STATS 2

WEEK 1

→ Joint PMF of two discrete random variables
  ↳ Two events can either be independent; or
  ↳ one could be influencing the other.

→ Types of PMF of Multiple Random Variables:
① Joint PMF
② Marginal PMF
③ Conditional PMF

① Joint PMF

→ X and Y → discrete random variables defined in the same probability space
Range of X and Y → T_X and T_Y
↳ Joint PMF of X and Y → f_{XY} → function from $T_X \times T_Y$ to $[0, 1]$

$$f_{XY}(t_1, t_2) = P(X = t_1 \text{ and } Y = t_2),\ t_2 \in T_X,\ t_2 \in T_Y$$

→ Usually written as a table or a matrix
↳

variable 1

variable 2

$P(x_1,x_2)$	$P(x_1,x_2)$	$P(x_1,x_2)$	...	..	..
...	...	...	...	...	...
...	...	...	..	...	...
...	...	...	...	..	..
...	...	...	...	...	...

② Marginal PMF

→ To obtain the individual PMF of a random discrete variable from a joint PMF.
↳ Suppose X and Y random discrete variables, and their joint PMF → f_{XY}, then:

$$f_X(t) = P(X = t) = \sum_{t' \in T_Y} f_{XY}(t, t')$$

$$f_Y(t) = P(Y = t) = \sum_{t' \in T_X} f_{XY}(t', t)$$

③ Conditional PMF

→ $f_{X|A}(t)$ → PMF of a random variable X conditional on an event A.
$P(X = t \mid A)$, $t \in T_X$

$$f_{X|A}(t) = \frac{P((X = t) \cap A)}{P(A)}$$

→ $f_{Y|X}(t)$ → conditional PMF of Y given $X = t$

$$f_{Y|X=t}(t') = P(Y = t' \mid X = t) = \frac{P(Y = t', X = t)}{P(X = t)} = \frac{f_{XY}(t, t')}{f_X(t)}$$

$$f_{XY}(t, t') = f_{Y|X=t}(t')\, f_X(t)$$

→ $f_{XY}(t_1, t_2) = f_{Y|X=t_1}(t_2)\, f_X(t_1) = f_{X|Y=t_2}(t_1)\, f_Y(t_2)$

Q. Let $N \sim \text{Poisson}(\lambda)$. Given $N=n$, toss a fair coin n times and denote the number of heads obtained by X. What is the distribution of X?

A. $f_N(n) = \frac{e^{-\lambda}\lambda^n}{n!}$; $n = 0, 1, 2, \dots$ $\quad (X|N=n) \sim \text{Bin}(n, 1/2)$ $\quad f_{X|N=n}(k) = \binom{n}{k}\left(\frac{1}{2}\right)^n$

$$f_{XN}(k,n) = \frac{e^{-\lambda}\lambda^n}{n!} \cdot \frac{n!}{k!(n-k)!} \cdot \left(\frac{1}{2}\right)^n = \frac{e^{-\lambda}\lambda^n}{k!(n-k)!}\left(\frac{1}{2}\right)^n$$

$$\boxed{f_X(k) = \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{k!(n-k)!}\left(\frac{1}{2}\right)^n} \Rightarrow \frac{e^{-\lambda}\lambda^k}{k!\,2^k} \sum_{n=k}^{\infty} \frac{(\lambda)^{n-k}}{(n-k)!\,(2)^{n-k}}$$

$$X \sim \text{Poisson}(\lambda/2) \Rightarrow \boxed{f_X(k) = \frac{e^{-\lambda/2}(\lambda/2)^k}{k!}}$$

→ Joint PMF of multiple random variables

↳ $X_1, X_2, X_3, \dots X_n$ are discrete random variables defined in the same probability space.

$$f_{X_1X_2\dots X_n}(t_1, t_2, \dots, t_n) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and} \cdots \text{and } X_n = t_n),\ t_i \in T_{X_i}$$

→ Marginal PMF with multiple random variables

↳ individual random variable ⇒ $f_{X_1}(t) = P(X_1 = t) = \sum_{t_2' \in T_2, t_3' \in T_3, \dots, t_n' \in T_n} f_{X_1,X_2,X_3\dots X_n}(t, t_2', t_3', \dots, t_n)$

↳ multiple random variables ⇒ $f_{X_1X_2}(t_1, t_2) = P(X_1 = t_1 \text{ and } X_2 = t_2) = \sum_{t_3' \in T_3} f_{X_1,X_2,X_3}(t_1, t_2, t')$

→ Conditional PMF with multiple random variables

↳ $(X_1, X_2 | X_3 = t_3) \sim f_{X_1,X_2|X_3=t_3}(t_1, t_2) = \frac{f_{X_1,X_2,X_3}(t_1, t_2, t_3)}{f_{X_3}(t_3)}$

↳ $(X_1 | X_2 = t_2, X_3 = t_3) \sim f_{X_1|X_2=t_2,X_3=t_3}(t_1) = \frac{f_{X_1,X_2,X_3}(t_1, t_2, t_3)}{f_{X_2X_3}(t_2, t_3)}$

↳ factors of a Joint PMF: $f_{X_1\dots X_4}(t_1, \dots, t_4) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$

$$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$$

$$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 | X_3 = t_3, X_4 = t_4) \cdot P(X_3 = t_3 | X_4 = t_4) \cdot P(X_4 = t_4)$$

$$f_{X_1\dots X_4}(t_1, \dots, t_4) = f_{X_1|X_2=t_2,X_3=t_3,X_4=t_4}(t_1) \cdot f_{X_2=t_2|X_3=t_3,X_4=t_4}(t_2) \cdot f_{X_3|X_4=t_4}(t_3) \cdot f_{X_4}(t_4)$$

WEEK 2

→ Independence of two random variables

↳ X and Y are independent if :

$$f_{XY}(t_1,t_2) = f_X(t_1)\cdot f_Y(t_2)$$

→ Joint PMF of f_{XY} is the product of the marginals PMF of X and Y.
→ Conditional PMF equals marginal PMF

→ Independence of multiple random variables

↳ $X_1, X_2, \dots, X_n$ are independent iff

$$f_{X_1,X_2,\dots X_n}(t_1,t_2,\dots,t_n) = f_{X_1}(t_1)\cdot f_{X_2}(t_2)\cdot\ \dots\ \cdot f_{X_n}(t_n)$$

→ Independent and Identically distributed (i.i.d.)

↳ Random variables $X_1, X_2, \dots X_n$ are i.i.d. if → ① they are independent,
② Marginal PMFs f_{X_i} are identical.

→ $X_1, X_2 \dots X_n \sim$ i.i.d. f_X

→ Memoryless property of Geometric distribution

→ Let $X \sim$ Geometric(p)

① $P(X>n) = (1-p)^n$

② $P(X>m+n \mid X>m) = \dfrac{P(X>m+n \cap X>m)}{P(X>m)} = \dfrac{P(X>m+n)}{P(X>m)} = \dfrac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n$

→ Functions of random variable

↳ One-to-one function → $P(Y=f(x)) = P(X=x)$; e.g. → $y=x-5$, $y=2^x$

↳ Many-to-one function → e.g.→ $y=(x-5)^2$, $y=x(1-x)$ → $y_0 = f(x_1) = f(x_2) = \dots = f(x_m)$ all the places $f(x)$ takes the value y_0

⇓

↳ sum over all the probabilities of X for wherever the function takes the 'y' value. ⟶ $P(Y=y_0) = P(X=x_1) + P(X=x_2) + \dots + P(X=x_m)$

→ Visualising function of 2 random variable

↳ $g(x,y)$: function

① Contours → values of (x,y) that result in $g(x,y)=c$
→ make a plot of those (x,y) for different c

② Regions → values of (x,y) that result in $g(x,y)\le c$
→ make a plot of those (x,y) for different c

→ **Function of 2 random variables**

→ Let $X,Y \sim f_{XY}$; let $Z = g(X,Y)$ be a function of X and Y

· What is the PMF of Z?

→ Step 1 : Find the range of Z

→ Step 2 : Add over all the contours

↳ Suppose z is a possible value taken by Z:

$$P(Z=z) = \sum_{(x,y):\,g(x,y)=z} f_{XY}(x,y)$$

→ Sum of 2 uniform random variables

$X,Y \sim$ iid Unif$\{1,2,\dots,n\}$, $W = X+Y$

range of $W = \{2,3,\dots,2n\}$ $W=w : (1,w-1),(2,w-2),\dots,(w-1,1)$

$W \in \{2,3,\dots,2n\}$

$$P(W=w) = \begin{cases} \dfrac{w-1}{n^2}, & 2\le w\le n+1 \\ \dfrac{2n-w+1}{n^2}, & n+2\le w\le 2n \end{cases}$$

→ Max of 2 uniform random variables $X, Y \sim iid\ Unif\{1,2,\dots,n\}$, $Z = \max(X,Y)$

$$Z \in \{1,2,\dots,n\}$$

$$P(Z=z) = \frac{2z-1}{n^2}$$

→ PMF of $g(X_1, X_2, \dots X_n)$ → The PMF $X = g(X_1, X_2, \dots, X_n)$ is given by

$$f_X(t) = P(g(X_1, X_2, \dots, X_n) = t) = \sum_{(t_1, t_2, \dots, t_n): g(t_1, t_2, \dots, t_n) = t} f_{X_1, X_2, \dots, X_n}(t_1, t_2, \dots, t_n)$$

e.g. →

$t_2 \cdot t_3$	$t_1+t_2+t_3$	t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1,t_2,t_3)$
0	0	0	0	0	1/9
0	1	0	0	1	1/9
0	2	0	0	2	1/9
1	2	0	1	1	1/9
2	3	0	1	2	1/9
0	1	1	0	0	1/9
0	3	1	0	2	1/9
0	2	1	1	0	1/9
1	3	1	1	1	1/9

① $X = g(X_1, X_2, X_3) = X_1 + X_2 + X_3$ $\quad g \in \{\overset{1/9}{0}, \overset{2/9}{1}, \overset{3/9}{2}, \overset{3/9}{3}\}$

② $Y = h(X_1, X_2, X_3) = X_2 \cdot X_3$ $\quad h \in \{\overset{6/9}{0}, \overset{2/9}{1}, \overset{1/9}{2}\}$

f_{XY}	0	1	2	3
0	1/9	2/9	2/9	1/9
1	0	0	1/9	1/9
2	0	0	0	1/9

→ sum of n independent Bernoulli(p) = Binomial(n,p)

→ suppose X and Y take integer values and their joint PMF = f_{XY}. Let $Z = X+Y$

Convolution

$$P(Z=z) = \sum_{x=-\infty}^{\infty} P(X=x, Y=z-x) = \sum_{x=-\infty}^{\infty} f_{XY}(x, z-x) = \sum_{y=-\infty}^{\infty} f_{XY}(z-y, y)$$

$$= \sum_{x=-\infty}^{\infty} f_X(x) f_Y(z-x) = \sum_{y=-\infty}^{\infty} f_Y(y) f_X(z-y)$$

→ Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent.

