MATH 2

WEEKS 1-4

1. Matrix multiplication

$$A \times B = C \quad ; \quad C[i,j] = \sum_{k=1}^{n} A[i,k] \times B[k,j]$$

2. Determinant

For $A_{2\times2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

For $A_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

3. $\det(A \cdot B) = \det(A) \cdot \det(B)$

4. $\det(A^n) = \det(A)^n \quad ; \quad \det(A^{-1}) = \det(A)^{-1}$

5. $\det(A)$ = product of diagnols if A is an upper/lower triangle matrix

6. $\det(A) = \det(A^T)$

7. Minor $\rightarrow$ (i,j)-th minor M_{ij} is the determinant of the submatrix after deleting i-th row and j-th column.
Cofactors $\rightarrow$ (i,j)-th cofactor C_{ij} is $(-1)^{i+j} \cdot M_{ij}$

8. $\det(A_{n\times n}) = \sum_{j=1}^{n} a_{1j} C_{1j}$

9. $\det(t \cdot A_{n\times n}) = (t)^n \cdot \det(A)$

10. Inverse matrix. If $\det(A) \neq 0$, then A is invertible

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

For A bigger than 2×2 $\rightarrow$ $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$, where $\operatorname{adj}(A) = C^T$ $\rightarrow$ transpose of cofactor matrix

For $A_{n\times n}$, $\det(\operatorname{adj}(A)) = (\det(A))^{n-1}$

11. Homogenous System of Linear Equations

$$Ax = b \quad , \text{ where } b = 0$$

$\rightarrow$ has a unique solution 0 if A is invertible

12. Elementary Row Operations

Type	Action	Example and notation	Effect on determinant
1	Interchange two rows	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = -\det(B)$
2	Scalar multiplication of a row by a constant t	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = t \cdot \det(B)$
3	Adding multiples of a row to another row	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_3} \begin{bmatrix} 3 & -19 & -2 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = \det(B)$

13. dependent variable $\rightarrow$ if the column corresponding to that variable has the leading entry of some row.

14. Properties of vectors:

i) $v + w = w + v$
ii) $(v+w) + u = v + (w+u)$
iii) $v + 0 = 0 + v = v$
iv) $v + (-v) = 0$
v) $1 \cdot (v) = v$
vi) $(ab)v = a(bv)$
vii) $a(v+w) = av + aw$
viii) $(a+b)v = av + bv$

(15) A set of vectors is linearly independent iff the matrix V obtained by arranging the vectors in columns is invertible, i.e., $\det(V) \neq 0$.

(16) If a set of vectors is linearly dependent, then every superset is linearly dependent.

(17) If a set of vectors contain a 0 vector, then the set is always linearly dependent.

(18) If you have more than n vectors in $\mathbb{R}^n$, then the homogenous system $Vx=0$ is always linearly dependent.
↳ more variables than equations means infinitely many solutions.

(19) Basis of a vector space is a linearly independent subset of the vector space that spans the vector space.

(20) Rank (A) = cardinality of a basis of the vector space.
↳ no. of non-zero rows in RREF.

WEEKS 5-8

① nullity (A) = no. of independent variables — set independent variables to 1 to find a basis of null space

② For an $m \times n$ matrix A, $\text{rank}(A) + \text{nullity}(A) = n$

③ Linear transformation properties: ① $f(x+y) = f(x) + f(y)$
② $f(c \cdot x) = c \cdot f(x)$

④ $f: V \to W$; $\ker(f) = \{v \in V \mid f(v) = 0\}$ → null space
$\text{Im}(f) = \{w \in W \mid \exists v \in V \text{ for which } f(v) = w\}$ → column space

⑤ f is injective iff $\ker(f) = 0$
f is surjective iff $\text{Im}(f) = W$

⑥ Basis of $\ker(f)$ → use basis of null space and basis of domain.
Basis of $\text{Im}(f)$ → use column of dependent variable (in the matrix representation) and basis of co-domain.

