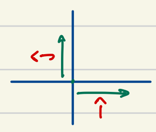
MATH 2

WEEK 1

→ Vectors

- can be thought of as a list.
- can be in a row or a column form.
- Addition of vectors:
  Add the corresponding entries. For e.g. → $(3,5) + (2,4) = (5,9)$
- Scalar multiplication:
  e.g. → $2(8,8,10,5) = (16,16,20,10)$
- Visualisation of a vector
  Point $(a,b) \equiv$ Vector $(a,b) \equiv a\hat{i} + b\hat{j}$ where, $\hat{i}$ → one unit in x-axis
  $\hat{j}$ → one unit " y-axis



  e.g. Point $(-1,-1) = -\hat{i} - \hat{j}$
- Vectors in $\mathbb{R}^n$ are lists with n real entries.

→ Matrices

- rectangular array of numbers
- (rows × columns) ; eg → $\begin{bmatrix} 5 & 7 & 10 \\ 3 & 5 & 2 \end{bmatrix}$ is a 2×3 matrix.
  - $(1,2)$ ith entry → 7
- square matrix → $N \times N$
  - Diagnol matrix → all entrices are 0 except the diagnol
    $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$
  - Scalar matrix → all entries have the same value
    $$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$
  - Indentity matrix → denoted by 'I'; scalar matrix with values = 1
    $$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Addition of matrices → must be of the same size: $\begin{bmatrix} 1 & 0 \\ 5 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 8 & 3 \\ 7 & 4 \end{bmatrix}$
- Scalar multiplication → multiply each number with the scalar.
- Matrix multiplication → $A \times B = C$; $C[i,j] = \sum_{k=1}^{n} A[i,k] \times B[k,j]$
  - no. of columns in first matrix $\underline{\text{must}}$ = no. of rows in 2^{nd} matrix
  - $A_{m \times n} \times B_{n \times p} = (AB)_{m \times p}$

    $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{jk}$
- Scalar multiplication is the same as multiplication by the scalar matrix
  ex. → $\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$
- Properties:
  1. $(A+B)+C = A+(B+C)$
  2. $(AB)C = A(BC)$
  3. $A+B = B+A$
  4. $AB \neq BA$
  5. $\lambda(A+B) = \lambda A + \lambda B$
  6. $\lambda(AB) = (\lambda A)B = A(\lambda B)$
  7. $A(B+C) = AB + AC$
  8. $(A+B)C = AC + BC$

→ System of Linear Equations

- collection of one or more linear equations involving the same set of variables.
- system of m linear equation with n variables :

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_n
\end{aligned}
$$

- system of lin. eq. in matrix form :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

$$\underbrace{A}_{\text{coefficient matrix}} \underbrace{x}_{\text{column of variables}} = \underbrace{b}_{\text{column of resulting values}}, \text{ where}$$

A = $m \times n$ matrix
x = column vector with n entries
b = column vector with m entries

- Solutions to a system of lin. eq. :
  1. Infinite solution
  2. Single unique solution
  3. No solution

→ Determinant

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad - bc$

e.g. $A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$ $\quad \det(A) = 20 - 18 = 2$

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det(A) = a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

- Determinant of Identity matrix $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  → $\det(I_2) = 1$
  → $\det(I_3) = 1$

- Determinant of a product of matrices
  - $\det(AB) = \det(A) \cdot \det(B)$; $\det(ABC) = \det(A) \cdot \det(B) \cdot \det(C)$
  - $\det(A^n) = \det(A)^n$
  - $\det(A^{-1}) = \det(A)^{-1}$

- Determinant of the inverse of a matrix

$$AA^{-1} = I \Rightarrow \det(AA^{-1}) = \det(I)$$

  - $\det(A^{-1}) = \dfrac{1}{\det(A)}$

- Switching rows

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\det(\tilde{A}) = cb - ad = -(ad - bc) = -\det(A)$$

$$\det(\tilde{A}) = -\det(A)$$

- Add multiple of a row/column to another row/column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} a + tc & b + td \\ c & d \end{bmatrix}$$

$$\det(\tilde{A}) = (a + tc)d - (b + td)c = ad + tcd - bc - tcd$$

$$\det(\tilde{A}) = \det(A)$$

↳ Scalar multiplication of a row/column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} ta & b \\ tc & d \end{bmatrix}$$

$$\det(\tilde{A}) = t \cdot \det(A)$$

↳ Upper | Lower triangle matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{bmatrix}$$ upper triangle matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 0 \\ 3 & 4 & 9 \end{bmatrix}$$ lower triangle matrix

→ determinant is the product of diagonal elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \det(A) = a_{11} \cdot a_{22} \cdot a_{33}$$

↳ Transpose of a matrix and its determinants

→ Transpose of $A_{m\times n}$ = $A^T_{n\times m}$ with (i,j)-th entry A_{ji}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2\times 3}$$

$$\det(A) = \det(A^T)$$

↳ Minors and cofactors

→ Minor of the entry in i-th row and j-th column is the determinant of the submatrix formed by deleting i-th row and j-th column.

e.g. → $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (1,1)-th minor; denoted by M_{11}

$$M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

→ Cofactor (i,j)th cofactor; $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$C_{11} = M_{11} \; ; \; C_{23} = -M_{23}$$

→ For $A_{3\times 3}$

$$\det(A) = (a_{11} \times C_{11}) + (a_{12} \times C_{12}) + (a_{13} \times C_{13})$$

For $A_{4\times 4}$

$$\det(A) = \sum_{j=1}^{4} a_{1j} C_{1j}$$

For $A_{n\times n}$; $\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$

WEEK 2

→ Determinant (continued)

↳ Expansion along any row | column

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } i$$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } j \text{ (determinant along a column)}$$

↳ Properties :

① Determinant of a product is product of the determinants.
↳ $\det(AB) = \det(BA) = \det(A) \cdot \det(B)$

② Switching two rows or columns changes the sign.

③ Adding multiples of a row to another row leaves the determinant unchanged.

④ Scalar multiplication of a row | column by a constant t multiplies the determinant by t.

⑤ $\det(t A_{n \times n}) = (t)^n \det(A)$

↳ Useful computational tips :

① The determinant of a matrix with a row or a column of zeros is 0.

② The determinant of a matrix in which one row (or column) is a linear combination of other rows (resp. columns) is 0.

③ Scalar multiplication of a row | column by a constant t multiplies the determinant by t.

④ While computing the determinant, you can choose to compute it using expansion along a suitable row or a column.

→ Cramer's rule

→ $Ax = b$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Cramer rule ⇒ $A_{x_1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{21} \end{bmatrix}$, $A_{x_2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$

↳ solutions for x_1 & x_2 ⇒ $x_1 = \dfrac{\det(A_{x_1})}{\det(A)}$; $x_2 = \dfrac{\det(A_{x_2})}{\det(A)}$

→ For a 3×3 matrix: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$A_{x_1} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$ $A_{x_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$ $A_{x_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$

$x_1 = \dfrac{\det(A_{x_1})}{\det(A)}$ $x_2 = \dfrac{\det(A_{x_2})}{\det(A)}$ $x_3 = \dfrac{\det(A_{x_3})}{\det(A)}$

Inverse Matrix

- The inverse of a matrix $A \to (A^{-1}) \Rightarrow A \cdot (A^{-1}) = I$
- if $\det(A) \neq 0$, then the matrix is invertible

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Adjucate of a matrix

  M_{ij} = determinant of submatrix after deleting i-th row and j-th column.

  $C_{ij} = (-1)^{i+j} M_{ij}$

  → Adjucate of a matrix → Transpose of Cofactor matrix

  $\mathrm{Adj}(A) = C^T$

  → $A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A)$ → $A_{n \times n}$, then $\det(\mathrm{adj}(A)) = (\det(A))^{n-1}$

- Solution to system of linear equations using inverse matrix.

  $Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$

Homogeneous System of Linear Equations

- $Ax = b$, where $b = 0 \Rightarrow Ax = 0$
  - → has unique solution 0 if A is invertible
  - → has infinite solutions if A is not invertible

The Row Echelon Form (reduced form)

- Matrix is in row echelon form if:
  - → first non-zero element in each row (leading entry) is 1
  - → each leading entry is in a column to the right of the leading entry in the previous row.
  - → rows with all zero elements, if any, are below rows having a non-zero element.

  e.g. → $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

  → reduced row echelon form → For a non-zero row, the leading entry in the row is the only non-zero entry in its column

  e.g. → $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

→ e.g. solution for $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow Ax = b$

$x_1 + 2x_2 = b_1 \Rightarrow x_1 = b_1 - 2x_2$

$x_3 = b_2$

$x_4 = b_3$

solution $\Rightarrow x = \begin{bmatrix} b_1 - 2c \\ c \\ b_2 \\ b_3 \end{bmatrix}$

→ If the i-th column has the leading entry of some row, we call x_i a dependent variable.
If the i-th column does not have the leading entry of any row, x_i → independent variable.

Row reduction

Elementary Row Operations

Type	Action	Example and notation	Effect on determinant
1	Interchange two rows	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = -\det(B)$
2	Scalar multiplication of a row by a constant t	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = t \cdot \det(B)$
3	Adding multiples of a row to another row	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_3} \begin{bmatrix} 3 & -19 & -2 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = \det(B)$

Row echelon form

Steps :
1. Find the left most non-zero column.
2. Use operations to get 1 in the top of that column.
3. Make entries below that 1 to 0.
4. Repeat the steps for the next row and onwards.

Reduced row echelon form

- start from the right most column and make all the non-leading terms 0s.

- e.g.

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_1/2} \begin{matrix} 1 & 2 & 1/2 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{matrix} \xrightarrow[R_3 - 5R_1]{R_2 - 3R_1} \begin{matrix} 1 & 2 & 1/2 \\ 0 & 2 & 11/2 \\ 0 & -4 & 13/2 \end{matrix} \xrightarrow[R_3 + 2R_2]{R_2/2} \begin{matrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 35 \end{matrix} \xrightarrow{R_3/35} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{matrix} \xrightarrow[R_2 - \frac{11}{4}R_3]{R_1 - \frac{1}{2}R_3} \begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det(A) = 35 \times 2 = 70$

Gaussian Elimination Method

- Augmented matrix → matrix of size $(m \times n+1)$
  - first n columns are from A and the last column is b.
  - denoted by $[A|b]$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- Bring the matrix A to reduced row echelon form.
  → Let R be the submatrix of obtained matrix of the first n columns and c be the submatrix of the obtained matrix of the last column.
  $[R|c]$, where R → reduced row echelon form of A
  Solutions of $Ax = b$ are presicely the solutions of $Rx = c$.