→ $Z = X + Y$

$$f_Z(z) = \sum_{x=0}^{\infty} f_X(x) \cdot f_Y(z-x) = \sum_{x=0}^{z} \frac{e^{-\lambda_1} \cdot \lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2} \lambda_2^{(z-x)}}{(z-x)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{z!} \underbrace{\sum_{x=0}^{z} \frac{\lambda_1^x \lambda_2^{z-x} \cdot z!}{x!(z-x)!}}_{(\lambda_1+\lambda_2)^z}$$

can be replaced with x going to z because after z this term will go to 0.

$$f_Z(z) = \frac{e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1+\lambda_2)^z}{z!} \Rightarrow Z \sim \text{Poisson}(\lambda_1+\lambda_2)$$

→ conditional of $X|Z$

$$P(X=k|Z=n) = \frac{P(x=k, Z=n)}{P(Z=n)} = \frac{P(x=k) \cdot P(Z=n|x=k)}{P(Z=n)} = \frac{P(X=k) \cdot P(Y=n-k)}{P(Z=n)}$$

$$= \frac{\frac{\cancel{e^{-\lambda_1}} \cdot \lambda_1^k}{k!} \cdot \frac{\cancel{e^{-\lambda_2}} \cdot \lambda_2^{n-k}}{(n-k)!}}{\frac{\cancel{e^{-\lambda_1-\lambda_2}} \cdot (\lambda_1+\lambda_2)^n}{n!}} = \frac{n!}{k!(n-k)!} \cdot \frac{\lambda_1^k \cdot \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}$$

$\longrightarrow (\lambda_1+\lambda_2)^n = (\lambda_1+\lambda_2)^k \cdot (\lambda_1+\lambda_2)^{n-k}$

$\longrightarrow \frac{n!}{k!(n-k)!} = {}^nC_k$

$$= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$X|Z = \text{Bin}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$Y|Z = \text{Bin}\left(n, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$

→ Functions of non-overlapping indepent random variables are also independent

→ If X and Y are independent, $g(X)$ and $h(Y)$ are independent for any two functions g and h

→ Min/Max of two random variables

↳ $X, Y \sim f_{XY}$ $\qquad Z = \min(X,Y)$

$$f_Z(z) = P(\min(X,Y)=z) = P\big((X=z) \text{ and } (Y=z) \text{ or } (X=z \text{ and } Y>z) \text{ or } (X>z \text{ and } Y=z)\big)$$
$$= f_{XY}(z,z) + \sum_{t_2>z} f_{XY}(z,t_2) + \sum_{t_1>z} f_{XY}(t_1,z)$$

↳ CDF of a random variable

↳ CDF of a random variable X is a function $F_X : \mathbb{R} \to [0,1]$ defined as:

↳ $F_X(x) = P(X \le x)$

e.g. → X and Y are independent. $Z = \max(X,Y)$

$$F_Z(z) = P(\max(X,Y) \le z) = F_X(z) \cdot F_Y(z)$$

↳ Let $X_1, X_2, \dots, X_n \sim$ i.i.d.

① Distribution of $\underline{\min(X_1, X_2, \dots, X_n)}$ → $P(\min(X_1, X_2, \dots, X_n) \ge z) = (P(X \ge z))^n$

② Distribution of $\underline{\max(X_1, X_2, \dots, X_n)}$ → $P(\max(X_1, X_2, \dots, X_n) \le z) = (P(X \le z))^n = (F_X(z))^n$

↳ Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(p)$ be independent. Find the distr. of $\underline{\min(X,Y)}$

$Z = \min(X,Y)$ $\qquad P(Z \ge z) = P(X \ge z, Y \ge z) = (1-p)^{z-1} \cdot (1-p)^{z-1}$

$$= \left((1-p)^2\right)^{z-1}$$

$$P(Z \ge z+1) = \left((1-p)^2\right)^z$$

$$P(Z = z) = P(Z \ge z) - P(Z \ge z+1)$$
$$= \left((1-p)^2\right)^{z-1} - \left((1-p)^2\right)^z = \left((1-p)^2\right)^{z-1}\left(1-(1-p)^2\right) \leftarrow \text{Geometric probability}$$

$\min(X,Y) \sim \text{Geometric}(1-(1-p)^2)$

Let $X_1 \sim \text{Geometric}(p_1)$ and $X_2 \sim \text{Geometric}(p_2)$

↳ $\min(X_1, X_2) \sim \text{Geometric}(1-(1-p_1)(1-p_2))$

$Z = \max(X_1, X_2, \dots, X_{10})$ $\qquad X \sim \text{Bin}(6, 1/2)$

$P(Z \le 2) = P(X_1 \le 2, X_2 \le 2, \dots, X_{10} \le 2)$

$$= \left(22 \cdot \left(\tfrac{1}{2}\right)^6\right)^{10}$$

X	0	1	2	3
	$\left(\frac{1}{2}\right)^6$	$6\left(\frac{1}{2}\right)^6$	$15\left(\frac{1}{2}\right)^6$	

$$\binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6 \times 5}{2} = 15$$

$X \le 2 = \left(\tfrac{1}{2}\right)^6 \cdot 22$

$Z = \min(X_1, X_2, \dots, X_{10})$ $\qquad X \sim \text{Bin}(6, 1/2)$

$F_Z(2) = 1 - P(X_1 > 2, X_2 > 2, \dots X_{10} > 2)$ $\qquad X > 2 = 42\left(\tfrac{1}{2}\right)^6$

$$= 1 - \left(\frac{42}{2^6}\right)^{10} = 1 - \left(\frac{21}{2^5}\right)^{10}$$

WEEK 3

→ Expected value of a discrete random variable:

↳ $X \to$ discrete rand. variable ; range $\to T_X$; PMF $\to f_X$

$$E[X] = \sum_{t \in T_X} t \cdot f_X(t) = \sum_{t \in T_X} t \cdot P(X=t)$$

↳ example ⇒ $X \sim \text{Bernoulli}(p) \Rightarrow E[X] = 0(1-p) + p = p$

$X \sim \text{Uniform}\{1,2,3,4,5,6\} \Rightarrow E[X] = \sum_{t \in T_X} t \cdot \frac{1}{6} = 3.5$

$X \sim \text{Uniform}\{a, a+1, \dots, b\} \Rightarrow E[X] = \frac{a}{b-a+1} + \frac{a+1}{b-a+1} + \dots + \frac{b}{b-a+1} = \frac{a+b}{2}$

Identity ⇒ $a + (a+1) + (a+2) + \dots + b = (b-a+1)\left(\frac{a+b}{2}\right)$

Simplify summations

① Difference Equation (DE): $a_{t+1} - r a_t = b_t \quad (r \neq 1)$ $\qquad \sum_{t=1}^{n} a_t = \frac{a_1 - r a_n}{1-r} + \frac{1}{1-r} \sum_{t=1}^{n-1} b_t$

② Geometric Progression (GP): $a_{t+1} - r a_t = 0 \quad (r \neq 1)$ $\qquad \sum_{t=1}^{n} a_t = \frac{a_1 - r a_n}{1-r} \xrightarrow[n \to \infty]{|r|<1} \frac{a_1}{1-r}$

③ Exponential Function : $\sum_{t=0}^{\infty} e^{-\lambda} \frac{\lambda^t}{t!} = 1 \qquad e^{\lambda} = \sum_{t=0}^{\infty} \frac{\lambda^t}{t!}$

④ Binomial Formula : $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$

↳ $X \sim \text{Geometric}(p)$ $\quad E[X] = \sum_{t=1}^{\infty} t \cdot (1-p)^{t-1} \cdot p$ → $E[X] = 1/p$

↳ GP : $a_1 = p$, $r = 1-p$, $b_t = r^t p$

↳ $X \sim \text{Poisson}(\lambda)$ $\quad E[X] = \sum_{t=0}^{\infty} t \cdot e^{-\lambda} \frac{\lambda^t}{t!}$ → $E[X] = \lambda$

↳ using exponent function →

↳ $X \sim \text{Binomial}(n,p)$ $\quad E[X] = \sum_{t=0}^{n} t \cdot \binom{n}{t} p^t \cdot (1-p)^{n-t}$

→ $E[X] = np$

→ Expected value of a function of random variables

↳ $X_1, X_2, \dots, X_n$ have joint PMF $f_{X_1, X_2, \dots X_n}$ with range X_i denoted T_{X_i}

Let $g : T_{X_1} \times T_{X_2} \times \dots \times T_{X_n} \to \mathbb{R}$ be a function, $Y = g(X_1, X_2, \dots, X_n)$; T_Y and f_Y

$$E[Y] = \sum_{t \in T_Y} t \cdot f_Y(t) = \sum_{t_i \in T_{X_i}} g(t_1, t_2, \dots, t_n) \cdot f_{X_1, X_2, \dots X_n}(t_1, t_2, \dots, t_n)$$

→ Linearity of Expected value :

① $E[cX] = cE[X]$ for a random variable X and a constant c

② $E[X+Y] = E[X] + E[Y]$ for any two random variables X and Y

③ $E[aX + bY] = aE[X] + bE[Y]$

→ Centering operation

↳ $Y = X - E[X]$ is a translated version of X and $E[Y] = 0$

$X - E[X]$ is a zero-mean random variable.

→ Variance of a random variable

↳ $Var(X) = E[(X - E[X])^2]$

$Var(X) = E[X^2] - (E[X])^2$

↳ If X and Y are independent:

— $E[XY] = E[X]E[Y]$

— $Var(X+Y) = Var(X) + Var(Y)$

Distribution	Expected Value	Variance
Bernoulli(p)	p	$p(1-p)$
Binomial(n,p)	$n \cdot p$	$np(1-p)$
Geometric(p)	$1/p$	$(1-p)/p^2$
Poisson(λ)	λ	λ
Uniform $\{1,2,\dots,n\}$	$(n+1)/2$	$(n^2-1)/12$

→ A random variable is standardised if $E(X)=0$ and $Var(X)=1$

↳ $Y = \dfrac{X - E[X]}{SD(X)}$, then Y is a standardised rand. var. obtained from X.

