MATH 2

WEEK 1

→ Vectors

- can be thought of as a list.
- can be in a row or a column form.

↳ Addition of vectors:

Add the corresponding entries. For e.g. → $(3,5)+(2,4)=(5,9)$

↳ Scalar multiplication:

e.g. → $2(8,8,10,5) = (16,16,20,10)$

↳ Visualisation of a vector

Point $(a,b) \equiv$ Vector $(a,b) \equiv a\hat{i} + b\hat{j}$ where, $\hat{i}$ → one unit in x-axis, $\hat{j}$ → one unit " y-axis

e.g.: Point $(-1,-1) = -\hat{i}-\hat{j}$

↳ Vectors in $\mathbb{R}^n$ are lists with n real entries.

→ Matrices

- rectangular array of numbers
- (rows × columns); eg → $\begin{bmatrix} 5 & 7 & 10 \\ 3 & 5 & 2 \end{bmatrix}$ is a 2×3 matrix.
  - $(1,2)$ ith entry → 7
- square matrix → $N \times N$
  - Diagonal matrix → all entries are 0 except the diagonal
    $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$
  - Scalar matrix → all entries have the same value
    $$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$
  - Indentity matrix → denoted by 'I'; scalar matrix with values = 1
    $$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Addition of matrices → must be of the same size: $\begin{bmatrix} 1 & 0 \\ 5 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 8 & 3 \\ 7 & 4 \end{bmatrix}$
- Scalar multiplication → multiply each number with the scalar.
- Matrix multiplication → $A \times B = C$; $C[i,j] = \sum_{k=1}^{n} A[i,k] \times B[k,j]$
  - no. of columns in first matrix <u>must</u> = no. of rows in 2nd matrix
  - $A_{m \times n} \times B_{n \times p} = (AB)_{m \times p}$

    $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{jk}$
- Scalar multiplication is the same as multiplication by the scalar matrix

  ex. → $\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$
- Properties:
  1. $(A+B)+C = A+(B+C)$
  2. $(AB)C = A(BC)$
  3. $A+B = B+A$
  4. $AB \neq BA$
  5. $\lambda(A+B) = \lambda A + \lambda B$
  6. $\lambda(AB) = (\lambda A)B = A(\lambda B)$
  7. $A(B+C) = AB + AC$
  8. $(A+B)C = AC + BC$

→ System of Linear Equations

- collection of one or more linear equations involving the same set of variables.
- system of m linear equation with n variables :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_n \end{aligned}$$

- system of lin. eq. in matrix form :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$$\underbrace{A}_{\text{coefficient matrix}} \underbrace{x}_{\text{column of variables}} = \underbrace{b}_{\text{column of resulting values}}$$

, where
A = $m \times n$ matrix
x = column vector with n entries
b = column vector with m entries

- Solutions to a system of lin. eq. :
  1. Infinite solution
  2. Single unique solution
  3. No solution

→ Determinant

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad - bc$

  e.g. $A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$ $\quad \det(A) = 20 - 18 = 2$

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad \det(A) = a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

- Determinant of Identity matrix $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - → $\det(I_2) = 1$
  - → $\det(I_3) = 1$

- Determinant of a product of matrices
  - $\det(AB) = \det(A) \cdot \det(B)$; $\det(ABC) = \det(A) \cdot \det(B) \cdot \det(C)$
  - $\det(A^n) = \det(A)^n$
  - $\det(A^{-1}) = \det(A)^{-1}$

- Determinant of the inverse of a matrix

  $$A A^{-1} = I \Rightarrow \det(AA^{-1}) = \det(I)$$

  - $\det(A^{-1}) = \dfrac{1}{\det(A)}$

- Switching rows

  $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

  $$\det(\tilde{A}) = cb - ad = -(ad - bc) = -\det(A)$$

  $$\det(\tilde{A}) = -\det(A)$$

- Add multiple of a row/column to another row/column

  $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$$

  $$\det(\tilde{A}) = (a+tc)d - (b+td)c = ad + tcd - bc - tcd$$

  $$\det(\tilde{A}) = \det(A)$$

↳ Scalar multiplication of a row/column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} ta & b \\ tc & d \end{bmatrix}$$

$$\det(\tilde{A}) = t \cdot \det(A)$$

↳ Upper | Lower triangle matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{bmatrix}$$ upper triangle matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 0 \\ 3 & 4 & 9 \end{bmatrix}$$ lower triangle matrix

→ determinant is the product of diagonal elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \det(A) = a_{11} \cdot a_{22} \cdot a_{33}$$

↳ Transpose of a matrix and its determinants

→ Transpose of $A_{m \times n}$ = $A^T_{n \times m}$ with (i,j)-th entry A_{ji}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2\times 3}$$

$$\det(A) = \det(A^T)$$

↳ Minors and cofactors

→ Minor of the entry in i-th row and j-th column is the determinant of the submatrix formed by deleting i-th row and j-th column.

e.g. → $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ (1,1)-th minor; denoted by M_{11}

$$M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

→ Cofactor (i,j)th cofactor; $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$C_{11} = M_{11} \; ; \; C_{23} = -M_{23}$$

→ For $A_{3\times 3}$

$$\det(A) = (a_{11} \times C_{11}) + (a_{12} \times C_{12}) + (a_{13} \times C_{13})$$

For $A_{4\times 4}$

$$\det(A) = \sum_{j=1}^{4} a_{1j} C_{1j}$$

For $A_{n\times n}$; $\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$

WEEK 2

→ Determinant (continued)

↳ Expansion along any row/column

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } i$$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } j \text{ (determinant along a column)}$$

↳ Properties:

① Determinant of a product is product of the determinants.
↳ $\det(AB) = \det(BA) = \det(A) \cdot \det(B)$

② Switching two rows or columns changes the sign.

