quad R_1 + 2R_3 \quad \begin{array}{r|rrr} 1 & 10 & -3 & 24 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{array} \quad R_3 + 3R_1 \quad \begin{array}{r|rrr} 1 & 10 & -3 & 24 \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 34 & -11 & 80 \end{array} \quad R_1 - 10R_2 \quad \begin{array}{r|rrr} 1 & 0 & y_3 & \frac{32}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & y_3 & \frac{104}{3} \end{array}$$

$$R_3 - 3R_2 \quad \begin{array}{r|rrr} 1 & 0 & y_3 & \frac{32}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 104 \end{array}$$

$$x = -24$$

$$y = 36 \quad x+y+z = 116$$

$$z = 104$$

$$\textcircled{11} \quad A = \begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix} \quad A^T A = \begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix} \begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 20 & 4 & 16 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{bmatrix} \quad R_{1/4} \quad \begin{bmatrix} 1 & 5 & 1 & 4 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{bmatrix} \quad R_2 - 20R_1 \quad \begin{bmatrix} 1 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\textcircled{12} \quad \begin{array}{cccc|ccccc|ccccc|ccccc} 4 & 2 & 1 & 4 & | & 1 & y_2 & y_4 & 1 & | & 1 & y_2 & y_4 & 1 & | & 1 & y_2 & y_4 & 1 \\ 9 & 3 & 1 & 6 & | & 9 & 3 & 1 & 6 & | & R_2 - 9R_1 & 0 & -3/2 & -5/4 & -3 & | & 0 & 3 & 5/2 & 6 & | & 0 & 3 & 5/2 & 6 \\ 16 & 4 & 1 & 10 & | & 16 & 4 & 1 & 10 & | & R_3 - 16R_1 & 0 & -4 & -3 & -6 & | & 0 & 4 & 3 & 6 & | & 0 & 0 & y_3 & 2 & | & 0 & 0 & 1 & 6 \end{array}$$

$$\begin{array}{l} a = 1 \\ b = -3 \\ c = 6 \end{array}$$

$$\textcircled{13} \quad \left| \begin{array}{cccc|cccccc} 2 & 3 & 0 & 0 & | & 1000 & R_{1/2} & 1 & 3/2 & 0 & 0 & 500 \\ 0 & 3 & 2 & 0 & | & 900 & R_4 - R_1 & 0 & 3 & 2 & 0 & 900 \\ 0 & 0 & 2 & 1 & | & C & & 0 & 0 & 2 & 1 & C \\ 2 & 0 & 0 & 1 & | & 400 & & 0 & -3 & 0 & 0 & -600 \end{array} \right. \quad \left| \begin{array}{cccc|cccccc} 1 & 3/2 & 0 & 0 & 500 & R_{2/3} & 1 & 3/2 & 0 & 0 & 500 \\ 0 & 1 & 3/2 & 0 & 300 & R_4 - 3R_2 & 0 & 1 & 3/2 & 0 & 300 \\ 0 & 0 & 2 & 1 & C & R_1 - \frac{3}{2}R_2 & 0 & 0 & 2 & 1 & C \\ 0 & 3 & 0 & 0 & 600 & & 0 & 0 & 2 & 0 & 300 \end{array} \right. \quad \left| \begin{array}{cccc|cccccc} 1 & 0 & -1 & 0 & 50 & R_{3/2} & 1 & 0 & -1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 200 & R_2 - \frac{1}{3}R_4 & 0 & 1 & 0 & 0 & 200 \\ 0 & 0 & 1 & 1/2 & 150 & -R_4 - 2R_3 & 0 & 0 & 1 & 1/2 & 150 \\ 0 & 0 & 0 & 1 & -300 & & 0 & 0 & 0 & 1 & -300 \end{array} \right.$$

$$(W \cdot N) u_1 = 200$$

$$(E \cdot N) u_2 = 200$$

$$(E \cdot S) u_3 = 150$$

$$(W \cdot S) u_4 = 0$$

$$C = 300$$

$$\min \rightarrow u_4 = 0 \rightarrow u_1 = 200, u_2 = 200, u_3 = 150$$

$$\max \rightarrow u_4 = 300 \rightarrow u_1 = 50, u_2 = 300, u_3 = 0$$

$$(W \cdot N) \rightarrow 400 \rightarrow 100$$

$$(E \cdot N) \rightarrow 900 \rightarrow 600$$

$$(E \cdot S) \rightarrow 300 \rightarrow 0$$

$$(W \cdot S) \rightarrow 300 \rightarrow 0$$

— X — X — X — X —

## WEEK 3 GA

$$\begin{aligned} -u &= -1 \\ -2y &= 1 \\ y &= 1 \\ u &= -1 \end{aligned}$$

$$\textcircled{2} \quad P = 30 \quad C = 11 \quad F = 53 \quad G = 213 \quad \text{Type A} = Ba, mi, al \quad \text{Type B} = Ap, mi, al$$

$$A = \begin{bmatrix} 2 & 4 & 6 & 30 \\ 3 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 5 & 100 & 1 & 213 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 6 & 30 \\ 2 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ b & 100 & 1 & 213 \end{bmatrix}$$

