MATH 2

WEEK 1

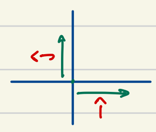
→ Vectors

↳ can be thought of as a list.
↳ can be in a row or a column form.

↳ **Addition of vectors:**
Add the corresponding entries. For e.g. → $(3,5) + (2,4) = (5,9)$

↳ **Scalar multiplication:**
e.g. → $2(8,8,10,5) = (16,16,20,10)$

↳ **Visualisation of a vector**
Point $(a,b) \equiv$ Vector $(a,b) \equiv a\hat{i} + b\hat{j}$ where, $\hat{i} \rightarrow$ one unit in x-axis
$\hat{j} \rightarrow$ one unit " y-axis



e.g.: Point $(-1,-1) = -\hat{i} - \hat{j}$

↳ Vectors in $\mathbb{R}^n$ are lists with n real entries.

→ Matrices

↳ rectangular array of numbers
↳ (rows × columns) ; eg → $\begin{bmatrix} 5 & 7 & 10 \\ 3 & 5 & 2 \end{bmatrix}$ is a 2×3 matrix.
↳ (1,2) ith entry → 7

↳ square matrix → $N \times N$

↳ **Diagnol matrix** → all entrices are 0 except the diagnol

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

↳ **Scalar matrix** → all entries have the same value

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

↳ **Indentity matrix** → denoted by 'I'; scalar matrix with values = 1

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↳ **Addition of matrices** → must be of the same size: $\begin{bmatrix} 1 & 0 \\ 5 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 8 & 3 \\ 7 & 4 \end{bmatrix}$

↳ **Scalar multiplication** → multiply each number with the scalar.

↳ **Matrix multiplication** → $A \times B = C$; $C[i,j] = \sum_{k=1}^{n} A[i,k] \times B[k,j]$
↳ no. of columns in first matrix $\underline{\text{must}}$ = no. of rows in 2^{nd} matrix
↳ $A_{m \times n} \times B_{n \times p} = (AB)_{m \times p}$

$$(AB)_{ij} = \Sigma_{k=1}^{n} A_{ik} B_{jk}$$

↳ Scalar multiplication is the same as multiplication by the scalar matrix
ex. → $\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

↳ Properties:
1. $(A+B)+C = A+(B+C)$
2. $(AB)C = A(BC)$
3. $A+B = B+A$
4. $AB \neq BA$
5. $\lambda(A+B) = \lambda A + \lambda B$
6. $\lambda(AB) = (\lambda A)B = A(\lambda B)$
7. $A(B+C) = AB + AC$
8. $(A+B)C = AC + BC$

→ System of Linear Equations

↳ collection of one or more linear equations involving the same set of variables.

↳ system of m linear equation with n variables :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_n \end{aligned}$$

↳ system of lin. eq. in matrix form :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \dots \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$$\underset{\text{coefficient matrix}}{A}\ \underset{\text{column of variables}}{x} = \underset{\text{column of resulting values}}{b}$$, where

A = $m \times n$ matrix

x = column vector with n entries

b = column vector with m entries

↳ Solutions to a system of lin. eq. :

1. Infinite solution
2. Single unique solution
3. No solution

→ Determinant

↳ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad - bc$

e.g. $A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$ $\quad \det(A) = 20 - 18 = 2$

↳ $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad \det(A) = a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

↳ Determinant of Identity matrix $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

→ $\det(I_2) = 1$

→ $\det(I_3) = 1$

↳ Determinant of a product of matrices

↳ $\det(AB) = \det(A) \cdot \det(B)$; $\det(ABC) = \det(A) \cdot \det(B) \cdot \det(C)$

↳ Determinant of the inverse of a matrix

$A A^{-1} = I \Rightarrow \det(A A^{-1}) = \det(I)$

↳ $\det(A^{-1}) = \dfrac{1}{\det(A)}$

↳ Switching rows

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\det(\tilde{A}) = cb - ad = -(ad - bc) = -\det(A)$

$\det(\tilde{A}) = -\det(A)$

↳ Add multiple of a row/column to another row/column

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$

$\det(\tilde{A}) = (a+tc)d - (b+td)c = ad + tcd - bc - tcd$

$\det(\tilde{A}) = \det(A)$

↳ Scalar multiplication of a row / column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} ta & b \\ tc & d \end{bmatrix}$$

$$\det(\tilde{A}) = t \cdot \det(A)$$

↳ Upper | Lower triangle matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{bmatrix}$$ upper triangle matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 0 \\ 3 & 4 & 9 \end{bmatrix}$$ lower triangle matrix

→ determinant is the product of diagonal elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \det(A) = a_{11} \cdot a_{22} \cdot a_{33}$$

↳ Transpose of a matrix and its determinants

→ Transpose of $A_{m \times n}$ = $A^T_{n \times m}$ with (i,j)-th entry A_{ji}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2\times 3}$$

$$\det(A) = \det(A^T)$$

↳ Minors and cofactors

→ Minor of the entry in i-th row and j-th column is the determinant of the submatrix formed by deleting i-th row and j-th column.