Homogenous system of linear equations

- 0 is always a solution a.k.a. trivial solution
- If there are more variables than equations, (i.e., more columns than rows) there will be infinite solutions because there must be some independent variables.

WEEK 3

→ Properties of vectors Let v, w and v' be vectors in $\mathbb{R}^n$ and $a, b \in \mathbb{R}$

i) $v + w = w + v$
ii) $(v+w) + v' = v + (w + v')$
iii) The 0 vector satisfies : $v + 0 = 0 + v = v$
iv) The vector $-v$ satifies : $v + (-v) = 0$
v) $1v = v$
vi) $(ab)v = a(bv)$
vii) $a(v+w) = av + aw$
viii) $(a+b)v = av + bv$

↳ Vector space → a set with two operations (addition and multiplication)
$+ : V \times V \to V$ and $\cdot : \mathbb{R} \times V \to V$
for each pair of elements v_1 and v_2 in V, there is a unique element $v_1 + v_2$.
for each $c \in \mathbb{R}$ and $v \in V$ there is a unique element $c \cdot v$ in V.

example of vector space → solutions of a homogenous system $Ax = 0$
↳ non-examples:
① define addition and scalar multiplication in $\mathbb{R}^2$ as:
→ $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$
→ $c(x_1, x_2) = (cx_1, cx_2)$

$v + w \neq w + v$ → i), iii) and viii) fail

② → $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0)$
→ $c(x_1, x_2) = (cx_1, 0)$
→ iii), iv) and v) fail

→ Cancellation law of vector addition

↳ If $v_1, v_2, v_3 \in V$ such that $v_1 + v_3 = v_2 + v_3$, then $v_1 = v_2$

↳ Corrolaries: ① The vector 0 described in iii) is unique. → $\cancel{v} + w = \cancel{v} + 0 \Rightarrow w = 0$
② The vector v' " " iv) " " and referred to as $-v$.
↳ $\cancel{v} + v' = 0 = \cancel{v} + v'' \Rightarrow v' = v''$

→ Some more properties:
① $0v = 0$ for each $v \in V$ → $(0+0)v = 0v + 0v \Rightarrow 0v = 0v + 0v$
$\Rightarrow \cancel{0v} + 0 = \cancel{0v} + 0v$

② $(-c)v = -(cv) = c(-v)$ for each $c \in \mathbb{R}$ and for each $v \in V$
↳ $(c + (-c))v = cv + (-c)v \Rightarrow 0 = cv + (-c)v \Rightarrow -cv = (-c)v$

③ $c0 = 0$ for each $c \in \mathbb{R}$

→ Linear combinations:
↳ Let V be a vector space and $v_1, v_2, \ldots, v_n \in V$.
Linear combinations of $v_1, v_2, \ldots, v_n$ with coefficients $a_1, a_2, \ldots, a_n \in \mathbb{R}$ is the vector $\sum_{i=1}^{n} a_i v_i \in V$.
↳ example: $(2,9)$ is a linear combination of vectors $(2,1)$ and $(-2,3)$:
$3(2,1) + 2(-2,3) = (6,3) + (-4,6) = (2,9)$
↳ $3(2,1) + 2(-2,3) - (2,9) = 0$
↳ 0 vector is a linear combinations of $(2,1), (-2,3)$, and $(2,9)$ with non-zero coefficients

↳ The plane of the two vectors $(0,2,1)$ and $(2,2,0)$ can be expressed by the equation $2x - 2y + 4z = 0$
↳ $(1,2,0)$ does not lie on the plane
↳ $(3,7,2)$ does lie on this plane

→ Linear Dependence
↳ a set of vectors $v_1, v_2, v_3, \dots, v_n \in V$ is said to be linearly dependent if there exists scalars $a_1, a_2, a_3, \dots, a_n$, not all zeros, such that ↳

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_n v_n = 0$$

↳ If a set of vectors are linearly depent, then every superset of it is linearly dependent.
↳ If v_1, v_2, v_3 are " " , then v_1, v_2, v_3, v_n are also linearly dependent.

→ Linear Independence
↳ a set of vectors if they are not linearly dependent
↳ $v_1, v_2, v_3, \dots, v_n$ are linearly independent if : $a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_n v_n = 0$
can only be satisfied when $a_i = 0$ for all $i = 1, 2, 3, \dots, n$

↳ If a set of vectors $v_1, v_2, \dots, v_n$ contain a vector $v_i = 0$, then the set is always linearly dependent.

↳ If two non-zero vectors are multiples of each other, then are linearly dependent.

$$a_1 v_1 + a_2 v_2 = 0 \Rightarrow c = -a_2/a_1 \text{ , then } v_1 = c v_2$$

↳ Two vectors are linearly independent when they are not multiples of each other.

↳ for three vectors v_1, v_2, v_3
Suppose $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0 \Rightarrow v_1 = -a_2/a_1 v_2 - a_3/a_1 v_3$

↳ v_1 is a linear combination of v_2 and v_3
↳ If three vectors are linearly independent then none of these vectors is a linear combination of the other two.

→ If you have more than n vectors in $\mathbb{R}^n$, then the homogenous system $Vx = 0$ has infinitely many solutions. → They are linear dependent
↳ This corresponds to having more variables than there are equations.

→ V is an $n \times n$ matrix obtained by arranging a set of vectors in columns.
$Vx = 0$ → homogenous system
V → square matrix, therefore $x = 0$ if and only if V is invertible
if and only if $\det(V) \neq 0$

WEEK 4

→ Span of a set of vectors

↳ Span of a vector space V → $\text{Span}(V) = \{ \sum_{i=1}^{n} a_i v_i \in V \mid a_1, a_2, a_3, \dots, a_n \in \mathbb{R} \}$

e.g. → Let $V = \{(1,0)\} \subset \mathbb{R}^2$. Then, $\text{Span}(V) = \{a(1,0) \mid a \in \mathbb{R}\} = \{(a,0) \mid a \in \mathbb{R}\}$

→ Let $V = \{(1,0,0), (0,1,0)\} \subset \mathbb{R}^3$. $\text{Span}(V) = \{a(1,0,0) + b(0,1,0) \mid a,b \in \mathbb{R}\} = \{(a,b,0) \mid a,b \in \mathbb{R}\}$

↳ Spanning set

→ A set $S \subseteq V$ is a spanning set for V if $\text{span}(S) = V$.

↳ If a vector $T \subseteq V$, then $\text{Span}(T) \subseteq \text{Span}(V)$

$T \subseteq \text{Span}(V)$ → then $\text{Span}(T) \subseteq \text{Span}(V)$

↳ Basis

↳ Basis B of a vector space V is a linearly independent subset of V that spans V.

↳ e.g. Standard Basis → Let $e_i \in \mathbb{R}^n$ be the vector with i^{th} coordinate 1 and all other coordinates 0. → $e_1 = (1,0,0,\dots,0)$

The set $\varepsilon = \{e_1, e_2, e_3, \dots, e_n\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$

→ How to find a basis:

① Start with $\emptyset$ and keep appending vectors which are not a linear combination of the set thus far obtained, until we obtain a spanning set.

② Take a spanning set and keep deleting vectors which are linear combinations of other vectors, until the remaining vectors satisfy that they are not a linear combination of the remaing ones.

→ Rank / dimension of a vector space

↳ size of (or cardinality) of a basis of the vector space.

↳ Rank of a matrix Let A be an $m \times n$ matrix.

→ column space → subspace of $\mathbb{R}^m$ spanned by column vectors of A.

row " → " $\mathbb{R}^n$ " " row " " .

Column rank = Row rank = Rank of A

number of non-zero rows when reduced to row echelon

e.g. → vector space W is spanned by $\{(1,0,1), (-2,-3,1), (3,3,0)\}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \qquad \text{RREF of } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank / dimension of W is 2 and a basis is given by $(1,0,1), (0,1,-1)$

↳ Row method → put the spanning vectors in rows and reduce the matrix. No. of non-zero rows is the rank and the rows themselves form a basis.

Column method → put the spanning vectors in columns and reduce the matrix. No. non-zero rows is the rank. The columns of original matrix corresponding to the columns of reduced matrix containing the pivots (i.e., dependent variables) form a basis.

WEEK 5

→ Null space

- Let A be an $m \times n$ matrix.
  Subspace $W = \{x \in \mathbb{R}^n \mid Ax = 0\}$ ⇒ solution space of homogenous system $Ax = 0$
  - a.k.a. null space of A.
  - dimension of null space ⇒ nullity of A.

- Finding nullity and basis of an $m \times n$ matrix A.
  - nullity(A) = no. of independent variables
    - by letting the independent variables vary in $\mathbb{R}$ and calculating the values of dependent variables, you get the solution space of $Ax = 0$.
  - basis ⇒ substitute one ind. variable as 1 and the others as 0.
    - all the vectors you get by fixing 1 for each independent variable form the basis.

→ Rank-nullity theorem

- For a matrix R in RREF, no. of non-zero rows (i.e., rank of R) is the number of dependent variables.
- For an $m \times n$ matrix A, rank(A) + nullity$(A) = n$
- To check whether n vectors form a basis for $\mathbb{R}^n$:
  - put n vectors as columns to form an $n \times n$ matrix
    - if determinant $\neq 0$, then this set of vectors form a basis.

→ Linear Mapping

- Linear mapping of f from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be defined as:

$$f(x_1, x_2, x_3, \dots, x_n) = \left(\sum_{j=1}^{n} a_{1j} x_j, \sum_{j=1}^{n} a_{2j} x_j, \dots, \sum_{j=1}^{n} a_{mj} x_j \right)$$

  - can be rewritten as Ax, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$

→ Linear Transformation

- in any linear transformation, $f(0) = 0$ because ⇒ $f(0+0) = f(0) + f(0)$
  $f(0) = f(0) + f(0)$ ⇒ $0 = f(0)$
- a function $f: V \to W$ is one-to-one → if $f(v_1) = f(v_2)$, then $v_1 = v_2$.
  " " $f: V \to W$ " onto → if for every $w \in W$, there exists $v \in V$ such that $f(v) = w$.
- Isomorphism → if a linear transformation is a bijection (both one-to-one and onto).

