STATS 2

WEEK 1

→ Joint PMF of two discrete random variables
↳ Two events can either be independent; or
↳ one could be influencing the other.

→ Types of PMF of Multiple Random Variables:
① Joint PMF
② Marginal PMF
③ Conditional PMF

① Joint PMF

→ X and Y → discrete random variables defined in the same probability space

Range of X and Y → T_X and T_Y

↳ Joint PMF of X and Y → f_{XY} → function from $T_X \times T_Y$ to $[0,1]$

$$f_{XY}(t_1, t_2) = P(X = t_1 \text{ and } Y = t_2),\ t_2 \in T_X,\ t_2 \in T_Y$$

→ Usually written as a table or a matrix
↳

variable 1 (across), variable 2 (down)

$P(x_1,x_2)$	$P(x_1,y_2)$	$P(x_1,x_2)$	...	..	..
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	..	..
...	...	...	...	...	...

② Marginal PMF

→ To obtain the individual PMF of a random discrete variable from a joint PMF,
↳ suppose X and Y random discrete variables, and their joint PMF → f_{XY}, then:

$$f_X(t) = P(X=t) = \sum_{t' \in T_Y} f_{XY}(t, t')$$

$$f_Y(t) = P(Y=t) = \sum_{t' \in T_X} f_{XY}(t', t)$$

③ Conditional PMF

→ $f_{X|A}(t)$ → PMF of a random variable X conditional on an event A.

$P(X = t \mid A),\ t \in T_X$

$$f_{X|A}(t) = \frac{P((X=t) \cap A)}{P(A)}$$

→ $f_{Y|X}(t)$ → conditional PMF of Y given $X=t$

$$f_{Y|X=t}(t') = P(Y=t' \mid X=t) = \frac{P(Y=t', X=t)}{P(X=t)} = \frac{f_{XY}(t,t')}{f_X(t)}$$

$$f_{XY}(t,t') = f_{Y|X=t}(t')\, f_X(t)$$

→ $f_{XY}(t_1,t_2) = f_{Y|X=t_1}(t_2)\, f_X(t_1) = f_{X|Y=t_2}(t_1)\, f_Y(t_2)$

Q. Let $N \sim \text{Poisson}(\lambda)$. Given $N=n$, toss a fair coin n times and denote the number of heads obtained by X. What is the distribution of X?

A. $f_N(n) = \dfrac{e^{-\lambda}\lambda^n}{n!}$; $n = 0, 1, 2, \dots$ $\quad (X|N=n) \sim \text{Bin}(n, 1/2) \quad f_{X|N=n}(k) = \binom{n}{k}\left(\frac{1}{2}\right)^n$

$$f_{XN}(k,n) = \frac{e^{-\lambda}\lambda^n}{n!} \cdot \frac{n!}{k!(n-k)!} \cdot \left(\frac{1}{2}\right)^n = \frac{e^{-\lambda}\lambda^n}{k!(n-k)!}\left(\frac{1}{2}\right)^n$$

$$\boxed{f_X(k) = \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{k!(n-k)!}\left(\frac{1}{2}\right)^n} \Rightarrow \frac{e^{-\lambda}\lambda^k}{k!\,2^k} \sum_{n=k}^{\infty} \frac{(\lambda)^{n-k}}{(n-k)!\,(2)^{n-k}}$$

$$X \sim \text{Poisson}(\lambda/2) \Rightarrow \boxed{f_X(k) = \frac{e^{-\lambda/2}(\lambda/2)^k}{k!}}$$

→ Joint PMF of multiple random variables

↳ $X_1, X_2, X_3, \dots X_n$ are discrete random variables defined in the same probability space.

$$f_{X_1X_2\dots X_n}(t_1, t_2, \dots, t_n) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and} \cdots \text{ and } X_n = t_n),\ t_i \in T_{X_i}$$

→ Marginal PMF with multiple random variables

↳ individual random variable ⇒ $f_{X_1}(t) = P(X_1 = t) = \sum_{t_2' \in T_2, t_3' \in T_3, \dots, t_n' \in T_n} f_{X_1, X_2, X_3 \dots X_n}(t, t_2', t_3', \dots, t_n)$

