

STATS 2



WEEK 1

→ Joint PMF of two discrete random variables
↳ Two events can either be independent; or
↳ one could be influencing the other.

→ Types of PMF of Multiple Random Variables:
① Joint PMF
② Marginal PMF
③ Conditional PMF

① Joint PMF

→ X and Y → discrete random variables defined in the same probability space

Range of X and Y → \mathcal{T}_X and \mathcal{T}_Y
↳ Joint PMF of X and Y → f_{XY} → function from $\mathcal{T}_X \times \mathcal{T}_Y$ to $[0, 1]$

$$f_{XY}(t_1, t_2) = P(X=t_1 \text{ and } Y=t_2), t_1 \in \mathcal{T}_X, t_2 \in \mathcal{T}_Y$$

→ Usually written as a table or a matrix

↳

		variable 1					
		P(X, X ₁)	P(X, X ₂)	P(X, X ₃)
	
	
	
	

② Marginal PMF

→ To obtain the individual PMF of a random discrete variable from a joint PMF.

↳ Suppose X and Y random discrete variables, and their joint PMF $\rightarrow f_{XY}$, then:

$$f_X(t) = P(X=t) = \sum_{t' \in \mathcal{T}_Y} f_{XY}(t, t')$$

$$f_Y(t) = P(Y=t) = \sum_{t' \in \mathcal{T}_X} f_{XY}(t', t)$$

③ Conditional PMF

→ $f_{X|A}(t) \rightarrow$ PMF of a random variable X conditional on an event A .
 $P(X=t|A), t \in \mathcal{T}_X$

$$f_{X|A}(t) = \frac{P((X=t) \cap A)}{P(A)}$$

→ $f_{Y|X}(t) \rightarrow$ conditional PMF of Y given $X=t$

$$f_{Y|X=t}(t') = P(Y=t'|X=t) = \frac{P(Y=t', X=t)}{P(X=t)} = \frac{f_{XY}(t, t')}{f_X(t)}$$

$$f_{XY}(t, t') = f_{Y|X=t}(t') f_X(t)$$

$$\rightarrow f_{XY}(t_1, t_2) = f_{Y|X=t_1}(t_2) f_X(t_1) = f_{X|Y=t_2}(t_1) f_Y(t_2)$$

Q. Let $N \sim \text{Poisson}(\lambda)$. Given $N=n$, toss a fair coin n times and denote the number of heads obtained by X . What is the distribution of X ?

A. $f_N(n) = \frac{e^{-\lambda} \lambda^n}{n!}; n=0, 1, 2, \dots \quad (X|N=n) \sim \text{Bin}(n, 1/2) \quad f_{X|N=n}(k) = \binom{n}{k} \left(\frac{1}{2}\right)^n$

$$f_{X|N}(k, n) = \frac{e^{-\lambda} \lambda^n}{n!} \cdot \frac{n!}{k!(n-k)!} \cdot \left(\frac{1}{2}\right)^n = \frac{e^{-\lambda} \lambda^n}{k! (n-k)!} \left(\frac{1}{2}\right)^n$$

$$f_X(k) = \sum_{n=k}^{\infty} \frac{e^{-\lambda} \lambda^n}{k! (n-k)!} \left(\frac{1}{2}\right)^n \Rightarrow \frac{e^{-\lambda} \lambda^k}{k! 2^k} \sum_{n=k}^{\infty} \frac{\lambda^{n-k}}{(n-k)! (2)^{n-k}}$$

$$X \sim \text{Poisson}(\lambda/2) \Rightarrow f_X(k) = \frac{e^{-\lambda/2} (\lambda/2)^k}{k!}$$

→ Joint PMF of multiple random variables

↳ $X_1, X_2, X_3, \dots, X_n$ are discrete random variables defined in the same probability space.

$$f_{X_1, X_2, \dots, X_n}(t_1, t_2, \dots, t_n) = P(X_1=t_1, \text{and } X_2=t_2, \text{and } \dots \text{and } X_n=t_n), t_i \in T_{X_i}$$

→ Marginal PMF with multiple random variables

↳ individual random variable $\Rightarrow f_{X_1}(t) = P(X_1=t) = \sum_{t'_2 \in T_2, t'_3 \in T_3, \dots, t'_n \in T_n} f_{X_1, X_2, X_3, \dots, X_n}(t, t'_2, t'_3, \dots, t'_n)$

↳ multiple random variables $\Rightarrow f_{X_1, X_2}(t_1, t_2) = P(X_1=t_1, \text{and } X_2=t_2) = \sum_{t'_3 \in T_3} f_{X_1, X_2, X_3}(t_1, t_2, t'_3)$

→ Conditional PMF with multiple random variables

$$\text{↳ } (X_1, X_2 | X_3=t_3) \sim f_{X_1, X_2 | X_3=t_3}(t_1, t_2) = \frac{f_{X_1, X_2, X_3}(t_1, t_2, t_3)}{f_{X_3}(t_3)}$$

$$\text{↳ } (X_1 | X_2=t_2, X_3=t_3) \sim f_{X_1 | X_2=t_2, X_3=t_3}(t_1) = \frac{f_{X_1, X_2, X_3}(t_1, t_2, t_3)}{f_{X_2, X_3}(t_2, t_3)}$$

↳ factors of a Joint PMF: $f_{X_1, \dots, X_n}(t_1, \dots, t_n) = P(X_1=t_1, \text{and } X_2=t_2, \text{and } X_3=t_3, \text{and } \dots, \text{and } X_n=t_n)$

$$= P(X_1=t_1 | X_2=t_2, X_3=t_3, \dots, X_n=t_n) \cdot P(X_2=t_2 | X_1=t_1, X_3=t_3, \dots, X_n=t_n)$$

$$= P(X_1=t_1 | X_2=t_2, X_3=t_3, \dots, X_n=t_n) \cdot P(X_2=t_2 | X_1=t_1, X_3=t_3, \dots, X_n=t_n) \cdot P(X_3=t_3 | X_1=t_1, X_2=t_2, \dots, X_n=t_n) \cdot \dots \cdot P(X_n=t_n | X_1=t_1, X_2=t_2, \dots, X_{n-1}=t_{n-1})$$

$$f_{X_1, \dots, X_n}(t_1, \dots, t_n) = f_{X_1 | X_2=t_2, X_3=t_3, \dots, X_n=t_n}(t_1) \cdot f_{X_2 | X_1=t_1, X_3=t_3, \dots, X_n=t_n}(t_2) \cdot f_{X_3 | X_1=t_1, X_2=t_2, \dots, X_n=t_n}(t_3) \cdot \dots \cdot f_{X_n | X_1=t_1, X_2=t_2, \dots, X_{n-1}=t_{n-1}}(t_n)$$



WEEK 2

→ Independence of two random variables

↳ X and Y are independent if:

$$f_{XY}(t_1, t_2) = f_X(t_1) \cdot f_Y(t_2)$$

→ Joint PMF of f_{XY} is the product of the marginal PMF of X and Y .

