STATS 2

WEEK 1

→ Joint PMF of two discrete random variables
  ↳ Two events can either be independent; or
  ↳ one could be influencing the other.

→ Types of PMF of Multiple Random Variables:
① Joint PMF
② Marginal PMF
③ Conditional PMF

① Joint PMF

→ X and Y → discrete random variables defined in the same probability space

Range of X and Y → T_X and T_Y

↳ Joint PMF of X and Y → f_{XY} → function from $T_X \times T_Y$ to $[0, 1]$

$$f_{XY}(t_1, t_2) = P(X = t_1 \text{ and } Y = t_2),\ t_2 \in T_X,\ t_2 \in T_Y$$

→ Usually written as a table or a matrix
↳

← variable 1 →

↑ variable 2 ↓

$P(x_1, x_2)$	$P(x_1, x_2)$	$P(x_1, x_2)$	...	..	..
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	..	..
...	...	...	...	...	...

② Marginal PMF

→ To obtain the individual PMF of a random discrete variable from a joint PMF.
↳ Suppose X and Y random discrete variables, and their joint PMF → f_{XY}, then:

$$f_X(t) = P(X = t) = \sum_{t' \in T_Y} f_{XY}(t, t')$$

$$f_Y(t) = P(Y = t) = \sum_{t' \in T_X} f_{XY}(t', t)$$

③ Conditional PMF

→ $f_{X|A}(t)$ → PMF of a random variable X conditional on an event A.

$P(X = t | A),\ t \in T_X$

$$f_{X|A}(t) = \frac{P((X = t) \cap A)}{P(A)}$$

→ $f_{Y|X}(t)$ → conditional PMF of Y given $X = t$

$$f_{Y|X=t}(t') = P(Y = t' | X = t) = \frac{P(Y = t', X = t)}{P(X = t)} = \frac{f_{XY}(t, t')}{f_X(t)}$$

$$f_{XY}(t, t') = f_{Y|X=t}(t')\, f_X(t)$$

→ $f_{XY}(t_1, t_2) = f_{Y|X=t_1}(t_2)\, f_X(t_1) = f_{X|Y=t_2}(t_1)\, f_Y(t_2)$

Q. Let $N \sim \text{Poisson}(\lambda)$. Given $N=n$, toss a fair coin n times and denote the number of heads obtained by X. What is the distribution of X?

A. $f_N(n) = \frac{e^{-\lambda}\lambda^n}{n!}$; $n = 0, 1, 2, \dots$ $\quad (X|N=n) \sim \text{Bin}(n, \tfrac{1}{2}) \quad f_{X|N=n}(k) = \binom{n}{k}\left(\frac{1}{2}\right)^n$

$$f_{XN}(k,n) = \frac{e^{-\lambda}\lambda^n}{n!} \cdot \frac{n!}{k!(n-k)!} \cdot \left(\frac{1}{2}\right)^n = \frac{e^{-\lambda}\lambda^n}{k!(n-k)!}\left(\frac{1}{2}\right)^n \qquad \boxed{f_X(k) = \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{k!(n-k)!}\left(\frac{1}{2}\right)^n} \Rightarrow \frac{e^{-\lambda}\lambda^k}{k!\,2^k}\sum_{n=k}^{\infty}\frac{(\lambda)^{n-k}}{(n-k)!\,(2)^{n-k}}$$

$$X \sim \text{Poisson}(\lambda/2) \Rightarrow \boxed{f_X(k) = \frac{e^{-\lambda/2}(\lambda/2)^k}{k!}}$$

→ Joint PMF of multiple random variables

↳ $X_1, X_2, X_3, \dots X_n$ are discrete random variables defined in the same probability space.

$$f_{X_1X_2\dots X_n}(t_1, t_2, \dots, t_n) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and} \cdots \text{ and } X_n = t_n),\ t_i \in T_{X_i}$$

→ Marginal PMF with multiple random variables

↳ individual random variable ⇒ $f_{X_1}(t) = P(X_1 = t) = \sum_{t_2' \in T_2, t_3' \in T_3, \dots, t_n' \in T_n} f_{X_1, X_2, X_3 \dots X_n}(t, t_2', t_3', \dots, t_n)$

↳ multiple random variables ⇒ $f_{X_1X_2}(t_1, t_2) = P(X_1 = t_1 \text{ and } X_2 = t_2) = \sum_{t_3' \in T_3} f_{X_1, X_2, X_3}(t_1, t_2, t')$

→ Conditional PMF with multiple random variables

↳ $(X_1, X_2 | X_3 = t_3) \sim f_{X_1, X_2|X_3=t_3}(t_1, t_2) = \frac{f_{X_1,X_2,X_3}(t_1, t_2, t_3)}{f_{X_3}(t_3)}$

↳ $(X_1 | X_2 = t_2, X_3 = t_3) \sim f_{X_1|X_2=t_2, X_3=t_3}(t_1) = \frac{f_{X_1,X_2,X_3}(t_1, t_2, t_3)}{f_{X_2X_3}(t_2, t_3)}$

↳ factors of a Joint PMF: $f_{X_1 \dots X_4}(t_1, \dots, t_4) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$

$$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$$

$$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 | X_3 = t_3, X_4 = t_4) \cdot P(X_3 = t_3 | X_4 = t_4) \cdot P(X_4 = t_4)$$

$$f_{X_1 \dots X_4}(t_1, \dots, t_4) = f_{X_1|X_2=t_2, X_3=t_3, X_4=t_4}(t_1) \cdot f_{X_2=t_2|X_3=t_3, X_4=t_4}(t_2) \cdot f_{X_3|X_4=t_4}(t_3) \cdot f_{X_4}(t_4)$$

WEEK 2

→ Independence of two random variables
↳ X and Y are independent if :

$$f_{XY}(t_1, t_2) = f_X(t_1) \cdot f_Y(t_2)$$

→ Joint PMF of f_{XY} is the product of the marginals PMF of X and Y.
→ Conditional PMF equals marginal PMF

→ Independence of multiple random variables
↳ $X_1, X_2, \ldots, X_n$ are independent iff

$$f_{X_1, X_2, \ldots X_n}(t_1, t_2, \ldots, t_n) = f_{X_1}(t_1) \cdot f_{X_2}(t_2) \cdot \ldots \cdot f_{X_n}(t_n)$$

→ Independent and Identically distributed (i.i.d.)
↳ Random variables $X_1, X_2, \ldots X_n$ are i.i.d. if → ① they are independent,
② Marginal PMFs f_{X_i} are identical.

