STATS 2

WEEK 1

→ Joint PMF of two discrete random variables
 ↳ Two events can either be independent; or
 ↳ one could be influencing the other.

→ Types of PMF of Multiple Random Variables:
① Joint PMF
② Marginal PMF
③ Conditional PMF

① Joint PMF

→ X and Y → discrete random variables defined in the same probability space

Range of X and Y → T_X and T_Y

↳ Joint PMF of X and Y → f_{XY} → function from $T_X \times T_Y$ to $[0,1]$

$$f_{XY}(t_1, t_2) = P(X = t_1 \text{ and } Y = t_2),\ t_2 \in T_X,\ t_2 \in T_Y$$

→ Usually written as a table or a matrix
↳

variable 1

variable 2

$P(x_1,x_2)$	$P(x_1,x_2)$	$P(x_1,x_2)$	...	..	..
...	...	...	...	...	...
...	...	...	..	...	...
...	...	...	...	..	..
...	...	...	...	...	...

② Marginal PMF

→ To obtain the individual PMF of a random discrete variable from a joint PMF,
↳ suppose X and Y random discrete variables, and their joint PMF → f_{XY}, then:

$$f_X(t) = P(X=t) = \sum_{t' \in T_Y} f_{XY}(t, t')$$

$$f_Y(t) = P(Y=t) = \sum_{t' \in T_X} f_{XY}(t', t)$$

③ Conditional PMF

→ $f_{X|A}(t)$ → PMF of a random variable X conditional on an event A.
$P(X=t \mid A)$, $t \in T_X$

$$f_{X|A}(t) = \frac{P((X=t) \cap A)}{P(A)}$$

→ $f_{Y|X}(t)$ → conditional PMF of Y given $X=t$

$$f_{Y|X=t}(t') = P(Y=t' \mid X=t) = \frac{P(Y=t', X=t)}{P(X=t)} = \frac{f_{XY}(t,t')}{f_X(t)}$$

$$f_{XY}(t,t') = f_{Y|X=t}(t')\, f_X(t)$$

→ $f_{XY}(t_1,t_2) = f_{Y|X=t_1}(t_2)\, f_X(t_1) = f_{X|Y=t_2}(t_1)\, f_Y(t_2)$

Q. Let $N \sim \text{Poisson}(\lambda)$. Given $N=n$, toss a fair coin n times and denote the number of heads obtained by X. What is the distribution of X?

A. $f_N(n) = \dfrac{e^{-\lambda}\lambda^n}{n!}$; $n = 0, 1, 2, \ldots$ $\quad (X|N=n) \sim \text{Bin}(n, 1/2) \quad f_{X|N=n}(k) = \binom{n}{k}\left(\frac{1}{2}\right)^n$

$$f_{XN}(k,n) = \frac{e^{-\lambda}\lambda^n}{n!} \cdot \frac{n!}{k!\,(n-k)!} \cdot \left(\frac{1}{2}\right)^n = \frac{e^{-\lambda}\lambda^n}{k!\,(n-k)!}\left(\frac{1}{2}\right)^n$$

$$\boxed{f_X(k) = \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{k!\,(n-k)!}\left(\frac{1}{2}\right)^n} \Rightarrow \frac{e^{-\lambda}\lambda^k}{k!\ 2^k} \sum_{n=k}^{\infty} \frac{(\lambda)^{n-k}}{(n-k)!\,(2)^{n-k}}$$

$$X \sim \text{Poisson}(\lambda/2) \Rightarrow \boxed{f_X(k) = \frac{e^{-\lambda/2}(\lambda/2)^k}{k!}}$$

→ Joint PMF of multiple random variables

↳ $X_1, X_2, X_3, \ldots X_n$ are discrete random variables defined in the same probability space.

$$f_{X_1X_2\cdots X_n}(t_1, t_2, \ldots, t_n) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and} \cdots \text{ and } X_n = t_n),\ t_i \in T_{X_i}$$