→ Conditional PMF equals marginal PMF

→ Independence of multiple random variables

↳ X_1, X_2, \dots, X_n are independent iff

$$f_{X_1, X_2, \dots, X_n}(t_1, t_2, \dots, t_n) = f_{X_1}(t_1) \cdot f_{X_2}(t_2) \cdot \dots \cdot f_{X_n}(t_n)$$

→ Independent and Identically distributed (i.i.d.)

↳ Random variables X_1, X_2, \dots, X_n are i.i.d. if → ① they are independent,

② Marginal PMFs f_{X_i} are identical.

$$\rightarrow X_1, X_2, \dots, X_n \sim \text{i.i.d. } f_X$$

→ Memoryless property of Geometric distribution

→ Let $X \sim \text{Geometric}(p)$

$$\textcircled{1} \quad P(X > n) = (1-p)^n$$

$$\textcircled{2} \quad P(X > m+n | X > m) = \frac{P(X > m+n \cap X > m)}{P(X > m)} = \frac{P(X > m+n)}{P(X > m)} = \frac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n$$

→ Functions of random variable

↳ One-to-one function → $P(Y = f(u)) = P(X = u)$; e.g. $\rightarrow y = n-5, y = 2^n$

↳ Many-to-one function → e.g. $\rightarrow y = (n-5)^2, y = u(1-u) \rightarrow y_0 = f(u_1) = f(u_2) = \dots = f(u_m)$

↳ sum over all the probabilities of X for whenever the function takes the 'y' value.

$$\longrightarrow$$

$$P(Y = y_0) = P(X = u_1) + P(X = u_2) + \dots + P(X = u_m)$$

all the places $f(u)$ takes the value y_0

→ Visualising function of 2 random variable

↳ $g(u, y)$: function

① Contours → values of (u, y) that result in $g(u, y) = c$

→ make a plot of those (u, y) for different c

② Regions → values of (u, y) that result in $g(u, y) \leq c$

→ make a plot of those (u, y) for different c

→ Function of 2 random variables

→ Let $X, Y \sim f_{XY}$; let $Z = g(X, Y)$ be a function of X and Y

• What is the PMF of Z ?

→ Step 1: Find the range of Z

→ Step 2: Add over all the contours

↳ Suppose z is a possible value taken by Z :

$$P(Z=z) = \sum_{(u, y) : g(u, y)=z} f_{XY}(u, y)$$

→ Sum of 2 uniform random variables

$X, Y \sim \text{iid Unif}\{1, 2, \dots, n\}, W = X + Y$

range of $W = \{2, 3, \dots, 2n\}$

$W \in \{2, 3, \dots, 2n\}$

$W = w \cdot (1, w-1), (2, w-2), \dots, (w-1, 1)$

$$P(W=w) = \begin{cases} \frac{w-1}{n^2}, & 2 \leq w \leq n+1 \\ \frac{2n-w+1}{n^2}, & n+2 \leq w \leq 2n \end{cases}$$

→ Max of 2 uniform random variables $X, Y \sim \text{iid} \text{Unif}\{1, 2, \dots, n\}$, $Z = \max(X, Y)$

$$Z \in \{1, 2, \dots, n\}$$

$$P(Z=z) = \frac{2z-1}{n^2}$$

→ PMF of $g(x_1, x_2, \dots, x_n) \rightarrow$ The PMF $X = g(x_1, x_2, \dots, x_n)$ is given by

$$f_X(t) = P(g(x_1, x_2, \dots, x_n) = t) = \sum_{(t_1, t_2, \dots, t_n) : g(t_1, t_2, \dots, t_n) = t} f_{x_1, x_2, \dots, x_n}(t_1, t_2, \dots, t_n)$$

$t_1 \cdot t_2 \cdot t_3$	$t_1 + t_2 + t_3$	t_1, t_2, t_3	$f_{x_1, x_2, x_3}(t_1, t_2, t_3)$
0	0	0 0 0	$\frac{1}{9}$
0	1	0 0 1	$\frac{1}{9}$
0	2	0 0 2	$\frac{1}{9}$
1	2	0 1 1	$\frac{1}{9}$
2	3	0 1 2	$\frac{1}{9}$
0	1	1 0 0	$\frac{1}{9}$
0	3	1 0 2	$\frac{1}{9}$
0	2	1 1 0	$\frac{1}{9}$
1	3	1 1 1	$\frac{1}{9}$

$$\textcircled{1} X = g(x_1, x_2, x_3) = x_1 + x_2 + x_3 \quad g \in \{0, 1, 2, 3\}$$

$$\textcircled{2} Y = h(x_1, x_2, x_3) = x_2 \cdot x_3 \quad h \in \{0, 1, 2\}$$

$$f_{xy} \begin{matrix} 0 & 1 & 2 & 3 \\ 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \\ 1 & 0 & 0 & \frac{1}{9} & \frac{1}{9} \\ 2 & 0 & 0 & 0 & \frac{1}{9} \end{matrix}$$

→ sum of n independent Bernoulli(p) = Binomial(n, p)

→ suppose X and Y take integer values and their joint PMF = f_{xy} . Let $Z = X + Y$

$$\begin{aligned} P(Z=z) &= \sum_{u=-\infty}^{\infty} P(X=u, Y=z-u) = \sum_{u=-\infty}^{\infty} f_{xy}(u, z-u) = \sum_{y=-\infty}^{\infty} f_{xy}(z-y, y) \\ &= \sum_{u=-\infty}^{\infty} f_x(u) f_y(z-u) = \sum_{y=-\infty}^{\infty} f_y(y) f_x(z-y) \end{aligned}$$

→ Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent.

$$\rightarrow Z = X + Y$$

$$f_Z(z) = \left(\sum_{u=0}^{\infty} f_x(u) \cdot f_y(z-u) \right) = \sum_{u=0}^{\infty} \frac{e^{-\lambda_1} \cdot \lambda_1^u}{u!} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2^{z-u}}{(z-u)!} = \frac{e^{-(\lambda_1+\lambda_2)} \sum_{u=0}^{\infty} \frac{\lambda_1^u \lambda_2^{z-u}}{u! (z-u)!}}{z!} \xrightarrow{(\lambda_1+\lambda_2)^z}$$

can be replaced with u going to z because after z this term will go to 0.

$$f_Z(z) = \frac{e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1+\lambda_2)^z}{z!} \Rightarrow Z \sim \text{Poisson}(\lambda_1+\lambda_2)$$

→ conditional of $X|Z$

$$\begin{aligned} P(X=k | Z=n) &= \frac{P(n=k, Z=n)}{P(Z=n)} = \frac{P(n=k) \cdot P(Z=n | n=k)}{P(Z=n)} = \frac{P(X=k) \cdot P(Y=n-k)}{P(Z=n)} \\ &= \frac{\frac{e^{-\lambda_1} \cdot \lambda_1^k}{k!} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2^{n-k}}{(n-k)!}}{\frac{e^{-\lambda_1-\lambda_2} \cdot (\lambda_1+\lambda_2)^n}{n!}} = \frac{n!}{k! (n-k)!} \frac{\lambda_1^k \cdot \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \xrightarrow{(\lambda_1+\lambda_2)^n = (\lambda_1+\lambda_2)^k \cdot (\lambda_1+\lambda_2)^{n-k}} \\ &= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2} \right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2} \right)^{n-k} \xrightarrow{\frac{n!}{k! (n-k)!} = \binom{n}{k}} \end{aligned}$$

$$X | Z = \text{Bin}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$Y | Z = \text{Bin}\left(n, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$

→ Functions of non-overlapping independent random variables are also independent

→ If X and Y are independent, $g(X)$ and $h(Y)$ are independent for any two functions g and h

→ Min/Max of two random variables

$$\hookrightarrow X, Y \sim f_{xy} \quad Z = \min(X, Y)$$

$$f_Z(z) = P(\min(X, Y) = z) = P((X=z) \text{ and } (Y=z) \text{ or } (X=z \text{ and } Y>z) \text{ or } (X>z \text{ and } Y=z)) \\ = f_{xy}(z, z) + \sum_{t_2 > z} f_{xy}(z, t_2) + \sum_{t_1 > z} f_{xy}(t_1, z)$$

↪ CDF of a random variable

↪ CDF of a random variable X is a function $F_X : \mathbb{R} \rightarrow [0, 1]$ defined as:

$$\hookrightarrow F_X(k) = P(X \leq k)$$

e.g. → X and Y are independent. $Z = \max(X, Y)$

$$F_Z(z) = P(\max(X, Y) \leq z) \\ = F_X(z) \cdot F_Y(z)$$

↪ Let $X_1, X_2, \dots, X_n \sim \text{i.i.d.}$

① Distribution of $\min(X_1, X_2, \dots, X_n) \rightarrow P(\min(X_1, X_2, \dots, X_n) \geq z) = (P(X \geq z))^n$

② Distribution of $\max(X_1, X_2, \dots, X_n) \rightarrow P(\max(X_1, X_2, \dots, X_n) \leq z) = (P(X \leq z))^n = (F_X(z))^n$

↪ Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(p)$ be independent. Find the dist. of $\min(X, Y)$

$$Z = \min(X, Y) \quad P(Z \geq z) = P(X \geq z, Y \geq z) = (1-p)^{z-1} \cdot (1-p)^{z-1} \\ = ((1-p)^2)^{z-1}$$

$$P(Z \geq z+1) = ((1-p)^2)^z$$

$$P(Z=z) = P(Z \geq z) - P(Z \geq z+1) \\ = ((1-p)^2)^{z-1} - ((1-p)^2)^z = ((1-p)^2)^{z-1} (1 - (1-p)^2) \leftarrow \begin{matrix} \text{Geometric} \\ \text{probability} \end{matrix}$$

$$\min(X, Y) \sim \text{Geometric}(1 - (1-p)^2)$$

Let $X_1 \sim \text{Geometric}(p_1)$ and $X_2 \sim \text{Geometric}(p_2)$

$$\hookrightarrow \min(X_1, X_2) \sim \text{Geometric}(1 - (1-p_1)(1-p_2))$$

$$Z = \max(X_1, X_2, \dots, X_{10}) \quad X \sim \text{Bin}(6, \frac{1}{2})$$

$$P(Z \leq 2) = P(X_1 \leq 2, X_2 \leq 2, \dots, X_{10} \leq 2) \\ = \left(2^2 \cdot \left(\frac{1}{2}\right)^6\right)^{10} \quad \begin{array}{c|ccccc} X & 0 & 1 & 2 & 3 \\ \left(\frac{1}{2}\right)^6 & 6 \cdot \left(\frac{1}{2}\right)^6 & 15 \cdot \left(\frac{1}{2}\right)^6 & & \\ \hline \end{array} \\ \binom{6}{2} = \frac{6!}{2! 4!} = \frac{6 \times 5}{2} = 15 \\ X \leq 2 = \left(\frac{1}{2}\right)^6 \cdot 2^2$$

$$Z = \min(X_1, X_2, \dots, X_{10}) \quad X \sim \text{Bin}(6, \frac{1}{2})$$

$$F_Z(2) = 1 - P(X_1 > 2, X_2 > 2, \dots, X_{10} > 2) \quad X > 2 = 42 \left(\frac{1}{2}\right)^6 \\ = 1 - \left(\frac{42}{2^6}\right)^{10} = 1 - \left(\frac{21}{2^5}\right)^{10}$$



WEEK 3

→ Expected value of a discrete random variable:

↪ $X \rightarrow$ discrete rand. variable; range $\rightarrow \mathbb{R}_X$; PMF $\rightarrow f_X$

$$E[X] = \sum_{t \in \mathbb{R}_X} t \cdot f_X(t) = \sum_{t \in \mathbb{R}_X} t \cdot P(X=t)$$