③ Adding multiples of a row to another row leaves the determinant unchanged.

④ Scalar multiplication of a row/column by a constant t multiplies the determinant by t.

⑤ $\det(t A_{n\times n}) = (t)^n \det(A)$

↳ Useful computational tips:

① The determinant of a matrix with a row or a column of zeros is 0.

② The determinant of a matrix in which one row (or column) is a linear combination of other rows (resp. columns) is 0.

③ Scalar multiplication of a row/column by a constant t multiplies the determinant by t.

④ While computing the determinant, you can choose to compute it using expansion along a suitable row or a column.

→ Cramer's rule

→ $Ax = b$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Cramer rule ⇒ $A_{x_1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{21} \end{bmatrix}$, $A_{x_2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$

↳ solutions for x_1 & x_2 ⇒ $x_1 = \dfrac{\det(A_{x_1})}{\det(A)}$; $x_2 = \dfrac{\det(A_{x_2})}{\det(A)}$

→ For a 3x3 matrix: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$A_{x_1} \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$ $A_{x_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$ $A_{x_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$

$x_1 = \dfrac{\det(A_{x_1})}{\det(A)}$ $x_2 = \dfrac{\det(A_{x_2})}{\det(A)}$ $x_3 = \dfrac{\det(A_{x_3})}{\det(A)}$

→ Inverse Matrix

↳ The inverse of a matrix $A \to (A^{-1}) \Rightarrow A \cdot (A^{-1}) = I$

↳ if $\det(A) \neq 0$, then the matrix is invertible

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

↳ Adjucate of a matrix

M_{ij} = determinant of submatrix after deleting i-th row and j-th column.

$C_{ij} = (-1)^{i+j} M_{ij}$

→ Adjucate of a matrix → Transpose of Cofactor matrix

$Adj(A) = C^T$

→ $A^{-1} = \frac{1}{\det(A)} \cdot adj(A)$ → $A_{n \times n}$, then $\det(adj(A)) = (\det(A))^{n-1}$

↳ Solution to system of linear equations using inverse matrix.

$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$

→ Homogeneous System of Linear Equations

↳ $Ax = b$, where $b = 0 \Rightarrow Ax = 0$

→ has unique solution 0 if A is invertible

→ has infinite solutions if A is not invertible

→ The Row Echelon Form (reduced form)

↳ Matrix is in row echelon form if:

→ first non-zero element in each row (leading entry) is 1

→ each leading entry is in a colum to the right of the leading entry in the previous row.

→ rows with all zero elements, if any, are below rows having a non-zero element.

e.g. → $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

→ reduced row echelon form → For a non-zero row, the leading entry in the row is the only non-zero entry in its column

e.g. → $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

→ e.g. solution for $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow Ax = b$

$x_1 + 2x_2 = b_1 \Rightarrow x_1 = b_1 - 2x_2$

$x_3 = b_2$

$x_4 = b_3$

solution $\Rightarrow x = \begin{bmatrix} b_1 - 2c \\ c \\ b_2 \\ b_3 \end{bmatrix}$

→ If the i-th column has the leading entry of some row, we call x_i a dependent variable.
If the i-th column does not have the leading entry of any row, x_i → independent variable.

→ Row reduction

Elementary Row Operations

Type	Action	Example and notation	Effect on determinant
1	Interchange two rows	$\begin{bmatrix}3&2&1&1\\1&1&0&0\\0&7&1&1\end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix}1&1&0&0\\3&2&1&1\\0&7&1&1\end{bmatrix}$	$\det(A) = -\det(B)$
2	Scalar multiplication of a row by a constant t	$\begin{bmatrix}3&2&1&1\\1&1&0&0\\0&7&1&1\end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix}1&2/3&1/3&1/3\\1&1&0&0\\0&7&1&1\end{bmatrix}$	$\det(A) = t\cdot\det(B)$
3	Adding multiples of a row to another row	$\begin{bmatrix}3&2&1&1\\1&1&0&0\\0&7&1&1\end{bmatrix} \xrightarrow{R_1-3R_3} \begin{bmatrix}3&-19&-2&-2\\1&1&0&0\\0&7&1&1\end{bmatrix}$	$\det(A) = \det(B)$

↳ Row echelon form

Steps :
1. Find the left most non-zero column.
2. Use operations to get 1 in the top of that column.
3. Make entries below that 1 to 0.
4. Repeat the steps for the next row and onwards.

↳ Reduced row echelon form
↳ start from the right most column and make all the non-leading terms 0s.