→ $X_1, X_2 \ldots X_n \sim$ i.i.d. f_X

→ Memoryless property of Geometric distribution
→ Let $X \sim$ Geometric(p)

① $P(X > n) = (1-p)^n$

② $P(X > m+n \mid X > m) = \dfrac{P(X > m+n \cap X > m)}{P(X > m)} = \dfrac{P(X > m+n)}{P(X > m)} = \dfrac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n$

→ Functions of random variable

↳ One-to-one function → $P(Y = f(x)) = P(X = x)$; e.g. → $y = x-5$, $y = 2^x$

↳ Many-to-one function → e.g. → $y = (x-5)^2$, $y = x(1-x)$ → $y_0 = f(x_1) = f(x_2) = \ldots = f(x_m)$ all the places $f(x)$ takes the value y_0

↳ sum over all the probabilities of X for wherever the function takes the 'y' value. ⟶ $P(Y = y_0) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_m)$

→ Visualising function of 2 random variable

↳ $g(x, y)$: function

① Contours → values of (x, y) that result in $g(x,y) = c$
→ make a plot of those (x,y) for different c

② Regions → values of (x,y) that result in $g(x,y) \leq c$
→ make a plot of those (x,y) for different c

→ Function of 2 random variables
→ Let $X, Y \sim f_{XY}$; let $Z = g(X, Y)$ be a function of X and Y
· What is the PMF of Z ?
→ Step 1 : Find the range of Z
→ Step 2 : Add over all the contours
↳ Suppose z is a possible value taken by Z :

$$P(Z = z) = \sum_{(x,y) : g(x,y) = z} f_{XY}(x, y)$$

→ Sum of 2 uniform random variables

$X, Y \sim$ iid Unif$\{1, 2, \ldots, n\}$, $W = X + Y$
range of $W = \{2, 3, \ldots, 2n\}$ $W = w : (1, w-1), (2, w-2), \ldots, (w-1, 1)$
$W \in \{2, 3, \ldots, 2n\}$

$$P(W = w) = \begin{cases} \dfrac{w-1}{n^2}, & 2 \leq w \leq n+1 \\ \dfrac{2n - w + 1}{n^2}, & n+2 \leq w \leq 2n \end{cases}$$

→ Max of 2 uniform random variables $X, Y \sim iid\ Unif\{1,2,\dots,n\}$, $Z = \max(X,Y)$

$Z \in \{1,2,\dots,n\}$

$P(Z=z) = \dfrac{2z-1}{n^2}$

→ **PMF of $g(X_1, X_2, \dots X_n)$** → The PMF $X = g(X_1, X_2, \dots, X_n)$ is given by

$$f_X(t) = P(g(X_1,X_2,\dots,X_n)=t) = \sum_{(t_1,t_2,\dots,t_n):\, g(t_1,t_2,\dots,t_n)=t} f_{X_1,X_2,\dots,X_n}(t_1,t_2,\dots,t_n)$$

e.g. →

$t_2 \cdot t_3$	$t_1+t_2+t_3$	t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1,t_2,t_3)$
0	0	0	0	0	1/9
0	1	0	0	1	1/9
0	2	0	0	2	1/9
1	2	0	1	1	1/9
2	3	0	1	2	1/9
0	1	1	0	0	1/9
0	3	1	0	2	1/9
0	2	1	1	0	1/9
1	3	1	1	1	1/9

① $X = g(X_1,X_2,X_3) = X_1+X_2+X_3$ $g \in \{\overset{1/9}{0}, \overset{2/9}{1}, \overset{3/9}{2}, \overset{3/9}{3}\}$

② $Y = h(X_1,X_2,X_3) = X_2 \cdot X_3$ $h \in \{\overset{6/9}{0}, \overset{2/9}{1}, \overset{1/9}{2}\}$

f_{XY}	0	1	2	3
0	1/9	2/9	2/9	1/9
1	0	0	1/9	1/9
2	0	0	0	1/9

→ sum of n independent Bernoulli(p) = Binomial(n,p)

→ suppose X and Y take integer values and their joint PMF = f_{XY}. Let $Z = X+Y$

Convolution

$$P(Z=z) = \sum_{x=-\infty}^{\infty} P(X=x, Y=z-x) = \sum_{x=-\infty}^{\infty} f_{XY}(x, z-x) = \sum_{y=-\infty}^{\infty} f_{XY}(z-y, y)$$

$$= \sum_{x=-\infty}^{\infty} f_X(x) f_Y(z-x) = \sum_{y=-\infty}^{\infty} f_Y(y) f_X(z-y)$$

→ Let $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$ be independent.

