CHAPTER 2

→ Problems:

③ Given, $\beta_A = 1.15$ $\quad \sigma_A = 35\%$
$\beta_B = 0.95$ $\quad \sigma_B = 33\%$
$\sigma_M = 20\%$ $\qquad$ $Corr(A,B)$? $\qquad \omega_A > \omega_B$?

$$Corr(A,B) = \frac{Cov(A,B)}{\sigma_A \cdot \sigma_B}$$

$$Cov(A,B) = \beta_A \beta_B \sigma_M^2 = 0.0437$$

$$Corr(A,B) = \frac{0.0437}{(0.35)(0.33)} = 0.378$$

$$\sigma_A^2 = \beta_A^2 \sigma_M^2 + \omega_A^2$$

$$0.1225 = (1.3225)(0.04) + \omega_A^2$$

$$\omega_A = 26.38\%$$

$$\omega_B = 26.98\%$$

→ Appendix:

↳ Assumptions:

① Risk-Free asset exists
② All first and second moments exist
③ Impossible to build a fully invested portfolio with zero risk
④ Expected excess return on portfolio C (fully invested with minimum risk) is positive.

↳ Characteristic Portfolios:

→ Assets have attributes — beta, P/E ratio, m-cap, etc.
→ Characteristic Portfolio → uniquely capture the defining attribute

→ $a^T = \{a_1, a_2, \ldots, a_N\}$ → vector of asset attributes
h_p = vector of risky asset holding (% weights)
Exposure of portfolio h_p to attribute a: $\quad a_p = \sum_n a_n h_{p,n}$ or $a_p = a_p^T h_p$

Minimum Var:
$\min \frac{1}{2} w^T \Sigma w$
s.t. $w^T i = 1$

$L(w,\lambda) = \frac{1}{2} w^T \Sigma w - \lambda(w^T i - 1)$

$\frac{\partial L}{\partial w} = \Sigma w - \lambda i = 0$

$w^* = \lambda \Sigma^{-1} i$
$(\lambda \Sigma^{-1} i)^T \cdot i = 1$
$\Rightarrow (i^T \Sigma^{-1} i) \cdot \lambda = 1$
$\Rightarrow \lambda = \frac{1}{i^T \Sigma^{-1} i}$

$\boxed{w^* = \frac{\Sigma^{-1} i}{i^T \Sigma^{-1} i}}$

→ Proposition 1

① For any attribute $a \neq 0$, unique portfolio h_a exists that has minimum risk and unit exposure to a

$$h_a = \frac{V^{-1} a}{a^T V^{-1} a} \qquad (2A.1)$$

V → covariance matrix of excess returns for the risky assets

→ includes long and short positions with leverage.

② Variance of characteristic portfolio:

$$\sigma_a^2 = h_a^T V h_a = \frac{1}{a^T V^{-1} a} \qquad (2A.2)$$

③ Beta of all assets w.r.t. portfolio h_a is a

$$a = \frac{V h_a}{\sigma_a^2} \qquad (2A.3)$$

β w.r.t portfolio $p = \frac{V h_p}{\sigma_p^2}$

④ Two attributes a and d with char. portfolios h_a and h_d.
a_d → exposure of portfolio h_d to characteristic a : $a^T h_d$
d_a → " " " h_a " " d : $d^T h_a$

$$\sigma_{a,d} = a_d \sigma_a^2 = d_a \sigma_d^2 \qquad (2A.4)$$

$\sigma_{a,d} = h_a^T V h_d = [\sigma_a^2 a^T] h_d = a_d \sigma_a^2$

⑤ κ → positive scalar. Characteristic portfolio of κa is $\frac{h_a}{\kappa}$

⑥ If a is weighted combination of d and f, then char. portfolio of a is weighted combination of h_d and h_f.
If $a = \kappa_d d + \kappa_f f$, then

$$h_a = \left[\frac{\kappa_d \sigma_a^2}{\sigma_d^2}\right] h_d + \left[\frac{\kappa_f \sigma_a^2}{\sigma_f^2}\right] h_f \qquad (2A.5)$$

where, $\frac{1}{\sigma_a^2} = \left[\frac{\kappa_d a_d}{\sigma_d^2}\right] + \left[\frac{\kappa_f a_f}{\sigma_f^2}\right]$

→ Portfolio C $\quad$ Suppose $e^T = \{1, 1, \ldots, 1\}$ is the attribute.
Portfolio C, the char. portfolio for e is the minimum risk fully invested portfolio.

$$h_c = \frac{V^{-1} e}{e^T V^{-1} e} \qquad \sigma_c^2 = \frac{1}{e^T V^{-1} e}$$

For any portfolio P, we have $\sigma_{P,C}^2 = e_P \sigma_C^2$

→ Portfolio B $\quad$ has β as the attribute. h_B = Benchmark portfolio. $\beta = \frac{V h_B}{\sigma_B^2}$
Relationship b/w portfolio B and C: $\sigma_{B,C}^2 = e_B \sigma_C^2 = \beta_C \sigma_B^2$

→ Portfolio q : expected excess returns f have portfolio q as their char. portfolio.

