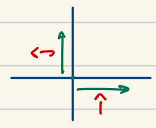
MATH 2

WEEK 1

→ Vectors

- can be thought of as a list.
- can be in a row or a column form.
- Addition of vectors:
  Add the corresponding entries. For e.g. → $(3,5)+(2,4)=(5,9)$
- Scalar multiplication:
  e.g. → $2(8,8,10,5) = (16,16,20,10)$
- Visualisation of a vector
  Point $(a,b) \equiv$ Vector $(a,b) \equiv a\hat{i} + b\hat{j}$ where, $\hat{i}$ → one unit in x-axis
  $\hat{j}$ → one unit " y-axis



  e.g.: Point $(-1,-1) = -\hat{i} - \hat{j}$
- Vectors in $\mathbb{R}^n$ are lists with n real entries.

→ Matrices

- rectangular array of numbers
- (rows × columns) ; eg → $\begin{bmatrix} 5 & 7 & 10 \\ 3 & 5 & 2 \end{bmatrix}$ is a 2×3 matrix.
  - $(1,2)$ ith entry → 7
- square matrix → $N \times N$
  - Diagnol matrix → all entrices are 0 except the diagnol
    $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$
  - Scalar matrix → all entries have the same value
    $$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$
  - Indentity matrix → denoted by 'I'; scalar matrix with values = 1
    $$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Addition of matrices → must be of the same size: $\begin{bmatrix} 1 & 0 \\ 5 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 8 & 3 \\ 7 & 4 \end{bmatrix}$
- Scalar multiplication → multiply each number with the scalar.
- Matrix multiplication → $A \times B = C$; $C[i,j] = \sum_{k=1}^{n} A[i,k] \times B[k,j]$
  - no. of columns in first matrix $\underline{\text{must}}$ = no. of rows in 2^{nd} matrix
  - $A_{m \times n} \times B_{n \times p} = (AB)_{m \times p}$
    $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{jk}$
- Scalar multiplication is the same as multiplication by the scalar matrix
  ex. → $\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$
- Properties:
  1. $(A+B)+C = A+(B+C)$
  2. $(AB)C = A(BC)$
  3. $A+B = B+A$
  4. $AB \neq BA$
  5. $\lambda(A+B) = \lambda A + \lambda B$
  6. $\lambda(AB) = (\lambda A)B = A(\lambda B)$
  7. $A(B+C) = AB + AC$
  8. $(A+B)C = AC + BC$

→ System of Linear Equations

- collection of one or more linear equations involving the same set of variables.
- system of m linear equation with n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_n \end{aligned}$$

- system of lin. eq. in matrix form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$$\underbrace{A}_{\text{coefficient matrix}} \underbrace{x}_{\text{column of variables}} = \underbrace{b}_{\text{column of resulting values}}$$, where

A = $m \times n$ matrix
x = column vector with n entries
b = column vector with m entries

- Solutions to a system of lin. eq.:
  1. Infinite solution
  2. Single unique solution
  3. No solution

→ Determinant

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad - bc$

e.g. $A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$ $\quad \det(A) = 20 - 18 = 2$

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad \det(A) = a_{11} \times \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

- Determinant of Identity matrix $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - → $\det(I_2) = 1$
  - → $\det(I_3) = 1$

- Determinant of a product of matrices
  - $\det(AB) = \det(A) \cdot \det(B)$; $\det(ABC) = \det(A) \cdot \det(B) \cdot \det(C)$
  - $\det(A^n) = \det(A)^n$
  - $\det(A^{-1}) = \det(A)^{-1}$

- Determinant of the inverse of a matrix

$$AA^{-1} = I \Rightarrow \det(AA^{-1}) = \det(I)$$

  - $\det(A^{-1}) = \dfrac{1}{\det(A)}$

- Switching rows

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\det(\tilde{A}) = cb - ad = -(ad - bc) = -\det(A)$$

$$\det(\tilde{A}) = -\det(A)$$

- Add multiple of a row/column to another row/column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$$

$$\det(\tilde{A}) = (a+tc)d - (b+td)c = ad + tcd - bc - tcd$$

$$\det(\tilde{A}) = \det(A)$$

↳ Scalar multiplication of a row/column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} ta & b \\ tc & d \end{bmatrix}$$

$$\det(\tilde{A}) = t \cdot \det(A)$$

↳ Upper | Lower triangle matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{bmatrix}$$ upper triangle matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 0 \\ 3 & 4 & 9 \end{bmatrix}$$ lower triangle matrix