↳ e.g.

$$A = \begin{bmatrix}2&4&1\\3&8&7\\5&6&9\end{bmatrix} \xrightarrow{R_1/2} \begin{matrix}1&2&1/2\\3&8&7\\5&6&9\end{matrix} \xrightarrow[R_3-5R_1]{R_2-3R_1} \begin{matrix}1&2&1/2\\0&2&11/2\\0&-4&13/2\end{matrix} \xrightarrow[R_3+2R_2]{R_2/2} \begin{matrix}1&2&1/2\\0&1&11/4\\0&0&35\end{matrix} \xrightarrow{R_3/35} \begin{bmatrix}1&2&1/2\\0&1&11/4\\0&0&1\end{bmatrix}$$

$$\begin{matrix}1&2&1/2\\0&1&11/4\\0&0&1\end{matrix} \xrightarrow[R_2-\frac{11}{4}R_3]{R_1-\frac{1}{2}R_3} \begin{matrix}1&2&0\\0&1&0\\0&0&1\end{matrix} \xrightarrow{R_1-2R_2} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\det(A) = 35 \times 2 = 70$$

→ Gaussian Elimination Method

↳ Augmented matrix → matrix of size $(m \times n+1)$
↳ first n columns are from A and the last column is b.
↳ denoted by $[A|b]$

$$\left[\begin{array}{cccc|c}a_{11}&a_{12}&\cdots&a_{1n}&b_1\\a_{21}&a_{22}&\cdots&a_{2n}&b_2\\\vdots&\vdots&\cdots&\vdots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}&b_m\end{array}\right]$$

↳ Bring the matrix A to reduced row echelon form.
→ Let R be the submatrix of obtained matrix of the first n columns and c be the submatrix of the obtained matrix of the last column.
$[R|c]$, where R → reduced row echelon form of A
Solutions of $Ax=b$ are presicely the solutions of $Rx=c$.

→ Homogenous system of linear equations

↳ 0 is always a solution a.k.a. trivial solution
↳ If there are more variables than equations, (i.e., more columns than rows) there will be infinite solutions because there must be some independent variables.

WEEK 3

→ Properties of vectors   Let v, w and v' be vectors in $\mathbb{R}^n$ and $a, b \in \mathbb{R}$

i) $v + w = w + v$
ii) $(v+w) + v' = v + (w + v')$
iii) The 0 vector satisfies : $v + 0 = 0 + v = v$
iv) The vector $-v$ satifies : $v + (-v) = 0$
v) $1v = v$
vi) $(ab)v = a(bv)$
vii) $a(v+w) = av + aw$
viii) $(a+b)v = av + bv$

↳ Vector space → a set with two operations (addition and multiplication)
$+ : V \times V \to V$ and $\cdot : \mathbb{R} \times V \to V$
for each pair of elements v_1 and v_2 in V, there is a unique element $v_1 + v_2$.
for each $c \in \mathbb{R}$ and $v \in V$ there is a unique element $c \cdot v$ in V.

example of vector space → solutions of a homogenous system $Ax = 0$
↳ non-examples:
① define addition and scalar multiplication in $\mathbb{R}^2$ as:
→ $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$
→ $c(x_1, x_2) = (cx_1, cx_2)$

$v + w \neq w + v$ → i), iii) and viii) fail

② → $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0)$
→ $c(x_1, x_2) = (cx_1, 0)$
→ iii), iv) and v) fail

→ Cancellation law of vector addition

↳ If $v_1, v_2, v_3 \in V$ such that $v_1 + v_3 = v_2 + v_3$, then $v_1 = v_2$

↳ Corrolaries: ① The vector 0 described in iii) is unique. → $\cancel{v} + w = \cancel{v} + 0 \Rightarrow w = 0$
② The vector v' " " iv) " " and referred to as $-v$.
↳ $\cancel{v} + v' = 0 = \cancel{v} + v'' \Rightarrow v' = v''$

→ Some more properties:

① $0v = 0$ for each $v \in V$ → $(0+0)v = 0v + 0v \Rightarrow 0v = 0v + 0v$
$\Rightarrow \cancel{0v} + 0 = \cancel{0v} + 0v$

② $(-c)v = -(cv) = c(-v)$ for each $c \in \mathbb{R}$ and for each $v \in V$
↳ $(c + (-c))v = cv + (-c)v \Rightarrow 0 = cv + (-c)v \Rightarrow -cv = (-c)v$

③ $c0 = 0$ for each $c \in \mathbb{R}$

→ Linear combinations:
↳ Let V be a vector space and $v_1, v_2, \dots, v_n \in V$.
Linear combinations of $v_1, v_2, \dots, v_n$ with coefficients $a_1, a_2, \dots, a_n \in \mathbb{R}$ is the vector $\sum_{i=1}^{n} a_i v_i \in V$.
↳ example: $(2,9)$ is a linear combination of vectors $(2,1)$ and $(-2,3)$:
$3(2,1) + 2(-2,3) = (6,3) + (-4,6) = (2,9)$
↳ $3(2,1) + 2(-2,3) - (2,9) = 0$
↳ 0 vector is a linear combinations of $(2,1)$, $(-2,3)$, and $(2,9)$ with non-zero coefficients

↳ The plane of the two vectors $(0,2,1)$ and $(2,2,0)$ can be expressed by the equation $2x - 2y + 4z = 0$
↳ $(1,2,0)$ does not lie on the plane
↳ $(3,7,2)$ does lie on this plane

→ Linear Dependence

↳ a set of vectors $v_1, v_2, v_3, \dots, v_n \in V$ is said to be linearly dependent if there exists scalars $a_1, a_2, a_3, \dots, a_n$, not all zeros, such that ↳ $a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n = 0$

↳ If a set of vectors are linearly depent, then every superset of it is linearly dependent.