Rough example

$$f(x,y) = x\text{-}y\,(2,0) + y(0,1)$$
$$= (2x-2y, 0) + (0,y) \Rightarrow (2x-2y, y)$$

- for a function to be linear transformation, it has to satisfy:
  1. $f(x+y) = f(x) + f(y)$; and
  2. $f(ax) = a \cdot f(x)$

WEEK 6

→ Example: $W = \{(x,y,z) \mid x+y+z=0\}$

basis ⇒ $(-1,1,0)$ and $(-1,0,1)$

$(x,y,z) \in W$ can be expressed as:
$(x,y,z) = y(-1,1,0) + z(-1,0,1)$ Hence $f: W \to \mathbb{R}^2$ is $f(x,y,z) = y(1,0) + z(0,1) = (y,z)$

→ Matrix corresponding to a linear transformation with respect to ordered bases

↳ Let $f: V \to W$ be a linear transformation
Let $\beta = v_1, v_2, \dots, v_n$ be an ordered basis of V and $\gamma = w_1, w_2, \dots, w_m$ ordered basis of W.

Each $f(v_i)$ can be written as a linear combination of w_j

$$f(v_1) = a_{11}w_1 + a_{21}w_2 + a_{31}w_3 \dots + a_{m1}w_m$$
$$\vdots$$
$$f(v_n) = a_{1n}w_1 + a_{2n}w_2 + a_{3n}w_3 + \dots + a_{mn}w_m$$

The matrix corresponding to the linear transformation f is given by $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

↳ example: Let $V = W = \mathbb{R}^2$, $\beta = \gamma = (1,0),(1,1)$ and $f(x,y) = f(2x,y)$

$$f(1,0) = (2,0) = 2(1,0) + 0(1,1)$$
$$f(1,1) = (2,1) = 1(1,0) + 1(1,1)$$

↳ the matrix corresponding to f w.r.t. the ordered bases $\{(1,0),(1,1)\}$ is $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

↳ Changing the ordered basis gives us different matrices corresponding to the same linear transformation.

Q. Let $W = \{(x,y,z) \mid x = 2y+z\}$. Let $\beta = \{(2,1,0),(1,0,1)\}$ be a basis of W. Let $T: W \to \mathbb{R}^2$ such that $T(2,1,0) = (1,0)$ and $T(1,0,1) = (0,1)$.

↳ Definition of T ⇒ $T(x,y,z) = (y,z)$ or $T(x,y,z) = (x-y-z, x-2y)$
↳ What will be the matrix representation of T wrt. β for W and $\gamma = \{(1,1),(1,-1)\}$ for $\mathbb{R}^2$?
↳ $T(2,1,0) = (1,0) = a(1,1) + b(1,-1)$ ⟶ $a+b=1$, $a-b=0$ ⇒ $a=b=\frac{1}{2}$
$T(1,0,1) = (0,1) = c(1,1) + d(1,-1)$ ⟶ $c+d=0$ ⇒ $c=-d$, $c-d=1$ ⇒ $-2d=1$ ⇒ $d=-\frac{1}{2}$, $c=\frac{1}{2}$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

→ Image and kernel of linear transformations
Let $f: V \to W$ be a linear transformation
↳ Kernel
$\ker(f) = \{v \in V \mid f(v) = 0\}$ ⟶ zeros of function

↳ Image
$\mathrm{Im}(f) = \{w \in W \mid \exists v \in V \text{ for which } f(v) = w\}$ ⟶ range of function

↳ A linear transformation f is 1-1 (injective) if and only if $\ker(f) = 0$
↳ A linear transformation f is onto (surjective) iff $\mathrm{Im}(f) = W$ for $f: V \to W$

↳ Kernel of a function is in the null space of matrix A, where A is the matrix corresponding to the transformation.
↳ A vector $w \in \mathrm{Im}(f)$ iff $d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$ is in the column space of A, where $w = \sum_{i=1}^{m} d_i w_i$

→ Bases for the kernel and image of a linear transformation

$T : V \to W$

↳ $B = RREF(A)$, where A is the matrix corresponding to T.

→ Basis for $\ker(T)$ = Basis for the null space of A if the basis for V and W were standard ordered basis.

→ Basis for $Im(T)$ = columns of A corresponding to the pivot columns in B.

↳ Example: Let $V = \mathbb{R}^2$, $W = \{(x,y,z) \mid x+y+z=0\}$. Respective ordered basis → $\beta = \{(1,1),(1,-1)\}$
Let $T : V \to W$, where $T(x,y) = (0, x+2y, -x-2y)$ $\quad \gamma = \{(-1,1,0),(-1,0,1)\}$

$T(1,1) = (0,3,-3) = a(-1,1,0) + b(-1,0,1)$
$T(1,-1) = (0,-1,1) = c(-1,1,0) + d(-1,0,1)$

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$$

$$RREF(A) = \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

$Rank(A) = 1$

Basis of null space $= \{(1/3, 1)\}$
Basis of $\ker(T) = (1/3)(1,1) + 1(1,-1)$
$= (4/3, -2/3)$

Basis of $Im(T) = 3(-1,1,0) - 3(-1,0,1)$
$= (0,3,-3)$

↳ Rank - nullity theorem for linear transformation

Let $T : V \to W$ be a linear transformation

→ $Rank(T) = \dim(Im(T))$
→ $Nullity(T) = \dim(\ker(T))$
→ $rank(T) + nullity(T) = \dim(V)$

WEEK 7

→ Equivalence of matrices

↳ Let A and B be two $m \times n$ matrices. A is equivalent to B if $B = QAP$ for some invertible $P_{m\times m}$ and $Q_{n\times n}$.
Characteristics:
① A be can be transformed into B by a combination elementary row and column operations.
② $\text{rank}(A) = \text{rank}(B)$

↳ Consider a linear transformation $T: V \to W$, two ordered bases β_1 and β_2 for V and γ_1 and γ_2 for W.
Let A → matrix corresponding to T w.r.t. β_1 and γ_1
B → " " " " " β_2 and γ_2

A is equivalent to B, where P → express β_2 in terms of β_1
Q → " γ_2 " " " γ_1

→ Similar matrices

↳ An $n\times n$ matrix A is similar to an $n \times n$ matrix B if there exists an $n\times n$ invertible matrix P such that $B = P^{-1}AP$

↳ Properties of similar matrices
① A and B are equivalent
② $\text{rank}(A) = \text{rank}(B)$
③ $\det(B) = \det(A)$
④ $\text{Trace}(A) = \text{Trace}(B)$

↳ Consider a linear transformation $T: V \to V$ and two ordered bases β and γ for V
Let A → matrix corresponding to T w.r.t. β
B → " " " " " γ

A is similar to B, where P → Express γ in terms of β.
P^{-1} → " β " " " γ.

→ Affine Subspace

↳ Let V be a vector space. An affine subspace of V is a subset L such that there exists $v \in V$ and a vector subspace $U \subseteq V$ such that
$$L = v + U := \{v + u \mid u \in U\}$$
↳ U is unique; v is not unique (can be any vector $v \in L$)
↳ Affine subspace are translates of a vector subspace of V.

↳ Solution set of system of linear equations
$Ax = b$
↳ If $b \in$ column space of A. Solution set L is an affine subspace of $\mathbb{R}^n$.
It can be described as → $L = v + \mathcal{N}(A)$ where v is any solution of $Ax = b$ and, $\mathcal{N}(A)$ → null space of A

example: Let $T(x,y,z) = (2x+3y+2,\ 4x-5y+3)$.
↳ This is not a linear transformation because $T(0,0,0) = (2,3) \neq (0,0)$

But this is an affine transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$
Let $T: V \to W$ be a lin. trans. and $w \in W$, the the mapping:
$T': V \to W$
$T'(v) = w + T(v)$, where $T(x,y,z) = (2x+3y,\ 4x-5y)$ and $w = (2,3)$

→ Length of a vector

↳ for any vector v , length = $\sqrt{\text{dot product of } v}$

example in $\mathbb{R}^2$

↳ $len(x,y) = \sqrt{(x,y)(x,y)} = \sqrt{x^2+y^2}$

in $\mathbb{R}^3$

↳ $len(x,y,z) = \sqrt{(x,y,z)(x,y,z)} = \sqrt{x^2+y^2+z^2}$

→ Angle between two vectors

↳ for any two vectors v and u , $\theta = \cos^{-1}\left(\frac{u \cdot v}{\sqrt{(v \cdot v) \times (u \cdot u)}}\right)$

$\theta = \cos^{-1} \frac{u \cdot v}{\|u\| \times \|v\|}$, where $\|u\| = \sqrt{u \cdot u}$

→ Inner product

↳ inner product on a vector space V is a function $\langle .,. \rangle : V \times V \to \mathbb{R}$

↳ $\langle v, v \rangle > 0$ for all $v \in V \setminus \{0\}$

$\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$

$\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$

$\langle c v_1, v_2 \rangle = \langle v_1, c v_2 \rangle = c \langle v_1, v_2 \rangle$

↳ example on $\mathbb{R}^2$:

$\langle .,. \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$\langle u, v \rangle = x_1 y_1 - (x_1 y_2 - x_2 y_1) + 2 x_2 y_2$, where $u = (x_1, x_2)$ and $v = (y_1, y_2)$

matrix form → $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

if this symmetric, then the function is an inner product.

→ Norm

↳ A norm on a vector space is a function

$\|.\| : V \to \mathbb{R}$

satisfies these conditions.

① $\|x + y\| \le \|x\| + \|y\| \quad \forall x, y \in V$

② $\|cx\| = |k| \|x\| \quad \forall c \in \mathbb{R}$ and $\forall x \in V$

③ $\|x\| \ge 0 \quad \forall x \in V$; $\|x\| = 0$ iff $x = 0$

↳ $\|v_i\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2}$, where $v_i = (x_1, x_2, x_3, \ldots, x_n)$

WEEK 8

→ Orthogonality and linear independence

- If the dot product of two vectors $= 0$, then they are orthogonal.
- Orthogonal set: $S = \{v_1, v_2, \dots, v_n\} \subseteq V$, S is an orthogonal set of vectors if:

$$\langle v_i, v_j \rangle = 0 \quad \forall\ i,j \in \{1, 2, \dots, n\} \text{ and } i \neq j$$

→ If a set of vectors is orthogonal, then they are linearly independent

- Orthogonal basis
  - A basis consisting of mutually orthogonal vectors.

  if $\dim(V) = n$, then
  orthogonal basis $\equiv$ orthogonal set of n vectors

→ Orthonormal basis

- orthogonal set of vectors such that the norm of each vector of the set $= 1$.