e.g. → $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (1,1)-th minor; denoted by M_{11}

$$M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

→ Cofactor (i,j)th cofactor; $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$C_{11} = M_{11} \; ; \; C_{23} = -M_{23}$$

→ For $A_{3\times 3}$

$$\det(A) = (a_{11} \times C_{11}) + (a_{12} \times C_{12}) + (a_{13} \times C_{13})$$

For $A_{4\times 4}$

$$\det(A) = \sum_{j=1}^{4} a_{1j} C_{1j}$$

For $A_{n\times n}$; $\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$

WEEK 2

WEEK 1 GA

(1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A - xI = \begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix}$

$$\det(A - xI) = (a-x)(d-x) - bc$$
$$= ad - ax - xd + x^2 - bc$$
$$= \underbrace{ad - bc}_{\det(A)} \underbrace{- x(a+d)}_{-x\,\mathrm{trace}(A)} + \underbrace{x^2}_{+\,x^2}$$

(2) $3A = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$ $\quad \det(3A) = 9ad - 9bc = (3)^2 \det(A)$

$3A = \begin{bmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$ $\quad |3A| = 3^3|A| = \boxed{27|A|}$

(3) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad I + A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad 5A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$

$5A + I = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$

$(I + A)^3 - (5A + I) = mA$

$\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$ $\quad \boxed{m = 8}$

(4) $A = \begin{bmatrix} 20 & 30 & 40 \\ 8 & 16 & 24 \\ 8 & 10 & 12 \end{bmatrix}$ $\quad \det(A) = 20 \det\begin{bmatrix} 16 & 24 \\ 10 & 12 \end{bmatrix} - 30 \det\begin{bmatrix} 8 & 24 \\ 8 & 12 \end{bmatrix} + 40 \det\begin{bmatrix} 8 & 16 \\ 8 & 10 \end{bmatrix}$

$$= 20(-48) - 30(-96) + 40(-48) = -960 + 2880 - 1920 = \boxed{0}$$

(5) $\det(A) = 3 \quad \det(B) = 3 \quad \det(B^{-1}) = 1/3$

$\det(3A^2B^{-1}) = (3)^3 \cdot (\det(A))^2 \cdot \left(\frac{1}{3}\right) = 9(3)^2 = \boxed{81}$

(6) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ $\quad A^3 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix}$ $\quad A^4 = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}$

$A^7 = \begin{bmatrix} 729 & 729 & 729 \\ 729 & 729 & 729 \\ 729 & 729 & 729 \end{bmatrix}$ $\quad$ sum of diagonals $= \boxed{2187}$

(7) $A = \begin{bmatrix} 12 & 19 & 26 \\ 17 & 24 & 31 \\ 22 & 29 & 36 \end{bmatrix}$ $\quad \det(A) = 12 \det\begin{bmatrix} 24 & 31 \\ 29 & 36 \end{bmatrix} - 19 \det\begin{bmatrix} 17 & 31 \\ 22 & 36 \end{bmatrix} + 26 \det\begin{bmatrix} 17 & 24 \\ 22 & 29 \end{bmatrix}$

$$= 12(-35) - 19(-70) + 26(-35) = \boxed{0}$$

(9)
$30x_1 + 20x_2 + 25x_3 = 670$ — (1)
$20x_1 + 35x_2 + 25x_3 = 730$ — (2)
$20x_1 + 10x_2 + 15x_3 = 400$ — (3)

$(1) - 2\cdot(3) \Rightarrow$ $30 \quad 20 \quad 25 \quad 670$
$-40 \quad -20 \quad -30 \quad -800$
$= -10x_1 - 5x_3 = -130$

$(2) - (3) \Rightarrow$ $20 \quad 35 \quad 25 \quad 730$
$-20 \quad -10 \quad -15 \quad -400$
$= 25x_2 + 10x_3 = 330$ — (4)
$10x_1 + 5x_3 = 130$ — (5)

$2(5) = 20x_1 + 10x_3 = 260 \Rightarrow 20x_1 = 260 - 10x_3$

$260 - 10x_3 + 10x_2 + 15x_3 = 400$
$\Rightarrow 10x_2 + 5x_3 = 140$ — (6)

$(4) - 2\cdot(6) \Rightarrow$ $0 \quad 25 \quad 10 \quad 330$
$0 \quad -20 \quad -10 \quad -280$
$= 5x_2 = 50$ $\quad \boxed{x_2 = 10}$

$250 + 10x_3 = 330 \Rightarrow \boxed{x_3 = 8}$

$10x_1 + 40 = 130 \Rightarrow \boxed{x_1 = 9}$

$x_1 + x_2 + x_3 = 9 + 10 + 8 = \boxed{27}$

(10) $A = \begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix}$