

MATH 2



WEEK 1

→ Vectors

- ↳ can be thought of as a list.
- ↳ can be in a row or a column form.
- ↳ Addition of vectors:

Add the corresponding entries. For e.g. $\rightarrow (3, 5) + (2, 4) = (5, 9)$

↳ Scalar multiplication:

$$\text{e.g. } \rightarrow 2(8, 8, 10, 5) = (16, 16, 20, 10)$$

↳ Visualisation of a vector

Point $(a, b) \equiv$ Vector $(a, b) \equiv a\hat{i} + b\hat{j}$ where, $\begin{matrix} \uparrow \\ \hat{i} \end{matrix} \rightarrow \text{one unit in } x\text{-axis}$ $\begin{matrix} \leftrightarrow \\ \hat{j} \end{matrix} \rightarrow \text{one unit in } y\text{-axis}$

$$\text{e.g.: Point } (-1, -1) = -\hat{i} - \hat{j}$$

- ↳ Vectors in \mathbb{R}^n are lists with n real entries.

→ Matrices

- ↳ rectangular array of numbers
- ↳ (rows × columns); e.g. $\begin{bmatrix} 5 & 7 & 10 \\ 3 & 5 & 2 \end{bmatrix}$ is a 2×3 matrix.

↳ (1, 2)ith entry $\rightarrow 7$

- ↳ square matrix $\rightarrow N \times N$

- ↳ Diagonal matrix \rightarrow all entries are 0 except the diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

- ↳ Scalar matrix \rightarrow all entries have the same value

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- ↳ Identity matrix \rightarrow denoted by 'I'; scalar matrix with values = 1

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ↳ Addition of matrices \rightarrow must be of the same size:

$$\begin{bmatrix} 1 & 0 \\ 5 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 8 & 3 \\ 7 & 4 \end{bmatrix}$$

- ↳ Scalar multiplication \rightarrow multiply each number with the scalar.

- ↳ Matrix multiplication $\rightarrow A \times B = C$; $C[i, j] = \sum_{k=1}^n A[i, k] \times B[k, j]$

↳ no. of columns in first matrix must = no. of rows in 2nd matrix

$$A_{m \times n} \times B_{n \times p} = (AB)_{m \times p}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

- ↳ Scalar multiplication is the same as multiplication by the scalar matrix

$$\text{ex. } \rightarrow \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

- ↳ Properties:
 - ① $(A + B) + C = A + (B + C)$
 - ② $(AB)L = A(BL)$
 - ③ $A + B = B + A$
 - ④ $AB \neq BA$

- ⑤ $\lambda(A + B) = \lambda A + \lambda B$
- ⑥ $\lambda(AB) = (\lambda A)B = A(\lambda B)$
- ⑦ $A(B + C) = AB + AC$
- ⑧ $(A + B)C = AC + BC$

→ System of Linear Equations

↪ collection of one or more linear equations involving the same set of variables.

↪ system of m linear equation with n variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

...

...

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

↪ system of lin. eq. in matrix form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

coefficient matrix
column of variables
column of resulting values

$$AX = b$$

, where $A = m \times n$ matrix

x = column vector with n entries

b = column vector with m entries

↪ Solutions to a system of lin. eq.:

- ① Infinite solution
- ② Single unique solution
- ③ No solution

→ Determinant

$$\hookrightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(A) = ad - bc$$

$$\text{e.g. } A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix} \quad \det(A) = 20 - 18 = 2$$

$$\hookrightarrow A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \det(A) = a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

↪ Determinant of Identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rightarrow \det(I_2) = 1$$

$$\rightarrow \det(I_3) = 1$$

↪ Determinant of a product of matrices

$$\hookrightarrow \det(AB) = \det(A) \cdot \det(B) ; \quad \det(ABC) = \det(A) \cdot \det(B) \cdot \det(C)$$

$$\hookrightarrow \det(A^n) = \det(A)^n \quad \hookrightarrow \det(A^{-1}) = \det(A)^{-1}$$

↪ Determinant of the inverse of a matrix

$$AA^{-1} = I \Rightarrow \det(AA^{-1}) = \det(I)$$

$$\hookrightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

↪ Switching rows

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\det(\tilde{A}) = cb - ad = -(ad - bc) = -\det(A)$$

$$\det(\tilde{A}) = -\det(A)$$

↪ Add multiple of a row/column to another row/column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$$

$$\det(\tilde{A}) = (a+tc)d - (b+td)c = ad + tcd - bc - tcd$$

$$\det(\tilde{A}) = \det(A)$$

↳ Scalar multiplication of a row / column

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} ta & b \\ tc & d \end{bmatrix}$$

$$\det(\tilde{A}) = t \cdot \det(A)$$

↳ Upper / Lower triangle matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{bmatrix} \quad \text{upper triangle matrix}$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 0 \\ 3 & 4 & 9 \end{bmatrix} \quad \text{lower triangle matrix}$$