→ Marginal PMF with multiple random variables

↳ individual random variable ⇒ $f_{X_1}(t) = P(X_1 = t) = \displaystyle\sum_{t_2' \in T_2, t_3' \in T_3, \ldots, t_n' \in T_n} f_{X_1,X_2,X_3\cdots X_n}(t, t_2', t_3', \ldots, t_n)$

↳ multiple random variables ⇒ $f_{X_1X_2}(t_1, t_2) = P(X_1 = t_1 \text{ and } X_2 = t_2) = \displaystyle\sum_{t_3' \in T_3} f_{X_1,X_2,X_3}(t_1, t_2, t')$

→ Conditional PMF with multiple random variables

↳ $(X_1, X_2 | X_3 = t_3) \sim f_{X_1,X_2|X_3=t_3}(t_1, t_2) = \dfrac{f_{X_1,X_2,X_3}(t_1, t_2, t_3)}{f_{X_3}(t_3)}$

↳ $(X_1 | X_2 = t_2, X_3 = t_3) \sim f_{X_1|X_2=t_2,X_3=t_3}(t_1) = \dfrac{f_{X_1,X_2,X_3}(t_1, t_2, t_3)}{f_{X_2X_3}(t_2, t_3)}$

↳ factors of a Joint PMF: $f_{X_1\ldots X_4}(t_1, \ldots, t_4) = P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$

$$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)$$

$$= P(X_1 = t_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4) \cdot P(X_2 = t_2 | X_3 = t_3, X_4 = t_4) \cdot P(X_3 = t_3 | X_4 = t_4) \cdot P(X_4 = t_4)$$

$$f_{X_1\ldots X_4}(t_1, \ldots, t_4) = f_{X_1|X_2=t_2,X_3=t_3,X_4=t_4}(t_1) \cdot f_{X_2=t_2|X_3=t_3,X_4=t_4}(t_2) \cdot f_{X_3|X_4=t_4}(t_3) \cdot f_{X_4}(t_4)$$

WEEK 2

→ Independence of two random variables

↳ X and Y are independent if :

$$f_{XY}(t_1, t_2) = f_X(t_1) \cdot f_Y(t_2)$$

→ Joint PMF of f_{XY} is the product of the marginals PMF of X and Y.

→ Conditional PMF equals marginal PMF

→ Independence of multiple random variables

↳ $X_1, X_2, \ldots, X_n$ are independent iff

$$f_{X_1, X_2, \ldots X_n}(t_1, t_2, \ldots, t_n) = f_{X_1}(t_1) \cdot f_{X_2}(t_2) \cdot \ldots \cdot f_{X_n}(t_n)$$

→ Independent and Identically distributed (i.i.d.)

↳ Random variables $X_1, X_2, \ldots X_n$ are i.i.d. if → ① they are independent,
② Marginal PMFs f_{X_i} are identical.

→ $X_1, X_2 \ldots X_n \sim$ i.i.d. f_X

→ Memoryless property of Geometric distribution

→ Let $X \sim$ Geometric(p)

① $P(X > n) = (1-p)^n$

② $P(X > m+n \mid X > m) = \dfrac{P(X > m+n \cap X > m)}{P(X > m)} = \dfrac{P(X > m+n)}{P(X > m)} = \dfrac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n$

→ Functions of random variable

↳ One-to-one function → $P(Y = f(x)) = P(X = x)$; e.g. → $y = x-5$, $y = 2^x$

↳ Many-to-one function → e.g. → $y = (x-5)^2$, $y = x(1-x)$ → $y_0 = f(x_1) = f(x_2) = \ldots = f(x_m)$ all the places $f(x)$ takes the value y_0

↳ sum over all the probabilities of X for wherever the function takes the 'y' value. ⟶ $P(Y = y_0) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_m)$

→ Visualising function of 2 random variable

↳ $g(x,y)$: function

① Contours → values of (x,y) that result in $g(x,y) = c$
→ make a plot of those (x,y) for different c

② Regions → values of (x,y) that result in $g(x,y) \leq c$
→ make a plot of those (x,y) for different c

→ **Function of 2 random variables**

→ Let $X, Y \sim f_{XY}$; let $Z = g(X, Y)$ be a function of X and Y

· What is the PMF of Z?