→ Covariance of two random variables

↳ X and Y are rand. var. on the same prob. space

$$Cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]\cdot E[Y]$$

↳ $Cov(X, aY+bZ) = a\,Cov(X,Y) + b\,Cov(X,Z)$

↳ $Cov(aX+bY, Z) = a\,Cov(X,Z) + b\,Cov(Y,Z)$

↳ If X and Y are independent, then $Cov(X,Y) = 0$

If X and Y are independent, then they are uncorrelated.
If X and Y are uncorrelated, they might still be dependent.

→ Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{SD(X)\,SD(Y)}$$

→ Markov's Inequality

$P(X \geq c) \leq \frac{\mu}{c}$, where $\mu = E(X)$

→ Chebyshev's Inequality

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

WEEK 4

→ Cumulative Distribution Function (CDF): $F_X(x) = P(X \leq x)$

Properties:

→ $F_X(b) - F_X(a) = P(a < X \leq b)$
→ As $x \to -\infty$, F_X goes to 0
→ As $x \to \infty$, F_X goes to 1

↳ if F_X jumps at a point x, then $P(X=x) > 0$.
if F_X is continuous at a point x, then $P(X=x) = 0$.

→ Integral

Indefinite : $F(x) = \int f(x)\,dx$

Definite :

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

→ Probability Density Function

↳ $F_X(x_0) = \int_{-\infty}^{x_0} f_X(x)\,dx$

↳ Properties: ① $f(x) \geq 0$ ② $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ③ $f(x)$ is a piecewise function

→ $\text{supp}(X) = \{x : f_X(x) > 0\}$ → interval in which X can fall with positive probability

→ Common Distributions:

↳ $X \sim \text{Exp}(\lambda)$ →

PDF $f_X(x) \begin{cases} \lambda \exp(-\lambda x) & x > 0 \\ 0 & \text{otherwise} \end{cases}$

CDF $F_X(x) \begin{cases} 0 & x \leq 0 \\ 1 - \exp(-\lambda x) & x > 0 \end{cases}$

↳ $X \sim \text{Normal}(\mu, \sigma^2)$ →

PDF $f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(\dfrac{-(x-\mu)^2}{2\sigma^2}\right)$

CDF $F_X(x) = \int_{-\infty}^{x} f_X(v)\,dv$

Standardization: if $X \sim \text{Normal}(\mu, \sigma^2)$, then $(X-\mu)/\sigma \sim \text{Normal}(0,1)$

↳ $Z \sim \text{Normal}(0,1)$ PDF : $f_Z(z) = \dfrac{1}{\sqrt{2\pi}} \exp(-z^2/2)$

CDF : $F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)\,du$

WEEK 5

→ Function of Continuous Random Variable :

↳ Example : Suppose $X \sim \text{Uniform}[0,1]$, and $Y = 2X \in [0,2]$

For $y \in [0,2]$, $CDF_Y(y) = P(Y \le y) = P(2X \le y) = P(X \le y/2)$

$$P(X \le y/2) = \int_0^{y/2} f_X(x)\,dx = x\Big|_0^{y/2} = \frac{y}{2}$$

So, $CDF_Y(y) = \frac{y}{2}$

PDF of Y → derivative of CDF_Y

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2}$$; So, $Y \sim \text{Uniform}[0,2]$

↳ Always find CDF first

↳ General case : CDF of $g(X)$

→ Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function
$Y = g(X)$

$$F_Y(y) = P(Y \le y) = P(g(x) \le y) = P(X \in \{x : g(x) \le y\})$$

↳ Monotonic, differentiable function → Let X be a continuous rand. var.

↳ Let $g(x)$ be monotonic (increasing or decreasing) for $x \in \text{supp}(X)$

PDF of $Y = g(X)$ is

$$f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))$$

example:

Translation ① $Y = X + a$

→ $g(x) = x + a$ $g^{-1}(y) = y - a$ $g'(x) = 1$ $g'(g^{-1}(y)) = 1$

⇒ $f_Y(y) = f_X(g^{-1}(y)) = f_X(y-a)$

scaling ② $Y = aX$

→ $g(x) = ax$ $g^{-1}(y) = y/a$ $g'(x) = a$

⇒ $f_Y(y) = \frac{1}{|a|} f_X(y/a)$

Affine ③ $Y = aX + b$

→ $g(x) = ax + b$ $g^{-1}(y) = (y-b)/a$ $g'(x) = a$

⇒ $f_Y(y) = \frac{1}{|a|} f_X((y-b)/a)$

↳ Normal distribution

↳ affine transformation of a normal dist. is normal.

↳ Let $X \sim \text{Exp}(\lambda)$. Find PDF of X^2 $Y = X^2$

$f_X(x) = \lambda e^{-\lambda x}$ $g(x) = x^2$ $g^{-1}(y) = \sqrt{y}$ $g'(x) = 2x$ $g'(g^{-1}(y)) = 2\sqrt{y}$

$$f_Y(y) = \frac{1}{|2\sqrt{y}|} f_X(\sqrt{y}) = \frac{1}{2\sqrt{y}} \lambda e^{-\lambda\sqrt{y}}$$

→ Expectation of a function of Continuous Random Variable

↳ Let X → cont. rand. var. , and $g : \mathbb{R} \to \mathbb{R}$ be a function

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$ → may diverge to $\pm\infty$

→ Expectation of a Continuous Random Variable

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

Variance → $Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$

→ Joint Distribution

↳ (X, Y) are jointly distributed. For each $x \in T_X$, we have continuous Y_x with PDF $f_{Y_x}(y)$

Y_x : Y given $X = x$, $(Y \mid X = x)$

Marginal density of Y → $f_Y(y) = \sum_{x \in T_x} f_X(x) \cdot f_{Y|X=x}(y)$

↳ Example: Let $X \sim \text{Uniform}\{0,1,2\}$. $Y|X=0 \sim \text{Normal}(5, 0.4)$
$Y|X=1 \sim \text{Normal}(6, 0.5)$ and $Y|X=2 \sim \text{Normal}(7, 0.6)$

Marginal of Y?

$$f_Y(y) = \frac{1}{3} \cdot \frac{1}{\sqrt{2\pi}} \left(\frac{1}{0.4} \cdot \exp\left(\frac{-(y-5)^2}{2 \cdot (0.4)^2}\right) + \frac{1}{0.5} \cdot \exp\left(\frac{-(y-6)^2}{2 \cdot (0.5)^2}\right) + \frac{1}{0.6} \cdot \exp\left(\frac{-(y-7)^2}{2 \cdot (0.6)^2}\right) \right)$$

↳ Conditional probability of discrete given continuous

$$P(X = x \mid Y = y_0) = \frac{f_X(x) \cdot f_{Y|X=x}(y_0)}{f_Y(y_0)}$$

, where f_Y → marginal density of Y, and $Y = y_0 \in \text{supp}(Y)$

↳ Example: Let $X \sim \text{Uniform}\{-1, 1\}$. $Y|X=-1 \sim \text{Uniform}(-2, 2)$ and $Y|X=1 \sim \text{Exp}(5)$

→ Find $f_{X|Y=-1}$, $f_{X|Y=1}$ and $f_{X|Y=3}$

$$f_Y(y) = \frac{1}{2} f_{Y|X=-1}(y) + \frac{1}{2} f_{Y|X=1}(y) = \begin{cases} 0 & , y < -2 \\ \frac{1}{2} \cdot \frac{1}{4} & , -2 \le y < 0 \\ \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} 5e^{-5y} & , 0 \le y \le 2 \\ \frac{1}{2} 5e^{-5y} & , y > 2 \end{cases}$$

$$P(X = -1 \mid Y = -1) = \frac{f_X(-1) \cdot f_{Y|X=-1}(-1)}{f_Y(-1)} = \frac{(1/2) \cdot (1/4)}{(1/2) \cdot (1/4)} = 1$$

$$P(X = 1 \mid Y = -1) = \frac{f_X(1) \cdot f_{Y|X=1}(-1)}{f_Y(-1)} = 0$$

$$P(X = -1 \mid Y = 1) = \frac{f_X(-1) \cdot f_{Y|X=-1}(1)}{f_Y(1)} = \frac{(1/2) \cdot (1/4)}{(1/2) \cdot (1/4) + (1/2) 5e^{-5}}$$

WEEK 6

→ Joint density

- A function $f(x,y)$ is a joint density function if:
  - → $f(x,y) \geq 0$
  - → $\iint_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

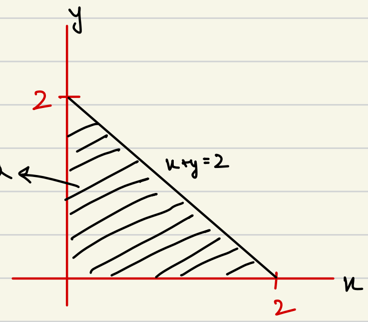
- For any two dimensional region A,
  $$P((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy$$

- $\text{supp}(X,Y) = \{(x,y) : f_{XY}(x,y) > 0\}$

- example:

  Let $(X,Y) \sim \text{Uniform}(D)$, where $D = \{(x,y) : x+y < 2,\ x>0,\ y>0\}$
  Sketch the support and find $P(X+Y<1)$, $P(X+2Y>1)$

  Area of support $=2$. uniform means $P(X,Y) = \frac{1}{2}$

  $$f_{XY}(x,y) = \begin{cases} 1/2 & ,(x,y) \in D \\ 0 & , \text{otherwise} \end{cases}$$

  $$P(X+Y<1) = \frac{\text{Area of } x+y<1}{\text{Area of } x+y<2} = \frac{1/2}{2} = \frac{1}{4}$$

  only works for uniform

  $$P(X+2Y>1) = 1 - P(X+2Y<1) = 1 - \frac{1/4}{2} = 1 - \frac{1}{8} = \frac{7}{8}$$

- example:

  Let (X,Y) have joint density:
  $$f_{XY}(x,y) = \begin{cases} x+y & , 0<x,y<1 \\ 0 & , \text{otherwise} \end{cases}$$

  Show it is a valid density. Find $P(X<1/2, Y<1/2)$, and $P(X+Y<1)$

  $$\int_{y=0}^{1}\int_{x=0}^{1} (x+y)\,dx\,dy = \int_{y=0}^{1} \frac{1}{2} + y\,dy = 1 \quad \checkmark \text{ valid density}$$

  $$P(X<1/2, Y<1/2) = \int_{y=0}^{1/2}\int_{x=0}^{1/2} x+y\,dx\,dy = \int_{y=0}^{1/2} (1/8 + y/2)\,dy = y/8 + y^2/4 \Big|_0^{1/2} = \frac{1}{8}$$

  $P(X+Y<1) \rightarrow$ when $Y=y$, $x<1-y$

  $$= \int_{y=0}^{1}\int_{x=0}^{1-y} (x+y)\,dx\,dy = \int_{y=0}^{1} \left(\frac{x^2}{2} + yx\Big|_0^{1-y}\right) dy = \int_{y=0}^{1} \frac{(1-y)^2}{2} + y - y^2\,dy = \int_0^1 \frac{1}{2} - \frac{y^2}{2}\,dy = \frac{1}{3}$$

  $\frac{1}{2} + \frac{y^2}{2} - y + y - y^2$

→ Marginal Density of jointly distributed continuous random variables

- $f_{XY}(x,y)$ is joint continuous density

  $$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy \qquad f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$$

- Joint density determines the marginal density but marginal density can be the same for different joint densities so you cannot determine the joint density from marginal densities alone.