$$\textcircled{5} \quad A = \begin{bmatrix} c & 1 & -1 \\ -1 & 0 & -3 \\ -2 & -1 & C \end{bmatrix} \quad \begin{aligned} -1(-c-6) + -3c - 1 &= 0 \\ c+6 - 3c - 1 &= 0 \\ -2c + 5 &= 0 \\ c &= 2.5 \end{aligned}$$

$$\textcircled{6} \quad A = \begin{bmatrix} 7 & 5 & 11 \\ 9 & 6 & 15 \\ 2 & 2 & C \end{bmatrix} \quad \begin{aligned} \det(A) &= 7(6C-30) - 5(9C-30) + 11(18-12) \\ &= 42C - 210 - 45C + 150 + 66 \\ 0 &= -3C + 6 \\ C &= 2 \end{aligned}$$

$$\textcircled{8} \quad (1, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0), (0, 1, 0)$$

$$\textcircled{9} \quad X_0 = (y_3, y_3, y_3) \quad X_1 = X_0 \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} = (\frac{1}{2}, \frac{1}{2}, 0) \quad X_2 = X_1 \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} = (\frac{3}{4}, \frac{1}{4}, 0)$$

$$\textcircled{10} \quad X_0 = (a, b, 1-a-b) \quad X_1 = (a+b/2, 1-a-\frac{b}{2}, 0)$$

$$X_1 = (a, b, 1-a-b) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left( \frac{2a+b}{2}, \frac{b+2-2a-2b}{2}, 0 \right) = \left( \frac{2a+b}{2}, \frac{2-2a-b}{2}, 0 \right)$$

$$X_2 = \left( \frac{2a+b}{2}, \frac{2-2a-b}{2}, 0 \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left( \frac{4a+2b+2-2a-b}{4}, \frac{2-2a-b}{4}, 0 \right) = \left( \frac{2a+b+2}{4}, \frac{2-2a-b}{4}, 0 \right)$$

$$\begin{bmatrix} a & \frac{2a+b+2}{4} \\ b & \frac{2-2a-b}{4} \\ 1-a-b & 0 \end{bmatrix}$$

$$\frac{4au + 2ay + by + 2y}{4} = 0 \quad \textcircled{1} \Rightarrow y(2a+b+2) = -4au$$

$$\frac{4bu + 2y - 2ay - by}{4} = 0 \quad \textcircled{2} \Rightarrow y(2-2a-b) = -4bu$$

$$u - au - bu = 0 \Rightarrow u - bu = au$$

$$\textcircled{1} - \textcircled{2} \Rightarrow \frac{4au - 4bu + 4}{4} = 0 \Rightarrow au - bu + 1 = 0$$

$$u - bu - bu + 1 = 0$$

$$u - 2bu + 1 = 0$$

— X — X — X —

## MOCK WEEK 1 - 2

$$\textcircled{3} \quad \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} \quad \lambda = 6$$

$$\textcircled{4} \quad a+4 = 2a+2 \Rightarrow 2=a \\ 3b=b+2 \Rightarrow 2b=2 \Rightarrow b=1 \quad a+2b=4$$

$$\textcircled{5} \quad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & | & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & | & 0 \end{bmatrix} \xrightarrow{\substack{R_2 + R_1 \\ R_1 \cdot \sqrt{2}}} \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & \frac{2}{\sqrt{2}} & | & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \quad \text{adj}(A) = C^T \quad \det(A) = \frac{1}{2} + \frac{1}{2} = 1$$

$$C^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = A^T = A^{-1}$$

$$\textcircled{7} \quad \det(A) = b+a$$

$$\textcircled{8} \quad \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \\ -1 & 6 & 7 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ -1 & 6 & 7 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\textcircled{13} \quad \begin{bmatrix} 2 & 0.5 & 1 & 80 \\ 4 & 1 & 2 & 100 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_1/2}} \begin{bmatrix} 1 & 0.25 & 0.5 & 40 \\ 0 & 0 & 0 & -60 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 0.25 & 0.5 & 40 \\ 0 & 0 & 0 & -60 \\ 0 & 0.5 & 1 & 0 \end{bmatrix}$$

$$\textcircled{14} \quad \begin{bmatrix} 2 & 2 & 1 & 80 \\ 4 & 1 & 2 & d \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_1/2}} \begin{bmatrix} 1 & 1 & \frac{1}{2} & 40 \\ 0 & -3 & 0 & d-160 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow{\substack{R_3 - 2R_1 \\ -R_2/3}} \begin{bmatrix} 1 & 1 & \frac{1}{2} & 40 \\ 0 & 1 & 0 & -\frac{d}{3} + \frac{160}{3} \\ 0 & -1 & 1 & 0 \end{bmatrix} \quad u_2 = u_3$$

$$\textcircled{15} \quad \begin{bmatrix} 2 & 3 & 1 & 80 \\ 4 & 1 & 2 & 100 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_1/2}} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 40 \\ 0 & -5 & 0 & -60 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow{\substack{-R_2/5 \\ R_3 - 2R_1}} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 40 \\ 0 & 1 & 0 & 12 \\ 0 & -2 & 1 & 0 \end{bmatrix} \quad u_2 = 12 \\ u_3 = 2u_2 = 24$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 24 \end{bmatrix}$$

$$u_1 = 40 - \frac{3}{2}u_2 - \frac{1}{2}u_3 \Rightarrow u_1 = 40 - \frac{3}{2}(12) - \frac{1}{2}(24) \\ = 40 - 18 - 12 = 10$$