↳ If v_1, v_2, v_3 are " " , then v_1, v_2, v_3, v_n are also linearly dependent.

→ Linear Independence

↳ a set of vectors if they are not linearly dependent

↳ $v_1, v_2, v_3, \dots, v_n$ are linearly independent if : $a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n = 0$ can only be satisfied when $a_i = 0$ for all $i = 1, 2, 3, \dots, n$

↳ If a set of vectors $v_1, v_2, \dots, v_n$ contain a vector $v_i = 0$, then the set is always linearly dependent.

↳ If two non-zero vectors are multiples of each other, then are linearly dependent.

$a_1v_1 + a_2v_2 = 0 \Rightarrow c = -a_2/a_1$, then $v_1 = cv_2$

↳ Two vectors are linearly independent when they are not multiples of each other.

↳ for three vectors v_1, v_2, v_3

Suppose $a_1v_1 + a_2v_2 + a_3v_3 = 0 \Rightarrow v_1 = -a_2/a_1 v_2 - a_3/a_1 v_3$

↳ v_1 is a linear combination of v_2 and v_3

↳ If three vectors are linearly independent then none of these vectors is a linear combination of the other two.

→ If you have more than n vectors in $\mathbb{R}^n$, then the homogenous system $Vx = 0$ has infinitely many solutions. → They are linear dependent

↳ This corresponds to having more variables than there are equations.

→ V is an $n \times n$ matrix obtained by arranging a set of vectors in columns.

$Vx = 0$ → homogenous system

V → square matrix , therefore $x = 0$ if and only if V is invertible

if and only if $\det(V) \neq 0$

WEEK 4

→ Span of a set of vectors

↳ Span of a vector space V → $\text{Span}(V) = \{ \sum_{i=1}^{n} a_i v_i \in V \mid a_1, a_2, a_3, \dots, a_n \in \mathbb{R} \}$

e.g. → Let $V = \{(1,0)\} \subset \mathbb{R}^2$. Then, $\text{Span}(V) = \{a(1,0) \mid a \in \mathbb{R}\} = \{(a,0) \mid a \in \mathbb{R}\}$

→ Let $V = \{(1,0,0), (0,1,0)\} \subset \mathbb{R}^3$. $\text{Span}(V) = \{a(1,0,0) + b(0,1,0) \mid a, b \in \mathbb{R}\} = \{(a,b,0) \mid a, b \in \mathbb{R}\}$

↳ Spanning set

→ A set $S \subseteq V$ is a spanning set for V if $\text{span}(S) = V$.

↳ If a vector $T \subseteq V$, then $\text{Span}(T) \subseteq \text{Span}(V)$

$T \subseteq \text{span}(V)$ → then $\text{Span}(T) \subseteq \text{Span}(V)$

↳ Basis

↳ Basis B of a vector space V is a linearly independent subset of V that spans V.

↳ e.g. Standard Basis → Let $e_i \in \mathbb{R}^n$ be the vector with i^{th} coordinate 1 and all other coordinates 0. → $e_1 = (1,0,0,\dots,0)$

The set $\varepsilon = \{e_1, e_2, e_3, \dots, e_n\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$

→ How to find a basis:

1. Start with $\emptyset$ and keep appending vectors which are not a linear combination of the set thus ~~for~~ obtained, until we obtain a spanning set.

2. Take a spanning set and keep deleting vectors which are linear combinations of other vectors, until the remaining vectors satisfy that they are not a linear combination of the remaing ones.

→ Rank / dimension of a vector space

↳ size of (or cardinality) of a basis of the vector space.

↳ Rank of a matrix Let A be an $m \times n$ matrix.

→ column space → subspace of $\mathbb{R}^m$ spanned by column vectors of A.

row " → " " $\mathbb{R}^n$ " " row " " .

Column rank = Row rank = Rank of A

→ number of non-zero rows when reduced to row echelon

e.g. → vector space W is spanned by $\{(1,0,1), (-2,-3,1), (3,3,0)\}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \qquad \text{RREF of } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

↳ Rank / dimension of W is 2 and a basis is given by $(1,0,1), (0,1,-1)$

↳ Row method → put the spanning vectors in rows and reduce the matrix. No. of non-zero rows is the rank and the rows themselves form a basis.

Column method → put the spanning vectors in columns and reduce the matrix. No. non-zero rows is the rank. The columns of original matrix corresponding to the columns of reduced matrix containing the pivots (i.e., dependent variables) form a basis.