→ Conditional density

↳ $f_{X|Y=b}(x) = \dfrac{f_{XY}(x,b)}{f_X(a)}$

$f_{XY}(x,y) = f_X(x) \cdot f_{Y|X=x}(y) = f_Y(y) \cdot f_{X|Y=y}(x)$

WEEK 7

→ Empirical distribution

↳ Let $X_1, X_2, \ldots, X_n \sim X$ be i.i.d. samples. $\#(X_i = t) \rightarrow$ no. of time t occurs.
The empirical distribution is the discrete distribution with PMF: $p(t) = \dfrac{\#(X_i = t)}{n}$

↳ Sample mean denoted $\bar{X}$

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

↳ Expected value and variance of the sample mean
Let $X_1, X_2, X_3, \ldots, X_n \sim X$ be iid with a finite mean μ and variance σ^2. Sample mean $\rightarrow \bar{X} = \dfrac{X_1 + X_2 + X_3 + \ldots X_n}{n}$

$$E[\bar{X}] = \mu \qquad Var(\bar{X}) = \frac{\sigma^2}{n}$$

↳ Sample variance denoted S^2

$$S^2 = \frac{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \ldots + (X_n - \bar{X})^2}{(n-1)}$$

$$E[S^2] = \sigma^2$$

↳ Sample proportion
Let A be an event and $P(A)$ be the probability of A.

$$S(A) = \frac{\#(X_i \text{ for which } A \text{ is true})}{n} \qquad E[S(A)] = P(A) \qquad Var(S(A)) = \frac{P(A)(1-P(A))}{n}$$

↳ Sum of random variables
Let $X_1, X_2, \ldots, X_n$ be random variables. Let $S = X_1 + X_2 + \ldots + X_n$ be their sum.

$$E[S] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

→ If $X_1, X_2, \ldots, X_n$ are pairwise uncorrelated, then → $E[X_i X_j] = E[X_i] \cdot E[X_j]$ for all $i, j, i \neq j$

$$Var(S) = Var(X_1) + Var(X_2) + \ldots + Var(X_n)$$

↳ Weak law of large numbers
Let $X_1, X_2, \ldots, X_n \sim$ iid X, where $E[X] = \mu$ and $Var(X) = \sigma^2$
Let $\bar{X} = (X_1 + X_2 + \ldots + X_n)/n$, then $E[\bar{X}] = \mu$ and $Var(\bar{X}) = \sigma^2/n$

$$P(|\bar{X} - \mu| > \delta) \leq \frac{\sigma^2}{n\delta^2}$$ → similar to Markov's inequality

WEEK 8

Moment Generating Function

- Let X be a zero-mean random variable. MGF of X, denoted $M_X(\lambda)$ is $f: \mathbb{R} \to \mathbb{R}$

$$M_X(\lambda) = E[e^{\lambda X}]$$

  - $X \to$ discrete $\Rightarrow$ $M_X(\lambda) = f_X(x_1)e^{\lambda x_1} + f_X(x_2)e^{\lambda x_2} + \ldots + f_X(x_n)e^{\lambda x_n}$, where $supp(X) = \{x_1, x_2, \ldots, x_n\}$

    $X \to$ continuous $\Rightarrow$ $M_X(\lambda) = \int_{x \in T_X} f_X(x)e^{\lambda x}\,dx$

    $X \sim Normal(0, \sigma^2)$
    - $M_X(\lambda) = e^{\lambda^2\sigma^2/2}$

- $E[e^{\lambda X}] = E[1 + \lambda X + \lambda^2X^2/2! + \lambda^3X^3/3! + \ldots]$

$$= 1 + \underbrace{\lambda E[X]}_{1^{st}\text{ moment}} + \underbrace{\frac{\lambda^2}{2!}E[X^2]}_{2^{nd}\text{ moment}} + \underbrace{\frac{\lambda^3}{3!}E[X^3]}_{3^{rd}\text{ moment}} + \ldots$$

- Sum of i.i.d. random variables

  Let $Y = X_1 + X_2$, where $X_1, X_2 \sim$ iid X

  example: $X \sim$ iid centralised Bernoulli (p)

  $\to M_X(\lambda) = (1-p)e^{-p\lambda} + pe^{(1-p)\lambda}$

  $\to M_Y(\lambda) = (M_X(\lambda))^2 = (1-p)^2e^{-2p\lambda} + 2p(1-p)e^{(1-2p)\lambda} + p^2e^{2(1-p)\lambda}$

Y	$-2p$	$1-2p$	$2(1-p)$
$f_Y(y)$	$(1-p)^2$	$2p(1-p)$	p^2

Central Limit Theorem

- Let $X_1, X_2, \ldots, X_n \sim$ iid X with $E[X]=0$ and $Var(X) = \sigma^2$

  Let $Y = \dfrac{X_1 + X_2 + \ldots X_n}{\sqrt{n}}$, Then,

$$M_Y(\lambda) \to e^{\lambda^2\sigma^2/2}$$

  - Y is said to converge to $Normal(0, \sigma^2)$

- Application:

  $X_1, X_2, \ldots X_n \sim$ iid X Let $\mu = E[X]$ and $Var(X) = \sigma^2$

  Let $Y = X_1 + X_2 + \ldots + X_n$ What is $P(Y - \mu n > \delta\mu n)$?

  $\to E[Y] = \mu n$

  $(Y - \mu n)/\sqrt{n} \approx Normal(0, \sigma^2)$

  $(Y - \mu n)/(\sqrt{n}\sigma) \approx Normal(0, 1)$

$$P(Y - \mu n > \delta\mu n) = 1 - F\left(\frac{\delta\mu\sqrt{n}}{\sigma}\right)$$

Linear Combination of independent Normals

- Let $X_1, X_2, \ldots, X_n \sim$ Normal. Let $X_i \sim Normal(\mu_i, \sigma_i^2)$

  Suppose $Y = a_1X_1 + a_2X_2 + \ldots + a_nX_n$, then

  $Y \sim Normal(\mu, \sigma^2)$, where $\mu = a_1\mu_1 + a_2\mu_2 + \ldots + a_n\mu_n$

  $\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots + a_n^2\sigma_n^2$

  Linear combinations of normals is normally distributed

Gamma distribution

- $X \sim Gamma(\alpha, \beta)$. PDF $f_X(x) \propto (x^{\alpha-1}e^{-\beta x})$, $x > 0$

  $\to$ Sum of n iid $Exp(\beta) = Gamma(\alpha, \beta)$

  $\to$ Square of $Normal(0, \sigma^2) = Gamma(1/2, 1/2\sigma^2)$

- $\alpha > 0$: shape parameter

  $\beta > 0$: rate parameter

  $\theta = 1/\beta$: scale parameter

→ Cauchy distribution

↳ $X \sim \text{Cauchy}(\theta, \alpha^2)$, where θ : location parameter
$\alpha > 0$: scale parameter

Suppose $X, Y \sim$ iid Normal$(0, \sigma^2)$. Then,

$\frac{X}{Y} \sim \text{Cauchy}(0, 1)$

→ Beta distribution

↳ $X \sim \text{Beta}(\alpha, \beta)$ if PDF $f_X(x) \propto (x)^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$

↳ has finite support $\alpha > 0, \beta > 0$: shape parameters

→ Sum of n independent Gamma(α, β) is Gamma$(n\alpha, \beta)$

Sum of squared Normal$(0,1)$ iid. $\sim$ Gamma$(n/2, 1/2)$

↳ a.k.a. chi-square distribution with n degrees of freedom

→ Sample mean and variance of normal samples

↳ Suppose $X_1, X_2, \ldots, X_n \sim$ Normal(μ, σ^2). Then,

1. $\bar{X} \sim$ Normal$(\mu, \sigma^2/n)$
2. $\frac{(n-1)(S^2)}{\sigma^2} \sim \chi^2_{n-1}$, Chi-square with $n-1$ deg. freedom
3. $\bar{X}$ and S^2 are indepent.