↪ example $\Rightarrow X \sim \text{Bernoulli}(p) \Rightarrow E[X] = 0(1-p) + p = p$

$$X \sim \text{Uniform}\{1, 2, 3, 4, 5, 6\} \Rightarrow E[X] = \sum_{t=1}^6 t \cdot \frac{1}{6} = 3.5$$

$$X \sim \text{Uniform}\{a, a+1, \dots, b\} \Rightarrow E[X] = \frac{a}{b-a+1} + \frac{a+1}{b-a+1} + \dots + \frac{b}{b-a+1} = \frac{a+b}{2}$$

$$\text{Identity} \Rightarrow a + (a+1) + (a+2) + \dots + b = (b-a+1)\left(\frac{a+b}{2}\right)$$

Simplify summations

① Difference Equation (DE): $a_{t+1} - r a_t = b_t \quad (r \neq 1) \quad \sum_{t=1}^n a_t = \frac{a_1 - r a_n}{1-r} + \frac{1}{1-r} \sum_{t=1}^{n-1} b_t$

② Geometric Progression (GP): $a_{t+1} - r a_t = 0 \quad (r \neq 1) \quad \sum_{t=1}^n a_t = \frac{a_1 - r a_n}{1-r} \xrightarrow[r<1]{n \rightarrow \infty} \frac{a_1}{1-r}$

③ Exponential Function: $\sum_{t=0}^{\infty} e^{-t} \frac{\lambda^t}{t!} = 1 \quad e^{\lambda} = \sum_{t=0}^{\infty} \frac{\lambda^t}{t!}$

④ Binomial Formula: $\sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = (a+b)^n$

↪ $X \sim \text{Geometric}(p) \quad E[X] = \sum_{t=1}^{\infty} t \cdot (1-p)^{t-1} \cdot p$
 ↪ GP: $a_1 = p, r = 1-p, b_t = r^t p \quad \rightarrow E[X] = 1/p$

↪ $X \sim \text{Poisson}(\lambda) \quad E[X] = \sum_{t=0}^{\infty} t \cdot e^{-\lambda} \frac{\lambda^t}{t!}$
 ↪ using exponent function $\rightarrow E[X] = \lambda$

↪ $X \sim \text{Binomial}(n, p) \quad E[X] = \sum_{t=0}^n t \cdot \binom{n}{t} p^t (1-p)^{n-t}$
 ↪ $E[X] = np$

→ Expected value of a function of random variables

↪ x_1, x_2, \dots, x_n have joint PMF f_{x_1, x_2, \dots, x_n} with range \mathbb{R}_{x_i} denoted \mathbb{R}_{X_i}
 let $g: \mathbb{R}_{x_1} \times \mathbb{R}_{x_2} \times \dots \times \mathbb{R}_{x_n} \rightarrow \mathbb{R}$ be a function, $y = g(x_1, x_2, \dots, x_n); \mathbb{R}_y$ and f_y

$$E[Y] = \sum_{t \in \mathbb{R}_Y} t \cdot f_Y(t) = \sum_{t_1 \in \mathbb{R}_{X_1}} \dots \sum_{t_n \in \mathbb{R}_{X_n}} g(t_1, t_2, \dots, t_n) \cdot f_{x_1, x_2, \dots, x_n}(t_1, t_2, \dots, t_n)$$

→ Linearity of Expected value:

① $E[cX] = cE[X]$ for a random variable X and a constant c

② $E[X+Y] = E[X] + E[Y]$ for any two random variables X and Y

③ $E[aX+bY] = aE[X] + bE[Y]$

→ Centering operation

↪ $y = X - E[X]$ is a translated version of X and $E[Y] = 0$
 $X - E[X]$ is a zero-mean random variable.

→ Variance of a random variable

$$\hookrightarrow \text{Var}(X) = E[(X - E[X])^2]$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

↪ If X and Y are independent:

$$- E[XY] = E[X]E[Y]$$

$$- \text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Distribution	Expected Value	Variance
Bernoulli(p)	p	$p(1-p)$
Binomial(n, p)	np	$np(1-p)$
Geometric(p)	$\frac{1}{p}$	$(1-p)/p^2$
Poisson(λ)	λ	λ
Uniform{1, 2, ..., n}	$(n+1)/2$	$(n^2-1)/12$

→ A random variable is standardised if $E(X)=0$ and $\text{Var}(X)=1$

$$\hookrightarrow Y = \frac{X - E[X]}{\text{SD}(X)}, \text{ then } Y \text{ is a standardised rand. var. obtained from } X.$$

→ Covariance of two random variables

↪ X and Y are rand. var. on the same prob. space

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X] \cdot E[Y]$$

$$\hookrightarrow \text{Cov}(X, aY + bZ) = a\text{Cov}(X, Y) + b\text{Cov}(X, Z)$$

$$\hookrightarrow \text{Cov}(aX + bY, Z) = a\text{Cov}(X, Z) + b\text{Cov}(Y, Z)$$

↪ If X and Y are independent, then $\text{Cov}(X, Y) = 0$

If X and Y are independent, then they are uncorrelated.

If X and Y are uncorrelated, they might still be dependent.

→ Correlation

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\text{SD}(X)\text{SD}(Y)}$$

→ Markov's Inequality

$$P(X \geq c) \leq \frac{\mu}{c}, \text{ where } \mu = E(X)$$

→ Chebychev's Inequality

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$



WEEK 1 GRA

① $T_x = \{0, 1, 2, 3\}$ $T_y = \{-1, 1, 2, 3\}$ $X \sim \text{Bin}(3, \frac{1}{2})$ $P(X < 3) = 1 - P(X=3) = \frac{7}{8}$ $P(X=3) = \binom{3}{3} \cdot (\frac{1}{2})^3 = \frac{1}{8}$

$$P(Y \leq 1) = P(Y=1) + P(Y=-1) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

$$f_{XY}(t_x < 3, t_y \leq 1) = \frac{5}{8} \times \frac{7}{8} = \frac{35}{64} \approx 0.5469$$