— ✕ — ✕ — ✕ —

WEEK 1 GA

(1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A - xI = \begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix}$

$$\det(A - xI) = (a-x)(d-x) - bc$$
$$= ad - ax - xd + x^2 - bc$$
$$= \underbrace{ad - bc}_{\det(A)} \underbrace{- x(a+d)}_{-x\,\text{trace}(A)} + \underbrace{x^2}_{x^2}$$

(2) $3A = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$ $\det(3A) = 9ad - 9bc = (3)^2 \det(A)$

$3A = \begin{bmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$ $|3A| = 3^3|A| = \boxed{27|A|}$

(3) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $I + A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $5A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$

$5A + I = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$

$(I + A)^3 - (5A + I) = mA$

$\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$ $\boxed{m = 8}$

(4) $A = \begin{bmatrix} 20 & 30 & 40 \\ 8 & 16 & 24 \\ 8 & 10 & 12 \end{bmatrix}$ $\det(A) = 20 \det\begin{bmatrix} 16 & 24 \\ 10 & 12 \end{bmatrix} - 30 \det\begin{bmatrix} 8 & 24 \\ 8 & 12 \end{bmatrix} + 40 \det\begin{bmatrix} 8 & 16 \\ 8 & 10 \end{bmatrix}$

$= 20(-48) - 30(-96) + 40(-48) = -960 + 2880 - 1920 = \boxed{0}$

(5) $\det(A) = 3$ $\det(B) = 3$ $\det(B^{-1}) = 1/3$

$\det(3A^2B^{-1}) = (3)^3 \cdot (\det(A))^2 \cdot \left(\frac{1}{3}\right) = 9(3)^2 = \boxed{81}$

(6) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ $A^3 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix}$ $A^4 = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}$

$A^7 = \begin{bmatrix} 729 & 729 & 729 \\ 729 & 729 & 729 \\ 729 & 729 & 729 \end{bmatrix}$ sum of diagonals $= \boxed{2187}$

(7) $A = \begin{bmatrix} 12 & 19 & 26 \\ 17 & 24 & 31 \\ 22 & 29 & 36 \end{bmatrix}$ $\det(A) = 12 \det\begin{bmatrix} 24 & 31 \\ 29 & 36 \end{bmatrix} - 19 \det\begin{bmatrix} 17 & 31 \\ 22 & 36 \end{bmatrix} + 26 \det\begin{bmatrix} 17 & 24 \\ 22 & 29 \end{bmatrix}$

$= 12(-35) - 19(-70) + 26(-35) = \boxed{0}$

(9) $30x_1 + 20x_2 + 25x_3 = 670$ — (1)
$20x_1 + 35x_2 + 25x_3 = 730$ — (2)
$20x_1 + 10x_2 + 15x_3 = 400$ — (3)

(1) − 2·(3) ⇒ 30 20 25 670
−40 −20 −30 −800
$= -10x_1 - 5x_3 = -130$

(2) − (3) ⇒ 20 35 25 730
−20 −10 −15 −400
$= 25x_2 + 10x_3 = 330$ — (4)
$10x_1 + 5x_3 = 130$ — (5)

2(5) $= 20x_1 + 10x_3 = 260 \Rightarrow 20x_1 = 260 - 10x_3$

$260 - 10x_3 + 10x_2 + 15x_3 = 400$
$\Rightarrow 10x_2 + 5x_3 = 140$ — (6)

(4) − 2·(6) ⇒ 0 25 10 330
0 −20 −10 −280
$= 5x_2 = 50$ $\boxed{x_2 = 10}$

$250 + 10x_3 = 330 \Rightarrow \boxed{x_3 = 8}$

$10x_1 + 40 = 130 \Rightarrow \boxed{x_1 = 9}$

$x_1 + x_2 + x_3 = 9 + 10 + 8 = \boxed{27}$

(10) $A = \begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix}$

WEEK 2 GA

① $P(1) = -45$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -45 \\ 4 & 2 & 1 & -19 \\ 9 & 3 & 1 & 3 \end{array}\right] \xrightarrow[R_3-9R_1]{R_2-4R_1} \begin{array}{cccc} 1 & 1 & 1 & -45 \\ 0 & -2 & -3 & 161 \\ 0 & -6 & -8 & 408 \end{array} \xrightarrow{R_2-\frac{1}{2}R_3} \begin{array}{cccc} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & -6 & -8 & 408 \end{array} \xrightarrow{R_3+6R_2} \begin{array}{cccc} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{array} \xrightarrow[-R_3/2]{R_1-R_2} \begin{array}{cccc} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 1 & -75 \end{array} \xrightarrow{R_2-R_3} \begin{array}{cccc} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 32 \\ 0 & 0 & 1 & -75 \end{array}$$

$P(x) = -2x^2 + 32x - 75$ $\qquad P'(x) = -4x + 32$

$\frac{-32}{-4} = x \qquad x = 8$

$P(8)$ is peak

$P(8) = 53$

④
$$\begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & c & d \end{array} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array} \xrightarrow[R_1-2R_2]{R_3+3R_2} \begin{array}{cccc} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

if $c=7$, $d=4000$

comic book = ?

if $c=2$, $d=4000$

$$\begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 4000 \end{array} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -2 & 0 \end{array}$$

if $c=7$, $d=3000$

$$\begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 7 & 3000 \end{array} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & -1000 \end{array}$$

if $c=2$, $d=3000$

$$\begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 3000 \end{array} \xrightarrow[-R_3-4R_1]{R_2-2R_1} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 1000 \end{array} \xrightarrow{R_3-3R_2} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 1000 \end{array} \Rightarrow \begin{pmatrix} 400 \\ 200 \\ 200 \end{pmatrix}$$

⑥ $\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) + a/6 = -1 \Rightarrow a = 0$

$\left(-\frac{1}{3}\right) + b/6 = 0 \Rightarrow \frac{b}{6} = \frac{1}{3} \Rightarrow b = 2$

