Foundations of Machine Learning

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Lecture 4: Concept Checks

Convexity

Optional Learning Objectives

Convex optimization and Lagrangian duality will not covered on the midterm exam, so in some sense these objectives are optional.

- Define a convex set, a convex function, and a strictly convex function. (Don't forget that the domain of a convex function must be a convex set!)
- For an optimization problem, define the terms feasible set, feasible point, active constraint, optimal value, and optimal point.
- Give the form for a general inequality-constrained optimization problem (there are many ways to do this, but our convention is to have inequality constraints of the form $f_i(x) \leq 0$).
- Define the Lagrangian for this optimization problem, and explain how the Lagrangian encodes all the information in the original optimization problem.
- Write the primal and dual optimization problem in terms of the Lagrangian.

Convexity Concept Check Problems

1. If $A, B \subseteq \mathbb{R}^n$ are convex, then $A \cap B$ is convex.
2. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be convex. Show that $af + bg$ is convex if $a, b \geq 0$.
3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Prove that if $\nabla f(x) = 0$ then x is a global minimizer.
4. Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex and x is a global minimizer, then it is the unique global minimizer.

5. Prove that any affine function $f : \mathbb{R}^n \to \mathbb{R}$ is both convex and concave.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and let $g : \mathbb{R}^m \to \mathbb{R}^n$ be affine. Then $f \circ g$ is convex.

7. ($\star\star$)

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be convex. Show that f has one-sided left and right derivatives at every point.

   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Show that f has one-sided directional derivatives at every point.

   (c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Show that if x is not a minimizer of f then f has a descent direction at x (i.e., a direction whose corresponding one-sided directional derivative is negative).

Convex Optimization Problems

1. Suppose there are mn people forming m rows with n columns. Let a denote the height of the tallest person taken from the shortest people in each column. Let b denote the height of the shortest person taken from the tallest people in each row. What is the relationship between a and b?

2. Let $x_1, \ldots, x_n \in \mathbb{R}^d$ be given data. You want to find the center and radius of the smallest sphere that encloses all of the points. Express this problem as a convex optimization problem.

3. Suppose $x_1, \ldots, x_n \in \mathbb{R}^d$ and $y_1, \ldots, y_n \in \{-1, 1\}$. Here we look at y_i as the label of x_i. We say the data points are linearly separable if there is a vector $v \in \mathbb{R}^d$ and $a \in \mathbb{R}$ such that $v^T x_i > a$ when $y_i = 1$ and $v^T x_i < a$ for $y_i = -1$. Give a method for determining if the given data points are linearly separable.

4. Consider the Ivanov form of ridge regression:

$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2^2 \\ \text{subject to} & \|x\|_2^2 \le r^2, \end{array}$$

   where $r > 0$, $y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$ are fixed.

   (a) What is the Lagrangian?

   (b) What do you get when you take the supremum of the Lagrangian over the feasible values for the dual variables?