  If $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, S is orthonormal set if:

$$\langle v_i, v_j \rangle = 0 \quad \forall\ i,j \in \{1, 2, \dots, k\} \text{ and } i \neq j$$
$$\text{and} \quad \|v_i\| = 1 \quad \forall\ i \in \{1, 2, \dots, k\}$$

- Orthonormal basis
  - it is an orthogonal basis where the norm of each vector $= 1$.
  - is a maximal orthonormal set.

- to obtain an orthonormal set from an orthogonal set: divide each vector by its norm.

  → Let $\Gamma = \{v_1, v_2, \dots, v_k\}$ be an orthogonal set.
  Let β be an orthonormal set.

$$B = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots, \frac{v_k}{\|v_k\|} \right\}$$

example: orthogonal basis $= \{(1,3), (-3,1)\}$

orthonormal basis $= \left\{ \frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right\}$

- orthonormal basis is a very useful way to find the coefficients that make any vector v.

explanation → Suppose $\Gamma = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

To find $c_1, c_2, \dots, c_n$, we need to write a system of linear equations and solve. But since Γ is an orthonormal basis, we can use the inner product to find $c_i = \langle v, v_i \rangle$

Proof → $\langle v, v_i \rangle = \langle c_1 v_1 + c_2 v_2 + \dots + c_i v_i + \dots + c_n v_n, v_i \rangle$

$= c_1 \cancelto{0}{\langle v_1, v_i \rangle} + c_2 \cancelto{0}{\langle v_2, v_i \rangle} + \dots + c_i \langle v_i, v_i \rangle + \dots + c_n \cancelto{0}{\langle v_n, v_i \rangle}$ ← because they're orthogonal

$= c_i \langle v_i, v_i \rangle$ ← because they're orthonormal

$= c_i$

→ Projection of a vector to a subspace

orthonormal

↳ $V \to$ inner product space. $W \subseteq V \to$ a subspace. Projection of $v \in V$ onto W is the vector in W, denoted by $proj_W(v)$, computed as:

Find an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ for W

Define $proj_W(v) = \sum_{i=1}^{n} \langle v, v_i \rangle v_i$

example: $V = \mathbb{R}^2$, $W = \langle (3,1) \rangle$, $v = (1,3)$

$\left\{ \frac{1}{\sqrt{10}} (3,1) \right\}$ is the orthonormal basis for W

$$proj_W(v) = \langle v, \tfrac{1}{\sqrt{10}}(3,1) \rangle \tfrac{1}{\sqrt{10}}(3,1) = \langle (1,3), (3,1) \rangle \tfrac{1}{10} (3,1)$$

$$= \tfrac{6}{10}(3,1) = (1.8, 0.6)$$

orthogonal

↳ if $\{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis for a subspace W, then $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_n}{\|v_n\|} \right\}$ is an orthonormal basis for W and hence:

$$proj_W(v) = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} v_i = \sum_{i=1}^{n} proj_{v_i}(v)$$

example: W is 2-dim. subspace of $V = \mathbb{R}^3$ spanned by orthogonal vectors $v_1 = (1,2,1)$ and $v_2 = (1,-1,1)$. what is projection of $v = (-2,2,2)$ on W?

$$proj_{v_1}(v) = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \frac{4}{6}(1,2,1) = \frac{2}{3}(1,2,1)$$

$$proj_{v_2}(v) = \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = \frac{-2}{3}(1,-1,1)$$

$$proj_W(v) = proj_{v_1}(v) + proj_{v_2}(v)$$

$$= \frac{2}{3}(1,2,1) - \frac{2}{3}(1,-1,1)$$

$$= (0,2,0)$$

↳ Properties of the projection P_W

1. $P_W(v) = v \quad \forall\, v \in W$
2. $Range(P_W) = W$
3. $W^\perp = \{v \mid v \in V, \text{ such that } \langle v, w \rangle = 0 \; \forall\, w \in W\}$ is the null space of P_W.
4. $P_W^2 = P_W$
5. $\|P_W(v)\| \leq \|v\|$

→ Projection matrix iff: $A^T = A$ and $A^2 = A$

→ Gram-Schmidt process

↳ Let V be an inner product space with a basis $\{u_1, u_2, \ldots, u_n\}$.
Define orthogonal basis $\{v_1, v_2, \ldots, v_n\}$ and orthonormal basis $\{w_1, w_2, \ldots, w_n\}$ as:

$$v_1 = u_1 \; ; \; w_1 = v_1 / \|v_1\|$$
$$v_2 = u_2 - \langle u_2, w_1 \rangle w_1 \; ; \; w_2 = v_2 / \|v_2\|$$
$$v_3 = u_3 - \langle u_3, w_1 \rangle w_1 - \langle u_3, w_2 \rangle w_2 \; ; \; w_3 = v_3 / \|v_3\|$$
$$\vdots$$
$$v_n = u_n - \langle u_n, w_1 \rangle w_1 - \langle u_n, w_2 \rangle w_2 - \ldots - \langle u_n, w_{n-1} \rangle w_{n-1} \; ; \; w_n = v_n / \|v_n\|$$

→ Orthogonal Transformations

↳ Let $T: V \to V$. T is said to be orthogonal transformation if:

$$\langle Tv, Tw \rangle = \langle v, w \rangle \quad \forall\, v, w \in V$$

↳ Preserves the lengths and angles.

→ Orthogonal matrix iff: $AA^T = A^T A = I$

→ Rotation → preserves the lengths and angles

↳ Rotation in $\mathbb{R}^2$

$$T_\theta(1,0) = (\cos\theta, \sin\theta)$$
$$T_\theta(0,1) = (-\sin\theta, \cos\theta)$$

matrix → $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

↳ Rotation in $\mathbb{R}^3$

along the Z-axis $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

along the Y-axis $R_\theta = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$

along the X-axis $R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$

WEEK 9

→ Scalar - valued multivariable functions

↳ a function $f: D \to \mathbb{R}$, where D is a domain in $\mathbb{R}^n$ where $n > 1$.

examples → linear transformations, polynomial function

→ Vector - valued multivariable functions

↳ $f: D \to \mathbb{R}^m$, where D is a domain in $\mathbb{R}^n$ where $m, n > 1$.

example of $\mathbb{R}^3 \to \mathbb{R}^2$: $f(x,y,z) = (x^2 + z^2, y^2)$

→ Partial Derivatives

↳ If $\underset{\sim}{a}$ is a point in $\mathbb{R}^n$, then an open ball of radius r around $\underset{\sim}{a}$ is the set defined as:

$$\{x \in \mathbb{R}^n \mid \|x - a\| < r\}$$

↳ $e_1, e_2, \dots, e_n$ is the standard ordered basis of $\mathbb{R}^n$.

↳ Let $f(x_1, x_2, \dots, x_n)$ be a function on a domain D in $\mathbb{R}^n$ containing a point $\underset{\sim}{a}$ and an open ball around it.

The rate of change of f at $\underset{\sim}{a}$ w.r.t. the variable x_i is

$$\lim_{h \to 0} \frac{f(\underset{\sim}{a} + he_i) - f(\underset{\sim}{a})}{h}$$

$\Downarrow$

Let $g(h) = f(\underset{\sim}{a} + he_i)$, then

$$\lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

example: $f(x,y) = x + y$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f((x,y) + h(1,0)) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{x+h+y-x-y}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = 1$$

↳ To calculate partial derivative w.r.t. x_i, think of f only as a function of x_i and treat all other variables as constants.

example:

→ $f(x,y,z) = xy + yz + zx$

$$\frac{\partial f}{\partial x}(x,y,z) = y + 0 + z = y + z$$

$$\frac{\partial f}{\partial y}(x,y,z) = x + z$$

→ $f(x,y) = \sin(xy)$

$$\frac{\partial f}{\partial x} = y \cdot \cos(xy)$$

$$\frac{\partial f}{\partial y} = x \cdot \cos(xy)$$

→ Directional derivative

↳ to calculate rate of change in a particular direction at a point
→ choose a unit vector $u \in \mathbb{R}^n$ in the direction we want and compute:

$$\lim_{h\to 0} \frac{f(\underset{\sim}{a} + hu) - f(\underset{\sim}{a})}{h}$$

examples:

→ rate of change of $f(x,y) = x + y$ at $(0,0)$ in the direction of line $x = y$.
$u = (1/\sqrt{2}, 1/\sqrt{2})$

$$\lim_{h\to 0} \frac{f(0 + h/\sqrt{2}, 0 + h/\sqrt{2}) - f(0,0)}{h} = \lim_{h\to 0} \frac{2h/\sqrt{2}}{h} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

→ rate of change of $f(x,y,z) = xy + yz + zx$ at $(1,2,3)$ in the direction of vector $(4,3,0)$.
$u = (4/5, 3/5, 0)$

$$\lim_{h\to 0} \frac{f(1 + 4h/5, 2 + 3h/5, 3) - f(1,2,3)}{h} = \frac{8 + 3 + 9 + 12}{5} = \frac{32}{5}$$

↳ directional derivative of f in the direction of the unit vector u is the function $f_u(\underset{\sim}{x})$ and defined as

$$f_u(\underset{\sim}{x}) = \lim_{h\to 0} \frac{f(\underset{\sim}{x} + hu) - f(\underset{\sim}{x})}{h}$$

Properties:

① $(cf + g)_u(\underset{\sim}{a}) = cf_u(\underset{\sim}{a}) + g_u(\underset{\sim}{a})$

② $(fg)_u(\underset{\sim}{a}) = f_u(\underset{\sim}{a})g(\underset{\sim}{a}) + f(\underset{\sim}{a})g_u(\underset{\sim}{a})$

③ $(f/g)_u(\underset{\sim}{a}) = \dfrac{f_u(\underset{\sim}{a})g(\underset{\sim}{a}) - f(\underset{\sim}{a})g_u(\underset{\sim}{a})}{g(\underset{\sim}{a})^2}$

example:

→ $f(x,y) = x + y$ $\quad u = (u_1, u_2)$

$$f_u(x,y) = \lim_{h\to 0} \frac{f(x + hu_1, y + hu_2) - f(x,y)}{h} = \lim_{h\to 0} \frac{x + hu_1 + y + hu_2 - (x+y)}{h}$$

$$= \lim_{h\to 0} \frac{hu_1 + hu_2}{h} = u_1 + u_2$$

→ Limits of scalar-valued multivariable function at a point

↳ Let f → scalar-valued multi-variable function and $\underset{\sim}{a}$ be a point such that there exists a sequence in the domain that converges to $\underset{\sim}{a}$. If there exists L, $L \in \mathbb{R}$ such that $f(\underset{\sim}{a}_n) \to L$ for all sequences $\underset{\sim}{a}_n$ such that $\underset{\sim}{a}_n \to \underset{\sim}{a}$, then limit of f at $\underset{\sim}{a}$ exists and equals L.