→ determinant is the product of diagonal elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \det(A) = a_{11} \cdot a_{22} \cdot a_{33}$$

↳ Transpose of a matrix and its determinants

→ Transpose of $A_{m \times n}$ = $A^T_{n \times m}$ with (i,j)-th entry A_{ji}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2\times 3}$$

$$\det(A) = \det(A^T)$$

↳ Minors and cofactors

→ Minor of the entry in i-th row and j-th column is the determinant of the submatrix formed by deleting i-th row and j-th column.

e.g. → $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(1,1)-th minor; denoted by M_{11}

$$M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

→ Cofactor (i,j)th cofactor; $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$C_{11} = M_{11} \; ; \; C_{23} = -M_{23}$$

→ For $A_{3\times 3}$

$$\det(A) = (a_{11} \times C_{11}) + (a_{12} \times C_{12}) + (a_{13} \times C_{13})$$

For $A_{4\times 4}$

$$\det(A) = \sum_{j=1}^{4} a_{1j} C_{1j}$$

For $A_{n\times n}$; $$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$$

WEEK 2

→ Determinant (continued)

↳ Expansion along any row/column

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } i$$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } j \text{ (determinant along a column)}$$

↳ **Properties:**

① Determinant of a product is product of the determinants.
↳ $\det(AB) = \det(BA) = \det(A) \cdot \det(B)$

② Switching two rows or columns changes the sign.

③ Adding multiples of a row to another row leaves the determinant unchanged.

④ Scalar multiplication of a row/column by a constant t multiplies the determinant by t.

⑤ $\det(t A_{n\times n}) = (t)^n \det(A)$

↳ **Useful computational tips:**

① The determinant of a matrix with a row or a column of zeros is 0.

② The determinant of a matrix in which one row (or column) is a linear combination of other rows (resp. columns) is 0.

③ Scalar multiplication of a row/column by a constant t multiplies the determinant by t.

④ While computing the determinant, you can choose to compute it using expansion along a suitable row or a column.

→ Cramer's rule

→ $Ax = b$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Cramer rule ⇒ $A_{x_1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{21} \end{bmatrix}$, $A_{x_2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$

↳ solutions for x_1 & x_2 ⇒ $x_1 = \dfrac{\det(A_{x_1})}{\det(A)}$; $x_2 = \dfrac{\det(A_{x_2})}{\det(A)}$

→ For a 3x3 matrix: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$A_{x_1} \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$ $A_{x_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$ $A_{x_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$

$x_1 = \dfrac{\det(A_{x_1})}{\det(A)}$ $x_2 = \dfrac{\det(A_{x_2})}{\det(A)}$ $x_3 = \dfrac{\det(A_{x_3})}{\det(A)}$

→ Inverse Matrix

↳ The inverse of a matrix $A \to (A^{-1}) \Rightarrow A \cdot (A^{-1}) = I$

↳ if $\det(A) \neq 0$, then the matrix is invertible

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

↳ Adjucate of a matrix

M_{ij} = determinant of submatrix after deleting i-th row and j-th column.

$C_{ij} = (-1)^{i+j} M_{ij}$

→ Adjucate of a matrix → Transpose of Cofactor matrix

$Adj(A) = C^T$

→ $A^{-1} = \frac{1}{\det(A)} \cdot adj(A)$ → $A_{n \times n}$, then $\det(adj(A)) = (\det(A))^{n-1}$

↳ Solution to system of linear equations using inverse matrix.

$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$

→ Homogenous System of Linear Equations

↳ $Ax = b$, where $b = 0 \Rightarrow Ax = 0$

→ has unique solution 0 if A is invertible

→ has infinite solutions if A is not invertible

→ The Row Echelon Form (reduced form)

↳ Matrix is in row echelon form if:

→ first non-zero element in each row (leading entry) is 1

→ each leading entry is in a colum to the right of the leading entry in the previous row.