## WEEK 4 GA

$$\textcircled{4} \quad \text{if } A_{4 \times 4}, \text{ then } A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \text{ then } \text{rank}(A) = 4$$

$$\text{if } A_{2021 \times 2021}, \text{ then } \text{rank}(A) = 2021$$

$$\textcircled{5} \quad 8u_1 + 10u_5 = 0 \quad 9u_2 + 6u_4 = 0$$

$$\textcircled{6} \quad A = \begin{bmatrix} 2 & -4 & 1 & -9 \\ 1 & -3 & 1 & -1 \\ 3 & -7 & 2 & -6 \end{bmatrix} \xrightarrow{\substack{R_1 \leftrightarrow R_2 \\ -R_2 - 2R_1 \\ -R_3 - 2R_1}} \begin{bmatrix} 1 & -3 & 1 & -1 \\ 0 & 10 & 1 & 11 \\ 0 & 16 & 1 & 9 \end{bmatrix}$$

$$\textcircled{7} \quad A = \begin{bmatrix} -10 & 0 & -3 \\ -5 & -4 & -5 \\ -8 & 8 & -3 \end{bmatrix} \xrightarrow{\substack{-R_1/10 \\ R_2 + 5R_1 \\ R_3 + 8R_1}} \begin{bmatrix} 1 & 0 & \frac{3}{10} \\ 0 & -4 & -\frac{35}{10} \\ 0 & 8 & -\frac{6}{10} \end{bmatrix} \xrightarrow{\substack{-R_2/4 \\ R_3 + 2R_2}} \begin{bmatrix} 1 & 0 & \frac{3}{10} \\ 0 & 1 & \frac{35}{40} \\ 0 & 0 & -\frac{7}{10} \end{bmatrix} \quad \left| \quad B = \begin{bmatrix} -10 & -5 & -8 \\ 0 & -4 & 8 \\ -3 & -5 & -3 \end{bmatrix} \xrightarrow{\substack{-R_1/10 \\ R_2 + 3R_1 \\ -R_2/4}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{4}{5} \\ 0 & 1 & -2 \\ 0 & -\frac{7}{2} & -\frac{3}{5} \end{bmatrix} \right.$$

$$\textcircled{8} \quad v_1 = (-1, 0, 1) \quad v_2 = (2, 1, -1)$$

$$v_1 + v_2 = (1, 1, 0) \quad 3v_1 + 0v_2 = (-3, 0, 3)$$

$$(5, 2, -3), (8, 5, -1), (-\cancel{5}/6, \cancel{5}/6, \cancel{5}/6), (-2, 3, 3) \\ \cancel{b=2} \quad \cancel{b=5} \quad \cancel{b=\frac{5}{6}} \quad \cancel{b=3} \quad 2b-a \neq -2 \\ \alpha-b=-3 \quad \alpha-b=-4 \quad \alpha-b=\cancel{\frac{5}{6}} \quad \alpha-b=3 \\ \alpha=-1 \quad \alpha=1 \quad \alpha=\cancel{\frac{10}{6}} \quad \alpha=b \\ 2b-a=5 \quad 2b-a \neq 8 \quad 2b-a \neq \cancel{-\frac{5}{6}}$$

$$\textcircled{9} \quad v_1 = (1, 0, 1) \quad v_2 = (0, 1, -1) \quad \alpha v_1 + b v_2 = (20, -13, c) \quad \alpha = 20 \quad b = -13 \quad \alpha-b=c \\ 20 - (-13) = 33$$

$$\textcircled{10} \quad (u, y, z) = (0, q, 1), \text{ then } 1 = 0 + 8q \quad q = 1/8 \quad p = yq$$

$$q/p = \frac{1}{72} \quad \frac{1}{q/p} = 72$$

$$\textcircled{11} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & -a_{11}-a_{22}-a_{33} \end{bmatrix}$$

## WEEK 5 GrA

$$\begin{aligned} \textcircled{2} \quad u_1C + 2u_1D + 2u_2H + u_2O &= bu_3C + 12u_3H + bu_3D + 2u_4O \\ u_1C + (2u_1+u_2)O + 2u_2H &= bu_3C + 12u_3H + (bu_3+2u_4)O \end{aligned}$$

$$\begin{aligned} u_1 &= bu_3 \\ 2u_1 + u_2 &= bu_3 + 2u_4 \\ u_2 &= bu_3 \implies u_1 = u_2 = bu_3 \\ 3u_1 &= bu_3 + 2u_4 \Rightarrow 2u_1 = 2u_4 \Rightarrow u_1 = u_4 \\ u_1 &= u_2 = u_4 = bu_3 \end{aligned}$$

$$\textcircled{3} \quad T(v) = \begin{bmatrix} -18u - 11y \\ -15y \end{bmatrix} \quad \text{for some } v, \quad T(v) = \begin{bmatrix} z \\ w \end{bmatrix} \quad z = -18u - 11y \rightarrow z = -18u + \frac{11w}{15} \rightarrow -\frac{z + 11w}{18} = u \\ w = -15y \rightarrow y = -\frac{w}{15} \end{math>$$