WEEK 9

WEEK 1 GA

① $T_x = \{0,1,2,3\}$ $\quad T_y = \{-1,1,2,3\}$ $\quad X \sim Bin(3, ½)$ $\quad P(X<3) = 1 - P(X=3) = 7/8$ $\quad P(X=3) = \binom{3}{3} \cdot (1/2)^3 = 1/8$

$P(Y \leq 1) = P(Y=1) + P(Y=-1)$ $\quad P(Y=1) = 1/2$ $\quad P(Y=-1) = 1/8$ $\quad P(Y \leq 1) = 1/2 + 1/8 = 5/8$

$f_{XY}(t_x < 3, t_y \leq 1) = \frac{5}{8} \times \frac{7}{8} = \frac{35}{64} \approx \boxed{0.5469}$

② $T_x = T_y = T_z = \{0,1,2\}$

X	Y	Z
0	0	2
2	0	0
1	0	1

$= \boxed{1/3}$

$f_{XY}(2,0) = \sum_{t_z=0}^{2} f_{XYZ}(2,0,t_z) = P(2,0,0) = 1/9$

$f_Y(0) = \sum_{t_x \in T_x, t_z \in T_z} f_{XYZ}(t_x, 0, t_z) = P(0,0,2) + P(1,0,1) + P(2,0,0) = 3/9$

$f_{X|Y=0}(2) = \frac{f_{XY}(2,0)}{f_Y(0)}$

$f_{X|Y=0}(2) = \frac{1/9}{3/9} = \boxed{1/3}$

③ $1/4 + 1/4 + 1/8 + 1/8 + k = 1$ $\quad k = 1 - (2/4 + 1/4) = 1 - 3/4 = 1/4$

$f_{Y|X=1}(2) = \frac{f_{YX}(2,1)}{f_X(1)} = \frac{1/4}{1/8 + 1/8 + 1/4} = \frac{1/4}{1/2} = \frac{1}{4} \times 2 = \boxed{\frac{1}{2}}$

⑤ $f_Y(1) = \sum_{t' \in T_x} f_{XY}(t', 1)$ $\quad f_X(1) = 12/13$ $\quad f_X(0) = 1/13$

$Y \sim Bin(n, p)$

$f_{Y|X=1}(1) = \begin{matrix} n=3 \\ p = 7/20 \end{matrix}$

$= \binom{3}{1}\left(\frac{7}{20}\right)\left(\frac{13}{20}\right)^2$

$= \frac{3 \times 7 \times (13)^2}{(20)^3}$

$= 0.443625$

$f_{Y|X=0}(1) = \begin{matrix} n=2 \\ p = 4/15 \end{matrix}$

$= \binom{2}{1}\left(\frac{4}{15}\right)\left(\frac{11}{15}\right)$

$= \frac{2 \times 4 \times 11}{(15)^2}$

$= 0.39111$

$f_{XY}(0,1) = f_{Y|X=0}(1) \cdot f_X(0)$

$= 0.03008547$

$f_{XY}(1,1) = f_{Y|X=1}(1) \cdot f_X(1)$

$= 0.4095$

$f_Y(1) = f_{XY}(0,1) + f_{XY}(1,1) \approx \boxed{0.4346}$

⑦ $f_{XY}(x,y) = a(bx+y)$

x \ y	0	1	2	3
0	0	a	2a	3a
1	ab	ab + a	ab + 2a	ab + 3a
2	2ab	2ab + a	2ab + 2a	2ab + 3a
	3ab	3ab + 2a	3ab + 4a	

$3\left(12a + 3ab = \frac{3}{7}\right) = 36a + 9ab = \frac{9}{7}$

$2\left(6a + 9ab = \frac{4}{7}\right) = 12a + 18ab = \frac{8}{7}$

$-12a - 3ab = -\frac{3}{7}$

$15ab = \frac{5}{7}$

$36a + 9ab = \frac{9}{7}$

$-6a - 9ab = -\frac{4}{7}$

$30a = \frac{5}{7}$

$\boxed{a = \frac{1}{42}}$

$ab = \frac{1}{21}$

$\frac{b}{42} = \frac{1}{21} \Rightarrow \boxed{b = 2}$

$f_{XY}(1,1) = \frac{1}{42}(2+1) = \boxed{\frac{1}{14}}$

⑧ $f_{Y|X=2}(0) = \frac{f_{YX}(0,2)}{f_X(2)}$

$f_X(2) = \binom{6}{2}(1/2)^6$

$f_{X|Y=0}(2) = \binom{5}{2}(1/2)^5$; $f_Y(0) = 1/2$; $f_{YX}(0,2) = \binom{5}{2}(1/2)^6$

$f_{Y|X=2}(0) = \frac{\binom{5}{2} \cdot (1/2)^6}{\binom{6}{2} \cdot (1/2)^6} = \frac{\binom{5}{2}}{\binom{6}{2}}$

$= \frac{5!}{2!\,3!} \times \frac{2!\,4!}{6!} = \frac{5!}{6!} \times \frac{4!}{3!} = \frac{4}{6} = \boxed{0.666}$

(9) $\dfrac{\binom{5}{1}\times\binom{4}{1}}{\binom{12}{2}} = \dfrac{5\times 4}{\frac{12!}{10!\,2!}} = \dfrac{5\times 4}{\cancel{12}_{3}\times 11}\times 2 = \dfrac{10}{33} = \boxed{0.30}$

(10) $N \sim Bin(7, 1/2) \qquad X \sim Bin(n, 1/2)$

$X \backslash N$	0	1	2	3	4
0	0	${}^7C_1(1/2)^8$			$\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11}$
1	0	${}^7C_1(1/2)^8$		$\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10}$	
2	0	0	$\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9}$		
3	0	0	0		
4	0	0	0	0	

$\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11} = \dfrac{7!}{4!\,3!}\times\dfrac{1}{2048} = \dfrac{7\times 6\times 5}{3\times 2}\times\dfrac{1}{2048} = \dfrac{35}{2048}$

$\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10} = \dfrac{7!}{3!\,4!}\times\dfrac{3}{1024} = \dfrac{210}{2048}$

$\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9} = \dfrac{7!}{2!\,5!}\times\dfrac{1}{512} = \dfrac{7\times 6}{2}\times\dfrac{1}{512} = \dfrac{84}{2048}$

$\text{Total} = \dfrac{35 + 210 + 84}{2048} = \dfrac{329}{2048} = \boxed{0.161}$

WEEK 2 GA

(2)

	0	1	2	
0	0.06	0.18	0.12	0.36
1	0.04	0.12	0.48	0.64
	0.1	0.3	0.6	

	0	1	2	
0	2/24	1/24	1/24	4/24
1	4/24	2/24	3/24	9/24
2	6/24	3/24	2/24	11/24
	12/24	6/24	6/24	

	0	1	2	
0	1/10	2/10	2/10	5/10
1	1/10	1/10	3/10	5/10
	2/10	3/10	5/10	

	0	1	
0	1/10	1.5/10	2.5/10
1	2/10	3/10	5/10
2	1/10	1.5/10	2.5/10
	4/10	6/10	

(3) $X \sim Bernoulli(0.2) \quad Y \sim Bernoulli(0.4) \qquad Z = X + Y$

$f_{X|Z=1}(1) = \dfrac{P(X=1, Z=1)}{P(Z=1)} = \dfrac{P(X=1, Y=0)}{P(X=1, Y=0 \text{ or } X=0, Y=1)} = \dfrac{(0.2)(0.6)}{(0.2)(0.6) + (0.8)(0.4)} = \dfrac{0.12}{0.12 + 0.32} = \boxed{0.2727}$

(4) $Z = X + Y \qquad f_{XY}(x, y) = \dfrac{9}{16\cdot(4)^{x+y}}$

$f_Z(k) = P(X = x, Y = k - x) = \sum_{x=0}^{k} f_{XY}(x, k-x) = \boxed{\dfrac{(k+1)\cdot 9}{16\cdot(4)^k}}$

(5) $Z = \max(X, Y)$

$f_Z(k) = P(X = k, Y \le k \text{ or } X \le k, Y = k) = \sum_{y=0}^{k} f_{XY}(k, y) + \sum_{x=0}^{k} f_{XY}(x, k)$

$= 2\cdot\sum_{x=0}^{k} \dfrac{9}{16\cdot(4)^{x+x}} =$

(6)

$y \backslash x$	1	2	3	4	5	6
1				*		
2					*	
3						*
4						
5						
6						

$f_X(4)\cdot f_Y(1) + f_X(5)\cdot f_Y(2) + f_X(6)\cdot f_Y(3)$

$= \left(\dfrac{1}{36}\right)\times 3 = \dfrac{3}{36} = \dfrac{1}{12}$

(7) $X \sim \text{Geometric}(p)$ $\quad Y \sim \text{Geometric}(p)$ iid $\quad Z = X + Y$

$f_Y(k) = p(1-p)^{k-1}$

$f_Z(k) = P(X=n, Y=k-n) = \sum_{n=1}^{k-1} p(1-p)^{n} \cdot p(1-p)^{k-n}$

$= p^2(1-p)^k (k-1)$

$f_Z(18) = p^2(1-p)^{18}(17) \qquad f_Z(19) = p^2(1-p)^{19}(18)$

$f_Z(18) < f_Z(19) \Rightarrow \cancel{p^2}(1-p)^{18}(17) < \cancel{p^2}(1-p)^{19}(18)$

$\frac{17}{18} < 1-p$

$\boxed{p < 0.056}$

(8) $X \sim \text{Poisson}(2)$ $\quad f_X(1) = e^{-2} \cdot 2 \quad f_X(2) = e^{-2} \cdot 2 \quad f_X(\neq 1 \text{ or } 2) = 1 - 4e^{-2}$

X_1	X_2	X_3
1	2	
1		2
2	1	
2		1
	2	1
	1	2

$6 \cdot (2e^{-2})^2 (1 - 4e^{-2}) = 0.439575(1 - 4e^{-2}) = \boxed{0.2016}$

(9) $X \sim \text{Bernoulli}(0.8)$ $\quad Y \sim \text{Bernoulli}(0.3)$ $\quad Z = X + Y - XY$

$f_Z(1) = P(X=1, Y=0 \text{ or } X=0, Y=1 \text{ or } X=1, Y=1) = 1 - P(X=0, Y=0) = 1 - (0.2)(0.7) = 0.86$

$Z \sim \text{Bernoulli}(0.86)$

(10) $X, Y \sim \text{Geometric}(0.8)$ $\quad P(X=1 \mid X+Y=2) = \frac{P(X=1, X+Y=2)}{P(X+Y=2)} = \frac{P(X=1, Y=1)}{P(X=1, Y=1)}$

(11) $X \sim \text{Poisson}(5)$ $\quad Y \sim \text{Poisson}(1)$ $\quad Z = X + Y$

$Z \sim \text{Poisson}(6)$

$Y \mid Z=4 \sim \text{Binomial}(4, 1/6)$

$Y \mid Z=4\,(3) = \binom{4}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right) = \frac{4 \cdot 5}{(6)^4} = \boxed{0.0154}$

WEEK 3 GA

(2) $X \sim \text{Bin}(2, 6/14)$ $\quad Y \sim \text{Bin}(2, 8/14)$ $\quad \rho(X,Y) = \frac{\text{Cov}(X,Y)}{SD(X)\,SD(Y)}$

