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Indira Gandhi National Open University
School of Sciences

BMTC-131
CALCULUS

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5
INTEGRATION

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BLOCK 5 INTEGRATION

The ancients knew how to find the area and circumference of a circle. They also knew how to find the area of many other regular figures like polygons. However, they were not able to find the areas, lengths and volumes of irregular figures with the methods available to them. These problems were solved by the Europeans only after renaissance. The important concept of infinitesimally small objects and how to add them up was found only in this period. Many of the insights were gained from the study of problems of Astronomy, Geometry and physical sciences. In the seventeenth century with the insights of their predecessors like Isaac Barrow and others, Newton and Leibniz laid the foundations of Calculus. Their work was improved upon by their successors like the Bernoulli family, Euler, Lagrange, Laplace and later on by Cauchy, Riemann and others.

In the second, third and fourth blocks of this course, we discussed the process of differentiation. We also saw how differentiation is a useful tool in Mathematics. In this block, we will study integration which can be viewed as the reverse process of differentiation.

In Unit 17, we introduce you to integration. In this unit we will see how the area below the graph of a function can be approximated by sums of areas of rectangles. As the number of rectangles increase and their sizes grow smaller and smaller their sum approaches the area. We will also see the Fundamental Theorem of Calculus which links the process of integration and differentiation. We will also see how to integrate some simple functions here.

In Unit 18, we focus on techniques for integrating a variety of functions like rational functions, trigonometric and functions that involve square root of a degree two polynomial. The back bone of all these methods is the method of substitution that we discuss at the beginning of this unit. We will also see some other techniques like integration by parts. We also see how to use partial fractions to integrate rational functions.

In Unit 19, we will discuss reduction formulas. The reduction formulas help us in integrating some common functions that we come across in applications. The main idea behind a reduction formula is to progressively reduce a parameter, usually the power of x or a trigonometric function in the integrand, using integration by parts. This results in a simpler integrand which we can integrate using standard methods.

In Unit 20, we will see how we can find the areas bounded by plane curves and the length of plane curves by a variety of methods. We begin by discussing the method for finding the area between the graphs of two functions when the graph is given in cartesian coordinates. We then discuss how to find the areas of curves in plane given in polar and parametric form. We will also discuss how to find the length of the curves given in cartesian, polar and parametric forms.

Finally, at the end of this Block, we have given some miscellaneous examples and exercises. We hope that they will be useful in reinforcing your understanding of the material in the units.

A word about some signs used in the unit! Throughout each unit, you will find theorems, examples and exercises. To signify the end of the proof of a theorem, we use the sign ■. To show the end of an example, we use $***$. Further, equations that need to be referred to are numbered sequentially within a unit, as are exercises and figures. E1, E2 etc. denote the exercises and Fig. 1, Fig. 2, etc. denote the figures.

NOTATIONS AND SYMBOLS

$L(P, f)$ Lower Sum, page 10

$U(P, f)$ Upper Sum, page 10

$\mathcal{P}([a, b])$ Set of partitions on $[a, b]$, page 8

$\overline{\int_a^b} f(x)dx$ Upper integral of the function $f(x)$ over the interval $[a, b]$., page 15

$\int f(x)\, dx$ Indefinite Integral of $f(x)$, page 25

$\underline{\int_a^b} f(x)\, dx$ Lower integral of the function over the interval $[a, b]$., page 15

$\mathcal{L}(f)$ $\{L(P, f) | P$ is a partition of $[a, b]\}$, page 15

$\mathcal{U}(f)$ $\{U(P, f) | P$ is a partition of $[a, b]\}$, page 15

For other notations and symbols, please refer to the lists in the previous blocks.

UNIT 17

INTRODUCTION TO INTEGRATION

Structure

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17.3	Fundamental Theorem of Calculus	21
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17.1 INTRODUCTION

As we have seen in the introduction to this block, from ancient times, measuring areas and volumes was essential for construction purposes. We have also seen that it was necessary to measure the lands for the purpose of levying taxes. Formulae for the areas of plane figures like squares and rectangles and squares were mentioned in ancient manuscripts, but no justifications were given.

In the third century B.C., the Greek mathematician Archimedes found a formula for the area of a circle by what is now known as **method of exhaustion**. His solution contained the seeds of the present day integral calculus. But it was only later, in the seventeenth century, that Newton and Leibniz were able to generalise Archimedes' method. They also established the link between differential and integral calculus.

A. L. Cauchy
1789-1857

The French mathematician A. L. Cauchy gave a definition of the integral under which only the continuous functions are integrable. Later, the German mathematician Riemann gave a definition of the integral under which a larger class of functions are integrable.

Other mathematicians like Stieltjes and Lebesgue have defined integrals which improve upon the Riemann Integral. However, the most popular way of introducing integration is through the Riemann Integral and we will follow this practice. In our discussion, we will use an equivalent definition of the integral due to Darboux, a French Mathematician.

G.B. Riemann
1826-1866

In Sec. 17.2, we discuss upper and lower sums introduced by Darboux. This lead us to a definition of the definite integral and some of its basic properties. In Sec. 17.3, we will state the Fundamental Theorem of Integral Calculus and explain how to use it evaluate some definite integrals. In Sec. 17.3 we will also discuss the indefinite integral. In Sec. 17.4, we will find the indefinite integrals of some common functions like polynomial, rational functions, trigonometric functions and functions like exponential, logarithm and inverse trigonometric functions.

Objectives

After studying this unit, you should be able to:

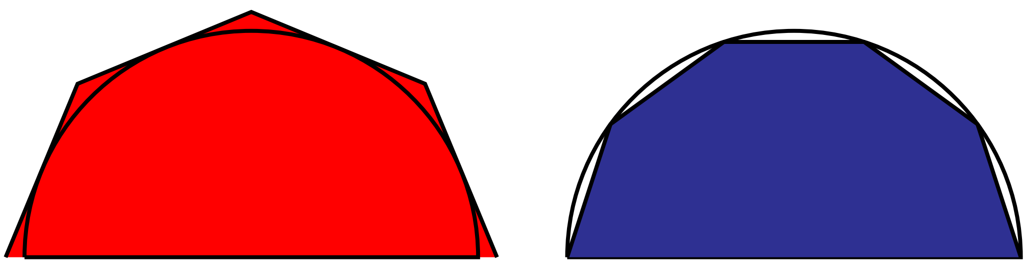
J. G. Darboux
1842-1917

- define and calculate the lower and upper sums of some simple functions defined on $[a, b]$, corresponding to a partition of $[a, b]$;
- define the upper and lower integrals of a function;
- define the definite integral of a given function and check whether a given function is integrable or not in simple cases;
- state and apply the Fundamental Theorem of Calculus;
- define the primitive or antiderivative of a function;
- use the Fundamental Theorem to calculate the definite integral of an integrable function; and
- find the indefinite integrals of some common functions.

17.2 INTEGRABILITY

In the introduction, we mentioned that Archimedes found the area of a circle using the **method of exhaustion.** Let us briefly discuss this method with the help of an example. Suppose that we want to find the area of a semi-circle. We bound the area from the above and below by regular polygons. See Fig. 1.

Our discussion is only for the purpose of motivation. We don't claim that we are being faithful to the history of Mathematics.



(a) Upper bound for the area of a semi-circle. **(b) Lower bound for the area of a semicircle.**

Fig. 1: Bounds for the area of a semicircle.

In Fig. 1a, the outer polygon, coloured red, gives an upper bound for the area of the semicircle. In Fig. 1b, the outer polygon, coloured blue gives a lower bound for the area of the semicircle.

In Fig. 2, the yellow coloured region gives the difference between the area of the semicircle and the area of the inner polygon. The red coloured region gives the difference between the area of the outer polygon and the area of the semicircle. You can see in the figure that as we increase the number of sides of the inner and outer polygons, the red and yellow coloured regions become smaller and smaller. In other words, the difference between the areas become smaller and the areas of the outer and inner polygons become closer and closer to the area of the semicircle. In the limiting case, they give the area of the circle.

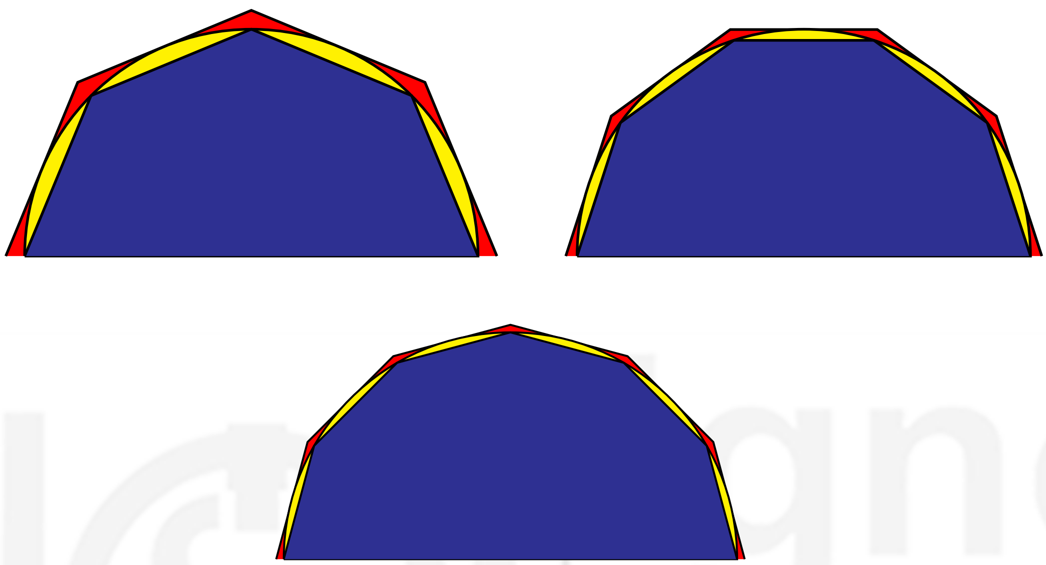


Fig. 2: Area of a semi-circle by method of exhaustion.

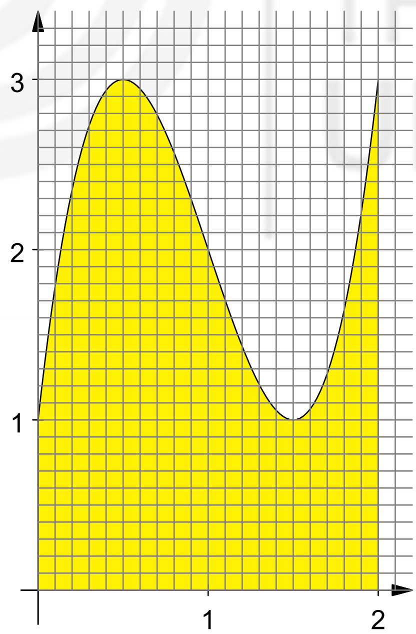


Fig. 3: Area under the curve f(x) = $4x^3 - 12x^2 + 9x + 1$.

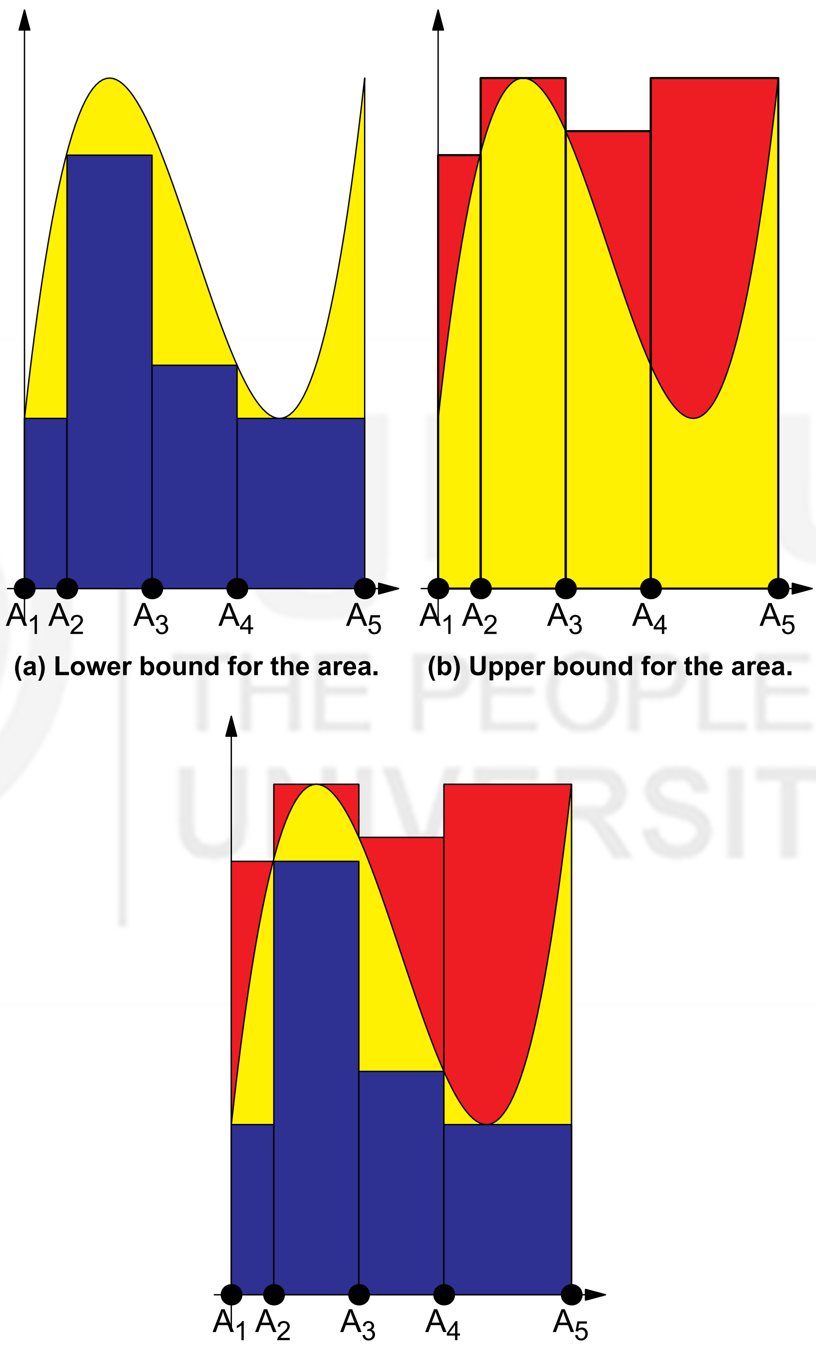
On the other hand suppose that we want to find the area under the curve $f(x) = 4x^3 - 12x^2 + 9x + 1$ from $x = 0$ to $x = 2$. You learnt how to find the maxima and minima of functions in the third block. You can use that knowledge to check the following:

1. In the interval $[0, 2]$, the function $f(x)$ is increasing between $x = 0$ and

$x = \frac{1}{2}$ and decreasing between $x = \frac{1}{2}$ and $x = \frac{3}{2}$. It is increasing between $x = \frac{3}{2}$ and $x = 2$.

2. The function has a maxima at $x = \frac{1}{2}$ and $x = 2$ in the interval $[0, 2]$; it has a minima at $x = \frac{3}{2}$.

See Fig. 3. Unlike the semi-circle, the area under f(x) cannot be closely approximated by regular polygons. So, instead of a single figure like a regular polygon, we use several rectangles. See Fig. 4.



(a) Lower bound for the area. **(b) Upper bound for the area.**

(c) Differences in area.

Fig. 4: Approximating the area under the curve using rectangles.

Here, we subdivide the interval $[0, 2]$ into four parts at the points $x = \frac{1}{4}$, $x = \frac{3}{4}$, $x = \frac{5}{4}$. So, A_1 denotes the point $(0, 0)$, A_2 denotes the point $\left(\frac{1}{4}, 0\right)$, A_3 denotes the point $\left(\frac{3}{4}, 0\right)$, A_4 denotes the point $\left(\frac{5}{4}, 0\right)$ and A_5 denotes the point $(2, 0)$.

Consider the blue coloured rectangles with the line segments A_1A_2, A_2A_3, A_3A_4 and A_4A_5 as bases. You can see that the height of each rectangle is the *infimum* value of the function f(x) in the interval given by the base. For example, the height of the blue coloured rectangle with base A_1A_2 is

$$\inf\left\{f(x)\,\middle|\,x \in \left[0, \frac{1}{4}\right]\right\}.$$

You are probably wondering how we can find the infimum of f(x). As we mentioned earlier, f(x) goes on increasing from x = 0 all the way up to $x = \frac{1}{4}$. Because of this the value of f(x) at the starting point x = 0 is less than the value of f(x) at any point in $\left]0, \frac{1}{4}\right]$. So, the infimum value is attained at the starting point x = 0 and the infimum value is f(0) = 1.

As you can see in Fig. 4(a), the sum of the areas of these blue rectangles is less than the area under the curve. So, the sum of the areas of the blue rectangles give a *lower bound* for the area of the curve y = f(x).

Similarly, the height of each red coloured, outer, rectangle in Fig. 4(b) is the *supremum* value of f(x) in the interval given by the base. For example, the height of the outer rectangle with base A_1A_2 is

$$\sup\left\{f(x)\,\middle|\,x \in \left[0, \frac{1}{4}\right]\right\}.$$

Again, the fact that f(x) is increasing from x = 0 to $x = \frac{1}{4}$ comes in useful in finding the supremum. Because it is increasing, the value at $x = \frac{1}{4}$ is greater than all the values of f(x) in the interval $\left[0, \frac{1}{4}\right[$. So, the supremum value is attained at $x = \frac{1}{4}$ and this is $f\left(\frac{1}{4}\right) = \frac{41}{16} = 2\frac{9}{16}$ units.

As you can see in Fig. 4(b), the sum of the areas of the red coloured rectangles is *greater than* the area under the curve. So, the sum of the red coloured rectangles give an *upper bound* for the area under the curve.

You may have noticed that, in Fig. 4((c),the yellow coloured portion gives the difference between the area under the curve and the areas of the smaller, blue coloured, rectangles. The red coloured portion gives the difference between the area under the curve and the areas of the larger, red coloured, rectangles. As in the case of the method of exhaustion, the sum of the areas of the outer rectangles and the sum of the areas of the inner rectangles approach closer and closer to the area under f(x).

In the next subsection, we will put our discussion on a formal footing.

17.2.1 Upper and Lower Sums

We begin this subsection with the definition of a partition.

Definition 1: Suppose $n \geq 1$ and let

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < b = x_n$$

be points in an interval $[a, b] \subset \mathbb{R}$. Then, the ordered set

$$\{x_0, x_1, \ldots, x_n\} \subset [a, b]$$

is called a **partition** of [a, b]. We will denote the set of all partitions of [a, b] by $\mathcal{P}([a, b])$.

By an **ordered set**, we mean that the order of listing the elements is important. Thus, $\{1, 2\} \neq \{2, 1\}$ as ordered sets. The ordered sets are special kind of sets and should not be confused with the usual sets where the order is not important.

Example 1: Give a partition of the interval $[0, 2]$.

Solution: The set of points we used to subdivide the interval $[0, 2]$ in our earlier discussion, namely $\left\{0, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, 2\right\}$, gives a partition of the interval $[0, 2]$. Note that, here $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$, $x_3 = \frac{5}{4}$ and $x_4 = 2$.

* * *

Next, we introduce the concepts of upper and lower sums.

Definition 2: Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function and let

$$P = \{a = x_0 < x_1 < \ldots < x_n = b\} \subset [a, b]$$

be a partition. For $1 \le i \le n$, let us write

$$\Delta_i = x_i - x_{i-1} \qquad \ldots (1)$$

$$m_i = \inf\{f(x) | x \in [x_{i-1}, x_i]\} \qquad \ldots (2)$$

$$M_i = \sup\{f(x) | x \in [x_{i-1}, x_i]\} \qquad \ldots (3)$$

Then, we define the **upper sum** $U(P, f)$ by

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta_i \qquad \ldots (4)$$

and the **lower sum** $L(P, f)$ by

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta_i \qquad \ldots (5)$$

These sums are called **Darboux sums** after the French mathematician, Jean Gaston Darboux who defined them.

Let us now look at an example to understand upper and lower sums.

Example 2: For the function $f(x) = 4x^3 - 12x^2 + 9x + 1$ on the interval $[0, 2]$, find the upper and lower sums with respect to the partition $\left\{0, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, 2\right\}$.

Solution: Here, $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$, $x_3 = \frac{5}{4}$, and $x_4 = 2$. Let us now calculate the Δ_i's.

$$\left.\begin{aligned} \Delta_1 &= x_1 - x_0 = \frac{1}{4} - 0 = \frac{1}{4}, & \Delta_2 &= x_2 - x_1 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \\ \Delta_3 &= x_3 - x_2 = \frac{5}{4} - \frac{3}{4} = \frac{1}{2}, & \Delta_4 &= x_4 - x_3 = 2 - \frac{5}{4} = \frac{3}{4} \end{aligned}\right\} \qquad \ldots (6)$$

Let us now compute the values of M_i, m_i, $1 \le i \le 4$. Let us use Fig. 5 to find these values.

We see that $f(x)$ is increasing on $\left[0, \frac{1}{4}\right]$. So, the infimum value is attained at $x = 0$ and the supremum is attained at $x = \frac{1}{4}$. The infimum value is the y-coordinate of P_1 and the supremum value is the y-coordinate of P_2. So, we get the following values for m_1 and M_1.

$$\left.\begin{aligned} m_1 &= \inf\left\{f(x) \,\middle|\, x \in \left[0, \frac{1}{4}\right]\right\} = f(0) = 1 \\ M_1 &= \sup\left\{f(x) \,\middle|\, x \in \left[0, \frac{1}{4}\right]\right\} = f\left(\frac{1}{4}\right) = \frac{41}{16} \end{aligned}\right\} \qquad \ldots (7)$$

You can see from Fig. 5 that, in the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$, the function f(x) increases from $x = \frac{1}{4}$ to $x = \frac{1}{2}$ and decreases from $x = \frac{1}{2}$ to $x = \frac{3}{4}$. In this interval, f(x) attains its infimum value at $x = \frac{1}{4}$ and its supremum value at $x = \frac{1}{2}$. The infimum value is the y-coordinate of $P_2 = \left(\frac{1}{4}, \frac{41}{16}\right)$ and the supremum value is the y-coordinate of $P_3 = \left(\frac{1}{2}, 3\right)$. So, we get the following values for m_2 and M_2.

$$\left.\begin{aligned} m_2 &= \inf\left\{f(x) \,\middle|\, x \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\} = f\left(\frac{1}{4}\right) = \frac{41}{16} \\ M_2 &= \sup\left\{f(x) \,\middle|\, x \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\} = f\left(\frac{1}{2}\right) = 3 \end{aligned}\right\} \qquad \ldots(8)$$

In the interval $\left[\frac{3}{4}, \frac{5}{4}\right]$, f(x) decreases from $x = \frac{3}{4}$ to $x = \frac{5}{4}$. The function attains its infimum value at $x = \frac{5}{4}$ and the infimum value is the y-coordinate of the point $P_5 = \left(\frac{5}{4}, \frac{21}{16}\right)$. The function attains its supremum value at $x = \frac{3}{4}$ and the supremum value is the y-coordinate of the point $P_4 = \left(\frac{3}{4}, \frac{43}{16}\right)$. So, we get the following values for m_3 and M_3.

$$\left.\begin{aligned} m_3 &= \inf\left\{f(x) \,\middle|\, x \in \left[\frac{3}{4}, \frac{5}{4}\right]\right\} = f\left(\frac{5}{4}\right) = \frac{21}{16} \\ M_3 &= \sup\left\{f(x) \,\middle|\, x \in \left[\frac{3}{4}, \frac{5}{4}\right]\right\} = f\left(\frac{3}{4}\right) = \frac{43}{16} \end{aligned}\right\} \qquad \ldots(9)$$

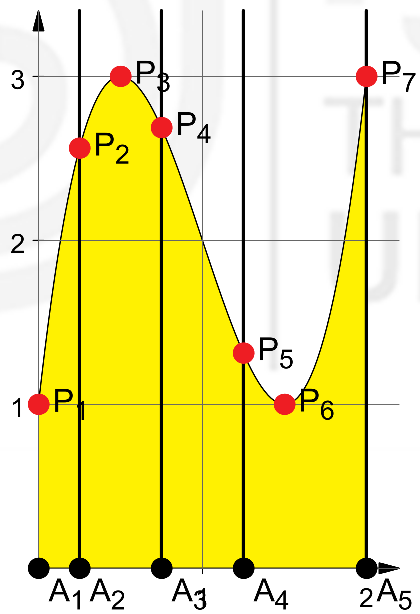


Fig. 5: Infimum and Supremum values of the function $f(x) = 4x^3 - 12x^2 + 9x + 1$ in the interval $[0, 1]$.

In the interval $\left[\frac{5}{4}, 2\right]$, f(x) decreases from $x = \frac{5}{4}$ to $x = \frac{5}{2}$ and increases from $x = \frac{5}{2}$ to $x = 2$. The infimum value is the y-coordinate of $P_6 = \left(\frac{5}{2}, 1\right)$ and the supremum value is the y-coordinate of $P_7 = (2, 3)$. So, we get the following values for m_4 and M_4.

$$\left.\begin{aligned} m_4 &= \inf\left\{f(x) \,\middle|\, x \in \left[\frac{5}{4}, 2\right]\right\} = f\left(\frac{5}{2}\right) = 1 \\ M_4 &= \sup\left\{f(x) \,\middle|\, x \in \left[\frac{5}{4}, 2\right]\right\} = f(2) = 3 \end{aligned}\right\} \qquad \ldots(10)$$

From Eqn. (7), Eqn. (8), Eqn. (9) and Eqn. (10), we see that

$$\begin{aligned} U(P, f) &= M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3 + M_4\Delta_4 \\ &= \frac{41}{16}\cdot\frac{1}{4} + 3\cdot\frac{1}{2} + \frac{43}{16}\cdot\frac{1}{2} + 3\cdot\frac{3}{2} = \frac{511}{64} \\ L(P, f) &= m_1\Delta_1 + m_2\Delta_2 + m_2\Delta_3 + m_4\Delta_4 \\ &= 1\cdot\frac{1}{4} + \frac{41}{16}\cdot\frac{1}{2} + \frac{21}{16}\cdot\frac{1}{2} + \frac{3}{2}\cdot 1 = \frac{59}{16} \end{aligned}$$

* * *

You may have noticed that , $m_1\Delta_1$, $m_2\Delta_2$, $m_3\Delta_3$ and $m_4\Delta_4$ are respectively, the areas of the blue coloured rectangles in Fig. 4(a) with bases A_1A_2, A_2A_3, A_3A_4 and A_4A_5. Similarly, $M_1\Delta_1$, $M_2\Delta_2$, $M_3\Delta_3$ and $M_4\Delta_4$ are, respectively, the areas of the red coloured rectangles in Fig. 4(b) with bases A_1A_2, A_2A_3, A_3A_4 and A_4A_5. You may have noticed that the upper sum we have found is the sum of the areas of the red coloured rectangles and the lower sum we have found is the sum of the areas of the blue coloured rectangles. Thus, intuitively, the upper sums provide an upper bound for the area below the curve

$$y = f(x) = 4x^3 - 12x^2 + 9x + 1$$

and the lower sums give a lower bound for the area.

Try the next exercise to check your understanding of upper and lower sums.

E1) Find the U(P, f) and L(P, f) for the function $f(x) = \frac{1}{1+x}$ in the interval [0, 1] with respect to the partition $\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{5}, 1\right\}$.

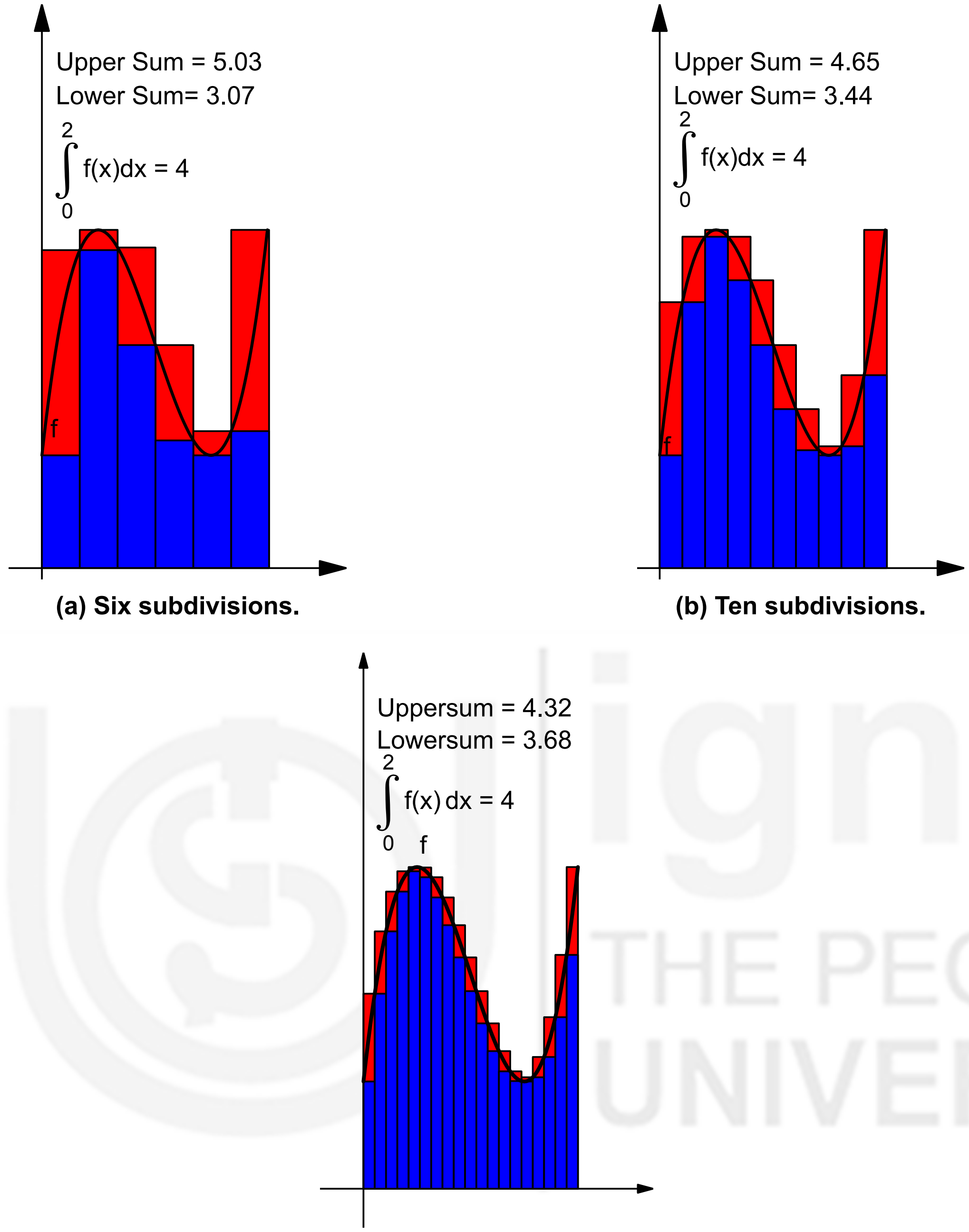
In this subsection, we defined what a partition of an interval is and defined the upper and lower sums of a function with respect to a partition. We conclude our discussion of upper and lower sums and we will begin the discussion of upper and lower integrals in the next subsection.

17.2.2 Upper and Lower Integrals

Let us now return to the problem of finding the area under the curve $4x^3 - 12x^2 + 9x + 1$ from x = 0 to x = 2. As we have already noticed, the upper sums provide an upper bound for the area and the lower sums provide a lower bound for the area. What happens to these sums as we increase the number of intervals in the partition? Let us investigate this now with the same example.

As we increase the number of subdivisions, the areas of the inner rectangles become larger and larger and the areas of the outer rectangles become smaller and smaller. The upper sums approach closer and closer to a value that we call the **upper integral**. The lower sums approach closer and closer to a value that we call the **lower integral**.

In the situation where the upper and lower integrals are equal, both the upper and lower sums approach closer and closer to a common limiting value, which is the area under the curve. The upper sums approach this value from above and the lower sums approach the value from below. See Fig. 6. The purpose of the figure is to give an intuitive feel. You will see a formal proof of this fact in your Real Analysis course.



(a) Six subdivisions.

(b) Ten subdivisions.

(c) Twenty subdivisions.

Fig. 6: Upper and Lower Sums.

We will now proceed to formally define the upper and lower integrals of a real valued function which is bounded in a closed interval. We start with a simple theorem that provides the basic foundation for the definition of these concepts.

Theorem 1: Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function and suppose that

$$m = \inf\{f(x) | x \in [a, b]\} \text{ and } M = \sup\{f(x) | x \in [a, b]\}$$

Then, for any partition P of [a, b] we have

Since f(x) is bounded, m and M exist.

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a) \qquad \ldots (11)$$

Proof: Let

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

If A and B are subsets of $\mathbb{R}$ such that $A \subset B$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$. If $a < b$ and $c > 0$, then $ac < bc$.

be a partition of $[a, b]$. Recall the definition of M_i in Eqn. (3) and the definition of $U(P, f)$ in Eqn. (4). Since $[x_{i-1}, x_i] \subset [a, b]$, we have $M_i \leq M$. Since $\Delta_i > 0$, we have $M_i\Delta_i \leq M\Delta_i$. So, we have

$$U(P, f) = \sum_{i=1}^{n} M_i\Delta_i \leq \sum_{i=1}^{n} M\,(x_i - x_{i-1}) = M \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$\begin{array}{l} b = x_n - x_{n-1} \\ x_{n-1} - x_{n-2} \\ x_{n-2} - x_{n-3} \\ \vdots \\ x_2 - x_1 \\ x_1 - x_0 = a \end{array}$$

Fig. 7: Telescoping Sum.

The last sum in the previous line is what is called a **telescopic sum** because it folds up like a telescope. Writing out the terms of the sum in reverse order, we see that all the other terms except the first term and last term cancel out. See Fig. 7: (The arrows show the terms that are cancelled out.) So, we have

$$U(P, f) \leq M \sum_{i=1}^{n} (x_i - x_{i-1}) \leq M(b - a)$$

and this proves the first part of Eqn. (11). We can prove the inequality for $m(b - a) \leq L(P, f)$ along similar lines. It follows from the fact that $m_i \geq m$. Let us now prove the middle inequality, namely $L(P, f) \leq U(P, f)$. We have $m_i \leq M_i$, $1 \leq i \leq n$. Since $\Delta_i > 0$, it follows that $m_i\Delta_i \leq M_i\Delta_i$. Summing up both sides of this inequality from i = 1 to n, we get $L(P, f) \leq U(P, f)$. ■

Let us write

$$\mathcal{L}(f) = \{L(P, f) | P \text{ is a partition of } [a, b]\}$$
$$\mathcal{U}(f) = \{U(P, f) | P \text{ is a partition of } [a, b]\}$$

From Theorem 1 it follows that the $\mathcal{U}(f)$ is bounded below by $m(b - a)$. So, the infimum of this set is finite. Similarly, $\mathcal{L}(f)$ is bounded above by $M(b - a)$ and therefore the supremum of this set is finite. So, the following definition of upper and lower integrals is meaningful.

Definition 3: Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function. Then the **upper integral** is defined by

$$\overline{\int_a^b} f(x)dx = \inf(\mathcal{U}(f)) = \inf\{U(P, f) | P \text{ is a partition of } [a, b]\} \qquad \ldots(12)$$

and the **lower integral** is defined by

$$\underline{\int_a^b} f(x)\,dx = \sup(\mathcal{L}(f)) = \sup\{L(P, f) | P \text{ is a partition of } [a, b]\} \qquad \ldots(13)$$

P. G. L. J. Dirichlet 1805-1859

We will now look at some examples.

Example 3: Let us consider the function f(x) defined by

$$f(x) = \begin{cases} 0, & \text{if x is irrational.} \\ 1, & \text{if x is rational.} \end{cases}$$

This function is called the **Dirichlet function** after the famous German mathematician Lejeune Dirichlet who defined this. Find the upper and lower integrals of the function f(x).

Solution: Let

$$P = \{0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1\}$$

be any partition of [0, 1]. Each sub-interval $[x_i, x_{i-1}]$ contains both rational as well has irrational numbers. On each interval $[x_{i-1}, x_i]$, for rational values of x, f(x) takes the value 1 and f(x) takes the value 0 for irrational values of x. So, $M_i = 1$, $m_i = 0$ for each i.

Thus

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} (1)(x_i - x_{i-1}) = 1 - 0 = 1$$

and

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (0)(x_i - x_{i-1}) = 0$$

Since P is an arbitrary partition of [0, 1], this means that $U(P, f) = 1$, $L(P, f) = 0$ for any partition of [0, 1]. It follows that

$$\inf\{U(P, f) | P \text{ is a partition of } [0, 1]\} = \{1\}$$

and

$$\sup\{L(P, f) | P \text{ is a partition of } [0, 1]\} = \{0\}.$$

So,

$$\overline{\int_0^1} f(x)\, dx = \inf\{U(P, f) | P \text{ is a partition of } [0, 1]\} = 1$$

and

$$\underline{\int_0^1} f(x)\, dx = \sup\{L(P, f) | P \text{ is a partition of } [0, 1]\} = 0$$

* * *

Let us now look at another example, the upper and lower integrals of a **step function**.

Example 4: Find the upper and lower integrals for the function f(x) defined by

$$f(x) = \begin{cases} 0, & \text{if } -1 \le x < 0 \\ 1, & \text{if } 0 \le x \le 1 \end{cases}$$

in the interval [–1, 1] .

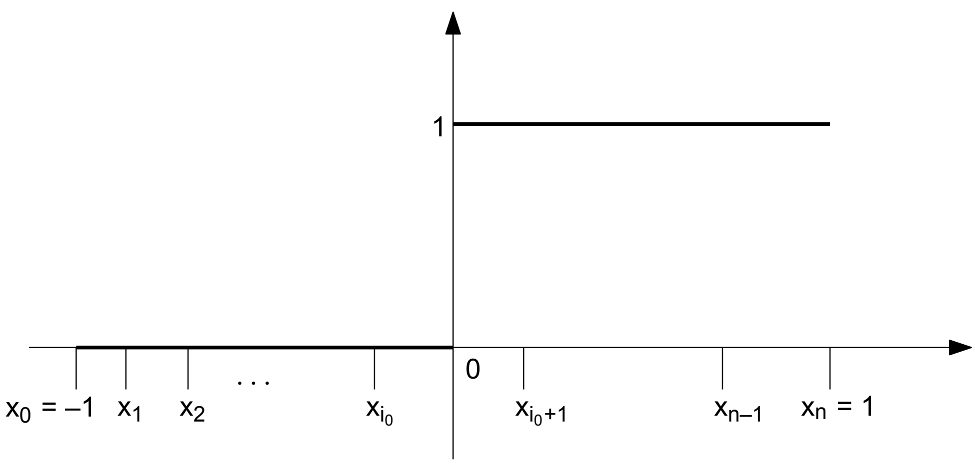


Fig. 8: Graph of f(x).

Solution: Let

$$P = \{-1 = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_{n-1} < x_n = 1\}$$

be a partition of $[-1, 1]$. Let i_0 be such that $x_{i_0} < 0 \le x_{i_0+1}$. Note that

$$m_k = \begin{cases} 0 & \text{if } 0 \le k \le i_0 + 1 \\ 1 & \text{if } i_0 + 1 < k \le n \end{cases}$$

Can you see why this is so? In the subinterval $\left[-1, x_{i_0}\right]$, $f(x) = 0$, so $m_k = 0$ for all $1 \le k \le i_0$. Also, $f(x) = 0$ in $\left[x_{i_0}, 0\right[$. So, $m_k = 0$ for $k = i_0 + 1$ also. Since $f(x) = 1$ in $\left[x_{i_0+1}, x_n\right]$, $m_k = 1$ for $k > i_0 + 1$. So,

$$L(P, f) = \sum_{k=1}^{i_0+1} m_k \Delta_k + \sum_{k=i_0+2}^{n} m_k \Delta_k = \sum_{k=i_0+2}^{n} (x_k - x_{k-1})$$

Since the last sum is a telescopic sum, we have

$$L(P, f) = x_n - x_{i_0+1} = 1 - x_{i_0+1}$$

If we write $d = x_{i_0+1}$, then d is the distance between 0 and the point x_{i_0+1}. Let us write

$$\Delta(P) = \max\{\Delta_i | 1 \le i \le n\}$$

Then as $\Delta(P)$ approaches zero, d also approaches zero and $L(P, f)$ approaches 1 from below. So, $\underline{\int_{-1}^{1}} f(x)\,dx = 1$.

Again, as we did in the case of m_k, check that

$$M_k = \begin{cases} 0, & \text{if } 0 \le k \le i_0 \\ 1, & \text{if } i_0 + 1 \le k \le n \end{cases}$$

So, we have

$$U(P, f) = \sum_{k=1}^{i_0} M_k \Delta_k + \sum_{k=i_0+1}^{n} M_k \Delta_k = \sum_{k=i_0+1}^{n} (x_k - x_{k-1}) = 1 - x_{i_0}$$

Writing $x_{i_0} = -e$, $e > 0$ is the distance between x_{i_0} and 0. So, $U(P, f) = 1 + e \ge 1$ and $1 + e$ approaches closer and closer to 1 as $\Delta(P)$ approaches zero. So,

$$\overline{\int_{-1}^{1}} f(x)\,dx = 1$$

.

* * *

Here is an exercise to check your understanding of what we have discussed so far.

E2) Find $\overline{\int_0^1} f(x)\,dx$ and $\underline{\int_0^1} f(x)\,dx$ for the function defined by $f(x) = 2$ on $[0, 1]$.

In this subsection, we defined the concepts of upper and lower sums. We also saw that, as we refine a partition by adding more and more points the upper sums and lower sums become closer and closer to certain values that we call upper and lower integrals, respectively. In some cases, the upper and the lower integral are equal, say α. In this case, the upper and lower sums converge to this common value α. In the next subsection we will study the situations in which the upper and lower integrals coincide.

17.2.3 The Definite Integral

In Exercise 2, we found the upper and lower integrals of the constant function f(x) = 2 over the interval [0, 1]. You would have seen that the values coincide and the common value is 2. Note that, the 'area' under the function f(x) = 2 over the interval [0, 1] is 2 and this is the area of the rectangle of sides 1 and 2 units. (Here, we put 'area' in quotes because this could be negative. You will see more about this in the last unit of this block.) This agrees with our intuition that the upper sums approach closer and closer to the 'area' from the above and the lower sums approach closer and closer to the 'area' from below. So, the common limit is the 'area' of the region enclosed by the function. Motivated by this, we define the integral of a real valued function over an interval as follows:

Definition 4: Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function. We say that f is **integrable** if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

We call the common value the **integral** of f and denote it by $\int_a^b f(x)\,dx$. We call f the **integrand**. We also call b and a, the **upper** and **lower limits of integration**, respectively.

In some textbooks, you will come across Riemann's definition of the integral which is different from Definition 4. However, our definition and Riemann's definition are equivalent definitions of integrability in the sense that a function that is integrable according to one definition is also integrable according to the other definition. Further, the value of the integral will be the same.