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = L$$

Rules: If $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = F$ and $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} g(\underset{\sim}{x}) = G$, and $c \in \mathbb{R}$

→ $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} (cf+g)(\underset{\sim}{x}) = cF + G$

→ $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} (fg)(\underset{\sim}{x}) = FG$

→ $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} (f/g)(\underset{\sim}{x}) = \frac{F}{G}$

example:

→ $h(x,y,z) = x^2y^3 + y^2z^2 + xyz$

$\lim_{(x,y,z) \to (1,2,3)} h(x,y,z) = (1)^2(2)^3 + (2)^2(3)^2 + (1)(2)(3) = 86$

Composition → If $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = F$, $\lim_{x \to F} g(x) = L$

$\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} (g \circ f)(\underset{\sim}{x}) = L$

Sandwich → If $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = L$, $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} g(\underset{\sim}{x}) = L$ and $f(\underset{\sim}{x}) \leq h(\underset{\sim}{x}) \leq g(\underset{\sim}{x})$

$\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} h(\underset{\sim}{x}) = L$

→ Limit of a vector-valued function at a point

↳ $f : D \to \mathbb{R}^m$, $D \in \mathbb{R}^k$

If f_i is i^{th} component of the function, then f_i is a scalar-valued function $D \to \mathbb{R}$

example:

→ $\lim_{\underset{\sim}{x} \to (1,2)} \left(x^2y + y^3, e^{xy}, \frac{x^2-1}{y^2-2} \right) = \left(\lim_{\underset{\sim}{x} \to (1,2)} x^2y + y^2, \lim_{\underset{\sim}{x} \to (1,2)} e^{xy}, \lim_{\underset{\sim}{x} \to (1,2)} \frac{x^2-1}{y^2-2} \right)$

$= (10, e^2, 0)$

→ To find the limit of a function, find the limit of that function at the chosen point, along different curves and if they match, the limit exists.

Tip: check along the axes, check along $x_1 = x_2 = x_3 = \dots = x_n$ in $\mathbb{R}^n$

→ Directional derivative in terms of the gradient

↳ ∇f at $\underset{\sim}{a}$ is the vector $(f_{x_1}(\underset{\sim}{a}), f_{x_2}(\underset{\sim}{a}), \dots, f_{x_n}(\underset{\sim}{a}))$ in $\mathbb{R}^n$ $\quad \nabla f(\underset{\sim}{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$ in $\mathbb{R}^n$

example:

→ $f(x,y) = \sin(xy)$ $\quad \frac{\partial f}{\partial x} = y \cdot \cos(xy)$ $\quad \frac{\partial f}{\partial y} = x \cdot \cos(xy)$

$\nabla f(x,y) = (y \cdot \cos(xy), x \cdot \cos(xy))$; $\nabla f(\pi, 1) = (\cos(\pi), \pi \cos(\pi))$
$= (-1, -\pi)$

↳ Properties

→ $\nabla(cf+g)(\underset{\sim}{x}) = c\nabla f(\underset{\sim}{x}) + \nabla g(\underset{\sim}{x})$

→ $\nabla(fg)(\underset{\sim}{x}) = \nabla f(\underset{\sim}{x}) g(\underset{\sim}{x}) + f(\underset{\sim}{x}) \nabla g(\underset{\sim}{x})$

→ $\nabla(f/g)(\underset{\sim}{x}) = \frac{\nabla f(\underset{\sim}{x}) g(\underset{\sim}{x}) - f(\underset{\sim}{x}) \nabla g(\underset{\sim}{x})}{(g(\underset{\sim}{x}))^2}$

↳ Direction derivative at point $\underset{\sim}{a}$ in the direction of u: $f_u(\underset{\sim}{a}) = \nabla f(\underset{\sim}{a}) \cdot u$

——— ✗ ——— ✗ ——— ✗ ———

WEEK 1 GA

(1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A - xI = \begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix}$

$$\det(A - xI) = (a-x)(d-x) - bc$$
$$= ad - ax - xd + x^2 - bc$$
$$= \underbrace{ad - bc}_{\det(A)} \underbrace{- x(a+d)}_{-x\,\mathrm{trace}(A)} + \underbrace{x^2}_{x^2}$$

(2) $3A = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$ $\quad \det(3A) = 9ad - 9bc = (3)^2 \det(A)$

$3A = \begin{bmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$ $\quad |3A| = 3^3|A| = \boxed{27|A|}$

(3) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad I + A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad 5A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$

$5A + I = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$

$(I + A)^3 - (5A + I) = mA$

$\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$ $\quad \boxed{m = 8}$

(4) $A = \begin{bmatrix} 20 & 30 & 40 \\ 8 & 16 & 24 \\ 8 & 10 & 12 \end{bmatrix}$ $\quad \det(A) = 20 \det\begin{bmatrix} 16 & 24 \\ 10 & 12 \end{bmatrix} - 30 \det\begin{bmatrix} 8 & 24 \\ 8 & 12 \end{bmatrix} + 40 \det\begin{bmatrix} 8 & 16 \\ 8 & 10 \end{bmatrix}$

$$= 20(-48) - 30(-96) + 40(-48) = -960 + 2880 - 1920 = \boxed{0}$$

(5) $\det(A) = 3$ $\quad \det(B) = 3$ $\quad \det(B^{-1}) = 1/3$

$$\det(3A^2B^{-1}) = (3)^3 \cdot (\det(A))^2 \cdot \left(\frac{1}{3}\right) = 9(3)^2 = \boxed{81}$$

(6) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ $\quad A^3 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix}$ $\quad A^4 = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}$

$A^7 = \begin{bmatrix} 729 & 729 & 729 \\ 729 & 729 & 729 \\ 729 & 729 & 729 \end{bmatrix}$ $\quad$ sum of diagonals $= \boxed{2187}$

(7) $A = \begin{bmatrix} 12 & 19 & 26 \\ 17 & 24 & 31 \\ 22 & 29 & 36 \end{bmatrix}$ $\quad \det(A) = 12 \det\begin{bmatrix} 24 & 31 \\ 29 & 36 \end{bmatrix} - 19 \det\begin{bmatrix} 17 & 31 \\ 22 & 36 \end{bmatrix} + 26 \det\begin{bmatrix} 17 & 24 \\ 22 & 29 \end{bmatrix}$

$$= 12(-35) - 19(-70) + 26(-35) = \boxed{0}$$

(9)
$$30x_1 + 20x_2 + 25x_3 = 670 \quad \text{— (1)}$$
$$20x_1 + 35x_2 + 25x_3 = 730 \quad \text{— (2)}$$
$$20x_1 + 10x_2 + 15x_3 = 400 \quad \text{— (3)}$$

$(2) - (3) \Rightarrow$ $\begin{array}{cccc} 20 & 35 & 25 & 730 \\ -20 & -10 & -15 & -400 \end{array}$

$= 25x_2 + 10x_3 = 330$ — (4)

$10x_1 + 5x_3 = 130$ — (5)

$(1) - 2\cdot(3) \Rightarrow$ $\begin{array}{cccc} 30 & 20 & 25 & 670 \\ -40 & -20 & -30 & -800 \end{array}$

$= -10x_1 - 5x_3 = -130$

$2(5) = 20x_1 + 10x_3 = 260 \Rightarrow 20x_1 = 260 - 10x_3$

$260 - 10x_3 + 10x_2 + 15x_3 = 400$

$\Rightarrow 10x_2 + 5x_3 = 140$ — (6)

$(4) - 2\cdot(6) \Rightarrow$ $\begin{array}{cccc} 0 & 25 & 10 & 330 \\ 0 & -20 & -10 & -280 \end{array}$

$= 5x_2 = 50$ $\quad \boxed{x_2 = 10}$

$250 + 10x_3 = 330 \Rightarrow \boxed{x_3 = 8}$

$10x_1 + 40 = 130 \Rightarrow \boxed{x_1 = 9}$

$x_1 + x_2 + x_3 = 9 + 10 + 8 = \boxed{27}$

(10) $A = \begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix}$

WEEK 2 GA

1) $P(1) = -45$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -45 \\ 4 & 2 & 1 & -19 \\ 9 & 3 & 1 & 3 \end{array}\right] \xrightarrow{\substack{R_2-4R_1 \\ R_3-9R_1}} \begin{matrix} 1 & 1 & 1 & -45 \\ 0 & -2 & -3 & 161 \\ 0 & -6 & -8 & 408 \end{matrix} \xrightarrow{R_2-\frac{1}{2}R_2} \begin{matrix} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & -6 & -8 & 408 \end{matrix} \xrightarrow{R_3+6R_2} \begin{matrix} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{matrix} \xrightarrow{\substack{R_1-R_2 \\ -R_3/2}} \begin{matrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 1 & -75 \end{matrix} \xrightarrow{R_2-R_3} \begin{matrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 32 \\ 0 & 0 & 1 & -75 \end{matrix}$$

$P(x) = -2x^2 + 32x - 75$ $\qquad$ $P'(x) = -4x + 32$

$\frac{-32}{-4} = x \qquad x = 8$

$P(8)$ is peak

$P(8) = 53$

4)

$$\begin{matrix} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & c & d \end{matrix} \xrightarrow{\substack{R_2-2R_1 \\ R_3-4R_1}} \begin{matrix} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{matrix} \xrightarrow{\substack{R_3+3R_2 \\ R_1-2R_2}} \begin{matrix} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

if $c = 7$, $d = 4000$

comic book = ?

if $c=2$, $d=4000$

$$\begin{matrix} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 4000 \end{matrix} \xrightarrow{\substack{R_2-2R_1 \\ R_3-4R_1}} \begin{matrix} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -2 & 0 \end{matrix}$$

if $c=7$, $d=3000$

$$\begin{matrix} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 7 & 3000 \end{matrix} \xrightarrow{\substack{R_2-2R_1 \\ R_3-4R_1}} \begin{matrix} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & -1000 \end{matrix}$$

if $c=2$, $d=3000$

$$\begin{matrix} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 3000 \end{matrix} \xrightarrow{\substack{R_2-2R_1 \\ -R_3-4R_1}} \begin{matrix} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 1000 \end{matrix} \xrightarrow{R_3-3R_2} \begin{matrix} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 1000 \end{matrix} \Rightarrow \begin{pmatrix} 400 \\ 200 \\ 200 \end{pmatrix}$$