Y \ X	0	1	2
0	0	0	30/182
1	0	96/182	0
2	56/182	0	0

$E(X) = 156/182 \qquad E(X^2) = 216/182 \qquad Var(X) = 14976/33124$

$E(Y) = 208/182 \qquad E(Y^2) = 320/182 \qquad Var(Y) = 14976/33124$

$SD(X)\,SD(Y) = 14976/33124 \qquad E(XY) = 96/182$

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{96}{182} - \frac{156 \cdot 208}{(182)^2} = \frac{-14976}{33124}$

$\rho(X,Y) = \frac{-14976}{33124} \times \frac{33124}{14976} = \boxed{-1}$

(3) $8^{th} \rightarrow P(G) = 0.4 \qquad G_8 \sim \text{Bin}(2, 0.4)$

$9^{th} \rightarrow 0.4 \qquad G_9 \sim \text{Bin}(2, 0.4)$

$10^{th} \rightarrow 0.6 \qquad G_{10} \sim \text{Bin}(2, 0.6)$

$E(G_{total}) = E(G_8) + E(G_9) + E(G_{10})$

$E(G_8) = E(G_9) = 0.8 \qquad E(G_{10}) = 1.2$

$E(G_{total}) = 0.8 + 0.8 + 1.2 = \boxed{2.8}$

(5) $\sigma = \sqrt{2}$ $\quad 7 = \mu - k\sigma \Rightarrow \left|\frac{7-15}{\sqrt{2}}\right| = k \Rightarrow k \simeq 5.66$ $\quad P(7 < X < 23) = 1 - \frac{1}{(5.66)^2} \simeq 0.969$

(6) $X \sim \text{Geometric}(0.04)$ $\quad Var(X) = (1-p)/p^2 = 600$

(8) $Var(XY) = Var(X)\cdot Var(Y) + Var(Y)(E(X))^2 + Var(X)(E(Y))^2$

$= 15 + 3969 + 605 = 4589$

(9)

X	Y	Z	P(X,Y,Z)	R
0	1	2	0.4	4
0	2	3	0.3	6
1	0	-2	0.3	-4

$R = XY + 2Z$

$E(R) = 2.2 \quad E(R^2) = 12.4$

$Var(R) = 12.4 - (2.2)^2 = 7.56$

(10) both losing $= \left(\frac{498 \times 497}{500 \times 499}\right) \times ₹0$

one wins 500 $= \left(\frac{1}{500} \times \frac{498}{499} \times 2\right) \times ₹500 = 1.99$

one wins 2000 $= \left(\frac{1}{500} \times \frac{498}{499} \times 2\right) \times ₹2000 = 7.98$

both tickets win $= \left(\frac{2}{500} \times \frac{1}{499}\right) \times ₹2500 = 0.02$

total $= 10$

(11)

$P(UU) = (0.4)(0.6)$ $\quad$ Gain(UU) = 2400 $\rightarrow$ 576 $\quad$ 1050

$P(UD) = (0.4)(0.4)$ $\quad$ " (UD) = 150 $\rightarrow$ 24 $\quad$ 300

$P(DU) = (0.6)(0.6)$ $\quad$ " (DU) = -1200 $\rightarrow$ -432 $\quad$ -450

$P(DD) = (0.6)(0.4)$ $\quad$ " (DD) = -2550 $\rightarrow$ -612 $\quad$ -1200

+

-36

48

-162

MOCK WEEK 1-2

(1)

X	Y	S	Z_1	Z_2
1	1	2	1	1
1	2	3	2	1
2	1	3	2	1
2	2	4	2	2
3	1	4	3	1
3	2	5	3	2

S	2	3	4	5
P(S)	0.1	0.4	0.4	0.1

Z_2	1	2
$P(Z_2)$	0.6	0.4

Z_1	1	2	3
$P(Z_1)$	0.1	0.7	0.2

(4)

y\x	0	1	2	
0	2/24	3/24	1/24	1/4
1	6/24	9/24	3/24	3/4
	1/3	1/2	1/6	

$P(Y=1 \mid X=2) = \frac{3}{24} \times 6 = \frac{3}{4}$

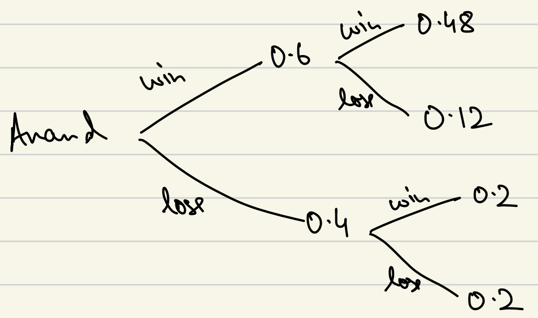
$P(X \le 1, Y=0) = 5/24$ $\quad P(X=2, Y=1) = 3/24$

sum $= 8/24 = 1/3$ $\quad \frac{1}{\text{sum}} = \boxed{3}$

⑥ $X \sim \text{Poisson}(2)$ $\quad Y \sim \text{Poisson}(3)$ $\quad Z = X+Y$ $\qquad Z \sim \text{Poisson}(5)$ $\qquad X|Z=5 \sim \text{Bin}(5, 2/5)$

$Y|Z=5 \sim \text{Bin}(5, 3/5)$

$$P(X=1|Z=5) = \binom{5}{1}\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^4 = 0.2592$$

⑧ 

$X_2 \backslash X_1$	0	1	
0	0.2	0.12	0.32
1	0.2	0.48	0.68
	0.4	0.6	

⑪ $P(X=3, Y=3) = P(N=6) \text{ and } P(X=3)$

$$P(N=6) = \frac{e^{-5}5^6}{6!} = 0.146223$$

$$P(X=3|N=6) = \binom{6}{3}(0.3)^3(0.7)^3 = 20(0.21)^3 = 0.18522$$

$$P(X=3, N=6) = \boxed{0.0271}$$

⑬ $$P(X_1=0|Y=8) = \frac{P(X_1=0, Y=8)}{P(Y=8)} \qquad P(Y=8) = \binom{10}{8}\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)^2$$

$$P(X_1=0, Y=8) = (1/3)\cdot\binom{9}{8}\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)$$

$$\frac{\cancel{(1/3)^2}\,\cancel{(2/3)^8}\;{}^9C_8}{\cancel{(1/3)^2}\,\cancel{(2/3)^8}\;{}^{10}C_8} = \frac{9!}{8!} \times \frac{8!\,2!}{10!} = \frac{2}{10}$$

⑭ $$f_{Z|Y=15}(50) = \frac{P(X=10, Y=5 \text{ or } X=5, Y=10)}{f_{XY}(10,5) \text{ or } (5,10) \text{ or } (7,8) \text{ or } (8,7) \text{ or } (6,9) \text{ or } (9,6)}$$

$$f_{XY}(10,5) = f_{XY}(5,10) = \binom{10}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5\cdot\binom{10}{10}\left(\frac{1}{3}\right)^{10} = \binom{10}{5}\left(\frac{1}{3}\right)^{15}\left(\frac{2}{3}\right)^5$$

$$f_{XY}(6,9) = f_{XY}(9,6) = \binom{10}{6}\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^4\cdot\binom{10}{9}\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right) = \binom{10}{6}\binom{10}{9}\left(\frac{1}{3}\right)^{15}\left(\frac{2}{3}\right)^5$$

$$f_{XY}(7,8) = f_{XY}(8,7) = \binom{10}{8}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^2\cdot\binom{10}{7}\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^3 = \binom{10}{8}\binom{10}{7}\left(\frac{1}{3}\right)^{15}\left(\frac{2}{3}\right)^5$$

$$\frac{10!}{5!\,5!} = \frac{\cancel{10}^{2}\times 9\times \cancel{8}^{2}\times 7\times \cancel{6}}{\cancel{5\times4\times3\times2}} = 28\times 9$$

$$\frac{10!\times 10!}{6!\,4!\,9!} = \frac{10\times \cancel{9}^{3}\times\cancel{8}\times 7}{\cancel{4}\times\cancel{3}\times\cancel{2}}\times 10 = 100\times3\times7$$

$$\frac{10!\times 10!}{8!\,2!\,7!\,3!} = \frac{10\times 9}{\cancel{2}}\times\frac{10\times\cancel{9}^{3}\times\cancel{8}^{2}}{\cancel{3}\times\cancel{2}} = 600\times 9$$

$$\frac{\binom{10}{5}\cancel{\left(\frac{1}{3}\right)^{15}}\cancel{\left(\frac{2}{3}\right)^5}}{\cancel{\left(\frac{1}{3}\right)^{15}}\cancel{\left(\frac{2}{3}\right)^5}\left[\binom{10}{5}+\binom{10}{6}+\binom{10}{8}\right]} = \frac{\frac{10!}{5!\,5!}}{\frac{10!}{5!\,5!}+\frac{10!}{6!\,4!}+\frac{10!}{8!\,2!}}$$

⑮ $X \sim \text{Uniform}\{0,1,\dots,9\}$ $\quad Y \sim \text{Uniform}\{0,1,\dots,9\}$

$$P(Z \geq 2) = 1 - P(Z=1) - P(Z=0)$$

$P(Z=0) \rightarrow X = Y = 1/10$ $\qquad P(Z=1) \rightarrow |X-Y| = 1 = 18/100$

$$P(Z\geq 2) = 1 - \frac{28}{100} = \frac{72}{100}$$

01	43	78
10	45	87
12	54	89
21	56	98
23	65	
32	67	
34	76	

WEEK 4 GA

② $P(-19 < X < 16) = P(X < 16) = 1 - \exp(-832)$

③ $X \sim \text{Exponential}(\lambda = 1) \qquad k = \lambda = 1$

④ $P(45 < X < 48) = F_X(48) - F_X(45) \qquad F_X(48) = 1 - \exp(-48) \qquad F_X(45) = 1 - \exp(-45)$

$$= 1 - \exp(-48) - 1 + \exp(-45)$$
$$= \exp(-45) - \exp(-48) = e^{-45} - e^{-48}$$