→ rows with all zero elements, if any, are below rows having a non-zero element.

e.g. → $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

→ reduced row echelon form → For a non-zero row, the leading entry in the row is the only non-zero entry in its column

e.g. → $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

→ e.g. solution for $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow Ax = b$

$x_1 + 2x_2 = b_1 \Rightarrow x_1 = b_1 - 2x_2$

$x_3 = b_2$

$x_4 = b_3$

$$\text{solution} \Rightarrow x = \begin{bmatrix} b_1 - 2c \\ c \\ b_2 \\ b_3 \end{bmatrix}$$

→ If the i-th column has the leading entry of some row, we call x_i a dependent variable.

If the i-th column does not have the leading entry of any row, x_i → independent variable.

→ Row reduction

Elementary Row Operations

Type	Action	Example and notation	Effect on determinant
1	Interchange two rows	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = -\det(B)$
2	Scalar multiplication of a row by a constant t	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = t \cdot \det(B)$
3	Adding multiples of a row to another row	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_3} \begin{bmatrix} 3 & -19 & -2 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = \det(B)$

↳ **Row echelon form**

Steps :
1. Find the left most non-zero column.
2. Use operations to get 1 in the top of that column.
3. Make entries below that 1 to 0.
4. Repeat the steps for the next row and onwards.

↳ **Reduced row echelon form**
- ↳ start from the right most column and make all the non-leading terms 0s.

↳ e.g.

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_1/2} \begin{matrix} 1 & 2 & 1/2 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{matrix} \xrightarrow[R_3 - 5R_1]{R_2 - 3R_1} \begin{matrix} 1 & 2 & 1/2 \\ 0 & 2 & 11/2 \\ 0 & -4 & 13/2 \end{matrix} \xrightarrow[R_3 + 2R_2]{R_2/2} \begin{matrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 35 \end{matrix} \xrightarrow{R_3/35} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{matrix} \xrightarrow[R_2 - \frac{11}{4}R_3]{R_1 - \frac{1}{2}R_3} \begin{matrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 35 \times 2 = 70$$

→ Gaussian Elimination Method

↳ **Augmented matrix** → matrix of size $(m \times n+1)$
- ↳ first n columns are from A and the last column is b.
- ↳ denoted by $[A|b]$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

↳ Bring the matrix A to reduced row echelon form.
- → Let R be the submatrix of obtained matrix of the first n columns and c be the submatrix of the obtained matrix of the last column.

$[R|c]$, where R → reduced row echelon form of A

Solutions of $Ax = b$ are presicely the solutions of $Rx = c$.

→ Homogenous system of linear equations

- ↳ 0 is always a solution a.k.a. trivial solution
- ↳ If there are more variables than equations, (i.e., more columns than rows) there will be infinite solutions because there must be some independent variables.

WEEK 3

WEEK 1 GA

1. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A - xI = \begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix}$

$\det(A - xI) = (a-x)(d-x) - bc$
$= ad - ax - xd + x^2 - bc$
$= \underbrace{ad - bc}_{\det(A)} \underbrace{- x(a+d)}_{-x\,\mathrm{trace}(A)} + \underbrace{x^2}_{x^2}$

2. $3A = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$ $\det(3A) = 9ad - 9bc = (3)^2 \det(A)$

$3A = \begin{bmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$ $|3A| = 3^3|A| = \boxed{27|A|}$

3. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $I + A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $5A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$

$5A + I = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$

$(I + A)^3 - (5A + I) = mA$

$\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$ $\boxed{m = 8}$

4. $A = \begin{bmatrix} 20 & 30 & 40 \\ 8 & 16 & 24 \\ 8 & 10 & 12 \end{bmatrix}$ $\det(A) = 20 \det\begin{bmatrix} 16 & 24 \\ 10 & 12 \end{bmatrix} - 30 \det\begin{bmatrix} 8 & 24 \\ 8 & 12 \end{bmatrix} + 40 \det\begin{bmatrix} 8 & 16 \\ 8 & 10 \end{bmatrix}$

$= 20(-48) - 30(-96) + 40(-48) = -960 + 2880 - 1920 = \boxed{0}$

5. $\det(A) = 3$ $\det(B) = 3$ $\det(B^{-1}) = 1/3$

$\det(3A^2B^{-1}) = (3)^3 \cdot (\det(A))^2 \cdot \left(\frac{1}{3}\right) = 9(3)^2 = \boxed{81}$

6. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ $A^3 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix}$ $A^4 = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}$

$A^7 = \begin{bmatrix} 729 & 729 & 729 \\ 729 & 729 & 729 \\ 729 & 729 & 729 \end{bmatrix}$ sum of diagonals $= \boxed{2187}$

7. $A = \begin{bmatrix} 12 & 19 & 26 \\ 17 & 24 & 31 \\ 22 & 29 & 36 \end{bmatrix}$ $\det(A) = 12 \det\begin{bmatrix} 24 & 31 \\ 29 & 36 \end{bmatrix} - 19 \det\begin{bmatrix} 17 & 31 \\ 22 & 36 \end{bmatrix} + 26 \det\begin{bmatrix} 17 & 24 \\ 22 & 29 \end{bmatrix}$

$= 12(-35) - 19(-70) + 26(-35) = \boxed{0}$

9.
$30x_1 + 20x_2 + 25x_3 = 670$ — 1
$20x_1 + 35x_2 + 25x_3 = 730$ — 2
$20x_1 + 10x_2 + 15x_3 = 400$ — 3

1 − 2·3 ⇒ 30 20 25 670
−40 −20 −30 −800
$= -10x_1 - 5x_3 = -130$

2 − 3 ⇒ 20 35 25 730
−20 −10 −15 −400
$= 25x_2 + 10x_3 = 330$ — 4
$10x_1 + 5x_3 = 130$ — 5

2·5 $= 20x_1 + 10x_3 = 260 \Rightarrow 20x_1 = 260 - 10x_3$

$260 - 10x_3 + 10x_2 + 15x_3 = 400$
$\Rightarrow 10x_2 + 5x_3 = 140$ — 6

$x_1 + x_2 + x_3 = 9 + 10 + 8 = \boxed{27}$

4 − 2·6 ⇒ 0 25 10 330
0 −20 −10 −280
$= 5x_2 = 50$ $\boxed{x_2 = 10}$

10. $A = \begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix}$

$250 + 10x_3 = 330 \Rightarrow \boxed{x_3 = 8}$

$10x_1 + 40 = 130 \Rightarrow \boxed{x_1 = 9}$

WEEK 2 GA

(1) $P(1) = -45$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -45 \\ 4 & 2 & 1 & -19 \\ 9 & 3 & 1 & 3 \end{array}\right] \xrightarrow{\substack{R_2-4R_1 \\ R_3-9R_1}} \begin{array}{cccc} 1 & 1 & 1 & -45 \\ 0 & -2 & -3 & 161 \\ 0 & -6 & -8 & 408 \end{array} \xrightarrow{R_2-\frac{1}{2}R_3} \begin{array}{cccc} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & -6 & -8 & 408 \end{array} \xrightarrow{R_3+6R_2} \begin{array}{cccc} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{array} \xrightarrow{\substack{R_1-R_2 \\ -R_3/2}} \begin{array}{cccc} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 1 & -75 \end{array} \xrightarrow{R_2-R_3} \begin{array}{cccc} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 32 \\ 0 & 0 & 1 & -75 \end{array}$$

$P(n) = -2n^2 + 32n - 75$ $\qquad P'(n) = -4n + 32$

$\frac{-32}{-4} = n \qquad n = 8$

$P(8)$ is peak
$P(8) = 53$

(4)
$$\begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & c & d \end{array} \xrightarrow{\substack{R_2-2R_1 \\ R_3-4R_1}} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array} \xrightarrow{\substack{R_3+3R_2 \\ R_1-2R_2}} \begin{array}{cccc} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

comic book = ?

if $c = 7$, $d = 4000$

if $c = 2$, $d = 4000$
$$\begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 4000 \end{array} \xrightarrow{\substack{R_2-2R_1 \\ R_3-4R_1}} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -2 & 0 \end{array}$$

if $c = 7$, $d = 3000$
$$\begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 7 & 3000 \end{array} \xrightarrow{\substack{R_2-2R_1 \\ R_3-4R_1}} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & -1000 \end{array}$$

if $c = 2$, $d = 3000$
$$\begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 3000 \end{array} \xrightarrow{\substack{R_2-2R_1 \\ -R_3-4R_1}} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 1000 \end{array} \xrightarrow{R_3-3R_2} \begin{array}{cccc} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 1000 \end{array} \Rightarrow \begin{pmatrix} 400 \\ 200 \\ 200 \end{pmatrix}$$