↳ multiple random variables ⇒ $f_{X_1X_2}(t_1, t_2) = P(X_1 = t_1 \text{ and } X_2 = t_2) = \sum_{t_3' \in T_3} f_{X_1, X_2, X_3}(t_1, t_2, t')$

→ Conditional PMF with multiple random variables

↳ $(X_1, X_2 | X_3 = t_3) \sim f_{X_1, X_2 | X_3 = t_3}(t_1, t_2) = \dfrac{f_{X_1, X_2, X_3}(t_1, t_2, t_3)}{f_{X_3}(t_3)}$

↳ $(X_1 | X_2 = t_2, X_3 = t_3) \sim f_{X_1 | X_2 = t_2, X_3 = t_3}(t_1) = \dfrac{f_{X_1, X_2, X_3}(t_1, t_2, t_3)}{f_{X_2X_3}(t_2, t_3)}$

↳ factors of a Joint PMF: $f_{X_1 \dots X_4}(t_1, \dots, t_4) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$

$$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$$

$$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 | X_3 = t_3, X_4 = t_4) \cdot P(X_3 = t_3 | X_4 = t_4) \cdot P(X_4 = t_4)$$

$$f_{X_1 \dots X_4}(t_1, \dots, t_4) = f_{X_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4}(t_1) \cdot f_{X_2 = t_2 | X_3 = t_3, X_4 = t_4}(t_2) \cdot f_{X_3 | X_4 = t_4}(t_3) \cdot f_{X_4}(t_4)$$

WEEK 1 GA

① $T_X = \{0,1,2,3\}$ $\quad T_Y = \{-1,1,2,3\}$ $\quad X \sim Bin(3, 1/2)$ $\quad P(X<3) = 1 - P(X=3) = 7/8$ $\quad P(X=3) = \binom{3}{3}\cdot(1/2)^3 = 1/8$

$P(Y\le 1) = P(Y=1) + P(Y=-1)$ $\quad P(Y=1) = 1/2$ $\quad P(Y=-1) = 1/8$ $\quad P(Y\le 1) = 1/2 + 1/8 = 5/8$

$f_{XY}(t_x<3, t_y\le 1) = \frac{5}{8} \times \frac{7}{8} = \frac{35}{64} \approx \boxed{0.5469}$

② $T_X = T_Y = T_Z = \{0,1,2\}$

X	Y	Z
0	0	2
2	0	0
1	0	1

$= \boxed{1/3}$

$f_{XY}(2,0) = \sum_{t_z=0}^{2} f_{XYZ}(2,0,t_z) = P(2,0,0) = 1/9$

$f_Y(0) = \sum_{t_x\in T_x, t_z\in T_z} f_{XYZ}(t_x,0,t_z) = P(0,0,2) + P(1,0,1) + P(2,0,0) = 3/9$

$f_{X|Y=0}(2) = \frac{f_{XY}(2,0)}{f_Y(0)}$

$f_{X|Y=0}(2) = \frac{1/9}{3/9} = \boxed{1/3}$

③ $1/4 + 1/4 + 1/8 + 1/8 + k = 1$ $\quad k = 1 - (2/4 + 1/4) = 1 - 3/4 = 1/4$

$f_{Y|X=1}(2) = \frac{f_{YX}(2,1)}{f_X(1)} = \frac{1/4}{1/8+1/8+1/4} = \frac{1/4}{1/2} = \frac{1}{4}\times 2 = \boxed{\frac{1}{2}}$

⑤ $f_Y(1) = \sum_{t'\in T_X} f_{XY}(t', 1)$ $\quad f_X(1) = 12/13$ $\quad f_X(0) = 1/13$

$Y \sim Bin(n,p)$

$f_{Y|X=1}(1) =$ $n=3$, $p = 7/20$
$= \binom{3}{1}\left(\frac{7}{20}\right)\left(\frac{13}{20}\right)^2$
$= \frac{3\times 7\times (13)^2}{(20)^3}$
$= 0.443625$