For convenience, we adopt the following convention: We let

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad \ldots (14)$$

There is a well known criterion for integrability of functions. We state this now without proof.

Theorem 2: Let $f\colon [a, b] \to \mathbb{R}$ be a bounded real valued function. Then, f is integrable if and only if, for every $\epsilon > 0$, there exists a partition P_ϵ such that $|U(P_\epsilon, f) - L(P_\epsilon, f)| < \epsilon$. ■

While it is difficult to show that the upper and lower integrals are equal in particular cases, there are general results that prove that certain classes of functions are integrable. You will see the proofs of such results in your Real Analysis course. Here, we give one such result without proof.

Theorem 3(Integrability): Let $f\colon [a, b] \to \mathbb{R}$ be any function. If f is bounded and monotonic or if f is continuous, then f is integrable. ■

If a function is integrable, it is necessarily bounded. We can use this **necessary** condition often to prove that certain functions are **not integrable** over certain intervals.

On the other hand, Theorem 3 gives us a **sufficient condition** for a function to be integrable and we use this to prove that certain functions **are integrable** over an interval. It is worth noting here that any function which is continuous on a closed interval is bounded on that interval.

Let us now look at some examples to understand how to apply the necessary and sufficient conditions for integrability of a real valued function.

Example 5: Discuss the integrability of the following functions:

i) $\sin x$ ii) $\tan x$ iii) $\ln|x|$ iv) $\frac{3}{2+x^2}$

v) $\frac{2+x}{x^2-1}$ vi) $\frac{x^2-1}{x^3-x^2+x+1}$ vii) $\frac{\sin x}{x}$

Solution:

By a closed and bounded interval of $\mathbb{R}$, we mean an interval of the form $[a, b]$, $a, b \in \mathbb{R}$.

i) The trigonometric function $\sin x$ is continuous in any closed and bounded interval. We can apply Theorem 3 to conclude that $\sin x$ is integrable in every closed and bounded subinterval of $\mathbb{R}$.

ii) The function $\tan x$ is not integrable in any closed interval containing an odd multiple of $\frac{\pi}{2}$, i.e a real number of the form $(2k + 1)\frac{\pi}{2}$, $k \in \mathbb{Z}$, because it is unbounded in any interval that contains $(2k + 1)\frac{\pi}{2}$, $k \in \mathbb{Z}$.

Of course, the first difficulty is that the function itself is not defined at $(2k + 1)\frac{\pi}{2}$. Since $\cos(2k + 1)\frac{\pi}{2}$ is zero, and division by zero is not defined, the function $\tan x$ is not defined at $x = (2k + 1)\frac{\pi}{2}$. We can try to rectify the situation by assigning some value to the function at $(2k + 1)\frac{\pi}{2}$. As you will see in more advanced courses, even if we change the value of an integrable function at finitely many points in a closed and bounded interval $[a, b]$, the function remains integrable in that interval. Also, changing the values of an integrable function at finitely many points of a closed and bounded interval doesn't affect the value of the integral of the function over that interval. So, we can define a function f(x) as follows:

$$f(x) = \begin{cases} 0, & \text{if } x = (2k + 1)\frac{\pi}{2},\ k \in \mathbb{Z}. \\ \tan x, & \text{Otherwise.} \end{cases}$$

When $x \neq (2k + 1)\frac{\pi}{2}$, $f(x) = \tan x = \frac{\sin x}{\cos x}$ and $\cos x \to 0$ as $x \to (2k + 1)\frac{\pi}{2}$, but $\sin x \to 1$. So, their ratio tends to ∞ as $x \to (2k + 1)\frac{\pi}{2}$. Therefore, the function f(x) is also unbounded in any interval that contains $(2k + 1)\frac{\pi}{2}$. So, f(x) is not integrable in any interval containing $(2k + 1)\frac{\pi}{2}$. On the other hand, $\tan x$ is integrable on every bounded interval not containing $(2k + 1)\frac{\pi}{2}$.

iii) The function $\ln|x|$ is not defined at 0. We can define the value of $\ln|x|$ to be 1 at $x = 0$ as we did in the case of $\tan x$. However, because $\ln|x| \to -\infty$ as $x \to 0$ the function $\ln|x|$ is unbounded in any closed and bounded interval of $\mathbb{R}$ that contains 0. Therefore, it is not integrable on any closed and bounded interval of $\mathbb{R}$ that contains 0. The function $\ln|x|$ is continuous in any closed and bounded interval of $\mathbb{R}$ not containing 0. So, it will be integrable on all closed and bounded intervals of $\mathbb{R}$ that do not contain 0.

iv) The function $\frac{2}{3+x^2}$ is a rational function, i.e. a ratio of two polynomials. All rational functions whose denominator doesn't vanish in an interval are integrable over that interval. Can you see why? If $f(x) = \frac{g(x)}{h(x)}$ is a ratio of two polynomials and $h(x) \neq 0$ in an interval, then $\frac{1}{h(x)}$ is also continuous in the interval. Since g(x) is continuous because it is a polynomial, the product $g(x) \cdot \frac{1}{h(x)}$ is also continuous. Since $3 + x^2 \geq 3 > 0$, the denominator is always positive So, in particular, the rational function $f(x) = \frac{2}{3+x^2}$ is integrable over any bounded interval in $\mathbb{R}$.

v) In this case, solving the equation $x^2 - 1 = 0$, we get $x = \pm 1$. Further, as $x \to 1$ and $x \to -1$, the numerator approaches a finite limit other than 0 and the denominator approaches 0. So the function is not bounded in any closed and bounded interval containing ± 1. So, it is not integrable on any closed and bounded interval that contains ± 1 It is integrable on any closed and bounded interval that doesn't contain ± 1.

vi) Here, the denominator $x^3 - x^2 + x + 1 = (x - 1)\left(x^2 + 1\right)$ is zero at x = 1 and non-zero at every other point on the real line. Both the numerator and the denominator approaches zero as $x \to 1$, but the numerator and the denominator have a common factor $x - 1$. We have

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x+1}{x^2+1} = 1.$$

We can define a function $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} \frac{x^2-1}{x^3-x^2+x+1}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$$

The function f(x) is continuous in any interval containing 1. Further, the denominator of f(x) doesn't vanish at any other point, so f(x) remains remains continuous in any closed and bounded subinterval of $\mathbb{R}$. Hence f(x) is integrable on every closed and bounded subinterval of $\mathbb{R}$.

vii) The denominator is 0 at x = 0. Both the numerator and the denominator approaches 0 as $x \to 0$. However, there is no common factor that we can remove. We have, from Example 25, Unit 7,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

We define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ \frac{\sin x}{x} & \text{if } x \neq 0 \end{cases}$$

So, the function remains continuous at 0. At all other points, the denominator neither vanishes nor tends to 0. So, this function is integrable on every closed and bounded interval of $\mathbb{R}$.

* * *

Remark 1: You will find that being unbounded is one of the most common reasons for a function being non-integrable in most of the cases that we encounter. One of the reasons is that the function tends to infinity at some point. Again, a common reason for this could be that function is defined as a ratio of two functions, and the function in the denominator vanishes at some point in the interval. But in these cases also, we have to carefully examine the limit of the function as the value in the domain approaches the possibly troublesome point. If the limit exists and is finite, the vanishing of the denominator is not a problem. We can always redefine the values at such points provided that there are only finitely many points.

Here are some exercises for you to test your understanding of the discussion so far.

E3) Which of the following functions are integrable? Justify your answer.

i) $f(x) = x^2 + 2x + 1$ in the interval $[1, 2]$.

ii) $f(x) = \begin{cases} 0 & \text{if } x = 1 \\ \frac{x^2+x+1}{x^2-4x+3} & \text{if } 1 < x \leq 2 \end{cases}$

in the interval $[1, 2]$.

iii) $f(x) = \begin{cases} \frac{1-\sin x}{\cos x} & \text{if } 0 \leq x < \frac{\pi}{2} \\ 0 & \text{if } x = \frac{\pi}{2} \end{cases}$

in the interval $\left[0, \frac{\pi}{2}\right]$.

iv) $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \ln x & \text{if } x > 0 \end{cases}$

in the interval $[0, 1]$.

E4) Is the Dirichlet function defined in Example 3 bounded? Is it integrable? Justify your answer.

In the next theorem, we list some of the properties of the integral without proof. You will see proof of these results in your Real Analysis course.

Theorem 4: a) If $f: [a, b] \to \mathbb{R}$ is a constant function $f(x) = k$, $k \in \mathbb{R}$, then

$$\int_a^b f(x)dx = k(b-a)$$

b) If f is a real valued function which is defined and integrable on $[a, c]$ and $[c, b]$, it is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

c) **Algebra of Integrals**: Let $f_1 : [a, b] \to \mathbb{R}$ and $f_2 : [a, b] \to \mathbb{R}$ be integrable functions. Then, for any constants a_1 and $a_2 \in \mathbb{R}$ the function $(a_1f_1 + a_2f_2) : [a, b] \to \mathbb{R}$ defined by

$$(a_1f_1 + a_2f_2)(x) = a_1f_1(x) + a_2f_2(x)$$

is also integrable and we have

$$\int_a^b (a_1f_1 + a_2f_2)(x)\,dx = a_1\int_a^b f_1(x)\,dx + a_2\int_a^b f_2(x)\,dx \qquad \ldots(15)$$

The function $(f_1f_2) : [a, b] \to \mathbb{R}$, defined by $(f_1f_2)(x) = f_1(x)f_2(x)$ is integrable.

■

We will not discuss examples of applications of Theorem 4 now because we will prove analogous results for indefinite integrals later and we will discuss examples there.

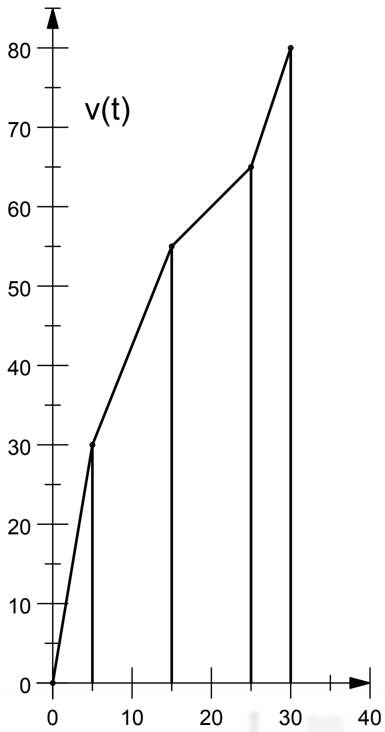


Fig. 9: Speed-time function.

You may have noticed that, while we discussed the existence of integrals of various functions, we never actually found the integrals of anything but the simplest of functions. It is generally difficult to find the integral of a function from first principles. In the next section we will discuss a method for actually finding the integral in many cases.

17.3 FUNDAMENTAL THEOREM OF CALCULUS

So far in our discussion, we have not seen any connection between differentiation and integration. For all we know, they may be unrelated. But, are they really unrelated? Let us discuss a situation before we actually present the relationship between integration and differentiation.

Let us suppose that you are making a long trip by car and you would like to know the distance you have covered every half an hour. Although your Odometer (the device for measuring the distance) is not working, your speedometer is working. Suppose your friend is taking readings of the speedometer from time to time and the values are as follows:

Time in minutes	5	15	25	30
Speed in km/hr	30	50	65	80

Can you find the approximate distance you have travelled in first half an hour?

Let us bring in Mathematics now. We have two functions, s(t) that gives the distance travelled in t seconds and v(t) gives the speed in the t^{th} second.

To simplify things, let us suppose that the acceleration was constant in the given time intervals. In this case, the graph of the function is as in Fig. 9. We have

Distance travelled = Speed $\times$ Time Taken

The distance travelled in first 5 minutes is $s(5)-s(0)$. Since the **minimum speed** in this interval is $v(0)$, the distance travelled in this interval is **at least** $v(0)\cdot\frac{5}{60}$, i.e.

$$v(0)\cdot\frac{5}{60}\le s(5)-s(0)$$

On the other hand, the **maximum speed** is $v(5)$. So, the distance travelled in this interval is **at most** $v(5)\cdot\frac{5}{60}$, i.e.

$$s(5)-s(0)\le v(5)\cdot\frac{5}{60}$$

Combining the inequalities, we get the following upper bounds and lower bounds for the distance travelled in each of the time intervals.

$$v(0)\cdot\frac{5}{60}\le s(5)-s(0)\le v(5)\cdot\frac{5}{60}$$

Similarly, for each of the remaining time intervals we will get bounds for the distance travelled in that time interval. The following are the bounds:

$$\left.\begin{array}{rcl} v(0)\cdot\frac{5}{60}\le & s(5)-s(0) & \le v(5)\cdot\frac{5}{60} \\ v(5)\cdot\frac{10}{60}\le & s(15)-s(5) & \le v(15)\cdot\frac{10}{60} \\ v(15)\cdot\frac{10}{60}\le & s(25)-s(15) & \le v(25)\cdot\frac{10}{60} \\ v(25)\cdot\frac{5}{60}\le & s(30)-s(25) & \le v(30)\cdot\frac{5}{60} \end{array}\right\} \quad \ldots(16)$$

Adding up all the inequalities in Eqn. (16), we get

$$S_1\le s(30)-s(0)\le S_2,$$

where

$$S_1 = v(0)\cdot\frac{5}{60}+v(5)\cdot\frac{5}{60}+v(15)\cdot\frac{10}{60}+v(25)\cdot\frac{5}{60}$$
$$S_2 = v(5)\cdot\frac{5}{60}+v(15)\cdot\frac{10}{60}+v(25)\cdot\frac{10}{60}+v(30)\cdot\frac{5}{60}$$

Did you notice that S_1 is the lower sum $L(P, v)$ of the function v with respect to the partition

$$P=\{x_0=0, x_1=5, x_2=15, x_3=25, x_4=30\}$$

of the interval $[0, 30]$? Also, we have,

$$v(0)=\inf\{v(t)\,|[x_0, x_1]\} \qquad v(5)=\inf\{v(t)\,|[x_1, x_2]\}$$
$$v(15)=\inf\{v(t)\,|[x_2, x_3]\} \qquad v(25)=\inf\{v(t)\,|[x_3, x_4]\}$$

Similarly, check that $S_2 = U(P, v)$.

As we increase the number of partitions, i,e, increase value of n, S_1 approaches closer and closer to $\underline{\int_a^b} f(x)\,dx$ and S_2 approaches closer and closer to $\overline{\int_a^b} f(x\,dx)$ and we have

$$\underline{\int_a^b} v(t)\,dt\le s(30)-s(0)\le\overline{\int_a^b} v(t)\,dt$$

Since v(t) is continuous, it is integrable, so the upper integral and the lower integral of v(t) are equal. Therefore,

$$s(30) - s(0) = \underline{\int_a^b} v(t)\,dt = \overline{\int_a^b} v(t)\,dt = \int_0^{30} v(t)\,dt \qquad \ldots(17)$$

looks plausible.

Now, recall the relationship between distance travelled and the speed. We know that, speed is the rate of change of distance. In other words,

$$\frac{ds}{dt} = v(t).$$

This suggests that, to find the integral $\int_a^b f(x)\,dx$, we should find a function F such that $F'(x) = f(x)$ for all $x \in [a, b]$. Then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

This last statement is true. We will now state the relation formally, clearly mentioning the conditions on the functions f and F under which this relation is true. Before we formally state this relationship we need some terminology.

Definition 5: Suppose that f and F are two real valued functions defined on (a, b) for some $a, b \in \mathbb{R}$, $a < b$. We say that F is the **antiderivative** of f on (a, b) if:

1) The function F is differentiable in (a, b) and continuous on $[a, b]$.
2) The function f is continuous on $[a, b]$.
3) $F'(x) = f(x)$ for all $x \in (a, b)$.

We also call F a **primitive** of f.

In the next example, we give some examples of functions and their antiderivatives

Example 6: Check the following:

i) The function F defined by $F(x) = \frac{x^2}{2}$ is the antiderivative of the function f given by $f(x) = x$ on any subinterval of $\mathbb{R}$.

ii) The function F defined by $F(x) = x|x|$ is the antiderivative of f, defined by $f(x) = 2|x|$ on every subinterval of $\mathbb{R}$.

Solution:

i) We know from Unit 9 of Block 3 that $F(x) = x^2/2$ is a polynomial so, it is differentiable and continuous on every subinterval of $\mathbb{R}$. Also, $f(x) = x$ is a differentiable function since it is a polynomial. In particular, it is also continuous. Further, $F'(x) = f(x)$. So, all the three conditions are satisfied and F is the antiderivative of f.

ii) From example 3 in Unit 11, we know that F is differentiable and its derivative is f. Further, we know that $2|x|$ is a continuous function on any subinterval of $\mathbb{R}$. So, it follows that F is the antiderivative of f.

$* * *$

Here are some exercises for you to check your understanding of the concept of antiderivative

E5) Check the following:

i) The function $\cos x + x \sin x$ is the antiderivative of $x \cos x$ on any subinterval of $\mathbb{R}$.

ii) The function $x \ln|x| - x$ is the antiderivative of $\ln|x|$ on any subinterval of $\mathbb{R}$ not containing 0.

After going through the definition of antiderivative and looking at the examples, one natural question that you may have is the following: 'In the case of part a) of Example 6, the function F_1 defined by $F_1(x) = \frac{x^2}{2} + 1$ also satisfies the three conditions in the definition. So, it is also an antiderivative of f. How many antiderivatives can a function have and how are the different antiderivatives of a function related to each other?'. The next theorem answers this question.

Theorem 5: Suppose F and G are two antiderivatives of f on $[a, b]$, i.e. we have $F' = f$ and $G' = f$ on (a, b). Then, $F(x) = G(x) + C$ for some constant function C. Conversely, if F is the antiderivative of a function f then $F + C$ is also an antiderivative of f where C is a constant function.

Proof: We have

$$\frac{d}{dx}(F(x) - G(x)) = F'(x) - G'(x) = f(x) - f(x) = 0$$

So, $F - G$ is a constant function, say C. Then, $F(x) = G(x) + C$. Conversely, if F is an antiderivative of f on $[a, b]$, for any constant function C we have

$$\frac{d}{dx}(F(x) + C) = F'(x) = f(x).$$

So, $F + C$ is also an antiderivative of $f(x)$ for any constant C. ■

Let us take the set S to be set of all functions that are differentiable on $[a, b]$ with a continuous derivative. We define a relation on S by $F \sim G$ if $F'(x) = G'(x)$. Then, you can easily check that this is an equivalence relation. So, the set of all antiderivatives of a continuous function forms an equivalence class. Let us summarise the discussion in the form of a theorem.

Theorem 6: Let S to be set of all functions that are differentiable on $[a, b]$ with a continuous derivative. We define a relation on S by $F \sim G$ if $F'(x) = G'(x)$. Then, $\sim$ is an equivalence relation on S. ■

The next theorem tells us that antiderivatives exist for all continuous functions.

Theorem 7: Let $f: [a, b] \to \mathbb{R}$ be a continuous function.
Part 1: The function $F: [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\,dt \qquad \ldots(18)$$

is an antiderivative of $f(x)$, that is, $F'(x) = f(x)$ for all $x \in [a, b]$.

Part 2: If F is an antiderivative of f in (a, b), then

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \ldots(19)$$

■

Theorem 7 is known as the Fundamental Theorem of Integral Calculus or Fundamental Theorem of Calculus.

Let us now look at some applications of Theorem 7.

Example 7: Find the following:

i) $\frac{d}{dx}\left(\int_0^{x^2} \sin t\,dt\right)$ ii) $\frac{d}{dx}\left(\int_1^{\sin x} \left(1-t^2\right)\,dt\right)$ iii) $\frac{d}{dt}\left(\int_x^{x^2} \left(t^2+t\right)\,dt\right)$

Solution: We use chain rule to find these derivatives.

i) We set $v = x^2$, and write $F(x) = \int_0^{x^2} \sin t\,dt$, $f(t) = \sin t$. Then,

$$\frac{d}{dx}(F(x)) = \frac{d}{dv}(F(v))\frac{dv}{dx}$$

Using Eqn. (18), we have $\frac{d}{dv}(F(v)) = f(v) = \sin x^2$. Also, $\frac{dv}{dx} = 2x$. Therefore, $\frac{d}{dx}\int_0^{x^2} \sin t\,dt = 2x\sin x^2$.

ii) Setting $v = \sin x$ and $F(x) = \frac{d}{dx}\int_1^{\sin x}(1-t^2)\,dt$, $f(t) = 1-t^2$ we get

$$\frac{d}{dx}(F(x)) = \frac{d}{dv}(F(v))\frac{d}{dx}(v) = f(v)\cos x = (1-\sin^2 x)\cos x = \cos^3 x.$$

iii) From part b) of Theorem 4 we have

$$\int_1^{x^2}\left(t^2+t\right)\,dt = \int_1^{x}\left(t^2+t\right)\,dt + \int_x^{x^2}\left(t^2+t\right)\,dt$$

So,

$$\int_x^{x^2}\left(t^2+t\right)\,dt = \int_1^{x^2}\left(t^2+t\right)\,dt - \int_1^{x}\left(t^2+t\right)\,dt = F_1(x) - F_2(x) \text{ (say).}$$

We have

$$\frac{d}{dx}(F_1(x)) = \left(x^4+x^2\right)2x \text{ and } \frac{d}{dx}(F_2(x)) = x^2+x.$$

Therefore,

$$\int_x^{x^2}\left(t^2+t\right)\,dt = 2x^5+2x^3-x^2-x.$$

* * *

If you have understood the above example, you will have no difficulty in proving the following general result:

Theorem 8: Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function and let

$$F(x) = \int_{g_1(x)}^{g_2(x)} f(t)\,dt$$

where $g_1(x)\colon [c,d] \to [a,b]$ and $g_2(x)\colon [c,d] \to [a,b]$ are differentiable functions on $[c,d]$. Then, we have

$$F'(x) = F'(g_2(x))g_2'(x) - F'(g_1(x))g_1'(x). \qquad \ldots(20)$$

$$\text{i.e. } \frac{d}{dx}\left(\int_{g_1(x)}^{g_2(x)} f(t)\,dt\right) = f(g_2(x))\,g_2'(x) - f(g_1(x))\,g_1'(x) \qquad \ldots(21)$$

Proof: Using Theorem 4, part b), we have

$$\int_a^{g_2(x)} f(x)\,dx = \int_a^{g_1(x)} f(x)\,dx + \int_{g_1(x)}^{g_2(x)} f(x)\,dx \qquad \ldots(22)$$

We write

$$F(x) = \int_{g_1(x)}^{g_2(x)} f(t)\,dt = \int_a^{g_2(x)} f(x)\,dx - \int_a^{g_1(x)} f(x)\,dx \text{ using Eqn. (22)}$$

We have

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\int_a^{g_2(x)} f(x)\,dx\right) - \frac{d}{dx}\left(\int_a^{g_1(x)} f(x)\,dx\right).$$

Let us consider the first term $\frac{d}{dx}\left(\int_a^{g_2(x)} f(x)\,dx\right)$. As before, we set $v = g_2(x)$.

Then,

$$\frac{d}{dx}\left(\int_a^{g_2(x)} f(x)\,dx\right) = \frac{d}{dv}(F(v))\frac{d}{dx}(g_2(x)) = F'(g_2(x))g_2'(x) \qquad \ldots(23)$$

Similarly, we get

$$\frac{d}{dx}\left(\int_a^{g_1(x)} f(t)\,dt\right) = F'(g_1(x))g_1'(x). \qquad \ldots(24)$$

The result in Eqn. (20) follows from Eqn. (23) and Eqn. (24) and Eqn. (21) is a rephrasing of Eqn. (20). ■

Here are some exercises to check your understanding of the previous example.

E6) Find the following:

i) $\frac{d}{dx}\int_0^{\sqrt{x}} \left(1+t^4\right)\,dt$ ii) $\frac{d}{dx}\int_1^{\tan x} \left(1+t^2\right)\,dt$ iii) $\frac{d}{dx}\int_{\sqrt{x}}^{x} \frac{dt}{1+t^2}$

Note that any function in the set of antiderivatives of $f(x)$ on the interval $[a,b]$ will give the same value $\int_a^b f(x)\,dx$ when we use Eqn. (19) for evaluating definite integrals. If $F_1(x)$ is any other antiderivative of f(x), we saw that $F_1(x) = F(x) + C$ for some constant C. From Eqn. (19), we have, for any constant C,

$$\int_a^b f(x)\,dx = \{F_1(b) - F_1(a)\} = \{F(b) + C\} - \{F(a) + C\} = F(b) - F(a)$$

which is independent of C.

So, in some sense, we can regard any two antiderivatives of a function over a closed interval the 'same' as far as finding the integral of the function over that interval is concerned. More formally, let S be the set of all functions on [a, b] whose derivative is continuous. Define a relation $\sim$ by '$F \sim G$ if they have the same derivative, i.e. $F'(x) = G'(x)$ in the interval [a, b]'. Note that this means that F and G are the antiderivatives of the same function. We leave it to you to check that $\sim$ defines an equivalence relation.

Definition 6: Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function. If F(x) is an antiderivative of f(x), we write

$$\int f(x)\,dx = F(x) + C.$$

where C is an arbitrary constant, and call $\int f(x)\,dx$ it the **indefinite integral** of f(x).

We conclude this section with an exercise for you.

E7) Check that the relation $\sim$ that we defined earlier is an equivalence relation.

In the next section, we will discuss the indefinite integrals of some common functions that we have encountered so far.

17.4 STANDARD INTEGRALS

You would have noticed that the functions that we commonly encounter are either polynomials, rational functions or functions involving trigonometric functions. Once we know how to integrate a certain set of functions, we can integrate many other functions using the following result, which is analogous to Theorem 4, part c). You will see that the results are a direct consequence of the results on differentiation.

Theorem 9: a) Let $F\colon [a, b] \to \mathbb{R}$ be the antiderivative of $f\colon [a, b] \to \mathbb{R}$. Then, the antiderivative of $(\alpha f)(x)$ exists for any $\alpha \in \mathbb{R}$ and equals $\alpha F(x)$. Also,

$$\int (\alpha f)(x)\,dx = \alpha \int f(x)\,dx. \qquad \ldots(25)$$

b) If $G\colon [a, b] \to \mathbb{R}$ is the antiderivative of $g\colon [a, b] \to \mathbb{R}$, then antiderivative of $(f + g)(x)$ exists and equals $(F + G)(x)$. Also,

$$\int (f + g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad \ldots(26)$$

Proof: a) If $\alpha = 0$, there is nothing to prove since both the RHS and LHS of Eqn. (25) are 0. So, let us assume that $\alpha \neq 0$. Since f(x) is continuous, $\alpha f(x)$ is also continuous. Further, since F(x) is differentiable on (a, b), $(\alpha F)(x)$ is also differentiable on (a, b) and we have

$$\frac{d}{dx}((\alpha F)(x)) = \alpha \frac{d}{dx}(F(x)) = (\alpha f)(x).$$

Therefore, by definition of indefinite integral, $(\alpha F)(x)$ is an antiderivative of $\alpha(f(x))$ and we have

$$\int (\alpha f)(x)\, dx = (\alpha F)(x) + C,\ C \in \mathbb{R} \qquad \ldots (27)$$

By the definition of the indefinite integral, we have

$$\begin{aligned}\alpha \int f(x)\, dx &= \alpha\left(F(x) + C'\right),\ C' \in \mathbb{R}\\ &= \alpha F(x) + \alpha C',\ C' \in \mathbb{R}\end{aligned}$$

Since $\alpha \neq 0$, as C' takes all possible values in $\mathbb{R}$, $\alpha C'$ also takes all possible values in of $\mathbb{R}$. If we write C for $\alpha C'$, C takes all the values in R. So,

$$\begin{aligned}\alpha \int f(x)\, dx &= \alpha F(x) + C,\ C \in \mathbb{R}\\ &= \int (\alpha f)(x)\, dx \text{ from Eqn. (27).}\end{aligned}$$

b) The proof of part b) is along similar lines. We have

$$\begin{aligned}\frac{d}{dx}\{(F+G)(x)\} &= \frac{d}{dx}\{F(x) + G(x)\}\\ &= \frac{d}{dx}(F(x)) + \frac{d}{dx}(G(x))\\ &= f(x) + g(x) = (f+g)(x).\end{aligned}$$

Therefore,

$$\int (f+g)(x)\, dx = (F+G)(x) + C,\ C \in \mathbb{R} \qquad \ldots (28)$$

We have

$$\begin{aligned}\int f(x)\, dx + \int g(x)\, dx &= F(x) + C_1 + G(x) + C_2,\ C_1, C_2 \in \mathbb{R}\\ &= F(x) + G(x) + C_1 + C_2, C_1, C_2 \in \mathbb{R}\end{aligned}$$

As C_1 and C_2 takes values over the whole of $\mathbb{R}$, $C_1 + C_2$ also takes values over the whole of $\mathbb{R}$. Writing C for $C_1 + C_2$, we have

$$\begin{aligned}\int f(x)\, dx + \int g(x)\, dx &= \{(F+G)(x) + C | C \in \mathbb{R}\}\\ &= \int (f+g)(x)\, dx \text{ from Eqn. (28).}\end{aligned}$$

■

Corollary 1: Let $f_1, f_2, \ldots, f_n$ be continuous functions on [a, b] and $c_1, c_2, \ldots, c_n \in \mathbb{R}$ be constants. Then,

$$\int (c_1 f_1 + c_2 f_2 + \cdots + c_n f_n)(x)\, dx = \sum_{i=1}^{n} c_i \int f_i(x)\, dx \qquad \ldots (29)$$

■

Table 1: Table of indefinite integrals

No.	f(x)	$\int f(x)\,dx$
1.	$x^n,\ n \neq -1,$	$\frac{x^{n+1}}{n+1} + C$
2.	$\frac{1}{x}$	$\ln\lvert x\rvert + C$
3.	$\sin x$	$-\cos x + C$
4.	$\cos x$	$\sin x + C$
5.	$\tan x$	$\ln\lvert\sec x\rvert + C$
6.	$\cot x$	$\ln\lvert\sin x\rvert + C$
7.	$\sec x$	$\ln\lvert\sec x + \tan x\rvert + C$
8.	$\operatorname{cosec} x$	$\ln\lvert\operatorname{cosec} x - \cot x\rvert + C$
9.	e^x	$e^x + C$
10.	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
11.	$\frac{1}{\sqrt{x^2-1}}$	$\ln\left\lvert x + \sqrt{x^2-1}\right\rvert + C$
12.	$\frac{1}{\sqrt{x^2+1}}$	$\ln\left\lvert x + \sqrt{x^2+1}\right\rvert + C$
13.	$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + C$
14.	$\frac{1}{1+x^2}$	$\tan^{-1} x + C$

We now compile a table of functions and their indefinite integrals for future reference in Table 1. We have not explicitly mentioned the intervals on which the function in the second column is the indefinite integral of the first column. So, before using these results to evaluate definite integrals over an interval, check that the conditions for the antiderivatives are satisfied over the interval you are integrating before you apply the formula!

You can check the table by differentiating the second column in the table and see if you get the first column. We will check some of the values in the next example,

Example 8: Check that:

a) The indefinite integral of $\tan x$ is $\ln\lvert\sec x\rvert + C$ in any subinterval of $\mathbb{R}$ not containing $(2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$.

b) The indefinite integral of $\sec x$ is $\ln|\sec x + \tan x| + C$ in any subinterval of $\mathbb{R}$ not containing $(2k+1)\frac{\pi}{2}$.

Solution: Indeed, we have $\frac{d}{dx}(\ln|\sec x|) = \frac{1}{\sec x}\cdot \sec x \cdot \tan x = \tan x$. You can check that $\tan x$ is continuous in any interval not containing $(2k+1)\frac{\pi}{2}$. So, $\ln|\sec x|$ is the antiderivative of $\tan x$. It follows that $\ln|\sec x| + C$ is the indefinite integral of $\tan x$.

Let us now check that the antiderivative of $\sec x$ is $\ln|\sec x + \tan x|$. Check that $\sec x + \tan x \neq 0$ whenever $\sec x$ and $\tan x$ are defined. We have

$$\begin{aligned}\frac{d}{dx}(\ln|\sec x + \tan x|) &= \frac{1}{\sec x + \tan x}\cdot\left(\sec x \tan x + \sec^2 x\right)\\ &= \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} = \sec x\end{aligned}$$

* * *

E8) Check the remaining values of Table 1.

Example 9: Calculate the following integrals:

i) $\int x^3\,dx$ ii) $\int x^{\frac{3}{2}}\,dx$ iii) $\int_0^2 \frac{1}{x^5}\,dx$

iv) $\int \frac{1}{x^{7/5}}\,dx$ v) $\int\left(x^5 + 3x^3 - 2x\right)\,dx$ vi) $\int \frac{(x+1)^3}{x^2}\,dx$

Solution:

i) From the first entry in Table 1, $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$. So, we have $\int x^3\,dx = \frac{x^4}{4} + C$.

ii) Again using first entry in Table 1, we get

$$\int x^{\frac{3}{2}}\,dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{2}{5}x^{\frac{5}{2}}\,dx + C$$

iii) We have

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{x^{-5+1}}{-5+1} + C = \frac{x^{-4}}{-4} + C = -\frac{1}{4x^4} + C.$$

iv) We have

$$\int \frac{1}{x^{\frac{7}{5}}}\,dx = \int x^{-\frac{7}{5}}\,dx = \frac{x^{-\frac{7}{5}+1}}{-\frac{7}{5}+1} + C = -\frac{5}{2x^{\frac{5}{2}}} + C$$

v) We now apply Eqn. (29) with $f_1 = x^5$, $c_1 = 1$, $f_2 = x^3$, $c_2 = 3$, $f_3 = x$ and $c_3 = -2$. We have

$$\begin{aligned}\int\left(x^5 + 3x^3 - 2x\right)\,dx &= \int x^5\,dx + 3\int x^3\,dx - 2\int x\,dx\\ &= \frac{x^6}{6} + C_1 + 3\frac{x^4}{4} + C_2 - 3\frac{x^2}{2} - C_3\\ &= \frac{x^6}{6} + 3\frac{x^4}{4} - 3\frac{x^2}{2} + C\end{aligned}$$

where we write C for $C_1 + C_2 - C_3$. In what follows, we will frequently combine two or more constants into a single constant without explaining in detail.

vi) We have $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$. Therefore,

$$\frac{(x+1)^3}{x^2} = x + 3 + \frac{3}{x} + \frac{1}{x^2}.$$

Using Eqn. (29), we get

$$\begin{aligned}\int \frac{(x+1)^3}{x^2}\,dx &= \int \left(x + 3 + \frac{3}{x} + \frac{1}{x^2}\right) dx \\ &= \int x\,dx + 3\int dx + 3\int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx \\ &= \frac{x^2}{2} + 3x + 3\ln|x| + \frac{x^{-2+1}}{-2+1} + C = -\frac{1}{x} + 3x + \frac{x^2}{2} + 3\ln|x| + C\end{aligned}$$

* * *

Here are some problems for you to try.

E9) Calculate the following integrals:

i) $\int x^4\,dx$ ii) $\int x^{\frac{5}{2}}\,dx$ iii) $\int_0^2 \frac{1}{x^7}\,dx$

iv) $\int \frac{dx}{x^{5/3}}$ v) $\int \left(x^6 + 3x^4 - 2x^2\right) dx$ vi) $\int \frac{(x-1)^4}{x^2}\,dx$

Let us now look at some examples that involve trigonometric functions.

Example 10: Calculate the following integrals:

i) $\int(3\sin x + 4\cos x)\,dx$ ii) $\int \sec^2 x\,dx$ iii) $\int(x + \sin x)\,dx$ iv) $\int \frac{(2+3\sin x)^2}{\cos^2 x}\,dx$

Solution:

i) Using Eqn. (29), we get

$$\begin{aligned}\int (3\sin x + 4\cos x\,dx &= 3\int \sin x\,dx + 4\int \cos x\,dx \\ &= -3\cos x + 4\sin x + C\end{aligned}$$

ii) We have

$$\begin{aligned}\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

$$\therefore \int \sec^2 x\,dx = \tan x + C.$$

iii) Using Eqn. (29), we get

$$\int (x + \sin x)\,dx = \int x\,dx + \int \sin x\,dx = \frac{x^2}{2} - \cos x + C$$

iv) We have

$$\frac{(2+3\sin x)^2}{\cos^2 x} = \frac{4+9\sin^2 x+12\sin x}{\cos^2 x}$$
$$= 4\sec^2 x + 9\tan^2 x + 12\sec x \tan x$$
$$= 4\sec^2 x + 9(\sec^2 x - 1) + 12\sec x \tan x$$
$$= 13\sec^2 x + 12\sec x\tan x - 9$$

Therefore, using Eqn. (29), we get

$$\int \frac{(2+3\sin x)^2}{\cos^2 x}\,dx = 13\int \sec^2 x\,dx + 12\int \sec x\tan x\,dx - 9\int dx$$

We have $\int \sec^2 x\,dx = \tan x + C_1$ from part ii) of this example. Also $\frac{d}{dx}(\sec x) = \sec x \tan x$.. Therefore,

$\int \sec x \tan x\,dx = \sec x + C_2$. Thus,

$$\int \frac{(2+3\sin x)^2}{\cos^2 x}\,dx = 13\tan x + 12\sec x - 9x + C$$

* * *

Try the following exercises to check your understanding of the above example.

E10) Calculate the following integrals:

i) $\int (4\cos x + 3\tan x)\,dx$ ii) $\int \operatorname{cosec}^2 x\,dx$ iii) $\int \frac{(3+2\cos x)^2}{\sin^2 x}\,dx$

We close this unit here. In the following units, we will develop methods, based on Theorem 7, for evaluating integrals. You may like to go through a summary, given in the next section, of the main topics discussed in this unit.

17.5 SUMMARY

In this Unit, we have:

1) defined the lower and upper sums of functions defined on [a, b] corresponding to a partition of [a, b] and computed them for some simple functions;
2) defined the upper and lower integrals of a function and calculated them for simple functions;
3) defined the definite integral of a given function and
4) stated the Fundamental Theorem of Calculus; and
5) used the Fundamental Theorem to calculate the definite integral of some integrable functions.

17.6 SOLUTIONS/ANSWERS

E1) Let $f(x) = \frac{1}{1+x}$. Then, f(x) is monotonic decreasing in [0, 1]. So,

$$\inf\{f(x) \mid x \in [x_i, x_{i+1}]\} = f(x_{i+1})$$

and

$$\sup\{f(x)\,|\,x \in [x_i, x_{i+1}]\} = f(x_i)$$

for any partition $\{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}$ of [0, 1]. We have $\Delta_1 = \frac{1}{4}$, $\Delta_2 = \frac{1}{4}$, $\Delta_3 = \frac{1}{10}$, $\Delta_4 = \frac{2}{5}$. Further, we have $m_1 = f\left(\frac{1}{4}\right) = \frac{4}{5}$, $m_2 = f\left(\frac{1}{2}\right) = \frac{3}{2}$, $m_3 = f\left(\frac{3}{5}\right) = \frac{5}{8}$ and $m_4 = f(1) = \frac{1}{2}$. So,

$$L(P, f) = \frac{1}{4}\cdot\frac{4}{5} + \frac{1}{4}\cdot\frac{2}{3} + \frac{1}{10}\cdot\frac{5}{8} + \frac{2}{5}\cdot\frac{1}{2} = \frac{151}{240}$$

Also, $M_1 = f(0) = 1$, $M_2 = f\left(\frac{1}{4}\right) = \frac{4}{5}$, $M_3 = f\left(\frac{1}{2}\right) = \frac{2}{3}$ and $M_4 = f\left(\frac{3}{5}\right) = \frac{5}{8}$.
So, $U(P, f) = 1\cdot\frac{1}{4} + \frac{4}{5}\cdot\frac{1}{4} + \frac{2}{3}\cdot\frac{1}{10} + \frac{5}{8}\cdot\frac{2}{5} = \frac{23}{30}$

E2) Let $P = \{0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1\}$ be any partition of [0, 1]. Since f(x) is a constant function on [0, 1] it follows that $M_i = 2$, $m_i = 2$ for any interval $[x_i, x_{i-1}]$. So, we have

$$U(P, f) = \sum_{i=1}^{n} M_i\,(x_i - x_{i-1}) = 2\sum_{i=1}^{n}(x_i - x_{i-1}) = 2(1-0) = 2$$

and

$$L(P, f) = \sum_{i=1}^{n} m_i\,(x_i - x_{i-1}) = 2\sum_{i=1}^{n}(x_i - x_{i-1}) = 2(1-0) = 2$$

Therefore,

$$\inf\{U(P, f)|P \text{ is a partition of } [0, 1]\} = \{2\}$$

and

$$\sup\{L(P, f)|P \text{ is a partition of } [0, 1]\} = \{2\}$$

So,

$$\underline{\int_0^1} f(x)\,dx = \sup\{U(P, f)|P \text{ is a partition of } [0, 1]\} = \sup\{2\} = 2.$$

Similarly $\underline{\int_0^1} f(x)\,dx = 2$

E3) i) In this case, since $f(x) = x^2 + 2x + 1$ is a polynomial, it is continuous. So, it is integrable.

ii) In this case, the denominator factors as $(x-1)(x-3)$, so we examine the points 1 and 3. The numerator has no real roots. So, there is no possibility of removing common factors. The domain doesn't contain 3, so we need not bother about it. However, 1 is in the domain and the function becomes unbounded as $x \to 1$. So, the function is not integrable.

iii) Here, the numerator and the denominator tends to 0 as $x \to \frac{\pi}{2}$. Applying L'hospital's rule, we get

$$\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\sin x} = 0.$$

At all other points, $1 - \sin x$ and $\cos x$ are also continuous. Further, $\cos x \neq 0$ at all the points other than $\frac{\pi}{2}$. Therefore, the function is continuous at all the points other than $x = \frac{\pi}{2}$. So, the function is integrable over the interval $\left[0, \frac{\pi}{2}\right]$.

iv) Here $\ln x$ tends to $-\infty$ as $x \to 0$. But

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{-\ln\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0} \frac{-x\left(\frac{-1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to 0} x = 0.$$

So, the function is continuous at $x = 0$. Since $\ln x$ and x are continuous at all other points, their product is also continuous at all the points other than 0. So, this function is continuous in the interval $[0, 1]$ and hence integrable.

E4) The Dirichlet function bounded because the only values it takes are 0 and 1 and so $|f(x)| \leq 1$. However, as we saw in Example 3, the upper and lower integrals are not equal and therefore it is not integrable.

E5) i) The functions x and $\sin x$ are differentiable in every subinterval of $\mathbb{R}$. So, $x\sin x$ is also differentiable in every subinterval of $\mathbb{R}$. Since $\cos x$ is also differentiable in every subinterval of $\mathbb{R}$, it follows that $\cos x + x\sin x$ is differentiable in every subinterval of $\mathbb{R}$. Further, $\frac{d}{dx}(\cos x + x\sin x) = -\sin x - \sin x + x\cos x = x\cos x$. Since the function $\cos x$ is continuous it follows that $\cos x + x\sin x$ is the antiderivative of $x\cos x$.