→ Step 1 : Find the range of Z

→ Step 2 : Add over all the contours

↳ Suppose z is a possible value taken by Z :

$$P(Z = z) = \sum_{(x,y):\, g(x,y) = z} f_{XY}(x,y)$$

→ Sum of 2 uniform random variables

$X, Y \sim$ iid Unif$\{1, 2, \ldots, n\}$, $W = X + Y$

range of $W = \{2, 3, \ldots, 2n\}$

$W \in \{2, 3, \ldots, 2n\}$

$W = w : (1, w-1), (2, w-2), \ldots, (w-1, 1)$

$$P(W = w) = \begin{cases} \dfrac{w-1}{n^2}, & 2 \leq w \leq n+1 \\ \dfrac{2n-w+1}{n^2}, & n+2 \leq w \leq 2n \end{cases}$$

→ Max of 2 uniform random variables $X, Y \sim \text{iid Unif}\{1,2,\dots,n\}$, $Z = \max(X,Y)$

$$Z \in \{1,2,\dots,n\}$$

$$P(Z=z) = \frac{2z-1}{n^2}$$

→ **PMF of $g(x_1, x_2, \dots x_n)$** → The PMF $X = g(X_1, X_2, \dots, X_n)$ is given by

$$f_X(t) = P(g(x_1,x_2,\dots,x_n)=t) = \sum_{(t_1,t_2,\dots,t_n):\, g(t_1,t_2,\dots,t_n)=t} f_{X_1,X_2,\dots,X_n}(t_1,t_2,\dots,t_n)$$

e.g. →

$t_2 \cdot t_3$	$t_1+t_2+t_3$	t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1,t_2,t_3)$
0	0	0	0	0	1/9
0	1	0	0	1	1/9
0	2	0	0	2	1/9
1	2	0	1	1	1/9
2	3	0	1	2	1/9
0	1	1	0	0	1/9
0	3	1	0	2	1/9
0	2	1	1	0	1/9
1	3	1	1	1	1/9

① $X = g(X_1,X_2,X_3) = X_1+X_2+X_3$ $\quad g \in \{\overset{1/9}{0}, \overset{2/9}{1}, \overset{3/9}{2}, \overset{3/9}{3}\}$

② $Y = h(X_1,X_2,X_3) = X_2 \cdot X_3$ $\quad h \in \{\overset{6/9}{0}, \overset{2/9}{1}, \overset{1/9}{2}\}$

f_{XY}	0	1	2	3
0	1/9	2/9	2/9	1/9
1	0	0	1/9	1/9
2	0	0	0	1/9

→ sum of n independent Bernoulli(p) = Binomial (n,p)

→ suppose X and Y take integer values and their joint PMF = f_{XY}. Let $Z = X+Y$

Convolution

$$P(Z=z) = \sum_{x=-\infty}^{\infty} P(X=x, Y=z-x) = \sum_{x=-\infty}^{\infty} f_{XY}(x, z-x) = \sum_{y=-\infty}^{\infty} f_{XY}(z-y, y)$$

$$= \sum_{x=-\infty}^{\infty} f_X(x) f_Y(z-x) = \sum_{y=-\infty}^{\infty} f_Y(y) f_X(z-y)$$

→ Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent.

→ $Z = X + Y$

$$f_Z(z) = \sum_{x=0}^{\infty} f_X(x) \cdot f_Y(z-x) = \sum_{x=0}^{z} \frac{e^{-\lambda_1} \cdot \lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2^{(z-x)}}{(z-x)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{z!} \underbrace{\sum_{x=0}^{z} \frac{\lambda_1^x \lambda_2^{z-x} \cdot z!}{x!\,(z-x)!}}_{\to (\lambda_1+\lambda_2)^z}$$

can be replaced with x going to z because after z this term will go to 0.

$$f_Z(z) = \frac{e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1+\lambda_2)^z}{z!} \Rightarrow Z \sim \text{Poisson}(\lambda_1+\lambda_2)$$