→ $Z = X + Y$

$$f_Z(z) = \sum_{x=0}^{\infty} f_X(x)\cdot f_Y(z-x) = \sum_{x=0}^{z} \frac{e^{-\lambda_1}\cdot \lambda_1^x}{x!}\cdot\frac{e^{-\lambda_2}\cdot\lambda_2^{(z-x)}}{(z-x)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{z!}\underbrace{\sum_{x=0}^{z}\frac{\lambda_1^x\,\lambda_2^{z-x}\cdot z!}{x!\,(z-x)!}}_{\to (\lambda_1+\lambda_2)^z}$$

can be replaced with x going to z because after z this term will go to 0.

$$f_Z(z) = \frac{e^{-(\lambda_1+\lambda_2)}\cdot(\lambda_1+\lambda_2)^z}{z!} \Rightarrow Z \sim Poisson(\lambda_1+\lambda_2)$$

→ conditional of $X|Z$

$$P(X=k|Z=n) = \frac{P(X=k, Z=n)}{P(Z=n)} = \frac{P(X=k)\cdot P(Z=n|X=k)}{P(Z=n)} = \frac{P(X=k)\cdot P(Y=n-k)}{P(Z=n)}$$

$$= \frac{\dfrac{\cancel{e^{-\lambda_1}}\cdot\lambda_1^k}{k!}\cdot\dfrac{\cancel{e^{-\lambda_2}}\cdot\lambda_2^{n-k}}{(n-k)!}}{\dfrac{\cancel{e^{-\lambda_1-\lambda_2}}\cdot(\lambda_1+\lambda_2)^n}{n!}} = \frac{n!}{k!\,(n-k)!}\,\frac{\lambda_1^k\cdot\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}$$

$\longrightarrow (\lambda_1+\lambda_2)^n = (\lambda_1+\lambda_2)^k\cdot(\lambda_1+\lambda_2)^{n-k}$

$\longrightarrow \dfrac{n!}{k!\,(n-k)!} = {}^nC_k$

$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$X|Z = Bin\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$Y|Z = Bin\left(n, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$

→ Functions of non-overlapping indepent random variables are also independent

→ If X and Y are independent, $g(X)$ and $h(Y)$ are independent for any two functions g and h

→ Min/Max of two random variables

↳ $X, Y \sim f_{XY}$ $\qquad Z = \min(X, Y)$

$f_Z(z) = P(\min(x,y) = z) = P((x=z) \text{ and } (y=z) \text{ or } (x=z \text{ and } y>z) \text{ or } (x>z \text{ and } y=z))$

$= f_{XY}(z,z) + \sum_{t_2>z} f_{XY}(z,t_2) + \sum_{t_1>z} f_{XY}(t_1,z)$

↳ CDF of a random variable

↳ CDF of a random variable X is a function $F_X : \mathbb{R} \to [0,1]$ defined as:

↳ $F_X(x) = P(X \le x)$

e.g. → X and Y are independent. $Z = \max(X,Y)$

$F_Z(z) = P(\max(X,Y) \le z)$

$= F_X(z) \cdot F_Y(z)$

↳ Let $X_1, X_2, \ldots, X_n \sim$ i.i.d.

① Distribution of $\underline{\min(X_1, X_2, \ldots, X_n)} \to P(\min(X_1, X_2, \ldots, X_n) \ge z) = (P(X \ge z))^n$

② Distribution of $\underline{\max(X_1, X_2, \ldots, X_n)} \to P(\max(X_1, X_2, \ldots, X_n) \le z) = (P(X \le z))^n = (F_X(z))^n$

↳ Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(p)$ be independent. Find the distr. of $\underline{\min(X,Y)}$

$Z = \min(X,Y)$ $\qquad P(Z \ge z) = P(X \ge z, Y \ge z) = (1-p)^{z-1} \cdot (1-p)^{z-1}$

$= ((1-p)^2)^{z-1}$

$P(Z \ge z+1) = ((1-p)^2)^z$

$P(Z = z) = P(Z \ge z) - P(Z \ge z+1)$

$= ((1-p)^2)^{z-1} - ((1-p)^2)^z = ((1-p)^2)^{z-1}(1-(1-p)^2)$ ← Geometric probability

$\min(X,Y) \sim \text{Geometric}(1-(1-p)^2)$

Let $X_1 \sim \text{Geometric}(p_1)$ and $X_2 \sim \text{Geometric}(p_2)$

↳ $\min(X_1, X_2) \sim \text{Geometric}(1-(1-p_1)(1-p_2))$

$Z = \max(X_1, X_2, \ldots, X_{10})$ $\qquad X \sim \text{Bin}(6, ½)$

$P(Z \le 2) = P(X_1 \le 2, X_2 \le 2, \ldots, X_{10} \le 2)$

$= \left(22 \cdot \left(\frac{1}{2}\right)^6\right)^{10}$

X	0	1	2	3
	$\left(\frac{1}{2}\right)^6$	$6\left(\frac{1}{2}\right)^6$	$15\left(\frac{1}{2}\right)^6$	

$\binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6 \times 5}{2} = 15$

$X \le 2 = \left(\frac{1}{2}\right)^6 \cdot 22$

$Z = \min(X_1, X_2, \ldots, X_{10})$ $\qquad X \sim \text{Bin}(6, ½)$

$F_Z(2) = 1 - P(X_1 > 2, X_2 > 2, \ldots X_{10} > 2)$ $\qquad X > 2 = 42\left(\frac{1}{2}\right)^6$

$= 1 - \left(\frac{42}{2^6}\right)^{10} = 1 - \left(\frac{21}{2^5}\right)^{10}$

WEEK 3

→ Expected value of a discrete random variable:

↳ $X \to$ discrete rand. variable ; range $\to T_X$; PMF $\to f_X$

$$E[X] = \sum_{t \in T_X} t \cdot f_X(t) = \sum_{t \in T_X} t \cdot P(X=t)$$

↳ example $\Rightarrow$ $X \sim \text{Bernoulli}(p) \Rightarrow E[X] = 0(1-p) + p = p$