The functions $f(x) = x\ln|x|$ is differentiable in any interval of $\mathbb{R}$ not containing 0 and $g(x) = x$ is differentiable in any interval of $\mathbb{R}$. So, $x\ln|x| - x$ is differentiable on every interval of $\mathbb{R}$ that doesn't contain 0. Further,

$$\frac{d}{dx}(x\ln|x| - x) = \ln|x| + x\frac{1}{x} - 1 = \ln|x|$$

The function $\ln|x|$ is continuous in every subinterval of $\mathbb{R}$ not containing 0. It follows that $x\ln|x| - x$ is the antiderivative of $\ln|x|$ in any interval of $\mathbb{R}$ not containing 0.

E6) i) Here $F(x) = \int_0^{\sqrt{t}} \left(t^2 + 1\right) dt$, $f(t) = \left(1 + t^2\right)$. We set $v = \sqrt{x}$.

$$\frac{d}{dx}(F(x)) = \frac{d}{dv}(F(v))\frac{dv}{dx} = f(v)\frac{1}{2\sqrt{x}} = \frac{x^2 + 1}{2\sqrt{x}}$$

ii) Here $F(x) = \int_1^{\tan x} \left(1 + t^2\right) dt$, $f(x) = \left(1 + t^2\right)$. We set $v = \tan x$.

$$\frac{d}{dx}(F(x)) = \frac{d}{dv}(F(v))\frac{dv}{dx} = f(v)\sec^2 x$$
$$= \left(1 + \tan^2 x\right)\sec^2 x = \sec^4 x.$$

iii) We have

$$\int_1^x \frac{dt}{1+t^2} = \int_1^{\sqrt{x}} \frac{dt}{1+t^2} + \int_{\sqrt{x}}^x \frac{dt}{1+t^2}$$

$$\int_{\sqrt{x}}^x \frac{dt}{1+t^2} = \int_1^x \frac{dt}{1+t^2} - \int_1^{\sqrt{x}} \frac{dt}{1+t^2}$$

Let us write $F(x) = \int_{\sqrt{x}}^{x} \frac{dt}{1+t^2}$, $F_1(x) = \int_{1}^{x} \frac{dt}{1+t^2}$, $F_2(x) = \int_{1}^{\sqrt{x}} \frac{dt}{1+t^2}$. We have

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}(F_1(x)) - \frac{d}{dx}(F_2(x)) = f(x)\frac{d}{dx}(x) - f\left(\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{1}{1+x^2} - \frac{1}{2\sqrt{x}(1+x)}$$

E7) We have $F'(x) = F'(x)$, so $F \equiv F$ and the relation is reflexive. If $F \equiv G$, we have $F'(x) = G'(x)$, $G'(x) = F'(x)$. So, $G'(x) = F'(x)$, that is $G \equiv F$ and the relation is symmetric. IF $F \equiv G$ and $G \equiv H$, then $F'(x) = G'(x)$ and $G'(x) = H'(x)$, $F'(x) = H'(x)$, that is $F \equiv H$. So, the relation is transitive. Thus, the relation is an equivalence relation.

E8) 1. When $n \neq -1$ have

$$\frac{d}{dx}\left(\frac{x^n}{n+1}\right) = \frac{1}{n+1}\frac{d}{dx}\left(x^{n+1}\right) = \frac{1}{n+1}\{(n+1)x^{n+1-1}\} = x^n.$$

2. We have

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

3. We have

$$\frac{d}{dx}(-\cos x) = -\frac{d}{dx}(\cos x) = -(-\sin x) = \sin x.$$

4. We have $\frac{d}{dx}(\sin x) = \cos x$.

6. We have $\frac{d}{dx}(\ln|\operatorname{cosec} x|) = \frac{1}{\operatorname{cosec} x}\operatorname{cosec} x \cot x = \cot x$.

8. We have

$$\frac{d}{dx}(\ln|\operatorname{cosec} x - \cot x|)$$
$$= \frac{1}{\operatorname{cosec} x - \cot x}\left(-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x\right) = \operatorname{cosec} x$$

9. We have $\frac{d}{dx}(e^x) = e^x$.

10. We have $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$. See Unit 9.

11. We have

$$\frac{d}{dx}\left(\ln\left|x + \sqrt{x^2-1}\right|\right) = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{1}{2}\frac{1}{\sqrt{x^2-1}}(2x)\right)$$
$$= \frac{1}{\sqrt{x^2-1}}$$

12. Similar to 11.

13. We have $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$. See Unit 9.

14. We have $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$. See Unit 9.

E9) i) Using the first entry in Table 1, we get

$$\int x^4\,dx = \frac{x^{4+1}}{4+1}\,dx = \frac{x^5}{5} + C$$

ii) We have $\int \dfrac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1}\,dx = \dfrac{2}{7}x^{\frac{7}{2}} + C.$

iii) We have $\int \dfrac{1}{x^7}\,dx = \int x^{-7}\,dx = \dfrac{x^{-7+1}}{-7+1} = -\dfrac{1}{6x^6}.$

iv) Using Eqn. (29), we have

$$\int \left(x^6 + 3x^4 - 2x^2\right) dx = \int x^6\,dx + 3\int x^4\,dx - 2\int x^2\,dx$$
$$= \frac{x^7}{7} + 3\frac{x^5}{5} - 2\frac{x^3}{3} + C$$

vi) Expanding $(x-1)^4$ using binomial theorem, we get

$$\int \frac{(x-1)^4}{x^2}\,dx = \int \frac{x^4 - 4x^3 + 6x^2 - 4x + 1}{x^2}\,dx$$
$$= \int \left(x^3 - 4x + 6 - \frac{4}{x} + \frac{1}{x^2}\right) dx$$

E10) i) We have

$$\int (4\cos x + 3\tan x)\,dx = 4\int \cos x + 3\int \tan x$$
$$= 4\sin x + \ln|\sec x| + C$$

ii) We have $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$. Therefore, $\int \operatorname{cosec}^2 x\,dx = -\cot x + C.$

iii) Expanding the numerator, we get

$$\int \frac{9 + 12\cos x + 4\cos^2 x}{\sin^2 x}\,dx = 9\int \operatorname{cosec}^2 x\,dx + 12\int \operatorname{cosec} x \cot x\,dx$$
$$+ 4\int \cot^2 x\,dx$$
$$= -9\cot x - 12\operatorname{cosec} x$$
$$+ 4\int \left(\operatorname{cosec}^2 x - 1\right) dx$$
$$= -9\cot x - 12\operatorname{cosec} x - 4\cot x - 4x + C$$
$$= -13\cot x - 12\operatorname{cosec} x - 4x + C$$

UNIT 18

METHODS OF INTEGRATION

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18.1 INTRODUCTION

In the last unit we have seen that the definite integral $\int_a^b f(x)\, dx$ is the signed area bounded by the curve y = f(x), the x-axis and the lines x = a and x = b. The Fundamental Theorem of Calculus gives us an easy way of evaluating such an integral, by first finding the antiderivative of the given function, whenever it exists.

Starting from this unit, we shall study various methods and techniques of integration. In this unit, we shall consider method of substitution and the method of integration by parts in Sec. 18.2 and Sec. 18.3. In Sec. 18.4 we shall discuss the method of partial fractions for integrating rational functions. In Sec. 18.5, we shall see various methods for integration of rational trigonometric functions. In Sec. 18.6 we shall consider the integration of irrational functions.

Objectives

After studying this unit, you should be able to:

- use the method of substitution to simplify and evaluate certain integrals;

- integrate by parts a product of two functions whenever it is possible;
- integrate a rational function using partial fraction expansion of the function whenever it is possible;
- integrate rational trigonometric functions whenever it is possible; and
- integrate irrational functions whenever it is possible.

18.2 INTEGRATION BY SUBSTITUTION

You have already seen how certain substitutions can help in simplifying the task of finding derivatives. In this section we shall see how the method of substitution helps in integration. In contrast to the Differential Calculus, where substitution played a marginal role, we will see that it is one of the most commonly used techniques of integration. We shall illustrate its application through a number of examples.

Before we proceed further, we reiterate what we said in the previous units. In the results we will prove about indefinite integrals, neither will we mention the interval nor will we check that the functions involved satisfy the conditions for integrability in that interval.

18.2.1 Method of Substitution

We begin our subsection with a theorem that provides the backbone of the method of substitution.

Theorem 1: Suppose u(x) has a continuous derivative in the interval [c, d] and u([c, d]) = [a, b]. Further, suppose that f(x) is continuous on [a, b] and F(x) is an antiderivative of f in [a, b]. Then, we have

$$\int f(u(x))u'(x)dx = F(u(x)) + C \qquad \ldots (1)$$

in the interval [c, d]. Further,

$$\int_c^d f[u(x)]u'(x)\,dx = \int_{u(c)}^{u(d)} f(v)\,dv \qquad \ldots (2)$$

Proof: We shall make use of the chain rule for derivatives (Unit 9) to prove this theorem. Since F is the antiderivative of f, we can write $\frac{dF(u)}{du} = f(u)$. Now,

$$\begin{aligned}\frac{d}{dx}F[u(x)] &= \frac{dF[u(x)]}{du(x)}.\frac{du(x)}{dx} \text{ by chain rule}\\ &= f[u(x)].\frac{du(x)}{dx}\\ &= f[u(x)].u'(x)\end{aligned}$$

This shows that F[u(x)] is an antiderivative of f[u(x)]u′(x). This means that

$$\int f[u(x)]u'(x)\,dx = F[u(x)] + C$$

Another way of writing Eqn. (1) is

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du. \qquad \ldots(3)$$

where $u = u(x)$. We interpret Eqn. (3) to mean that 'The indefinite integral of $f(u)\frac{du}{dx}$ is obtained by replacing u by $u = u(x)$ in the indefinite integral of $f(u)$.'.

Example 1: Evaluate the following integrals:
i) $\int (2x+1)^9\,dx$. ii) $\int \sin 2x\,dx$. iii) $\int \frac{2x}{(x^2+1)^5}\,dx$. iv) $\int x\sqrt{3x+1}\,dx$.

Solution:

i) We look at the table of indefinite integrals in Unit 17. We find that the $(2x+1)^n$ doesn't figure in the table, but the table gives the integral for x^n. So, the substitution $u(x) = 2x + 1$ looks promising. We have $\frac{d}{dx}(u) = 2$. So, substituting $u(x) = 2x + 1$,

$$\int (2x+1)^9\,dx = \frac{1}{2}\int (2x+1)^9\,2dx = \frac{1}{2}\int \underbrace{(2x+1)^9}_{u^9}\underbrace{2}_{\frac{du}{dx}}\,dx$$

$$= \frac{1}{2}\int u^9\,du = \frac{1}{2}\int u^9\,du$$

We have

$$\int u^9 du = \frac{u^{10}}{10} + C$$

Substituting $u = u(x) = 2x + 1$, we get

$$\int (2x+1)^9\,dx = \frac{(2x+1)^{10}}{20} + C.$$

ii) We set $u(x) = 2x$. Then, $\frac{du}{dx} = 2$. So,

$$\int \sin 2x\,dx = \frac{1}{2}\int \sin 2x\,2dx = \frac{1}{2}\int \underbrace{\sin 2x}_{\sin u}\underbrace{2}_{\frac{du}{dx}}\,dx$$

$$= \frac{1}{2}\int \sin u\,du = -\frac{\cos u}{2} + C.$$

Substituting $u = u(x) = 2x$, we get

$$\int \sin 2x\,dx = -\frac{\cos 2x}{2} + C$$

iii) We set $u(x) = x^2 + 1$. Then, $\frac{du}{dx} = 2x$. We have

$$\int \frac{2x}{(x^2+1)^5}\,dx = \int \underbrace{\frac{1}{(x^2+1)^5}}_{\frac{1}{u^5}}\underbrace{2x}_{\frac{du}{dx}}\,dx = \int \frac{1}{u^5}\,du = -\frac{1}{4u^4} + C.$$

On substituting $u = u(x) = x^2 + 1$, we get

$$\int \frac{2x}{(x^2+1)^5}\,dx = -\frac{1}{4(x^2+1)^4} + C$$

iv) The substitution u(x) = 3x + 1 looks promising. Then, $\frac{du}{dx} = 3$. We have,

$$\int x\sqrt{3x+1}\,dx = \frac{1}{3}\int \underbrace{x}_{?}\underbrace{\sqrt{3x+1}}_{\sqrt{u}}\underbrace{3}_{\frac{du}{dx}}\,dx$$

We are almost successful. We just need to write x also in terms of u. From u = 3x + 1, we get $x = \frac{u-1}{3}$. So,

$$\int x\sqrt{3x+1}\,dx = \frac{1}{3}\int \frac{u-1}{3}\sqrt{u}\,du = \frac{1}{9}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$$

$$= \frac{1}{9}\left(\int u^{\frac{3}{2}}\,du - \int u^{\frac{1}{2}}\,du\right) = \frac{1}{9}\left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{3}}}{\frac{3}{2}}\right) + C$$

Substituting u = 3x + 1, we get

$$\int x\sqrt{3x+1}\,dx = \frac{2}{9}\left(\frac{(3x+1)^{\frac{5}{2}}}{5} - \frac{(3x+1)^{\frac{3}{2}}}{3}\right) + C$$

* * *

We make a special mention of the following two cases which follow from Theorem 1.

Case i) If $f(u) = u^n$, $n \neq -1$ and u = u(x), then, $f(u(x))u'(x) = (u(x))^n u'(x)$. By the formula for the indefinite integral of x^n in Unit 17, we get $F(u) = \frac{u^{n+1}}{n+1}$. So,

$$\int [u(x)]^n u'(x)\,dx = F(u(x)) + C = \frac{u(x)^{n+1}}{n+1} + C \qquad \ldots(4)$$

Case ii) If $f(u) = \frac{1}{u}$ and u = u(x), then by formula 11 of Table 1 in Unit 17, we have $F(u) = \ln|u|$. So,

$$\int \frac{u'(x)}{u(x)}\,dx = F(u(x)) + C = \ln|u(x)| + C. \qquad \ldots(5)$$

You can see that the cases above are very useful from the examples that follow.

Example 2: Integrate $(2x+1)\left(x^2+x+1\right)^5$.

Solution: For this we observe that $\frac{d}{dx}\left(x^2+x+1\right) = 2x+1$. Thus $\int(2x+1)\left(x^2+x+1\right)^5$ is of the form $\int[u(x)]^n u'(x)\,dx$ where $u(x) = x^2+x+1$. So, we can evaluate it using Eqn. (4). Therefore,

$$\int(2x+1)\left(x^2+x+1\right)^5 dx = \frac{1}{6}\left(x^2+x+1\right)^6 + C$$

Alternatively, we set $u = x^2+x+1$. Then, $du = (2x+1)\,dx$. Solving for dx, we have $dx = \frac{du}{2x+1}$. (See below.)

$$\int(2x+1)\left(x^2+x+1\right)^5 dx = \int(2x+1)\left(x^2+x+1\right)^5 \frac{du}{2x+1}$$

$$= \int u^5\,du = \frac{u^6}{6} + C.$$

Substituting $u = x^2 + x + 1$, we get

$$\int (2x+1)\left(x^2+x+1\right)^5 dx = \frac{\left(x^2+x+1\right)^6}{6} + C.$$

* * *

Note that,we have used the 'differential notation' in the previous example. This will help you in carrying out substitutions in a routine fashion. Although $\frac{du}{dx}$ is not a fraction, we can justify 'solving for dx' as follows: Suppose that we are trying to integrate $\int h(x)\,dx$ by the substitution $u = u(x)$. Then, solving $\frac{du}{dx} = u'(x)$ for dx amounts to replacing dx by $\frac{du}{u'(x)}$ and hence writing

$$\int h(x)\,dx = \int h(x)\frac{du}{u'(x)} \qquad \ldots(6)$$

Suppose, $f(u)$ is such that $\frac{h(x)}{u'(x)} = f(u(x))$. Then, we are saying that $h(x) = f(u)u'(x) = f(u(x))u'(x)$ and we can rewrite Eqn. (6) as

$$\int f(u(x))u'(x)\,dx = \int f(u)\,du$$

This is basically Eqn. (3).

Example 3: Evaluate $\int (ax+b)^n\,dx$.

Solution: We substitute $u = (ax+b)$. We have $du = a\,dx$. So,

$$\int \underbrace{(ax+b)^n}_{u^n} dx = \int u^n \frac{du}{a}$$

$$= \frac{1}{a}\int u^n\,du = \frac{u^{n+1}}{a(n+1)} + C = \frac{(ax+b)^{n+1}}{a(n+1)} + C \text{ if } n \neq -1.$$

If $n = -1$,

$$\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{du}{u} = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C$$

* * *

Example 4: Evaluate the definite integral $\displaystyle\int_0^2 \frac{x+1}{x^2+2x+3}\,dx$.

Solution: Comparing with Eqn. (2), here, $c = 0$, $d = 2$. We put

$$x^2 + 2x + 3 = u(x) = u.$$

Note that, $u(x)$ is a polynomial, so it has a continuous derivative everywhere on the real line. We have $u(x) = 3$ when $x = 0$ and when $x = 2$, $u(x) = 11$. Further, since $u'(x) > 0$, in $(0, 2)$, $u(x)$ is increasing in $[0, 2]$. Hence, $u([0, 2]) = [3, 11]$. So, $a = 3$, $b = 11$. (See Fig. 1.) We have $f(u) = \frac{1}{u}$ is continuous on $[3, 11]$, $F(u) = \ln|u|$ is an antiderivative of f and $u'(x) = 2(x+1)$. Thus,

$$\int_0^2 \frac{x+1}{x^2+2x+3}\,dx = \frac{1}{2}\int_0^2 \underbrace{\frac{1}{x^2+2x+3}}_{f(u)=\frac{1}{u}}\underbrace{2(x+1)}_{u'(x)}\,dx = \frac{1}{2}\int_0^2 f(u(x))u'(x)\,dx$$

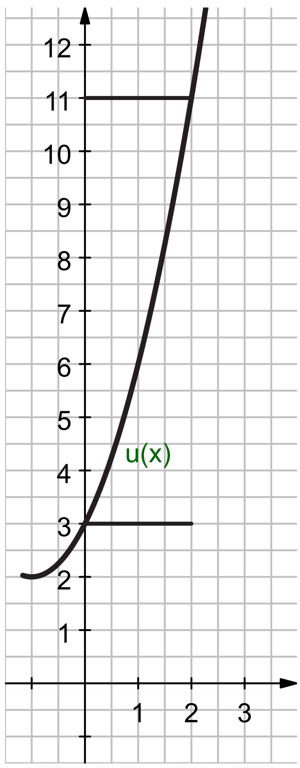

Fig. 1: Range of $u(x) = x^2 + 2x + 3$.

From Eqn. (2),

$$\frac{1}{2}\int_0^2 f(u(x))u'(x)\,dx = \frac{1}{2}\int_{u(c)}^{u(d)} f(u)\,du = \frac{1}{2}\int_3^{11} \frac{1}{u}\,du$$

Since $u \neq 0$ in $[3, 11]$, $\frac{1}{u}$ is a continuous function of u and $F(u) = \ln|u|$ is an antiderivative of $\frac{1}{u}$. Therefore, by Fundamental Theorem of Calculus.

$$\frac{1}{2}\int_3^{11} \frac{1}{u}\,du = \frac{1}{2}(F(11) - F(3))$$

So,

$$\int_0^2 \frac{x+1}{x^2+2x+3}\,dx = \frac{1}{2}\int_3^{11} \frac{1}{u}\,du = \frac{1}{2}(\ln 11 - \ln 3) = \frac{1}{2}\ln\frac{11}{3}.$$

Another approach is to find the indefinite integral of $\frac{x+1}{x^2+2x+3}$ and use Fundamental Theorem of Integral Calculus. Note that, the discriminant of the equation $x^2 + 2x + 3 = 0$ is negative, i.e. $2^2 - 4 \cdot 1 \cdot 3 = -8 < 0$. So, the equation doesn't have a real root, i.e. $x^2 + 2x + 3 \neq 0$ on $\mathbb{R}$. Hence, f is continous on $\mathbb{R}$. We set $u(x) = x^2 + 2x + 3$. Then, $du = 2(x+1)\,dx$, so $dx = \frac{du}{2(x+1)}$ dx. We have

$$\int \frac{x+1}{x^2+2x+3}\,dx = \int \frac{x+1}{u}\frac{du}{2(x+1)}$$
$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+2x+3\right| + C.$$

So, $\frac{1}{2}\ln\left|x^2+2x+3\right|$ is an antiderivative of $\frac{x+1}{x^2+2x+3}$. Recall that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

if f is continous on [a, b] and F is an antiderivative of f. So,

We write $F(x)|_a^b$ for $F(b) - F(a)$.

$$\int_0^2 \frac{x+1}{x^2+2x+3}\,dx = \frac{1}{2}\ln\left|x^2+2x+3\right|\Big|_0^2$$

$$= \frac{1}{2}\ln\left|2^2+2\cdot 2+3\right| - \frac{1}{2}\ln\left|0^2+0\cdot 2+3\right|$$

$$= \frac{1}{2}\ln|11| - \frac{1}{2}\ln|3| = \frac{1}{2}\ln\left|\frac{11}{3}\right|.$$

* * *

Example 5: Evaluate the integral $\int xe^{2x^2}\,dx$.

Solution: We set $u = 2x^2$. Then, $du = 4x\,dx$. We have

$$\int xe^{2x^2}\,dx = \int e^u x\frac{du}{4x} = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C = \frac{1}{4}e^{2x^2} + C.$$

* * *

On the basis of the rules discussed in this section, you will be able to solve this exercise.

E1) Evaluate the following integrals:

i) $\int\sqrt{5x-3}\,dx$ ii) $\int(2x+1)^6\,dx$ iii) $\int_1^3 \frac{dx}{4+5x}\,dx$

iv) $\int\frac{5}{10x+7}\,dx$ v) $\int\frac{x+1}{x^2+2x+7}\,dx$ vi) $\int_2^3\frac{3x^2+2x+1}{x^3+x^2+x-8}\,dx$

vii) $\int x^{1/3}\sqrt{x^{4/3}-1}\,dx$ viii) $\int\frac{x}{\sqrt{1-3x^2}}\,dx$

Now, we shall use the method of substitution to integrate some integrals involving trigonometric functions.

Example 6: Evaluate the following:

i) $\int\sin(ax+b)\,dx$ ii) $\int\cot(ax+b)\,dx$ iii) $\int\tan(ax+b)\,dx$ iv) $\int\cos\left(x^2\right)x\,dx$

Solution:

i) We proceed in the same manner as we did for $\int\sin 2x\,dx$ in Example 1. We set $u = ax+b$. This gives $du = a\,dx$. Thus

$$\int\sin(ax+b)\,dx = \int\sin u\frac{du}{a} = \frac{1}{a}\int\sin u\,du$$

$$= -\frac{1}{a}\cos u + C = -\frac{\cos(ax+b)}{a} + C.$$

ii) We make the substitution $u = ax+b$. Then, $du = a\,dx$.

$$\int\cot(ax+b)\,du = \int\cot u\frac{du}{a} = \frac{1}{a}\int\cot u\,du = \frac{1}{a}\int\frac{\cos u}{\sin u}\,du.$$

Here, the trick is to notice that the numerator is the derivative of the denominator and Eqn. (5) applies here. So, the integral in this case is $\frac{1}{a}\ln|\sin u| + C = \frac{1}{a}\ln|\sin(ax+b)| + C$.

Table 1: Table of indefinite integrals

No.	f(x)	$\int f(x)\,dx$
1.	$(ax+b)^n,\ n \neq -1,$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$
2.	$\frac{1}{(ax+b)}$	$\frac{1}{a}\ln\lvert ax+b\rvert + C$
3.	$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + C$
4.	$\cos(ax+b)$	
5.	$\tan(ax+b)$	$\frac{1}{a}\ln\lvert\sec(ax+b)\rvert + C$
6.	$\cot(ax+b)$	$\frac{1}{a}\ln\lvert\sin(ax+b)\rvert + C$
7.	$\sec(ax+b)$	
8.	$\operatorname{cosec}(ax+b)$	
9.	e^{ax+b}	

iii) We make the substitution $u = ax + b$.Then, $du = a\,dx$. So,

$$\int \tan(ax+b)\,dx = \int \frac{\sin u}{\cos u}\frac{du}{a} = \frac{1}{a}\int \frac{\sin u}{\cos u}\,du.$$

We have $\frac{d}{du}\cos u = -\sin u$. Here, the numerator is almost the derivative of the denominator, but for the sign difference. So, we write

$$\frac{1}{a}\int \frac{\sin u}{\cos u}\,du = -\frac{1}{a}\int \frac{1}{\cos u}(-\sin u)\,du$$

Recall that $-\ln|x| = \ln\left|\frac{1}{x}\right|$. We can now apply Eqn. (5) to get

$$\frac{1}{a}\int \frac{\sin u}{\cos u}\,du = -\frac{1}{a}\int \frac{-\sin u}{\cos u}\,du = -\frac{1}{a}\ln|\cos u| + C = \frac{1}{a}\ln|\sec u| + C$$
$$= \frac{1}{a}\ln|\sec(ax+b)| + C$$

iv) We set $u = x^2$. Then, $du = 2x\,dx$. So, we have

$$\int x\cos\left(x^2\right)\,dx = \int x\cos u\frac{du}{2x} = \frac{1}{2}\int \cos u\,du$$
$$= \frac{1}{2}\sin u + C = \frac{1}{2}\sin\left(x^2\right) + C$$

* * *

Try the exercises below to check you understanding of the discussion so far.

E2) Fill in the blanks in Table 1.

E3) Evaluate the following integrals:

i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot 2x \operatorname{cosec}^2 2x \, dx$ ii) $\int \sin 2\theta e^{\cos 2\theta} \, d\theta$

iii) $\int_0^{\frac{\pi}{2}} \sin\theta \left(1 + \cos^4\theta\right) d\theta$ iv) $\int (1 + \cos\theta)^4 \sin\theta \, d\theta$

v) $\int \frac{\sec^2\theta}{(1 - 5\tan\theta)^3} d\theta$ vi) $\int_0^{\frac{\pi}{4}} \sec\theta \tan\theta (1 + \sec\theta)^3 \, d\theta$

18.2.2 Integrals Using Trigonometric Formulae

We will now see how to use trigonometric identities for evaluating integrals. We need the following trigonometric formulae you studied in your higher secondary Mathematics course.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \ldots(7)$$

$$\sec^2\theta = 1 + \tan^2\theta \qquad \ldots(8)$$

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta \qquad \ldots(9)$$

$$\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B)) \qquad \ldots(10)$$

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B)) \qquad \ldots(11)$$

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B)) \qquad \ldots(12)$$

Here is an example to show you how to apply some of these formulae.

Example 7: Evaluate the following integrals:

i) $\int \sin 3x \cos 4x \, dx$ ii) $\int \sin 4x \sin 7x \, dx$ iii) $\int \cos 4x \cos 6x \, dx$

iv) $\int \cos 3x \cos 4x \sin 6x \, dx$

Solution:

i) From Eqn. (10), we have

$$\sin 3x \cos 4x = \frac{1}{2}(\sin(3x + 4x) + \sin(3x - 4x))$$
$$= \frac{1}{2}(\sin 7x + \sin(-x)) = \frac{1}{2}(\sin 7x - \sin x)$$

So,

$$\int \sin 3x \cos 4x \, dx = \int \frac{1}{2}(\sin 7x - \sin x) \, dx = \frac{1}{2}\left(-\frac{1}{7}\cos 7x + \cos x\right) + C$$
$$= -\frac{1}{14}\cos 7x + \frac{1}{2}\cos x + C$$

ii) By Eqn. (12) We have

$$\sin 4x \sin 7x = \frac{1}{2}(\cos(4x - 7x) - \cos(4x + 7x)) = \frac{1}{2}(\cos(-3x) + \cos 11x)$$
$$= \frac{1}{2}(\cos 3x + \cos 11x)$$

So, we have

$$\int \sin 4x \sin 7x\, dx = \int \frac{1}{2}(\cos 3x + \cos 11x)\, dx$$
$$= \frac{1}{6}\sin 3x + \frac{1}{22}\sin 11x + C$$

iii) From Eqn. (12), we get

$$\cos 4x \cos 6x = \frac{1}{2}(\cos(4x+6x) + \cos(4x-6x)) = \frac{1}{2}(\cos 10x + \cos(-2x))$$
$$= \frac{1}{2}(\cos 10x + \cos 2x)$$

So, we have

$$\int \cos 4x \cos 6x\, dx = \int \frac{1}{2}(\cos 10x + \cos 2x)\, dx$$
$$= \frac{1}{20}\sin 10x + \frac{1}{4}\sin 2x + C$$

iv) We have

$$\cos 3x \cos 4x = \frac{1}{2}(\cos 7x + \cos x) \text{ using Eqn. (11)}$$

So, we have

$$\cos 3x \cos 4x \sin 6x = \frac{1}{2}(\cos 7x \sin 6x + \cos x \sin 6x)$$
$$= \frac{1}{2}\left\{\frac{1}{2}(\sin(6x+7x) + \sin(6x-7x)) + \frac{1}{2}(\sin(6x+x) + \sin(6x-x))\right\}$$
$$= \frac{1}{4}(\sin 13x - \sin x + \sin 7x + \sin 5x)$$

So,

$$\int \cos 3x \cos 4x \sin 6x\, dx = \frac{1}{4}\left(-\frac{1}{13}\cos 13x + \cos x - \frac{1}{7}\cos 7x - \frac{1}{5}\cos 5x\right) + C$$
$$= -\frac{1}{52}\cos 13x + \frac{1}{4}\cos x - \frac{1}{28}\cos 7x - \frac{1}{20}\cos 5x + C$$

* * *

We will now see how to integrate powers of $\sin x$ and $\cos x$. Using Eqn. (7), we can write

$$\int \sin^{2n+1} x\, dx = \int \left(1-\cos^2 x\right)^n \sin x\, dx \qquad \ldots(13)$$

and

$$\int \cos^{2n+1} x\, dx = \int \left(1-\sin^2 x\right)^n \cos x\, dx \qquad \ldots(14)$$

To evaluate the integral in Eqn. (13), we substitute $u = \cos x$. The integral in Eqn. (13) becomes $-\int \left(1-u^2\right)^n du$. We expand $\left(1-u^2\right)^n$ using binomial theorem and integrate it term by term. In case of Eqn. (14), the substitution $u = \sin x$ reduces it to $\int \left(1-u^2\right)^n du$.

Note that we can evaluate integrals of the form $\int \sin^m x \cos^n x\, dx$ similarly if one of m or n is odd. If m is odd we use the substitution $u = \cos x$. If n is odd, we use the substitution $u = \sin x$.

We will now look at an example that illustrates all the methods we have discussed above.

Example 8: Evaluate the following integrals:

i) $\int \sin^5 x\, dx$ ii) $\int \cos^7 x\, dx$ iii) $\int \sin^2 x \cos^3 x\, dx$

v) $\int \sin^3 x \cos^4 x\, dx$ iv) $\int \cos^2 x\, dx$

Solution:

i) We have

$$\int \sin^5 x\, dx = \int \sin^4 x \sin x\, dx = \int \left(1-\cos^2 x\right)^2 \sin x\, dx$$

using Eqn. (7). We set $u = \cos x$. Then, $du = -\sin x\, dx$. So, we have

$$\int \sin^5 x\, dx = -\int \left(1-u^2\right)^2 du = -\int \left(1-2u^2+u^4\right) du$$

$$= -u + 2\frac{u^3}{3} - \frac{u^5}{5} + C = -\frac{\cos^5 x}{5} + 2\frac{\cos^3 x}{3} - \cos x + C$$

ii) We use Eqn. (7) to rewrite the integral as

$$\int \cos^7 x\, dx = \int \cos^6 x \cos x\, dx = \int \left(1-\sin^2 x\right)^3 \cos x\, dx.$$

Now, we set $v = \sin x$. Then, $dv = \cos x\, dx$. So, we have

$$\int \cos^7 x\, dx = \int \left(1-v^2\right)^3 dv$$

$$= \int \left(1-3v^2+3v^4-v^6\right) dv = v - v^3 + 3\frac{v^5}{5} - \frac{v^7}{7} + C$$

$$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{\sin^7 x}{7} + C$$

iii) We have

$$\sin^2 x \cos^3 x = \sin^2 x \left(1-\sin^2 x\right) \cos x$$

So,

$$\int \sin^2 x \cos^3 x\, dx = \int \left(\sin^2 x - \sin^4 x\right) \cos x\, dx$$

We set $u = \sin x$. Then, $du = \cos x\, dx$. Therefore,

$$\int \left(\sin^2 x - \sin^4 x\right) \cos x\, dx = \int \left(u^2 - u^4\right) du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

iv) We have

$$\int \sin^3 x \cos^4 x\, dx = \int \left(1-\cos^2 x\right)\cos^4 x \sin x\, dx$$

We set $u = \cos x$. Then, $du = -\sin x\, dx$. So, we have

$$\int \left(1-\cos^2 x\right)\cos^4 x \sin x\, dx = -\int \left(u^4 - u^6\right) du$$
$$= -\frac{u^5}{5} + \frac{u^7}{7} + C$$
$$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$

v) We have $\cos 2x = 2\cos^2 x - 1$. (Put $A = B = x$ in Eqn. (11)). So, $\cos^2 x = \frac{1+\cos 2x}{2}$. Therefore,

$$\int \cos^2 x\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{1}{2}\left(\int dx + \int \cos 2x\, dx\right)$$
$$= \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$$

* * *

Remark 1: We will see how to integrate powers of $\sin x$ and $\cos x$, even or odd, using reduction formulae in the next Unit. But, the substitution method is a quicker method for integrating odd powers of $\sin x$ and $\cos x$.

Here are some exercises for you to check your understanding of the example.

E4) Evaluate the following:

i) $\int \sin^7 x\, dx$ ii) $\int \cos^5 x\, dx$ iii) $\int \cos^2 x \sin^3 x\, dx$

iv) $\int \sin 5x \cos 3x\, dx$ v) $\int \cos 3x \cos 4x\, dx$ vi) $\int \sin 4x \sin 3x\, dx$

vii) $\int \sin 2x \sin 3x \sin 5x\, dx$

A trigonometric substitution is generally used to integrate expressions involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ or $a^2 + x^2$. We suggest the substitutions in Table 2.

Table 2: Trigonmetric substitutions

Expression involved	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$
$a^2 + x^2$	$x = a\tan\theta$

We now derive the formulae for integration for the functions $\frac{1}{\sqrt{a^2 \pm x^2}}$ and $\frac{1}{\sqrt{x^2 - a^2}}$ using Table 2 and Eqn. (7), Eqn. (8) and Eqn. (9).

Theorem 2: We have

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad \ldots(15)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \ldots(16)$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left|\frac{x + \sqrt{x^2 + a^2}}{a}\right| + C \qquad \ldots(17)$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + C \qquad \ldots(18)$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C \qquad \ldots(19)$$

Proof: If we put $x = a\sin\theta$, we have $dx = a\cos\theta d\theta$. Further,

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\left(1 - \sin^2\theta\right)} = a\cos\theta$$

So, we have

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta + C$$

From $x = a\sin\theta$, we have $\theta = \sin^{-1}\frac{x}{a}$. So,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

proving Eqn. (15).

Let us now prove Eqn. (16). From the Table 2, we know that we need to substitute $x = a\tan\theta$. We get $dx = a\sec^2\theta\, d\theta$. We have

$$\int \frac{1}{a^2 + x^2}\,dx = \int \frac{a\sec^2\theta}{a^2\left(1 + \tan^2\theta\right)}\,d\theta = \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

since $\sec^2\theta = 1 + \tan^2\theta$.

$$\therefore \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$$
$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Let us now prove Eqn. (17). Setting $x = a\tan\theta$, we get $dx = a\sec^2\theta\, d\theta$. We have

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{a\sec^2\theta}{\sqrt{\left(a^2 + a^2\tan^2\theta\right)}}\,d\theta = \int \frac{a\sec^2\theta}{a\sqrt{1 + \tan^2\theta}}\,d\theta$$
$$= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C \qquad \ldots(20)$$

We have

$$\sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + \left(\frac{x}{a}\right)^2} = \frac{\sqrt{a^2 + x^2}}{a}$$

Also,

$$\tan\theta = \left(\frac{x}{a}\right)$$

$$\therefore\ \sec\theta + \tan\theta = \frac{x + \sqrt{x^2 + a^2}}{a}$$

So, from Eqn. (20), we have

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left|\frac{x + \sqrt{x^2 + a^2}}{a}\right| + C$$

We leave Eqn. (18) and Eqn. (19) to you as exercises. ■

We now look at some easy applications of Theorem 2 in the next example.

Example 9: Evaluate the following integrals:

i) $\int \frac{dx}{\sqrt{4 - x^2}}$ ii) $\int \frac{dx}{9 + x^2}$ iii) $\int \frac{dx}{\sqrt{25 + x^2}}$ iv) $\int \frac{dx}{\sqrt{1 - 2x^2}}$ v) $\int \frac{dx}{1 + 4x^2}$

Solution:

i) We have

$$\int \frac{dx}{\sqrt{4 - x^2}} = \int \frac{dx}{\sqrt{2^2 - x^2}}$$

Applying Eqn. (15) with a = 2 to the RHS of the above equation, we get

$$\int \frac{dx}{\sqrt{4 - x^2}} = \sin^{-1}\left(\frac{x}{2}\right) + C$$

ii) We have

$$\int \frac{dx}{9 + x^2} = \int \frac{dx}{3^2 + x^2}$$

Applying Eqn. (16) with a = 3 to the RHS of the above equation, we get

$$\int \frac{dx}{9 + x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

iii) We have

$$\int \frac{dx}{\sqrt{25 + x^2}} = \int \frac{dx}{\sqrt{5^2 + x^2}}$$

Using Eqn. (17), we get

$$\int \frac{dx}{\sqrt{25 + x^2}} = \ln\left|\frac{x + \sqrt{25 + x^2}}{5}\right| + C$$

iv) We have

$$\int \frac{1}{\sqrt{1 - 2x^2}}\,dx = \int \frac{dx}{\sqrt{2\left(\frac{1}{2} - x^2\right)}} = \int \frac{dx}{\sqrt{2}\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - x^2}}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\frac{1}{\sqrt{2}}} + C = \frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}x) + C$$

v) We have

$$\int \frac{dx}{\sqrt{1+4x^2}} = \int \frac{dx}{\sqrt{4\left(\frac{1}{4}+x^2\right)}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 + x^2}}$$

$$= \frac{1}{2}\left(\ln\left|\frac{x+\sqrt{x^2+\frac{1}{4}}}{\frac{1}{2}}\right|\right) + C = \frac{1}{2}\ln\left|2x+\sqrt{4x^2+1}\right| + C$$

* * *

Here are some exercises that may help you in testing your understanding of the discussion in the above example.

E5) We evaluated the integral of $\frac{1}{\sqrt{a^2-x^2}}$ using the substitution $x = a\sin\theta$. Evaluate the integral using the substitution $x = a\cos\theta$. You will get the answer $-\cos^{-1}\left(\frac{x}{a}\right) + C$. But we got the answer $\sin^{-1}\left(\frac{x}{a}\right) + C$ when we used the substitution $x = a\sin\theta$. How would you explain this difference in the answers ?

E6) Prove the results in Eqn. (18) and Eqn. (19).

E7) Evaluate the following integrals:

i) $\int \frac{dx}{\sqrt{3-x^2}}$ ii) $\int \frac{dx}{\sqrt{1-5x^2}}$ iii) $\int \frac{dx}{x^2+5}$ iv) $\int \frac{dx}{3x^2+1}$

v) $\int \frac{dx}{\sqrt{x^2+7}}$ vi) $\int \frac{dx}{\sqrt{1+5x^2}}$ vii) $\int \frac{dx}{\sqrt{x^2-9}}$ viii) $\int \frac{dx}{x\sqrt{x^2-4}}$

ix) $\int \frac{dx}{\sqrt{9x^2-1}}$ x) $\int \frac{dx}{x\sqrt{2x^2-1}}$

We conclude our discussion of substitution for now and discuss another method, the method of integration by parts, in the next section. However, we are not through with the use of substitution yet! We will come back to the method in the later sections of this unit.

18.3 INTEGRATION BY PARTS

In this section we shall evolve a method for evaluating integrals of the type $\int u(x)v(x)\,dx$, in which the integrand $u(x)v(x)$ is the product of two functions. In other words, we shall first evolve the integral analogue of

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$$

and then use that result to evaluate some standard integrals.

18.3.1 Integrals of a Product of Two Functions

We can calculate the derivative of the product of two functions by the formula

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x).$$

Let us rewrite this as

$$u(x)\frac{d}{dx}v(x) = \frac{d}{dx}[u(x)v(x)] - v(x)\frac{d}{dx}u(x)$$

Integrating both the sides with respect to x, we have
$\int u(x)\frac{d}{dx}(v(x))dx = \int \frac{d}{dx}(u(x)v(x))dx - \int v(x)\frac{d}{dx}(u(x))dx$, or

$$\int u(x)\frac{d}{dx}(v(x))dx = u(x)v(x) - \int v(x)\frac{d}{dx}(u(x))dx \qquad \ldots(21)$$

To express this in a more symmetrical form, we replace u(x) by f(x), and put $\frac{d}{dx}v(x) = g(x)$. This means $v(x) = \int g(x)dx$.

As a result of this substitution, Eqn. (21) takes the form

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int \left\{f'(x)\int g(x)dx\right\} dx$$

This formula may be read as:

The integral of the product of two functions = First factor $\times$ integral of second factor – integral of (derivative of first factor $\times$ integral of second factor)

It is called the **formula for integration by parts**. This formula may appear a little complicated to you. But the success of this method depends upon choosing the first factor in such a way that the second term on the right-hand side may be easy to evaluate. It is also essential to choose the second factor such that it can be easily integrated.

The following examples will show you the wide variety of integrals which can be evaluated by this technique. You should carefully study our choice of first and second functions in each example. You may also try to evaluate the integrals by reversing the order of functions. This will make you realise why we have chosen these functions the way we have.

Example 10: Evaluate the following integrals:

i) $\int \ln|x|\, dx$ ii) $\int xe^x\, dx$ iii) $\int_0^{\frac{\pi}{2}} x^2 \cos x\, dx$ iv) $\int x\ln|x|\, dx$

Solution:

i) We can find $\int \ln x\, dx$ by taking $\ln x$ as the first factor and 1 as the second factor. Thus,

$$\int \ln x\, dx = \int (\ln x)\,(1)\,dx = \ln x\int 1dx - \int \left(\frac{1}{x}\int 1dx\right) dx$$
$$= (\ln x)\,(x) - \int \frac{1}{x}(x)\,dx = x\ln x - \int dx = x\ln x - x + C$$

ii) In the integrand xe^x we choose x as the first factor and e^x as the second factor. Thus, we get

$$\int xe^x dx = x\int e^x dx - \int \left\{\frac{d}{dx}(x)\int e^x dx\right\} dx = xe^x - \int e^x dx$$
$$= xe^x - e^x + C.$$

iii) We shall take x^2 as the first factor and $\cos x$ as the second. Let us first evaluate the corresponding indefinite integral.

$$\int x^2 \cos x\, dx = x^2\int \cos x\, dx - \int \left\{\frac{d}{dx}(x^2)\int \cos x\, dx\right\} dx$$
$$= x^2 \sin x - \int 2x\, \sin x\, dx = x^2 \sin x - 2\int x \sin x\, dx$$

We shall again use integration by parts to evaluate $\int x \sin x\, dx$. We have,

$$\int x \sin x\, dx = x(-\cos x) - \int (1)(-\cos x) dx \quad (f(x) = x,\ g(x) = \sin x)$$

$$= -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + C$$

Hence,

$$\int x^2 \cos x\, dx = x^2 \sin x + 2x\cos x - 2\sin x + C$$

Note that we have written the arbitrary constant as C instead of 2C.
Now

$$\int_0^{\pi/2} x^2 \cos x\ dx = \left(x^2 \sin x + 2x\cos x - 2\sin x + C\right)\Big|_0^{\pi/2} = \frac{\pi^2}{4} - 2.$$

iv) Here we take $\ln|x|$ as the first factor since it can be differentiated easily, but cannot be integrated that easily. We shall take x to be the second factor.

$$\int x \ln|x|\, dx = \int (\ln|x|) x\, dx = \left(\ln|x| \frac{x^2}{2} - \int \left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right)\right) dx$$

$$= \frac{1}{2}x^2 \ln|x| - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2 \ln|x| - \frac{1}{4}x^2 + C$$

* * *

Try the following exercises to see if you have understood example You will be able to solve the following exercises by using the method of integration by parts.