→ **Proposition 2**

→ The expected excess returns f have portfolio q as their characteristic portfolio.

$$h_q = \frac{V^{-1}f}{f^T V^{-1} f} \qquad (2A.23)$$

① $SR_q = \max\{SR_P | P\} = (f^T V^{-1} f)^{1/2}$ (2A.24)

② $f_q = 1$ (2A.25) $\sigma_q^2 = \frac{1}{f^T V^{-1} f}$ (2A.26)

③ $f = \frac{V h_q}{\sigma_q^2} = \left[\frac{V h_q}{\sigma_q}\right] SR_q$ (2A.27) From ③ in Proposition 1

④ If $\rho_{P,q}$ is the correlation b/w portfolios P and q,

$SR_P = \rho_{P,q}\, SR_q$ (2A.28)

⑤ Fraction of q invested in risky assets:

$e_q = \frac{f_C \sigma_q^2}{\sigma_C^2}$ (2A.29) Follows from (2A.4): $\sigma_{q,C} = e_q \sigma_C^2 = f_C \sigma_q^2$

→ Portfolio A Define alpha as $\alpha = f - \beta f_B$. Let h_A be the char. portfolio for alpha, minimum risk with 100% alpha.

According to equation (2A.5), we can express h_A in terms of h_B and h_q.

From (2A.4), relationship b/w alpha and beta: $\sigma_{B,A} = \alpha_B \sigma_A^2 = \beta_A \sigma_B^2$.

However, $\alpha_B = 0$ by construction, so portfolios $A \perp B$ and $\beta_A = 0$.

→ **Proposition 3**

→ Assume $f_C > 0$

① Portfolio q is net long : $e_q > 0$ (2A.33)

Let portfolio Q be the char. portfolio of $e_q f$.

$h_Q = h_q / e_q$ from item ⑤ in Proposition 1.

② $\frac{f_C}{\sigma_C^2} = \frac{f_Q}{\sigma_Q^2}$ (2A.35)

$f = f_Q \left[\frac{V h_Q}{\sigma_Q^2}\right] = f_Q \beta_{w.r.t.Q}$ (2A.36) → specifies how portfolio Q explains expected returns

$f = \frac{V h_q}{\sigma_q^2}$, $\quad f_C \sigma_q^2 = e_q \sigma_C^2 \Rightarrow \frac{1}{\sigma_q^2} = \frac{f_C}{e_q \sigma_C^2}$

$f = f_C \left[\frac{V h_q}{\sigma_C^2 e_q}\right] = f_C \left[\frac{V h_Q}{\sigma_C^2}\right]$ "premultiply this by h_Q" ???

③ $\beta_Q = \frac{f_B \sigma_Q^2}{f_Q \sigma_B^2}$ (2A.37) $\quad \frac{h_q}{\sigma_q^2} = \frac{h_Q}{\sigma_Q^2} f_Q$

Multiply eq. (2A.27) by h_B: $h_B^T f = h_B^T \left[\frac{V h_q}{\sigma_q^2}\right] \rightarrow f_B = h_B^T V h_Q \left[\frac{f_Q}{\sigma_Q^2}\right] = \frac{h_B^T V h_Q}{\sigma_B^2}\left[\frac{\sigma_B^2 f_Q}{\sigma_Q^2}\right]$

$f_B = \beta_Q \left[\frac{\sigma_B^2 f_Q}{\sigma_Q^2}\right]$

④ $\beta_Q = \frac{\beta_C f_B}{f_C}$ (2A.38)

$e_B = 1$ and $\sigma_{B,C} = e_B \sigma_C^2 = \beta_C \sigma_B^2$ imply $\beta_C = \sigma_C^2 / \sigma_B^2$

when this is combined with $\frac{f_C}{\sigma_C^2} = \frac{f_Q}{\sigma_Q^2}$, we get

→

	Characteristic	Portfolio	Description
①	f	$h_q = \frac{V^{-1} f}{f^T V^{-1} f}$	Characteristic portfolio of expected excess returns with minimum risk. Max SR.
②	$e_q f$	$h_Q = \frac{h_q}{e_q}$	Portfolio q is net long. $e_q > 0$
③	β	$h_B = \frac{V^{-1}\beta}{\beta^T V^{-1} \beta}$	The Benchmark portfolio i.e., minimum risk portfolio with a beta of 1.
④	e	$h_C = \frac{V^{-1} e}{e^T V^{-1} e}$	The minimum-risk fully invested portfolio
⑤	$\alpha = f - \beta f_B$	h_A	The minimum-risk portfolio with alpha of 100%.