$X \sim \text{Uniform}\{1,2,3,4,5,6\} \Rightarrow E[X] = \sum_{t \in T_X} t \cdot \frac{1}{6} = 3.5$

$X \sim \text{Uniform}\{a, a+1, \dots, b\} \Rightarrow E[X] = \frac{a}{b-a+1} + \frac{a+1}{b-a+1} + \dots + \frac{b}{b-a+1} = \frac{a+b}{2}$

Identity $\Rightarrow$ $a + (a+1) + (a+2) + \dots + b = (b-a+1)\left(\frac{a+b}{2}\right)$

Simplify summations

① Difference Equation (DE): $a_{t+1} - r a_t = b_t \ (r \neq 1)$ $\qquad \sum_{t=1}^{n} a_t = \frac{a_1 - r a_n}{1-r} + \frac{1}{1-r} \sum_{t=1}^{n-1} b_t$

② Geometric Progression (GP): $a_{t+1} - r a_t = 0 \ (r \neq 1)$ $\qquad \sum_{t=1}^{n} a_t = \frac{a_1 - r a_n}{1-r} \xrightarrow[n \to \infty]{|r|<1} \frac{a_1}{1-r}$

③ Exponential Function: $\sum_{t=0}^{\infty} e^{-\lambda} \frac{\lambda^t}{t!} = 1$ $\qquad e^{\lambda} = \sum_{t=0}^{\infty} \frac{\lambda^t}{t!}$

④ Binomial Formula: $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$

↳ $X \sim \text{Geometric}(p)$ $\quad E[X] = \sum_{t=1}^{\infty} t \cdot (1-p)^{t-1} \cdot p$

↳ GP : $a_1 = p$, $r = 1-p$, $b_t = r^t p$ $\qquad E[X] = 1/p$

↳ $X \sim \text{Poisson}(\lambda)$ $\quad E[X] = \sum_{t=0}^{\infty} t \cdot e^{-\lambda} \frac{\lambda^t}{t!}$

↳ using exponent function $\to$ $E[X] = \lambda$

↳ $X \sim \text{Binomial}(n,p)$ $\quad E[X] = \sum_{t=0}^{n} t \cdot \binom{n}{t} p^t \cdot (1-p)^{n-t}$

↳ $E[X] = np$

→ Expected value of a function of random variables

↳ $X_1, X_2, \dots, X_n$ have joint PMF $f_{X_1, X_2, \dots X_n}$ with range X_i denoted T_{X_i}

Let $g : T_{X_1} \times T_{X_2} \times \dots \times T_{X_n} \to \mathbb{R}$ be a function, $Y = g(X_1, X_2, \dots, X_n)$; T_Y and f_Y

$$E[Y] = \sum_{t \in T_Y} t \cdot f_Y(t) = \sum_{t_i \in T_{X_i}} g(t_1, t_2, \dots, t_n) \cdot f_{X_1, X_2, \dots X_n}(t_1, t_2, \dots, t_n)$$

→ Linearity of Expected value:

① $E[cX] = cE[X]$ for a random variable X and a constant c

② $E[X+Y] = E[X] + E[Y]$ for any two random variables X and Y

③ $E[aX + bY] = aE[X] + bE[Y]$

→ Centering operation

↳ $Y = X - E[X]$ is a translated version of X and $E[Y] = 0$

$X - E[X]$ is a zero-mean random variable.

→ Variance of a random variable

↳ $Var(X) = E[(X - E[X])^2]$

$Var(X) = E[X^2] - (E[X])^2$

↳ If X and Y are independent:

– $E[XY] = E[X]E[Y]$

– $Var(X+Y) = Var(X) + Var(Y)$

Distribution	Expected Value	Variance
Bernoulli(p)	p	$p(1-p)$
Binomial(n,p)	$n \cdot p$	$np(1-p)$
Geometric(p)	$1/p$	$(1-p)/p^2$
Poisson(λ)	λ	λ
Uniform$\{1,2,\dots,n\}$	$(n+1)/2$	$(n^2-1)/12$

→ A random variable is standardised if $E(X)=0$ and $Var(X)=1$

↳ $Y = \dfrac{X - E[X]}{SD(X)}$, then Y is a standardised rand. var. obtained from X.

→ Covariance of two random variables

↳ X and Y are rand. var. on the same prob. space

$$Cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]\cdot E[Y]$$

↳ $Cov(X, aY + bZ) = a\,Cov(X,Y) + b\,Cov(X,Z)$

↳ $Cov(aX + bY, Z) = a\,Cov(X,Z) + b\,Cov(Y,Z)$

↳ If X and Y are independent, then $Cov(X,Y) = 0$

If X and Y are independent, then they are uncorrelated.
If X and Y are uncorrelated, they might still be dependent.

→ Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{SD(X)\,SD(Y)}$$

→ Markov's Inequality

$P(X \geq c) \leq \frac{\mu}{c}$, where $\mu = E(X)$

→ Chebyshev's Inequality

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

WEEK 4

→ Cumulative Distribution Function (CDF): $F_X(x) = P(X \le x)$

Properties:

→ $F_X(b) - F_X(a) = P(a < X \le b)$
→ As $x \to -\infty$, F_X goes to 0
→ As $x \to \infty$, F_X goes to 1

↳ if F_X jumps at a point x, then $P(X=x) > 0$.
if F_X is continuous at a point x, then $P(X=x) = 0$.