→ conditional of $X|Z$

$$P(X=k|Z=n) = \frac{P(x=k, Z=n)}{P(Z=n)} = \frac{P(x=k) \cdot P(Z=n|x=k)}{P(Z=n)} = \frac{P(X=k) \cdot P(Y=n-k)}{P(Z=n)}$$

$$= \frac{\frac{\cancel{e^{-\lambda_1}} \cdot \lambda_1^k}{k!} \cdot \frac{\cancel{e^{-\lambda_2}} \cdot \lambda_2^{n-k}}{(n-k)!}}{\frac{\cancel{e^{-\lambda_1-\lambda_2}} \cdot (\lambda_1+\lambda_2)^n}{n!}} = \frac{n!}{k!\,(n-k)!} \cdot \frac{\lambda_1^k \cdot \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}$$

$\longrightarrow (\lambda_1+\lambda_2)^n = (\lambda_1+\lambda_2)^k \cdot (\lambda_1+\lambda_2)^{n-k}$

$\longrightarrow \frac{n!}{k!\,(n-k)!} = {}^nC_k$

$$= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$X|Z = \text{Bin}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$Y|Z = \text{Bin}\left(n, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$$

→ Functions of non-overlapping indepent random variables are also independent

→ If X and Y are independent, $g(X)$ and $h(Y)$ are independent for any two functions g and h

→ Min/Max of two random variables

↳ $X, Y \sim f_{XY}$ $\quad\quad Z = \min(X, Y)$

$$f_Z(z) = P(\min(X,Y) = z) = P((X=z) \text{ and } (Y=z) \text{ or } (X=z \text{ and } Y>z) \text{ or } (X>z \text{ and } Y=z))$$
$$= f_{XY}(z,z) + \sum_{t_2>z} f_{XY}(z,t_2) + \sum_{t_1>z} f_{XY}(t_1,z)$$

↳ CDF of a random variable

↳ CDF of a random variable X is a function $F_X : \mathbb{R} \to [0,1]$ defined as:

↳ $F_X(x) = P(X \le x)$

e.g. ⇒ X and Y are independent. $Z = \max(X,Y)$

$$F_Z(z) = P(\max(X,Y) \le z)$$
$$= F_X(z) \cdot F_Y(z)$$

↳ Let $X_1, X_2, \dots, X_n \sim$ i.i.d.

① Distribution of $\underline{\min(X_1, X_2, \dots, X_n)} \to P(\min(X_1,X_2,\dots,X_n) \ge z) = (P(X \ge z))^n$

② Distribution of $\underline{\max(X_1, X_2, \dots, X_n)} \to P(\max(X_1,X_2,\dots,X_n) \le z) = (P(X \le z))^n = (F_X(z))^n$

↳ Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(p)$ be independent. Find the distr. of $\underline{\min(X,Y)}$

$Z = \min(X,Y)$ $\quad\quad P(Z \ge z) = P(X \ge z, Y \ge z) = (1-p)^{z-1} \cdot (1-p)^{z-1}$

$$= ((1-p)^2)^{z-1}$$

$$P(Z \ge z+1) = ((1-p)^2)^z$$

$$P(Z = z) = P(Z \ge z) - P(Z \ge z+1)$$
$$= ((1-p)^2)^{z-1} - ((1-p)^2)^z = ((1-p)^2)^{z-1}(1-(1-p)^2)$$ ← Geometric probability

$$\min(X,Y) \sim \text{Geometric}(1-(1-p)^2)$$

Let $X_1 \sim \text{Geometric}(p_1)$ and $X_2 \sim \text{Geometric}(p_2)$

↳ $\min(X_1, X_2) \sim \text{Geometric}(1-(1-p_1)(1-p_2))$

$Z = \max(X_1, X_2, \dots, X_{10})$ $\quad\quad X \sim \text{Bin}(6, 1/2)$

$$P(Z \le 2) = P(X_1 \le 2, X_2 \le 2, \dots, X_{10} \le 2)$$
$$= \left(22 \cdot \left(\tfrac{1}{2}\right)^6\right)^{10}$$