6)

$(-\frac{1}{2}) + (-\frac{1}{2}) + a/6 = -1 \Rightarrow a = 0$

$(-\frac{1}{3}) + b/6 = 0 \Rightarrow \frac{b}{6} = \frac{1}{3} \Rightarrow b = 2$

$(\frac{1}{2}) + (-\frac{1}{6}) + c/6 = 0 \Rightarrow \frac{c-1}{6} = -\frac{1}{2} \Rightarrow c-1 = -3 \Rightarrow c = -2$

$a + b + c = 0$

7)

$$\begin{matrix} 1 & 3k & 3k+4 & 61 \\ 1 & k+4 & 4k+2 & 65 \\ 1 & 2k+2 & 3k+4 & 66 \end{matrix} \xrightarrow{\substack{R_2-R_1 \\ R_3-R_1}} \begin{matrix} 1 & 3k & 3k+4 & 61 \\ 0 & -2k+4 & k-2 & 4 \\ 0 & -k+2 & 0 & 5 \end{matrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{matrix} 1 & 3 & 7 & 61 \\ 0 & 1 & 0 & 5 \\ 0 & 2 & -1 & 4 \end{matrix} \xrightarrow{-R_3+2R_2} \boxed{\left[\begin{array}{ccc|c} 1 & 3 & 7 & 61 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right]}$$

$\frac{4k+2-3k-4}{k-2}$ $\qquad$ $-k+2 = 1$

$k = 1$

8)

$$\begin{matrix} 2 & 3 & 5 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 44 \end{matrix} \xrightarrow{\substack{R_1-R_2 \\ R_2-R_3}} \begin{matrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 1 & 1 & 2 & 44 \end{matrix} \xrightarrow{R_3-R_1} \begin{matrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 0 & 44 \end{matrix}$$

→ shows no solution

9)

$$\begin{matrix} 1 & 3 & 0 & 0 \\ 4 & 1 & 5 & 34 \\ 2 & 2 & 7 & 32 \\ 3 & 9 & 0 & 0 \end{matrix} \xrightarrow{\substack{R_2-4R_1 \\ R_3-2R_1 \\ R_4-3R_1}} \begin{matrix} 1 & 3 & 0 & 0 \\ 0 & -11 & 5 & 34 \\ 0 & -4 & 7 & 32 \\ 0 & 0 & 0 & 0 \end{matrix}$$

R_4 shows $\det(A) = 0$

10)

$$\begin{matrix} 7 & 2 & 1 & 8 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{matrix} \xrightarrow{R_1+2R_3} \begin{matrix} 1 & 10 & -3 & 24 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{matrix} \xrightarrow{\substack{R_3+3R_1 \\ R_2/3}} \begin{matrix} 1 & 10 & -3 & 24 \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 34 & -11 & 80 \end{matrix} \xrightarrow{\substack{R_1-10R_2 \\ R_3-34R_2}} \begin{matrix} 1 & 0 & \frac{1}{3} & \frac{32}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & \frac{1}{3} & \frac{104}{3} \end{matrix} \xrightarrow{3R_3} \begin{matrix} 1 & 0 & \frac{1}{3} & \frac{32}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 104 \end{matrix}$$

$x = -24$

$y = 36$ $\qquad$ $x + y + z = 116$

$z = 104$

(11) $A = \begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix}$ $\quad A^T A = \underset{4\times1}{\begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix}} \underset{1\times4}{\begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix}} = \begin{matrix} 4 & 20 & 4 & 16 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{matrix} \xrightarrow{R_1/4} \begin{matrix} 1 & 5 & 1 & 4 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{matrix} \xrightarrow[\substack{R_3-4R_1 \\ R_4-16R_1}]{R_2-20R_1} \begin{matrix} 1 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$

(12) $\begin{matrix} 4 & 2 & 1 & 4 \\ 9 & 3 & 1 & 6 \\ 16 & 4 & 1 & 10 \end{matrix} \xrightarrow{R_1/4} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 9 & 3 & 1 & 6 \\ 16 & 4 & 1 & 10 \end{matrix} \xrightarrow[R_3-16R_1]{R_2-9R_1} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & -3/2 & -5/4 & -3 \\ 0 & -4 & -3 & -6 \end{matrix} \xrightarrow[-R_3]{-2R_2} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & 3 & 5/2 & 6 \\ 0 & 4 & 3 & 6 \end{matrix} \xrightarrow{-R_3 \frac{4}{3}R_2} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & 3 & 5/2 & 6 \\ 0 & 0 & 1/3 & 2 \end{matrix} \xrightarrow{3R_3} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & 3 & 5/2 & 6 \\ 0 & 0 & 1 & 6 \end{matrix}$

$a = 1$ $\qquad a - b + c = 10$
$b = -3$
$c = 6$

(13) $\left[\begin{matrix} 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{matrix}\,\middle|\,\begin{matrix} 1000 \\ 900 \\ c \\ 400 \end{matrix}\right] \xrightarrow[R_4-R_1]{R_1/2} \begin{matrix} 1 & 3/2 & 0 & 0 & 500 \\ 0 & 3 & 2 & 0 & 900 \\ 0 & 0 & 2 & 1 & c \\ 0 & -3 & 0 & 0 & -600 \end{matrix} \xrightarrow{R_2/3} \begin{matrix} 1 & 3/2 & 0 & 0 & 500 \\ 0 & 1 & 2/3 & 0 & 300 \\ 0 & 0 & 2 & 1 & c \\ 0 & 3 & 0 & 0 & 600 \end{matrix} \xrightarrow[R_1-\frac{3}{2}R_2]{-R_4-3R_2} \begin{matrix} 1 & 0 & -1 & 0 & 50 \\ 0 & 1 & 2/3 & 0 & 300 \\ 0 & 0 & 2 & 1 & c \\ 0 & 0 & 2 & 0 & 300 \end{matrix} \xrightarrow{R_3/2} \begin{matrix} 1 & 0 & -1 & 0 & 50 \\ 0 & 1 & 2/3 & 0 & 300 \\ 0 & 0 & 1 & 1/2 & c/2 \\ 0 & 0 & 2 & 0 & 300 \end{matrix}$

$\xrightarrow[\substack{R_2-\frac{1}{3}R_4 \\ -R_4-2R_3}]{R_1+\frac{1}{2}R_4} \begin{matrix} 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & 200 \\ 0 & 0 & 1 & 1/2 & c/2 \\ 0 & 0 & 0 & 1 & -300+c \end{matrix} \xrightarrow{R_3-\frac{1}{2}R_4} \begin{matrix} 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & 200 \\ 0 & 0 & 1 & 0 & 150 \\ 0 & 0 & 0 & 1 & c-300 \end{matrix}$

(W.N.) $x_1 = 200$
(E.N.) $x_2 = 200$ $\qquad c = 300$
(E.S.) $x_3 = 150$
(W.S.) $x_4 = 0$

min $\rightarrow x_4 = 0 \rightarrow x_1 = 200, x_2 = 200, x_3 = 150$
max $\rightarrow x_4 = 300 \rightarrow x_1 = 50, x_2 = 300, x_3 = 0$

(W.N.) $\rightarrow$ 400 $\rightarrow$ 100
(E.N.) $\rightarrow$ 900 $\rightarrow$ 600
(E.S.) $\rightarrow$ 300 $\rightarrow$ 0
(W.S.) $\rightarrow$ 300 $\rightarrow$ 0

WEEK 3 GA

(2) $-x = -1$
$-2y = 1$
$y = 1$
$x = -1$

(3) $P = 30$ $\quad C = 11$ $\quad F = 53$ $\quad Ca = 213$ $\qquad$ Type A = Ba, mi, al $\qquad$ Type B = Ap, mi, al

$A = \begin{bmatrix} 2 & 4 & 6 & 30 \\ 3 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 5 & 100 & 1 & 213 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 6 & 30 \\ 2 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 6 & 100 & 1 & 213 \end{bmatrix}$

(5) $A = \begin{bmatrix} c & 1 & -1 \\ -1 & 0 & -3 \\ -2 & -1 & c \end{bmatrix}$

$-1(-c-6) + -3c - 1 = 0$
$c + 6 - 3c - 1 = 0$
$-2c + 5 = 0$
$c = 2.5$

(6) $A = \begin{bmatrix} 7 & 5 & 11 \\ 9 & 6 & 15 \\ 2 & 2 & c \end{bmatrix}$

$\det(A) = 7(6c-30) - 5(9c-30) + 11(18-12)$
$= 42c - 210 - 45c + 150 + 66$
$0 = -3c + 6$
$c = 2$

(8) $(1,0,0), (1/2, 1/2, 0), (0,1,0)$

(9) $X_0 = (1/3, 1/3, 1/3)$ $\qquad X_1 = X_0 \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (1/2, 1/2, 0)$ $\qquad X_2 = X_1 \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (3/4, 1/4, 0)$

(10) $X_0 = (a, b, 1-a-b) \qquad X_1 = (a + b/2,\ 1-a-\frac{b}{2},\ 0)$

$$X_1 = (a, b, 1-a-b)\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \left(\frac{2a+b}{2}, \frac{b+2-2a-2b}{2}, 0\right) = \left(\frac{2a+b}{2}, \frac{2-2a-b}{2}, 0\right)$$

$$X_2 = \left(\frac{2a+b}{2}, \frac{2-2a-b}{2}, 0\right)\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \left(\frac{4a+2b+2-2a-b}{4}, \frac{2-2a-b}{4}, 0\right) = \left(\frac{2a+b+2}{4}, \frac{2-2a-b}{4}, 0\right)$$

$$\begin{bmatrix} a & \frac{2a+b+2}{4} \\ b & \frac{2-2a-b}{4} \\ 1-a-b & 0 \end{bmatrix}$$

$\frac{4au + 2ay + by + 2y}{4} = 0 \quad — ① \quad \Rightarrow \quad y(2a+b+2) = -4au$

$\frac{4bu + 2y - 2ay - by}{4} = 0 \quad — ② \quad \Rightarrow \quad y(2-2a-b) = -4bu$

$u - au - bu = 0 \quad \Rightarrow \quad u - bu = au$

$① - ② \quad \Rightarrow \quad \frac{4au - 4bu + 4}{4} = 0 \quad \Rightarrow \quad au - bu + 1 = 0$