E8) Evaluate the following integrals:

i) $\int x^2 \ln|x|\ dx$ (Take $f(x) = \ln|x|$ and $g(x) = x^2$)

ii) $\int (1 + x)e^x dx$ (Take $f(x) = 1 + x$ and $g(x) = e^x$)

iii) $\int \left(1 + x^2\right) e^x dx$

iv) $\int x^2 \sin x \cos x\ dx$ (Take $f(x) = x^2$ and $g(x) = \sin x \cos x = \frac{1}{2}\sin 2x$.)

E9) Evaluate the following integrals by choosing 1 as the second factor.

i) $\int \sin^{-1} x\, dx$ ii) $\int_0^1 \tan^{-1} x\, dx$ iii) $\int \cot^{-1} x\ dx$

E10) Integrate $x \ln\left|\left(1 + x^2\right)\right|$ w.r.t. x.

18.3.2 Evaluation of $\int e^{ax}\sin bx\ dx$ and $\int e^{ax}\cos bx\ dx$

To evaluate $\int e^{ax} \sin bx\ dx$ and $\int e^{ax} \cos bx\ dx$, we use the formula for integration by parts.

$$\int e^{ax} \sin bx\ dx = \left(e^{ax}\right)\ \left(-\frac{1}{b}\cos bx\right) - \int \left(ae^{ax}\right)\ \left(-\frac{1}{b}\cos bx\right) dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\ dx$$
$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left[(e^{ax})\ \left(\frac{1}{b}\sin bx\right) - \int \left(e^{ax}\frac{a}{b}\sin bx\right) dx\right]$$
$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\ dx$$

You would have noted that the last integral on the right hand side is the same as the integral on the left hand side. Now we transfer the third term on the right to the left hand side, and obtain,

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\sin bx\, dx = e^{ax}\left(\frac{a}{b^2}\sin bx - \frac{1}{b}\cos bx\right)$$

This means,

$$\int e^{ax}\sin bx\ dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx) + C$$

We leave it to you as an exercise to show that

$$\int e^{ax}\cos bx\, dx = \frac{1}{a^2+b^2}e^{ax}(a\cos bx + b\sin bx) + C$$

If we put $a = r\cos\theta$, $b = r\sin\theta$, these formulas become

$$\int e^{ax}\sin bx\ dx = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin(bx-\theta) + C \qquad \ldots(22)$$
$$\int e^{ax}\cos bx\ dx = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\cos(bx-\theta) + C \qquad \ldots(23)$$

Here is an example to help you understand how to apply Eqn. (22) and Eqn. (23).

Example 11: Use Eqn. (22) and Eqn. (23) to evaluate the following integrals:
i) $\int e^x \sin x\, dx$ ii) $\int e^x \cos\sqrt{3}x\, dx$ iii) $\int e^{-x}\cos x\, dx$

Solution:

i) Let us apply Eqn. (22) with a = 1, b = 1. Then, $a^2 + b^2 = 2$. Also, $a^2 + b^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$ or $r = \sqrt{2}$. So, $\cos\theta = \frac{a}{r} = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{b}{r} = \frac{1}{\sqrt{2}}$. Therefore, it follows that $\theta = \frac{\pi}{4}$. So, using Eqn. (22), we get

$$\int e^x\sin x\, dx = \frac{1}{\sqrt{2}}e^x\sin\left(x - \frac{\pi}{4}\right) + C$$

ii) Let us apply Eqn. (23) with a = 1, $b = \sqrt{3}$. Then, as before $r = \sqrt{a^2+b^2} = \sqrt{4} = 2$. Further, $\cos\theta = \frac{a}{r} = \frac{1}{2}$, $\sin\theta = \frac{b}{r} = \frac{\sqrt{3}}{2}$. So, $\theta = \frac{\pi}{3}$. So, we have

$$\int e^x\cos\sqrt{3}x\ dx = \frac{1}{2}e^x\cos\left(\sqrt{3}x - \frac{\pi}{3}\right) + C$$

iii) Here $a = -1$, $b = 1$, so $r = \sqrt{a^2 + b^2} = \sqrt{2}$, $\cos\theta = \frac{a}{r} = -\frac{1}{\sqrt{2}}$, $\sin\theta = \frac{b}{r} = \frac{1}{\sqrt{2}}$. Since $\sin\theta$ is positive and $\cos\theta$ is negative, θ is in the second quadrant. We take $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. So, we have

$$\int e^{-x}\cos bx\,dx = \frac{1}{\sqrt{2}}e^{-x}\cos\left(x - \frac{3\pi}{4}\right) + C$$

* * *

Here are some exercises for you to try.

E11) Evaluate the following:
i) $\int e^{3x}\cos 4x\,dx$ ii) $\int e^{4x}\sin 3x\,dx$ iii) $\int e^{-4x}\cos 4x\,dx$

Let us look at some more examples that illustrate the method of integration by parts.

Example 12: Evaluate $\int e^{2x}\sin x\cos 2x\ dx$.

Solution: We shall first write $\sin x\cos 2x = \frac{1}{2}(\sin 3x - \sin x)$ as in Sec. 18.2. Therefore,

$$\int e^{2x}\sin x\cos 2x\ dx = \frac{1}{2}\int e^{2x}\sin 3x\ dx - \frac{1}{2}\int e^{2x}\sin x\ dx$$

Now the two integrals on the right hand side can be evaluated. We see that

$$\int e^{2x}\sin 3x\ dx = \frac{1}{\sqrt{13}}e^{2x}\sin\ \left(3x - \tan^{-1}\frac{3}{2}\right) + C$$

and

$$\int e^{2x}\sin x\ dx = \frac{1}{\sqrt{5}}e^{2x}\sin\ \left(x - \tan^{-1}\frac{1}{2}\right) + C'$$

Hence

$$\int e^{2x}\sin x\cos 2x\,dx = \frac{e^{2x}}{2}\left[\frac{1}{\sqrt{13}}\sin\left(3x - \tan^{-1}\frac{3}{2}\right)\right.$$
$$\left.-\frac{1}{\sqrt{5}}\sin\left(x - \tan^{-1}\frac{1}{2}\right)\right] + C.$$

* * *

Example 13: Evaluate $\int x^3\sin(a\ln x)dx$.

Solution: Let $\ln x = u$. This implies $x = e^u$ and $du/dx = 1/x$. Then,

$$\int x^3\sin(a\ln x)\,dx = \int x^4\sin(a\ln x)\,(1/x)\,dx = \int e^{4u}\sin au\ du$$
$$= \frac{1}{\sqrt{16+a^2}}e^{4u}\sin(au - \tan^{-1}(a/4)) + C$$
$$= \frac{1}{\sqrt{16+a^2}}x^4\sin\left(a\ln x - \tan^{-1}\frac{a}{4}\right) + C$$

* * *

We have already seen how to evaluate integrals of the form $\int \frac{dx}{\sqrt{x^2 \pm a^2}}$ and $\int \frac{dx}{\sqrt{a^2 \pm x^2}}$. In the next subsection, we will see how we can combine this knowledge with the technique of integration by parts and use them to evaluate integrals of the form $\int \sqrt{x^2 \pm a^2}\, dx$ and $\int \sqrt{a^2 \pm x^2}\, dx$. But, before we move on to this topic, check if you have understood the technique of integrarion by parts by trying the following exercises now.

E12) Evaluate the following integrals

i) $\int e^{4x} \cos x \cos 2x \; dx$ ii) $\int e^{2x} \cos^2 x \, dx$ iii) $\int x e^{ax} \sin bx \; dx$.

18.3.3 Evaluation of $\int \sqrt{a^2 - x^2}\, dx$, $\int \sqrt{a^2 + x^2}\, dx$, and $\int \sqrt{x^2 - a^2}\, dx$

In this sub-section, we shall see that we can evaluate integrals like $\int \sqrt{a^2 - x^2} dx$, $\int \sqrt{a^2 + x^2} dx$ and $\int \sqrt{x^2 - a^2} dx$ with the help of the formula for integration by parts and Theorem 2.

$$\begin{aligned}
\int \sqrt{a^2 - x^2} dx &= \int \sqrt{a^2 - x^2} (1)\, dx \\
&= \sqrt{a^2 - x^2} \times x - \int \left(\frac{-x}{\sqrt{a^2 - x^2}} \times x \right) dx \\
&= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}} dx \\
&= x\sqrt{a^2 - x^2} - \int \frac{(a^2 - x^2) - a^2}{\sqrt{a^2 - x^2}} dx \\
&= x\sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int \sqrt{a^2 - x^2}\, dx
\end{aligned}$$

Shifting the last term on the right hand side to the left we get,

$$2\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

Using the formula,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C,$$

we obtain

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C \qquad \ldots(24)$$

We leave it to you as an exercise to prove

$$\int \sqrt{a^2 + x^2}\, dx = \frac{1}{2} x\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln\left|\frac{x + \sqrt{a^2 + x^2}}{a}\right| + C \qquad \ldots(25)$$

and

$$\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + C \qquad \ldots(26)$$

Let us look at some examples now.

Example 14: Evaluate the following integrals:

i) $\int \sqrt{4-x^2}\,dx$ ii) $\int \sqrt{25+x^2}\,dx$ iii) $\int \sqrt{x^2-9}\,dx$

Solution:

i) We write the integral in the form

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{2^2-x^2}\,dx$$

and apply Eqn. (24) with a = 2. We have

$$\int \sqrt{2^2-x^2}\,dx = \frac{1}{2}x\sqrt{2^2-x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} + C$$
$$= \frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$$

ii) We apply Eqn. (25) with a = 5. We have

$$\int \sqrt{25+x^2}\,dx = \int \sqrt{5^2+x^2}\,dx$$
$$= \frac{1}{2}x\sqrt{25+x^2} + \frac{25}{2}\ln\left|\frac{x+\sqrt{x^2+5^2}}{5}\right| + C$$

iii) We use Eqn. (26) with a = 3. We have

$$\int \sqrt{x^2-9}\,dx = \int \sqrt{x^2-3^2}\,dx = \frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C$$

* * *

Let us now look at an elaborate example that involves more than one of the types of integrands we have seen so far.

Example 15: Evaluate the integral

$$\int x\sqrt{\frac{a-x}{a+x}}\,dx$$

Solution: We have

$$\int x\sqrt{\frac{a-x}{a+x}}\,dx$$
$$= \int \frac{x(a-x)}{\sqrt{a^2-x^2}}\,dx = \int \frac{\left(a^2-x^2\right)-a^2+ax}{\sqrt{a^2-x^2}}\,dx$$
$$= \int \sqrt{a^2-x^2}\,dx - a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \frac{a}{2}\int \frac{-2x}{\sqrt{a^2-x^2}}\,dx$$
$$= \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - a^2\sin^{-1}\frac{x}{a} - a\sqrt{a^2-x^2} + C$$

* * *

Check if you have understood the discussion so far in this subsection by trying the following exercises now.

E13) Prove that:

i) $\int \sqrt{a^2 + x^2}\, dx = \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln\frac{x + \sqrt{a^2 + x^2}}{a} + C$

ii) $\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\frac{x + \sqrt{x^2 - a^2}}{a} + C.$

E14) Evaluate the following integrals:

i) $\int \sqrt{3 - x^2}\, dx$ ii) $\int \sqrt{1 - 5x^2}\, dx$ iii) $\int \sqrt{x^2 + 7}\, dx$

iv) $\int \sqrt{1 + 5x^2}\, dx$ v) $\int \sqrt{x^2 - 9}\, dx$ vi) $\int \sqrt{9x^2 - 1}\, dx$

vii) $\int x\sin^{-1} x\, dx$

We conclude this section on the method of integration by parts here. In the next section, we will see how to integrate rational functions.

18.4 INTEGRATION OF RATIONAL FUNCTIONS

If you have gone through Appendix 2 of Block 3, you may already know what a rational function is. You also know that we can split a proper rational function into a sum of simple rational functions. We will use this knowledge to integrate rational functions.

We begin by considering some simple types of proper rational functions of the form $\frac{1}{(x-b)^k}$ and $\frac{Ax+B}{ax^2+bx+c}$. Later we will use the fact that any proper rational function can be written as a sum of these simple types of functions and use it to integrate general rational functions.

We already know how to integrate function of type $\frac{1}{ax+b}$. So, let us see how to integrate functions of the type $\frac{Ax+B}{ax^2+bx+c}$.

Before we take up the integration of functions of the form $\frac{Ax+B}{ax^2+bx+c}$ we take up a particular case of this type in the next example; we will see later that this particular case is useful in integrating the general case.

Example 16: Evaluate $\int \frac{dx}{x^2 - a^2}$.

Solution: We have $x^2 - a^2 = (x - a)(x + a)$. So, we write

$$\frac{1}{x^2 - a^2} = \frac{\alpha}{x - a} + \frac{\beta}{x + a}. \qquad \ldots (27)$$

Multiplying both sides of the equation by $x^2 - a^2$ we get

$$1 = \alpha(x + a) + \beta(x - a).$$

Putting $x = a$, we get $1 = 2\alpha a$ or $\alpha = \frac{1}{2a}$. Similarly, putting $x = -a$, we get $1 = -2\beta a$ or $\beta = -\frac{1}{2a}$. So, We have

$$\frac{1}{x^2 - a^2} = \frac{\frac{1}{2a}}{x - a} - \frac{\frac{1}{2a}}{x + a}$$

So,

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a}$$
$$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C$$
$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Let us now assign a number to the final result for reference. We have

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \qquad \ldots (28)$$

* * *

We will now see how to integrate rational functions of the form $\frac{Ax+B}{ax^2+bx+c}$. We first write the numerator $Ax + B$ in the form $p(2ax + b) + q$, i.e. we set

$$Ax + B = p(2ax + b) + q$$

and solve for p and q. We have

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx = p\int \frac{2ax+b}{ax^2+bx+c}\,dx + q\int \frac{dx}{ax^2+bx+c} \qquad \ldots (29)$$

The first integral in the RHS of Eqn. (29) is of the form $\int \frac{f'(x)}{f(x)}\,dx$ where $f(x) = ax^2 + bx + c$. So,

$$\int \frac{2ax+b}{ax^2+bx+c}\,dx = \ln\left|ax^2+bx+c\right| + C.$$

Let us now consider the second integral. Note that, we can write $ax^2 + bx + c$ in the form $a\left(x^2+\alpha^2\right)$ or $a\left(x^2-\beta^2\right)$ where $\alpha > 0$, $\beta > 0$. More precisely,

$$ax^2+bx+c = \begin{cases} a\left\{\left(x+\frac{b}{2a}\right)^2 + \left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2\right\} & \text{if } b^2 \le 4ac \\ a\left\{\left(x+\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right\} & \text{if } b^2 > 4ac \end{cases} \qquad \ldots (30)$$

So, on substituting $u = x + \frac{b}{2a}$ we can rewrite the second integral in Eqn. (29) as

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{u^2+\alpha^2},$$

or

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{u^2-\beta^2}.$$

We already know how to integrate $\int \frac{dx}{a^2+x^2}\,dx$ from Eqn. (16). We also know how to integrate $\int \frac{dx}{x^2-a^2}$ from Eqn. (28). So, we can integrate any function of the form $\frac{Ax+B}{ax^2+bx+c}$.

Example 17: Integrate the function $f(x) = \frac{2x+3}{x^2-4x+5}$.

Solution: As we discussed above, we write

$$2x + 3 = p(2x - 4) + q = 2px + (q - 4p)$$

Comparing the coefficient of x on both sides, we get $p = 1$. Comparing the constant terms on both sides, we get $q - 4p = 3$. Substituting $p = 1$, we get $q = 7$. So, we can write $\int \frac{2x+3}{x^2-4x+5}dx$ as

$$\int \frac{2x-4}{x^2-4x+5}dx + \int \frac{7}{x^2-4x+5}dx.$$

As we noted in our earlier discussion, the first integral is of the form $\frac{f'(x)}{f(x)}$; and we know that

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C.$$

Thus,

$$\int \frac{2x-4}{x^2-4x+5}dx = \ln|x^2-4x+5| + C_1$$

To evaluate the second integral, we write

$$\int \frac{1}{x^2-4x+5}dx = \int \frac{1}{(x^2-4x+4)+1}dx = \int \frac{1}{(x-2)^2+1}dx$$

Now, if we put $x - 2 = u$,

$$\int \frac{1}{x^2-4x+5}dx = \int \frac{1}{u^2+1}du = \tan^{-1}u + C_2 = \tan^{-1}(x-2) + C_2$$

This implies,

$$\int \frac{2x+3}{x^2-4x+5}dx = \ln|x^2-4x+5| + 7\tan^{-1}(x-2) + C.$$

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Let us now look at an example where Eqn. (28) is useful.

Example 18: Evaluate $\int \frac{2x+1}{x^2+8x+1}dx$.

Solution: We have $\frac{d}{dx}\left(x^2+8x+1\right) = 2x+8$. We can write $2x+1 = 2x+8-7$. So, we have

$$\int \frac{2x+1}{x^2+8x+1}dx = \int \frac{2x+8}{x^2+8x+1}dx - 7\int \frac{dx}{x^2+8x+1}$$

The first integral is of the form $\int \frac{f'(x)}{f(x)}dx$, with $f(x) = x^2 + 8x + 1$. So,

$$\int \frac{2x+8}{x^2+8x+1}dx = \ln\left|x^2+8x+1\right| + C_1$$

We now evaluate the second integral.

$$\int \frac{dx}{x^2+8x+1}dx = \int \frac{dx}{(x+4)^2-\left(\sqrt{15}\right)^2}$$

Substituting u = x + 4, we get

$$\int \frac{dx}{x^2+8x+1} = \int \frac{du}{u^2-\left(\sqrt{15}\right)^2}$$

$$= \frac{1}{2\sqrt{15}} \ln\left|\frac{u-\sqrt{15}}{u+\sqrt{15}}\right| + C_2, \text{ using Eqn. (27).}$$

$$= \frac{1}{2\sqrt{15}} \ln\left|\frac{x+4-\sqrt{15}}{x+4+\sqrt{15}}\right| + C_2$$

So,

$$\int \frac{2x+1}{x^2+8x+1}\,dx = \ln\left|x^2+8x+1\right| - \frac{7}{2\sqrt{15}} \ln\left|\frac{x+4-\sqrt{15}}{x+4+\sqrt{15}}\right| + C$$

* * *

Let us look at some more examples now.

Example 19: Evaluate the following integrals:

i) $\int \frac{3x+5}{x^2+7x+10}\,dx$ ii) $\int \frac{4x+1}{3x^2+4x+1}\,dx$

Solution:

i) We have $\frac{d}{dx}\left(x^2+7x+10\right) = 2x+7$. Writing $3x+5 = p(2x+7)+q$ and solving for p and q, we get $p = \frac{3}{2}$, $7p+q = 5$ or $q = 5-7p = 5-\frac{21}{2} = -\frac{11}{2}$. So, we have

$$\int \frac{3x+5}{x^2+7x+10}\,dx = \frac{3}{2}\int \frac{2x+7}{x^2+7x+10}\,dx - \frac{11}{2}\int \frac{dx}{x^2+7x+10}$$

As before,

$$\int \frac{2x+7}{x^2+7x+10}\,dx = \ln\left|x^2+7x+10\right| + C_1$$

We have

$$\int \frac{dx}{x^2+7x+10} = \int \frac{dx}{\left(x+\frac{7}{2}\right)^2+10-\frac{49}{4}} = \int \frac{dx}{\left(x+\frac{7}{2}\right)^2-\left(\frac{3}{2}\right)^2}$$

Subsituting $u = x+\frac{7}{2}$ and using Eqn. (28), we get

$$\int \frac{dx}{x^2+7x+10} = \frac{1}{3}\ln\left|\frac{x+\frac{7}{2}-\frac{3}{2}}{x+\frac{7}{2}+\frac{3}{2}}\right| + C_2 = \frac{1}{3}\ln\left|\frac{x+2}{x+5}\right| + C_2$$

Therefore,

$$\int \frac{3x+5}{x^2+7x+10}\,dx = \frac{3}{2}\ln\left|x^2+7x+10\right| - \frac{11}{6}\ln\left|\frac{x+2}{x+5}\right| + C$$

ii) We have $\frac{d}{dx}\left(3x^2+4x+2\right) = 6x+4$. Writing $4x+1 = p(6x+4)+q$ and solving for p and q, we get $p = \frac{2}{3}$, $q = 1-4p = 1-\frac{8}{3} = -\frac{5}{3}$. So, we have

$$\int \frac{4x+1}{3x^2+4x+1}\,dx = \frac{2}{3}\int \frac{6x+4}{3x^2+4x+1}\,dx - \frac{5}{3}\int \frac{dx}{3x^2+4x+1}$$

We have

$$\int \frac{6x+4}{3x^2+4x+1}\,dx = \ln\left|3x^2+4x+1\right| + C_1$$

Also,

$$\int \frac{dx}{3x^2+4x+1} = \int \frac{dx}{3\left(x^2+\frac{4}{3}x+\frac{1}{3}\right)} = \frac{1}{3}\int \frac{dx}{\left\{\left(x+\frac{2}{3}\right)^2+\frac{1}{3}-\frac{4}{9}\right\}}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2}$$

$$= \frac{1}{3}\int \frac{du}{u^2-\left(\frac{1}{3}\right)^2} \text{ on substituting } u = x+\frac{2}{3}.$$

$$= \frac{1}{2}\ln\left|\frac{u-\frac{1}{3}}{u+\frac{1}{3}}\right| + C_2 = \frac{1}{2}\ln\left|\frac{x+\frac{1}{3}}{x+1}\right| + C_2$$

Therefore,

$$\int \frac{4x+1}{3x^2+4x+1}\,dx = \frac{2}{3}\ln\left|3x^2+4x+1\right| - \frac{5}{6}\ln\left|\frac{x+\frac{1}{3}}{x+1}\right| + C$$

* * *

Before we take up the discussion of integration of general rational functions, you may find it useful to try the following exercises to check your understanding of what we have discussed so far.

E15) Evaluate the following integrals:

i) $\int \frac{dx}{2x-3}$ ii) $\int \frac{dt}{(t+5)^2}$ iii) $\int \frac{4x+1}{x^2+x+2}\,dx$ iv) $\int \frac{5x-1}{x^2-1}dx$

v) $\int \frac{3x+1}{x^2-6x+3}\,dx$

Let us now discuss the integration of general rational functions using partial fractions.

Example 20: Evaluate $\int \frac{3x+1}{2x^3+3x^2-3x-2}\,dx$.

Solution: The denominator of the integrand has factors as $(2x+1)(x-1)(x+2)$. Let us write

$$\frac{3x+1}{2x^3+3x^2-3x-2} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{x+2}$$

Multiplying throughout by $2x^3+3x^2-3x-2$, we get

$$3x+1 = A(x-1)(x+2) + B(2x+1)(x+2) + C(2x+1)(x-1) \qquad \ldots(31)$$

Note that $-\frac{1}{2}$, 1, –2 are the roots of the polynomial in the denominator. We can easily solve for A, B and C by succesively putting $x = -\frac{1}{2}$, $x = 1$ and $x = -2$. For example, if we put $x = -\frac{1}{2}$, all the terms in the RHS of Eqn. (31), except the first term, will vanish. We get $-\frac{3}{2}+1 = A(-\frac{1}{2}-1)(-\frac{1}{2}+2)$. So,

$$A = \frac{-\frac{3}{2}+1}{\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}+2\right)} = \frac{2}{9}.$$

Similarly, we get B = $\frac{4}{9}$, C = $-\frac{5}{9}$. So, we have

$$\int \frac{3x+1}{2x^3+3x^2-3x-2}dx = \frac{2}{9}\int \frac{dx}{2x+1} + \frac{4}{9}\int \frac{dx}{x-1} - \frac{5}{9}\int \frac{dx}{x+2}$$
$$= \frac{1}{9}\ln|2x+1| + \frac{4}{9}\ln|x-1| - \frac{5}{9}\ln|x+2| + C$$

* * *

Let us now look at an example where there are repeated linear factors in the denominator.

Example 21: Evaluate $\int \frac{x}{x^3-3x+2}dx$.

Solution: The denominator of the integrand factors into $(x-1)^2(x+2)$. The linear factor $(x-1)$ is repeated twice in the decomposition of x^3-3x+2.

In this case we write

$$\frac{x}{x^3-3x+2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

From this point we proceed as before to find A, B and C. Multiplying both sides of the equation above by x^3-3x+2 we get

$$x = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

We put x = 1 and x = –2 and get C = 1/3 and A = –2/9.

Then to find B, let us put any other convenient value, say x = 0. This gives us $0 = A - 2B + 2C$ or, $0 = -\frac{2}{9} - 2B + \frac{2}{3}$ or $2B = \frac{4}{9}$. This implies B = 2/9. Thus,

$$\int \frac{x}{x^3-3x+2}dx = -\frac{2}{9}\int \frac{1}{x+2}dx + \frac{2}{9}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{1}{(x-1)^2}dx$$
$$= \frac{-2}{9}\ln|x+2| + \frac{2}{9}\ln|x-1| - \frac{1}{3}\left(\frac{1}{x-1}\right) + C$$
$$= \frac{2}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C.$$

* * *

In our next example, we shall consider the case when the denominator of the integrand contains an irreducible quadratic factor (i.e., a quadratic factor which cannot be further factored into linear factors).

Example 22: Evaluate

$$\int \frac{6x^3-11x^2+5x-4}{x^4-2x^3+x^2-2x}dx,$$

Solution: We factorise $x^4-2x^3+x^2-2x$ as $x(x-2)(x^2+1)$. Then we write

$$\frac{6x^3-11x^2+5x-4}{x^4-2x^3+x^2-2x} = \frac{A}{x} + \frac{B}{x-2} + \frac{Cx+D}{x^2+1}.$$

Thus, $6x^3-11x^2+5x-4 = A(x-2)(x^2+1) + Bx(x^2+1) + (Cx+D)x(x-2)$.

Next, we substitute x = 0 and x = 2 to get A = 2 and B = 1.

Then we put x = 1 and x = –1 (some convenient values) to get C = 3 and D = –1.

Thus

$$\int \frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x}dx = 2\int \frac{1}{x}dx + \int \frac{1}{x-2}dx + \int \frac{3x-1}{x^2+1}\,dx$$
$$= 2\ln|x| + \ln|x-2| + \frac{3}{2}\int \frac{2x}{x^2+1}dx - \int \frac{dx}{x^2+1}$$
$$= 2\ln|x| + \ln|x-2| + \frac{3}{2}\ln|x^2+1| - \tan^{-1}x + c.$$

Thus, you see, once we decompose our integrand, which is a proper rational function, into partial fractions, then the given integral can be written as the sum of some integrals of the type discussed in Examples 1, 2, and 3.

* * *

All the functions which we integrated till now were proper rational functions. Now we shall take up an example of an improper rational function.

Example 23: Evaluate $\int \frac{x^3 + 2x}{x^2 - x + 2}dx$.

Solution: Since the integrand is an improper rational function, we shall first write it as the sum of a polynomial and a proper rational function.

Thus,

$$\frac{x^3 + 2x}{x^2 - x + 2} = x + 1 + \frac{x-2}{x^2 - x + 2}.$$

Therefore,

$$\int \frac{x^3 + 2x}{x^2 - x + 2}dx = \int x dx + \int dx + \int \frac{x-2}{x^2-x+2}dx = \frac{x^2}{2} + x + \int \frac{x-2}{x^2-x+2}dx$$

The polynomial $x^2 - x + 2$ is irreducible over $\mathbb{R}$ since it has complex roots. It is possible to factor this polynomial over $\mathbb{C}$ and split the function $\frac{x-2}{x^2-x+2}$ into partial fractions with complex coefficients, but this will lead to integration of complex valued functions. The integration of complex valued functions of a real variable is not very difficult. Since we haven't discussed the integration of such functions, we avoid integration of complex valued functions by using the methods used in Example 17, Example 18 and Example 19.

We have $\frac{d}{dx}\left(x^2 - x + 2\right) = 2x - 1$. Writing $x - 2 = p(2x - 1) + q$ and solving for p and q, we get $p = \frac{1}{2}$, $q = -2 + p = -\frac{3}{2}$. So, we have

$$\int \frac{x-2}{x^2-x+2}\,dx = \frac{1}{2}\int \frac{2x-1}{x^2-x+2}dx - \frac{3}{2}\int \frac{dx}{x^2-x+2}$$
$$= \frac{1}{2}\ln\left|x^2 - x + 2\right| - \frac{3}{2}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 2}$$
$$= \frac{1}{2}\ln\left|x^2 - x + 2\right| - \frac{3}{2}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$$
$$= \frac{1}{2}\ln\left|x^2 - x + 2\right| - \frac{6}{\sqrt{7}}\tan^{-1}\frac{2x}{\sqrt{7}} + C$$

Therefore,

$$\int \frac{x^3+2x}{x^2-x+2}dx = \frac{x^2}{2} + x + \frac{1}{2}\ln\left|x^2-x+2\right| - \frac{6}{\sqrt{7}}\tan^{-1}\frac{2x}{\sqrt{7}} + C$$

* * *

Try to do the following exercise now. You will find that each integrand falls in one of the various types we have seen so far.

E16) Evaluate the following integrals:

i) $\int \frac{2}{x^2+2x}\,dx$ ii) $\int \frac{x}{x^2-2x-3}\,dx$

iii) $\int \frac{3x-13}{x^2+3x-10}\,dx$ iv) $\int \frac{6x^2+22x-23}{(2x-1)\,(x^2+x-6)}\,dx.$

v) $\int \frac{3x^3}{x^2+x-2}\,dx$ vi) $\int \frac{x^2+x-1}{(x-1)\,(x^2-x+1)}\,dx$

vii) $\int \frac{x^3-4x}{(x^2+1)^2}\,dx$

18.4.1 Method of Substitution

The method of partial fraction decomposition which we studied in the last sub-section can be applied to all rational functions. We can say this because the Fundamental theorem of algebra guarantees the factorisation of any polynomial into linear and quadratic factors. But the actual process of factorising a polynomial is sometimes not quite simple. In such cases it would be a good idea to carefully examine the integrand to check if the method of substitution can be applied. We will now give two examples to show how we can sometimes integrate a given rational function with the help of a suitable substitution.

Example 24: Integrate $\frac{1}{x(x^5+1)}$ with respect to x.

Solution: For this we write

$$\int \frac{dx}{x(x^5+1)} = \int \frac{x^4dx}{x^5(x^5+1)}$$

Now let us write $x^5 = t$. Then $\frac{dt}{dx} = 5x^4$.

$$\int \frac{x^4dx}{x^5(x^5+1)} = \frac{1}{5}\int \frac{dt}{t(t+1)} = \frac{1}{5}\int \left[\frac{1}{t} - \frac{1}{t+1}\right] dt = \frac{1}{5}\ln\left|\frac{t}{t+1}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{x^5}{x^5+1}\right| + C.$$

* * *

Example 25: Integrate $\frac{x^2-1}{x^4+x^2+1}$ w.r.t. x

Solution: We have

$$\int \frac{x^2-1}{x^4+x^2+1}dx = \int \frac{\left(1-1/x^2\right)}{x^2+1+1/x^2}dx \text{ on division by } x^2.$$

$$= \int \frac{(1-1/x^2)}{(x+1/x)^2-1}dx = \int \frac{dt}{t^2-1} \text{ if we put } t = x + \frac{1}{x}.$$

$$= \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{x^2 - x + 1}{x^2 + x + 1}\right| + C.$$

* * *

In Example 24 and Example 26 you must have noted that the denominators of the integrands were not easily factorisable. The method of substitution provided an easier alternative. See if you can solve this exercise now.

E17) Integrate the following functions w.r.t. x:

i) $\frac{x^2 - 1}{1 + x^4}$ ii) $\frac{1 + x^2}{1 + x^2 + x^4}$.

The exercises in this section have given you a fair amount of practice in integrating rational functions. In the next section we take up the case of rational trigonometric functions.

18.5 INTEGRATION OF RATIONAL TRIGONOMETRIC FUNCTIONS

We begin by discussing of integration of rational trigonometric functions.

18.5.1 INTEGRATION OF RATIONAL TRIGONOMETRIC FUNCTIONS

You know that a polynomial in two variables x and y is an expression of the form

$$P(x,\ y) = \sum_{n=0}^{k}\sum_{m=0}^{p} a_{m,n}x^m y^n,\ a_{m,n} \in R$$

Accordingly, a polynomial in $\sin x$ and $\cos x$ is an expression of the form

$$P(\sin x,\ \cos x) = \sum_{n=0}^{k}\sum_{m=0}^{p} a_{m,n}\sin^m x\cos^n x,\ a_{m,n} \in R.$$

An expression, which is the ratio of two polynomials, $P(\sin x,\ \cos x)$ and $Q(\sin x,\ \cos x)$ is called a **rational function of $\sin x$ and $\cos x$**. In this section we shall discuss the integration of some simple rational functions in $\sin x$ and $\cos x$. We shall first indicate a general method for integrating these functions.

Let $f(\sin x,\ \cos x)$ be a rational function in $\sin x$ and $\cos x$. The first step in the evaluation of the integral of f is to make the substitution $\tan\frac{x}{2} = t$.

Thus, $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1+t^2}{2}$. Since

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

and

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

we get

$$\int f(\sin x, \cos x)dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt = \int F(t)\ dt,$$

where

$$F(t) = f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}$$

is a rational function of t. Now we can use the method of partial fraction decomposition to integrate F(t). In principle then, we can integrate any rational function in sin x and cos x. But in actual practice we find that the rational function F(t) is often complicated, and it is not feasible to apply the method of partial fractions. In this unit, however, we shall restrict ourselves to a few simple rational functions only.

Example 26: Integrate $\frac{1}{a+b\cos x}$.

Solution: Now

$$a + b\cos x = a\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + b\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)$$
$$= (a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}$$

Therefore,

$$\int\frac{dx}{a+b\cos x} = \int\frac{\sec^2\frac{x}{2}dx}{(a+b)+(a-b)\tan^2\frac{x}{2}} = \int\frac{\sec^2\frac{x}{2}dx}{(a-b)\left[\frac{a+b}{a-b}+\tan^2\frac{x}{2}\right]}$$

If we put $\tan\frac{x}{2} = t$, we get

$$\int\frac{dx}{a+b\cos x} = 2\int\frac{dt}{(a-b)\left(\frac{a+b}{a-b}+t^2\right)} = \frac{2}{a-b}\int\frac{dt}{\frac{a+b}{a-b}+t^2}$$

If $a > b > 0$, then $\frac{a+b}{a-b} > 0$, and we get

$$\int\frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(t\sqrt{\frac{a-b}{a+b}}\right)$$
$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) \qquad \ldots(32)$$

If $0 < a < b$, then $\frac{a+b}{a-b} < 0$, and

$$\int\frac{dx}{a+b\cos x} = \frac{2}{a-b}\int\frac{dt}{t^2-\left(\sqrt{\frac{b+a}{b-a}}\right)^2} = \frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{\sqrt{b+a}+\sqrt{b-a}\,t}{\sqrt{b+a}-\sqrt{b-a}\,t}\right| + C$$
$$= \frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}\right| + C \qquad \ldots(33)$$

* * *

Example 27: Evaluate the following:

i) $\int\frac{dx}{a+b\sin x}$ ii) $\int\frac{dx}{a+b\cos x+c\sin x}$

Solution:

i) We use the substitution $x = \frac{\pi}{2} + y$. Then,

$$\int\frac{dx}{a+b\sin x} = \int\frac{dy}{a+b\sin\left(\frac{\pi}{2}+y\right)} = \int\frac{dy}{a+b\cos y}.$$

which is of the form $\int\frac{dx}{a+\cos x}$.

ii) We have

$$b\cos x + c\sin x = \sqrt{b^2+c^2}\cos\left(x - \tan^{-1}\frac{b}{c}\right)$$

So, we can reduce this integral also to an integral of the form

$$\int \frac{dx}{a + b\cos x}.$$

* * *

Example 28: Evaluate $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx.$

Solution: We write

$1 + \cos x = 2\cos^2\frac{x}{2}$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$$\begin{aligned}\int \frac{1+\sin x}{\sin x(1+\cos x)}dx &= \int \frac{dx}{\sin x(1+\cos x)} + \int \frac{dx}{1+\cos x}\\ &= \frac{1}{4}\int \frac{dx}{\sin\frac{x}{2}\cos^3\frac{x}{2}} + \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2}}\\ &= \frac{1}{4}\int \frac{\sec^4\frac{x}{2}}{\tan\frac{x}{2}}dx + \frac{1}{2}\int \sec^2\frac{x}{2}dx\\ &= \frac{1}{2}\int \frac{1+t^2}{t}dt + \int dt \quad \left(\tan\frac{x}{2} = t\right)\\ &= \frac{1}{2}\left[\int \frac{1}{t}dt + \int t\,dt\right] + \int dt = \frac{1}{2}\left[\ln|t| + \frac{t^2}{2}\right] + t + C.\end{aligned}$$

Thus,

$$\int \frac{1+\sin x}{\sin x(1+\cos x)}dx = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + C.$$

* * *

Now proceeding exactly as in Example 26 and Example 27, you can do these exercises.

E18) Integrate the following w.r.t. x:

i) $\frac{1}{4+5\cos x}$ ii) $\frac{\cos x}{2-\cos x}$

By now you have seen and applied many different methods of integration. The crux of the matter lies in choosing the appropriate method for integrating a given function. For example, suppose we ask you to integrate the function $\frac{\sin x\cos x}{1+\sin^2 x}$. Realising that this is a rational function in $\sin x$ and $\cos x$, you may put $\tan\frac{x}{2} = t$ and proceed:

$$\int \frac{\sin x\cos x}{1+\sin^2 x}dx = 4\int \frac{t\left(1-t^2\right)dt}{\left(1+t^2\right)\left(1+6t^2+t^4\right)}$$

Now $1 + 6t^2 + t^4 = (3 + \sqrt{8} + t^2)(3 - \sqrt{8} + t^2)$. By this step you will realise that it is going to be a tough job. But don't worry. There is an easy way out.

In $\int \frac{\sin x \cos x}{1+\sin^2 x} dx$, if we make the substitutiong $1 + \sin^2 x = t$, we get

$$\int \frac{\sin x \cos x dx}{1+\sin^2 x} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2} \ln|t| + C.$$
$$= \frac{1}{2} \ln(1+\sin^2 x) + C.$$

Thus, the choice of the method is very crucial. And only practice can help you make a good choice.

We shall now illustrate some techniques used in integrating irrational functions.

18.6 INTEGRATION OF IRRATIONAL FUNCTIONS

The task of integrating functions gets tougher if the given function is an irrational one, that is, it is not of the form $\frac{P(x)}{Q(x)}$. In this section, we shall give some tips for evaluating some particular types of irrational functions. In most cases, we will try to arrive at a rational function through an appropriate substitution. We can integrate this rational function using the techniques in the previous section.

I) Integration of functions containing only fractional powers of x

In this case we put $x = t^n$, where n is the lowest common multiple (l.c.m.) of the denominators of powers of x. This substitution reduces the function to a rational function of t. Look at the following example.

Example 29: Evaluate $\int \frac{2x^{1/2}+3x^{1/3}}{1+x^{1/3}} dx.$

Solution: We put $x = t^6$, as 6 is the l.c.m. of 2 and 3. We get

$$\int \frac{2x^{1/2}+3x^{1/3}}{1+x^{1/3}} dx = 6\int \frac{2t^3+3t^2}{1+t^2} t^5 \, dt$$
$$= 6\int \frac{2t^8+3t^7}{1+t^2} dt$$
$$= 6\int \left[2t^6+3t^5-2t^4-3t^3+2t^2+3t-2-\frac{3t-2}{1+t^2}\right] dt$$
$$= 6\left[\frac{2}{7}t^7+\frac{1}{2}t^6-\frac{2}{5}t^5-\frac{3}{4}t^4+\frac{2}{3}t^3+\frac{3}{2}t^2-2t\right.$$
$$\left.-\frac{3}{2}\ln(1+t^2)+2\tan^{-1} t\right] + C$$
$$= \frac{12}{7}x^{7/6}+3x-\frac{12}{5}x^{5/6}-\frac{9}{2}x^{2/3}+4x^{1/2}+9x^{1/3}-12x^{1/6}$$
$$-9\ln|1+x^{1/2}|+12\tan^{-1} x^{1/6}+C.$$

* * *

Here is an exercise to test your understanding of Example 29.

E19) Integrate $\int \frac{\sqrt{x}}{1+\sqrt[4]{x}} dx.$

II) Integral of the type $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Here we shall have to consider two cases: (i) a > 0 and (ii) a < 0. In each case we will try to put the given integrand in a form which we have already seen how to integrate.

i) If a > 0, from Eqn. (30), we have

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)}}$$

If we put t = x + b/2a, we get

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)}}$$

If $b^2 > 4ac$, we have

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{t^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2}} \qquad \ldots(34)$$

If $b^2 < 4ac$, we have

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{t^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}} \qquad \ldots(35)$$

We can evaluate both the integrals in the RHS of Eqn. (34) and Eqn. (35) using Theorem 2.

ii) a < 0: If we put –a = d, then d > 0, and we can write

$$ax^2+bx+c = \frac{1}{d}\left\{\frac{4cd+b^2}{4d^2}-\left(x-\frac{b}{d}\right)^2\right\} \qquad \ldots(36)$$

If $4cd+b^2 < 0$, the RHS of Eqn. (36) will be negative for all values of x and $\frac{1}{\sqrt{ax^2+bx+c}}$ will be a complex valued function. We haven't developed the mathematical concepts required to handle such functions, so we will discuss only the case $4cd+b^2 \geq 0$. In this case, if we substitute $t = x - b/d$ we have

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{d}}\int \frac{dx}{\sqrt{\left(\frac{\sqrt{b^2+4dc}}{2d}\right)^2-\left(x-\frac{b}{d}\right)^2}} = \frac{1}{\sqrt{d}}\int \frac{dt}{\sqrt{\alpha^2-t^2}}$$

with $\alpha = \frac{\sqrt{b^2+4dc}}{2d}$. This is again in one of the standard forms.