⑦ $\text{Rank}(f) = \dim(\text{Im}(f))$
$\text{Nullity}(f) = \dim(\ker(f))$
$\text{rank}(f) + \text{nullity}(f) = \dim(V)$

⑧ $T: V \to W$ is a linear transformation
β_1, β_2 are ordered bases of V — A = matrix of T w.r.t. β_1 and γ_1
γ_1, γ_2 " " " " W — B = " " " " β_2 and γ_2

$B = QAP$, where P → expresses β_2 in terms of β_1
Q → " γ_2 " " " γ_1

⑨ Similar matrices
① $PB = AP$ — ③ $\det(B) = \det(A)$
② $\text{rank}(A) = \text{rank}(B)$ — ④ $\text{trace}(A) = \text{trace}(A)$

⑩ Affine subspace → translation of a vector subspace

⑪ Norm
$\|.\| : V \to \mathbb{R}$
① $\|x+y\| \leq \|x\| + \|y\| \quad \forall\, x, y \in V$
② $\|cx\| = |c|\,\|x\| \quad \forall c \in \mathbb{R},\ \forall x \in V$
③ $\|x\| \geq 0 \quad \forall x \in V$; $\|x\| = 0$ iff $x = 0$

example: length of vector $\text{len}(x,y) = \sqrt{(x,y)(x,y)} = \sqrt{x^2+y^2}$
$\text{len}(x,y,z) = \sqrt{x^2+y^2+z^2}$

⑫ Angle b/w vectors v and u

$$\cos(\theta) = \frac{u \cdot v}{\|u\| \times \|v\|}, \text{ where } \|u\| = \sqrt{u \cdot u}$$

(13) Inner product $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$

① $\langle v, v \rangle > 0 \quad \forall\, v \in V \setminus \{0\}$

② $\langle v_1 + v_2 , v_3 \rangle = \langle v_1 , v_3 \rangle + \langle v_2 , v_3 \rangle$

③ $\langle v_1 , v_2 \rangle = \langle v_2 , v_1 \rangle$

④ $\langle c v_1 , v_2 \rangle = \langle v_1 , c v_2 \rangle = c \langle v_1 , v_2 \rangle$

(14) v_1, v_2 are orthogonal if $\langle v_1 , v_2 \rangle = 0$

(15) Orthonormal basis S, $S = \{v_1, v_2, \dots, v_n\} \subseteq V$ is othornormal if: $\langle v_i , v_j \rangle = 0 \quad \forall\, i,j \in \{1,2,\dots,n\}, i \neq j$

$\|v_i\| = 1 \quad \forall\, i \in \{1,2,\dots,n\}$

(16) $S = \{v_1, v_2, \dots, v_k\}$ is an orthonormal basis of V.

$v = c_1 v_1 + c_2 v_2 + \dots + c_k v_k \quad , v \in V$

$c_i = \langle v, v_i \rangle$

(17) Projection of v onto w, where $v \in V, w \in W, W \subseteq V$

Let $\{v_1, v_2, \dots v_n\}$ be an orthonormal basis for W

$$\mathrm{proj}_W(v) = \sum_{i=1}^{n} \langle v, v_i \rangle v_i$$

① $P_W(v) = v \quad \forall\, v \in W$

② $\mathrm{Range}(P_W) = W$

③ $W^{\perp} = \{ v \mid v \in V$, such that $\langle v, w \rangle = 0 \ \forall\, w \in W \}$ is the null space of P_W

④ $P_W^2 = P_W$

⑤ $\|P_W(v)\| \le \|v\|$

(18) Orthogonal transformation $T: V \to V$ iff: $\langle Tv, Tw \rangle = \langle v, w \rangle \quad \forall\, v,w \in V$

↳ preserves lengths and angles

↳ matrix A iff: $A^T A = A A^T = I$

WEEKS 9-11

① Partial Derivative

$$\frac{\partial f}{\partial x_i}(\underset{\sim}{a}) = \lim_{h \to 0} \frac{f(\underset{\sim}{a} + h e_i) - f(\underset{\sim}{a})}{h}$$