$$\textcircled{4} \quad \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1/3 \\ R_2 - R_1 \\ R_3 - R_1 \\ R_2/5 \end{array} \begin{bmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_2 + R_3 \\ R_3/3 \\ R_1 - R_3/3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{nullity}(A) = 1$$

$$\begin{array}{c} 0 \ 1 \ 0 \ 0 \\ 0 \ 1 \ 0 \ 0 \\ 1 \ 0 \ 0 \ 0 \\ 1 \ 0 \ 0 \ 0 \\ 0 \ 0 \ 1 \ 0 \end{array}$$


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## WEEK 6 GrA

$$\begin{aligned} \textcircled{2} \quad T(1,0,1) &= (1,0,0) \\ T(0,1,0) &= (0,1,0) \\ T(0,0,1) &= (1,0,1) \end{aligned}$$

$$(1,0,0) = (1,0,1) - (0,0,1) \Rightarrow T(1,0,0) = (1,0,0) - (1,0,1) = (0,0,-1)$$

$$(0,1,0) = (0,1,1) \Rightarrow T(0,1,0) = (0,1,0)$$

$$(0,0,1) = (0,0,1) \Rightarrow T(0,0,1) = (1,0,1)$$

$$\textcircled{3} \quad \text{i)} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ y \\ 1-u-y \end{bmatrix} = \begin{bmatrix} u \\ y \\ -u-y+3 \end{bmatrix} \quad \begin{array}{l} z = -u-y+3 \\ 3 = u+y+z \end{array} \quad \text{i)} \rightarrow \text{b})$$

$$\text{ii)} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ y \\ 1-u-y \end{bmatrix} = \begin{bmatrix} u \\ y \\ -1+u+y \end{bmatrix} \quad \begin{array}{l} z = -1+u+y \\ 1 = u+y-z \end{array} \quad \text{ii)} \rightarrow \text{d})$$

$$\text{iii)} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ 1-u \end{bmatrix} = \begin{bmatrix} u \\ -1+u \end{bmatrix} \quad \begin{array}{l} y = -1+u \\ 1 = u-y \end{array} \quad \text{iii)} \rightarrow \text{a}) \rightarrow 3$$

$$\text{iv)} \quad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ 1-u \end{bmatrix} = \begin{bmatrix} u \\ 2u+1-u \end{bmatrix} = \begin{bmatrix} u \\ u+1 \end{bmatrix} \quad \begin{array}{l} y = u+1 \\ u-y = -1 \end{array} \quad \text{iv)} \rightarrow \text{c}) \rightarrow 1$$

$$\textcircled{4} \quad S\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad S\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad S\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad S\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\textcircled{5} \quad \left[ \begin{array}{ccc|c} 1 & 0 & r \\ 0 & \alpha & r \\ 0 & \beta & \frac{\gamma\beta}{\alpha} \end{array} \right] \xrightarrow{R_3/\beta} \left[ \begin{array}{ccc|c} 1 & 0 & r \\ 0 & \alpha & r \\ 0 & 1 & \frac{\gamma}{\alpha} \end{array} \right] \xrightarrow{R_3 \cdot \alpha} \left[ \begin{array}{ccc|c} 1 & 0 & r \\ 0 & \alpha & r \\ 0 & 0 & \gamma \end{array} \right] \xrightarrow{R_3 - R_2} \left[ \begin{array}{ccc|c} 1 & 0 & r \\ 0 & \alpha & r \\ 0 & 0 & 0 \end{array} \right] \quad \text{Rank}(T) = 2$$

$$T(u, v, w) = (2u - z, 3v - 2w, z, 0)$$

$$\left[ \begin{array}{ccc} 2 & 0 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 + R_3} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right] \quad \text{ind. vars. } \rightarrow u_1, u_2, u_4$$

stand. basis. of image =  $\{(1, 0, 0), (0, 0, 1), (0, 1, 0)\}$

$$\textcircled{6} \quad T(2, 3, 1) = 6 \quad T(3, 4, 2) = 8 \quad T(5, 5, 3) = 10$$

$$\left[ \begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 3 & 4 & 2 & 8 \\ 5 & 5 & 3 & 10 \end{array} \right] \quad \text{matrix of } T = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}$$

$$\textcircled{11} \quad T(u, v, w) = (2u + 7v, 4w)$$

$$\begin{aligned} T(1, 0, 0) &= (2, 0) = a(1, 0) + b(0, 1) \\ T(0, 1, 0) &= (7, 0) = c(1, 0) + d(0, 1) \\ T(0, 0, 1) &= (0, 4) = e(1, 0) + f(0, 1) \end{aligned}$$

$$\text{matrix of } T = A = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} 2 & 7 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 7/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Basis of null space} = \{-\frac{7}{2}, 1, 0\}$$

$$\text{Im}(T) = 2(1, 0) + 0(0, 1) = (2, 0) \\ 0(1, 0) + 4(0, 1) = (0, 4)$$

$$\text{Basis for Range}(T) = \{(2, 0), (0, 4)\}$$



## WEEK 7 GRA

$$\textcircled{1} \quad \text{i) matrix} \rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{iv) } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\textcircled{2} \quad T(1, 0) = (2, 1) = 2(1, 0) + 1(0, 1) \\ T(0, 1) = (1, 1) = 1(1, 0) + 1(0, 1)$$

$$S(1, 0) = (1, 1) = 1(1, 0) + 1(0, 1) \\ S(0, 1) = (1, 2) = 1(1, 0) + 2(0, 1)$$

$$T \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\text{if } c=1, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$PT = SP$$