$\left(\frac{1}{2}\right) + \left(-\frac{1}{6}\right) + c/6 = 0 \Rightarrow \frac{c-1}{6} = -\frac{1}{2} \Rightarrow c - 1 = -3 \Rightarrow c = -2$

$a + b + c = 0$

⑦
$$\begin{array}{cccc} 1 & 3k & 3k+4 & 61 \\ 1 & k+4 & 4k+2 & 65 \\ 1 & 2k+2 & 3k+4 & 66 \end{array} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{array}{cccc} 1 & 3k & 3k+4 & 61 \\ 0 & -2k+4 & k-2 & 4 \\ 0 & -k+2 & 0 & 5 \end{array} \xrightarrow{R_3 \leftrightarrow R_2} \begin{array}{cccc} 1 & 3 & 7 & 61 \\ 0 & 1 & 0 & 5 \\ 0 & 2 & -1 & 4 \end{array} \xrightarrow{-R_3+2R_2} \boxed{\left[\begin{array}{ccc|c} 1 & 3 & 7 & 61 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right]}$$

$4k+2-3k-4$

$k-2$

$-k+2 = 1$

$k = 1$

⑧
$$\begin{array}{cccc} 2 & 3 & 5 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 44 \end{array} \xrightarrow[R_2-R_3]{R_1-R_2} \begin{array}{cccc} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 1 & 1 & 2 & 44 \end{array} \xrightarrow{R_3-R_1} \begin{array}{cccc} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 0 & 44 \end{array}$$

→ shows no solution

⑨
$$\begin{array}{cccc} 1 & 3 & 0 & 0 \\ 4 & 1 & 5 & 34 \\ 2 & 2 & 7 & 32 \\ 3 & 9 & 0 & 0 \end{array} \xrightarrow{R_2-4R_1,\ R_3-2R_1,\ R_4-3R_1} \begin{array}{cccc} 1 & 3 & 0 & 0 \\ 0 & -11 & 5 & 34 \\ 0 & -4 & 7 & 32 \\ 0 & 0 & 0 & 0 \end{array}$$

R_4 shows $\det(A) = 0$

⑩
$$\begin{array}{cccc} 7 & 2 & 1 & 8 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{array} \xrightarrow{R_1+2R_3} \begin{array}{cccc} 1 & 10 & -3 & 24 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{array} \xrightarrow[R_2/3]{R_3+3R_1} \begin{array}{cccc} 1 & 10 & -3 & 24 \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 34 & -11 & 80 \end{array} \xrightarrow[R_3-34R_2]{R_1-10R_2} \begin{array}{cccc} 1 & 0 & \frac{1}{3} & \frac{32}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & \frac{1}{3} & \frac{104}{3} \end{array} \xrightarrow{3R_3} \begin{array}{cccc} 1 & 0 & \frac{1}{3} & \frac{32}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 104 \end{array}$$

$x = -24$

$y = 36$ $\qquad x + y + z = 116$

$z = 104$

(11) $A = \begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix}$ $A^TA = \underset{4\times1}{\begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix}} \underset{1\times4}{\begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix}} = \begin{matrix} 4 & 20 & 4 & 16 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{matrix} \xrightarrow{R_1/4} \begin{matrix} 1 & 5 & 1 & 4 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{matrix} \begin{matrix} \\ R_2-20R_1 \\ R_3-4R_1 \\ R_4-16R_1 \end{matrix} \begin{matrix} 1 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$

(12) $\begin{matrix} 4 & 2 & 1 & 4 \\ 9 & 3 & 1 & 6 \\ 16 & 4 & 1 & 10 \end{matrix} \xrightarrow{R_1/4} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 9 & 3 & 1 & 6 \\ 16 & 4 & 1 & 10 \end{matrix} \begin{matrix} R_2-9R_1 \\ R_3-16R_1 \end{matrix} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & -3/2 & -5/4 & -3 \\ 0 & -4 & -3 & -6 \end{matrix} \begin{matrix} -2R_2 \\ -R_3 \end{matrix} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & 3 & 5/2 & 6 \\ 0 & 4 & 3 & 6 \end{matrix} \xrightarrow{-R_3\frac{4}{3}R_2} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & 3 & 5/2 & 6 \\ 0 & 0 & 1/3 & 2 \end{matrix} \xrightarrow{3R_3} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & 3 & 5/2 & 6 \\ 0 & 0 & 1 & 6 \end{matrix}$

$a = 1$ $a - b + c = 10$
$b = -3$
$c = 6$

(13) $\left[\begin{matrix} 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 1000 \\ 900 \\ c \\ 400 \end{matrix}\right] \begin{matrix} R_1/2 \\ R_4-R_1 \end{matrix} \begin{matrix} 1 & 3/2 & 0 & 0 & 500 \\ 0 & 3 & 2 & 0 & 900 \\ 0 & 0 & 2 & 1 & c \\ 0 & -3 & 0 & 0 & -600 \end{matrix} \xrightarrow{R_2/3} \begin{matrix} 1 & 3/2 & 0 & 0 & 500 \\ 0 & 1 & 2/3 & 0 & 300 \\ 0 & 0 & 2 & 1 & c \\ 0 & 3 & 0 & 0 & 600 \end{matrix} \begin{matrix} -R_4-3R_2 \\ R_1-\frac{3}{2}R_2 \end{matrix} \begin{matrix} 1 & 0 & -1 & 0 & 50 \\ 0 & 1 & 2/3 & 0 & 300 \\ 0 & 0 & 2 & 1 & c \\ 0 & 0 & 2 & 0 & 300 \end{matrix} \xrightarrow{R_3/2} \begin{matrix} 1 & 0 & -1 & 0 & 50 \\ 0 & 1 & 2/3 & 0 & 300 \\ 0 & 0 & 1 & 1/2 & c/2 \\ 0 & 0 & 2 & 0 & 300 \end{matrix}$