(6) $\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) + a/6 = -1 \Rightarrow a = 0$

$\left(-\frac{1}{3}\right) + b/6 = 0 \Rightarrow \frac{b}{6} = \frac{1}{3} \Rightarrow b = 2$

$\left(\frac{1}{2}\right) + \left(-\frac{1}{6}\right) + c/6 = 0 \Rightarrow \frac{c-1}{6} = -\frac{1}{2} \Rightarrow c - 1 = -3 \Rightarrow c = -2$

$a + b + c = 0$

(7)
$$\begin{array}{cccc} 1 & 3k & 3k+4 & 61 \\ 1 & k+4 & 4k+2 & 65 \\ 1 & 2k+2 & 3k+4 & 66 \end{array} \xrightarrow{\substack{R_2-R_1 \\ R_3-R_1}} \begin{array}{cccc} 1 & 3k & 3k+4 & 61 \\ 0 & -2k+4 & k-2 & 4 \\ 0 & -k+2 & 0 & 5 \end{array} \xrightarrow{R_3 \leftrightarrow R_2} \begin{array}{cccc} 1 & 3 & 7 & 61 \\ 0 & 1 & 0 & 5 \\ 0 & 2 & -1 & 4 \end{array} \xrightarrow{-R_3+2R_2} \boxed{\left[\begin{array}{ccc|c} 1 & 3 & 7 & 61 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right]}$$

$4k + 2 - 3k - 4$
$k - 2$

$-k + 2 = 1$
$k = 1$

(8)
$$\begin{array}{cccc} 2 & 3 & 5 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 44 \end{array} \xrightarrow{\substack{R_1-R_2 \\ R_2-R_3}} \begin{array}{cccc} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 1 & 1 & 2 & 44 \end{array} \xrightarrow{R_3-R_1} \begin{array}{cccc} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 0 & 44 \end{array}$$

→ shows no solution

(9)
$$\begin{array}{cccc} 1 & 3 & 0 & 0 \\ 4 & 1 & 5 & 34 \\ 2 & 2 & 7 & 32 \\ 3 & 9 & 0 & 0 \end{array} \xrightarrow{\substack{R_2-4R_1 \\ R_3-2R_1 \\ R_4-3R_1}} \begin{array}{cccc} 1 & 3 & 0 & 0 \\ 0 & -11 & 5 & 34 \\ 0 & -4 & 7 & 32 \\ 0 & 0 & 0 & 0 \end{array}$$

R_4 shows $\det(A) = 0$

(10)
$$\begin{array}{cccc} 7 & 2 & 1 & 8 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{array} \xrightarrow{R_1+2R_3} \begin{array}{cccc} 1 & 10 & -3 & 24 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{array} \xrightarrow{\substack{R_3+3R_1 \\ R_3/3}} \begin{array}{cccc} 1 & 10 & -3 & 24 \\ 0 & 1 & -1/3 & 4/3 \\ 0 & 34 & -11 & 80 \end{array} \xrightarrow{\substack{R_1-10R_2 \\ R_3-34R_2}} \begin{array}{cccc} 1 & 0 & 1/3 & 32/3 \\ 0 & 1 & -1/3 & 4/3 \\ 0 & 0 & 1/3 & 104/3 \end{array} \xrightarrow{3R_3} \begin{array}{cccc} 1 & 0 & 1/3 & 32/3 \\ 0 & 1 & -1/3 & 4/3 \\ 0 & 0 & 1 & 104 \end{array}$$

$x = -24$
$y = 36$ $\qquad x + y + z = 116$
$z = 104$

(11) $A = \begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix}$ $\quad A^T A = \begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix}_{4\times 1} \begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix}_{1\times 4} = \begin{matrix} 4 & 20 & 4 & 16 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{matrix}$

$$\xrightarrow{R_1/4} \begin{matrix} 1 & 5 & 1 & 4 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{matrix} \quad \begin{matrix} R_2-20R_1 \\ R_3-4R_1 \\ R_4-16R_1 \end{matrix} \quad \begin{matrix} 1 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