Let us now look at some examples that will help in understanding our discussion so far.

Example 30: Evaluate the following integrals:

i) $\int \frac{1}{\sqrt{8-2x-x^2}}dx$ ii) $\int \frac{dx}{\sqrt{x^2+4x+5}}$ iii) $\int \frac{dx}{\sqrt{2x^2+4x+1}}$

Solution:

i) Completing the square, we get $x^2+2x = (x+1)^2-1$. So,

$$8-x^2-2x = 8-(x+1)^2+1 = 9-(x+1)^2.$$

We can rewrite the integral as

$$\int \frac{dx}{\sqrt{8-2x-x^2}} = \int \frac{dx}{\sqrt{9-(x+1)^2}} = \int \frac{dx}{\sqrt{3^2-(x+1)^2}}$$

Setting $u = x + 1$, we get $du = dx$. So,

$$\int \frac{dx}{\sqrt{8-2x-x^2}} = \int \frac{du}{\sqrt{3^2-u^2}} = \sin^{-1}\left(\frac{u}{3}\right) + C = \sin^{-1}\left(\frac{x+1}{3}\right) + C$$

ii) Completing the square, we get $x^2 + 4x = (x+2)^2 - 4$. So,

$x^2 + 4x + 5 = (x+2)^2 + 1$.

We have

$$\int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{dx}{\sqrt{(x+2)^2+1}}$$

Setting $u = x + 2$, we get $du = dx$. So,

$$\int \frac{dx}{\sqrt{(x+2)^2+1}} = \int \frac{du}{\sqrt{u^2+1}} = \ln\left|u + \sqrt{u^2+1}\right| + C$$

$$= \ln\left|x + 2 + \sqrt{x^2+4x+5}\right| + C$$

iii) We have $2x^2 + 4x + 1 = 2\left(x^2 + 2x + \frac{1}{2}\right)$. Also, $x^2 + 2x = (x+1)^2 - 1$. So, $x^2 + 2x + \frac{1}{2} = (x+1)^2 - \frac{1}{2}$. We have

$$\int \frac{dx}{\sqrt{2x^2+4x+1}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x+1)^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

Substituting $u = x + 1$, we have

$$\int \frac{dx}{\sqrt{2x^2+4x+1}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{u^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= \frac{1}{\sqrt{2}} \ln\left|\frac{u + \sqrt{u^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}{\frac{1}{\sqrt{2}}}\right| + C$$

$$= \frac{1}{\sqrt{2}} \ln\left|\sqrt{2}(x+1) + \sqrt{2x^2+4x+1}\right| + C$$

* * *

You may like to test your understanding of the above example by trying the following exercise.

E20) Evaluate the following integrals:

i) $\int \frac{dx}{\sqrt{3x^2+4x+1}}$ ii) $\int \frac{dx}{\sqrt{3x^2-4x+3}}$ iii) $\int \frac{dx}{\sqrt{5-2x-x^2}}$

Let us now move on to the next type.

III) Integration of $\frac{(Ax+B)}{\sqrt{ax^2+bx+c}}$

We break Ax + B into two parts such that the first part is a constant multiple of the differential coefficient of $ax^2 + bx + c$, that is, $2ax + b$, and the second part is independent of x. Thus, $Ax + B = \frac{A}{2a}(2ax + b) + B - \frac{Ab}{2a}$ and

$$\int \frac{(Ax + B)dx}{\sqrt{ax^2 + bx + c}} = \frac{A}{2a}\int \frac{(2ax + b)dx}{\sqrt{ax^2 + bx + c}} + \frac{(2aB - Ab)}{2a}\int \frac{dx}{\sqrt{ax^2 + bx + c}}$$
$$= \frac{A}{a}\sqrt{ax^2 + bx + c} + \frac{(2aB - Ab)}{2a}\int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

Evaluation of the last integral has already been discussed in II). Let us look at an example.

Example 31: Evaluate $\int \frac{2x + 1}{\sqrt{8 - 2x - x^2}}\,dx$.

Solution: We have $\frac{d}{dx}(8 - 2x - x^2) = -2 - 2x$. Writing $2x + 1 = A(-2x - 2) + B$ and solving for A and B, we get A = –1, B = –1. So, we have

$$\int \frac{2x + 1}{\sqrt{8 - 2x - x^2}}\,dx = -\int \frac{-2 - 2x}{\sqrt{8 - 2x - x^2}}\,dx - \int \frac{dx}{\sqrt{8 - 2x - x^2}}\,dx$$

The first integral is of the form $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx$. So, we have

$$\int \frac{-2 - 2x}{\sqrt{8 - 2x - x^2}}\,dx = 2\sqrt{8 - 2x - x^2} + C_1$$

From Example 30, we have

$$\int \frac{dx}{\sqrt{8 - 2x - x^2}} = \sin^{-1}\left(\frac{x + 1}{3}\right) + C_2$$

So,

$$\int \frac{2x + 1}{\sqrt{8 - 2x - x^2}}\,dx = -2\sqrt{8 - 2x - x^2} - \sin^{-1}\left(\frac{x + 1}{3}\right) + C$$

* * *

Example 32: Evaluate $\int \frac{x + 2}{\sqrt{x^2 + 2x + 3}}\,dx$.

Solution: We note that $x + 2 = \frac{1}{2}(2x + 2) + 1$, and write

$$\int \frac{x + 2}{\sqrt{x^2 + 2x + 3}}\,dx = \frac{1}{2}\int \frac{(2x + 2)}{\sqrt{x^2 + 2x + 3}}\,dx + \int \frac{dx}{\sqrt{x^2 + 2x + 3}}$$
$$= \sqrt{x^2 + 2x + 3} + \int \frac{dx}{\sqrt{(x + 1)^2 + 2}}$$
$$= \sqrt{x^2 + 2x + 3} + \ln\left|\frac{x + 1 + \sqrt{2} + \sqrt{x^2 + 2x + 3}}{\sqrt{2}}\right| + C.$$

* * *

Here is an exercise for you.

E21) Evaluate the following integrals:

i) $\int \frac{3x+1}{\sqrt{3x^2+4x+1}}\,dx$. ii) $\int \frac{2x-3}{\sqrt{3x^2-4x+3}}\,dx$ iii) $\int \frac{x+2}{\sqrt{5-2x-x^2}}\,dx$

IV) Integration of $\int \sqrt{ax^2+bx+c}\,dx$

The method is similar to the one we used for evaluating integrals of the form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$. If a > 0 we can write these integrals in the form $\int \sqrt{x^2 \pm a^2}\,dx$ or $\int \sqrt{a^2 \pm x^2}\,dx$ and use Eqn. (24), Eqn. (25) and Eqn. (26). As before, if a < 0, we can evaluate the integral only in the case $b^2 > 4ac$.

Here is an example to illustrate the method.

Example 33: Evaluate the following integrals:

i) $\int \sqrt{8-2x-x^2}\,dx$. ii) $\int \sqrt{x^2+2x+2}\,dx$ iii) $\int_0^1 \sqrt{x+x^2}\,dx$.

Solution:

i) We have

$$\int \sqrt{8-2x-x^2}\,dx = \int \sqrt{3^2-(x+1)^2}\,dx$$

On substituting u = x + 1, we get

$$\int \sqrt{8-2x-x^2}\,dx = \int \sqrt{3^2-u^2}\,du$$

$$= \frac{1}{2}u\sqrt{3^2-u^2} + \frac{3^2}{2}\sin^{-1}\frac{u}{3} + C \text{ from Eqn. (24)}$$

$$= \frac{1}{2}(x+1)\sqrt{8-2x-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+1}{3}\right) + C$$

ii) We have

$$\int \sqrt{x^2+2x+2}\,dx = \int \sqrt{(x+1)^2+1}\,dx$$

Substituting u = x + 1, we get

$$\int \sqrt{x^2+2x+2}\,dx = \int \sqrt{u^2+1}\,dx$$

$$= \frac{1}{2}u\sqrt{u^2+1} + \frac{1}{2}\ln\left|u+\sqrt{u^2+1}\right| + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+2}$$

$$+ \frac{1}{2}\ln\left|x+1+\sqrt{x^2+2x+2}\right| + C$$

iii) Now

$$\int_0^1 \sqrt{x+x^2}\,dx = \int_0^1 \sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}\,dx$$

Let $x + \frac{1}{2} = u$. Then,

$$\int_0^1 \sqrt{x+x^2}\,dx = \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{u^2 - \frac{1}{4}}\,du = \left[\left\{\frac{1}{2}u\sqrt{u^2-\frac{1}{4}} - \frac{1}{8}\ln\frac{u+\sqrt{u^2-\frac{1}{4}}}{\frac{1}{2}}\right\}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{3\sqrt{2}}{4} - \frac{1}{8}\ln\left(3+2\sqrt{2}\right).$$

***

Here are some exercises for you to try.

E22) Evaluate the following integrals:

i) $\int \sqrt{3x^2+4x+1}\,dx$ ii) $\int \sqrt{3x^2-4x+3}\,dx$ iii) $\int \sqrt{5-2x-x^2}\,dx$

V) Integration of $\frac{1}{(fx+e)\sqrt{ax^2+bx+c}}$.

Substituting $fx + e = \frac{1}{y}$ or $y = \frac{1}{fx+e}$, we get $dy = -\frac{f}{(fx+e)^2}dx = -fy^2dx$ or $dx = -\frac{dy}{fy^2}$. Letting $h(x) = ax^2 + bx + c$, we write $h(x) = A(fx + e)^2 + B(fx + e) + C$. We have

$$C = h\left(-\frac{e}{f}\right), fB = h'\left(-\frac{e}{f}\right), 2f^2A = h''\left(-\frac{e}{f}\right).$$

So, we can find A, B and C such that

$$ax^2 + bx + c = \frac{A}{y^2} + \frac{B}{y} + C = \frac{A + By + Cy^2}{y^2}.$$

So, we have

$$\int \frac{1}{(fx+e)\sqrt{ax^2+bx+c}}dx = \int \frac{-\frac{dy}{fy^2}}{\frac{1}{y}\frac{\sqrt{A+By+Cy^2}}{y}} = -\frac{1}{f}\int \frac{dy}{\sqrt{A+By+Cy^2}}.$$

You may recall that we have already seen how to evaluate the last integral in the equation above in Example 30.

We will illustrate the method through an example.

Example 34: Evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+4x+2}}$.

Solution: Let us put $x + 1 = 1/y$. Then $\frac{-1}{y^2}\frac{dy}{dx} = 1$.

Now we will express $x^2 + 4x + 2$ in terms of $\frac{1}{y}$. We let

$$h(x) = x^2 + 4x + 1 = A(x + 1)^2 + B(x + 1) + C.$$

We have $h(-1) = (-1)^2 + 4(-1) + 2 = -1$. So, $-1 = h(-1) = C$, $2 = h'(-1) = B$, $2 = h''(-1) = A$. Therefore, we have

$$x^2 + 4x + 2 = (x+1)^2 + 2(x+1) - 1 = \frac{1}{y^2} + \frac{2}{y} - 1 = \frac{1+2y-y^2}{y^2}$$

Therefore,

$$\int \frac{dx}{(x+1)\sqrt{x^2+4x+2}} = \int \frac{-\frac{1}{y^2}dy}{\frac{1}{y}\sqrt{\frac{1+2y-y^2}{y^2}}} = -\int \frac{dy}{\sqrt{1+2y-y^2}}$$

$$= -\int \frac{dy}{\sqrt{2-(y-1)^2}} = \cos^{-1}\left(\frac{y-1}{\sqrt{2}}\right) + C$$

$$= \cos^{-1}\left[\frac{-x}{(x+1)\sqrt{2}}\right] + C.$$

* * *

Here is an exercise for you to check your understanding of the above example.

E23) Evaluate the following integrals:

i) $\int \frac{dx}{(x-2)\sqrt{8+2x-x^2}}$ ii) $\int \frac{dx}{(x+2)\sqrt{x^2+6x+10}}$

Let us move over to the next type now.

VI) Integration of $(Ax+B)\sqrt{ax^2+bx+c}$

We break $Ax + B$ as we did in IV), and obtain

$$\int (Ax+B)\sqrt{ax^2+bx+c}\,dx$$
$$= \frac{A}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \frac{B2a-Ab}{2a}\int \sqrt{ax^2+bx+c}\,dx$$
$$= \frac{A}{3a}(ax^2+bx+c)^{3/2} + \frac{2aB-Ab}{2a}\int \sqrt{ax^2+bx+c}\,dx.$$

We have already seen how to evaluate the integral on the right hand side in part IV).

Let us use these methods to solve some examples now.

Example 35: Evaluate the following integrals:

i) $\int (2x+1)\sqrt{8-2x-x^2}\,dx$ ii) $\int (x+2)\sqrt{x^2+2x+2}\,dx$

Solution:

i) We have $\frac{d}{dx}\left(8-2x-x^2\right) = -2x-2$ and $2x+1 = -(-2x-2)-1$. So,

$$\int (2x+1)\sqrt{8-2x-x^2}\,dx = -\int (-2x-2)\sqrt{8-2x-x^2}\,dx$$
$$-\int \sqrt{8-2x-x^2}\,dx$$

The first integral is of the form $\int f'(x)\sqrt{f(x)}\,dx$. So, substituting $u = f(x)$, it becomes $\int \sqrt{u}\,du$. Therefore,

$$\int (-2x-2)\sqrt{8-2x-x^2}\,dx = \frac{2}{3}\left(8-2x-x^2\right)^{3/2} + C_1$$

From Example 33, we know that

$$\int \sqrt{8-2x-x^2}\,dx = \frac{1}{2}(x+1)\sqrt{8-2x-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+1}{3}\right) + C_2$$

Therefore,

$$\int (2x+1)\sqrt{8-2x-x^2}\,dx = -\frac{2}{3}\left(8-2x-x^2\right)^{3/2} - \frac{1}{2}(x+1)\sqrt{8-2x-x^2} - \frac{9}{2}\sin^{-1}\left(\frac{x+1}{3}\right) + C$$

ii) We have $\frac{d}{dx}\left(x^2+2x+2\right) = 2x+2$. Further, $x+2 = \frac{1}{2}(2x+2)+1$. So,

$$\begin{aligned}\int (x+2)\sqrt{x^2+2x+2}\,dx &= \frac{1}{2}\int (2x+2)\sqrt{x^2+2x+2}\,dx + \int \sqrt{x^2+2x+2}\,dx \\ &= \frac{1}{3}\left(x^2+2x+2\right)^{3/2} + \int \sqrt{(x+1)^2+1}\,dx \\ &= \frac{1}{3}\left(x^2+2x+2\right)^{3/2} + \frac{1}{2}(x+1)\sqrt{x^2+2x+2} + \frac{1}{2}\ln\left|x+1+\sqrt{x^2+2x+2}\right| + C\end{aligned}$$

* * *

Here is an exercise for you to try.

E24) Evaluate the following integrals:

i) $\int (2x-1)\sqrt{3x^2+4x+1}\,dx$ ii) $\int (x-3)\sqrt{5-2x-x^2}\,dx$

When you are faced with a new integrand, the following suggestions furnish a thread through the labyrinth of methods.

1) Check the integrand to see if it fits one of the patterns $\int u^n du$ or $\int \frac{du}{u}$

2) See if the integrand fits any one of the patterns obtained by the reversal of differentiation formulas. (We have considered these in Unit 17).

3) If none of these patterns is appropriate, and if the integrand is a rational function, then our theory of partial fractions enables us to integrate it.

4) If the integrand is a rational function of $\sin x$ and $\cos x$, and simpler methods of previous units fail, the substitution $t = \tan\frac{x}{2}$ will make the integrand into a rational function of t, which can then be evaluated.

5) If the integrand is a radical of one of the forms $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$, then the trigonometric substitutions $x = a\sin\theta$, $x = a\cos\theta$ and $x = a\sec\theta$ will reduce the integrand to a rational function of $\sin\theta$ and $\cos\theta$. If the radical is of the form $\sqrt{ax^2+bx+c}$, a square completion $\sqrt{a(x+b/2a)^2+c-b^2/4a}$ will reduce it essentially to one of the above radicals.

6) If the integrand is an irrational function of x, try to express it as a rational function or anintegrable radical through appropriate substitutions.

7) Inspect the integrand to see if it will yield to integration by parts.Finally, we would like to remind you again that a lot of practice is essential if you want to master the various techniques of integration. We have already mentioned that a proper choice of the method of integration is the key to the correct evaluation of any integral. Now let us briefly recall what we have covered in this unit.

18.7 SUMMARY

In this unit we have covered the following points:

1) A rational function f of x is given by f(x) = P(x)/Q(x), where P(x) and Q(x) are polynomials in x. It is called proper if the degree of P(x) is less than the degree of Q(x). Otherwise it is called improper.

2) A proper rational expression can be resolved into partial fractions with linear or quadratic denominators.

3) A rational function can be integrated by the method of partial fractions.

4) Integration of a rational function of $\sin x$ and $\cos x$ can be done by putting $t = \tan \frac{x}{2}$.

5) Integration of irrational fucntions of the following types is discussed.

   i) Integrand contains fractional power of x,

   ii) $\frac{1}{\sqrt{ax^2+bx+c}}$

   iii) $\frac{1}{(fx+e)\sqrt{ax^2+bx+c}}$

   iv) $\frac{Ax+B}{\sqrt{ax^2+bx+c}}$

   v) $(Ax + B)\sqrt{ax^2 + bx + c}$

6) A check list of points to be considered while evaluating any integral is given.

18.8 SOLUTIONS/ANSWERS

E1) i) $\int \sqrt{5x-3}\,dx = \int (5x-3)^{\frac{1}{2}}\,dx$. Putting $u = 5x - 3$, we have

$$\int \sqrt{5x-3}\,dx = \frac{1}{5}\int \underbrace{\sqrt{5x-3}}_{u^{\frac{1}{2}}}\,\underbrace{5}_{\frac{du}{dx}}\,dx = \frac{1}{5}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{5}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2(5x-3)^{\frac{3}{2}}}{15} + C$$

ii) Putting $x = 2x + 1$, we have $\frac{du}{dx} = 2$ We can write the integral as

$$\frac{1}{2}\int \underbrace{(2x+1)^6}_{u^6}\underbrace{2}_{\frac{du}{dx}}\, dx = \frac{1}{2}\int u^6 du = \frac{1}{2}\frac{u^7}{7} + C = \frac{(2x+1)^7}{14} + C$$

iii) Putting $u = 4 + 5x$, $du = 5dx$. We can write the integral as

$$\frac{1}{5}\int \underbrace{\frac{1}{4+5x}}_{\frac{1}{u}}\underbrace{5}_{\frac{du}{dx}}\, dx = \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|4+5x| + C.$$

$$\therefore \int_1^3 \frac{dx}{4+5x} = \frac{1}{5}\ln|4+5x|\Big|_1^3 = \frac{1}{5}(\ln 19 - \ln 9) = \frac{1}{5}\ln\frac{19}{9}.$$

iv) Putting $u = 10x + 7$, we have $\frac{du}{dx} = 10$. We can rewrite the integral the integral as

$$\frac{5}{10}\int \underbrace{\frac{1}{(10x+7)}}_{\frac{1}{u}}\underbrace{10}_{\frac{du}{dx}}\, dx = \frac{5}{10}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|10x+1| + C$$

v) Putting $u = x^2 + 2x + 7$, we have $du = (2x + 2)dx$. We can rewrite the integral as

$$\frac{1}{2}\int \underbrace{\frac{1}{x^2+2x+7}}_{\frac{1}{u}}\underbrace{(2x+2)}_{\frac{du}{dx}}\, dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln\left|x^2+2x+7\right| + C.$$

vi) Putting $u = x^3 + x^2 + x - 8$, we have $du = \left(3x^2 + 2x + 1\right) dx$. We can rewrite the integral as

$$\int \underbrace{\frac{1}{x^3+x^2+x-8}}_{\frac{1}{u}}\underbrace{3x^2+2x+1}_{\frac{du}{dx}}\, dx = \frac{1}{2}\int \frac{du}{u} = \ln|u| + C$$

$$= \ln\left|x^3+x^2+x-8\right| + C$$

Therefore,

$$\int_2^3 \frac{3x^2+2x+1}{x^3+x^2+x-8}dx = \frac{1}{2}\ln\left|x^3+x^2+x-8\right|\Big|_2^3$$

$$= \frac{1}{2}(\ln|27+9+3-8| - \ln|8+4+2-8|)$$

$$= \frac{1}{2}(\ln 31 - \ln 6) = \frac{1}{2}\ln\frac{31}{6}$$

.

vii) Putting $u = x^{\frac{4}{3}} - 1, du = \frac{4}{3}x^{\frac{1}{3}}dx$. We can write the integral as

$$\frac{3}{4}\int \underbrace{\sqrt{x^{\frac{4}{3}}-1}}_{\sqrt{u}}\underbrace{\frac{4}{3}x^{\frac{1}{3}}}_{\frac{du}{dx}}\, dx = \frac{3}{4}\int \sqrt{u}\, du = \frac{3}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{\left(x^{4/3}-1\right)^{\frac{3}{2}}}{2} + C$$

viii) Putting $u = 1-3x^2$, we get $du = -6x dx$. We can write the integral as

$$-\frac{1}{6}\int \underbrace{\frac{1}{\sqrt{1-3x^2}}}_{\frac{1}{\sqrt{u}}} \underbrace{-6x}_{\frac{du}{dx}} dx = -\frac{1}{6}\int \frac{du}{\sqrt{u}} = -\frac{1}{6}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= -\frac{\sqrt{u}}{3} + C = -\frac{\left(1-3x^2\right)^{\frac{1}{2}}}{3} + C.$$

E2) 4. $\int \cos(ax+b)$. Putting $u = ax+b$, we have $du = a\,dx$, So, we can write the integral as

$$\frac{1}{a}\int \cos(ax+b)a\,dx = \frac{1}{a}\int \cos u\,du = \frac{1}{a}\sin u + c = \frac{\sin(ax+b)}{a} + C.$$

7. $\int \sec(ax+b)$. Putting $u = ax+b$, we have $du = adx$. We can write the integral as

$$\frac{1}{a}\int \sec(ax+b)a\,du = \frac{1}{a}\int \sec u\,dx = \frac{1}{a}\ln|\sec u + \tan u| + C$$
$$= \frac{1}{a}\ln|\sec(ax+b)+\tan(ax+b)| + C$$

8. $\int \operatorname{cosec}(ax+b)\,dx$. Putting $u = ax+b$, we have $du = a\,dx$. We can write the integral as

$$\frac{1}{a}\int \operatorname{cosec}(ax+b)\,a\,dx = \frac{1}{a}\int \operatorname{cosec} u\,du = \frac{1}{a}\ln|\operatorname{cosec} u - \cot u| + C$$
$$= \frac{1}{a}\ln|\operatorname{cosec}(ax+b) - \cot(ax+b)| + C$$

9. $\int e^{ax+b}$. Putting $u = ax+b$, we have $du = adx$. We can write the integral as

$$\frac{1}{a}\int e^{ax+b}adx = \frac{1}{a}\int e^u du = \frac{e^u}{a} + C = \frac{e^{ax+b}}{a} + C.$$

E3) i) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot 2x \operatorname{cosec}^2 2x\,dx$. Notice that $\operatorname{cosec}^2 2x$ is almost the derivative of $\cot 2x$, except for a constant factor 2. This suggests the substitution $u = \cot 2x$. Putting $u = \cot 2x$, we have $du = -2\operatorname{cosec}^2 2x dx$. We can write the integral as

$$-\frac{1}{2}\int u\,du = -\frac{u^2}{4} + C = -\frac{\cot^2 2x}{4} + C$$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot 2x \operatorname{cosec}^2 2x\,dx = -\left.\frac{\cot^2 2x}{4}\right|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\frac{1}{4}\left(\cot^2\frac{2\pi}{3} - \cot^2\frac{2\pi}{6}\right)$$

$$= -\frac{1}{4}\left(\frac{1}{3}-\frac{1}{3}\right) = 0.$$

ii) Putting $u = \cos 2\theta$, we get $du = -2\sin 2\theta d\theta$. So, we can write the integral as

$$-\frac{1}{2}\int e^{\cos 2\theta}(-2\sin 2\theta)d\theta = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{\cos 2\theta} + C$$

iii) Putting $u = \cos\theta$ we have $du = -\sin\theta d\theta$. So,

$$\int \sin\theta\left(1+\cos^4\theta\right)\, d\theta = -\int\left(1+\cos^4\theta\right)(-\sin\theta)\, d\theta$$

$$= -\int\left(1+u^4\right)\, du = -\left(u+\frac{u^5}{5}\right)+C$$

$$= -\left(\cos\theta+\frac{\cos^5\theta}{5}\right)+C$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin\theta\left(1+\cos^4\theta\right) d\theta = -\left(\cos\theta+\frac{\cos^5\theta}{5}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= 0-\left\{-\left(1+\frac{1}{5}\right)\right\} = \frac{6}{5}$$

iv) Putting $u = 1+\cos\theta$, we get $du = -\sin\theta d\theta$. We can write the integral as $-\int u^4 du = -\frac{u^5}{5}+C = -\frac{(1+\cos\theta)^5}{5}+C$.

v) Putting $u = 1-5\tan\theta$, we get $du = -5\sec^2\theta d\theta$. We can write the integral as $-\frac{1}{5}\int\frac{du}{u^3} = -\frac{1}{5}(-\frac{1}{2u^2})+C = \frac{1}{10(1-5\tan\theta)^2}+C$.

vi) Putting $u = 1+\sec\theta$ we get $du = \sec\theta\tan\theta d\theta$ So we have

$$\int \sec\theta\tan\theta(1+\sec\theta)^3 d\theta = \int u^3\, du = \frac{u^4}{4}+C = \frac{(1+\sec\theta)^4}{4}+C$$

$$\int_o^{\frac{\pi}{4}} \sec\theta\tan\theta(1+\sec\theta)^3 d\theta = \frac{1+\sec\theta}{4}\Bigg|_0^{\frac{\pi}{4}} = \left[\frac{1+\sqrt{2}}{4}-\frac{2}{4}\right]$$

$$= \frac{\sqrt{2}-1}{4}$$

E4) i) We have

$$\int \sin^7 x\, dx = \int\left(1-\cos^2 x\right)^3 \sin x\, dx$$

$$= \int\left(1-3\cos^2 x+3\cos^4 x-\cos^6 x\right)\sin x\, dx$$

Putting $u = \cos x$, we have $du = -\sin x\, dx$. The integral becomes

$$= -\int(1-3u^2+3u^4-u^6)du$$

$$= -\left(u-u^3+\frac{3u^5}{5}-\frac{u^7}{7}\right)+C$$

$$= -\cos x+\cos^3 x-\frac{3\cos^5 x}{5}+\frac{\cos^7 x}{7}+C$$

ii) We have $\int\cos^5 x\, dx = \int\left((1-\sin^2 x\right)^2\cos x\, dx$. Putting $u = \sin x$, we get $du = \cos x\, dx$ The integral becomes

$$\int\left(1-u^2\right)^2 du = \int\left(1+u^4-2u^2\right)du$$

$$= u+\frac{u^5}{5}-\frac{2u^3}{3}+C$$

$$= \sin x+\frac{\sin^5 x}{5}-\frac{2\sin^3 x}{3}+C$$

iii) We have $\int \cos^2 x \sin^3 x \, dx = \int \cos^2 x \left(1-\cos^2 x\right) \sin x \, dx$. Putting $u = \cos x$, we have $du = -\sin x \, dx$. The integral becomes

$$-\int \left(u^2 - u^4\right) du = -\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$

$$= \left(\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}\right) + C$$

iv) From Eqn. (10), we have

$$\sin 5x \cos 3x = \frac{1}{2}(\sin(3x+5x) - \sin(3x-5x)) = \frac{1}{2}(\sin 8x + \sin 2x).$$

So,

$$\int \sin 5x \cot 3x \; dx = \frac{1}{2}\left(\int \sin 8x \, dx + \int \sin 2x dx\right)$$

$$= \frac{1}{2}\left(-\frac{1}{8}\cos 8x\right) + \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + C$$

$$= -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C.$$

v) Using Eqn. (11) we get

$$\cos 3x \cos 4x = \frac{1}{2}(\cos(3x-4x) + \cos(3x+4x)) = \frac{1}{2}(\cos x + \cos 7x)$$

So

$$\int \cos 3x \cos 4x = \frac{1}{2}\left(\sin x + \frac{\cos 7x}{7}\right) + C$$

vi) We have

$$\int \sin 4x \sin 3x dx = \int \frac{1}{2}(\cos(4x-3x) - \cos(4x+3x))dx$$

$$= \frac{1}{2}\int (\cos x - \cos 7x)dx = \frac{1}{2}\left(\sin x - \frac{\sin 7x}{7}\right) + C$$

vii) We have

$$\sin 2x \sin 3x \sin 5x = \frac{\sin 2x}{2}(\cos(3x-5x) - \cos(3x+5x))$$

$$= \frac{\sin 2x}{2}(\cos 2x - \cos 8x)$$

$$= \frac{1}{2}(\sin 2x \cos 2x - \sin 2x \cos 8x)$$

$$= \frac{1}{4}(\sin 4x) - \frac{1}{2}(\sin 10x + \sin(-6x))$$

$$= \frac{\sin 4x}{4} - \frac{\sin 10x}{4} + \frac{\sin 6x}{4}.$$

$$\therefore \int \sin 2x \sin 3x \sin 5x \, dx = \frac{\cos 4x}{16} - \frac{\cos 10x}{40} + \frac{\cos 6x}{24} + C$$

E5) We have $\cos(\frac{\pi}{2} - a) = \sin a$. Let $x = \cos(\frac{\pi}{2} - a)$. Then $\cos^{-1} x = \frac{\pi}{2} - a$ or $a = \frac{\pi}{2} - \cos^{-1} x$. We also have $x = \sin a$. So, $a = \sin^{-1} x$ i.e.

$$\frac{\pi}{2} - \cos^{-1} x = \sin^{-1} x \text{ or } \sin^{-1} x - (-\cos^{-1} x) = \frac{\pi}{2}.$$

So, the two answers differ only by a constant. Since the primitive of a function is determined only up to a constant, both $\sin^{-1}\left(\frac{x}{a}\right)$ and $-\cos^{-1}\left(\frac{x}{a}\right)$ are primitives of $\frac{1}{\sqrt{a^2-x^2}}$.

E6) Let us prove Eqn. (18). Putting $x = a\sec\theta$, $dx = a\sec\theta\tan\theta d\theta$. Also

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$$

The integral becomes

$$\int \frac{a\sec\theta.\tan\theta}{a\tan\theta} d\theta = \ln|\sec\theta + \tan\theta| + C$$

from entry 8 in table 1 in unit 17. We have $\sec\theta = \frac{x}{a}$. So,

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{a}\right)^2 - 1} = \frac{\sqrt{x^2 - a^2}}{a}$$

Therefore,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C = \ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + C.$$

To prove (21) we use the substitution $x = a\sec\theta$ again. We have $dx = a\sec\theta\tan\theta d\theta$. The integral becomes $\int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta\tan\theta} = \int d\theta = \theta + C$.

$$\therefore \int \frac{dx}{x\sqrt{x^2 - a^2}} = \sec^{-1}\left(\frac{x}{a}\right) + C.$$

E7) i) $$\int \frac{dx}{\sqrt{3 - x^2}} = \int \frac{dx}{\sqrt{(\sqrt{3})^2 - x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

ii) We have

$$\int \frac{dx}{\sqrt{1 - 5x^2}} = \int \frac{dx}{\sqrt{5\left(\frac{1}{5} - x^2\right)}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 - x^2}}$$
$$= \frac{1}{\sqrt{5}}\sin^{-1}(\sqrt{5}x) + C$$

iii) $$\int \frac{dx}{x^2 + 5} = \int \frac{dx}{x^2 + (\sqrt{5})^2} = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

iv) We have

$$\int \frac{dx}{3x^2 + 1} = \frac{1}{3}\int \frac{dx}{x^2 + \frac{1}{3}} = \frac{1}{3}\int \frac{dx}{x^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{1}{3}\sqrt{3}\tan^{-1}(\sqrt{3}x) + C$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + C$$

v) We have

$$\int \frac{dx}{x^2 + 7} = \int \frac{dx}{\sqrt{x^2 + (\sqrt{7})^2}} = \ln\left|\frac{x + \sqrt{x^2 + 7}}{\sqrt{7}}\right| + C$$

vi)

$$\int \frac{dx}{\sqrt{5\left(\frac{1}{5}+x^2\right)}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + x^2}}$$

$$= \frac{1}{\sqrt{5}} \ln\left| x + \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + x^2}\right| + C$$

$$= \frac{1}{\sqrt{5}}\sqrt{5}\ln\left|\sqrt{5}x + \sqrt{1+5x^2}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\sqrt{5}x + \sqrt{1+5x^2}\right| + C.$$

vii) $\displaystyle\int \frac{dx}{\sqrt{x^2-9}} = \int \frac{dx}{\sqrt{x^2-3^2}} = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C$

viii) $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{dx}{x\sqrt{x^2-2^2}} = \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C$

ix) We have

$$\int \frac{dx}{\sqrt{9x^2-1}} = \int \frac{dx}{\sqrt{9\left(x^2-\frac{1}{9}\right)}} = \frac{1}{3}\int \frac{dx}{\sqrt{x^2-\left(\frac{1}{3}\right)^2}}$$

$$= \frac{1}{3}\ln\left|\frac{x+\sqrt{x^2-\left(\frac{1}{3}\right)^2}}{\frac{1}{3}}\right| + C = \frac{1}{3}\ln\left|3x+\sqrt{9x^2-1}\right| + C.$$

x) We have

$$\int \frac{dx}{x\sqrt{2x^2-1}} = \int \frac{dx}{x\sqrt{2\left(x^2-\frac{1}{2}\right)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{x\sqrt{x^2-\left(\frac{1}{2}\right)^2}}$$

$$= \sqrt{2}\sec^{-1}\left(\frac{x}{\frac{1}{\sqrt{2}}}\right) + C = \sqrt{2}\sec^{-1}(\sqrt{2}x) + C.$$

E8) i) $\displaystyle\int x^2 \ln|x|dx = \frac{x^3}{3}\ln|x| - \int \frac{x^3}{3}.\frac{1}{x}dx = x^3\ln|x| - \frac{x^3}{9} + C.$

ii) We have
$\displaystyle\int (1+x)e^x\,dx = (1+x)e^x - \int e^x\,dx = (1+x)e^x - e^x + C = xe^x + C.$

iii) We have

$$\int \left(1+x^2\right)e^x\,dx = \left(1+x^2\right)e^x - \int e^x\frac{d}{dx}\left(1+x^2\right)\,dx$$

$$= \left(1+x^2\right)e^x - 2\int xe^x\,dx$$

Integrating $\int xe^x\,dx$ by parts again, we get

$$\int xe^x dx = xe^x - \int e^x\,dx = xe^x - e^x$$

$$\therefore \int \left(1+x^2\right)e^x\,dx = \left(1+x^2\right)e^x - 2xe^x + 2e^x + C$$

iv) We have

$$\int x^2 \sin x \cos x \, dx = \int x^2 \frac{\sin 2x}{2} dx = \frac{1}{2}\int x^2 \sin 2x \, dx$$
$$= \frac{1}{2}\left\{x^2\left(-\frac{\cos 2x}{2}\right) + \int x \cos 2x \, dx\right\}.$$

Again,

$$\int x \cos 2x \, dx = \frac{x \sin 2x}{2} - \frac{1}{2}\int \sin 2x \, dx$$
$$= \frac{x \sin 2x}{2} - \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + C.$$
$$\therefore \int x^2 \sin x \cos x dx = -\frac{1}{2}\left(\frac{x^2 \cos 2x}{2} - \frac{x \sin 2x}{2} - \frac{\cos 2x}{4}\right)$$
$$= -\frac{x^2 \cos 2x}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C$$

E9) i) We have

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x \, dx}{\sqrt{1-x^2}}.$$

Putting $u = 1 - x^2$, we get $du = -2x \, dx$. The integral becomes

$$-\frac{1}{2}\int \frac{-2x \, dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = -\sqrt{1-x^2} + C.$$

So,

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$$

ii) We have

$$\int \tan^{-1} x dx = x \tan^{-1} x - \int x \frac{1}{1+x^2} dx.$$

Substituting $u = 1 + x^2$, $du = 2x \, dx$, we have

$$\int \frac{x}{1+x^2} dx = \frac{1}{2}\int \frac{2x \, dx}{1+x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln\left|(1+x^2)\right| + C.$$
$$\therefore \int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$

iii) We have

$$\int \cot^{-1} x \, dx = x \cot^{-1} x - \int x\left(\frac{-1}{1+x^2}\right) dx = x \cot^{-1} x + \int \frac{x \, dx}{1+x^2}$$
$$= x \cot^{-1} x + \frac{1}{2}\ln\left|1+x^2\right| + C$$

E10) We have

$$\int x \ln\left|\left(1+x^2\right)\right| dx = \frac{x^2}{2}\ln\left|\left(1+x^2\right)\right| - \frac{1}{2}\int x^2 \frac{1}{1+x^2} .2x \, dx$$
$$= \frac{x^2}{2}\ln\left|1+x^2\right| - \int \frac{x^3}{1+x^2} dx.$$

Further,

$$\int \frac{x^3}{1+x^2}dx = \int \frac{x\left(1+x^2\right)-x}{1+x^2}dx = \int x\,dx - \int \frac{x\,dx}{1+x^2}$$
$$= \frac{x^2}{2} - \frac{1}{2}\int \frac{2x}{1+x^2}dx = \frac{x^2}{2} - \frac{1}{2}\ln\left|\left(1+x^2\right)\right| + C.$$

$$\therefore \int x\ln\left|\left(1+x^2\right)\right|dx = \frac{\left(x^2+1\right)}{2}\ln\left|x^2+1\right| - \frac{x^2}{2} + C$$

E11) i) Comparing with Eqn. (23) , we see that
$a = 3, b = 4. \therefore r = \sqrt{3^2+4^2} = 5.$
$\therefore \int e^{3x}\cos 4x\,dx = \frac{1}{5}e^{3x}\cos(4x-\theta) + C$ where θ is such that
$\cos\theta = \frac{3}{5}, \sin\theta = \frac{4}{5}.$

ii) Comparing with Eqn. (23), we see that $a = 3$, $b = 4$. $\therefore r = 5$.
$\int e^{4x}\sin 3x\,dx = \frac{1}{5}e^{4x}\sin(3x-\theta) + C$ where θ is such that
$\cos\theta = \frac{4}{5}, \sin\theta = \frac{3}{5}.$

iii) $\int e^{-4x}\cos 4x dx$. Here, $a = -4$, $b = 4$. $r = \sqrt{32} = 4\sqrt{2}$. $\cos\theta = \frac{a}{r} = \frac{-1}{\sqrt{2}}$,
$\sin\theta = \frac{1}{\sqrt{2}}$. $\therefore$, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. So, we have

$$\int e^{-4x}\cos 4x\,dx = \frac{1}{4\sqrt{2}}e^{-4x}\cos\left(4x - \frac{3\pi}{4}\right) + C$$

E12) i) We have $\cos x\cos 2x = \frac{1}{2}(\cos 3x + \cos x)$. Therefore,
$\int e^{4x}\cos x\cos 2x\,dx = \frac{1}{2}\left(\int e^{4x}\cos 3x\,dx + \int e^{4x}\cos x\,dx\right)$. Both the integrals are standard forms that we have seen already.

ii) We have $\cos^2 x = \frac{1+\cos 2x}{2}$. Therefore

$$\int e^{2x}\cos^2 x\,dx = \frac{1}{2}\left(\int dx + \int \cos 2x\,dx\right).$$

Both the integrals are easy to evaluate.

iii) We have $\int e^{ax}\sin bx\,dx = e^{ax}\sin(bx-\theta)$ where θ is such that
$\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\cos\theta = \frac{b}{\sqrt{a^2+b^2}}$. Integrating by parts, we get

$$\int xe^{ax}\sin bx\,dx = xe^{ax}\sin(bx-\theta) - \int e^{ax}\sin(bx-\theta)\,dx$$

Substituting $u = bx - \theta$, we have $du = b\,dx$. Therefore,

$$\int e^{ax}\sin(bx-\theta)\,dx = \int e^{a\left(\frac{u+\theta}{b}\right)}\sin u\,du = e^{\frac{a\theta}{b}}\int e^{\frac{au}{b}}\sin u\,du$$

This is a standard form which we know how to integrate.