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2a+b & a+b \\ 2c+d & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$$

$$\begin{bmatrix} a+b-c & a+d \\ d-a & c-b-d \end{bmatrix} = 0$$

$$a+b-c = 0$$

$$a+d = 0$$

$$d-a = 0$$

$$c-b-d = 0$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$$\text{if } c=2, \quad S = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\textcircled{3} \quad \langle (3, 1), \gamma \rangle = 4, \quad 3\gamma_1 + \gamma_2 = 4 \quad \langle (6, 2), \gamma \rangle = 8, \quad 6\gamma_1 + 2\gamma_2 = 8$$

$$\begin{bmatrix} 3 & 1 & | & 4 \\ 6 & 2 & | & 8 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 3 & 1 & | & 4 \\ 0 & 0 & | & 0 \end{bmatrix} \quad \text{if } \gamma_1 = t, \text{ then } \gamma_2 = 4 - 3t$$

$$\textcircled{5} \quad L = \{0, 1, 1\} + v, \text{ where } v = \{(1, 1, 0), (1, 0, 1)\}$$

$$T(1, 0, 1) = (0, 0, 0)$$

$$T(1, 1, 0) = (1, 0, 0)$$

$$\textcircled{6} \quad \cos(\theta) = \frac{u \cdot v}{\|u\| \times \|v\|} \quad \|u\| = \sqrt{74} \quad \|v\| = \sqrt{41}$$

$$u \cdot v = 0$$

$$\textcircled{7} \quad c_1(1, 2, 0) + c_2(2, -1, 0) + c_3(0, 0, 2) = (u, y, \frac{3u+y}{5})$$

$$c_1 + 2c_2 = u$$

$$2c_1 - c_2 = y$$

$$2c_3 = \frac{3u+y}{5} \rightarrow \frac{3(c_1+2c_2) + 2c_1 - c_2}{10} = c_3$$

$$\frac{3c_1 + 6c_2 + 2c_1 - c_2}{10} = c_3 \Rightarrow \frac{c_1 + c_2}{2} = c_3 \Rightarrow c_3 = 2$$

$$\textcircled{8} \quad \langle (a, b), (9, 5) \rangle_1 = 9a - 5a - 9b + 20b = 119 \Rightarrow 4a + 11b = 119$$

$$\left[ \begin{array}{cc|c} 4 & 11 & 119 \\ 27 & 10 & 225 \end{array} \right]$$

$$\langle (a, b), (9, 5) \rangle_2 = 27a + 10b = 225$$

$$a=5 \quad b=9 \quad a+2b=23$$

$$\textcircled{9} \quad \|v\| = \sqrt{(x-1b)^2 + 5}$$

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## WEEK 8 GA

$$\textcircled{10} \quad b) \left\{ \frac{1}{\sqrt{2}}(1, 0, -1), \frac{1}{\sqrt{2}}(-1, 0, -1) \right\} \quad \|v_1\| = \langle (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}) \rangle = 1 \quad \langle v_1, v_2 \rangle = 0$$

$$\|v_2\| = \langle (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}) \rangle = 1$$

$$c) \left\{ (2, 3, 4), (-1, 2, -1), (0, 4, -3) \right\} \rightarrow i) \quad \langle v_1, v_2 \rangle = 0 \quad \langle v_1, v_3 \rangle = 0$$

$$d) \left\{ (2, 3, 4), (-1, 2, -1), (11, 2, -7) \right\} \rightarrow ii) \quad \langle v_1, v_2 \rangle = 0 \quad \langle v_1, v_3 \rangle = 0$$

$$\langle v_2, v_3 \rangle = 11$$

$$\langle v_2, v_3 \rangle = 0$$

$$\textcircled{11} \quad v = (1, 2, 2) \quad R_\theta \text{ along } x\text{-axis} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1+2\sin(60) & 1-2\sin(60) \\ 0 & 1+1+2\sin(60)-2\sin(60) & 1+1+1+2\sin(60)-2\sin(60) = 3 \end{bmatrix}$$

$$\textcircled{12} \quad \|C\| = \max(|\sqrt{7}u| + |\sqrt{7}y|, |\sqrt{7}u| + |\sqrt{7}y|) = |\sqrt{7}y| + |\sqrt{7}u|$$

$$\begin{bmatrix} u & \sqrt{16}u \\ -\sqrt{6}y & y \end{bmatrix} \begin{bmatrix} u & -\sqrt{6}y \\ \sqrt{6}u & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 17u^2 & 0 \\ 0 & 17y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$17u^2 = 1$$