(12)
$$\begin{matrix} 4 & 2 & 1 & 4 \\ 9 & 3 & 1 & 6 \\ 16 & 4 & 1 & 10 \end{matrix} \xrightarrow{R_1/4} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 9 & 3 & 1 & 6 \\ 16 & 4 & 1 & 10 \end{matrix} \quad \begin{matrix} R_2-9R_1 \\ R_3-16R_1 \end{matrix} \quad \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & -3/2 & -5/4 & -3 \\ 0 & -4 & -3 & -6 \end{matrix} \quad \begin{matrix} -2R_2 \\ -R_3 \end{matrix} \quad \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & 3 & 5/2 & 6 \\ 0 & 4 & 3 & 6 \end{matrix}$$

$$\xrightarrow{-R_3+\frac{4}{3}R_2} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & 3 & 5/2 & 6 \\ 0 & 0 & 1/3 & 2 \end{matrix} \xrightarrow{3R_3} \begin{matrix} 1 & 1/2 & 1/4 & 1 \\ 0 & 3 & 5/2 & 6 \\ 0 & 0 & 1 & 6 \end{matrix}$$

$a = 1$ $\qquad a - b + c = 10$
$b = -3$
$c = 6$

(13)
$$\left[\begin{matrix} 2 & 3 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 1 \end{matrix}\,\middle|\,\begin{matrix} 1000 \\ 900 \\ c \\ 400 \end{matrix}\right] \quad \begin{matrix} R_1/2 \\ R_4-R_1 \end{matrix} \quad \begin{matrix} 1 & 3/2 & 0 & 0 & 500 \\ 0 & 3 & 2 & 0 & 900 \\ 0 & 0 & 2 & 1 & c \\ 0 & -3 & 0 & 0 & -600 \end{matrix} \xrightarrow{R_2/3} \begin{matrix} 1 & 3/2 & 0 & 0 & 500 \\ 0 & 1 & 2/3 & 0 & 300 \\ 0 & 0 & 2 & 1 & c \\ 0 & 3 & 0 & 0 & 600 \end{matrix}$$

$$\begin{matrix} -R_4-3R_2 \\ R_1-\frac{3}{2}R_2 \end{matrix} \quad \begin{matrix} 1 & 0 & -1 & 0 & 50 \\ 0 & 1 & 2/3 & 0 & 300 \\ 0 & 0 & 2 & 1 & c \\ 0 & 0 & 2 & 0 & 300 \end{matrix} \xrightarrow{R_3/2} \begin{matrix} 1 & 0 & -1 & 0 & 50 \\ 0 & 1 & 2/3 & 0 & 300 \\ 0 & 0 & 1 & 1/2 & c/2 \\ 0 & 0 & 2 & 0 & 300 \end{matrix}$$

$$\begin{matrix} R_1+\frac{1}{2}R_4 \\ R_2-\frac{1}{3}R_4 \\ -R_4-2R_3 \end{matrix} \quad \begin{matrix} 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & 200 \\ 0 & 0 & 1 & 1/2 & c/2 \\ 0 & 0 & 0 & 1 & -300+c \end{matrix} \xrightarrow{R_3-\frac{1}{2}R_4} \begin{matrix} 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & 200 \\ 0 & 0 & 1 & 0 & 150 \\ 0 & 0 & 0 & 1 & c-300 \end{matrix}$$

(W.N.) $u_1 = 200$
(E.N.) $u_2 = 200$
(E.S.) $u_3 = 150$ $\qquad c = 300$
(W.S.) $u_4 = 0$

min $\rightarrow u_4 = 0 \rightarrow u_1 = 200,\ u_2 = 200,\ u_3 = 150$
max $\rightarrow u_4 = 300 \rightarrow u_1 = 50,\ u_2 = 300,\ u_3 = 0$

(W.N.) $\rightarrow 400 \rightarrow 100$
(E.N.) $\rightarrow 900 \rightarrow 600$
(E.S.) $\rightarrow 300 \rightarrow 0$
(W.S.) $\rightarrow 300 \rightarrow 0$