② $T_x = T_y = T_z = \{0, 1, 2\}$

0	0	2
2	0	0
1	0	1

$$f_{XYZ}(2, 0) = \sum_{t_z=0}^2 f_{XYZ}(2, 0, t_z) = P(2, 0, 0) = \frac{1}{9}$$

$$f_Y(0) = \sum_{t_x \in T_x, t_z \in T_z} f_{XYZ}(t_x, 0, t_z) = \frac{P(0, 0, 0) + P(1, 0, 0) + P(2, 0, 0)}{9} = \frac{3}{9}$$

$$f_{X|Y=0}(2) = \frac{f_{XY}(2, 0)}{f_Y(0)}$$

$$f_{X|Y=0}(2) = \frac{1/9}{3/9} = \frac{1}{3}$$

③ $\frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + k = 1$ $k = 1 - \left(\frac{1}{4} + \frac{1}{4}\right) = 1 - \frac{3}{4} = \frac{1}{4}$

$$f_{Y|X=1}(2) = \frac{f_{YX}(2, 1)}{f_X(1)} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{8} + \frac{1}{8}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \times 2 = \frac{1}{2}$$

⑤ $f_Y(1) = \sum_{t' \in T_x} f_{XY}(t', 1)$ $f_X(1) = \frac{1}{2}$ $f_X(0) = \frac{1}{2}$

$Y \sim \text{Bin}(n, p)$

$f_{Y X=1}(1) = \frac{n=3}{p=\frac{3}{20}}$	$f_{Y X=0}(1) = \frac{n=2}{p=\frac{4}{15}}$	$f_{XY}(0, 1) = f_{Y X=0}(1) \cdot f_X(0) = 0.03008547$
$= \binom{3}{1} \left(\frac{7}{20}\right) \left(\frac{13}{20}\right)^2$	$= \binom{2}{1} \left(\frac{4}{15}\right) \left(\frac{11}{15}\right)$	$f_{XY}(1, 1) = f_{Y X=1}(1) \cdot f_X(1) = 0.4095$
$= \frac{3 \times 7 \times (13)^2}{(20)^3}$	$= \frac{2 \times 4 \times 11}{(15)^2}$	
$= 0.443625$	$= 0.391111$	

$$f_Y(1) = f_{XY}(0, 1) + f_{XY}(1, 1) \approx 0.4346$$

⑦ $f_{XY}(x, y) = a(bx+ay)$

x \ y	0	1	2	3
0	0	a	2a	3a
1	\boxed{ab}	$\boxed{ab+a}$	$\boxed{ab+2a}$	$ab+3a$
2	$\boxed{2ab}$	$\boxed{2ab+a}$	$\boxed{2ab+2a}$	$2ab+3a$
	$3ab$	$3ab+2a$	$3ab+4a$	

$3\left(12a + 3ab = \frac{3}{7}\right) = 36a + 9ab = \frac{9}{7}$

$2\left(6a + 9ab = \frac{4}{7}\right) = 12a + 18ab = \frac{8}{7}$

$-12a - 3ab = -\frac{3}{7}$

$15ab = \frac{5}{7}$

$ab = \frac{1}{21}$

$\frac{b}{a} = \frac{1}{42} \Rightarrow b = 2$

$$f_{XY}(1, 1) = \frac{1}{42} (2+1) = \frac{1}{14}$$

⑧ $f_{Y|X=2}(0) = \frac{f_{YX}(0, 2)}{f_X(2)}$ $f_X(2) = \binom{6}{2} \left(\frac{1}{2}\right)^6$

$$f_{X|Y=0}(2) = \binom{5}{2} \left(\frac{1}{2}\right)^5 ; f_Y(0) = \frac{1}{2} ; f_{YX}(0, 2) = \binom{5}{2} \left(\frac{1}{2}\right)^6$$

$$f_{Y|X=2}(0) = \frac{\binom{5}{2} \cdot \left(\frac{1}{2}\right)^6}{\binom{6}{2} \cdot \left(\frac{1}{2}\right)^6} = \frac{\binom{5}{2}}{\binom{6}{2}}$$

$$= \frac{5!}{2! 3!} \times \frac{2! 4!}{6!} = \frac{5!}{6!} \times \frac{4!}{3!} = \frac{4}{6} = \frac{2}{3} = 0.666$$

$$\textcircled{9} \quad \frac{\binom{5}{1} \times \binom{4}{1}}{\binom{12}{2}} = \frac{5 \times 4}{\frac{12!}{10!2!}} = \frac{5 \times 4}{\frac{12 \times 11}{3} \times 2} = \frac{10}{33} = \boxed{0.30}$$

$$\textcircled{10} \quad N \sim \text{Bin}(7, 1/2) \quad X \sim \text{Bin}(n, 1/2)$$

$$\begin{array}{c} x/N \\ \hline 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & \cancel{\binom{7}{1}\binom{6}{2}} & \cancel{\binom{7}{2}\binom{6}{0}\binom{1}{2}} & \binom{7}{4}\binom{4}{0}\binom{1}{2}^0 = \frac{7!}{4!3!} \times \frac{1}{2048} = \frac{7 \times 6 \times 5}{2 \times 2} \times \frac{1}{2048} = \frac{35}{2048} \\ 1 & 0 & \cancel{\binom{7}{1}\binom{6}{2}} & \cancel{\binom{7}{3}\binom{4}{1}\binom{1}{2}} & \binom{7}{3}\binom{3}{1}\binom{1}{2}^0 = \frac{7!}{3!4!} \times \frac{3}{1024} = \frac{210}{2048} \\ 2 & 0 & 0 & \cancel{\binom{7}{2}\binom{5}{2}\binom{1}{2}} & \binom{7}{2}\binom{2}{2}\binom{1}{2}^0 = \frac{7!}{2!5!} \times \frac{1}{512} = \frac{7 \times 6}{2} \times \frac{1}{512} = \frac{84}{2048} \\ 3 & 0 & 0 & 0 & \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$\text{Total} = \frac{35 + 210 + 84}{2048} = \frac{329}{2048} = \boxed{0.161}$$