X	0	1	2	3
	$\left(\frac{1}{2}\right)^6$	$6\left(\frac{1}{2}\right)^6$	$15\left(\frac{1}{2}\right)^6$	

$$\binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6 \times 5}{2} = 15$$

$$X \le 2 = \left(\tfrac{1}{2}\right)^6 \cdot 22$$

$Z = \min(X_1, X_2, \dots, X_{10})$ $\quad\quad X \sim \text{Bin}(6, 1/2)$

$$F_Z(2) = 1 - P(X_1 > 2, X_2 > 2, \dots X_{10} > 2)$$ $\quad\quad X > 2 = 42\left(\tfrac{1}{2}\right)^6$

$$= 1 - \left(\frac{42}{2^6}\right)^{10} = 1 - \left(\frac{21}{2^5}\right)^{10}$$

WEEK 3

WEEK 1 GA

① $T_x = \{0,1,2,3\}$ $\quad T_y = \{-1,1,2,3\}$ $\qquad X \sim Bin(3, 1/2)$ $\qquad P(X<3) = 1 - P(X=3) = 7/8$ $\qquad P(X=3) = \binom{3}{3}\cdot(1/2)^3 = 1/8$

$P(Y\le 1) = P(Y=1) + P(Y=-1)$ $\qquad P(Y=1) = 1/2$ $\qquad P(Y=-1) = 1/8$ $\qquad P(Y\le 1) = 1/2 + 1/8 = 5/8$

$$f_{XY}(t_x<3,\ t_y\le 1) = \frac{5}{8}\times\frac{7}{8} = \frac{35}{64} \approx \boxed{0.5469}$$

② $T_x = T_y = T_z = \{0,1,2\}$

X	Y	Z
0	0	2
2	0	0
1	0	1

$= \boxed{1/3}$

$$f_{XY}(2,0) = \sum_{t_z=0}^{2} f_{XYZ}(2,0,t_z) = P(2,0,0) = 1/9$$

$$f_Y(0) = \sum_{t_x\in T_x,\ t_z\in T_z} f_{XYZ}(t_x,0,t_z) = P(0,0,2)+P(1,0,1)+P(2,0,0) = 3/9$$

$$f_{X|Y=0}(2) = \frac{f_{XY}(2,0)}{f_Y(0)}$$

$$f_{X|Y=0}(2) = \frac{1/9}{3/9} = \boxed{1/3}$$

③ $1/4 + 1/4 + 1/8 + 1/8 + k = 1$ $\qquad k = 1 - (2/4 + 1/4) = 1 - 3/4 = 1/4$

$$f_{Y|X=1}(2) = \frac{f_{YX}(2,1)}{f_X(1)} = \frac{1/4}{1/8+1/8+1/4} = \frac{1/4}{1/2} = \frac{1}{4}\times 2 = \boxed{\frac{1}{2}}$$

⑤ $f_Y(1) = \sum_{t'\in T_x} f_{XY}(t',1)$ $\qquad f_X(1) = 12/13$ $\qquad f_X(0) = 1/13$

$Y \sim Bin(n,p)$

$f_{Y|X=1}(1)$ = $n=3$, $p=7/20$

$$= \binom{3}{1}\left(\frac{7}{20}\right)\left(\frac{13}{20}\right)^2 = \frac{3\times 7\times(13)^2}{(20)^3} = 0.443625$$

$f_{Y|X=0}(1)$ = $n=2$, $p=4/15$

$$= \binom{2}{1}\left(\frac{4}{15}\right)\left(\frac{11}{15}\right) = \frac{2\times 4\times 11}{(15)^2} = 0.39111$$

$$f_{XY}(0,1) = f_{Y|X=0}(1)\cdot f_X(0) = 0.03008547$$

$$f_{XY}(1,1) = f_{Y|X=1}(1)\cdot f_X(1) = 0.4095$$

$$f_Y(1) = f_{XY}(0,1) + f_{XY}(1,1) \approx \boxed{0.4396}$$

⑦ $f_{XY}(x,y) = a(bx+y)$

x \ y	0	1	2	3
0	0	a	2a	3a
1	ab	ab+a	ab+2a	ab+3a
2	2ab	2ab+a	2ab+2a	2ab+3a
	3ab	3ab+2a	3ab+4a	