⑤ $X \sim \text{Exponential}(\lambda = 1/900) \qquad P(X > 700) = 1 - P(X \le 700) = \exp(-700/900)$

$$= \exp(-0.78)$$

⑥ $f_X(x) = 5x^4 \qquad F_X(x) = x^5 \qquad P(X < 6/9 \mid X > 1/9) = \dfrac{P(1/9 < X < 6/9)}{P(X > 1/9)} = \dfrac{F_X(6/9) - F_X(1/9)}{1 - F_X(1/9)}$

$$= \frac{(6/9)^5 - (1/9)^5}{1 - (1/9)^5} = \frac{(6)^5 - 1}{\cancel{(9)^5}} \times \frac{\cancel{(9)^5}}{(9)^5 - 1} = \frac{(6)^5 - 1}{(9)^5 - 1} = \frac{7775}{59048} \simeq 0.132$$

⑦ $Z = \dfrac{X - 206}{35} \Rightarrow -1 = \dfrac{X - 206}{35} \Rightarrow X = -35 + 206 = 171$

WEEK 5 GA

① $A = 7/37 \quad B = 30/37$

$A \sim \text{Exp}(1/7) \qquad B \sim \text{Exp}(1/15) \qquad f_Y(y) = \frac{7}{37}(1/7)\exp(-y/7) + \frac{30}{37}(1/15)\exp(-y/15)$

$$P(X = A \mid Y = 5) = \frac{f_{Y|X=A}(y) \cdot f_X(A)}{f_Y(y)} = \frac{(1/37) \cdot \exp(-5/7)}{\frac{7}{37}(1/7)\exp(-5/7) + \frac{30}{37}(1/15)\exp(-5/15)}$$

$$= \frac{\exp(-5/7)}{\exp(-5/7) + 2\exp(-1/3)} = 0.2546$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

② $Y = XZ + X \qquad X \sim \text{Uniform}\{1, 2, 3\} \qquad Z \sim \text{Normal}(1, 4)$

$$P(X = 2 \mid Y = 2) = \frac{f_{Y|X=2}(2) \cdot f_X(2)}{f_Y(2)} \qquad f_X(2) = 1/3 \qquad f_{Y|X=2}(2) = \frac{P(X = 2, Z = 0)}{P(X = 2)} = P(Z = 0) = \frac{1}{2\sqrt{2\pi}}\exp(-1/8)$$

$$= \frac{\exp(-1/8)}{1 + \exp(-1/8) + \exp(-2/9)}$$

$$f_Y(y) = \frac{1}{3}\frac{1}{2\sqrt{2\pi}}\left(\exp\left(\frac{-(y-2)^2}{8}\right) + \exp\left(\frac{-(y-4)^2}{32}\right) + \exp\left(\frac{-(y-6)^2}{72}\right)\right)$$

$$f_Y(2) = \frac{1}{6\sqrt{2\pi}} \cdot \left(\exp\left(\frac{-(2-2)^2}{8}\right) + \exp\left(\frac{-1}{8}\right) + \exp\left(\frac{-2}{9}\right)\right)$$

$$= 0.32889$$

③ $X = (6)(1-x)^5 \qquad Y = (1-X)^9 \qquad g^{-1}(y) = 1 - (y)^{1/9} \qquad g'(x) = 9(1-x)^8 \qquad g'(g^{-1}(y)) = 9(y)^{8/9}$

$$f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} \cdot f_X(g^{-1}(y)) = \frac{1}{9(y)^{8/9}}(6)(1 - (1 - (y)^{1/9}))^5$$
$$= \frac{2}{3}\frac{(y)^{5/9}}{(y)^{8/9}} = \frac{2}{3}(y)^{-3/9}$$

④ $\text{supp}(y) = \frac{24}{3} < y < \frac{33}{3} \qquad g(x) = \frac{27 - x}{3} \qquad g^{-1}(y) = 27 - 3y \qquad g'(x) = \frac{1}{3} \qquad f_X(g^{-1}(y)) = \frac{(27-3y)^2}{81}$

$$f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} \cdot f_X(g^{-1}(y)) = \frac{3(27-3y)^2}{81}$$

⑤ $E[X] = \int_0^1 x \cdot x(4x^2 + 3x - 2)\,dx = \int_0^1 4x^4 + 3x^3 + 2x^2\,dx = \left.\frac{4x^5}{5}\right|_0^1 + \left.\frac{3x^4}{4}\right|_0^1 + \left.\frac{2x^3}{3}\right|_0^1$

$$= \frac{4}{5} + \frac{3}{4} + \frac{2}{3} = \frac{48 + 45 + 40}{60} = \frac{133}{60}$$

$9 \cdot E[X] = 9 \cdot \dfrac{(133)}{60} = 19.95$

(6) $E[X] = \int_0^1 x^2\,dx + \int_1^2 2x - x^2\,dx = \frac{x^3}{3}\Big|_0^1 + x^2\Big|_1^2 - \frac{x^3}{3}\Big|_1^2 = \frac{1}{3} + 3 - \frac{7}{3} = \frac{-6+9}{3} = 1$

$Var(X) = \int_0^1 (x-1)^2 x\,dx + \int_1^2 (x-1)^2(2-x)\,dx$

$= \int_0^1 x^3 + x - 2x^2\,dx + \int_1^2 -x^3 + 4x^2 - 5x + 2\,dx = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\Big|_0^1 + -\frac{x^4}{4} + \frac{4x^3}{3} - \frac{5x^2}{2} + 2x\Big|_1^2$

$= \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) + \left(-4 + \frac{32}{3} - 10 + 4\right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{5}{2} + 2\right)$

$= \frac{1}{4} - \frac{2}{3} + \frac{2}{4} - \frac{16}{4} + \frac{32}{3} - \frac{40}{4} + \frac{16}{4} + \frac{1}{4} - \frac{4}{3} + \frac{10}{4} - \frac{8}{4}$

$= \frac{1+2-16-40+16+1+10-8}{4} + \frac{32-2-4}{3} = \frac{-34}{4} + \frac{26}{3} = \frac{104-102}{12} = \frac{1}{6}$

$Var(Y) = (48)^2(Var(X)) = 384$

(9) $g(x) = x^3 + 22 \quad g^{-1}(y) = (y-22)^{1/3} \quad g'(x) = 3x^2 \quad g'(g^{-1}(y)) = 3(y-22)^{2/3}$

$f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} \cdot f_X(g^{-1}(y)) = \frac{1}{3}(y-22)^{-2/3} \cdot \frac{1}{20} = \frac{(y-22)^{-2/3}}{60}$

$E[Y] = \int_{22}^{8022} \frac{(y-22)^{1/3}}{60}\,dx = \frac{3(y-22)^{4/3}}{(4)(60)}\Big|_{22}^{8022} = \frac{(8000)^{4/3}}{80} - \frac{1}{80} \approx 2000$

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$

(10) male $\sim$ Normal $(60, 25)$ $\quad X =$ Gender $\quad Y|X=M \sim$ Normal $(60, 25)$

female $\sim$ Normal $(55, 36)$ $\quad Y =$ Age $\quad Y|X=F \sim$ Normal $(55, 36)$

$P(X=M \mid Y=60) = \frac{f_{Y|X=M}(60) \cdot f_X(M)}{f_Y(60)}$

$= \frac{(3/5) \cdot (1/\sqrt{50\pi})}{f_Y(60)}$

$= \frac{3}{\cancel{5}} \cdot \frac{\cancel{15}\,3}{9 + 5\exp(\frac{-25}{72})}$

$= \frac{9}{9 + 5\exp(\frac{-25}{72})}$

$f_Y(y) = \frac{3}{5} \cdot \frac{1}{5\sqrt{2\pi}} \exp\left(\frac{-(y-60)^2}{50}\right) + \frac{2}{5} \cdot \frac{1}{6\sqrt{2\pi}} \exp\left(\frac{-(y-55)^2}{72}\right)$

$= \frac{1}{\sqrt{50\pi}}\left(\frac{3}{5}\exp\left(\frac{-(y-60)^2}{50}\right) + \frac{1}{3}\exp\left(\frac{-(y-55)^2}{72}\right)\right)$

$f_Y(60) = \frac{1}{\sqrt{50\pi}}\left(\frac{3}{5} + \frac{1}{3}\exp\left(\frac{-25}{72}\right)\right)$

$= \frac{1}{\sqrt{50\pi}}\left(\frac{9 + 5\exp\left(\frac{-25}{72}\right)}{15}\right)$

Mock for Quiz 1

(1)

$y \backslash x$	1	2	
1	2/8	3/8	5/8
2	2/8	1/8	3/8
	4/8	4/8	

(2) $P(X+Y=3) = f_{XY}(1,2) + f_{XY}(2,1) = 2/8 + 3/8 = 5/8$

(3) $P(\max(X,Y) \le 2) = f_{XY}(x \le 2, y \le 2) = 1$

(4) $P(X<2) = P(X>2) = 0.1$; $P(-2 < X < 2) = 1 - (P(X<2) + P(X>2)) = 0.8$

(5) Probability of getting same $= \frac{1}{6}$ $\sim$ Geometric $(1/6)$

$E[X] = 1/p = 6$

(7) $X \sim Exp(1/20)$ $\qquad F_X(15) = e^{-3/4} \approx 0.527633$

(8) $f_X(15 < X < 30) = F_X(30) - F_X(15) \approx 0.249236$

(9)

X	-10	20	30
P(X=x)	9/36	18/36	9/36

$\frac{9}{36}(30-10) + \frac{18}{36}(20) = 20\left(\frac{9+18}{36}\right) = 15$

11 13 15
33 31 35
55 51 53

12 14 16
21 23 25
32 34 36
41 43 45
52 54 56
61 63 65

(10) $P(Y>5 \mid X=2) \sim \text{Geometric}(0.5) = (1-0.5)^5 = 0.03125$

(11) $P(Y>5) = 0.5\left(\text{Geometric}(1/2) + \text{Geometric}(1/3)\right) \approx 0.1629$

(13) $P(X \geq 180) + P(X \leq 120) = 1 - P(120 \leq X \leq 180)$

$P(3\sigma \leq X \leq 3\sigma) \geq 1 - \frac{1}{9} \geq \frac{8}{9}$ $\qquad 1 - P(3\sigma \leq X \leq 3\sigma) \leq \frac{1}{9}$

(14) $k \cdot \int_0^1 (x)^{1/2}\, dx = 1 \Rightarrow \frac{2k(x)^{3/2}}{3}\Big|_0^1 = 1 \qquad \frac{2k}{3} = 1 \qquad k = 3/2$