$\begin{matrix} R_1+\frac{1}{2}R_4 \\ R_2-\frac{1}{3}R_4 \\ -R_4-2R_3 \end{matrix}$ $\begin{matrix} 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & 200 \\ 0 & 0 & 1 & 1/2 & c/2 \\ 0 & 0 & 0 & 1 & -300+c \end{matrix} \xrightarrow{R_3-\frac{1}{2}R_4} \begin{matrix} 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & 200 \\ 0 & 0 & 1 & 0 & 150 \\ 0 & 0 & 0 & 1 & c-300 \end{matrix}$

(W.N.) $x_1 = 200$
(E.N.) $x_2 = 200$
(E.S.) $x_3 = 150$
(W.S.) $x_4 = 0$

$c = 300$

min $\rightarrow x_4 = 0 \rightarrow x_1 = 200, x_2 = 200, x_3 = 150$
max $\rightarrow x_4 = 300 \rightarrow x_1 = 50, x_2 = 300, x_3 = 0$

(W.N.) $\rightarrow 400 \rightarrow 100$
(E.N.) $\rightarrow 900 \rightarrow 600$
(E.S.) $\rightarrow 300 \rightarrow 0$
(W.S.) $\rightarrow 300 \rightarrow 0$

WEEK 3 GA

(2) $-x = -1$
$-2y = 1$
$y = 1$
$x = -1$

(3) $P = 30$ $C = 11$ $F = 53$ $Ca = 213$ Type A = Ba, mi, al Type B = Ap, mi, al

$A = \begin{bmatrix} 2 & 4 & 6 & 30 \\ 3 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 5 & 100 & 1 & 213 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 & 6 & 30 \\ 2 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 6 & 100 & 1 & 213 \end{bmatrix}$

(5) $A = \begin{bmatrix} c & 1 & -1 \\ -1 & 0 & -3 \\ -2 & -1 & c \end{bmatrix}$ $-1(-c-6) + -3c - 1 = 0$
$c + 6 - 3c - 1 = 0$
$-2c + 5 = 0$
$c = 2.5$

(6) $A = \begin{bmatrix} 7 & 5 & 11 \\ 9 & 6 & 15 \\ 2 & 2 & c \end{bmatrix}$ $\det(A) = 7(6c-30) - 5(9c-30) + 11(18-12)$
$= 42c - 210 - 45c + 150 + 66$
$0 = -3c + 6$
$c = 2$

(8) $(1,0,0), (1/2, 1/2, 0), (0,1,0)$

(9) $X_0 = (1/3, 1/3, 1/3)$ $X_1 = X_0 \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (1/2, 1/2, 0)$ $X_2 = X_1 \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (3/4, 1/4, 0)$

⑩ $X_0 = (a, b, 1-a-b)$ $\qquad X_1 = (a + b/2,\ 1-a-\frac{b}{2},\ 0)$

$$X_1 = (a, b, 1-a-b)\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \left(\frac{2a+b}{2}, \frac{b+2-2a-2b}{2}, 0\right) = \left(\frac{2a+b}{2}, \frac{2-2a-b}{2}, 0\right)$$

$$X_2 = \left(\frac{2a+b}{2}, \frac{2-2a-b}{2}, 0\right)\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \left(\frac{4a+2b+2-2a-b}{4}, \frac{2-2a-b}{4}, 0\right) = \left(\frac{2a+b+2}{4}, \frac{2-2a-b}{4}, 0\right)$$

$$\begin{bmatrix} a & \frac{2a+b+2}{4} \\ b & \frac{2-2a-b}{4} \\ 1-a-b & 0 \end{bmatrix}$$

$\frac{4ax + 2ay + by + 2y}{4} = 0$ — ① $\Rightarrow y(2a+b+2) = -4ax$

$\frac{4bx + 2y - 2ay - by}{4} = 0$ — ② $\Rightarrow y(2-2a-b) = -4bx$

$x - ax - bx = 0 \implies x - bx = ax$

① − ② $\Rightarrow \frac{4ax - 4bx + 4}{4} = 0 \Rightarrow ax - bx + 1 = 0$

$\Downarrow$

$x - bx - bx + 1 = 0$

$x - 2bx + 1 = 0$

MOCK WEEK 1-2

③ $\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix}$ $\qquad \lambda = 6$

④ $a + 4 = 2a + 2 \Rightarrow 2 = a$

$3b = b + 2 \Rightarrow 2b = 2 \Rightarrow b = 1$ $\qquad a + 2b = 4$

⑥ $\left[\begin{array}{cc|c} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{array}\right] \xrightarrow[R_1 \cdot \sqrt{2}]{R_2 + R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2/\sqrt{2} & 0 \end{array}\right]$