$$3\left(12a + 3ab = \frac{3}{7}\right) = 36a + 9ab = \frac{9}{7}$$

$$2\left(6a + 9ab = \frac{4}{7}\right) = 12a + 18ab = \frac{8}{7}$$

$$-12a - 3ab = -\frac{3}{7}$$

$$15ab = \frac{5}{7}$$

$$36a + 9ab = \frac{9}{7}$$
$$-6a - 9ab = -\frac{4}{7}$$
$$30a = \frac{5}{7}$$
$$\boxed{a = \frac{1}{42}}$$

$$ab = \frac{1}{21}$$

$$\frac{b}{42} = \frac{1}{21} \Rightarrow \boxed{b=2}$$

$$f_{XY}(1,1) = \frac{1}{42}(2+1) = \boxed{\frac{1}{14}}$$

⑧ $$f_{Y|X=2}(0) = \frac{f_{YX}(0,2)}{f_X(2)}$$

$$f_X(2) = \binom{6}{2}(1/2)^6$$

$$f_{X|Y=0}(2) = \binom{5}{2}(1/2)^5 \ ;\ f_Y(0) = 1/2 \ ;\ f_{YX}(0,2) = \binom{5}{2}(1/2)^6$$

$$f_{Y|X=2}(0) = \frac{\binom{5}{2}\cdot(1/2)^6}{\binom{6}{2}\cdot(1/2)^6} = \frac{\binom{5}{2}}{\binom{6}{2}}$$

$$= \frac{5!}{2!\,3!}\times\frac{2!\,4!}{6!} = \frac{5!}{6!}\times\frac{4!}{3!} = \frac{4}{6} = \boxed{0.666}$$

⑨ $\dfrac{\binom{5}{1}\times\binom{4}{1}}{\binom{12}{2}} = \dfrac{5\times4}{\frac{12!}{10!\,2!}} = \dfrac{5\times4}{\cancel{12}_{3}\times 11}\times 2 = \dfrac{10}{33} = \boxed{0.30}$

⑩ $N \sim Bin(7, 1/2) \qquad X \sim Bin(n, 1/2)$

$x \backslash N$	0	1	2	3	4
0	0	${}^7C_1(1/2)^8$			$\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11}$
1	0	${}^7C_1(1/2)^8$		$\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10}$	
2	0	0	$\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9}$		
3	0	0	0		
4	0	0	0	0	

$\binom{7}{4}\binom{4}{0}\left(\frac{1}{2}\right)^{11} = \frac{7!}{4!\,3!}\times\frac{1}{2048} = \frac{7\times6\times5}{3\times2}\times\frac{1}{2048} = \frac{35}{2048}$

$\binom{7}{3}\binom{3}{1}\left(\frac{1}{2}\right)^{10} = \frac{7!}{3!\,4!}\times\frac{3}{1024} = \frac{210}{2048}$

$\binom{7}{2}\binom{2}{2}\left(\frac{1}{2}\right)^{9} = \frac{7!}{2!\,5!}\times\frac{1}{512} = \frac{7\times6}{2}\times\frac{1}{512} = \frac{84}{2048}$

Total $= \dfrac{35+210+84}{2048} = \dfrac{329}{2048} = \boxed{0.161}$

WEEK 2 GA

②

	0	1	2	
0	0.06	0.18	0.12	0.36
1	0.04	0.12	0.48	0.64
	0.1	0.3	0.6	

	0	1	2	
0	2/24	1/24	1/24	4/24
1	4/24	2/24	3/24	9/24
2	6/24	3/24	2/24	11/24
	12/24	6/24	6/24	