→ determinant is the product of diagonal elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \det(A) = a_{11} \cdot a_{22} \cdot a_{33}$$

↳ Transpose of a matrix and its determinants

→ Transpose of $A_{m \times n} = A^T_{n \times m}$ with (i,j) -th entry A_{ji}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3 \times 2} \quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2 \times 3}$$

$$\det(A) = \det(A^T)$$

↳ Minors and cofactors

→ Minor of the entry in i -th row and j -th column is the determinant of the submatrix formed by deleting i -th row and j -th column.

e.g. → $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$(1,1)$ -th minor; denoted by M_{11}

$M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

→ Cofactor (i,j) -th cofactor; $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$C_{11} = M_{11}; \quad C_{23} = -M_{23}$$

→ For $A_{3 \times 3}$

$$\det(A) = (a_{11} \times C_{11}) + (a_{12} \times C_{12}) + (a_{13} \times C_{13})$$

For $A_{4 \times 4}$

$$\det(A) = \sum_{j=1}^4 a_{1j} C_{1j}$$

$$\text{For } A_{n \times n}; \quad \det(A) = \sum_{j=1}^n a_{1j} C_{1j}$$



WEEK 2

→ Determinant (continued)

↳ Expansion along any row / column

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } i$$

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \cdot M_{ij} \quad \text{for a fixed } j \quad (\text{determinant along a column})$$

↳ Properties :

① Determinant of a product is product of the determinants.

$$\hookrightarrow \det(AB) = \det(BA) = \det(A) \cdot \det(B)$$

② Switching two rows or columns changes the sign.

③ Adding multiples of a row to another row leaves the determinant unchanged.

④ Scalar multiplication of a row/column by a constant t multiplies the determinant by t .

$$⑤ \det(tA_{nn}) = (t)^n \det(A)$$

↳ Useful computational tips:

① The determinant of a matrix with a row or a column of zeros is 0.

② The determinant of a matrix in which one row (or column) is a linear combination of other rows (resp. columns) is 0.

③ Scalar multiplication of a row/column by a constant t multiplies the determinant by t .

④ While computing the determinant, you can choose to compute it using expansion along a suitable row or a column.

→ Cramer's rule

$$\rightarrow A_n = b, \text{ where } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\text{Cramer's rule} \Rightarrow A_{u_1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{21} \end{bmatrix}, A_{u_2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

$$\hookrightarrow \text{solutions for } u_1 \text{ & } u_2 \Rightarrow u_1 = \frac{\det(A_{u_1})}{\det(A)} ; u_2 = \frac{\det(A_{u_2})}{\det(A)}$$

→ For a 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A_{u_1} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$A_{u_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$

$$A_{u_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

$$u_1 = \frac{\det(A_{u_1})}{\det(A)}$$

$$u_2 = \frac{\det(A_{u_2})}{\det(A)}$$

$$u_3 = \frac{\det(A_{u_3})}{\det(A)}$$

→ Inverse Matrix

↳ The inverse of a matrix $A \rightarrow (A^{-1}) \Rightarrow A \cdot (A^{-1}) = I$

↳ if $\det(A) \neq 0$, then the matrix is invertible

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

↳ Adjugate of a matrix

M_{ij} = determinant of submatrix after deleting i-th row and j-th column.

$$C_{ij} = (-1)^{i+j} M_{ij}$$

→ Adjugate of a matrix \rightarrow Transpose of cofactor matrix

$$\text{Adj}(A) = C^T$$

$$\rightarrow A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

$\rightarrow A_{n \times n}$, then $\det(\text{adj}(A)) = (\det(A))^{n-1}$

↳ Solution to system of linear equations using inverse matrix.

$$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$$

→ Homogeneous System of Linear Equations

↳ $Ax = 0$, where $b = 0 \Rightarrow Ax = 0$

→ has unique solution 0 if A is invertible

→ has infinite solutions if A is not invertible

→ The Row Echelon Form (reduced form)

↳ Matrix is in row echelon form if:

- first non-zero element in each row (leading entry) is 1
- each leading entry is in a column to the right of the leading entry in the previous row.
- rows with all zero elements, if any, are below rows having a non-zero element.

e.g. $\rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

→ reduced row echelon form \rightarrow For a non-zero row, the leading entry in the row is the only non-zero entry in its column

e.g. $\rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

→ e.g. solution for $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow Ax = b$

$$\begin{aligned} x_1 + 2x_2 &= b_1 \Rightarrow x_1 = b_1 - 2x_2 \\ x_3 &= b_2 \\ x_4 &= b_3 \end{aligned}$$

solution $\Rightarrow x = \begin{bmatrix} b_1 - 2b_3 \\ b_2 \\ b_2 \\ b_3 \end{bmatrix}$

→ If the i-th column has the leading entry of some row, we call x_i a dependent variable.
If the i-th column does not have the leading entry of any row, $x_i \rightarrow$ independent variable.

→ Row reduction

Elementary Row Operations

Type	Action	Example and notation	Effect on determinant
1 Interchange two rows		$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = -\det(B)$
2 Scalar multiplication of a row by a constant t		$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = t \cdot \det(B)$
3 Adding multiples of a row to another row		$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_3} \begin{bmatrix} 3 & -19 & -2 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$	$\det(A) = \det(B)$

↳ Row echelon form

Steps : ① Find the left most non-zero column.
 ② Use operations to get 1 in the top of that column.
 ③ Make entries below that 1 to 0.
 ④ Repeat the steps for the next row and onwards.

↳ Reduced row echelon form

↳ start from the right most column and make all the non-leading terms 0s.

↳ e.g. $A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 2 & 1/2 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 2 & 1/2 \\ 5 & 6 & 9 \end{bmatrix} \xrightarrow{R_3 - 5R_1} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 2 & 1/2 \\ 0 & -4 & 13/2 \end{bmatrix} \xrightarrow{R_2/2} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & -4 & 13/2 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 35 \end{bmatrix} \xrightarrow{R_3/35} \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - \frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 35 \times 2 = 70$$

→ Gaussian Elimination Method

↳ Augmented matrix → matrix of size $(m \times n+1)$

↳ First n columns are from A and the last column is b .

↳ denoted by $[A|b]$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$

↳ Bring the matrix A to reduced row echelon form.

→ let R be the submatrix of obtained matrix of the first n columns and c be the submatrix of the obtained matrix of the last column.

$[R|c]$, where $R \rightarrow$ reduced row echelon form of A

Solutions of $Au=b$ are precisely the solutions of $Ru=c$.

→ Homogeneous system of linear equations

↳ 0 is always a solution a.k.a. trivial solution

↳ If there are more variables than equations, (i.e., more columns than rows) there will be infinite solutions because there must be some independent variables.



WEEK 3

→ Properties of vectors Let v, w and v' be vectors in \mathbb{R}^n and $a, b \in \mathbb{R}$

- i) $v + w = w + v$
- ii) $(v + w) + v' = v + (w + v')$
- iii) The 0 vector satisfies: $v + 0 = 0 + v = v$
- iv) The vector $-v$ satisfies: $v + (-v) = 0$
- v) $1v = v$
- vi) $(ab)v = a(bv)$
- vii) $a(v+w) = av + aw$
- viii) $(a+b)v = av + bv$

↳ Vector space → a set with two operations (addition and multiplication)

$$\begin{array}{ccc} + : V \times V \rightarrow V & \text{and} & \cdot : \mathbb{R} \times V \rightarrow V \\ \text{for each pair of elements } v_1, v_2 \text{ in } V, \text{ there is} & & \text{for each } c \in \mathbb{R} \text{ and } v \in V \text{ there is a unique} \\ \text{a unique element } v_1 + v_2. & & \text{element } cv \text{ in } V. \end{array}$$

↳ Example of vector space → solutions of a homogeneous system $Ax=0$

↳ non-examples:

① Define addition and scalar multiplication in \mathbb{R}^2 as:

$$\begin{aligned} \rightarrow (u_1, u_2) + (y_1, y_2) &= (u_1 + y_1, u_2 - y_2) \\ \rightarrow c(u_1, u_2) &= (cu_1, cu_2) \end{aligned}$$

$$v + w \neq w + v \rightarrow \text{i), iii) and viii) fail}$$

$$\textcircled{2} \rightarrow (u_1, u_2) + (y_1, y_2) = (u_1 + y_1, 0)$$

$$\rightarrow c(u_1, u_2) = (cu_1, 0) \rightarrow \text{iii), iv) and v) fail}$$

→ Cancellation law of vector addition

↳ If $v_1, v_2, v_3 \in V$ such that $v_1 + v_3 = v_2 + v_3$, then $v_1 = v_2$

↳ Corollaries: ① The vector 0 described in iii) is unique. $\rightarrow v + w = v + 0 \Rightarrow w = 0$
 ② The vector v' " " iv) " " and referred to as $-v$.
 $\rightarrow v + v' = 0 = v + v'' \Rightarrow v' = v''$

→ Some more properties:

$$\textcircled{1} 0v = 0 \text{ for each } v \in V \rightarrow (0+0)v = 0v + 0v \Rightarrow 0v = 0v + 0v \Rightarrow 0v + 0 = 0v + 0v$$

$$\textcircled{2} (-c)v = -cv = c(-v) \text{ for each } c \in \mathbb{R} \text{ and for each } v \in V$$

$$\rightarrow (c + (-c))v = cv + (-c)v \Rightarrow 0 = cv + (-c)v \Rightarrow -cv = (-c)v$$

$$\textcircled{3} c0 = 0 \text{ for each } c \in \mathbb{R}$$

→ Linear combinations:

↳ Let V be a vector space and $v_1, v_2, \dots, v_n \in V$.

Linear combinations of v_1, v_2, \dots, v_n with coefficients $a_1, a_2, \dots, a_n \in \mathbb{R}$ is the vector $\sum_{i=1}^n a_i v_i \in V$.

↳ example: $(2, 9)$ is a linear combination of vectors $(2, 1)$ and $(-2, 3)$:

$$3(2, 1) + 2(-2, 3) = (6, 3) + (-4, 6) = (2, 9)$$

$$\rightarrow 3(2, 1) + 2(-2, 3) - (2, 9) = 0$$

↳ 0 vector is a linear combination of $(2, 1), (-2, 3)$, and $(2, 9)$ with non-zero coefficients

↳ The plane of the two vectors $(0, 2, 1)$ and $(2, 2, 0)$ can be expressed by the equation $2x - 2y + 4z = 0$

↳ $(1, 2, 0)$ does not lie on the plane

↳ $(3, 7, 2)$ does lie on this plane

→ Linear Dependence

↪ a set of vectors $v_1, v_2, v_3, \dots, v_n \in V$ is said to be linearly dependent if there exists scalars $a_1, a_2, a_3, \dots, a_n$, not all zeros, such that $\rightarrow a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n = 0$

↪ If a set of vectors are linearly dependent, then every superset of it is linearly dependent.
↪ If v_1, v_2, v_3 are " ", then v_1, v_2, v_3, v_n are also linearly dependent.

→ Linear Independence

↪ a set of vectors if they are not linearly dependent

↪ $v_1, v_2, v_3, \dots, v_n$ are linearly independent if : $a_1v_1 + a_2v_2 + a_3v_3 + \dots + a_nv_n = 0$
can only be satisfied when $a_i = 0$ for all $i = 1, 2, 3, \dots, n$

↪ If a set of vectors v_1, v_2, \dots, v_n contain a vector $v_i = 0$, then the set is always linearly dependent.

↪ If two non-zero vectors are multiples of each other, then are linearly dependent.

$$a_1v_1 + a_2v_2 = 0 \Rightarrow c = -a_2/a_1, \text{ then } v_1 = cv_2$$

↪ Two vectors are linearly independent when they are not multiples of each other.

↪ for three vectors v_1, v_2, v_3

$$\text{Suppose } a_1v_1 + a_2v_2 + a_3v_3 = 0 \Rightarrow v_1 = -a_2/a_1 v_2 - a_3/a_1 v_3$$

↪ v_1 is a linear combination of v_2 and v_3

↪ If three vectors are linearly independent then none of these vectors
is a linear combination of the other two.

→ If you have more than n vectors in \mathbb{R}^n , then the homogeneous system $Vu=0$ has infinitely many solutions. → They are linear dependent

↪ This corresponds to having more variables than there are equations.

→ V is an $n \times n$ matrix obtained by arranging a set of vectors in columns.