$$u = \frac{1}{\sqrt{17}} = y$$

$$\text{So, } |\sqrt{17}y| + |\sqrt{17}u| = 1+1 = 2$$

$$\textcircled{13} \quad \begin{bmatrix} \frac{a}{\sqrt{a^2+7^2}} & \frac{19}{\sqrt{b^2+19^2}} \\ \frac{7}{\sqrt{a^2+7^2}} & \frac{b}{\sqrt{b^2+19^2}} \end{bmatrix} \begin{bmatrix} \frac{a}{\sqrt{a^2+7^2}} & \frac{7}{\sqrt{a^2+7^2}} \\ \frac{19}{\sqrt{b^2+19^2}} & \frac{b}{\sqrt{b^2+19^2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{a^2}{a^2+7^2} + \frac{19^2}{b^2+19^2} = 1$$

$$= a^2(b^2+19^2) + 19^2(a^2+7^2) = (a^2+7^2)(b^2+19^2)$$

$$a^2b^2 + 19^2a^2 + 19^2a^2 + (19^2)(7^2) = a^2b^2 + 19^2a^2 + 7^2b^2 + (19^2)(7^2)$$

$$19^2a^2 = 7^2b^2$$

$$(19a)^2 - (7b)^2 = 0$$

$$(19a+7b)(19a-7b) = 0$$

⑧ basis for  $S_1 = \left\{ \frac{\sqrt{2}}{2} (1, 1, 0) \right\}$   
 $S_2 = \left\{ \frac{\sqrt{2}}{2} (1, -1, 0) \right\}$   
 $S_3 = \left\{ \frac{\sqrt{5}}{5} (2, 1, 0) \right\}$   
 $S_4 = \left\{ \frac{\sqrt{2}}{2} (1, 0, -1), \frac{\sqrt{2}}{2} (0, 1, -1) \right\}$

$v = (3, 4, 5)$

$\text{Proj}_{S_1}(v) = \langle (3, 4, 5), (1, 1, 0) \rangle \frac{1}{2} (1, 1, 0)$   
 $= \frac{7}{2} (1, 1, 0)$

$\text{Proj}_{S_2}(v) = \langle (3, 4, 5), (1, -1, 0) \rangle \frac{1}{2} (1, -1, 0)$   
 $= -\frac{1}{2} (1, -1, 0)$

$\text{Proj}_{S_3}(v) = \langle (3, 4, 5), (1, 0, -1) \rangle \frac{1}{2} (1, 0, -1)$   
 $+ \langle (3, 4, 5), (0, 1, -1) \rangle \frac{1}{2} (0, 1, -1)$   
 $= -1(1, 0, -1) - \frac{1}{2} (0, 1, -1)$   
 $= (-1, -\frac{1}{2}, \frac{3}{2})$

$a + 2b + 3c = -1 - 2 + \frac{9}{2} = \frac{-2 - 4 + 9}{2} = \frac{3}{2}$

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## WEEK 9 GRA

⑤  $(y_n, 0)$   $f(u, y) = \lim_{n \rightarrow \infty} \frac{y_n}{x_n} = 1$   
if  $y = 2u$ ,  $\lim_{(u, y) \rightarrow (0, 0)} f(u, y) = \frac{3u}{\sqrt{u^2 + y^2}} = \frac{3}{5}$

⑦  $\frac{\partial f}{\partial u}(u, y) = (u^2 + y^2)^{\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{(u^2 + y^2)^{\frac{1}{2}}} \cdot 2u = \frac{u}{\sqrt{u^2 + y^2}}$   
 $\frac{\partial f}{\partial y}(u, y) = \frac{y}{\sqrt{u^2 + y^2}}$

⑧  $f$  along  $y=u$   $(y_n, y_n)$   $\lim_{n \rightarrow \infty} \frac{(y_n)^3}{3(y_n)^2} = \frac{1}{3}$   
 $f$  along  $x$ -axis  $(y_n, 0)$   $\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^4 = 0 \rightarrow$  same along  $y$ -axis

⑨  $y=2u$   $f(u, y) = \frac{2u^2}{2u^2 + 4u^2} = \frac{2u^2}{6u^2} = \frac{1}{3}$

⑩  $\frac{\partial f_1}{\partial u}(-3, 1) = 1 \quad \frac{\partial f_1}{\partial y}(-3, 1) = u+2y = -1$   
 $\frac{\partial f_2}{\partial u}(-3, 1) = 1+y = 2 \quad \frac{\partial f_2}{\partial y}(-3, 1) = u = -3$   
 $\det(A) = (-3) - (-2) = -1$

⑪  $u=(0, 1) \quad \nabla f_{(0,1)}(4, 4) = \left( \frac{\partial f}{\partial u}, \frac{\partial f}{\partial y} \right) \cdot (0, 1) \quad \frac{\partial f}{\partial u}(4, 4) = 2ky^3 = 512 \quad \frac{\partial f}{\partial y}(4, 4) = 3u^2y^2 = 768$   
 $= 0(512) + 1(768) = 768$

⑫  $\frac{\partial f}{\partial u} = 2u + y \quad \frac{\partial f}{\partial y} = 2y + u \quad \text{if } \frac{\partial f}{\partial u} = \frac{\partial f}{\partial y}, \quad 2u + y = 2y + u \Rightarrow u = y$