→ Integral

Indefinite : $F(x) = \int f(x)\, dx$

Definite : $\int_a^b f(x)\, dx = F(b) - F(a)$

→ Probability Density Function

↳ $F_X(x_0) = \int_{-\infty}^{x_0} f_X(x)\, dx$

↳ Properties:

① $f(x) \ge 0$ ② $\int_{-\infty}^{\infty} f(x)\, dx = 1$ ③ $f(x)$ is a piecewise function

→ $supp(X) = \{x : f_X(x) > 0\}$ → interval in which X can fall with positive probability

→ Common Distributions:

↳ $X \sim Exp(\lambda)$ ↘ PDF $f_X(x) \begin{cases} \lambda \exp(-\lambda x) & x > 0 \\ 0 & \text{otherwise} \end{cases}$

CDF $F_X(x) \begin{cases} 0 & x \le 0 \\ 1 - \exp(-\lambda x) & x > 0 \end{cases}$

↳ $X \sim Normal(\mu, \sigma^2)$

↘ PDF $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$

CDF $F_X(x) = \int_{-\infty}^{x} f_X(v)\, dv$

Standardization:

if $X \sim Normal(\mu, \sigma^2)$, then $(X-\mu)/\sigma \sim Normal(0,1)$

↳ $Z \sim Normal(0,1)$ PDF : $f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$

CDF : $F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)\, du$

WEEK 5

WEEK 1 GA

(1) $T_x = \{0,1,2,3\}$ $T_y = \{-1,1,2,3\}$ $X \sim Bin(3, 1/2)$ $P(X<3) = 1 - P(X=3) = 7/8$ $P(X=3) = \binom{3}{3}\cdot(1/2)^3 = 1/8$

$P(Y \le 1) = P(Y=1) + P(Y=-1)$ $P(Y=1) = 1/2$ $P(Y=-1) = 1/8$ $P(Y \le 1) = 1/2 + 1/8 = 5/8$

$f_{XY}(t_x < 3, t_y \le 1) = \frac{5}{8} \times \frac{7}{8} = \frac{35}{64} \approx \boxed{0.5469}$

(2) $T_x = T_y = T_z = \{0,1,2\}$

X	Y	Z
0	0	2
2	0	0
1	0	1

$= \boxed{1/3}$

$f_{XY}(2,0) = \sum_{t_z=0}^{2} f_{XYZ}(2,0,t_z) = P(2,0,0) = 1/9$

$f_Y(0) = \sum_{t_x \in T_x, t_z \in T_z} f_{XYZ}(t_x, 0, t_z) = P(0,0,2) + P(1,0,1) + P(2,0,0) = 3/9$

$f_{X|Y=0}(2) = \frac{f_{XY}(2,0)}{f_Y(0)}$

$f_{X|Y=0}(2) = \frac{1/9}{3/9} = \boxed{1/3}$

(3) $1/4 + 1/4 + 1/8 + 1/8 + k = 1$ $k = 1 - (2/4 + 1/4) = 1 - 3/4 = 1/4$

$f_{Y|X=1}(2) = \frac{f_{YX}(2,1)}{f_X(1)} = \frac{1/4}{1/8 + 1/8 + 1/4} = \frac{1/4}{1/2} = \frac{1}{4} \times 2 = \boxed{\frac{1}{2}}$

(5) $f_Y(1) = \sum_{t' \in T_X} f_{XY}(t', 1)$ $f_X(1) = 12/13$ $f_X(0) = 1/13$

$Y \sim Bin(n, p)$

$f_{Y|X=1}(1) =$ $n=3$, $p = 7/20$

$= \binom{3}{1}\left(\frac{7}{20}\right)\left(\frac{13}{20}\right)^2$

$= \frac{3 \times 7 \times (13)^2}{(20)^3}$

$= 0.443625$

$f_{Y|X=0}(1) =$ $n=2$, $p = 4/15$

$= \binom{2}{1}\left(\frac{4}{15}\right)\left(\frac{11}{15}\right)$

$= \frac{2 \times 4 \times 11}{(15)^2}$

$= 0.39111$

$f_{XY}(0,1) = f_{Y|X=0}(1) \cdot f_X(0)$
$= 0.03008547$

$f_{XY}(1,1) = f_{Y|X=1}(1) \cdot f_X(1)$
$= 0.4095$

$f_Y(1) = f_{XY}(0,1) + f_{XY}(1,1) \approx \boxed{0.4346}$

(7) $f_{XY}(x,y) = a(bx+y)$

$x \backslash y$	0	1	2	3
0	0	a	2a	3a
1	ab	ab + a	ab + 2a	ab + 3a
2	2ab	2ab + a	2ab + 2a	2ab + 3a
	3ab	3ab + 2a	3ab + 4a	

$3\left(12a + 3ab = \frac{3}{7}\right) = 36a + 9ab = \frac{9}{7}$

$2\left(6a + 9ab = \frac{4}{7}\right) = 12a + 18ab = \frac{8}{7}$

$-12a - 3ab = -\frac{3}{7}$

$15ab = \frac{5}{7}$

$36a + 9ab = \frac{9}{7}$

$-6a - 9ab = -\frac{4}{7}$

$30a = \frac{5}{7}$

$\boxed{a = \frac{1}{42}}$

$ab = \frac{1}{21}$

$\frac{b}{42} = \frac{1}{21} \Rightarrow \boxed{b = 2}$

$f_{XY}(1,1) = \frac{1}{42}(2+1) = \boxed{\frac{1}{14}}$

(8) $f_{Y|X=2}(0) = \frac{f_{YX}(0,2)}{f_X(2)}$

$f_X(2) = \binom{6}{2}(1/2)^6$

$f_{X|Y=0}(2) = \binom{5}{2}(1/2)^5$; $f_Y(0) = 1/2$; $f_{YX}(0,2) = \binom{5}{2}(1/2)^6$

$f_{Y|X=2}(0) = \frac{\binom{5}{2}\cdot(1/2)^6}{\binom{6}{2}\cdot(1/2)^6} = \frac{\binom{5}{2}}{\binom{6}{2}}$