$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A)$ $\qquad \mathrm{Adj}(A) = C^T$ $\qquad \det(A) = \frac{1}{2} + \frac{1}{2} = 1$

$$C^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = A^T = A^{-1}$$

⑦ $\det(A) = b + a$

⑧ $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \\ -1 & 6 & 7 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ -1 & 6 & 7 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{bmatrix} \xrightarrow[R_2/2]{R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$

(13) $\begin{bmatrix} 2 & 0.5 & 1 & 80 \\ 4 & 1 & 2 & 100 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[R_1/2]{R_2-2R_1} \begin{bmatrix} 1 & 0.25 & 0.5 & 40 \\ 0 & 0 & 0 & -60 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & 0.25 & 0.5 & 40 \\ 0 & 0 & 0 & -60 \\ 0 & 0.5 & 1 & 0 \end{bmatrix}$

(14) $\begin{bmatrix} 2 & 2 & 1 & 80 \\ 4 & 1 & 2 & d \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[R_1/2]{R_2-2R_1} \begin{bmatrix} 1 & 1 & 1/2 & 40 \\ 0 & -3 & 0 & d-160 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[-R_2/3]{R_3-2R_1} \begin{bmatrix} 1 & 1 & 1/2 & 40 \\ 0 & 1 & 0 & -d/3+160/3 \\ 0 & -1 & 1 & 0 \end{bmatrix}$

$x_2 = x_3$

(15) $\begin{bmatrix} 2 & 3 & 1 & 80 \\ 4 & 1 & 2 & 100 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[R_1/2]{R_2-2R_1} \begin{bmatrix} 1 & 3/2 & 1/2 & 40 \\ 0 & -5 & 0 & -60 \\ 2 & 1 & 2 & 80 \end{bmatrix} \xrightarrow[R_3-2R_1]{-R_2/5} \begin{bmatrix} 1 & 3/2 & 1/2 & 40 \\ 0 & 1 & 0 & 12 \\ 0 & -2 & 1 & 0 \end{bmatrix}$

$x_2 = 12$
$x_3 = 2x_2 = 24$

$x_1 = 40 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \Rightarrow x_1 = 40 - \frac{3}{2}(12) - \frac{1}{2}(24)$
$= 40 - 18 - 12 = 10$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 24 \end{bmatrix}$

WEEK 4 GA

(4) if $A_{4\times4}$, then $A = \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{matrix}$, then rank(A) = 4

if $A_{2021\times2021}$, then rank(A) = 2021

(5) $8x_1 + 10x_3 = 0 \quad 9x_2 + 6x_4 = 0$

(6) $A = \begin{bmatrix} 2 & -4 & 1 & -9 \\ 1 & -3 & 1 & -1 \\ 3 & -7 & 2 & -6 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ -R_2-2R_1 \\ -R_3-3R_1 \end{matrix} \quad \begin{matrix} 1 & -3 & 1 & -1 \\ 0 & 10 & 1 & 11 \\ 0 & 16 & 1 & 9 \end{matrix}$

(7) $A = \begin{bmatrix} -10 & 0 & -3 \\ -5 & -4 & -5 \\ -8 & 8 & -3 \end{bmatrix} \begin{matrix} -R_1/10 \\ R_2+5R_1 \\ R_3+8R_1 \end{matrix} \quad \begin{matrix} 1 & 0 & 3/10 \\ 0 & -4 & -35/10 \\ 0 & 8 & -6/10 \end{matrix} \begin{matrix} -R_2/4 \\ R_3+2R_2 \end{matrix} \quad \begin{matrix} 1 & 0 & 3/10 \\ 0 & 1 & 35/40 \\ 0 & 0 & -76/10 \end{matrix}$

$B = \begin{bmatrix} -10 & -5 & -8 \\ 0 & -4 & 8 \\ -3 & -5 & -3 \end{bmatrix} \begin{matrix} -R_1/10 \\ R_3+3R_1 \\ -R_2/4 \end{matrix} \quad \begin{matrix} 1 & 1/2 & 4/5 \\ 0 & 1 & -2 \\ 0 & -7/2 & -3/5 \end{matrix}$

(8) $v_1 = (-1, 0, 1) \quad v_2 = (2, 1, -1)$

$v_1 + v_2 = (1, 1, 0) \qquad 3v_1 + 0v_2 = (-3, 0, 3)$

$(5, 2, -3)$, $(8, \cancel{5}, -4)$, $(-5/6, \cancel{5/6}, 5/6)$, $(-2, \cancel{3}, 3)$

$b = 2$, $a - b = -3$, $a = -1$, $2b - a = 5$

$6b = 5$, $a - b = -4$, $a = 1$, $2b - a \neq 8$

$b = 5/6$, $a - b = 5/6$, $a = 10/6$, $2b - a \neq -5/6$

$b = 3$, $a - b = 3$, $a = 6$

$2b - a \neq -2$

(9) $v_1 = (1, 0, 1) \quad v_2 = (0, 1, -1) \qquad av_1 + bv_2 = (20, -13, c) \qquad a = 20 \quad b = -13 \qquad a - b = c$
$20 - (-13) = 33$

(10) $(x, y, z) = (0, q, 1)$, then $1 = 0 + 8q \quad q = 1/8 \quad p = 1/9$

$q \cdot p = \frac{1}{72} \qquad \frac{1}{q \cdot p} = 72$

(11) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & -a_{11}-a_{22}-a_{33} \end{bmatrix}$