$f_{Y|X=0}(1) =$ $n=2$, $p=4/15$
$= \binom{2}{1}\left(\frac{4}{15}\right)\left(\frac{11}{15}\right)$
$= \frac{2\times 4\times 11}{(15)^2}$
$= 0.39111$

$f_{XY}(0,1) = f_{Y|X=0}(1)\cdot f_X(0) = 0.03008547$

$f_{XY}(1,1) = f_{Y|X=1}(1)\cdot f_X(1) = 0.4095$

$f_Y(1) = f_{XY}(0,1) + f_{XY}(1,1) \approx \boxed{0.4396}$

⑦ $f_{XY}(x,y) = a(bx+y)$

x \ y	0	1	2	3
0	0	a	2a	3a
1	ab	ab + a	ab + 2a	ab + 3a
2	2ab	2ab + a	2ab + 2a	2ab + 3a
	3ab	3ab + 2a	3ab + 4a	

$3\left(12a + 3ab = \frac{3}{7}\right) = 36a + 9ab = \frac{9}{7}$

$2\left(6a + 9ab = \frac{4}{7}\right) = 12a + 18ab = \frac{8}{7}$

$-12a - 3ab = -\frac{3}{7}$

$15ab = \frac{5}{7}$

$36a + 9ab = \frac{9}{7}$

$-6a - 9ab = -\frac{4}{7}$

$30a = \frac{5}{7}$

$\boxed{a = \frac{1}{42}}$

$ab = \frac{1}{21}$

$\frac{b}{42} = \frac{1}{21} \Rightarrow \boxed{b=2}$

$f_{XY}(1,1) = \frac{1}{42}(2+1) = \boxed{\frac{1}{14}}$

⑧ $f_{Y|X=2}(0) = \frac{f_{YX}(0,2)}{f_X(2)}$

$f_X(2) = \binom{6}{2}(1/2)^6$

$f_{X|Y=0}(2) = \binom{5}{2}(1/2)^5$; $f_Y(0) = 1/2$; $f_{YX}(0,2) = \binom{5}{2}(1/2)^6$

$f_{Y|X=2}(0) = \frac{\binom{5}{2}\cdot(1/2)^6}{\binom{6}{2}\cdot(1/2)^6} = \frac{\binom{5}{2}}{\binom{6}{2}}$

$= \frac{5!}{2!\,3!} \times \frac{2!\,4!}{6!} = \frac{5!}{6!} \times \frac{4!}{3!} = \frac{4}{6} = \boxed{0.666}$

⑨ $$\frac{\binom{5}{1}\times\binom{4}{1}}{\binom{12}{2}} = \frac{5\times 4}{\frac{12!}{10!\,2!}} = \frac{5\times\cancel{4}}{\cancel{12}_{3}\times 11}\times 2 = \frac{10}{33} = \boxed{0{\cdot}30}$$

⑩ $N \sim Bin(7, 1/2)$ $\quad X \sim Bin(n, 1/2)$

$x \backslash N$	0	1	2	3	4
0	0	${}^7C_1(1/2)^8$			$\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11}$
1	0	${}^7C_1(1/2)^8$		$\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10}$	
2	0	0	$\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9}$		
3	0	0	0		
4	0	0	0	0	

$$\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11} = \frac{7!}{4!\,3!}\times\frac{1}{2048} = \frac{7\times 6\times 5}{3\times 2}\times\frac{1}{2048} = \frac{35}{2048}$$

$$\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10} = \frac{7!}{3!\,4!}\times\frac{3}{1024} = \frac{210}{2048}$$

$$\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9} = \frac{7!}{2!\,5!}\times\frac{1}{512} = \frac{7\times 6}{2}\times\frac{1}{512} = \frac{84}{2048}$$

$$\text{Total} = \frac{35+210+84}{2048} = \frac{329}{2048} = \boxed{0{\cdot}161}$$

WEEK 2