(15) $P(X \geq 0.4 \mid X < 0.8) = \frac{P(0.4 \leq X < 0.8)}{P(X<0.8)} = \frac{\int_{0.4}^{0.8} 3/2 (x)^{1/2}\, dx}{\int_0^{0.8} 3/2 (x)^{1/2}\, dx} = \frac{(0.8)^{3/2} - (0.4)^{3/2}}{(0.8)^{3/2}} \approx 0.64645$

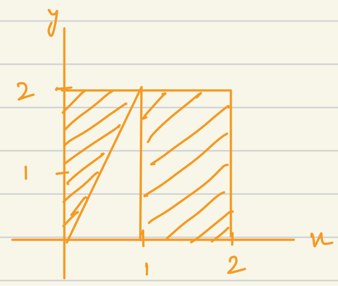
(18) $P(X_1 = 0 \mid Y=8) = \frac{P(X_1=0, Y=8)}{P(Y=8)} = \frac{\binom{9}{8}\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^2}{\binom{10}{8}\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^2} = \frac{9}{45} = 1/5 = 0.2$

WEEK 6 GA

(1) $f_X(x) = \int_0^5 \frac{4xy}{5^4}\, dy = \frac{4x}{5^4}\left(\frac{y^2}{2}\Big|_0^5\right) = \frac{2x}{5^2}$

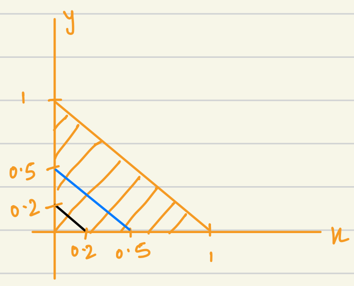
$f_Y(y) = \frac{2y}{5^2} \qquad f_X(x) \cdot f_Y(y) = \frac{4xy}{5^4} = f_{XY}(x,y)$

(3)



Total area $= \left(\frac{1}{2} \times 2 \times 1\right) + (1 \times 2) = 1 + 2 = 3$

(4)



$P(X+Y \leq 1/2) = \int_{y=0}^{1/2}\int_{x=0}^{1/2-y} 24xy\, dx\, dy = \int_{y=0}^{1/2} 24y\left(\frac{x^2}{2}\Big|_0^{1/2-y}\right) dy = \int_{y=0}^{1/2} 3y + 12y^3 - 12y^2\, dy = \frac{3y^2}{2} + 3y^4 - 4y^3\Big|_0^{1/2}$

$= \frac{3}{8} + \frac{3}{16} - \frac{4}{8} = \frac{3}{16} - \frac{2}{16} = \frac{1}{16}$

$P(X+Y \leq 1/5) = \int_{y=0}^{1/5}\int_{x=0}^{1/5-y} 24xy\, dx\, dy = \int_{y=0}^{1/5} 24y\left(\frac{x^2}{2}\Big|_0^{1/5-y}\right) dy = \int_{y=0}^{1/5} 24y\left(1/50 + y^2/2 - 1/5 y\right) dy = \frac{6y^2}{25} + 3y^4 - \frac{8y^3}{5}\Big|_0^{1/5}$

$= \frac{6}{625} + \frac{3}{625} - \frac{8}{625} = \frac{1}{625}$

(5) $f_{XY}(x,y) = 3xy - 3x^2y \qquad f_X(x) = \int_{y=0}^{2} 3xy - 3x^2y\, dy = \frac{3xy^2}{2} - \frac{3x^2y^2}{2}\Big|_0^2 = 6x - 6x^2 = 6x(1-x)$

$f_Y(y) = \int_{x=0}^{1} 3xy - 3x^2y\, dx = \frac{3x^2y}{2} - x^3y\Big|_0^1 = \frac{3y}{2} - y = \frac{y}{2}$

$f_{X|Y=1}(>1/3) = \frac{f_{XY}(x>1/3, 1)}{f_Y(1)} \qquad f_{XY}(x>1/3, 1) = \int_{1/3}^{1} 3x - 3x^2\, dx = \frac{3x^2}{2} - x^3\Big|_{1/3}^{1} = \frac{3}{2} - 1 - \frac{3}{18} + \frac{1}{27} = \frac{27-18-3}{18} + \frac{1}{27} = \frac{1}{3} + \frac{1}{27}$

$= \frac{10}{27}$

$= \frac{10/27}{1/2} = 20/27 \approx 0.74$

(6) $P(x-y<3) = P(y>x-3) = \int_{x=0}^{10}\int_{y=x-3}^{x} 1/50\, dy\, dx = \int_0^{10} \frac{y}{50}\Big|_{x-3}^{x} dx = \frac{3x}{50}\Big|_0^{10} = \frac{3}{5}$

(7) $\int_{x=6}^{\infty}\int_{y=0}^{8} ke^{-x}(e)^{-y}\, dy\, dx = \int_6^{\infty} ke^{-x}(1-e^{-8})\, dx = ke^{-6}(1-e^{-8})$

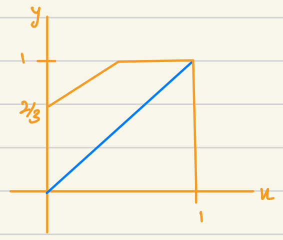
(8) $f_X(x) = \int_0^2 x/8 + y/8\, dy = \frac{xy}{8} + \frac{y^2}{16}\Big|_0^2 = \frac{x+1}{4} \qquad f_X(1/6) = \frac{7/6}{4} = 7/24 \qquad P(2/6 \leq Y \leq 2, X=1/6) = \int_{y=2/6}^{2} 1/48 + 6y/48\, dy = y/48 + 3y^2/48\Big|_{2/6}^{2}$

$= \frac{2}{48} + \frac{12}{48} - \frac{(1/3)}{48} - \frac{(1/3)}{48} = 5/18$

$P(2/6 \leq Y \leq 2 \mid X = 1/6) = \frac{5/18}{7/24} = \frac{5}{\cancel{18}_3} \times \frac{\cancel{24}^4}{7} = \frac{20}{21} \approx 0.95$

9 $f_X(x) = \int_0^1 \frac{2x}{3} + \frac{2}{3}\, dy = \frac{2xy}{3} + \frac{2y}{3}\Big|_0^1 = \frac{2x}{3} + \frac{2}{3}$

$f_Y(y) = \int_0^1 \frac{2x}{3} + \frac{2}{3}\, dx = \frac{x^2}{3} + \frac{2x}{3}\Big|_0^1 = \frac{1}{3} + \frac{2}{3} = 1$



$P(X>Y) = \int_{x=0}^{1}\int_{y=0}^{x} \frac{2x}{3} + \frac{2}{3}\, dy\, dx = \int_0^1 \frac{2x^2}{3} + \frac{2x}{3}\, dx = \frac{2x^3}{9} + \frac{x^2}{3}\Big|_0^1 = \frac{2}{9} + \frac{1}{3} = \frac{2+3}{9} = \frac{5}{9}$

WEEK 7 GA

1 $Var(A) = \sigma^2/3$ $\quad Var(B) = 0.46\sigma^2$ $\quad Var(C) = 0.38\sigma^2$

2 $\sigma^2 = 25$ $\quad n = 30$ $\quad Var(\text{sample mean}) = \sigma^2/n = 0.83$

3 $p = 1/6$ $\quad \sigma^2 = 5/36$ $\quad \mu = 1/6$ $\quad \delta = 0.6$ $\quad P(|\bar{X} - \mu| < \delta) \geq 1 - \frac{\sigma^2}{n\delta^2}$

$\geq 1 - \frac{5}{36} \times \frac{1}{(300)(0.36)} \Rightarrow \geq 1 - \frac{5}{3888} \Rightarrow \frac{3883}{3888}$

5 $Var(X_i) = \frac{1}{(i)^2}, \; i \in [1,9]$ $\quad Var(i \cdot X_i) = 1, \; i \in [1,9]$ $\quad Var(Y) = 9$

6 $\sigma^2 = 12$ $\quad P(|\bar{X} - \mu| < \delta) \geq 1 - \frac{\sigma^2}{n\delta^2}$, where $n = 50$, $\delta = 3$

7 $Var(S(A)) = 0.0004$ $\quad n = (0.45)(0.55)/0.0004 = 619$

8 $\mu = 2$ $\quad n = 50$ $\quad \delta = 1$ $\quad P(|\bar{X} - 2| < 1) \geq 1 - \frac{4}{(50)(1)}$

$\sigma^2 = 4$

≥ 0.92

WEEK 8 GA

1 $E[Y] = 2$ $\quad Var(Y) = 50(0.04) = 2$

$P(Y > 8) \rightarrow$ z-score $= \frac{6}{\sqrt{2}} =$

3 $E[Y] = 600$ $\quad Var(Y) = 1200(0.25) = 300$

z-score of $Y = 580 \rightarrow \frac{580 - 600}{\sqrt{300}} = -1.15$ $\quad F_Z(-1.15) \approx 0.125$

4 $E[Y] = 250/0.5 = 500$ $\quad Var(Y) = 1000$

$P(Y > 565) = 1 - P(Y < 565) = 1 - F_Z\left(\frac{565 - 500}{\sqrt{1000}}\right) = 1 - 0.98030 = 0.0197$

5 $E[X] = 2n$ $\quad Var(X) = 2n$

X is a sum of i.i.d. Gamma(2,1)

$E\left[\frac{X}{2n}\right] = 1$ $\quad Var\left(\frac{X}{2n}\right) = \frac{2n}{(2n)^2} = \frac{1}{2n}$ $\quad P\left(\frac{X - E[X]}{Var(X)} > 0.01\right) \leq \frac{1}{(0.01)^2}$

$\frac{Var(X/2n)}{(0.01)^2} = 0.01$

$\frac{1}{2n} = (0.01)^3$ $\quad n = \frac{1}{2(0.01)^3} = 500000$

⑦ $E[L_i] = 20$ $Var(L_i) = 400$ $SD(L_i) = 20$

$E[T] = 1000$ $Var(T) = 20,000$ $SD(T) = 141.4214$

$P(T > 1950) = 1 - P(T < 1950) = 1 - F_Z\left(\frac{1950-1000}{\sqrt{20000}}\right) = 1 - F_Z(6.7)$

⑧ $E[\bar{X}] = 36$ $Var(\bar{X}) = 4/35$ $SD(X) = 2/\sqrt{35}$

$P(27 < \bar{X} < 40) = F_Z\left(\frac{40-36}{2/\sqrt{35}}\right) - F_Z\left(\frac{27-36}{2/\sqrt{35}}\right)$