, where $\{e_1, e_2, \dots\}$ is standard ordered basis of $\mathbb{R}^n$

② Directional Derivative

↳ rate of change in the direction of the unit vector u.

$$f_u(\underset{\sim}{a}) = \lim_{h \to 0} \frac{f(\underset{\sim}{a} + hu) - f(\underset{\sim}{a})}{h}$$

Properties:

→ $(cf + g)_u(\underset{\sim}{a}) = c f_u(\underset{\sim}{a}) + g_u(\underset{\sim}{a})$

→ $(fg)_u(\underset{\sim}{a}) = f_u(\underset{\sim}{a}) g(\underset{\sim}{a}) + f(\underset{\sim}{a}) g_u(\underset{\sim}{a})$

→ $(f/g)_u(\underset{\sim}{a}) = \dfrac{f_u(\underset{\sim}{a}) g(\underset{\sim}{a}) - f(\underset{\sim}{a}) g_u(\underset{\sim}{a})}{(g(\underset{\sim}{a}))^2}$

③ To check for continuity of a function at a point, check the limit of that function along different curves at that point.

④ $\nabla f(\underset{\sim}{x}) = \left(\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \dots, \dfrac{\partial f}{\partial x_n} \right)$ in $\mathbb{R}^n$

⑤ Directional derivative at point $\underset{\sim}{a}$ in direction of u:

$$\nabla f_u(\underset{\sim}{a}) = \nabla f(\underset{\sim}{a}) \cdot u$$

⑥ $f_u \rightarrow$ directional derivative in the direction of u.

Steepest ascent $\rightarrow$ when $u = \dfrac{\nabla f}{\|\nabla f\|}$

Steepest descent $\rightarrow$ when $u = -\dfrac{\nabla f}{\|\nabla f\|}$

No change $\rightarrow$ when $\langle u \cdot \nabla f \rangle = 0$, i.e. orthogonal

⑦ Equation of tangent line at point (a,b) in direction of $u = (u_1, u_2)$

$x(t) = a + tu_1$; $y(t) = b + tu_2$; $z(t) = f(a,b) + t f_u(a,b)$, where $t f_u(a,b) = t(\nabla f(a,b) \cdot u)$

⑧ Tangent Plane :

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

⑨ Hessian matrix

Let $f(x_1, x_2, \dots, x_n)$ be defined on D in $\mathbb{R}^n$

The hessian matrix of f is defined as:

$$Hf = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

⑩ Classification of critical points using Hf

f in $\mathbb{R}^2 \Rightarrow$

If $\det(Hf(\underset{\sim}{a})) > 0$
- $f_{xx}(\underset{\sim}{a}) > 0 \rightarrow$ then $\underset{\sim}{a}$ — local minima
- $f_{xx}(\underset{\sim}{a}) < 0 \rightarrow$ then $\underset{\sim}{a}$ — local maxima

If $\det(Hf(\underset{\sim}{a})) < 0 \rightarrow$ then $\underset{\sim}{a}$ — saddle point

If $\det(Hf(\underset{\sim}{a})) = 0 \rightarrow$ then inconclusive

f in $\mathbb{R}^3 \Rightarrow$

If $\det(Hf(\underset{\sim}{a})) \overset{+ve}{>} 0$ AND $(f_{xx} f_{yy} - f_{xy}^2)(\underset{\sim}{a}) \overset{+ve}{>} 0$ AND $f_{xx}(\underset{\sim}{a}) \overset{+ve}{>} 0$ $\Rightarrow$ local minimum

If $\det(Hf(\underset{\sim}{a})) \overset{-ve}{<} 0$ AND $(f_{xx} f_{yy} - f_{xy}^2)(\underset{\sim}{a}) \overset{+ve}{>} 0$ AND $f_{xx}(\underset{\sim}{a}) \overset{-ve}{<} 0$ $\Rightarrow$ local minimum

If $\det(Hf(\underset{\sim}{a})) \neq 0$ AND cases 1 or 2 do not occur $\Rightarrow$ saddle point

If $\det(Hf(\underset{\sim}{a})) = 0$ $\Rightarrow$ inconclusive