$= \frac{5!}{2!\,3!} \times \frac{2!\,4!}{6!} = \frac{5!}{6!} \times \frac{4!}{3!} = \frac{4}{6} = \boxed{0.666}$

9. $$\frac{\binom{5}{1}\times\binom{4}{1}}{\binom{12}{2}} = \frac{5\times 4}{\frac{12!}{10!\,2!}} = \frac{5\times 4}{\cancel{12}_{3}\times 11}\times 2 = \frac{10}{33} = \boxed{0.30}$$

10. $N \sim Bin(7, 1/2)$ $\quad X \sim Bin(n, 1/2)$

$x \backslash N$	0	1	2	3	4
0	0	$^7C_1(1/2)^8$			$\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11}$
1	0	$^7C_1(1/2)^8$		$\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10}$	
2	0	0	$\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9}$		
3	0	0	0		
4	0	0	0	0	

$$\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11} = \frac{7!}{4!\,3!}\times\frac{1}{2048} = \frac{7\times 6\times 5}{3\times 2}\times\frac{1}{2048} = \frac{35}{2048}$$

$$\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10} = \frac{7!}{3!\,4!}\times\frac{3}{1024} = \frac{210}{2048}$$

$$\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9} = \frac{7!}{2!\,5!}\times\frac{1}{512} = \frac{7\times 6}{2}\times\frac{1}{512} = \frac{84}{2048}$$

$$\text{Total} = \frac{35+210+84}{2048} = \frac{329}{2048} = \boxed{0.161}$$

WEEK 2 GA

2.

	0	1	2	
0	0.06	0.18	0.12	0.36
1	0.04	0.12	0.48	0.64
	0.1	0.3	0.6	

	0	1	2	
0	2/24	1/24	1/24	4/24
1	4/24	2/24	3/24	9/24
2	6/24	3/24	2/24	11/24
	12/24	6/24	6/24	

	0	1	2	
0	1/10	2/10	2/10	5/10
1	1/10	1/10	3/10	5/10
	2/10	3/10	5/10	

	0	1	
0	1/10	1.5/10	2.5/10
1	2/10	3/10	5/10
2	1/10	1.5/10	2.5/10
	4/10	6/10	

3. $X \sim \text{Bernoulli}(0.2)$ $\quad Y \sim \text{Bernoulli}(0.4)$ $\qquad Z = X + Y$

$$f_{X|Z=1}(1) = \frac{P(X=1, Z=1)}{P(Z=1)} = \frac{P(X=1, Y=0)}{P(X=1,Y=0 \text{ or } X=0,Y=1)} = \frac{(0.2)(0.6)}{(0.2)(0.6)+(0.8)(0.4)} = \frac{0.12}{0.12+0.32} = \boxed{0.2727}$$

4. $Z = X + Y$ $\qquad f_{XY}(x,y) = \dfrac{9}{16\cdot(4)^{x+y}}$

$$f_Z(k) = P(X=x, Y=k-x) = \sum_{x=0}^{k} f_{XY}(x, k-x) = \boxed{\frac{(k+1)\cdot 9}{16\cdot(4)^k}}$$

5. $Z = \max(X, Y)$

$$f_Z(k) = P(X=k, Y\le k \text{ or } X\le k, Y=k) = \sum_{y=0}^{k} f_{XY}(k,y) + \sum_{x=0}^{k} f_{XY}(x,k)$$

$$= 2\cdot\sum_{x=0}^{k}\frac{9}{16\cdot(4)^{x+x}} =$$

6.

$y \backslash x$	1	2	3	4	5	6
1				*		
2					*	
3						*
4						
5						
6						

$$f_X(4)\cdot f_Y(1) + f_X(5)\cdot f_Y(2) + f_X(6)\cdot f_Y(3)$$

$$= \left(\frac{1}{36}\right)\times 3 = \frac{3}{36} = \frac{1}{12}$$

(7) $X \sim \text{Geometric}(p)$ $Y \sim \text{Geometric}(p)$ iid $Z = X + Y$

$f_Y(k) = p(1-p)^{k-1}$

$f_Z(k) = P(X=n, Y=k-n) = \sum_{n=1}^{k-1} p(1-p)^{n} \cdot p(1-p)^{k-n}$

$= p^2(1-p)^k (k-1)$

$f_Z(18) = p^2(1-p)^{18}(17)$ $f_Z(19) = p^2(1-p)^{19}(18)$

$f_Z(18) < f_Z(19) \Rightarrow \cancel{p^2}\cancel{(1-p)^{18}}(17) < \cancel{p^2}(1-p)^{19}(18)$

$\frac{17}{18} < 1-p$

$\boxed{p < 0.056}$

(8) $X \sim \text{Poisson}(2)$ $f_X(1) = e^{-2}2$ $f_X(2) = e^{-2}2$ $f_X(\neq 1 \text{ or } 2) = 1 - 4e^{-2}$

X_1	X_2	X_3
1	2	
1		2
2	1	
2		1
	2	1
	1	2

$6 \cdot (2e^{-2})^2 (1-4e^{-2}) = 0.439575(1-4e^{-2}) = \boxed{0.2016}$

(9) $X \sim \text{Bernoulli}(0.8)$ $Y \sim \text{Bernoulli}(0.3)$ $Z = X + Y - XY$

$f_Z(1) = P(X=1, Y=0 \text{ or } X=0, Y=1 \text{ or } X=1, Y=1) = 1 - P(X=0, Y=0) = 1-(0.2)(0.7) = 0.86$

$Z \sim \text{Bernoulli}(0.86)$

(10) $X, Y \sim \text{Geometric}(0.8)$ $P(X=1 \mid X+Y=2) = \frac{P(X=1, X+Y=2)}{P(X+Y=2)} = \frac{P(X=1, Y=1)}{P(X=1, Y=1)}$