$Vu=0 \rightarrow$ homogeneous system

↪ square matrix, therefore $u=0$ if and only if V is invertible
if and only if $\det(V) \neq 0$



WEEK 1 GA

① $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A - xI = \begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix}$

$$\det(A - xI) = (a-x)(d-x) - bc$$

$$= ad - ax - dx + x^2 - bc$$

$$= \underbrace{ad - bc}_{\det(A)} - \underbrace{x(a+d)}_{\text{trace}(A)} + \underbrace{x^2}_{c^2}$$

② $3A = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$ $\det(3A) = 9ad - 9bc = (3)^2 \det(A)$

$$3A = \begin{bmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$$

$$|3A| = 3^3 |A| = \boxed{27|A|}$$

③ $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $I + A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $5A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$

$$5A + I = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$$

$$(I + A)^3 - (5A + I) = mA$$

$$\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \quad m = 8$$

④ $A = \begin{bmatrix} 20 & 30 & 40 \\ 8 & 16 & 24 \\ 8 & 10 & 12 \end{bmatrix}$ $\det(A) = 20 \det \begin{bmatrix} 16 & 24 \\ 10 & 12 \end{bmatrix} - 30 \det \begin{bmatrix} 8 & 24 \\ 8 & 12 \end{bmatrix} + 40 \det \begin{bmatrix} 8 & 16 \\ 8 & 10 \end{bmatrix}$

$$= 20(-48) - 30(-96) + 40(-48) = -960 + 2880 - 1920 = \boxed{0}$$

⑤ $\det(A) = 3$ $\det(B) = 3$ $\det(B^{-1}) = \frac{1}{3}$

$$\det(3A^2B^{-1}) = (3)^3 \cdot (\det(A))^2 \cdot \left(\frac{1}{3}\right) = 9(3)^2 = \boxed{81}$$

⑥ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ $A^3 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix}$ $A^6 = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 27 & 27 \\ 27 & 27 & 27 \\ 27 & 27 & 27 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 729 & 729 & 729 \\ 729 & 729 & 729 \\ 729 & 729 & 729 \end{bmatrix} \quad \text{sum of diagonals} = \boxed{2187}$$

⑦ $A = \begin{bmatrix} 12 & 19 & 26 \\ 17 & 24 & 31 \\ 22 & 29 & 36 \end{bmatrix}$ $\det(A) = 12 \det \begin{bmatrix} 24 & 31 \\ 29 & 36 \end{bmatrix} - 19 \det \begin{bmatrix} 17 & 31 \\ 22 & 36 \end{bmatrix} + 26 \det \begin{bmatrix} 17 & 24 \\ 22 & 29 \end{bmatrix}$

$$= 12(-35) - 19(-70) + 26(-35) = \boxed{0}$$

⑧ $30u_1 + 20u_2 + 25u_3 = 670 - \textcircled{1}$
 $20u_1 + 35u_2 + 25u_3 = 730 - \textcircled{2}$
 $20u_1 + 10u_2 + 15u_3 = 400 - \textcircled{3}$

$$\textcircled{1} - 2 \cdot \textcircled{3} \Rightarrow 30u_1 + 20u_2 + 25u_3 - 20u_1 - 20u_2 - 30u_3 = 670 - 400 - 800$$

$$= -10u_1 - 5u_3 = -130$$

$$\textcircled{2} - \textcircled{3} \Rightarrow 20u_1 + 15u_2 + 10u_3 = 330$$

$$-20u_1 - 10u_2 - 15u_3 = -400$$

$$= 25u_2 + 10u_3 = 330 - \textcircled{4}$$

$$10u_1 + 5u_3 = 130 - \textcircled{5}$$

$$2 \cdot \textcircled{5} = 20u_1 + 10u_3 = 260 \Rightarrow 20u_1 = 260 - 10u_3$$

$$u_1 + u_2 + u_3 = 9 + 10 + 8 = \boxed{27}$$

$$260 - 10u_3 + 10u_2 + 15u_3 = 400$$

$$\Rightarrow 10u_2 + 5u_3 = 140 - \textcircled{6}$$

$$\textcircled{4} - 2 \cdot \textcircled{6} \Rightarrow 0 \ 25 \ 10 \ 330$$

$$0 \ -20 \ -10 \ -280$$

$$= 5u_2 = 50 \quad \boxed{u_2 = 10}$$

⑩ $A = \begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix}$

$$250 + 10u_3 = 330 \Rightarrow \boxed{u_3 = 8}$$

$$10u_1 + 40 = 130 \Rightarrow \boxed{u_1 = 9}$$

X

X

X

WEEK 2 GA

$$\textcircled{1} \quad P(1) = -45 \quad \begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 4 & 2 & 1 & -19 \\ 9 & 3 & 1 & 3 \end{array} \quad R_2 - 4R_1 \quad \begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 0 & -2 & -3 & 161 \\ 0 & -6 & -8 & 408 \end{array} \quad R_2 - \frac{1}{2}R_3 \quad \begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & -6 & -8 & 408 \end{array} \quad R_3 + 6R_2 \quad \begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{array}$$