E13) i) Integrating by parts, we have

$$\int \sqrt{a^2+x^2}dx = \int \sqrt{a^2+x^2}(1)dx$$
$$= x\sqrt{a^2+x^2} - \int x.\frac{1}{2}\frac{1}{\sqrt{a^2+X^2}}2x\,dx$$
$$= x\sqrt{a^2+x^2} - \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx$$
$$= x\sqrt{a^2+x^2} - \int \frac{(x^2+a^2)-a^2}{\sqrt{a^2+x^2}}dx$$
$$= x\sqrt{a^2+x^2} + a^2\int \frac{dx}{\sqrt{a^2+x^2}} - \int \sqrt{a^2+x^2}\,dx$$
$$\therefore 2\int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} + a^2\int \frac{dx}{\sqrt{a^2+x^2}}$$
$$= x\sqrt{a^2+x^2} + a^2\left[\ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right|\right] + C.$$
$$\therefore \int \sqrt{a^2+x^2}dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|\frac{x+\sqrt{a^2+x^2}}{a}\right| + C$$

ii) Integrating by parts,

$$\int \sqrt{x^2-a^2}\,dx = x\sqrt{x^2-a^2} - \int x\frac{1}{2}\frac{1}{\sqrt{x^2-a^2}}.2x\,dx$$
$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}dx$$
$$= x\sqrt{x^2-a^2} - \int \frac{\left(x^2-a^2\right)+a^2}{\sqrt{x^2-a^2}}dx$$
$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{dx}{\sqrt{x^2-a^2}}.$$
$$\therefore 2\int \sqrt{x^2-a^2}dx = x\sqrt{x^2-a^2} - a^2\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C.$$
$$\therefore \int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$$

E14) i) We have $\int \sqrt{\left(\sqrt{3}\right)^2 - x^2}\,dx = \frac{1}{2}x\sqrt{3-x^2} + \frac{3}{2}\sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

ii) We have

$$\int \sqrt{1-5x^2}dx = \sqrt{5}\int \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 - x^2}\,dx$$
$$= \sqrt{5}\left(\frac{x}{2}\sqrt{\frac{1}{5}-x^2} + \frac{1}{10}\sin^{-1}\left(\sqrt{5}x\right)\right) + C$$
$$= \frac{x}{2}\sqrt{1-5x^2} + \frac{1}{2\sqrt{5}}\sin^{-1}\left(\sqrt{5}x\right) + C.$$

iii) $\int \sqrt{x+\left(\sqrt{7}\right)^2}dx = \frac{x}{2}\sqrt{7+x^2} + \frac{7}{2}\ln\left|\frac{x+\sqrt{x^2+7}}{7}\right| + C.$

iv) We have

$$\int \sqrt{1+5x^2}\,dx = \sqrt{5}\int \sqrt{\frac{1}{\left(\sqrt{5}\right)^2}+x^2}\,dx$$

$$= \sqrt{5}\left(\frac{x}{2}\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2+x^2}+\frac{1}{10}\ln\left|\frac{x+\sqrt{\frac{1}{5}+x^2}}{\frac{1}{\sqrt{5}}}\right|\right)+C$$

$$= \frac{x}{2}\sqrt{1+5x^2}+\frac{1}{2\sqrt{5}}\ln\left|\sqrt{5}x+\sqrt{1+5x^2}\right|+C$$

v) We have

$$\int \sqrt{x^2-9}\,dx = \int \sqrt{x^2-3^2}\,dx$$

$$= \frac{x}{2}\sqrt{x^2-9}-\frac{9}{2}\ln\left|\frac{x+\sqrt{x^2-9}}{3}\right|+C$$

vi) We have

$$\int \sqrt{9x^2-1}\,dx = 3\int \sqrt{x^2-\frac{1}{9}}\,dx$$

$$= 3\left(\frac{x}{2}\sqrt{x^2-\left(\frac{1}{3}\right)^2}-\frac{1}{18}\ln\left|\frac{x+\sqrt{x^2-\left(\frac{1}{3}\right)^2}}{\frac{1}{3}}\right|\right).$$

$$= \frac{x}{2}\sqrt{9x^2-1}-\frac{1}{6}\ln\left|3x+\sqrt{9x^2-1}\right|+C$$

vii) Integrating by parts we have

$$\int x\sin^{-1}x\,dx = \frac{x^2\sin^{-1}x}{2}-\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

We have

$$\int \frac{x^2}{\sqrt{1-x^2}}dx = -\int \frac{\left(1-x^2\right)-1}{\sqrt{1-x^2}}dx$$

$$= -\int \sqrt{1-x^2}\,dx+\int \frac{dx}{\sqrt{1-x^2}}$$

$$= -\left(\frac{x}{2}\right)\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}\frac{x}{2}+\sin^{-1}\frac{x}{2}+C.$$

E15) i) We have

$$\int \frac{dx}{2x-3} = \frac{1}{2}\ln|2x-3|+C$$

using entry 2 in Table 1.

ii) Using entry 1 in Table 1, we have $\int \frac{dt}{(t+5)^2} = -\frac{1}{(t+5)}+C.$

iii) We have

$$\int \frac{4x+1}{x^2+x+2}dx = \frac{1}{2}\left(\int \frac{2x+1}{x^2+x+2}dx-\int \frac{dx}{x^2+x+2}\right)$$

$\int \frac{2x+1}{x^2+x+2}dx$ is of the form $\int \frac{f'(x)}{f(x)}dx$ where $f(x) = x^2 + x + 2$.

$$\therefore \int \frac{2x+1}{x^2+x+2}dx = \ln\left|x^2+x+2\right| + C_1.$$

We have

$$\int \frac{dx}{x^2+x+2} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\sqrt{\frac{7}{2}}\right)^2}$$

$$= \int \frac{du}{u^2 + \left(\sqrt{\frac{7}{2}}\right)^2} \text{ on putting } u = x + \frac{1}{2}.$$

$$\therefore \int \frac{dx}{x^2+x+2} = \left(\frac{2}{\sqrt{7}}\right)\tan^{-1}\left(\frac{2u}{\sqrt{7}}\right) + C_2$$

$$= \left(\frac{2}{\sqrt{7}}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C_2.$$

$$\therefore \int \frac{4x+1}{x^2+x+2}\,dx = 2\ln\left|x^2+x+2\right| - \frac{4}{\sqrt{7}}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C$$

iv) We have

$$\frac{5x-1}{x^2-1} = \frac{5x-1}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}.$$

Multiplying by x^2-1 both sides, we get $5x-1 = A(x-1) + B(x+1)$. Putting, $x = 1$, we get $4 = 2B$ or $B = 2$. Putting $x = -1$, we get $-6 = -2A$ or $A = 3$. Therefore, the integral becomes

$$\int \frac{5x-1}{x^2-1}dx = 3\int \frac{dx}{x+1} + 2\int \frac{dx}{x-1} = 3\ln|x+1| + 2\ln|x-1| + C.$$

v) We have $\frac{d}{dx}(x^2-6x+3) = 2x-6$. Writing $3x+1 = A(2x-6)+B$ and solving for the values of A and B by comparing the constant term and the coefficient of x both sides, we get $A = 3/2, B = 1+6A = 10$. So, we have

$$\int \frac{3x+1}{x^2-6x+3}dx = \frac{3}{2}\int \frac{2x-6}{x^2-6x+3}dx + 10\int \frac{dx}{x^2-6x+3}$$

$\int \frac{2x-6}{x^2-6x+3}dx$ is of the form $\int \frac{f'(x)}{f(x)}\,dx$ where $f(x) = x^2-6x+3$.

$$\therefore \int \frac{2x-6}{x^2-6x+3}dx = \ln\left|x^2-6x+3\right| + C_1.$$

We have $\int \frac{dx}{x^2-6x+3} = \int \frac{dx}{(x-3)^2-6}$. Putting $u = x-3$, the integral becomes

$$\int \frac{du}{u^2-(\sqrt{6})^2} = \frac{1}{2\sqrt{6}}\ln\left|\frac{u-\sqrt{6}}{u+\sqrt{6}}\right| + C.$$

$$\therefore \int \frac{3x+1}{x^2-3x+6}dx = \frac{3}{2}\ln\left|x^2-6x+3\right| + \frac{5}{\sqrt{6}}\ln\left|\frac{x-3-\sqrt{6}}{x-3+\sqrt{6}}\right| + C.$$

E16) i) We can integrate this by completing the square also. However, the denominator factors as $x(x+2)$. So, we can use the method of partial fractions. We have

$$\frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}.$$

Multiplying both sides by $x(x+2)$, we get $1 = A(x+2) + Bx$. Putting $x = 0$, we get $2A = 1$ or $A = \frac{1}{2}$. Putting $x = -2$, we get $1 = -2B$ or $B = -\frac{1}{2}$.

$$\therefore \int \frac{2}{x^2+2x}dx = 2\int \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)dx = \int \frac{dx}{x} - \int \frac{dx}{x+2}$$
$$= \ln|x| - \ln|x+2| + C = \ln\left|\frac{x}{x+2}\right| + C$$

ii) We have $x^2 - 2x - 3 = (x-3)(x+1)$. To split $\frac{x}{(x-3)(x+1)}$ into partial fractions, we write

$$\frac{x}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}.$$

Multiplying both sides by $(x-3)(x+1)$, we get $x = A(x+1) + B(x-3)$. Putting $x = -1$, we get $-1 = -4B$ or $B = \frac{1}{4}$. Putting $x = 3$, we get $3 = 4A$ or $A = \frac{3}{4}$.

$$\therefore \int \frac{x}{x^2-2x-3}dx = \frac{3}{4}\int \frac{dx}{x-3} + \frac{1}{4}\int \frac{dx}{x+1}$$
$$= \frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + C.$$

iii) We can factor the denominator as $(x+5)(x-2)$. We write

$$\frac{3x-13}{(x+5)(x-2)} = \frac{A}{(x+5)} + \frac{B}{x-2}.$$

Multiplying both sides by $(x+5)(x-2)$. we get $3x - 13 = A(x-2) + B(x+5)$. Putting $x = 2$, we get $-7 = 7B$ or $B = -1$. Putting $x = -5$, we get $-28 = -7A$ or $A = 4$.

$$\therefore \int \frac{3x-13}{x^2+3x-10}dx = 4\int \frac{dx}{x+5} - \int \frac{dx}{x-2} = 4\ln|x+5| - \ln|x-2| + C.$$

iv) We have $x^2 + x - 6 = (x+3)(x-2)$. We write

$$\frac{6x^2+22x-23}{(2x-1)(x+3)(x-2)} = \frac{A}{(2x-1)} + \frac{B}{x+3} + \frac{C}{x-2}$$

Multiplying both sides by $(2x-1)(x+3)(x-2)$, we get

$$6x^2 + 22x - 23 = A(x+3)(x-2) + B(2x-1)(x-2) + C(2x-1)(x+3)$$

Putting $x = \frac{1}{2}$, we get $21A = 42$ or $A = 2$. Putting $x = -3$, we get $-35 = 35B$ or $B = -1$. Putting $x = 2$, we get $5C = 15$ or $C = 3$. The integral becomes

$$\therefore \int \frac{6x^2+22x-23}{(2x-1)(x+3)(x-2)}dx = 2\int \frac{dx}{2x-1} - \int \frac{dx}{x+3} + 3\int \frac{dx}{x-2}$$
$$= \ln|2x-1| - \ln|x+3| + 3\ln|x-2| + C.$$

v) Check using long division $3x^3 = (3x-3)(x^2+x-2) + 9x - 6$. We can write the integral as

$$\int (3x-3)dx + \int \frac{9x-6}{x^2+x-2}dx = 3\int (x-1)dx + 3\int \frac{3x-2}{(x+2)(x-1)}dx.$$

Splitting $\frac{3x-2}{(x+2)(x-1)}$ into partial fractions, we get

$$\frac{3x-2}{(x+2)(x-1)} = \frac{8}{3(x+2)} + \frac{1}{3(x-1)}$$

$$\therefore \int \frac{3x-2}{(x+2)(x-1)}dx = \frac{8}{3}\int \frac{dx}{x+2} + \frac{1}{3}\int \frac{dx}{x-1}$$

$$= \frac{8}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C_1.$$

$$3\int (x-1)dx = \frac{3}{2}(x-1)^2 + C_2.$$

$$\therefore \int \frac{x^3}{x^2+x-2}dx = \frac{3(x-1)^2}{2} + \frac{8}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C.$$

vi) Notice that $x^2 - x + 1$ has complex roots. To split the expression into partial fractions we write

$$\frac{x^2+x-1}{(x-1)(x^2-x+1)} = \frac{A}{x-1} + \frac{Cx+D}{x^2-x+1}.$$

Multiplying both sides by $(x-1)(x^2-x+1)$, we get

$$x^2+x-1 = A(x^2-x+1) + (Cx+D)(x-1).$$

Putting $x = 1$, we get $1 = A$ or $A = 1$. Comparing the coefficients of x^2 both sides we get $1 = A + C$. $\therefore C = 0$. Comparing the constant terms both sides, we get $A - D = -1$,. $\therefore D = 2$. We can write the integral as

$$\int \frac{dx}{x-1} + 2\int \frac{dx}{x^2-x+1}.$$

$$\int \frac{dx}{x^2-x+1} = \int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}.$$

Substituting $u = x - \frac{1}{2}$, we get $du = dx$.

$$\therefore \int \frac{dx}{x^2-x+1} = \int \frac{du}{u^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + C_1 = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C_1.$$

$$\therefore \int \frac{x^2+x-1}{(x-1)(x^2-x+1)}dx = \ln|x-1| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

.

vii) Let

$$\frac{x^3-4x}{(x^2+1)^2} = \frac{(Ax+B)}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Multiplying both sides by $(x^2+1)^2$, we get

$$(x^3-4x) = (Ax+B)(x^2+1) + Cx + D.$$

Comparing coefficient of x^3 both sides, we get $A = 1$. Comparing coefficients of x^2 both sides, we get $B = 0$. Comparing coefficients of x both sides, we get A + C = –4, so that C = –5. Comparing constant terms both sides, we get B + D = 0, so that D = 0.

$$\therefore \int \frac{(x^3-4x)}{(x^2+1)^2}dx = \int \frac{x\,dx}{(x^2+1)} - 5\int \frac{5x}{(x^2+1)^2}\,dx.$$

Now

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{2x\,dx}{x^2+1}.$$

Substituting $u = x^2 + 1$, $du = 2\,dx$.

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+1\right| + C_1.$$

Substituting $u = x^2 + 1$, in $\int \frac{x\,dx}{(x^2+1)^2}$ we get $du = 2x\,dx$. So,

$$\int \frac{xdx}{(x^2+1)^2} = \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + C = \frac{-1}{2\,(x^2+1)} + C_2.$$

$$\therefore \int \frac{(x^3-4x)}{(x^2+1)^2}dx = \frac{1}{2}\ln\left|x^2+1\right| + \frac{5}{2\,(x^2+1)} + C.$$

E17) i) We have

$$\int \frac{x^2-1}{1+x^4}\,dx = \int \frac{1-\frac{1}{x^2}}{\frac{1}{x^2}+x^2}\,dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$$

Putting $t = x + \frac{1}{x}$, we get $dt = 1 - \frac{1}{x^2}$. Therefore,

$$\int \frac{x^2-1}{1+x^4}\,dx = \int \frac{dt}{t^2-2} = \frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

ii) We have

$$\int \frac{1+x^2}{1+x^2+x^4}\,dx = \int \frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+1+x^2}\,dx = \int \frac{\frac{1}{x^2}+1}{\left(x-\frac{1}{x}\right)^2+3}\,dx.$$

If $t = x - \frac{1}{x}$, $\frac{dt}{dx} = 1 + \frac{1}{x^2}$,

$$\int \frac{1+x^2}{1+x^2+x^4}\,dx = \int \frac{dt}{t^2+3} = \frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{1}{\sqrt{3}}\left(x-\frac{1}{x}\right)\right\} + C$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3x}}\right) + C$$

E18) i) Here a < b. Therefore

$$\int \frac{dx}{4+5\cos x} = \frac{1}{\sqrt{5^2-4^2}}\ln\left|\frac{\sqrt{4+5}+\sqrt{5-4}\tan\frac{x}{2}}{\sqrt{4+5}-\sqrt{5-4}\tan\frac{x}{2}}\right| + C = \frac{1}{3}\ln\left|\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right| + C.$$

ii) We have

$$\int \frac{\cos x}{2-\cos x}dx = -\int \frac{(2-\cos x)-2}{(2-\cos x)}dx = -\int dx + 2\int \frac{dx}{(2-\cos x)}.$$

We have $-\int dx = -x + C + C_1$. In the second integral $a = 2, b = -1$, So $a > b$. Therefore

$$\int \frac{dx}{(2-\cos x)} = \frac{2}{\sqrt{2^2-(-1)^2}}\tan^{-1}\left(\sqrt{\frac{2-(-1)}{2+(-1)}}\tan\frac{x}{2}\right) + C_2$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{x}{2}\right) + C_2$$

$$\therefore \int \frac{\cos x\, dx}{2-\cos x} = -x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan\frac{x}{2}\right) + C$$

E19) We have

$$\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}dx = \int \frac{t^2}{1+t}4t^3 dt \text{ if } t = \sqrt[4]{x}.$$

$$= 4\int \frac{t^5}{1+t}dt = 4\int \left[t^4 - t^3 + t^2 - t + 1 - \frac{1}{t+1}\right] dt$$

$$= 4\left[\frac{t^5}{5} - \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right] + C$$

$$= 4\left[\frac{x^{5/4}}{5} - \frac{x}{4} + \frac{x^{3/4}}{3} - \frac{x^{1/2}}{2} + x^{1/4} - \ln|x^{1/4}+1|\right] + C$$

E20) i) We have

$$\int \frac{dx}{\sqrt{3x^2+4x+1}} = \int \frac{dx}{\sqrt{3\left(x^2+\frac{4}{3}x+\frac{1}{3}\right)}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x+\frac{2}{3}\right)^2 - \frac{4}{9} + \frac{1}{3}}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x+\frac{2}{3}\right)^2 - \frac{1}{9}}}$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\left(x+\frac{2}{3}\right)^2 + \sqrt{\left(x+\frac{2}{3}\right)^2 - \frac{1}{9}}}{1/3}\right| + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|3x+2+\sqrt{3}\sqrt{3x^2+4x+1}\right| + C.$$

ii) We have

$$\int \frac{dx}{\sqrt{3x^2-4x+3}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x-\frac{2}{3}\right)^2 - \frac{4}{9} + 1}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{\left(x - \frac{2}{3}\right)^2 + \frac{5}{9}}}$$

$$= \frac{1}{\sqrt{3}} \ln \left| \frac{\left(x - \frac{2}{3}\right) + \sqrt{\left(x - \frac{2}{3}\right)^2 + \frac{5}{9}}}{\sqrt{5}/3} \right| + C$$

$$= \frac{1}{\sqrt{3}} \ln \left| \frac{3x - 2 + \sqrt{3}\sqrt{3x^2 - 4x + 3}}{\sqrt{5}} \right| + C$$

iii) We have

$$5 - 2x - x^2 = 5 - (2x + x^2) = 5 - \left\{(x+1)^2 - 1\right\} = 6 - (x+1)^2$$

therefore

$$\int \frac{dx}{\sqrt{5 - 2x - x^2}} = \int \frac{dx}{\sqrt{\left(\sqrt{6}\right)^2 - (x+1)^2}} = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C$$

E21) i) We have $\frac{d}{dx}(3x^2 + 4x + 1) = 6x + 4.$ We have $3x + 1 = A(6x + 4) + B.$ So, $A = \frac{1}{2}$. From $4A + B = 1$, $B = -1$.

$$\therefore \int \frac{3x+1}{\sqrt{3x^2 + 4x + 1}} dx = \frac{1}{2} \int \frac{6x+4}{\sqrt{3x^2 + 4x + 1}} dx - \int \frac{dx}{\sqrt{3x^2 + 4x + 1}}.$$

$\int \frac{6x+4}{\sqrt{3x^2+4x+1}}$ dx is of the form $\int \frac{f'(x)}{\sqrt{f(x)}}dx$ where $f(x) = 3x^2 + 4x + 1.$

$$\therefore \int \frac{6x+4}{\sqrt{3x^2 + 4x + 1}} dx = 2\sqrt{3x^2 + 4x + 1} + C_1.$$

From E) 20) i) We know the

$$\int \frac{dx}{\sqrt{3x^2 + 4x + 1}} = \frac{1}{\sqrt{3}} \ln\left|3x + 2 + \sqrt{3}\sqrt{3x^2 + 4x + 1}\right| + C_2.$$

$$\therefore \int \frac{3x+1}{3x^2 + 4x + 1} dx = \sqrt{3x^2 + 4x + 1} - \frac{1}{\sqrt{3}} \ln\left|3x + 2 + \sqrt{3}\sqrt{3x^2 + 4x + 1}\right| + C$$

ii) We have $\frac{d}{dx}(3x^2 - 4x + 3) = 6x - 4$. We write $2x - 3 = A(6x - 4) + B$. Comparing coefficients of x both sides, we get $A = \frac{1}{3}$. Comparing constant terms both sides, we get $B - 4A = -3$ or $B = -\frac{5}{3}$.

$$\therefore \int \frac{2x-3}{\sqrt{3x^2 - 4x + 3}} dx = \frac{1}{3} \int \frac{6x-4}{\sqrt{3x^2 - 4x + 3}} - \frac{5}{3} \int \frac{dx}{\sqrt{3x^2 - 4x + 3}}.$$

From 20) ii) we know that

$$\int \frac{dx}{\sqrt{3x^2 - 4x + 3}} dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{3x^2 - 4x + 3} + c.$$

Therefore

$$\int \frac{2x-3}{\sqrt{3x^2 - 4x + 3}} dx = \frac{2}{3}\sqrt{3x^2 - 4x + 3}$$

$$- \frac{5}{3\sqrt{3}} \ln \left| \frac{3x - 2 + \sqrt{3}\sqrt{3x^2 - 4x + 3}}{\sqrt{5}} \right| + C$$

iii) We have $\frac{d}{dx}(5-2x-x^2) = -2(x+1)$. Also $x+2 = -\frac{1}{2}(-2(x+1)) + 1$.

$$\int \frac{x+2}{\sqrt{5-2x-x^2}}\,dx = -\frac{1}{2}\int \frac{-2(x+1)}{\sqrt{5-2x-x^2}}dx + \int \frac{dx}{\sqrt{5-2x-x^2}}.$$

As before

$$\int \frac{-2(x+1)}{\sqrt{5-2x-x^2}}dx = 2\sqrt{5-2x-x^2} + C_1.$$

From 21) iii) we have

$$\int \frac{dx}{\sqrt{5-2x-x^2}} = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C_2.$$

Therefore

$$\int \frac{(x+2)}{\sqrt{5-2x-x^2}}dx = -\sqrt{5-2x-x^2} + \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C$$

E22) i) We have

$$\begin{aligned}
&\int \sqrt{3x^2+4x+1}dx \\
&= \int \sqrt{3\left(x^2+\frac{4}{3}x+\frac{1}{3}\right)}dx \\
&= \sqrt{3}\int \sqrt{\left(x+\frac{2}{3}\right)^2 - \frac{1}{9}}\,dx \\
&= \sqrt{3}\int \sqrt{u^2-\frac{1}{9}}du \left(\text{on substituting } u = x+\frac{2}{3}\right) \\
&= \sqrt{3}\left(\frac{u\sqrt{u^2-\frac{1}{9}}}{2} - \frac{\frac{1}{9}}{2}\ln\left|\frac{u+\sqrt{u^2-\frac{1}{9}}}{\frac{1}{3}}\right|\right) + C \\
&= \frac{\left(x+\frac{2}{3}\right)\sqrt{\left(x+\frac{2}{3}\right)^2-\frac{1}{9}}}{2} \\
&\quad - \frac{1}{6\sqrt{3}}\ln\left|3\left(x+\frac{2}{3}\right) + \sqrt{3}\sqrt{3\left(x+\frac{2}{3}\right)^2-\frac{1}{3}}\right| + C \\
&= \frac{(3x+2)\sqrt{3x^2+4x+1}}{6} \\
&\quad - \frac{1}{6\sqrt{3}}\ln\left|3x+2+\sqrt{3}\sqrt{3x^2+4x+1}\right| + C
\end{aligned}$$

ii) We have

$$\begin{aligned}
\int \sqrt{3x^2-4x+3}\,dx &= \sqrt{3}\int \sqrt{\left(x-\frac{2}{3}\right)^2 - \frac{4}{9}+1}\,dx \\
&= \sqrt{3}\int \sqrt{\left(x-\frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2}dx \\
&= \sqrt{3}\left\{\frac{\left(x-\frac{2}{3}\right)\sqrt{\left(x-\frac{2}{3}\right)^2-\frac{5}{9}}}{2}\right.
\end{aligned}$$

$$+\ \frac{1/9}{2}\ln\left|\frac{\left(x-\frac{2}{3}\right)+\sqrt{\left(x-\frac{2}{3}\right)^2+\frac{5}{9}}}{\sqrt{5}/3}\right|\Bigg\} + C$$

$$= \sqrt{3}\left\{\frac{(3x-2)}{6}\sqrt{3x^2-4x+1}\right.$$

$$\left.+\frac{5}{6\sqrt{3}}\ln\left|\frac{3x-2+\sqrt{3}\sqrt{3x^2-4x+1}}{\sqrt{5}}\right|+C\right\}$$

iii) We have

$$\int\sqrt{5-2x-x^2}dx = \int\sqrt{\left(\sqrt{6}\right)^2-(x+1)^2}dx$$

$$= \int\sqrt{\left(\sqrt{6}\right)^2-u^2}\,du \text{ (On substitutingu = x + 1)}$$

$$= \frac{u\sqrt{\left(\sqrt{6}\right)^2-u^2}}{2}+\frac{6}{2}\sin^{-1}\left(\frac{u}{\sqrt{6}}\right)+C$$

$$= \frac{(x+1)\sqrt{5-2x-x^2}}{2}+3\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C.$$

E23) i) Let

$$h(x) = 8+2x-x^2 = A+B(x-2)+C(x-2)^2$$

We have $8 = h(2) = A$. Since $h'(x) = 2-2x$, $-2 = h'(2) = B$. Also, $h''(x) = -2$, so $-2 = h''(2) = 2C$ and $C = -1$. Putting $x-2 = \frac{1}{y}$, the integral becomes

$$\int\frac{-\frac{1}{y^2}}{\frac{1}{y}\sqrt{\frac{8y^2-2y-1}{y^2}}}dy = -\int\frac{dy}{\sqrt{8y^2-2y-1}}$$

$$= -\frac{1}{2\sqrt{2}}\int\frac{dy}{\sqrt{y^2-\frac{y}{4}-\frac{1}{8}}}$$

$$= -\frac{1}{2\sqrt{2}}\int\frac{dy}{\sqrt{\left(y-\frac{1}{8}\right)^2-\frac{1}{64}-\frac{1}{8}}}$$

$$\therefore\int\frac{dx}{(x-2)\sqrt{8+2x-x^2}} = -\frac{1}{2\sqrt{2}}\int\frac{dy}{\sqrt{\left(y-\frac{1}{8}\right)^2-\frac{9}{64}}}$$

$$= \frac{-1}{2\sqrt{2}}\int\frac{dy}{\sqrt{\left(y-\frac{1}{8}\right)^2-\left(\frac{3}{8}\right)^2}}$$

Substituting $u = y-\frac{1}{8}$, we get

$$\int\frac{dx}{(x-2)\sqrt{8+2x-x^2}}$$

$$= -\frac{1}{2\sqrt{2}}\int\frac{du}{\sqrt{u^2-\left(\frac{3}{8}\right)^2}} = -\frac{1}{2\sqrt{2}}\ln\left|\frac{u+\sqrt{u^2-\left(\frac{3}{8}\right)^2}}{3/8}\right|+C$$

$$= -\frac{1}{2\sqrt{2}} \ln \left| \frac{\left(y-\frac{1}{8}\right) + \sqrt{\left(y-\frac{1}{8}\right)^2 - \left(\frac{3}{8}\right)^2}}{3/8} \right| + C$$

$$= -\frac{1}{2\sqrt{2}} \ln \left| \frac{8y-1+2\sqrt{2}\sqrt{8y^2-2y-1}}{3} \right| + C$$

$$= -\frac{1}{2\sqrt{2}} \ln \left| \frac{\frac{8}{x-2} - 1 + 2\sqrt{2}\sqrt{\frac{8}{(x-2)^2} - \frac{2}{(x-2)} - 1}}{3} \right| + C$$

$$= -\frac{1}{2\sqrt{2}} \ln \left| \frac{10-x+2\sqrt{2}\sqrt{8-2(x-2)-(x-2)^2}}{3(x-2)} \right| + C$$

$$= -\frac{1}{2\sqrt{2}} \ln \left| \frac{10-x+2\sqrt{2}\sqrt{8+2x-x^2}}{3(x-2)} \right| + C$$

ii) We have $x^2 + 6x + 10 = (x+2)^2 + 2(x+2) + 2$. Putting $y = \frac{1}{x+2}$, the integral becomes

$$-\int \frac{\frac{1}{y^2}}{\frac{1}{y}\sqrt{\frac{2y^2+2y+1}{y^2}}} dy$$

$$= -\int \frac{dy}{\sqrt{2y^2+2y+1}} = -\frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{y^2+y+\frac{1}{2}}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{(y+\frac{1}{2})^2 - \frac{1}{4} + \frac{1}{2}}} = -\frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{(y+\frac{1}{2})^2 + \frac{1}{4}}}$$

$$= -\frac{1}{\sqrt{2}} \ln \left| \frac{\left(y+\frac{1}{2}\right) + \sqrt{\left(y+\frac{1}{2}\right)^2 + \frac{1}{4}}}{1/2} \right| + C$$

$$= -\frac{1}{\sqrt{2}} \ln \left| 2\left(\frac{2y+1}{2}\right) + \sqrt{2}\sqrt{2\left\{\left(y+\frac{1}{2}\right)^2 + \frac{1}{4}\right\}} \right| + C$$

$$= -\frac{1}{\sqrt{2}} \ln \left| (2y+1) + \sqrt{2}\sqrt{2y^2+2y+1} \right| + C$$

$$= -\frac{1}{\sqrt{2}} \ln \left| \frac{2}{x+2} + 1 + \sqrt{2}\sqrt{\frac{2}{(x+2)^2} + \frac{2}{x+2} + 1} \right| + C$$

$$= -\frac{1}{\sqrt{2}} \ln \left| \frac{x+4+\sqrt{2}\sqrt{x^2+6x+10}}{x+2} \right| + C$$

E24) i) Let $f(x) = 3x^2 + 4x + 1$ Then $f'(x) = 6x + 4$. Writing $2x - 1 = A(6x+4) + B$ and comparing the coefficients of x both sides, $6A = 2$ or $A = \frac{1}{3}$. Comparing the constant terms both sides, we get $-1 = 4A + B$ or $B = -4A - 1 = -\frac{7}{3}$. We can write the integral as

$$\frac{1}{3}\int (6x+4)\sqrt{3x^2+4x+1}\,dx - \frac{7}{3}\int \sqrt{3x^2+4x+1}\,dx.$$

The $\int (6x+4)\sqrt{3x^2+4x+1}\,dx$ is of the form $\int f'(x)\sqrt{f(x)}\,dx$. Putting $u = f(x)$, the integral becomes

$$\int \sqrt{u}\,du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\left(3x^2+4x+1\right)^{3/2} + C_1$$

From Exercise 22) i), we have

$$\int \sqrt{3x^2+4x+1}\,dx = \frac{(3x+2)\sqrt{3x^2+4x+1}}{6} - \frac{1}{6\sqrt{3}} \ln\left|3x+2+\sqrt{3}\sqrt{3x^2+4x+1}\right| + C_1$$

$$\therefore \int (2x-1)\sqrt{3x^2+4x+1}\,dx$$

$$= \frac{2}{9}\left(3x^2+4x+1\right)^{\frac{3}{2}} - \frac{(21x+14)\sqrt{3x^2+4x+1}}{18} + \frac{7}{18\sqrt{3}} \ln\left|3x+2+\sqrt{3}\sqrt{3x^2+4x+1}\right| + +C$$

ii) Taking $f(x) = 5-2x-x^2$ we have $f'(x) = -2-2x$. Writing $x-3 = A(-2-2x)+B$ and comparing the coefficients of x both sides we get $-2A = 1$ or $A = -\frac{1}{2}$. Comparing the constant terms, we get $-2A+B = -3$ or $B = -3+2a = -3-1 = -4$. Therefore we can write the integral as

$$-\frac{1}{2}\int (-2-2x)\sqrt{5-2x-x^2}\,dx - 4\int \sqrt{5-2x-x^2}\,dx$$

The integral $\int (-2x-2)\sqrt{5-2x-x^2}\,dx$ is of the form $\int f'(x)\sqrt{f(x)}\,dx$.

$$\therefore \int (-2-2x)\sqrt{5-2x-x^2}\,dx = \frac{2}{3}\left(5-2x-x^2\right)^{\frac{3}{2}} + C_1.$$

$$\int \sqrt{5-2x-x^2}\,dx = \frac{(x+1)\sqrt{5-2x-x^2}}{2} + 3\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C_2$$

$$\therefore \int (x-3)\sqrt{5-2x-x^2}\,dx = -\frac{1}{3}(5-2x-x^2) - 2(x+1)\sqrt{5-2x-x^2} - 12\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C$$

UNIT 19

REDUCTION FORMULAS

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19.1 INTRODUCTION

In the first two units of this block we have introduced the concept of a definite integral and have obtained the values of integrals of some standard forms. We have also studied two important methods of evaluating integrals, namely, the method of substitution and the method of integration by parts. In the solution of many physical or engineering problems, we have to integrate some integrands involving powers or products of trigonometric functions. In this unit we shall devise a quicker method for evaluating these integrals. We shall consider some standard forms of integrands one by one, and derive formulas to integrate them.

The integrands which we will discuss here have one thing in common. They depend upon an integer parameter. By using the method of integration by parts we shall try to express such an integral in terms of another similar integral with a lower value of the parameter. You will see that by the repeated used of this technique, we shall be able to evaluate the given integral.

In Sec. 19.2, we introduce you to the idea of reduction formula with the help of some examples. In Sec. 19.3, we discuss reduction formulas for powers of trigonometric functions. In Sec. 19.4, we discuss reduction formula for products of powers of sine and cosine functions. We also discuss reduction formulas for integrating $e^{ax} \sin x$ and $e^{ax} \cos x$. Here are the

Objectives

After studying this unit, you should be able to:

- derive and apply the reduction formulas for $\int x^n e^x dx$;

- derive and apply the reduction formulas for $\int \sin^n x\,dx$, $\int \cos^n x\,dx$, $\int \tan^n x\,dx$, etc.;
- derive and apply the reduction formulas for $\int \sin^m x \cos^n x\,dx$;
- derive and apply the reduction formulas for $\int e^{ax} \sin^n x\,dx$;

19.2 REDUCTION FORMULA

Sometimes the integrand is not only a function of the independent variable, but also depends upon a number n (usually an integer). For example, in $\int \sin^n x\,dx$, the integrand $\sin^n x$ depends on x and n. Similarly, in $\int e^x \cos mx\,dx$, the integrand $e^x \cos mx$ depends on x and m. The numbers n and m in these two examples are called parameters. We shall discuss only integer parameters here.

In integrating by parts we sometimes obtain the value of the given integral in terms of another similar integral in which the parameter has a smaller value. Thus, after a number of steps we might arrive at an integrand which can be readily evaluated. Such a process is called the **method of successive reduction**, and a formula connecting an integral with parameter n to a similar integral with a lower value of the parameter, is called a **reduction formula**.

Definition 7: A formula of the form

$$\int f(x, n)dx = g(x) + \int f(x, k)\,dx,$$

where k < n, is called a **reduction formula**.

Consider the following example as an illustration.

Example 1: Derive a reduction formula for $\int x^n e^x\,dx$.

Solution: The integrand in $\int x^n e^x dx$ depends on x and also on the parameter n which is the exponent of x. Let

$$I_n = \int x^n e^x dx.$$

Integrating this by parts, with x^n as the first function and e^x as the second function gives us

$$I_n = x^n \int e^x dx - \int \left(n\,x^{n-1} \int e^x dx \right) dx$$

$$= x^n e^x - n \int x^{n-1} e^x\,dx$$

Note that the integrand in the integral on the right hand side is similar to the one we started with. The only difference is that the exponent of x is $n-1$. Or, we can say that the exponent of x is reduced by 1. Thus, we can write

$$I_n = x^n e^x - n\,I_{n-1} \qquad \ldots(1)$$

* * *

The formula in Eqn. (1) is an example of a reduction formula. Let us now look at an example to see how we use this formula.

Example 2: Evaluate $\int x^4 e^x dx$ using Eqn. (1).

Solution: Notice that $I_4 = \int x^4 e^x dx$. So, using Eqn. (1) we can write

$$I_4 = x^4 e^x - 4I_3$$

$$= x^4e^x - 4[x^3e^x - 3I_2] \text{ using Eqn. (1) for } I_3$$
$$= x^4e^x - 4x^3e^x + 12I_2$$
$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24I_1, \text{ using Eqn. (1) for } I_2$$
$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24I_0.$$

Now $I_0 = \int x^0e^xdx = \int e^xdx = e^x + c.$

Thus,the method of successive reduction gives us

$$\int x^4e^xdx = x^4e^x - 4x^3e^x - 12x^2e^x - 24xe^x + 24e^x + c$$

in five simple steps.

* * *

You may have noted in Example 2 that we were saved from having to integrate by parts four times. This became possible because of Eqn. (1). In this unit we shall derive many such reduction formulas. Here are some more examples.

Example 3: Find a reduction formula for $\int x^m (\ln|x|)^n\, dx$. Use it to find $\int x^3 (\ln|x|)^2\, dx$.

Solution: Let us write

$$I_n = \int x^m (\ln|x|)^n\, dx.. \qquad \ldots(2)$$

Then, taking $\ln(|x|)^n$ as the first factor and x^m as the second factor and integrating by parts, we have

$$I_n = \frac{x^{m+1}}{m+1}(\ln|x|)^n - \frac{1}{m+1}\int x^{m+1}\frac{d}{dx}(\ln|x|)^n\, dx$$
$$= \frac{x^{m+1}}{m+1}(\ln|x|)^n - \frac{1}{m+1}\int x^{m+1} n(\ln|x|)^{n-1}\frac{1}{x}\, dx$$
$$= \frac{x^{m+1}}{m+1}(\ln|x|)^n - \frac{1}{m+1}\int x^m n(\ln|x|)^{n-1}\, dx = \frac{x^{m+1}}{m+1}(\ln|x|)^n - \frac{n}{m+1}I_{n-1}$$

Thus,

$$I_n = \frac{x^{m+1}}{m+1}(\ln|x|)^n - \frac{n}{m+1}I_{n-1},\ m \neq -1 \qquad \ldots(3)$$

Again, note that $\int x^3 (\ln|x|)^2\, dx = I_2$ according to the way we have defined I_n in Eqn. (2). Applying Eqn. (3), we get

$$I_2 = \frac{x^4}{4}(\ln|x|)^2 - \frac{2}{4}I_1 = \frac{x^4}{4}(\ln|x|)^2 - \frac{2}{4}\left(\frac{x^4}{4}\ln|x| - \frac{1}{4}I_0\right)$$
$$= \frac{x^4}{4}\left\{(\ln|x|)^2 - \frac{1}{2}\ln|x|\right\} + \frac{1}{8}I_0$$

We have

$$I_0 = \int x^3\, dx = \frac{x^4}{4} + C$$

Therefore,

$$I_2 = \frac{x^4}{4}\left\{(\ln|x|)^2 - \frac{1}{2}\ln|x| + \frac{1}{8}\right\} + C$$

* * *

Here is another example.

Example 4: Find a reduction formula for $\int x^n(ax+b)^m\,dx$ where $m, n \in \mathbb{Z}$, $m \neq -1$ and $a, b \in \mathbb{R}$. Use it to integrate $\int x^3(2x+5)^{10}\,dx$.

Solution: Here, there are two choices. We can progressively reduce the power of x or the power of (ax + b), depending on which among n and m are smaller. In this example, we will see how to reduce the power of x. Since we want to reduce the power of x term by one, so we choose x^n as the second factor so that when we differentiate x^n the power is reduced by one. We have

$$\begin{aligned}\int (ax+b)^m x^n\,dx &= x^n \int (ax+b)^m\,dx - \int nx^{n-1}\left(\int (ax+b)^m\,dx\right)dx\\ &= x^n\frac{(ax+b)^{m+1}}{a(m+1)} - \frac{n}{a(m+1)}\int x^{n-1}(ax+b)^{m+1}\,dx\end{aligned}$$

Writing $I_{n,m} = \int x^n(ax+b)^m\,dx$, we get

$$I_{n,m} = \frac{1}{a(m+1)}\left(x^n(ax+b)^{m+1} - nI_{n-1,m+1}\right) \qquad \ldots(4)$$

Let us now use this to integrate $\int x^3(2x+5)^{10}\,dx$. Here, n = 3, m = 10 and a = 2. We have

$$\begin{aligned}\int x^3(2x+5)^{10}\,dx = I_{3,10} &= \frac{1}{2\cdot 11}\left(x^3(2x+5)^{11} - 3I_{2,11}\right)\\ I_{2,11} &= \frac{1}{2\cdot 12}\left(x^2(2x+5)^{12} - 2I_{1,12}\right)\\ I_{1,12} &= \frac{1}{2\cdot 13}\left(x(2x+5)^{13} - I_{0,13}\right)\\ I_{0,13} &= \int (2x+5)^{13}\,dx = \frac{(2x+5)^{14}}{28} + C\end{aligned}$$

Therefore,

$$\begin{aligned}\int x^3(2x+5)^{10}\,dx &= \frac{x^3(2x+5)^{11}}{22} - \frac{3}{22}\left\{\frac{1}{24}\left(x^2(2x+5)^{12} - 2I_{1,12}\right)\right\}\\ &= \frac{x^3(2x+5)^{11}}{22} - \frac{x^2(2x+5)^{12}}{176} + \frac{1}{1144}\left(x(2x+5)^{13} - I_{0,13}\right)\\ &= \frac{x^3(2x+5)^{11}}{22} - \frac{x^2(2x+5)^{12}}{176} + \frac{1}{1144}x(2x+5)^{13}\\ &\quad - \frac{1}{32032}(2x+5)^{14} + C\end{aligned}$$

* * *

The fact that parameter m in Eqn. (4) increases by one can be useful! Here is an example that illustrates this.

Example 5: Evaluate the integral $\int \frac{x^5}{(1+x)^3}\,dx$.

Solution: Suppose we use the method we discussed in Unit 18 for integrating rational functions. We have, by long division,

$$x^5 = \left(x^2 - 3x + 6\right)(1+x)^3 - (10x^2 + 15x + 6).$$

This computation itself is tedious. The next step is to write

$$\int \frac{x^5}{(1+x)^3}\,dx = \int \left(x^2 - 3x + 6\right)\,dx - \int \frac{10x^2 + 15x + 6}{(1+x)^3}$$

The first integral is straight forward. For the second integral, we need to use partial fractions.

However, using Eqn. (4), we have

$$\int \frac{x^5}{(1+x)^3}\,dx = I_{5,-3} = \frac{1}{(-3+1)}\left(\frac{x^5}{(1+x)^2} - 5I_{4,-2}\right)$$

$$= -\frac{1}{2}\left[\frac{x^5}{(1+x)^2} - 5\left\{\frac{1}{(-2+1)}\left(\frac{x^4}{(1+x)} - 4I_{3,-1}\right)\right\}\right]$$

$$= -\frac{1}{2}\left(\frac{x^5}{(1+x)^2} + 5\frac{x^4}{1+x} - 20I_{3,-1}\right)$$

$$I_{3,-1} = \int \frac{x^3}{1+x}\,dx = \int \frac{(u-1)^3}{u}\,du \text{ on putting } u = x+1$$

$$= \int \frac{u^3 - 3u^2 + 3u - 1}{u}\,du = \frac{u^3}{3} - 3\frac{u^2}{2} + 3u - \ln|u| + C$$

$$= \frac{(1+x)^3}{3} - 3\frac{(1+x)^2}{2} + 3(1+x) - \ln|1+x| + C$$

$$\therefore \int \frac{x^5}{(1+x)^3}\,dx = -\frac{x^5}{2(1+x)^2} - 5\frac{x^4}{2(1+x)} + 10\frac{(1+x)^3}{3} - 15(1+x)^2$$
$$+ 30(1+x) + 10\ln|1+x| + C$$

* * *

The formulas that we will now discuss will fall into two main categories according as the integrand

i) is a power of trigonometric functions,
ii) is a product of trigonometric functions.

We will take these up in the next two sections. Before we move on to the next section, try the following exercises:

E1) Find a reduction formula for the integral $\int x^n e^{ax}\,dx$. Use it to evaluate the integral $\int x^3 e^{2x}\,dx$.

E2) Evaluate the integral $\int \frac{x^2}{(1+2x)^3}\,dx$.

E3) The reduction formula in Example 4 reduces the power of x by one. Prove a reduction formula that reduces the power of (ax + b) by one. Use it to evaluate the integral $\int x^{10}(3x+1)^2\,dx$.

If you have gone through the discussion carefully, you would have noticed certain pattern in choosing the first function and second function while deriving the reduction formulas. We always choose the function which is a power of another function, such as a polynomial, as the second function because we differentiate the second function and this reduces the power. By repeatedly applying this, we remove this function from the integrand. Hopefully, the function that remains is a simpler function to integrate.