	0	1	2	
0	1/10	2/10	2/10	5/10
1	1/10	1/10	3/10	5/10
	2/10	3/10	5/10	

	0	1	
0	1/10	1.5/10	2.5/10
1	2/10	3/10	5/10
2	1/10	1.5/10	2.5/10
	4/10	6/10	

③ $X \sim Bernoulli(0.2) \quad Y \sim Bernoulli(0.4) \qquad Z = X+Y$

$f_{X|Z=1}(1) = \dfrac{P(X=1, Z=1)}{P(Z=1)} = \dfrac{P(X=1, Y=0)}{P(X=1, Y=0 \text{ or } X=0, Y=1)} = \dfrac{(0.2)(0.6)}{(0.2)(0.6)+(0.8)(0.4)} = \dfrac{0.12}{0.12+0.32} = \boxed{0.2727}$

④ $Z = X+Y \qquad f_{XY}(x,y) = \dfrac{9}{16\cdot(4)^{x+y}}$

$f_Z(k) = P(X=x, Y=k-x) = \sum_{x=0}^{k} f_{XY}(x, k-x) = \boxed{\dfrac{(k+1)\cdot 9}{16\cdot(4)^k}}$

⑤ $Z = \max(X, Y)$

$f_Z(k) = P(X=k, Y\le k \text{ or } X\le k, Y=k) = \sum_{y=0}^{k} f_{XY}(k,y) + \sum_{x=0}^{k} f_{XY}(x,k)$

$= 2\cdot\sum_{x=0}^{k} \dfrac{9}{16\cdot(4)^{k+x}} =$

⑥

$y \backslash x$	1	2	3	4	5	6
1				*		
2					*	
3						*
4						
5						
6						

$f_X(4)\cdot f_Y(1) + f_X(5)\cdot f_Y(2) + f_X(6)\cdot f_Y(3)$

$= \left(\frac{1}{36}\right)\times 3 = \frac{3}{36} = \frac{1}{12}$

(7) $X \sim \text{Geometric}(p)$ $\quad Y \sim \text{Geometric}(p)$ iid $\quad Z = X + Y$

$f_Y(k) = p(1-p)^{k-1}$

$f_Z(k) = P(X=u, Y=k-u) = \sum_{u=1}^{k-1} p(1-p)^{u} \cdot p(1-p)^{k-u}$

$= p^2(1-p)^k (k-1)$

$f_Z(18) = p^2(1-p)^{18}(17) \qquad f_Z(19) = p^2(1-p)^{19}(18)$

$f_Z(18) < f_Z(19) \Rightarrow \cancel{p^2(1-p)^{18}}(17) < \cancel{p^2}(1-p)^{19}(18)$

$\frac{17}{18} < 1-p$

$\boxed{p < 0.056}$

(8) $X \sim \text{Poisson}(2)$ $\quad f_X(1) = e^{-2} \cdot 2 \quad f_X(2) = e^{-2} \cdot 2 \quad f_X(\neq 1 \text{ or } 2) = 1 - 4e^{-2}$

X_1	X_2	X_3
1	2	
1		2
2	1	
2		1
	2	1
	1	2

$6 \cdot (2e^{-2})^2 (1-4e^{-2}) = 0.439575(1-4e^{-2}) = \boxed{0.2016}$

(9) $X \sim \text{Bernoulli}(0.8)$ $\quad Y \sim \text{Bernoulli}(0.3)$ $\quad Z = X + Y - XY$

$f_Z(1) = P(X=1, Y=0 \text{ or } X=0, Y=1 \text{ or } X=1, Y=1) = 1 - P(X=0, Y=0) = 1-(0.2)(0.7) = 0.86$

$Z \sim \text{Bernoulli}(0.86)$

(10) $X, Y \sim \text{Geometric}(0.8)$ $\quad P(X=1 \mid X+Y=2) = \frac{P(X=1, X+Y=2)}{P(X+Y=2)} = \frac{P(X=1, Y=1)}{P(X=1, Y=1)}$

(11) $X \sim \text{Poisson}(5)$ $\quad Y \sim \text{Poisson}(1)$ $\quad Z = X + Y$

$Z \sim \text{Poisson}(6)$

$Y \mid Z=4 \sim \text{Binomial}(4, 1/6)$

$Y \mid Z=4\,(3) = \binom{4}{3}(1/6)^3(5/6) = \frac{4 \cdot 5}{(6)^4} = \boxed{0.0154}$