(11) $X \sim \text{Poisson}(5)$ $Y \sim \text{Poisson}(1)$ $Z = X + Y$

$Z \sim \text{Poisson}(6)$

$Y \mid Z=4 \sim \text{Binomial}(4, 1/6)$

$Y \mid Z=4\,(3) = \binom{4}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right) = \frac{4 \cdot 5}{(6)^4} = \boxed{0.0154}$

WEEK 3 GA

(2) $X \sim \text{Bin}(2, 6/14)$ $Y \sim \text{Bin}(2, 8/14)$ $\rho(X,Y) = \frac{\text{Cov}(X,Y)}{SD(X)SD(Y)}$

Y \ X	0	1	2
0	0	0	30/182
1	0	96/182	0
2	56/182	0	0

$E(X) = 156/182$ $E(X^2) = 216/182$ $\text{Var}(X) = 14976/33124$

$E(Y) = 208/182$ $E(Y^2) = 320/182$ $\text{Var}(Y) = 14976/33124$

$SD(X)SD(Y) = 14976/33124$ $E(XY) = 96/182$

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{96}{182} - \frac{156 \cdot 208}{(182)^2} = \frac{-14976}{33124}$

$\rho(X,Y) = \frac{-14976}{33124} \times \frac{33124}{14976} = \boxed{-1}$

(3) $8^{th} \rightarrow P(G) = 0.4$ $G_8 \sim \text{Bin}(2, 0.4)$

$9^{th} \rightarrow 0.4$ $G_9 \sim \text{Bin}(2, 0.4)$

$10^{th} \rightarrow 0.6$ $G_{10} \sim \text{Bin}(2, 0.6)$

$E(G_{total}) = E(G_8) + E(G_9) + E(G_{10})$

$E(G_8) = E(G_9) = 0.8$ $E(G_{10}) = 1.2$

$E(G_{total}) = 0.8 + 0.8 + 1.2 = \boxed{2.8}$

⑤ $\sigma = \sqrt{2}$ $7 = \mu - k\sigma \Rightarrow \left|\frac{7-15}{\sqrt{2}}\right| = k \Rightarrow k \simeq 5.66$ $P(7 < X < 23) = 1 - \frac{1}{(5.66)^2} \simeq 0.969$

⑥ $X \sim \text{Geometric}(0.04)$ $Var(X) = (1-p)/p^2 = 600$

⑧ $Var(XY) = Var(X) \cdot Var(Y) + Var(Y)(E(X))^2 + Var(X)(E(Y))^2$

$= 15 + 3969 + 605 = 4589$

⑨

X	Y	Z	P(X,Y,Z)	R
0	1	2	0.4	4
0	2	3	0.3	6
1	0	−2	0.3	−4

$R = XY + 2Z$

$E(R) = 2.2$ $E(R^2) = 12.4$

$Var(R) = 12.4 - (2.2)^2 = 7.56$

⑩ both losing $= \left(\frac{498 \times 497}{500 \times 499}\right) \times ₹0$

one wins 500 $= \left(\frac{1}{500} \times \frac{498}{499} \times 2\right) \times ₹500 = 1.99$

one wins 2000 $= \left(\frac{1}{500} \times \frac{498}{499} \times 2\right) \times ₹2000 = 7.98$

both tickets win $= \left(\frac{2}{500} \times \frac{1}{499}\right) \times ₹2500 = 0.02$

total = 10

⑪

$P(UU) = (0.4)(0.6)$	Gain(UU) = 2400	→ 576	1050
$P(UD) = (0.4)(0.4)$	" (UD) = 150	→ 24	300
$P(DU) = (0.6)(0.6)$	" (DU) = −1200	→ −432	−450
$P(DD) = (0.6)(0.4)$	" (DD) = −2550	→ −612	−1200

+

−36
48
−162

MOCK WEEK 1-2

①

X	Y	S	Z_1	Z_2
1	1	2	1	1
1	2	3	2	1
2	1	3	2	1
2	2	4	2	2
3	1	4	3	1
3	2	5	3	2

S	2	3	4	5
P(S)	0.1	0.4	0.4	0.1

Z_2	1	2
$P(Z_2)$	0.6	0.4

Z_1	1	2	3
$P(Z_1)$	0.1	0.7	0.2

④

y\x	0	1	2	
0	2/24	3/24	1/24	1/4
1	6/24	9/24	3/24	3/4
	1/3	1/2	1/6	

$P(Y=1 \mid X=2) = \frac{3}{24} \times 6 = \frac{3}{4}$

$P(X \le 1, Y=0) = 5/24$ $P(X=2, Y=1) = 3/24$

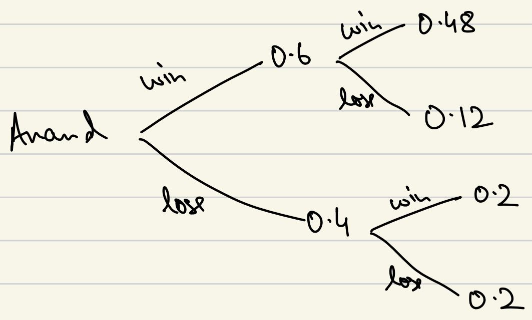
sum $= 8/24 = 1/3$ $\frac{1}{\text{sum}} = \boxed{3}$

(6) $X \sim \text{Poisson}(2) \quad Y \sim \text{Poisson}(3) \quad Z = X+Y \qquad Z \sim \text{Poisson}(5) \qquad X|Z=5 \sim \text{Bin}(5, 2/5)$

$Y|Z=5 \sim \text{Bin}(5, 3/5)$

$$P(X=1|Z=5) = \binom{5}{1}\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^4 = 0.2592$$

(8)