$\Downarrow$

$u - bu - bu + 1 = 0$

$u - 2bu + 1 = 0$

MOCK WEEK 1 - 2

(3) $\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} \qquad \lambda = 6$

(4) $a + 4 = 2a + 2 \quad \Rightarrow \quad 2 = a$

$3b = b + 2 \quad \Rightarrow \quad 2b = 2 \quad \Rightarrow \quad b = 1 \qquad a + 2b = 4$

(6) $\left[\begin{array}{cc|c} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{array}\right] \xrightarrow[R_1 \cdot \sqrt{2}]{R_2 + R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2/\sqrt{2} & 0 \end{array}\right]$

$A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A) \qquad \operatorname{Adj}(A) = C^T \qquad \det(A) = \frac{1}{2} + \frac{1}{2} = 1$

$C^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = A^T = A^{-1}$

(7) $\det(A) = b + a$

(8) $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \\ -1 & 6 & 7 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ -1 & 6 & 7 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{bmatrix} \xrightarrow[R_2/2]{R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$

(13) $\begin{bmatrix} 2 & 0.5 & 1 & 80 \\ 4 & 1 & 2 & 100 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[R_1/2]{R_2-2R_1} \begin{bmatrix} 1 & 0.25 & 0.5 & 40 \\ 0 & 0 & 0 & -60 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & 0.25 & 0.5 & 40 \\ 0 & 0 & 0 & -60 \\ 0 & 0.5 & 1 & 0 \end{bmatrix}$

(14) $\begin{bmatrix} 2 & 2 & 1 & 80 \\ 4 & 1 & 2 & d \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[R_1/2]{R_2-2R_1} \begin{bmatrix} 1 & 1 & \frac{1}{2} & 40 \\ 0 & -3 & 0 & d-160 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[-R_2/3]{R_3-2R_1} \begin{bmatrix} 1 & 1 & \frac{1}{2} & 40 \\ 0 & 1 & 0 & -\frac{d}{3}+\frac{160}{3} \\ 0 & -1 & 1 & 0 \end{bmatrix}$

$x_2 = x_3$

(15) $\begin{bmatrix} 2 & 3 & 1 & 80 \\ 4 & 1 & 2 & 100 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[R_1/2]{R_2-2R_1} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 40 \\ 0 & -5 & 0 & -60 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[R_3-2R_1]{-R_2/5} \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 40 \\ 0 & 1 & 0 & 12 \\ 0 & -2 & 1 & 0 \end{bmatrix}$

$x_2 = 12$

$x_3 = 2x_2 = 24$

$x_1 = 40 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \Rightarrow x_1 = 40 - \frac{3}{2}(12) - \frac{1}{2}(24)$

$= 40 - 18 - 12 = 10$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 24 \end{bmatrix}$

WEEK 4 GA

(4) if $A_{4\times4}$, then $A = \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{matrix}$, then rank(A) = 4

if $A_{2021\times2021}$, then rank(A) = 2021

(5) $8x_1 + 10x_5 = 0 \qquad 9x_2 + 6x_4 = 0$

(6) $A = \begin{bmatrix} 2 & -4 & 1 & -9 \\ 1 & -3 & 1 & -1 \\ 3 & -7 & 2 & -6 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \quad \begin{matrix} 1 & -3 & 1 & -1 \\ 0 & 10 & 1 & 11 \\ 0 & 16 & 1 & 9 \end{matrix}$

(7) $A = \begin{bmatrix} -10 & 0 & -3 \\ -5 & -4 & -5 \\ -8 & 8 & -3 \end{bmatrix} \begin{matrix} -R_1/10 \\ R_2+5R_1 \\ R_3+8R_1 \end{matrix} \quad \begin{matrix} 1 & 0 & 3/10 \\ 0 & -4 & -35/10 \\ 0 & 8 & -6/10 \end{matrix} \quad \begin{matrix} -R_2/4 \\ R_3+2R_2 \end{matrix} \quad \begin{matrix} 1 & 0 & 3/10 \\ 0 & 1 & 35/40 \\ 0 & 0 & -76/10 \end{matrix}$

$B = \begin{bmatrix} -10 & -5 & -8 \\ 0 & -4 & 8 \\ -3 & -5 & -3 \end{bmatrix} \begin{matrix} -R_1/10 \\ R_3+3R_1 \\ -R_2/4 \end{matrix} \quad \begin{matrix} 1 & 1/2 & 4/5 \\ 0 & 1 & -2 \\ 0 & -7/2 & -3/5 \end{matrix}$

(8) $v_1 = (-1,0,1) \quad v_2 = (2,1,-1)$

$v_1 + v_2 = (1,1,0) \qquad 3v_1 + 0v_2 = (-3,0,3)$

$(5,2,-3)$, $(8,\cancel{5,-4})$, $(-5/6, \cancel{5/6}, 5/6)$, $(-2,\cancel{3},3)$

$b=2$, $a-b=-3$, $a=-1$, $2b-a=5$

$b=5$, $a-b=-4$, $a=1$, $2b-a \neq 8$

$b=5/6$, $a-b=5/6$, $a=10/6$, $2b-a \neq -5/6$

$b=3$, $a-b=3$, $a=6$, $2b-a \neq -2$

(9) $v_1 = (1,0,1) \quad v_2 = (0,1,-1) \qquad av_1 + bv_2 = (20,-13,c) \qquad a = 20 \quad b = -13 \qquad a-b = c$

$20-(-13) = 33$

(10) $(x,y,z) = (0,q,1)$, then $1 = 0 + 8q \qquad q = 1/8 \qquad p = 1/9$

$q \cdot p = \frac{1}{72} \qquad \frac{1}{q \cdot p} = 72$

(11) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & -a_{11}-a_{22}-a_{33} \end{bmatrix}$

WEEK 5 GA

(2) $x_1 C + 2x_1 O + 2x_2 H + x_2 O = 6x_3 C + 12x_3 H + 6x_3 O + 2x_4 O$

$x_1 C + (2x_1 + x_2)O + 2x_2 H = 6x_3 C + 12x_3 H + (6x_3 + 2x_4)O$

$x_1 = 6x_3$

$2x_1 + x_2 = 6x_3 + 2x_4$

$x_2 = 6x_3 \implies x_1 = x_2 = 6x_3$

$3x_1 = 6x_3 + 2x_4 \Rightarrow 2x_1 = 2x_4 \Rightarrow x_1 = x_4$

$x_1 = x_2 = x_4 = 6x_3$

(5) $T(v) = \begin{bmatrix} -18x - 11y \\ -15y \end{bmatrix}$ for some v', $T(v) = \begin{bmatrix} z \\ w \end{bmatrix}$ $\quad z = -18x - 11y \rightarrow z = -18x + \frac{11w}{15} \rightarrow -\frac{z}{18} + \frac{11w}{(15)(18)} = x$

$w = -15y \rightarrow y = -\frac{w}{15}$

(6) $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} R_1/3 \\ R_3 - R_1 \\ R_2/5 \end{matrix}$ $\begin{matrix} 1 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$ $\begin{matrix} R_2 + R_3 \\ R_3/3 \\ R_1 - R_3/3 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ nullity$(A) = 1$

(8)

$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$$

WEEK 6 GA

(2) $T(1,0,1) = (1,0,0)$

$T(0,1,0) = (0,1,0)$

$T(0,0,1) = (1,0,1)$

$(1,0,0) = (1,0,1) - (0,0,1) \Rightarrow T(1,0,0) = (1,0,0) - (1,0,1) = (0,0,-1)$

$(0,1,0) = (0,1,0) \Rightarrow T(0,1,0) = (0,1,0)$

$(0,0,1) = (0,0,1) \Rightarrow T(0,0,1) = (1,0,1)$

(3) i) $\underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ 1-x-y \end{bmatrix}} = \begin{bmatrix} x \\ y \\ -x-y+3 \end{bmatrix}$ $\quad z = -x - y + 3$, $3 = x + y + z$ $\quad$ i) $\rightarrow$ b)

ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1-x-y \end{bmatrix} = \begin{bmatrix} x \\ y \\ -1+x+y \end{bmatrix}$ $\quad z = -1 + x + y$, $1 = x + y - z$ $\quad$ ii) $\rightarrow$ d)

iii) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ -1+x \end{bmatrix}$ $\quad y = -1 + x$, $1 = x - y$ $\quad$ iii) $\rightarrow$ a) $\rightarrow$ 3

iv) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 2x+1-x \end{bmatrix} = \begin{bmatrix} x \\ x+1 \end{bmatrix}$ $\quad y = x + 1$, $x - y = -1$ $\quad$ iv) $\rightarrow$ c) $\rightarrow$ 1

(4) $S\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad S\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad S\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad S\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5 $\begin{bmatrix} 1 & 0 & \gamma \\ 0 & \alpha & \gamma \\ 0 & \beta & \frac{\gamma\beta}{\alpha} \end{bmatrix} \xrightarrow{R_3/\beta} \begin{bmatrix} 1 & 0 & \gamma \\ 0 & \alpha & \gamma \\ 0 & 1 & \gamma/\alpha \end{bmatrix} \xrightarrow{R_3 \cdot \alpha} \begin{bmatrix} 1 & 0 & \gamma \\ 0 & \alpha & \gamma \\ 0 & \alpha & \gamma \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & \gamma \\ 0 & \alpha & \gamma \\ 0 & 0 & 0 \end{bmatrix}$ Rank(T) = 2

$T(x,y,z) = (2x-z, 3y-2z, z, 0)$

$\underset{4\times 3}{\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \xrightarrow[R_2+2R_3]{R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ind. vec. → u_1, u_2, u_4

stand. basis. of image $= \{(1,0,0), (0,0,1), (0,1,0)\}$

6 $T(2,3,1) = 6$ $T(3,4,2) = 8$ $T(5,5,3) = 10$

$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ 3 & 4 & 2 & 8 \\ 5 & 5 & 3 & 10 \end{array}\right]$ matrix of T $= \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}$

11 $T(x,y,z) = (2x+7y, 4z)$

$T(1,0,0) = (2,0) = a(1,0) + b(0,1)$
$T(0,1,0) = (7,0) = c(1,0) + d(0,1)$
$T(0,0,1) = (0,4) = e(1,0) + g(0,1)$

matrix of $T = A = \begin{bmatrix} a & c & e \\ b & d & g \end{bmatrix} = \begin{bmatrix} 2 & 7 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$RREF(A) = \begin{bmatrix} 1 & 7/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Basis of null space $= \{(-7/2, 1, 0)\}$