————— ✗ ————— ✗ ————— ✗ ————— ✗ —————

WEEK 2 GA

$$\textcircled{2} \quad \begin{array}{ccc} 0 & 1 & 2 \\ 0 & 0.06 & 0.18 & 0.12 & 0.36 \\ 1 & 0.04 & 0.12 & 0.48 & \underline{0.64} \\ 0.1 & 0.3 & 0.6 \end{array}$$

$$\begin{array}{ccc} 0 & 1 & 2 \\ 0 & \frac{1}{24} & \frac{3}{24} & \frac{1}{24} & \frac{7}{24} \\ 1 & \frac{7}{24} & \frac{3}{24} & \frac{3}{24} & \frac{9}{24} \\ 2 & \frac{3}{24} & \frac{3}{24} & \frac{2}{24} & \frac{11}{24} \\ \frac{1}{24} & \frac{3}{24} & \frac{3}{24} & \frac{7}{24} \end{array}$$

$$\begin{array}{ccc} 0 & 1 & 2 \\ 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{10} & \frac{5}{10} \\ 1 & \frac{1}{10} & \frac{1}{10} & \frac{3}{10} & \frac{5}{10} \\ 2 & \frac{3}{10} & \frac{3}{10} & \frac{5}{10} & \end{array}$$

$$\begin{array}{cc} 0 & 1 \\ 0 & \frac{1}{10} & \frac{1.5}{10} & \frac{2.5}{10} \\ 1 & \frac{2}{10} & \frac{3}{10} & \frac{5}{10} \\ 2 & \frac{1}{10} & \frac{1.5}{10} & \frac{2.5}{10} \\ \frac{4}{10} & \frac{6}{10} \end{array}$$

$$\textcircled{3} \quad X \sim \text{Bernoulli}(0.2) \quad Y \sim \text{Bernoulli}(0.4) \quad Z = X + Y$$

$$f_{X|Z=1}(1) = \frac{P(X=1, Z=1)}{P(Z=1)} = \frac{P(X=1, Y=0)}{P(X=1, Y=0 \text{ or } X=0, Y=1)} = \frac{(0.2)(0.6)}{(0.2)(0.6) + (0.8)(0.4)} = \frac{0.12}{0.12 + 0.32} = \boxed{0.2727}$$

$$\textcircled{4} \quad Z = X + Y \quad f_{xy}(x, y) = \frac{9}{16 \cdot (4)^{x+y}}$$

$$f_z(k) = P(X=u, Y=k-u) = \sum_{u=0}^k f_{xy}(u, k-u) = \boxed{\frac{(k+1) \cdot 9}{16 \cdot (4)^k}}$$

$$\textcircled{5} \quad Z = \max(x, y)$$

$$f_z(k) = P(X=k, Y \leq k \text{ or } X \leq k, Y=k) = \sum_{y=0}^k f_{xy}(k, y) + \sum_{u=0}^{k-1} f_{xy}(u, k) \\ = 2 \cdot \sum_{u=0}^{k-1} \frac{9}{16 \cdot (4)^{k-u}} =$$

$$\textcircled{6} \quad \begin{array}{cccccc} y \setminus u & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & * & & & & & \\ 2 & * & * & & & & \\ 3 & & * & & & & \\ 4 & & & & & & \\ 5 & & & & & & \\ 6 & & & & & & \end{array} \quad \begin{array}{l} f_x(u) \cdot f_y(1) + f_x(5) \cdot f_y(2) + f_x(6) \cdot f_y(3) \\ = \left(\frac{1}{36}\right) \times 3 = \frac{3}{36} = \frac{1}{12} \end{array}$$

$$\textcircled{7} \quad X \sim \text{Geometric}(p) \quad Y \sim \text{Geometric}(p) \quad \text{and} \quad Z = X + Y$$

$$f_y(k) = p(1-p)^{k-1}$$

$$f_z(k) = P(X=u, Y=k-u) = \sum_{u=1}^{k-1} p(1-p)^u \cdot p(1-p)^{k-u}$$

$$= p^2 (1-p)^k (k-1)$$

$$f_z(18) = p^2 (1-p)^{18} (17) \quad f_z(19) = p^2 (1-p)^{19} (18)$$

$$f_z(18) < f_z(19) \Rightarrow p^2 (1-p)^{18} (17) < p^2 (1-p)^{19} (18)$$

$$\frac{17}{18} < 1-p$$

$$p < 0.056$$

$$\textcircled{8} \quad X \sim \text{Poisson}(2) \quad f_x(1) = e^2 \cdot 2 \quad f_x(2) = e^2 \cdot 2 \quad f_x(\neq 1 \text{ or } 2) = 1 - 4e^2$$

$$\begin{matrix} X_1 & X_2 & X_3 \\ 1 & 2 & \\ 1 & & 2 \end{matrix}$$

$$6 \cdot (2e^2)^2 (1-4e^2) = 0.439575 (1-4e^2) = \boxed{0.2016}$$

$$\begin{matrix} 2 & 1 \\ 2 & & 1 \\ 2 & 1 \\ 1 & 2 \end{matrix}$$

$$\textcircled{9} \quad X \sim \text{Bernoulli}(0.8) \quad Y \sim \text{Bernoulli}(0.3) \quad Z = X+Y-XY$$

$$f_z(1) = P(X=1 \mid X+Y=2) = \frac{P(X=1, X+Y=2)}{P(X+Y=2)} = \frac{P(X=1, Y=1)}{P(X=1, Y=1)}$$

$$Z \sim \text{Bernoulli}(0.86)$$

$$\textcircled{10} \quad X, Y \sim \text{Geometric}(0.8) \quad P(X=1 \mid X+Y=2) = \frac{P(X=1, X+Y=2)}{P(X+Y=2)} = \frac{P(X=1, Y=1)}{P(X=1, Y=1)}$$

$$\textcircled{11} \quad X \sim \text{Poisson}(5) \quad Y \sim \text{Poisson}(1) \quad Z = X + Y$$

$$Z \sim \text{Poisson}(6)$$

$$Y/Z=4 \sim \text{Binomial}(4, 1/6)$$

$$Y/Z=4(3) = \binom{4}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^1 = \frac{4 \cdot 5}{6^3} = \boxed{0.0154}$$