$X_2 \backslash X_1$	0	1	
0	0.2	0.12	0.32
1	0.2	0.48	0.68
	0.4	0.6	

(11) $P(X=3, Y=3) = P(N=6)$ and $P(X=3)$

$$P(N=6) = \frac{e^{-5}5^6}{6!} = 0.146223$$

$$P(X=3|N=6) = \binom{6}{3}(0.3)^3(0.7)^3 = 20\,(0.21)^3 = 0.18522$$

$$P(X=3, N=6) = \boxed{0.0271}$$

(13) $$P(X_1=0|Y=8) = \frac{P(X_1=0, Y=8)}{P(Y=8)} \qquad P(Y=8) = \binom{10}{8}\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)^2$$

$$P(X_1=0, Y=8) = (1/3)\cdot\binom{9}{8}\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)$$

$$\frac{\cancel{(1/3)^2}\,\cancel{(2/3)^8}\;{}^9C_8}{\cancel{(1/3)^2}\,\cancel{(2/3)^8}\;{}^{10}C_8} = \frac{9!}{8!}\times\frac{8!\,2!}{10!} = \frac{2}{10}$$

(14) $$f_{Z|Y=15}(50) = \frac{P(X=10, Y=5 \text{ or } X=5, Y=10)}{f_{XY}(10,5) \text{ or } (5,10) \text{ or } (7,8) \text{ or } (8,7) \text{ or } (6,9) \text{ or } (9,6)}$$

$$f_{XY}(10,5) = f_{XY}(5,10) = \binom{10}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5\cdot\binom{10}{10}\left(\frac{1}{3}\right)^{10} = \binom{10}{5}\left(\frac{1}{3}\right)^{15}\left(\frac{2}{3}\right)^5 \qquad \frac{10!}{5!\,5!} = \frac{\cancel{10}^{2}\times 9\times\cancel{8}^{2}\times 7\times\cancel{6}}{\cancel{5\times4\times3\times2}} = 28\times 9$$

$$f_{XY}(6,9) = f_{XY}(9,6) = \binom{10}{6}\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^4\cdot\binom{10}{9}\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right) = \binom{10}{6}\binom{10}{9}\left(\frac{1}{3}\right)^{15}\left(\frac{2}{3}\right)^5 \qquad \frac{10!\times 10!}{6!\,4!\,9!} = \frac{10\times\cancel{9}^{3}\times\cancel{8}\times 7}{\cancel{4}\times\cancel{3}\times\cancel{2}}\times 10 = 100\times3\times7$$

$$f_{XY}(7,8) = f_{XY}(8,7) = \binom{10}{8}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^2\cdot\binom{10}{7}\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^3 = \binom{10}{8}\binom{10}{7}\left(\frac{1}{3}\right)^{15}\left(\frac{2}{3}\right)^5 \qquad \frac{10!\times10!}{8!\,2!\,7!\,3!} = \frac{10\times 9}{\cancel{2}}\times\frac{10\times\cancel{9}^{3}\times\cancel{8}^{2}}{\cancel{3}\times\cancel{2}} = 600\times 9$$

$$\frac{\binom{10}{5}\cancel{\left(\frac{1}{3}\right)^{15}}\cancel{\left(\frac{2}{3}\right)^5}}{\cancel{\left(\frac{1}{3}\right)^{15}}\cancel{\left(\frac{2}{3}\right)^5}\left[\binom{10}{5}+\binom{10}{6}+\binom{10}{8}\right]} = \frac{\frac{10!}{5!\,5!}}{\frac{10!}{5!\,5!}+\frac{10!}{6!\,4!}+\frac{10!}{8!\,2!}}$$

(15) $X \sim \text{Uniform}\{0,1,\dots,9\} \qquad Y \sim \text{Uniform}\{0,1,\dots,9\}$

$$P(Z \geq 2) = 1 - P(Z=1) - P(Z=0)$$

$P(Z=0) \rightarrow X = Y = 1/10 \qquad P(Z=1) \rightarrow |X-Y| = 1 = 18/100$

$$P(Z\geq 2) = 1 - \frac{28}{100} = \frac{72}{100}$$

01	43	78
10	45	87
12	54	89
21	56	98
23	65	
32	67	
34	76	

WEEK 4 GA

(2) $P(-19 < X < 16) = P(X < 16) = 1 - \exp(-832)$

(3) $X \sim \text{Exponential}(\lambda = 1)$ $\quad k = \lambda = 1$

(4) $P(45 < X < 48) = F_X(48) - F_X(45)$ $\quad F_X(48) = 1 - \exp(-48)$ $\quad F_X(45) = 1 - \exp(-45)$

$= 1 - \exp(-48) - 1 + \exp(-45)$

$= \exp(-45) - \exp(-48) = e^{-45} - e^{-48}$

(5) $X \sim \text{Exponential}(\lambda = 1/900)$ $\quad P(X > 700) = 1 - P(X \leq 700) = \exp(-700/900)$

$= \exp(-0.78)$

(6) $f_X(x) = 5x^4$ $\quad F_X(x) = x^5$ $\quad P(X < 6/9 \mid X > 1/9) = \dfrac{P(1/9 < X < 6/9)}{P(X > 1/9)} = \dfrac{F_X(6/9) - F_X(1/9)}{1 - F_X(1/9)}$

$= \dfrac{(6/9)^5 - (1/9)^5}{1 - (1/9)^5} = \dfrac{(6)^5 - 1}{\cancel{(9)^5}} \times \dfrac{\cancel{(9)^5}}{(9)^5 - 1} = \dfrac{(6)^5 - 1}{(9)^5 - 1} = \dfrac{7775}{59048} \simeq 0.132$

(7) $Z = \dfrac{X - 206}{35} \Rightarrow -1 = \dfrac{X - 206}{35} \Rightarrow X = -35 + 206 = 171$