For example, while deriving the reduction formula for $\int x^n e^x\,dx$, we chose x^n as the second function. After applying the reduction formula repeatedly, we arrived $\int e^x\,dx$ which is a simpler integral. With help of this insight, you can recall the derivation of reduction formulas without having to memorise them. You will find this idea applicable in the next section where we discuss reduction formulas for integrands involving powers of trigonometric functions.

19.3 INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS.

There are many occasions when we have to integrate product of a trigonometric function with a polynomial. In this section we shall indicate how to proceed in such cases. Let us look at an example.

Example 6: Find reduction formulas for $\int x^n \sin\alpha x\,dx$ and $\int x^n \cos\alpha x\,dx$ where $n \in \mathbb{N}$, $n \geq 2$ and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Use them to evaluate $\int x^2 \cos 4x\,dx$, and $\int x^2 \sin 6x\,dx$..

Solution: As we mentioned earlier, x^n is the right choice for the second function since the power of x decreases on differentiation. Let us write

$$I_n = \int x^n \cos\alpha x\,dx,\ \ I'_n = \int x^n \sin\alpha x\,dx$$

We have

$$\begin{aligned}
I_n &= \int x^n \cos\alpha x\,dx = x^n \int \cos\alpha x\,dx - \int \frac{d}{dx}x^n \left(\int \cos\alpha x\right)\,dx\\
&= \frac{x^n \sin\alpha x}{\alpha} - \frac{n}{\alpha}\int x^{n-1}\sin\alpha x\,dx\\
&= \frac{x^n \sin\alpha x}{\alpha} - \frac{n}{\alpha} I'_{n-1} && \ldots(5)\\
I'_n &= \int x^n \sin\alpha x\,dx = x^n \int \sin\alpha x\,dx - \int \frac{d}{dx}x^n \left(\int \sin\alpha x\right)\,dx\\
&= -\frac{x^n \cos\alpha x}{\alpha} + \frac{n}{\alpha}\int x^{n-1}\cos\alpha x\,dx = -\frac{x^n\cos\alpha x}{\alpha} + \frac{n}{\alpha} I_{n-1} && \ldots(6)
\end{aligned}$$

Using Eqn. (5), we get

$$\begin{aligned}
I'_n &= -\frac{x^n \cos\alpha x}{\alpha} + \frac{n}{\alpha}\left(\frac{x^{n-1}\sin\alpha x}{\alpha} - \frac{n-1}{\alpha} I'_{n-2}\right)\\
&= -\frac{x^n \cos\alpha x}{\alpha} + \frac{nx^{n-1}\sin\alpha x}{\alpha^2} - \frac{n(n-1)}{\alpha^2} I'_{n-2} && \ldots(7)
\end{aligned}$$

Similarly, using Eqn. (6) in Eqn. (5) we get

$$I_n = \frac{x^n \sin \alpha x}{\alpha} - \frac{n}{\alpha}\left(-\frac{x^{n-1}\cos \alpha x}{\alpha} + \frac{n-1}{\alpha} I_{n-2}\right)$$

$$= \frac{x^n \sin \alpha x}{\alpha} + \frac{n x^{n-1}\cos \alpha x}{\alpha^2} - \frac{n(n-1)}{\alpha^2} I_{n-2} \qquad \ldots (8)$$

Let us now evaluate $\int x^2 \cos 4x\, dx$ using Eqn. (8). We have

$$I_2 = \int x^2 \cos 4x\, dx = \frac{x^2 \sin 4x}{4} + \frac{2x \cos 4x}{16} - \frac{2}{16} I_0$$

$$= \frac{x^2 \sin 4x}{4} + \frac{x \cos 4x}{8} - \frac{2}{16}\int \cos 4x\, dx$$

$$= \frac{x^2 \sin 4x}{4} + \frac{x \cos 4x}{8} - \frac{\sin 4x}{32} + C$$

For $\int x^2 \sin 6x\, dx$, n = 2, $\alpha = 6$. Using Eqn. (7), we have

$$I_2' = \int x^2 \sin 6x\, dx = -\frac{x^2 \cos 6x}{6} + \frac{2x \sin 6x}{36} - \frac{2}{36} I_0'$$

$$= -\frac{x^2 \cos 6x}{6} + \frac{x \sin 6x}{18} - \frac{1}{18}\int \sin 6x\, dx$$

$$= -\frac{x^2 \cos 6x}{6} + \frac{x \sin 6x}{18} + \frac{\cos 6x}{108} + C$$

* * *

You may like to test your understanding of the example by trying this exercise.

E4) Evaluate the following integrals:
i) $\int x^4 \sin 3x\, dx$. ii) $\int x^4 \cos 5x\, dx$.

19.3.1 Reduction Formulas for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$

In this sub-section we will consider integrands which are powers of either sin x or cos x. Before we discuss general powers, recall that we have already seen how to integrate the odd powers of sin x and cos x using substitution in Example 8 in Unit 18. Let us now derive general reduction formulas that work for both even and odd powers of $\sin^n x$ and $\cos^n x$. We start with powers of cos x first. Let us now derive the reduction formula for $\int \cos^n x\, dx$. We write

$$I_n = \int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx, \; n > 1.$$

Integrating this integral by parts we get

$$I_n = \cos^{n-1} x \sin x - \int (n-1)\cos^{n-2} x(-\sin x).\sin x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\,(I_{n-2} - I_n)$$

By rearranging the terms we get

$$I_n = \int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2} \qquad \ldots (9)$$

Although this formula is valid for $n \neq 0$, this formula is really useful only when $n > 0$. When $n \leq 0$ we use the reduction formula for $\int \sec^n x\,dx$ that we discuss later.

Let us now look at an example to see how we apply Eqn. (9) for integrating powers of $\cos x$.

Example 7: Evaluate $\int \cos^6 x\,dx$ using Eqn. (9).

Solution: We have

$$\begin{aligned}
\int \cos^6 x\,dx = I_6 &= \frac{\cos^5 x \sin x}{6} + \frac{5}{6} I_4 \\
&= \frac{\cos^5 x \sin x}{6} + \frac{5}{6}\left(\frac{\cos^3 x \sin x}{4} + \frac{3}{4} I_2\right) \\
&= \frac{\cos^5 x \sin x}{6} + \frac{5}{24}\cos^3 x \sin x + \frac{15}{24}\left(\frac{\cos x \sin x}{2} + \frac{1}{2} I_0\right) \\
&= \frac{\cos^5 x \sin x}{6} + \frac{5}{24}\cos^3 x \sin x + \frac{15}{48}\cos x \sin x + \frac{15}{48} x + C
\end{aligned}$$

* * *

You may like to check your understanding of our discussion so far by trying the next exercise.

E5) Evaluate the following integrals:
i) $\int \cos^7 x\,dx$. ii) $\int \cos^8 x\,dx$.

In the next example, we will derive a reduction formula for integrating $\sin^n x$.

Example 8: For evaluating $\int \sin^n x dx$, we write

$$I'_n = \int \sin^n x\,dx = \int \sin^{n-1} x \ \sin x \ dx, \text{ if } n \in \mathbb{Z},\ n \neq 0.$$

Taking $\sin^{n-1} x$ as the first function and $\sin x$ as the second and integrating by parts, we get

$$\begin{aligned}
I'_n &= -\sin^{n-1} x \cos x - (n-1)\int \sin^{n-2} x \cos x(-\cos x)\,dx \\
&= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx \\
&= -\sin^{n-1} x \cos x + (n-1)\left[\int \sin^{n-2} x(1-\sin^2 x)\,dx\right] \\
&= -\sin^{n-1} x \cos x + (n-1)\left[\int \sin^{n-2} x\,dx - \int \sin^n x\,dx\right] \\
&= -\sin^{n-1} x \cos x + (n-1)\,[I'_{n-2} - I'_n]
\end{aligned}$$

Hence,

$$I'_n + (n-1)I'_n = -\sin^{n-1} x \cos x + (n-1)I'_{n-2}.$$

That is,

$$nI'_n = -\sin^{n-1} x \cos x + (n-1)I'_{n-2}. \text{ Or,}$$

$$I'_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I'_{n-2} \qquad \ldots (10)$$

This is the reduction formula for $\int \sin^n x\, dx$ (valid for $n \in \mathbb{Z}$, $n \neq 0$).

* * *

Here is an example to help you understand the application of Eqn. (10)

Example 9: Evaluate $\int \sin^6 x\, dx$ using Eqn. (10).

Solution: Using Eqn. (10) we have

$$\begin{aligned}\int \sin^6 x\, dx = I_6 &= -\frac{\sin^5 x \cos x}{6} + \frac{5}{6} I'_4 \\ &= -\frac{\sin^5 x \cos x}{6} + \frac{5}{6}\left(-\frac{\sin^3 x \cos x}{4} + \frac{3}{4} I'_2\right) \\ &= -\frac{\sin^5 x \cos x}{6} - \frac{5}{24}\sin^3 x \cos x + \frac{15}{24}\left(-\frac{\sin x \cos x}{2} + \frac{1}{2} I'_0\right) \\ &= -\frac{\sin^5 x \cos x}{6} - \frac{5}{24}\sin^3 x \cos x - \frac{15}{48}\sin x \cos x + \frac{15}{48}x + C\end{aligned}$$

* * *

Try the following exercise to check your understanding of the above example.

E6) Evaluate the following integrals: i) $\int \sin^7 x\, dx$. ii) $\int \sin^8 x\, dx$.

Let us derive reduction formulas for $\int_0^{\frac{\pi}{2}} \sin^n x\, dx$ and $\int_0^{\frac{\pi}{e}} \cos^n x\, dx$. A natural question you may have is 'Why can't we use the formula we derived in to find the primitives for these integrands using and apply the Fundamental Theorem of Calculus?'. To answer this question, before we take up the general case, let us look at a particular case.

Example 10: Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\, dx$.

Solution: We can carry out computation that we did in Example 9 to get

$$\int \sin^6 x\, dx = -\frac{\sin^5 x \cos x}{6} - \frac{5}{24}\sin^3 x \cos x - \frac{15}{48}\sin x \cos x + \frac{15}{48}x + C$$

Notice that, since $\sin 0 = 0$ and $\cos\frac{\pi}{2} = 0$ all the expressions of the form $\sin^p x \cos^q x$, p, q > 0, vanish at 0 as well as $\frac{\pi}{2}$.

$$\therefore \int_0^{\frac{\pi}{2}} \sin^6 x\,dx = \left\{\left(\frac{15}{48}\right)\left(\frac{\pi}{2}\right) - \left(\frac{15}{48}\right)\cdot 0\right\} = \frac{5\pi}{16}$$

As we saw, $\sin^p x \cos^q x$, p, q > 0 vanishes at both the upper and lower limits of the integration, namely $\frac{\pi}{2}$ and 0, respectively. Let us see if we can use this fact to simplify the computation of the definite integral $\int_0^{\frac{\pi}{2}} \sin^6 x\,dx$. We first observe that

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \left.\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx,\ \ n \geq 2 \qquad \ldots(11)$$

Applying Eqn. (11) repeatedly, we get

$$\int_0^{\frac{\pi}{2}} \sin^6 x\,dx = \frac{5}{6}\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} dx = \frac{5}{16}\pi$$

* * *

Example 11: Derive reduction formulas for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ and $\int \cos^n x\,dx$.

Solution: As we saw in Eqn. (11),

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\ dx,\ \ n \geq 2$$

Using this formula repeatedly we get

$$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin x\,dx & \text{if n an odd number } n \geq 3 \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} dx & \text{if n is an even number } n \geq 2 \end{cases}$$

This means

$$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3} & \text{if n an odd number } n \geq 3 \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\,\frac{\pi}{2} & \text{if n is an even number } n \geq 2 \end{cases}$$

We can reverse the order of the factors, and write this as

$$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{2}{3}\frac{4}{5}\cdots\frac{n-3}{n-2}\,\frac{n-1}{n} = \frac{2^{2k}(k!)^2}{(2k+1)!} & \text{n is odd, } n = 2k+1, n \geq 3 \\ \frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}\,\frac{\pi}{2} = \frac{(2k)!}{2^{2k}(k!)^2} & \text{n is even, } n = 2k,\ n \geq 2 \end{cases} \quad \ldots (12)$$

Making the substitution $x = \frac{\pi}{2} - u$, we get $du = -dx$. When $x = 0$, $u = \frac{\pi}{2}$. When $x = \frac{\pi}{2}$, $u = 0$. Therefore,

John Wallis (1616–1703)

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = -\int_{\frac{\pi}{2}}^{0} \sin^n\left(\frac{\pi}{2} - u\right) du = \int_0^{\frac{\pi}{2}} \cos^n u\,du.$$

So, we have

$$\int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{2}{3}\frac{4}{5}\cdots\frac{n-3}{n-2}\,\frac{n-1}{n} = \frac{2^{2k}(k!)^2}{(2k+1)!} & \text{n is odd, } n = 2k+1, n \geq 3 \\ \frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}\,\frac{\pi}{2} = \frac{(2k)!}{2^{2k}(k!)^2}\frac{\pi}{2} & \text{n is even, } n = 2k,\ n \geq 2 \end{cases} \quad \ldots (13)$$

The formulas Eqn. (12) and Eqn. (13) are known as Wallis' formulas after the English Mathematician John Wallis.

* * *

Let us now look at an example to see how we apply these formulas.

Example 12: Evaluate the following integrals:

i) $\int_0^{\frac{\pi}{2}} \sin^8 x\,dx$ ii) $\int_0^{\frac{\pi}{2}} \cos^7 x\,dx$

Solution:

i) We have

$$\int_0^{\frac{\pi}{2}} \sin^8 x\,dx = \frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\frac{\pi}{2} = \frac{35}{256}\pi$$

ii) We have

$$\int_0^{\frac{\pi}{2}} \cos^7 x\,dx = \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3\cdot 1} = \frac{16}{35}$$

* * *

E7) Evaluate i) $\int_0^{\pi/2} \cos^9 x\,dx$ ii) $\int_0^{\pi/2} \sin^6 x\,dx$

19.3.2 Reduction Formulas for $\int \tan^n x\, dx$ and $\int \sec^n x\, dx$

In this sub-section we will take up two other trigonometric functions: $\tan x$ and $\sec x$. That is, we will derive the reduction formulas for $\int \tan^n x\, dx$ and $\int \sec^n x\, dx$. To derive a reduction formula for $\int \tan^n x\, dx$, $n \in \mathbb{Z}$, we start in a slightly different manner.

Instead of writing $\tan^n x = \tan x \tan^{n-1} x$, as we did in the case of $\sin^n x$ and $\cos^n x$, we shall write $\tan^n x = \tan^{n-2} x \tan^2 x$. You will shortly, see the reason behind this. So, we write

$$
\begin{aligned}
I_n = \int \tan^n x\, dx &= \int \tan^{n-2} x \tan^2 x\, dx. \\
&= \int \tan^{n-2} x(\sec^2 x - 1)\, dx \\
&= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx \qquad \ldots (14)
\end{aligned}
$$

You must have observed that the second integral on the right hand side is I_{n-2}. Now in the first integral on the right hand side, the integrand is of the form $[f(x)]^m.f'(x)$ As we have seen in Unit 18,

$$\int [f(x)]^m f'(x) dx = \frac{[f(x)]^{m+1}}{m+1} + c$$

Thus,

$$\int \tan^{n-2} x \sec^2 x\, dx = \frac{\tan^{n-1} x}{n-1} + C$$

provided $n \neq 1$. Therefore, Eqn. (14) gives $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

Thus the reduction formula for $\int \tan^n x\, dx$ is

$$\int \tan^n x\, dx = I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}, \text{ if } n \neq 1 \qquad \ldots (15)$$

We already know that, if $n = 1$, $\int \tan x\, dx = \ln|\sec x| + C$

To derive the reduction formula for $\int \sec^n x\, dx$, $(n > 2)$, we first write $\sec^n x = \sec^{n-2} x \sec^2 x$, and then integrate by parts. Thus,

$$
\begin{aligned}
I_n = \int \sec^n x\, dx &= \int \sec^{n-2} x \sec^2 x\, dx \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-3} x \sec x \tan^2 x\, dx \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\, dx \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x(\sec^2 x - 1) dx \\
&= \sec^{n-2} x \tan x - (n-2)\, (I_n - I_{n-2})
\end{aligned}
$$

After rearranging the terms we get

$$\int \sec^n x\, dx = I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2} \text{ if } n \neq 1 \qquad \ldots (16)$$

We know already that if $n = 1$, $\int \sec x\, dx = \ln|\sec x + \tan x| + C$.

Let us now look at some examples.

Example 13: Evaluate the following integrals:

i) $\int_0^{\pi/4} \tan^5 x\, dx$ ii) $\int_0^{\pi/4} \sec^6 x\, dx$.

Solution:

i) We have

$$\int_0^{\pi/4} \tan^5 x\,dx = \left.\frac{\tan^4 x}{4}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^3 x\,dx = \frac{1}{4} - \left.\frac{\tan^2 x}{2}\right]_0^{\pi/4} + \int_0^{\pi/4} \tan x\,dx$$

$$= \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \frac{\sin x}{\cos x}dx = -\frac{1}{4} - \ln|\cos x|\Big]_0^{\pi/4}$$

$$= -\frac{1}{4} - \ln\frac{1}{\sqrt{2}} + \ln 1 = -\frac{1}{4} + \ln\sqrt{2}$$

ii) We have

$$\int_0^{\pi/4} \sec^6 x\,dx = \left.\frac{\sec^4 x \tan x}{5}\right]_0^{\pi/4} + \frac{4}{5}\int_0^{\pi/4} \sec^4 x\,dx$$

$$= \frac{4}{5} + \frac{4}{5}\left\{\left.\frac{\sec^2 x \tan x}{3}\right]_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sec^2 x\,dx\right\}$$

$$= \frac{4}{5} + \frac{8}{15} + \frac{8}{15}\int_0^{\pi/4} \sec^2 x\,dx = \frac{4}{3} + \frac{8}{15}\tan x\Big]_0^{\pi/4} = \frac{28}{15}$$

* * *

On the basis of our discussion in this section you will be able to solve these exercises.

E8) Derive the following reduction formulas for $\int \cot^n x\,dx$ and $\int \operatorname{cosec}^n x\,dx$:

i) $\int \cot^n x\,dx = I_n = -\frac{1}{n-1}\cot^{n-1} x - I_{n-2}$

ii) $\int \operatorname{cosec}^n x\,dx = I_n = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

E9) Evaluate the following integrals:

i) $\int_{\pi/4}^{\pi/2} \operatorname{cosec}^3 x\,dx$ ii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4 x\,dx$ iii) $\int_0^{\frac{\pi}{4}} \tan^6 x\,dx$ iv) $\int_0^{\frac{\pi}{4}} \sec^5 x\,dx$

We conclude this section here. In the next section, we will discuss the problem of integrating power of cos and sin functions.

19.4 INTEGRALS INVOLVING PRODUCTS OF TRIGONOMETRIC FUNCTIONS

In the last section we have seen the reduction formulas for the case where integrands were powers of a single trigonometric function. Here we shall consider some integrands involving products of powers of trigonometric functions. The technique of finding a reduction formula basically involves integration by parts. Since there can be more than one way of writing the integrand as a product of two functions, you will see that we can have many reduction formulas for the same integral. We start with the first one of the two types of integrands which we shall study in this section.

19.4.1 Integrand of the Type $\sin^m x \cos^n x$

In Unit 18, example 8, we saw that we can integrate the powers of the sin and cos functions and products of powers of cos and sin functions using substitution if the power is odd. In this subsection, we will consider the general case. First, we derive reduction formulae for integrals involving products of powers of the sin and cos functions.

The function $\sin^m x \cos^n x$ depends on two parameters m and n. To find a reduction formula for $\int \sin^m x \cos^n x \, dx$, let us first write

$$I_{m,n} = \int \sin^m x \cos^n x \, dx.$$

Let us now discuss three forms of the reduction formulae.

Theorem 3: We have

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2} \text{ if } m \neq -1,\ m+n \neq 0. \qquad \ldots(17)$$

$$I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \text{ if } n \neq -1,\ m+n \neq 0. \qquad \ldots(18)$$

$$I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n+2} \text{ if } n \neq -1 \qquad \ldots(19)$$

You are probably wondering why we need three different reduction formulas. Before we proceed with the derivation of these formulas, let us see why.

We use Eqn. (17) if n < m and we want to reduce the power of $\cos x$. Using this, we can reduce $I_{m,n}$ to $I_{m,1}$ if n is odd and to the integral $\int \sin^m x \, dx$ if n is even. If n = 1,

$$I_{m,1} = \int \sin^m x \cos x \, dx = \begin{cases} \frac{\sin^{m+1} x}{m+1} + C & \text{if } m \neq -1 \\ \ln|\sin x| + C & \text{if } m = -1 \end{cases} \qquad \ldots(20)$$

We use Eqn. (18) if n > m and we want to reduce the power of $\sin x$. In this case we end up with $\int \sin x \cos^n x \, dx$ if m is odd and $\int \cos^n x \, dx$ if m is even. If m = 1,

$$I_{1,n} = \int \sin x \cos^n x \, dx = \begin{cases} -\frac{\cos^{n+1} x}{n+1} + C & \text{if } n \neq -1 \\ \ln|\operatorname{cosec} x| + C & \text{if } n = -1 \end{cases} \qquad \ldots(21)$$

We use Eqn. (19) to get rid of negative powers of $\cos x$. For example, consider the integral $\int \sin^4 x \tan^4 x \, dx$. We can write this in the form $\int \sin^8 x \cos^{-4} x \, dx$ and apply Eqn. (19) twice to reduce the problem to finding $\int \sin^4 x \, dx$. We already know how to handle this integral.

Proof of Theorem 3: We derive Eqn. (17) first.

$$I_{m,n} = \int \sin^m x \cos^n x \, dx = \int \cos^{n-1} x (\sin^m x \cos x)\, dx$$

Integrating by parts we get

$$\begin{aligned} I_{m,n} &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} \\ &\quad - \int (n-1)\cos^{n-2} x(-\sin x)\frac{\sin^{m+1} x}{m+1} dx, \text{ if } m \neq -1 \\ &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x \left(1-\cos^2 x\right) dx \\ &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\left(I_{m,n-2} - I_{m,n}\right) \end{aligned}$$

Rearranging the terms, we get

$$\frac{m+n}{m+1} I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

This gives us

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}, \text{ if } m \neq -1, m+n \neq 0.$$

which is Eqn. (17).

To derive Eqn. (18), we write

$$I_{m,n} = \int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x (\cos^n x \sin x)\, dx.$$

Integrating this by parts we get

$$\begin{aligned} I_{m,n} &= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} \\ &\quad - (m-1)\int \sin^{m-2} x \cos x \frac{(-\cos^{n+1} x)}{n+1} dx \text{ for } n \neq -1. \\ &= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x (1-\sin^2 x)\, dx \\ &= \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\left(I_{m-2,n} - I_{m,n}\right) \end{aligned}$$

Rearranging the terms,

$$\frac{m+n}{n+1} I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + I_{m-2,n}$$

From this we obtain

$$I_{m,n} = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \text{ if } m+n \neq 0, n \neq -1.$$

For deriving Eqn. (19) we write

$$\begin{aligned} \int \sin^m x \cos^n x \, dx &= -\int \sin^{m-1} x \cos^n x (-\sin x)\, dx \\ &= -\left(\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} \right. \\ &\quad \left. - \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x \, dx\right) \text{ if } n \neq -1. \end{aligned}$$

$$I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n+2}, \text{ if } n \neq -1.$$

■

Note that, if $m + n = 0$ we can use the reduction formula for $\int \tan^m x\,dx$ if $m > 0$ and the reduction formula for $\int \cot^n x\,dx$ if $n > 0$.

Let us now look at an example to understand the application of Theorem 3.

Example 14: Evaluate the following integrals:

i) $\int \sin^4 x \cos^8 x\,dx$ ii) $\int \sin^8 x \cos^4 x\,dx$ iii) $\int \sin^4 x \tan^4 x\,dx$

Solution:

i) Applying Eqn. (18), we get

$$\int \sin^4 x \cos^8 x\,dx = I_{4,8} = -\frac{1}{12}\sin^3 x \cos^9 x + \frac{3}{12} I_{2,8}$$
$$= -\frac{1}{12}\sin^3 x \cos^9 x + \frac{3}{12}\left(-\frac{1}{10}\sin x \cos^9 x + \frac{1}{10} I_{0,8}\right)$$
$$= -\frac{1}{12}\sin^3 x \cos^9 x - \frac{1}{40}\sin x \cos^9 x + \frac{1}{40}\int \cos^8 x\,dx$$

From Exercise 5)b), we have

$$\int \cos^8 x\,dx = \frac{\cos^7 x \sin x}{8} + \frac{7\cos^5 x \sin x}{48} + \frac{35\cos^3 x \sin x}{192} + \frac{105}{384}\cos x \sin x + \frac{105}{384}x + C$$

$$\therefore \int \sin^4 x \cos^8 x\,dx = -\frac{1}{12}\sin^3 x \cos^9 x - \frac{1}{40}\sin x \cos^9 x + \frac{1}{320}\sin x \cos^7 x + \frac{7}{1920}\sin x \cos^5 x + \frac{7}{1536}\sin x \cos^3 x + \frac{21}{3072}\sin x \cos x + \frac{21}{3072}x + C$$

ii) We have

$$\int \sin^8 x \cos^4 x\,dx = \frac{1}{12}\sin^9 x \cos^3 x + \frac{3}{12} I_{8,2}$$
$$= \frac{1}{12}\sin^9 x \cos^3 x + \frac{3}{12}\left(\frac{1}{10}\cos x \sin^9 x + \frac{1}{10} I_{8,0}\right)$$
$$= \frac{1}{12}\cos^3 x \sin^9 x + \frac{1}{40}\cos x \sin^9 x + \frac{1}{40}\int \sin^8 x\,dx$$

From Exercise 6, we get

$$\int \sin^8 x\,dx = -\frac{1}{8}\sin^7 x \cos x - \frac{7}{48}\sin^5 x \cos x - \frac{35}{192}\sin^3 x \cos x - \frac{105}{384}\sin x \cos x + \frac{105}{384}x + C$$

$$\int \sin^8 x \cos x\,dx = \frac{1}{12}\cos^3 x \sin^9 x + \frac{1}{40}\cos x \sin^9 x - \frac{1}{320}\sin^7 x \cos x - \frac{7}{1920}\sin^5 x \cos x - \frac{7}{1536}\sin^3 x \cos x - \frac{21}{3072}\sin x \cos x - \frac{21}{3072}x + C$$

iii) We have

$$\int \sin^4 x \tan^4 x\,dx = \int \sin^4 x \sin^4 x(\cos x)^{-4}\,dx = \int \sin^8 x(\cos x)^{-4}\,dx$$

Here, the magnitude of the power of $\cos x$ is smaller. The power of $\cos x$ is negative and Eqn. (19) actually **adds** two to the power of $\cos x$ term. So, we can get rid of the cos term using Eqn. (19).

$$\begin{aligned}\int \sin^4 x \tan^4 x\,dx &= -\frac{1}{(-3)}\sin^7 x(\cos x)^{-3} + \frac{7}{(-3)}I_{6,-2}\\ &= \frac{1}{3}\sin^7 x(\cos x)^{-3} - \frac{7}{3}\left(-\frac{1}{(-1)}\sin^5 x(\cos x)^{-1}\right.\\ &\qquad \left. + \frac{5}{-1}I_{4,0}\right)\\ &= \frac{1}{3}\sin^7 x(\cos x)^{-3} - \frac{7}{3}\sin^5 x(\cos x)^{-1} + \frac{35}{3}I_{4,0}\end{aligned}$$

Using Eqn. (10), we get

$$\begin{aligned}\int \sin^4 x\,dx &= -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx\\ &= -\frac{1}{4}\sin^3 x\cos x - \frac{3}{4}\left\{\frac{1}{2}\left(-\sin x\cos x + \int dx\right)\right\}\\ &= -\frac{1}{4}\sin^3 x\cos x + \frac{3}{8}\sin x\cos x - \frac{3}{8}x + C\end{aligned}$$

$$\begin{aligned}\therefore \int \sin^4 x\tan^4 x\,dx &= \frac{1}{3}\sin^7 x(\cos x)^{-3} - \frac{7}{3}\sin^5 x(\cos x)^{-1}\\ &\quad - \frac{35}{12}\sin^3 x\cos x - \frac{105}{24}\sin x\cos x + \frac{105}{24}x + C\end{aligned}$$

* * *

E10) Evaluate the integrals:

i) $\int \sin^4 x\cos^6 x\,dx$. ii) $\int \sin^6 x\cos^4 x\,dx$. iii) $\int \sin^4 x\tan^2 x\,dx$.

iv) $\int \sec x\,\mathrm{cosec}\,x\,dx$

19.4.2 Integrand of the Type $e^a x \sin^n x$

In this sub-section we will consider the evaluation of those integrals, where the integrand is a product of a power of a trigonometric function and an exponential function. That is, we will consider integrands of the type $e^{ax}\sin^n x$.

Example 15: Derive the reduction formulae for integrating $\int e^{ax}\sin^n x\,dx$.

Solution: Let us denote $\int e^{ax}\sin^n x\,dx$ by L_n, and integrate it by parts, taking $\sin^n x$ as the first function and e^{ax} as the second function. This gives us

$$L_n = \frac{1}{a}e^{ax}\sin^n x - \frac{n}{a}\int e^{ax}\sin^{n-1} x\cos x\,dx.$$

We shall now evaluate the integral on the right hand side, again by parts, with $\sin^{n-1} x\cos x$ as the first function and e^{ax} as the second one. We get

$$\begin{aligned}\int e^{ax}\sin^{n-1} x\cos x\,dx &= \frac{e^{ax}}{a}\sin^{n-1}\cos x\\ &\quad - \frac{1}{a}\int e^{ax}\left((n-1)\sin^{n-2} x\cos^2 x - \sin^n x\right)\,dx\end{aligned}$$

$$= \frac{e^{ax}}{a}\sin^{n-1}\cos x - \frac{n-1}{a}\int e^{ax}\sin^{n-2}x\left(1-\sin^2 x\right)\,dx$$
$$+\frac{1}{a}\int e^{ax}\sin^n x\,dx$$
$$= \frac{e^{ax}}{a}\sin^{n-1}x\cos x - \frac{n-1}{a}\int e^{ax}\sin^{n-2}x\,dx$$
$$+\frac{n-1}{a}\int e^{ax}\sin^n x\,dx + \frac{1}{a}\int e^{ax}\sin^n x\,dx$$
$$= \frac{e^{ax}}{a}\sin^{n-1}x\cos x + \frac{n-1}{a}L_{n-2} + \frac{n-1}{a}L_n + \frac{1}{a}L_n$$
$$= \frac{e^{ax}}{a}\sin^{n-1}\cos x + \frac{n-1}{a}L_{n-2} + \frac{n}{a}L_n$$
$$\therefore\ L_n = \frac{1}{a}e^{ax}\sin^n x - \frac{n}{a}\left(\frac{e^{ax}}{a}\sin^{n-1}\cos x\right.$$
$$\left.+\frac{n-1}{a}L_{n-2} + \frac{n}{a}L_n\right)$$

This means

$$L_n = \frac{e^{ax}}{a}\sin^n x - \frac{ne^{ax}}{a^2}\sin^{n-1}x\cos x + \frac{n(n-1)}{a^2}L_{n-2} - \frac{n^2}{a^2}L_n$$

Rearranging the terms,

$$\left(1+\frac{n^2}{a^2}\right)L_n = (a\sin x - n\cos x)\,\frac{e^{ax}\sin^{n-1}x}{a^2} + \frac{n(n-1)}{a^2}L_{n-2}$$

Multiplying both sides of this equation by $\frac{a^2}{n^2+a^2}$, we get

$$L_n = (a\sin x - n\cos x)\,\frac{e^{ax}\sin^{n-1}x}{n^2+a^2} + \frac{n(n-1)}{n^2+a^2}L_{n-2}. \qquad \ldots(22)$$

Given any L_n, we use this reduction formula repeatedly, till we get L_1 or L_0 (depending on whether n is odd or even). As we will see, L_0 and L_1 is easy to evaluate using the methods we developed in Unit 18. This means that L_n can be evaluated for any positive integer n.

* * *

Let us now look at an example to see how we can use the reduction formula that we just derived.

Example 16: Evaluate the following integrals:
i) $\int e^{3x}\sin^4 x\,dx$ ii) $\int e^{2x}\sin^5 x\,dx$

Solution:

i) We apply Eqn. (22) with a = 3, n = 4 to get

$$\int e^{3x}\sin^4 x\,dx = L_4 = \frac{(3\sin x - 4\cos x)e^{3x}\sin^3 x}{16+9} + \frac{4(4-1)}{16+9}L_2$$

Applying Eqn. (22) with n = 2, a = 3, we get

$$L_2 = \frac{(3\sin x - 2\cos x)e^{3x}\sin x}{4+9} + \frac{2}{4+9}L_0$$

$$L_0 = \int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$$

$$\therefore\ L_2 = \frac{(3\sin x - 2\cos x)e^{3x}\sin x}{13} + \frac{2}{39}e^{3x} + C$$

$$\therefore L_4 = \frac{(3\sin x - 4\cos x)e^{3x}\sin^3 x}{25} + \frac{12(3\sin x - 2\cos x)e^{3x}\sin x}{325} + \frac{8}{325} + C$$

ii) Applying Eqn. (22) repeatedly, we have

$$\int e^{2x}\sin^5 x\,dx = L_5 = \frac{(2\sin x - 5\cos x)e^{2x}\sin^4 x}{29} + \frac{20}{29}L_3$$

$$L_3 = \frac{(2\sin x - 3\cos x)e^{2x}\sin^2 x}{13} + \frac{6}{13}L_1$$

From Unit 18, Eqn.(24), we have

$$\int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx) + C$$

$$\therefore\ L_1 = \frac{1}{1+4}e^{2x}(2\sin x - \cos x) + C$$

$$\therefore\ L_3 = \frac{(2\sin x - 3\cos x)e^{2x}\sin^2 x}{13} + \frac{6}{13}\left(\frac{e^{2x}}{5}(2\sin x - \cos x)\right) + C$$

$$\therefore\ L_5 = \frac{(2\sin x - 5\cos x)e^{2x}\sin^4 x}{29} + \frac{20}{29}\left\{\frac{(2\sin x - 3\cos x)e^{2x}\sin^2 x}{13} + \frac{6e^{2x}}{65}(2\sin x - \cos x)\right\} + C$$

$$= \frac{(2\sin x - 5\cos x)e^{2x}\sin^4 x}{29} + \frac{20(2\sin x - 3\cos x)e^{2x}\sin^2 x}{377} + \frac{24}{377}(2\sin x - \cos x) + C$$

* * *

Remark 1: If we put a = 0 in L_n it reduces to the integral $\int \sin^n x\,dx$. This suggests that the reduction formula for $\int \sin^n x\,dx$ which we have derived in Sec. 19.3 is a special case of reduction formula for L_n.

If you have followed the arguments in this sub-section closely, you should be able to do the exercises below.

E11) Evaluate the following integrals:

i) $\int e^{4x}\sin^3 x\,dx$ ii) $\int e^{5x}\sin^2 x\,dx$

E12) Prove: if $C_n = \int e^{ax}\cos^n x\,dx$, then

$$C_n = \frac{(a\cos x + n\sin x)e^{ax}\cos^{n-1}x}{n^2+a^2} + \frac{n(n-1)}{n^2+a^2}C_{n-2}$$

E13) Evaluate the following integrals:

i) $\int e^{3x}\cos^2 x\,dx$ ii) $\int e^{2x}\cos^3 x\,dx$

E14) Verify that the reduction formula for $\int \cos^n x\,dx$ is a special case of the formula in Exercise 12.

That brings us to the end of this unit. We shall now summarise what we have covered in it.

19.5 SUMMARY

A reduction formula is one which links an integral dependent on a parameter with a similar integral with a lower value of the parameter.

In this unit we have derived a number of reduction formulas.

1) $\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx.$

2) $\int \sin^n x\,dx = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx,\ n \geq 2.$

3) $\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx,\ n \geq 2.$

4) $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx,\ n > 2.$

5) $\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx,\ n > 2.$

6) We have

$$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \begin{cases}\frac{2}{3}\,\frac{4}{5}\cdots\frac{n-1}{n} & \text{if n is odd, } n \geq 3\\ \frac{1}{2}\,\frac{3}{4}\cdots\frac{n-1}{n}\,\frac{\pi}{2}, & \text{if n is even, } n \geq 2\end{cases}$$

7) If $n \neq -1$ and $m + n \neq 0$, we have

$$\int \sin^m x\cos^n x\,dx = \frac{\cos^{n-1}x\sin^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx,$$

$$= \frac{-\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx,\ m > 1$$

If $n \neq -1$, we have

$$\int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}I_{m-2,n+2},\ \text{if } n \neq -1.$$

8) $$\int e^{ax}\sin^n x\,dx = \frac{ae^{ax}\sin^n x}{n^2+a^2} - \frac{ne^{ax}\sin^{n-1}x\cos x}{n^2+a^2} + \frac{n(n-1)}{n^2+a^2}\int e^{ax}\sin^{n-2}x\,dx$$

We have noted that the prime technique of deriving reduction formulas involves integration by parts. We have also observed that many more reduction formulas involving other trigonometric and hyperbolic functions can be derived using the same technique.