$Im(T) = 2(1,0) + 0(0,1) = (2,0)$
$0(1,0) + 4(0,1) = (0,4)$

Basis for Range(T) $= \{(2,0), (0,4)\}$

WEEK 7 GA

1 i) matrix → $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ iv) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2 $T(1,0) = (2,1) = 2(1,0) + 1(0,1)$
$T(0,1) = (1,1) = 1(1,0) + 1(0,1)$

$S(1,0) = (1,1) = 1(1,0) + 1(0,1)$
$S(0,1) = (c,2) = c(1,0) + 2(0,1)$

$T \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $S \rightarrow \begin{bmatrix} 1 & c \\ 1 & 2 \end{bmatrix}$

if $c = 1$, $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $PT = SP$ $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} 2a+b & a+b \\ 2c+d & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$

$a+b-c = 0$
$a+d = 0$
$d-a = 0$
$c-b-d = 0$

$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$

$\begin{bmatrix} a+b-c & a+d \\ d-a & c-b-d \end{bmatrix} = 0$

if $c = 2$, $S = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

3 $\langle (3,1), \gamma \rangle = 4$, $3\gamma_1 + \gamma_2 = 4$ $\langle (6,2), \gamma \rangle = 8$, $6\gamma_1 + 2\gamma_2 = 8$

$\left[\begin{array}{cc|c} 3 & 1 & 4 \\ 6 & 2 & 8 \end{array}\right] \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ if $\gamma_1 = t$, then $\gamma_2 = 4-3t$

(5) $L = (0,1,1) + U$, where $U = \{(1,1,0),(1,0,1)\}$

$T(1,0,1) = (0,0,0)$
$T(1,1,0) = (1,0,0)$

(6) $\cos(\theta) = \dfrac{u \cdot v}{\|u\| \times \|v\|}$ $\quad \|u\| = \sqrt{74} \quad \|v\| = \sqrt{41}$

$u \cdot v = 0$

(7) $c_1(1,2,0) + c_2(2,-1,0) + c_3(0,0,2) = \left(x, y, \frac{3x+y}{5}\right)$

$c_1 + 2c_2 = x$
$2c_1 - c_2 = y$
$2c_3 = \dfrac{3x+y}{5} \rightarrow \dfrac{3(c_1+2c_2) + 2c_1 - c_2}{10} = c_3$

$\dfrac{3c_1 + 6c_2 + 2c_1 - c_2}{10} = c_3 \Rightarrow \dfrac{c_1 + c_2}{2} = c_3 \Rightarrow c_3 = 2$

(8) $\langle (a,b),(9,5) \rangle_1 = 9a - 5a - 9b + 20b = 119 \Rightarrow 4a + 11b = 119$ $\quad \left[\begin{array}{cc|c} 4 & 11 & 119 \\ 27 & 10 & 225 \end{array}\right]$
$\langle (a,b),(9,5) \rangle_2 = 27a + 10b = 225$
$a = 5 \quad b = 9 \qquad a + 2b = 23$

(10) $\|v\| = \sqrt{(x-16)^2 + 5}$

WEEK 8 GA

(1) b) $\left\{\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{2}}(-1,0,-1)\right\}$ $\quad \|v_1\| = \langle (1/\sqrt{2}, 0, -1/\sqrt{2}), (1/\sqrt{2}, 0, -1/\sqrt{2}) \rangle = 1$ $\quad \langle v_1, v_2 \rangle = 0$
$\rightarrow$ iv) $\quad \|v_2\| = \langle (-1/\sqrt{2}, 0, -1/\sqrt{2}), (-1/\sqrt{2}, 0, -1/\sqrt{2}) \rangle = 1$

c) $\{(2,3,4),(-1,2,-1),(0,4,-3)\} \rightarrow$ i)
$\langle v_1, v_2 \rangle = 0 \quad \langle v_1, v_2 \rangle = 0$
$\langle v_2, v_3 \rangle = 11$

d) $\{(2,3,4),(-1,2,-1),(11,2,-7)\} \rightarrow$ ii)
$\langle v_1, v_2 \rangle = 0 \quad \langle v_1, v_3 \rangle = 0$
$\langle v_2, v_3 \rangle = 0$

(5) $v = (1,2,2)$ $\quad R_\theta$ along x-axis $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sin(60) \\ 0 & \sin(60) & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1+2\sin(60) & 1-2\sin(60) \end{bmatrix}$

$a + b + c = 1 + 1 + 1 + 2\sin(60) - 2\sin(60) = 3$

(6) $\|C\| = \max\left(|\sqrt{17}x| + |\sqrt{17}y|, |\sqrt{17}x| + |\sqrt{17}y|\right) = |\sqrt{17}y| + |\sqrt{17}x|$

$\begin{bmatrix} x & \sqrt{16}x \\ -\sqrt{16}y & y \end{bmatrix} \begin{bmatrix} x & -\sqrt{16}y \\ \sqrt{16}x & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 17x^2 & 0 \\ 0 & 17y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad 17x^2 = 1$

$x = \dfrac{1}{\sqrt{17}} = y$ $\quad$ So, $|\sqrt{17}y| + |\sqrt{17}x| = 1 + 1 = 2$

(7) $\begin{bmatrix} \frac{a}{\sqrt{a^2+7^2}} & \frac{19}{\sqrt{b^2+19^2}} \\ \frac{7}{\sqrt{a^2+7^2}} & \frac{b}{\sqrt{b^2+19^2}} \end{bmatrix} \begin{bmatrix} \frac{a}{\sqrt{a^2+7^2}} & \frac{7}{\sqrt{a^2+7^2}} \\ \frac{19}{\sqrt{b^2+19^2}} & \frac{b}{\sqrt{b^2+19^2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\dfrac{a^2}{a^2+7^2} + \dfrac{19^2}{b^2+19^2} = 1$

$= a^2(b^2+19^2) + 19^2(a^2+7^2) = (a^2+7^2)(b^2+19^2)$
$\cancel{a^2b^2} + \cancel{19^2a^2} + 19^2a^2 + \cancel{(19^2)(7^2)} = \cancel{a^2b^2} + \cancel{19^2a^2} + 7^2b^2 + \cancel{(19^2)(7^2)}$
$19^2a^2 = 7^2b^2$
$(19a)^2 - (7b)^2 = 0$
$(19a + 7b)(19a - 7b) = 0$

(8) basis for $S_1 = \{ 1/\sqrt{2} (1,1,0) \}$
$S_2 = \{ 1/\sqrt{2} (1,-1,0) \}$
$S_3 = \{ 1/\sqrt{5} (2,1,0) \}$
$S_4 = \{ 1/\sqrt{2} (1,0,-1), 1/\sqrt{2} (0,1,-1) \}$

$v = (3,4,5)$

$\text{Proj}_{S_3}(v) = \langle (3,4,5), (2,1,0) \rangle \frac{1}{5} (2,1,0)$

$= 2(2,1,0)$

$\text{Proj}_{S_4}(v) = \langle (3,4,5), (1,0,-1) \rangle \frac{1}{2} (1,0,-1)$
$+ \langle (3,4,5), (0,1,-1) \rangle \frac{1}{2} (0,1,-1)$
$= -1(1,0,-1) - \frac{1}{2} (0,1,-1)$
$= (-1, -1/2, 3/2)$
$a + 2b + 3c = -1 - 2 + \frac{9}{2} = \frac{-2-4+9}{2} = 3/2$

$\text{Proj}_{S_1}(v) = \langle (3,4,5), (1,1,0) \rangle \frac{1}{2} (1,1,0)$

$= \frac{7}{2} (1,1,0)$

$\text{Proj}_{S_2}(v) = \langle (3,4,5), (1,-1,0) \rangle \frac{1}{2} (1,-1,0)$

$= -\frac{1}{2} (1,-1,0)$

WEEK 9 GA

(5) $(1/n, 0)$ $f(x,y) = \lim_{n\to\infty} \frac{1/n}{1/n} = 1$

if $y = 2x$, $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{3x}{\sqrt{5}\,x} = \frac{3}{5}$

(7) $\frac{\partial f}{\partial x}(x,y) = (x^2+y^2)^{1/2} = \frac{1}{2} \cdot \frac{1}{(x^2+y^2)} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}}$

(8) f along $y = x$ $(1/n, 1/n)$ $\lim_{n\to\infty} \frac{(1/n)^3}{3(1/n)^3} = \frac{1}{3}$

g along x-axis $(1/n, 0)$ $\lim_{n\to\infty} \left(\frac{1}{n}\right)^4 = 0$ → same along y-axis

(9) $y = 2x$ $f(x,y) = \frac{2x^2}{2x^2 + 4x^2} = \frac{2x^2}{6x^2} = \frac{1}{3}$

(10) $\frac{\partial f_1}{\partial x}(-3,1) = 1$ $\frac{\partial f_1}{\partial y}(-3,1) = x + 2y = -1$

$\frac{\partial f_2}{\partial x}(-3,1) = 1 + y = 2$ $\frac{\partial f_2}{\partial y}(-3,1) = x = -3$

$\det(A) = (-3) - (-2) = -1$

(11) $u = (0,1)$ $\nabla f_{(0,1)}(4,4) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (0,1)$ $\frac{\partial f}{\partial x}(4,4) = 2xy^3 = 512$ $\frac{\partial f}{\partial y}(4,4) = 3x^2y^2 = 768$

$= 0(512) + 1(768) = 768$

(13) $\frac{\partial f}{\partial x} = 2x + y$ $\frac{\partial f}{\partial y} = 2y + x$ if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$, $2x + y = 2y + x \Rightarrow x = y$

(14) $f(cx, cy) = c^2x^2 + c^2xy + c^2y^2 = c^2 f(x,y)$
$\lim_{(x,y)\to(5,0)} x^2 + xy + y^2 =$

(15) $\nabla f(x,y) = (2x+y, 2y+x)$
$(2x+y, 2y+x)\left(1/\sqrt{1+m^2}, m/\sqrt{1+m^2}\right) = \frac{2x+y}{\sqrt{1+m^2}} + \frac{2my + mx}{\sqrt{1+m^2}} = \frac{1}{\sqrt{1+m^2}} \left(\overset{=k}{(2+m)}x + \overset{=l}{(2m+1)}y\right)$

$2k - l = 2(2+m) - (2m+1) = 4 + 2m - 2m - 1 = 3$