⑬  $f(cu, cy) = c^2 u^2 + c^2 uy + c^2 y^2 = c^2 f(u, y)$   
 $\lim_{(u, y) \rightarrow (0, 0)} u^2 + uy + y^2 =$

⑭  $\nabla f(u, y) = (2u+uy, 2y+u)$   
 $(2u+uy, 2y+u) \left( \frac{1}{\sqrt{1+u^2}}, \frac{u}{\sqrt{1+u^2}} \right) = \frac{2u+y}{\sqrt{1+u^2}} + \frac{2my+mu}{\sqrt{1+u^2}} = \frac{1}{\sqrt{1+u^2}} \left( (2+m)u + (2m+1)y \right)$   
 $2k-l = 2(2+m) - (2m+1) = 4+2m-2m-1 = 3$

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## Mock Quiz 2

$$\textcircled{3} \quad \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & -1 & 0 & 0 \end{array} \right] \xrightarrow{R_3 - R_1} \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{array} \right] \xrightarrow{R_2 + R_3} \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{array} \right] \quad \dim(V_1) = 2$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 3 & 6 \end{array} \right] \xrightarrow{R_3 - 3R_1} \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \dim(V_2) = 1$$

$$\textcircled{4} \quad T(-2, 1, 0) = (1, 0, -2)$$

$$T(-2, 0, 1) = (0, 0, 0)$$

$$\textcircled{5} \quad S\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\textcircled{6} \quad T = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

basis for  $\ker(T) = (-4, 0, 1)$

$$T(1, 0, 0) = (2, 4)$$

$$T(0, 1, 0) = (3, 9)$$

$$T(0, 0, 1) = (8, 16)$$

$$\begin{bmatrix} 2 & 3 & 8 \\ 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} = 10(1, 0) + 20(0, 1)$$

$$\text{``} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 25 \end{bmatrix} = 11(1, 0) + 25(0, 1)$$

$$\text{``} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix} = 5(1, 0) + 13(0, 1)$$

$$A = \begin{bmatrix} 10 & 11 & 5 \\ 20 & 25 & 13 \end{bmatrix}$$

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## WEEK 10 GRA

$$\textcircled{1} \quad \nabla f(u, y) = (6uy - 6u, 3u^2 + 3y^2 - 6y) \quad \begin{aligned} 6uy - 6u &= 0 \\ 3u^2 + 3y^2 - 6y &= 0 \end{aligned} \quad \begin{aligned} 6uy = 6u \\ y = 1 \end{aligned} \quad \begin{aligned} 3u^2 + 3 - 6 &= 0 \\ u^2 &= 1 \\ u &= \pm 1 \end{aligned}$$

$$\textcircled{2} \quad \nabla f(u, y) = \left( \frac{y^6 - u^6 y^2}{(u^2 + y^2)^2}, \frac{2u^5 y - 2u y^5}{(u^2 + y^2)^2} \right)$$

$$\textcircled{3} \quad \nabla f(u, y) = (ye^u - u/2, e^u - y/2) \quad f(0, 1) = 3/4$$

$$\nabla f(0, 1) = (1, y_2)$$

$$\begin{aligned} z &= f(0, 1) + (u - 0) + y_2(y - 1) \\ z &= \frac{3}{4} + u + \frac{1}{2} - \frac{3}{4} = u + \frac{1}{2}y + \frac{1}{4} \end{aligned}$$

$$\textcircled{4} \quad \nabla f(u, y) = \left( \frac{2u}{\sqrt{u^2 + uy}}, \frac{4}{\sqrt{u^2 + uy}} \right) \quad \nabla f(-2, 3) = (-1, 1)$$

$$f_u(-2, 3) = -u_1 + u_2$$

$$u_1 = u_2$$

$$\begin{aligned} u_1 &= u_2 \\ \sqrt{2u_1^2} &= 1 \end{aligned}$$

$$u_1 = \frac{1}{\sqrt{2}}$$

$$u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

$$\textcircled{5} \quad \nabla f(x, y) = (ye^x + xye^x, xe^x)$$

$$f(1, 1) = e \quad \nabla f(1, 1) = (2e, e)$$

$$L_f(x, y) = e + 2e(x-1) + e(y-1) \\ = e + 2xe - 2e + ey - e \\ = 2xe + ye - 2e \quad L_f(1.2, 0.9) = 1.3e$$

$$\textcircled{6} \quad f(x, y, z) = (e^{-(x+y+z)})$$

$$\nabla f(x, y, z) = (-e^{-x-y-z}, -e^{-x-y-z}, -e^{-x-y-z})$$

$$\nabla f(0, 0, 0) = (0, 0, 0)$$

$$\textcircled{7} \quad f(1, 0, 0) = e^{-1} \quad u = (\frac{3}{10}, \frac{6}{10}, 0)$$

$$\nabla f(1, 0, 0) = (-2e^{-1}, 0, 0)$$

$$\nabla f_u(1, 0, 0) = (-2e^{-1}, 0, 0) \cdot (\frac{3}{10}, \frac{6}{10}, 0) = -\frac{16e^{-1}}{10} \quad 10Ae = -16$$