$$\begin{array}{r|rrr} 1 & 1 & 1 & -45 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & -2 & 150 \end{array} \quad R_1 - R_2 \quad \begin{array}{r|rrr} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 1 & -75 \end{array} \quad R_2 - R_3 \quad \begin{array}{r|rrr} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 32 \\ 0 & 0 & 1 & -75 \end{array}$$

$$P(n) = -2n^2 + 32n - 75 \quad P'(n) = -4n + 32$$

$$\frac{-32}{-4} = n \quad n=8$$

$P(8)$ is peak
 $P(8) = 53$

$$\textcircled{4} \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & c & d \end{array} \quad R_2 - 2R_1 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array} \quad R_3 + 3R_2 \quad \begin{array}{r|rrr} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \quad \text{comic book} = ?$$

$\frac{\text{if } c=2}{d=4000} \rightarrow$

$$\text{if } c=2 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 4000 \end{array} \quad R_2 - 2R_1 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -2 & 0 \end{array}$$

$$\text{if } c=7 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 7 & 3000 \end{array} \quad R_2 - 2R_1 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & -1000 \end{array}$$

$$\text{if } c=2 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & 2 & 3000 \end{array} \quad R_2 - 2R_1 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 1000 \end{array} \quad R_3 - 3R_2 \quad \begin{array}{r|rrr} 1 & 2 & 1 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 1000 \end{array} \Rightarrow \begin{pmatrix} 400 \\ 200 \\ 200 \end{pmatrix}$$

$$\textcircled{6} \quad (-\frac{1}{2}) + (-\frac{1}{2}) + \alpha/6 = -1 \Rightarrow \alpha = 0$$

$$(-\frac{1}{3}) + b/6 = 0 \Rightarrow \frac{b}{6} = \frac{1}{3} \Rightarrow b = 2$$

$$(\frac{1}{2}) + (-\frac{1}{6}) + c/6 = 0 \Rightarrow \frac{c-1}{6} = -\frac{1}{2} \Rightarrow c-1 = -3 \Rightarrow c = -2$$

$$\alpha + b + c = 0$$

$$\textcircled{7} \quad \begin{array}{r|rrr} 1 & 3k & 3k+4 & 61 \\ 1 & k+4 & 4k+2 & 65 \\ 1 & 2k+2 & 3k+4 & 66 \end{array} \quad R_2 - R_1 \quad \begin{array}{r|rrr} 1 & 3k & 3k+4 & 61 \\ 0 & -2k+4 & k-2 & 4 \\ 0 & -k+2 & 0 & 5 \end{array} \quad R_3 \leftrightarrow R_2 \quad \begin{array}{r|rrr} 1 & 3 & 7 & 61 \\ 0 & 1 & 0 & 5 \\ 0 & 2 & -1 & 4 \end{array} \quad -R_3 + 2R_2 \quad \boxed{\begin{array}{r|rr} 1 & 3 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \left| \begin{array}{r} 61 \\ 5 \\ 6 \end{array} \right.}$$

$\frac{4k+2-3k-4}{k-2} = 1$
 $k=1$

$$\textcircled{8} \quad \begin{array}{r|rrr} 2 & 3 & 5 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 44 \end{array} \quad R_1 - R_2 \quad \begin{array}{r|rrr} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 1 & 1 & 2 & 44 \end{array} \quad R_3 - R_1 \quad \begin{array}{r|rrr} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -43 \\ 0 & 0 & 0 & 44 \end{array} \rightarrow \text{shows no solution}$$

$$\textcircled{9} \quad \begin{array}{r|rrr} 1 & 3 & 0 & 0 \\ 4 & 1 & 5 & 34 \\ 2 & 2 & 7 & 32 \\ 3 & 9 & 0 & 0 \end{array} \quad R_2 - 4R_1 \quad \begin{array}{r|rrr} 1 & 3 & 0 & 0 \\ 0 & -11 & 5 & 34 \end{array}$$

$$R_3 - 2R_1 \quad \begin{array}{r|rrr} 1 & 3 & 0 & 0 \\ 0 & -4 & 7 & 32 \end{array}$$