————— X ————— X ————— X ————— X —————

WEEK 3 GA

$$\textcircled{2} \quad X \sim \text{Bin}(2, 6/14) \quad Y \sim \text{Bin}(2, 8/14) \quad \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\text{SD}(X)\text{SD}(Y)}$$

$$\begin{array}{c|ccc} Y \setminus X & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & \frac{30}{182} \\ 1 & 0 & \frac{96}{182} & 0 \\ 2 & \frac{56}{182} & 0 & 0 \end{array}$$

$$\text{E}(X) = \frac{156}{182} \quad \text{E}(X^2) = \frac{216}{182} \quad \text{Var}(X) = \frac{14976}{33124}$$

$$\text{E}(Y) = \frac{208}{182} \quad \text{E}(Y^2) = \frac{320}{182} \quad \text{Var}(Y) = \frac{14976}{33124}$$

$$\text{SD}(X)\text{SD}(Y) = \frac{14976}{33124} \quad \text{E}(XY) = \frac{96}{182}$$

$$\text{Cov}(X, Y) = \text{E}(XY) - \text{E}(X)\text{E}(Y) = \frac{96}{182} - \frac{156 \cdot 208}{(182)^2} = \frac{-14976}{33124}$$

$$\rho(X, Y) = -\frac{14976}{33124} \times \frac{33124}{14976} = \boxed{-1}$$

$$8^{\text{th}} \rightarrow P(G_7) = 0.4$$

$$9^{\text{th}} \rightarrow 0.4$$

$$10^{\text{th}} \rightarrow 0.6$$

$$G_8 \sim \text{Bin}(2, 0.4)$$

$$G_9 \sim \text{Bin}(2, 0.4)$$

$$G_{10} \sim \text{Bin}(2, 0.6)$$

$$E(G_{\text{total}}) = E(G_8) + E(G_9) + E(G_{10})$$

$$E(G_8) = E(G_9) = 0.8 \quad E(G_{10}) = 1.2$$

$$E(G_{\text{total}}) = 0.8 + 0.8 + 1.2 = \boxed{2.8}$$

$$\textcircled{5} \quad \sigma = \sqrt{2} \quad z = \mu - k\sigma \Rightarrow \left| \frac{z-15}{\sqrt{2}} \right| = k \Rightarrow k \approx 5.66 \quad P(z < x < 23) = 1 - \frac{1}{(5.66)^2} \approx 0.969$$

$$\textcircled{6} \quad X \sim \text{Geometric}(0.04) \quad \text{Var}(X) = (1-p)/p^2 = 600$$

$$\textcircled{7} \quad \text{Var}(XY) = \text{Var}(X) \cdot \text{Var}(Y) + \text{Var}(Y)(E(X))^2 + \text{Var}(X)(E(Y))^2 \\ = 15 + 3969 + 605 = 4589$$

$$\textcircled{8} \quad \begin{array}{ccccc} X & Y & Z & P(X,Y,Z) & R \\ 0 & 1 & 2 & 0.4 & 4 \\ 0 & 2 & 3 & 0.3 & 6 \\ 1 & 0 & -2 & 0.3 & -4 \end{array} \quad R = XY + 2Z$$

$$E(R) = 2 \cdot 2 = 12.4 \quad E(R^2) = 12.4 \\ \text{Var}(R) = 12.4 - (2 \cdot 2)^2 = 7.56$$

$$\textcircled{9} \quad \text{both losing} = \left(\frac{498 \times 497}{500 \times 499} \right) \times \text{₹}0$$

$$\text{one wins 500} = \left(\frac{1}{500} \times \frac{498}{499} \times 2 \right) \times \text{₹}500 = 1.99$$

$$\text{one wins 2000} = \left(\frac{1}{500} \times \frac{498}{499} \times 2 \right) \times \text{₹}2000 = 7.98$$

$$\text{both tickets win} = \left(\frac{2}{500} \times \frac{1}{499} \right) \times \text{₹}2500 = 0.02$$

$$\text{total} = 10$$

$$\textcircled{10} \quad \begin{array}{lllll} P(UU) = (0.4)(0.6) & \text{Gain}(UU) = 2400 & \rightarrow 576 & 1050 \\ P(UD) = (0.4)(0.4) & " (UD) = 150 & \rightarrow 24 & 300 \\ P(DU) = (0.6)(0.6) & " (DU) = -1200 & \rightarrow -432 & -450 \\ P(DD) = (0.6)(0.4) & " (DD) = -2550 & \rightarrow -612 & -1200 \\ & & & + \\ & & & -36 \\ & & & 48 \\ & & & -162 \end{array}$$

————— X ————— X ————— X ————— X —————

MOCK WEEK 1 - 2

$$\textcircled{1} \quad \begin{array}{cccccc} X & Y & S & Z_1 & Z_2 & \\ 1 & 1 & 2 & 1 & 1 & \\ 1 & 2 & 3 & 2 & 1 & \\ 2 & 1 & 3 & 2 & 1 & \\ 2 & 2 & 4 & 2 & 2 & \\ 3 & 1 & 4 & 3 & 1 & \\ 3 & 2 & 5 & 3 & 2 & \end{array} \quad \begin{array}{ccccc} S & 2 & 3 & 4 & 5 \\ P(S) & 0.1 & 0.4 & 0.4 & 0.1 \end{array} \quad \begin{array}{ccccc} Z_2 & 1 & 2 \\ P(Z_2) & 0.6 & 0.4 \end{array}$$

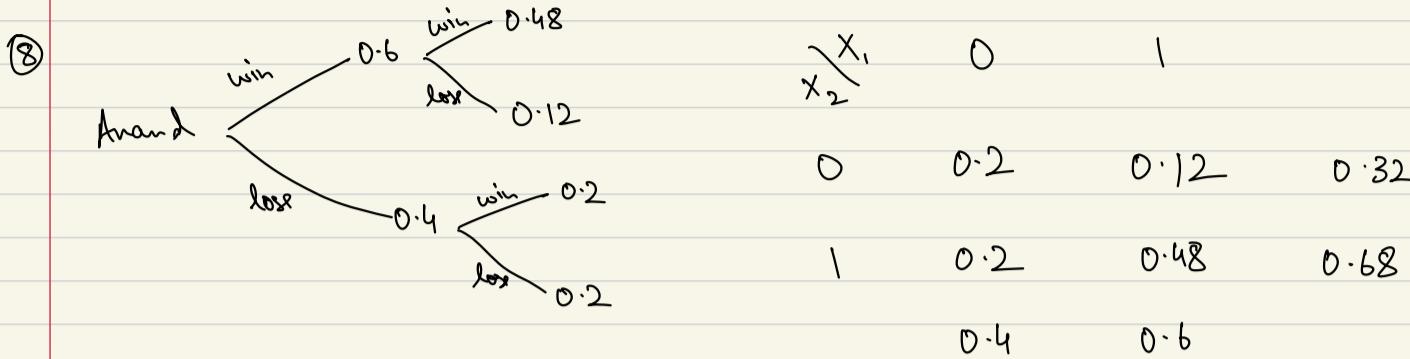
$$\textcircled{2} \quad \begin{array}{cccc} Y \setminus X & 0 & 1 & 2 \\ 0 & \frac{2}{24} & \frac{3}{24} & \frac{1}{24} & \frac{1}{4} \\ 1 & \frac{6}{24} & \frac{9}{24} & \frac{3}{24} & \frac{3}{4} \end{array} \quad P(Y=1 | X=2) = \frac{3}{24} \times 6 = \frac{3}{4}$$