$$\textcircled{8} \quad \nabla f(a, b, c) = \left( \frac{-2a}{e^{a^2+b^2+c^2}}, \frac{-2b}{e^{a^2+b^2+c^2}}, \frac{-2c}{e^{a^2+b^2+c^2}} \right) \times \frac{e^{a^2+b^2+c^2}}{2\sqrt{a^2+b^2+c^2}}$$

$$\textcircled{9} \quad f_1(x, y) = y^6 e^{xy}$$

$$\nabla f_1(x, y) = (y^6 e^{xy}, 6y^5 e^{xy})$$

$$\nabla f_1(0, 0) = (0, 0)$$

$$\textcircled{10} \quad \nabla f_2(x, y) = (-12x+6, -4y)$$

$$\nabla f_2(0, 0) = (6, 0) \quad u = (1, 0)$$

$$f_u(0, 0) = (6, 0) \cdot (1, 0) = 6$$

$$\textcircled{11} \quad \nabla f_3(x, y) = (2\sin(6x) + 12x\cos(6x), 6\cos(2y) - 12y\sin(2y))$$

$$\nabla f_3(0, 0) = (0, 6) \quad u = (0, 1)$$

$$f_u(0, 0) = (0, 6) \cdot (0, 1) = 6$$

$$\textcircled{12} \quad f(x, y) = 5 - x^5 - y^5 \quad f(1, 1) = 3$$

$$\nabla f(x, y) = (-4x^4, -4y^4) \quad \nabla f(1, 1) = (-4, -4)$$

$$z = 3 + (-4)(u-1) + (-4)(v-1)$$

$$= -4u - 4v + 11$$



## WEEK 11 GRA

$$\textcircled{1} \quad \frac{\partial F}{\partial u} = 2xz + y^5 e^{xy} + yz \cos(yxz)$$

$$\frac{\partial^2 F}{\partial u^2} = 2z + y^5 e^{xy} - y^2 z^2 \sin(yxz)$$

$$\textcircled{2} \quad f = x^6 - y^2 e^x - 2z^2 e^x$$

$$\nabla f = (2xe^x + x^2 e^x - y^2 e^x - 2z^2 e^x, -2ye^x, -4ze^x) \\ = e^x (2x + x^2 - y^2 - 2z^2, -2y, -4z)$$

critical points =  $\{(0, 0, 0), (-2, 0, 0)\}$

$$2x + x^2 = 0$$

$$x(2+x) = 0 \Rightarrow 2+x=0 \Rightarrow x=-2$$

$$\textcircled{3} \quad \nabla f = \frac{1}{\sqrt{x^2+y^2}} (u, v)$$

$$\textcircled{4} \quad xyz = 1000 \Rightarrow x = 1000/yz \quad \text{S.A.} = 2\left(\frac{1000}{2} + yz + \frac{1000}{y}\right)$$

$$\nabla f(y, z) = (z - 1000/y^2, y - 1000/z^2)$$

$$z = \frac{1000}{y^2} \quad y = \frac{1000}{z^2} \quad y^2 = \frac{(1000)^2}{z^2}$$

$$z = \frac{z^4}{1000}$$

$$z^3 = 1000$$

$$z = 10$$

$$\textcircled{5} \quad \nabla f(x, y, z) = (3x^2 - 4, 9y^2 - 6, 3z^2 - 4)$$

$$H_f = \begin{bmatrix} bu & 0 & 0 \\ 0 & 18y & 0 \\ 0 & 0 & bz \end{bmatrix}$$

$$H_f(0, 1, 1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & b \end{bmatrix}$$

$$⑥ \nabla f(x, y) = (e^x - e^{x+y}, e^y - e^{x+y})$$

$$H_f = \begin{bmatrix} e^x - e^{x+y} & -e^{x+y} \\ -e^{x+y} & e^y - e^{x+y} \end{bmatrix} \quad H_f(1,1) = \begin{bmatrix} e - e^2 & -e^2 \\ -e^2 & e - e^2 \end{bmatrix} \quad H_f(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$⑦ \nabla f(x, y) = (-4x + 2y + 12, 2x - 2y - 4)$$

critical points = {(4, 2)}

$$H_f(4,2) = \begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix} \quad \det = 8 - 4 = 4 \quad (4,2) \rightarrow \max$$

$$f(4,2) = 13$$

$$⑧ \nabla f(x, y) = (2(x-y)\sin(y-x) - \frac{(x-y)^2 \cos(y-x)}{x^2}, -2(x-y)\sin(y-x) + \frac{(x-y)^2 \cos(y-x)}{y^2})$$

$$⑨ P(x, y) = F(x, y) - G(x, y)$$

$$P(x, y) = -2x^2 + 2xy - y^2 + 11x - 3y - 7$$

$$\nabla P(x, y) = (-4x + 2y + 11, 2x - 2y - 3)$$

critical point = {(4, 5/2)}

$$P(4, 5/2) = -2(4)^2 + 2(4)(5/2) - (5/2)^2 + 11(4) - 3(5/2) - 7 \\ = -32 + 20 - \frac{25}{4} + 44 - \frac{15}{2} - 7 = 25 - 6.25 - 7.5 = 11.25$$