$$R_4 - 3R_1 \quad \begin{array}{r|rrr} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \quad R_4 \text{ shows } \det(A) = 0$$

$$\textcircled{10} \quad \begin{array}{r|rrr} 7 & 2 & 1 & 8 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{array} \quad R_1 + 2R_3 \quad \begin{array}{r|rrr} 1 & 10 & -3 & 24 \\ 0 & 3 & -1 & 4 \\ -3 & 4 & -2 & 8 \end{array} \quad R_3 + 3R_1 \quad \begin{array}{r|rrr} 1 & 10 & -3 & 24 \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 34 & -11 & 80 \end{array} \quad R_1 - 10R_2 \quad \begin{array}{r|rrr} 1 & 0 & y_3 & \frac{32}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & y_3 & \frac{10y_3}{3} \end{array}$$

$$R_3 - 3R_2 \quad \begin{array}{r|rrr} 1 & 0 & y_3 & \frac{32}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 10y_3 \end{array}$$

$$x = -24$$

$$y = 36 \quad x+y+z = 116$$

$$z = 104$$

$$\textcircled{11} \quad A = \begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix} \quad A^T A = \begin{bmatrix} 2 \\ 10 \\ 2 \\ 8 \end{bmatrix} \begin{bmatrix} 2 & 10 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 20 & 4 & 16 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{bmatrix} \quad R_{1/4} \quad \begin{bmatrix} 1 & 5 & 1 & 4 \\ 20 & 100 & 20 & 80 \\ 4 & 20 & 4 & 16 \\ 16 & 80 & 16 & 64 \end{bmatrix} \quad R_2 - 20R_1 \quad \begin{bmatrix} 1 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\textcircled{12} \quad \begin{array}{cccc|ccccc|ccccc|ccccc} 4 & 2 & 1 & 4 & | & 1 & y_2 & y_4 & 1 & | & 1 & y_2 & y_4 & 1 & | & 1 & y_2 & y_4 & 1 \\ 9 & 3 & 1 & 6 & | & 9 & 3 & 1 & 6 & | & R_2 - 9R_1 & 0 & -3/2 & -5/4 & -3 & | & 0 & 3 & 5/2 & 6 & | & 0 & 3 & 5/2 & 6 \\ 16 & 4 & 1 & 10 & | & 16 & 4 & 1 & 10 & | & R_3 - 16R_1 & 0 & -4 & -3 & -6 & | & 0 & 4 & 3 & 6 & | & 0 & 0 & y_3 & 2 & | & 0 & 0 & 1 & 6 \end{array}$$

$$\begin{array}{l} a = 1 \\ b = -3 \\ c = 6 \end{array}$$

$$\textcircled{13} \quad \left| \begin{array}{cccc|cccccc} 2 & 3 & 0 & 0 & | & 1000 & R_{1/2} & 1 & 3/2 & 0 & 0 & 500 \\ 0 & 3 & 2 & 0 & | & 900 & R_4 - R_1 & 0 & 3 & 2 & 0 & 900 \\ 0 & 0 & 2 & 1 & | & C & & 0 & 0 & 2 & 1 & C \\ 2 & 0 & 0 & 1 & | & 400 & & 0 & -3 & 0 & 0 & -600 \end{array} \right. \quad \left| \begin{array}{cccc|cccccc} 1 & 3/2 & 0 & 0 & 500 & R_{2/3} & 1 & 3/2 & 0 & 0 & 500 \\ 0 & 1 & 3/2 & 0 & 300 & R_4 - 3R_2 & 0 & 1 & 3/2 & 0 & 300 \\ 0 & 0 & 2 & 1 & C & R_1 - \frac{3}{2}R_2 & 0 & 0 & 2 & 1 & C \\ 0 & 3 & 0 & 0 & 600 & & 0 & 0 & 2 & 0 & 300 \end{array} \right. \quad \left| \begin{array}{cccc|cccccc} 1 & 0 & -1 & 0 & 50 & R_{3/2} & 1 & 0 & -1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 200 & R_2 - \frac{1}{3}R_4 & 0 & 1 & 0 & 0 & 200 \\ 0 & 0 & 1 & 1/2 & 150 & -R_4 - 2R_3 & 0 & 0 & 1 & 1/2 & 150 \\ 0 & 0 & 0 & 1 & -300 & & 0 & 0 & 0 & 1 & -300 \end{array} \right.$$