$$\frac{1}{3} \quad \frac{1}{2} \quad \frac{1}{6} \quad P(X \leq 1, Y=0) = \frac{5}{24} \quad P(X=2, Y=1) = \frac{3}{24}$$

$$\text{sum} = \frac{8}{24} = \frac{1}{3} \quad \frac{1}{\text{sum}} = \boxed{3}$$

$$\textcircled{6} \quad X \sim \text{Poisson}(2) \quad Y \sim \text{Poisson}(3) \quad Z = X+Y \quad Z \sim \text{Poisson}(5) \quad X/2=5 \sim \text{Bin}(5, 2/5) \quad Y/2=5 \sim \text{Bin}(5, 3/5)$$

$$P(X=1 | Z=5) = \binom{5}{1} \left(\frac{2}{5}\right) \left(\frac{3}{5}\right)^4 = 0.2592$$



$$\textcircled{9} \quad P(X=3, Y=3) = P(N=6) \text{ and } P(X=3)$$

$$P(N=6) = \frac{e^5 5^6}{6!} = 0.146223$$

$$P(X=3 | N=6) = \binom{6}{3} (0.3)^3 (0.7)^3 = 20 (0.21)^3 = 0.18522$$

$$P(X=3, N=6) = \boxed{0.0271}$$

\textcircled{10} $P(X_1=0 | Y=8) = \frac{P(X_1=0, Y=8)}{P(Y=8)}$ $P(Y=8) = \binom{10}{8} \left(\frac{2}{3}\right)^8 \left(\frac{1}{3}\right)^2$

$P(X_1=0, Y=8) = \left(\frac{1}{3}\right) \cdot \binom{9}{8} \left(\frac{2}{3}\right)^8 \left(\frac{1}{3}\right)$ $\rightarrow \frac{\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^8 \binom{9}{8}}{\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^8 \binom{10}{8}} = \frac{9!}{8!} \times \frac{8! 2!}{10!} = \frac{2}{10}$

\textcircled{11} $f_{Z|Y=15}(50) = \frac{P(X=10, Y=5 \text{ or } X=5, Y=10)}{f_{XY}(10, 5) \text{ or } (5, 10) \text{ or } (7, 8) \text{ or } (8, 7) \text{ or } (6, 9) \text{ or } (9, 6)}$

$$f_{XY}(10, 5) = f_{XY}(5, 10) = \binom{10}{5} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^5 \cdot \binom{10}{10} \left(\frac{1}{3}\right)^{10} = \binom{10}{5} \left(\frac{1}{3}\right)^{15} \left(\frac{2}{3}\right)^5$$

$$f_{XY}(6, 9) = f_{XY}(9, 6) = \binom{10}{6} \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^4 \cdot \binom{10}{9} \left(\frac{1}{3}\right)^9 \left(\frac{2}{3}\right) = \binom{10}{6} \left(\frac{1}{3}\right)^{15} \left(\frac{2}{3}\right)^5$$

$$f_{XY}(7, 8) = f_{XY}(8, 7) = \binom{10}{8} \left(\frac{1}{3}\right)^8 \left(\frac{2}{3}\right)^2 \cdot \binom{10}{7} \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right)^3 = \binom{10}{8} \left(\frac{1}{3}\right)^{15} \left(\frac{2}{3}\right)^5$$

$$\frac{10!}{5! 5!} = \frac{\cancel{10}^2 \times \cancel{9}^2 \times \cancel{8}^2 \times \cancel{7}^2 \times \cancel{6}^2}{\cancel{5}^3 \times \cancel{4}^2 \times \cancel{3}^2} = 28 \times 9$$

$$\frac{10! \times 10!}{6! 4! 9!} = \frac{10 \times \cancel{9} \times \cancel{8} \times \cancel{7}}{\cancel{6} \times \cancel{4} \times \cancel{2}} \times 10 = 100 \times 3 \times 7$$

$$\frac{10! \times 10!}{8! 2! 7! 3!} = \frac{10 \times 9 \times \cancel{8} \times \cancel{7}^2}{\cancel{8} \times \cancel{7} \times \cancel{6}} = 600 \times 9$$

$$\frac{\binom{10}{5} \left(\frac{1}{3}\right)^{15} \left(\frac{2}{3}\right)^5}{\left(\frac{1}{3}\right)^{15} \left(\frac{2}{3}\right)^5 \left[\binom{10}{5} + \binom{10}{6} + \binom{10}{8} \right]} = \frac{\frac{10!}{5! 5!}}{\frac{10!}{5! 5!} + \frac{10!}{6! 4!} + \frac{10!}{8! 2!}}$$

\textcircled{12} $X \sim \text{Uniform}\{0, 1, \dots, 9\} \quad Y \sim \text{Uniform}\{0, 1, \dots, 9\}$

$$P(Z \geq 2) = 1 - P(Z=1) - P(Z=0)$$

$$P(Z=0) \rightarrow X=Y = \frac{1}{10} \quad P(Z=1) \rightarrow |X-Y|=1 = \frac{18}{100}$$

$$P(Z \geq 2) = 1 - \frac{28}{100} = \frac{72}{100}$$

01	43	78
10	45	87
12	54	89
21	56	98
23	65	
32	67	
34	76	

— X — X — X — X —