$$(W \cdot N) u_1 = 200$$

$$(E \cdot N) u_2 = 200$$

$$(E \cdot S) u_3 = 150$$

$$(W \cdot S) u_4 = 0$$

$$C = 300$$

$$\min \rightarrow u_4 = 0 \rightarrow u_1 = 200, u_2 = 200, u_3 = 150$$

$$\max \rightarrow u_4 = 300 \rightarrow u_1 = 50, u_2 = 300, u_3 = 0$$

$$(W \cdot N) \rightarrow 400 \rightarrow 100$$

$$(E \cdot N) \rightarrow 900 \rightarrow 600$$

$$(E \cdot S) \rightarrow 300 \rightarrow 0$$

$$(W \cdot S) \rightarrow 300 \rightarrow 0$$

— X — X — X — X —

WEEK 3 GA

$$\begin{aligned} -u &= -1 \\ -2y &= 1 \\ y &= 1 \\ u &= -1 \end{aligned}$$

$$\textcircled{2} \quad P = 30 \quad C = 11 \quad F = 53 \quad G = 213 \quad \text{Type A} = Ba, mi, al \quad \text{Type B} = Ap, mi, al$$

$$A = \begin{bmatrix} 2 & 4 & 6 & 30 \\ 3 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 5 & 100 & 1 & 213 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 6 & 30 \\ 2 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ b & 100 & 1 & 213 \end{bmatrix}$$

$$\textcircled{5} \quad A = \begin{bmatrix} c & 1 & -1 \\ -1 & 0 & -3 \\ -2 & -1 & C \end{bmatrix} \quad \begin{aligned} -1(-c-6) + -3c - 1 &= 0 \\ c+6 - 3c - 1 &= 0 \\ -2c + 5 &= 0 \\ c &= 2.5 \end{aligned}$$

$$\textcircled{6} \quad A = \begin{bmatrix} 7 & 5 & 11 \\ 9 & 6 & 15 \\ 2 & 2 & C \end{bmatrix} \quad \begin{aligned} \det(A) &= 7(6C-30) - 5(9C-30) + 11(18-12) \\ &= 42C - 210 - 45C + 150 + 66 \\ 0 &= -3C + 6 \\ C &= 2 \end{aligned}$$

$$\textcircled{8} \quad (1, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0), (0, 1, 0)$$

$$\textcircled{9} \quad X_0 = (y_3, y_3, y_3) \quad X_1 = X_0 \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} = (\frac{1}{2}, \frac{1}{2}, 0) \quad X_2 = X_1 \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} = (\frac{3}{4}, \frac{1}{4}, 0)$$

$$\textcircled{10} \quad X_0 = (a, b, 1-a-b) \quad X_1 = (a+b/2, 1-a-\frac{b}{2}, 0)$$

$$X_1 = (a, b, 1-a-b) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left(\frac{2a+b}{2}, \frac{b+2-2a-2b}{2}, 0 \right) = \left(\frac{2a+b}{2}, \frac{2-2a-b}{2}, 0 \right)$$

$$X_2 = \left(\frac{2a+b}{2}, \frac{2-2a-b}{2}, 0 \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left(\frac{4a+2b+2-2a-b}{4}, \frac{2-2a-b}{4}, 0 \right) = \left(\frac{2a+b+2}{4}, \frac{2-2a-b}{4}, 0 \right)$$

$$\begin{bmatrix} a & \frac{2a+b+2}{4} \\ b & \frac{2-2a-b}{4} \\ 1-a-b & 0 \end{bmatrix}$$

$$\frac{4au + 2ay + by + 2y}{4} = 0 \quad \textcircled{1} \Rightarrow y(2a + b + 2) = -4au$$

$$\frac{4bu + 2y - 2ay - by}{4} = 0 \quad \textcircled{2} \Rightarrow y(2 - 2a - b) = -4bu$$

$$u - au - bu = 0 \Rightarrow u - bu = au$$

$$\textcircled{1} - \textcircled{2} \Rightarrow \frac{4au - 4bu + 4}{4} = 0 \Rightarrow au - bu + 1 = 0$$

$$\begin{aligned} u - bu - bu + 1 &= 0 \\ u - 2bu + 1 &= 0 \end{aligned}$$

If $b=0$ and $a=0$, $c=1$, then:

$$X_0 = (0, 0, 1) \text{ and } X_2 = (0, 0, 0)$$

$$a_1 X_0 + a_2 X_2 = (0, 0, 0) \text{ if } a_1 = 0 \text{ and } a_2 = 1$$

— X — X — X — X —