19.6 SOLUTIONS/ANSWERS

E1) Proceeding exactly as in Example 1, we get

$$\int x^n e^{ax}\,dx = x^n \int e^{ax}\,dx - \int \left(nx^{n-1}\int e^{ax}\,dx\right)dx$$
$$= x^n\left(\frac{e^{ax}}{a}\right) - n\int x^{n-1}\left(\frac{e^{ax}}{a}\right)dx.$$

Writing $I_n = \int x^n e^{ax}\,dx$, we get

$$I_n = \frac{1}{a}\left(x^n e^{ax} - nI_{n-1}\right)$$

We have

$$I_3 = \frac{1}{2}\left(x^3e^{2x} - 3I_2\right)$$
$$I_2 = \frac{1}{2}\left(x^2e^{2x} - 2I_1\right)$$
$$I_1 = \frac{1}{2}\left(xe^{x} - I_0\right)$$
$$I_0 = \int e^{2x}\,dx = \frac{e^{2x}}{2} + C$$
$$\therefore\ I_3 = \frac{x^3e^{2x}}{2} - \frac{3}{2}\left\{\frac{1}{2}\left(x^2e^{2x} - 2I_1\right)\right\}$$
$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + 3\left\{\frac{1}{2}\left(xe^{2x} - I_0\right)\right\}$$
$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3xe^{2x}}{2} - \frac{3e^{2x}}{4} + C$$

E2) We have

$$\int \frac{x^2}{(1+2x)^3}\,dx = I_{2,-3} = \frac{1}{2(-3+1)}\left(\frac{x^2}{(1+2x)^2} - 2I_{1,-2}\right)$$
$$= -\frac{1}{4}\left[\frac{x}{(1+2x)^2} - 2\left\{\frac{1}{(-2+1)}\left(\frac{x}{1+2x} - I_{0,-1}\right)\right\}\right]$$
$$= -\frac{1}{4}\left\{\frac{x}{(1+2x)^2} + \frac{x}{1+2x} + 2\int \frac{dx}{1+2x}\right\}$$
$$= -\frac{1}{4}\left\{\frac{x}{(1+2x)^2} + \frac{x}{1+2x} + \ln|1+2x| + C\right\}$$

E3) We want to reduce the power of $(ax+b)$ by one so we choose $(ax+b)^m$ as the second factor, so that when we differentiate it, the power is reduced by one. We assume that $m, n \in \mathbb{Z}$, $n \neq -1$. We have

$$\int x^n(ax+b)^m = (ax+b)^m \int x^n\,dx - \int \frac{d}{dx}(ax+b)^m\left(\int x^n\,dx\right)dx$$
$$= \frac{x^{n+1}(ax+b)^m}{n+1} - \frac{am}{n+1}I_{n+1,m-1}$$

Therefore,

$$I_{n,m} = \frac{x^{n+1}(ax+b)^m}{n+1} - \frac{am}{n+1} I_{n+1,m-1} \qquad \ldots (23)$$

Using this formula, we have

$$\int x^{10}(3x+1)^2\,dx = I_{10,2} = \frac{(3x+1)^2 x^{11}}{11} - \frac{6}{11} I_{11,1}$$

$$I_{11,1} = \int x^{11}(3x+1)\,dx = 3\int x^{12}\,dx + \int x^{11}\,dx$$

$$= \frac{3x^{13}}{13} + \frac{x^{12}}{12} + C$$

$$\therefore\ I_{10,2} = \frac{(3x+1)^2 x^{11}}{11} - \frac{18x^{13}}{143} - \frac{6x^{12}}{132} + C$$

E4) i) We have

$$\int x^4 \sin 3x\,dx = I_4' = -\frac{x^4\cos 3x}{3} + \frac{4x^3 \sin 3x}{9} - \frac{12}{9} I_2'$$

$$= -\frac{x^4\cos 3x}{3} + \frac{4x^3 \sin 3x}{9}$$

$$- \frac{4}{3}\left(-\frac{x^2\cos 3x}{3} + \frac{2x\sin 3x}{9} - \frac{2}{9} I_0'\right)$$

$$= \frac{-x^4\cos 3x}{3} + \frac{4x^3\sin 3x}{9} + \frac{4x^2\cos 3x}{9}$$

$$- \frac{8x\sin 3x}{27} - \frac{8}{81}\cos 3x + C$$

ii) We have

$$\int x^4\cos 5x\,dx = I_4 = \frac{x^4\sin 5x}{5} + \frac{4x^3\cos 5x}{25} - \frac{12}{25} I_2'$$

$$= \frac{x^4\sin 5x}{5} + \frac{4x^3\cos 5x}{25} - \frac{12}{25}\left(\frac{x^2\sin 5x}{5}\right.$$

$$\left. + \frac{2x\cos 5x}{25} - \frac{2}{25}\int \cos 5x\,dx\right)$$

$$= \frac{x^4\sin 5x}{5} + \frac{4x^3\cos 5x}{25} - \frac{12x^2\sin 5x}{125}$$

$$- \frac{24x\cos 5x}{625} + \frac{24}{3125}\sin 5x$$

E5) i) We have

$$\int \cos^7 x\,dx = I_7 = \frac{1}{7}\cos^6 x\sin x + \frac{6}{7} I_5$$

$$= \frac{1}{7}\cos^6 x\sin x + \frac{6}{7}\left(\frac{1}{5}\cos^4 x\sin x + \frac{4}{5} I_3\right)$$

$$= \frac{1}{7}\cos^6 x\sin x + \frac{6}{35}\cos^4 x\sin x$$

$$+ \frac{24}{35}\left(\frac{\cos^2 x\sin x}{3} + \frac{2}{3} I_1\right)$$

$$= \frac{1}{7}\cos^6 x\sin x + \frac{6}{35}\cos^4 x\sin x + \frac{8}{35}\cos^2 x\sin x$$

$$+ \frac{16}{35}\sin x + C$$

You may like to compare this method with the one we used in Example 8) in Unit 18. You will also find it interesting to check that the answer we got here is the same as the answer we got in Example 8 in Unit 18.

ii) We have

$$\begin{aligned}\int \cos^8 x\,dx = I_8 &= \frac{\cos^7 x \sin x}{8} + \frac{7}{8} I_6 \\ &= \frac{\cos^7 x \sin x}{8} + \frac{7\cos^5 x \sin x}{48} + \frac{35}{48} I_4 \\ &= \frac{\cos^7 x \sin x}{8} + \frac{7\cos^5 x \sin x}{48} \\ &\quad + \frac{35\cos^3 x \sin x}{192} + \frac{105}{192} I_2 \\ &= \frac{\cos^7 x \sin x}{8} + \frac{7\cos^5 x \sin x}{48} + \frac{35\cos^3 x \sin x}{192} \\ &\quad + \frac{105}{384}\cos x \sin x + \frac{105}{384} x + C\end{aligned}$$

E6) i) We have

$$\begin{aligned}\int \sin^7 x\,dx = I_7' &= -\frac{1}{7}\sin^6 x \cos x + \frac{6}{7} I_5' \\ &= -\frac{1}{7}\sin^6 x \cos x + \frac{6}{7}\left(-\frac{1}{5}\sin^4 x \cos x + \frac{4}{5} I_3'\right) \\ &= -\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x \\ &\quad + \frac{24}{35}\left(-\frac{1}{3}\sin^2 x \cos x + \frac{2}{3} I_1'\right) \\ &= -\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x \\ &\quad - \frac{8}{35}\sin^2 x \cos x + \frac{16}{35}\cos x + C\end{aligned}$$

ii) We have

$$\begin{aligned}\int \sin^8 x\,dx = I_8 &= -\frac{1}{8}\sin^7 x \cos x + \frac{7}{8} I_6 \\ &= -\frac{1}{8}\sin^7 x \cos x + \frac{7}{8}\left(-\frac{1}{6}\sin^5 x \cos x + \frac{5}{6} I_4\right) \\ &= -\frac{1}{8}\sin^7 x \cos x - \frac{7}{48}\sin^5 x \cos x \\ &\quad + \frac{35}{48}\left(-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4} I_2\right) \\ &= -\frac{1}{8}\sin^7 x \cos x - \frac{7}{48}\sin^5 x \cos x - \frac{35}{192}\sin^3 x \cos x \\ &\quad - \frac{105}{384}\sin x \cos x + \frac{105}{384} x + C\end{aligned}$$

E7) i) We have

$$\int_0^{\frac{\pi}{2}} \cos^9 x\,dx = \frac{8\cdot 6\cdot 4\cdot 2}{9\cdot 7\cdot 5\cdot 3} = \frac{128}{315}$$

ii) We have

$$\int_0^{\frac{\pi}{2}} \sin^6 x\,dx = \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\frac{\pi}{2} = \frac{5\pi}{32}$$

E8) i) We have

$$\begin{aligned} I_n &= \int \cot^n x\,dx = \int \cot^{n-2} x \cot^2 x\,dx \\ &= \int \cot^{n-2} x \left(\operatorname{cosec}^2 x - 1\right)\,dx \\ &= \int \cot^{n-2} x \operatorname{cosec}^2 x\,dx - \int \cot^{n-2} x\,dx \\ &= \int \cot^{n-2} x \operatorname{cosec}^2 x\,dx - I_{n-2} \end{aligned}$$

If $n-2 \neq -1$, i.e. if $n \neq 1$ we have

$$\int \cot^{n-2} x \operatorname{cosec}^2 x\,dx = -\frac{\cot^{n-1} x}{n-1}.$$

$$\therefore\ I_n = -\frac{\operatorname{cosec}^{n-1} x}{n-1} - I_{n-2} \text{ if } n \neq 1.$$

ii) We have

$$\begin{aligned} I_n &= \int \operatorname{cosec}^n x\,dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x\,dx \\ &= \operatorname{cosec}^{n-2} x(-\cot x) \\ &\quad - (n-2)\int \operatorname{cosec}^{n-3} x(-\operatorname{cosec} x \cot x)(-\cot x)\,dx \\ &= -\operatorname{cosec}^{n-2} \cot x \\ &\quad - (n-2)\int \operatorname{cosec}^{n-2} x \left(\operatorname{cosec}^2 x - 1\right)\,dx \\ &= -\operatorname{cosec}^{n-2} x \cot x - (n-2)I_n + (n-2)I_{n-2} \end{aligned}$$

$$\therefore\ (n-1)I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2)I_{n-2}$$

If $n \neq 1$,

$$I_n = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

E9) i) We have

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec}^3 x\,dx &= -\frac{1}{2}\operatorname{cosec} x \cot x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec} x\,dx \\ &= -\frac{1}{2}(0-\sqrt{2}) - \frac{1}{2}\ln|\operatorname{cosec} x + \cot x|\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \frac{1}{\sqrt{2}} - \frac{1}{2}\left(\ln 1 - \ln\left|1+\sqrt{2}\right|\right) = \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(1+\sqrt{2}\right) \end{aligned}$$

ii) We have

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4 x\,dx = -\frac{\cot^3 x}{3}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - I_2$$

$$= \left\{\left(-\frac{0}{3}\right) - \left(-\frac{1}{3}\right)\right\} + \cot x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + I_0$$

$$= \frac{1}{3} + (0-1) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{2}{3}$$

iii) We have

$$\int_0^{\frac{\pi}{4}} \tan^6 x\,dx = I_6 = \frac{\tan^5 x}{5}\Bigg|_0^{\frac{\pi}{4}} - I_4 = \frac{1}{5} - \left(\frac{\tan^3 x}{3}\Bigg|_0^{\frac{\pi}{4}} - I_2\right)$$

$$= \frac{1}{5} - \frac{1}{3} + \left(\tan x\Big|_0^{\frac{\pi}{4}} - I_0\right)$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4}$$

iv) We have

$$\int_0^{\frac{\pi}{4}} \sec^5 x\,dx = I_5 = \frac{\sec^3 x \tan x}{4}\Bigg|_0^{\frac{\pi}{4}} + \frac{3}{4} I_3$$

$$= \left(\frac{\left(\sqrt{2}\right)^3}{4} - 0\right) + \frac{3}{4}\left(\frac{\sec x \tan x}{2}\Bigg|_0^{\frac{\pi}{4}} + \frac{1}{2} I_1\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{3}{4}\left(\frac{\sqrt{2}}{2} - 0\right) + \frac{3}{8}\left(\ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{4}}\right)$$

$$= \frac{7}{4\sqrt{2}} + \frac{3}{8}\ln(1+\sqrt{2})$$

E10) i) Using Eqn. (18), we have

$$\int \sin^4 x \cos^6 x\,dx = I_{4,6} = -\frac{1}{10}\sin^3 x \cos^7 x + \frac{3}{10} I_{2,6}$$

$$= -\frac{1}{10}\sin^3 x\cos^7 x + \frac{3}{10}\left(-\frac{1}{8}\sin x \cos^7 x + \frac{1}{8} I_{2,6}\right)$$

$$= -\frac{1}{10}\sin^3 x \cos^7 x - \frac{3}{80}\sin x \cos^7 x + \frac{3}{80} I_{0,6}$$

From Example 7, we have

$$I_{0,6} = \int \cos^6 x\,dx = \frac{1}{6}\sin x\cos^5 x + \frac{5}{24}\cos^3 x \sin x + \frac{15}{48}\cos x \sin x + \frac{15}{48}x + C$$

$$\therefore \int \sin^4 x \cos^6 x\,dx = -\frac{1}{10}\sin^3 x\cos^7 x - \frac{3}{80}\sin x\cos^7 x + \frac{1}{160}\sin x \cos^5 x + \frac{1}{128}\sin x\cos^3 x + \frac{3}{256}\sin x\cos x + \frac{3}{256}x + C$$

ii) Using Eqn. (17), we have

$$\int \sin^6 x \cos^4 x\, dx = \frac{1}{10}\sin^7 x \cos^3 x + \frac{3}{10} I_{6,2}$$

$$= \frac{1}{10}\sin^7 x \cos^3 x + \frac{3}{10}\left(\frac{1}{8}\sin^7 x \cos x + \frac{1}{8} I_{6,0}\right)$$

$$= \frac{1}{10}\sin^7 x \cos^3 x + \frac{3}{80}\sin^7 x \cos x + \frac{3}{80} I_{6,0}$$

From Example 9, we have

$$I_{6,0} = \int \sin^6 x\, dx = -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x + \frac{15}{48}\sin x \cos x + \frac{15}{48}x + C$$

$$\therefore \int \sin^6 x \cos^4 x\, dx = \frac{1}{10}\sin^7 x \cos^3 x + \frac{3}{80}\sin^7 x \cos x - \frac{1}{160}\sin^5 x \cos x - \frac{1}{128}\sin^3 x \cos x + \frac{3}{256}\sin x \cos x + \frac{3}{256}x + C$$

iii) We have

$$\int \sin^4 x \tan^2 x\, dx = \int \sin^6 x (\cos x)^{-2}\, dx = I_{6,-2}$$

$$= -\frac{1}{(-1)}\sin^5 x (\cos x)^{-1} + \frac{3}{(-1)} I_{4,0}$$

From Example 14)c) we have

$$I_{4,0} = -\frac{1}{4}\sin^3 x \cos x + \frac{3}{8}\sin x \cos x - \frac{3}{8}x + C$$

$$\therefore \int \sin^4 x \tan^2 x\, dx = \sin^5 x \sec x + \frac{3}{4}\sin^3 x \cos x - \frac{9}{8}\sin x \cos x + \frac{9}{8}x + C$$

iv) Here, the reduction formulas we derived will not work because both m and n are –1. However, we can integrate this using a small manipulation.

$$\int \sec x \operatorname{cosec} x\, dx = \int \frac{1}{\sin x \cos x} dx = \int \frac{2}{2 \sin x \cos x} dx$$

$$= 2\int \operatorname{cosec} 2x\, dx = -\ln|\operatorname{cosec} 2x + \cot 2x|\, dx + C$$

E11) i) We have

$$\int e^{4x} \sin^3 x\, dx = \frac{(4\sin x - 3\cos x)e^{4x}\sin^2 x}{25} + \frac{6}{25} L_1$$

$$L_1 = \frac{1}{25}e^{4x}(4\sin x - \cos x) + C$$

$$\therefore \int e^{4x}\sin^2 x\, dx = \frac{(4\sin x - 3\cos x)e^{4x}\sin^2 x}{25} + \frac{6}{625}e^{4x}(4\sin x - \cos x) + C$$

ii) We have

$$\int e^{5x}\sin^2 x\,dx = \frac{(5\sin x - 2\cos x)e^{5x}\sin x}{29} + \frac{2}{29}L_0$$
$$= \frac{(5\sin x - 2\cos x)e^{5x}\sin x}{29} + \frac{2}{145}e^{5x} + C$$

E12) We have

$$\int e^{ax}\cos^n x\,dx = \frac{e^{ax}}{a}\cos^n x - \frac{1}{a}\int e^{ax} n\cos^{n-1} x(-\sin x)\,dx$$
$$\int e^{ax}\cos^{n-1} x\sin x\,dx = \frac{e^{ax}}{a}\cos^{n-1} x\sin x - \frac{1}{a}\int e^{ax}\frac{d}{dx}\left(\cos^{n-1} x\sin x\right)\,dx$$

We have

$$\frac{d}{dx}\left(\cos^{n-1} x\sin x\right) = (n-1)\cos^{n-2} x\left(-\sin^2 x\right) + \cos^n x$$
$$= -(n-1)\cos^{n-2} x\left(1-\cos^2 x\right) + \cos^n x$$
$$= -(n-1)\cos^{n-2} x + (n-1)\cos^n x + \cos^n x$$
$$= -(n-1)\cos^{n-2} x + n\cos^n x$$
$$\therefore \int e^{ax}\cos^{n-1} x\sin x\,dx = \frac{ne^{ax}}{a}\cos^{n-1} x\sin x + \frac{n(n-1)}{a}\int e^{ax}\cos^{n-2} x\,dx - \frac{n}{a}\int e^{ax}\cos^n x\,dx$$
$$\therefore \int e^{ax}\cos^n x\,dx = \frac{e^{ax}}{a}\cos^n x + \frac{e^{ax}}{a^2}\cos^{n-1} x\sin x + \frac{n(n-1)}{a^2}\int e^{ax}\cos^{n-2} x\,dx - \frac{n^2}{a^2}\int e^{ax}\cos^n x\,dx$$

Re-arranging the terms, we have

$$\left(1+\frac{n^2}{a^2}\right)C_n = \frac{(a\cos x + n\sin x)}{a^2}e^{ax}\cos^{n-1} x + \frac{n(n-1)}{a^2}C_{n-2}$$

Multiplying throughout by $\frac{a^2}{n^2+a^2}$,

$$C_n = \frac{(a\cos x + n\sin x)}{n^2+a^2}e^{ax}\cos^{n-1} x + \frac{n-1}{n^2+a^2}C_{n-2} \qquad \ldots(24)$$

E13) i) Using Eqn. (24) with a = 3, n = 2, we get

$$\int e^{3x}\cos^2 x\,dx = \frac{(3\cos x + 2\sin x)}{13}e^{3x} + \frac{2}{13}L_0$$
$$L_0 = \int e^{3x}\,dx = \frac{e^{3x}}{3} + C$$
$$\therefore \int e^{3x}\cos^2 x\,dx = \frac{(3\cos x + 2\sin x)}{13}e^{3x} + \frac{2}{39}e^{3x} + C$$

ii) Using Eqn. (24) with a = 2, n = 3, we get

$$\int e^{2x}\cos^3 x\,dx = \frac{(2\cos x + 3\sin x)}{13}e^{2x}\cos^2 x + \frac{6}{13}\int e^{2x}\cos x\,dx$$

From Unit 18, we have

$$\int e^{ax} \cos bx \, dx = \frac{e^{2x}}{a^2 + b^2}(a \cos x + b \sin bx) + C$$

$$\therefore \int e^{2x} \cos x \, dx = \frac{e^{2x}}{4+1}(2 \cos x + \sin x) + C$$

$$\therefore \int e^{2x} \cos^3 x \, dx = \frac{(2 \cos x + 3 \sin x)}{13} e^{2x} \cos^2 x + \frac{6e^{2x}}{65}(2 \cos x + \sin x) + C$$

E14) Putting a = 0 in the formula

$$C_n = \frac{(a \cos x + n \sin x) e^{ax} \cos^{n-1} x}{n^2 + a^2} + \frac{n(n-1)}{n^2 + a^2} C_{n-2}$$

we get

$$\int e^{0x} \cos^n x \, dx = \frac{(0 \cos x + n \sin x) e^{0x} \cos^{n-1} x}{n^2 + 0^2} + \frac{n(n-1)}{n^2 + 0^2} \int e^{0x} \cos^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{n \sin x \cos^{n-1} x}{n^2} + \frac{n(n-1)}{n^2} \int \cos^{n-2} x \, dx$$

$$= \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

and this is the reduction formula for $\int \cos^n x \, dx$.

UNIT 20

APPLICATIONS OF INTEGRATION

Structure

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20.1 INTRODUCTION

You may recall that we discussed the relationship between the definite integral of a function and the area under the curve defined by the function in Unit 17. There, we saw how we can use the process of integration and add a large number of infinitesimally small rectangles to get the area bounded by the graph of sum function. We pick up the thread again in this Unit. In Sec. 20.2 of this Unit we will see how we can use integration to find the area enclosed by a curve in the plane. We will do this for curves represented using cartesian coordinates, polar coordinates and in parametric form. In Sec. 20.3, we use the same ideas to find the length of the curves in the plane. Here also, we will discuss methods for finding the length of a plane curve represented any of the three forms, cartesian, polar or parametric form.

Objectives

After studying this unit, you should be able to:

- explain the concept of area under a curve and between two curves;
- find the area under a curve and the area between some well known plane curves;
- explain the concept of length of a curves; and
- find the length of the arc of some well known plane curves.

20.2 AREA UNDER A CURVE

Recall that, in Unit 17, we saw that the area under a curve can be approximated by rectangles. We saw that we can get closer and closer approximations to the area by increasing the number of rectangles we use in approximation. We had put the idea on a formal footing using the concept of upper and lower sums associated with a partition. We saw that the upper sums gives approximations to the area from above and the lower sums gives approximations to the area from below. When the functions is integrable both of them approach a common value which is the area bounded by the curve and the x-axis.

In this section we will find the area under a curve using integration. Before we start our discussion we need to establish some conventions that facilitate our discussion.

You would have noticed that all the curves that we looked at in Unit 17 were in the part of the cartesian plane where the y-coordinate was greater than or equal to 0. What happens when we have a curve like the one given in Fig. 1 where the function can take negative values also?

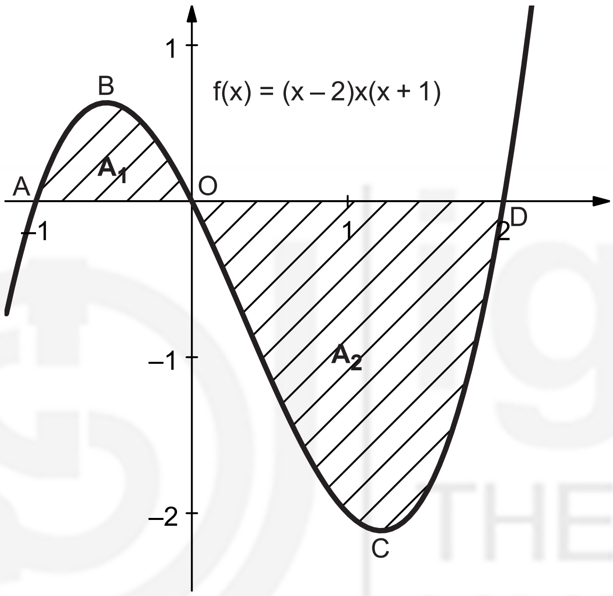


Fig. 1: Area bounded by the curve $f(x) = (x-2)x(x+1)$ and x–axis.

Let us find out by integrating $f(x) = (x-2)x(x+1) = x^3 - x^2 - 2x$ from -1 to 2. We have

$$\int_{-1}^{2}\left(x^3 - x^2 - 2x\right)\,dx = \left.\frac{x^4}{4} - \frac{x^3}{3} - x^2\right|_{-1}^{2} = \left(4 - \frac{8}{3} - 4\right) - \left(\frac{1}{4} + \frac{1}{3} - 1\right) = -\frac{9}{4}$$

But, area is always a positive quantity. How can our answer be negative?

The answer lies in the way we defined upper and lower sums. Recall from Unit 17 that, given a function $f\colon [a, b] \to \mathbb{R}$ and a partition

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\},$$

we define the upper sum of f with respect to the partition P by

$$U(P, f) = \sum_{i=1}^{n} M_k \Delta_i$$

where $\Delta_i = x_i - x_{i-1}$. If $f(x) \le 0$ for all $x \in [a, b]$ then $M_k \le 0$ for all k, $1 \le k \le n$. Since $\Delta_i > 0$, $M_k\Delta_k \le 0$. So, the $\sum_{i=1}^{n} M_k\Delta_i \le 0$. Therefore, it is not surprising that the 'area', which is a limiting value of the upper sums, is negative.

With this insight, let us split up the integral into two parts. In one part the function takes negative values and it takes positive values in the other part.

$$\int_{-1}^{0}\left(x^3 - x^2 - 2x\right)\,dx = \frac{x^4}{4} - \frac{x^3}{3} - x^2\Bigg|_{-1}^{0} = 0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right) = \frac{5}{12}$$

$$\int_{0}^{2}\left(x^3 - x^2 - 2x\right)\,dx = \frac{x^4}{4} - \frac{x^3}{3} - x^2\Bigg|_{0}^{2} = \left(4 - \frac{8}{3} - 4\right) - 0 = -\frac{8}{3}$$

Here, $\frac{5}{12}$ is the area A_1 in Fig. 1 and $-\frac{8}{3}$ is the area A_2 in Fig. 1 with a negative sign. So, the actual area is $\frac{5}{12} + \frac{8}{3} = \frac{37}{12}$.

In general, if $f(x) \leq 0$ **in an interval** $[a, b]$**, we take** $-\int_a^b f(x)\,dx$ **to be the area under the curve** $y = f(x)$ **from** $x = a$ **to** $x = b$**.**

While we find the area, wherever possible, we will exploit the symmetry of curve under consideration and confine ourselves to only that portion of the curve that lies above x-axis. Let us now look at an example.

Example 1: Find the area bounded by the curve $x^2 + y^2 - 2x - 3 = 0$.

Solution: This is a second degree equation and the coefficients of x^2 and y^2 are equal. So, this is the equation of a circle. Completing the square, we get

$$(x-1)^2 + y^2 - 4 = 0 \text{ or } (x-1)^2 + y^2 = 4.$$

This is the equation of a circle with centre $(1, 0)$ and radius 2.

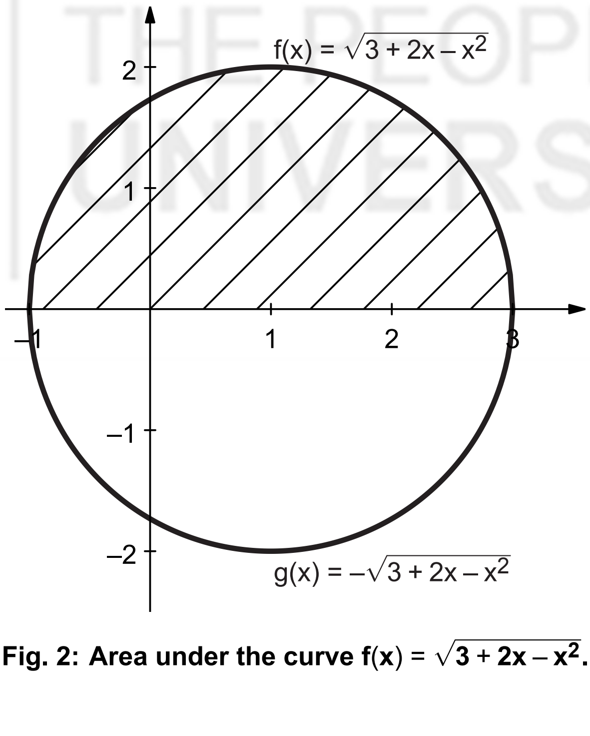


Fig. 2: Area under the curve f(x) = $\sqrt{3 + 2x - x^2}$.

We can re-write the equation as $y^2 = 3 + 2x - x^2$ or $y = \pm\sqrt{3 + 2x - x^2}$. Each choice of sign gives a different portion of the curve. The equation $y = \sqrt{3 + 2x - x^2}$ gives the portion of the curve in the upper half plane. The equation $y = -\sqrt{3 + 3x - x^2}$ gives the portion of the curve in the lower half plane, i.e. the portion of the cartesian plane where the y-coordinate is ≤ 0.

As the curve is symmetric about x-axis, the area bounded by the curve is twice the area in the upper half plane, i.e. the portion of the cartesian plane where the y-coordinate is non-negative. So, we have to find the area of the shaded region in Fig. 2 and multiply it by two find the area bounded by the circle. The shaded region is in the upper half plane. From the figure, the upper and lower limits of integration are, respectively, 3 and –1. So the required area is

$$A = 2\int_{-1}^{3} \sqrt{3 + 2x - x^2}\, dx = 2\int_{-1}^{3} \sqrt{4 - (x-1)^2}\, dx$$

Substituting $u = x - 1$, the new limits are $u = -2$ and $u = 2$. We then have

$$A = 2\int_{-2}^{2} \sqrt{2^2 - u^2}\, du$$

Substituting $u = 2\sin\theta$, we get $du = 2\cos\theta\, d\theta$. When $u = -2$, $2\sin\theta = -2$ or $\sin\theta = -1$. So, $\theta = -\frac{\pi}{2}$ is the lower limit of integration. When $u = 2$, $2\sin\theta = 1$, so $\theta = \frac{\pi}{2}$ is the upper limit. Also, since $\cos\theta \geq 0$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\sqrt{2^2 - u^2} = \sqrt{2^2 - 2^2\sin^2\theta} = 2\sqrt{1 - \sin^2\theta} = 2\cos\theta$$

Therefore,

$$A = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)(2\cos\theta)\, d\theta = 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta = 4\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4\left\{\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - \left(-\frac{\pi}{2} + \frac{-\sin\pi}{2}\right)\right\} = 4\pi = \pi \times 2^2.$$

The answer agrees with the value we get from applying the formula for the area of a circle of radius 2 from high school mensuration.

* * *

In the next example, we will find the area between two curves.

Example 2: Find the area bounded by the curves $y = x^2 - 4x + 7$, $y = -3 + 4x - x^2$ and the lines $x = 1$ and $x = 3$.

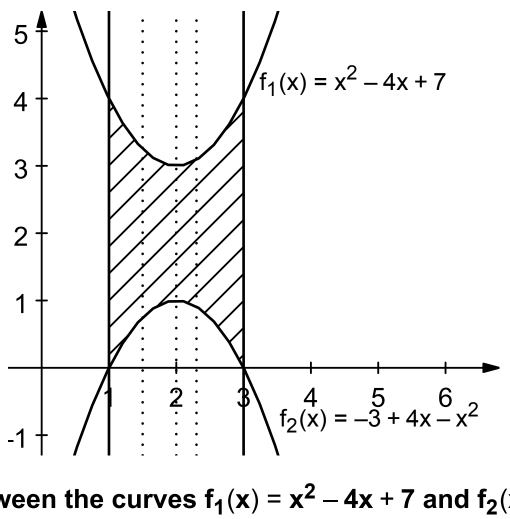


Fig. 3: Area between the curves $f_1(x) = x^2 - 4x + 7$ and $f_2(x) = -3 + 4x - x^2$.

Solution: We have

$$x^2 - 4x + 7 - (-3 + 4x - x^2) = 2x^2 - 8x + 10$$

The discriminant of $2x^2 - 8x + 10$ is $64 - 80 = -16 < 0$, therefore, $y = x^2 - 4x + 7$ and $y = -3 + 4x - x^2$ don't intersect. Further, $2x^2 - 8x + 10 \geq 0$ always. Writing $f_1(x) = x^2 - 4x + 7$ and $f_2(x) = -3 + 4x - x^2$, we have $f_1(x) \geq f_2(x)$, i.e. $f_1(x)$ lies above $f_2(x)$. The area bounded by the curves is the shaded region in Fig. 3.

The region is an example of a **vertically simple region.** We say that a region between the graphs of any two functions $f_1(x)$ and $f_2(x)$ is **vertically simple** if any vertical line intersects the region in either a single point or in a line segment with its lower end point on $y = f_2(x)$ and upper end point on $y = f_1(x)$. You can see another example in Fig. 4.

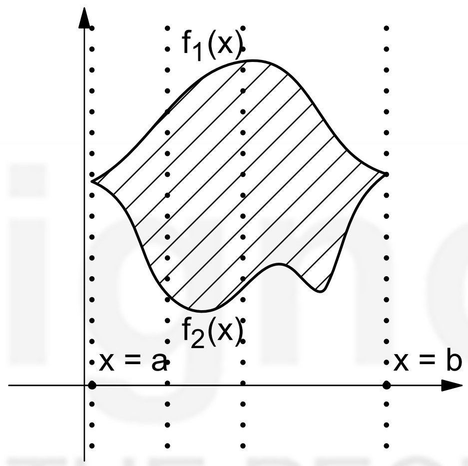


Fig. 4: Example of a vertically simple region.

As you can see in this figure, the dotted lines at the left and right extremes intersect the area between the curves $y = f_1(x)$ and $y = f_2(x)$ at a single point. The other two dotted lines intersect the area between these curves along a line segment with its lower end point on $y = f_2(x)$ and the upper end point on $y = f_1(x)$. In such cases, the area between the curves is

$$\int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx$$

In our case, the area between the curves is

$$\int_1^3 f_1(x)\,dx - \int_1^3 f_2(x)\,dx$$

We have

$$\int_1^3 f_1(x)\,dx = \int_1^3 \left(x^2 - 4x + 7\right)\,dx = \left.\frac{x^3}{3} - 4\frac{x^2}{2} + 7x\right|_1^3$$

$$= (9 - 18 + 21) - \left(\frac{1}{3} - 2 + 7\right) = 12 - \frac{16}{3} = \frac{20}{3}$$

Also,

$$\int_1^3 f_2(x)\,dx = \int_1^3 \left(-3+4x-x^2\right)\,dx = -3x+4\frac{x^2}{2}-\frac{x^3}{3}\Big|_1^3$$

$$= (-9+18-9)-\left(-3+2-\frac{1}{3}\right)=\frac{4}{3}$$

So, the area between the curves is $\frac{16}{3}$.

* * *

Note that, in the previous example, $f_1(x) \geq 0$ and $f_2(x) \geq 0$. Our method for calculating the area between two curves work even if $f_1(x)$ and $f_2(x)$ are not non-negative. Why? We can always shift the functions by adding a sufficiently large constant C to both of them.

See Fig. 5. Here, we want to find the area between the curves $f_1(x)$ and $f_2(x)$ and both have negative values in the interval $[a, b]$. To make them non-negative, we add the **same** constant C to $f_1(x)$ and $f_2(x)$. Note that, shifting doesn't change the area.

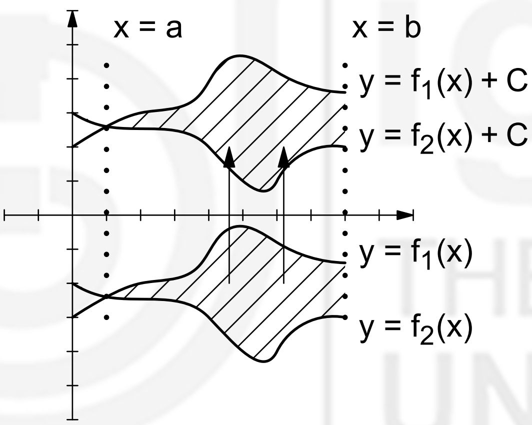


Fig. 5: Shifting the functions $f_1(x)$ and $f_2(x)$ to make them non-negative.

So, we have

$$\text{Area between the curves} = \int_a^b ((f_1(x)+C)-(f_2(x)+C))\,dx$$

$$= \int_a^b (f_1(x)-f_2(x))\,dx$$

So, our method works even if the $f_1(x)$ and $f_2(x)$ take negative values.

Let us look at an example where $f_1(x)$ and $f_2(x)$ takes negative values.

Example 3: Find the area between the curves $y = 2x - x^2$ and $y = 2 - x$ from $x = -1$ to $x = 2$.

Solution: The x-coordinate of the point of intersection of the curves is given by $2x - x^2 = x - 2$ or $x^2 - x - 2 = 0$. Factorising the quadratic $x^2 - x - 2$, we get $(x-2)(x+1) = 0$. So, the curves intersect at $x = -1$ and $x = 2$. The points of intersection are C and D. See Fig. 6.

Let us take $f_1(x) = 2x - x^2$ and $f_2(x) = x - 2$. Note that, although $f_1(x)$ takes negative values in the interval $[-1, 0]$ and $f_2(x)$ is negative in $[-1, 0]$ the required area is vertically simple.

Let us verify that $f_1(x) \geq f_2(x)$ in $[-1, 2]$. We have

$$h(x) = 2x - x^2 - (x - 2) = -x^2 + x + 2 = -(x^2 - x - 2) = -(x - 2)(x + 1).$$

Note that, $x - 2 \leq 0$ in $[-1, 2]$ because $x \leq 2$ in $[-1, 2]$. Similarly, $x \geq -1$ in $[-1, 2]$, so $x + 1 \geq 0$ in $[-1, 2]$. It follows that $(x - 2)(x + 1) \leq 0$ in $[-1, 2]$. Thus, $-(x - 2)(x + 1) \geq 0$.

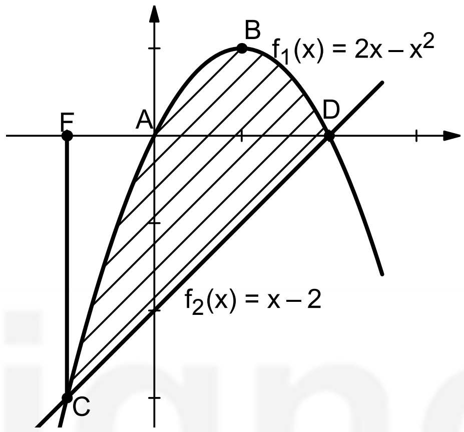


Fig. 6: Area between the curves $f_1(x) = 2x - x^2$ and $f_2(x) = x - 2$.

The required area is

$$\int_{-1}^{2} (f_1(x) - f_2(x))\, dx = \int_{-1}^{2} \left(-x^2 + x + 2\right) dx = \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Bigg|_{-1}^{2}$$

$$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) = \frac{10}{3} - \left(-\frac{7}{2}\right) = \frac{41}{6}$$

* * *

Let us look at another example.

Example 4: Find the upper portion of the area between the Strophoid $y^2 = x^2\left(\frac{5-x}{5+x}\right)$ and the line $\frac{x}{4} - \frac{y}{6} = 1$.

Solution: We leave it to you trace the curve. See Fig. 7. Eliminating y from the equation of the Strophoid using the value of y from the equation of the line, we have, for $x \neq -5$,

$$\left(6\left(\frac{x}{4} - 1\right)\right)^2 = x^2\left(\frac{5 - x}{5 + x}\right) \text{ or } (5 + x)\left(6\left(\frac{x}{4} - 1\right)\right)^2 = x^2(5 - x)$$

Simplifying the last equation, the points of intersection of the curve and the line are given by

$$\frac{13}{4}x^3 - \frac{47}{4}x^2 - 54x + 180 = \frac{1}{4}(x - 3)(x + 4)(13x - 60) = 0$$

Note that $x > 0$ for all the points on the loop of the Strophoid because the loop corresponds to the values $x \in [0, 5]$. So, the point of intersection of the line with the loop in Strophoid are $A\left(\frac{60}{13}, \frac{12}{13}\right)$ and $B\left(3, -\frac{3}{2}\right)$.

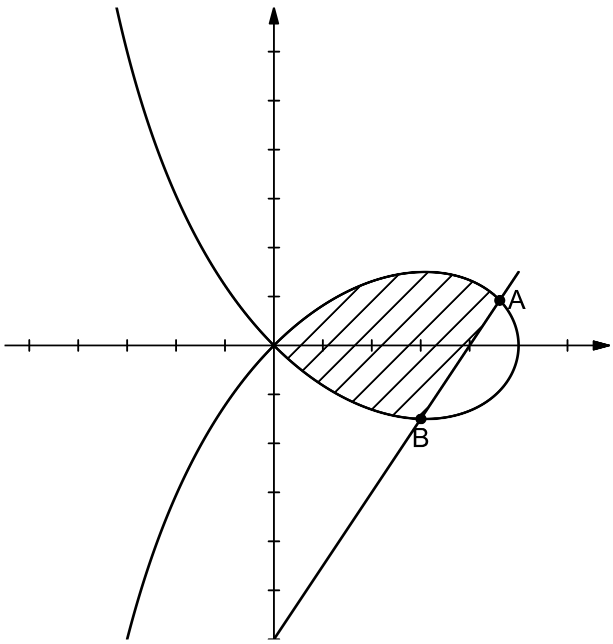


Fig. 7: Area between the Strophoid $y^2 = x^2\left(\frac{5-x}{5+x}\right)$ and $y = 6\left(\frac{x}{4} - 1\right)$.

We take $f_1(x)$ such that the graph of $y = f_1(x)$ is the upper part of the loop from $x = 0$ to $x = \frac{60}{13}$. We choose $f_2(x)$ such that $y = f_2(x)$ from $x = 0$ to $x = \frac{60}{13}$ is the lower part of the loop from the origin to B followed by the line $y = 6\left(\frac{x}{4} - 1\right)$from B to A. So, we define

$$f_1(x) = x\sqrt{\frac{5-x}{5+x}} \text{ for } x \in \left[0, \frac{60}{13}\right]$$

and

$$f_2(x) = \begin{cases} -x\sqrt{\frac{5-x}{5+x}} & 0 \le x \le 3 \\ 6\left(\frac{x}{4} - 1\right) & 3 \le x \le \frac{60}{13} \end{cases}$$

So, the required area is

$$\int_0^{\frac{60}{13}} (f_1(x) - f_2(x))\, dx = \int_0^3 (f_1(x) - f_2(x))\, dx + \int_3^{\frac{60}{13}} (f_1(x) - f_2(x))$$

$$= 2\int_0^3 x\sqrt{\frac{5-x}{5+x}}\, dx + \int_3^{\frac{60}{13}} x\sqrt{\frac{5-x}{5+x}}\, dx - 6\int_3^{\frac{60}{13}} \left(\frac{x}{4} - 1\right)\, dx \qquad \ldots(1)$$

Using the substitution $x = 5\sin\theta$, we get

$$\int_0^3 x\sqrt{\frac{5-x}{5+x}}\, dx = \int_0^{\alpha} (5\sin\theta)\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}(5\cos\theta)\, d\theta$$

where $0 < \alpha < \frac{\pi}{2}$ is such that $\sin\alpha = \frac{3}{5}$. On mutiplying the numerator and the denominator by $1 - \sin\theta$.

We can do this because $0 < \alpha < \frac{\pi}{2}$, so $1 - \sin\theta \neq 0$.

$$\int_0^3 x\sqrt{\frac{5-x}{5+x}}\, dx = 25\int_0^{\alpha} \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}\, \sin\theta\cos\theta\, d\theta$$

Since $1 - \sin\theta > 0$ in $[0, \alpha] \subset \left[0, \frac{\pi}{2}\right]$,

$$\int_0^3 x\sqrt{\frac{5-x}{5+x}}\, dx = 25\int_0^\alpha \left(\sin\theta - \sin^2\theta\right)\, d\theta$$

$$= 25\left(-\cos\theta - \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Bigg|_0^\alpha$$

$$= 25\left\{\left(-\cos\alpha - \frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\right) - (-1)\right\}$$

We have

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{9}{25}} = \frac{4}{5} \text{ and } \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}.$$

Therefore,

$$\int_0^3 x\sqrt{\frac{5-x}{5+x}}\, dx = -20 - \frac{25\alpha}{2} + 6 + 25 = 11 - \frac{25\alpha}{2}$$

Similarly,

$$\int_3^{\frac{60}{13}} x\sqrt{\frac{5-x}{5+x}}\, dx = 25\int_\alpha^\beta \left(\sin\theta - \sin^2\theta\right)\, d\theta$$

where $0 < \beta < \frac{\pi}{2}$ is such that $\sin\beta = \frac{12}{13}$.

$$\int_3^{\frac{60}{13}} x\sqrt{\frac{5-x}{5+x}}\, dx = 25\left(-\cos\theta - \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Bigg|_\alpha^\beta$$

$$= 25\left\{\left(-\cos\beta - \frac{\beta}{2} + \frac{\sin 2\beta}{4}\right) - \left(-\cos\alpha - \frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\right)\right\}$$

We have

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{144}{169}} = \frac{5}{13}$$

and

$$\sin 2\beta = 2\sin\beta\cos\beta = 2\cdot\frac{12}{13}\cdot\frac{5}{13}$$

Therefore,

$$\int_3^{\frac{60}{13}} x\sqrt{\frac{5-x}{5+x}}\, dx = 25\left(-\frac{5}{13} - \frac{\beta}{2} + \frac{30}{169} + \frac{4}{5} + \frac{\alpha}{2} - \frac{6}{25}\right) = \frac{1491}{169} - 25\frac{(\beta-\alpha)}{2}$$

We have

$$\int_3^{\frac{60}{13}} \left(\frac{x}{4} - 1\right)\, dx = \frac{x^2}{8} - x\Bigg|_3^{\frac{60}{13}} = -\frac{105}{1352}$$

Therefore, the area is

$$\int_0^{\frac{60}{13}} (f_1(x) - f_2(x))\, dx = 2\left(11 - \frac{25}{2}\alpha\right) + \frac{1491}{169} - \frac{25(\beta - \alpha)}{2} - 6\left(-\frac{105}{1352}\right)$$
$$= \frac{1627}{52} - 25\frac{(\alpha + \beta)}{2}$$

* * *

Let us now look at an example where the area we want to evaluate is in two parts. In these two parts, $f_1(x)$ and $f_2(x)$ are interchanged.

Example 5: Find the area between the curves $y = x^2 + 3x - 4$ and $y = \left(\frac{x}{3} - 3\right)$ from $x = -3$ to $x = 3$ shown in Fig. 8.

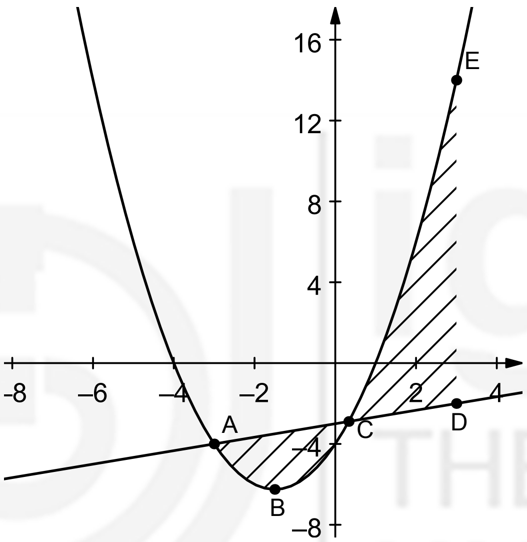


Fig. 8: Area between the curves $y = x^2 + 3x - 4$ and $y = \left(\frac{x}{3} - 3\right)$.

Solution: The x-coordinates of the points of intersection of the curves $y = x^2 + 3x - 4$ and $y = \left(\frac{x}{3} - 3\right)$ are given by $x^2 + 3x - 4 = \left(\frac{x}{3} - 3\right)$ or $x^2 + \frac{8}{3}x - 1 = 0$. Factorising $x^2 + \frac{8}{3}x - 1$, we get

$$x^2 + \frac{8}{3}x - 1 = \frac{1}{3}(x + 3)(3x - 1).$$

So, the points of intersection are $A(-3, -4)$ and $C\left(\frac{1}{3}, -\frac{26}{9}\right)$. Let $B = \left(-\frac{3}{2}, -\frac{25}{4}\right)$, $D = (3, -2)$ and $E = (3, 14)$.

In this example, we find the areas of the regions ABC and CDE separately. While finding the area of the region ABC we choose $\left(\frac{x}{3} - 3\right)$ as $f_1(x)$ and $x^2 + 3x - 4$ as $f_2(x)$. In $\left[-3, \frac{1}{3}\right]$, we have

$$f_1(x) - f_2(x) = \left(\frac{x}{3} - 3\right) - \left(x^2 + 3x - 4\right) = -\frac{1}{3}(x + 3)(3x - 1).$$

Let us now check whether $f_1(x) - f_2(x) \geq 0$ in $\left[-3, \frac{1}{3}\right]$. Since $x \geq -3$, $x + 3 \geq 0$. Since $x \leq \frac{1}{3}$ in $\left[-3, \frac{1}{3}\right]$, $3x - 1 \leq 0$. So, $\frac{1}{3}(x + 3)(3x - 1) \leq 0$ in $\left[-3, \frac{1}{3}\right]$, therefore $-\frac{1}{3}(x + 3)(3x - 1) \geq 0$ in $\left[-3, \frac{1}{3}\right]$.

To find the area ABC we integrate $f_1(x) - f_2(x)$ from $x = -3$ to $x = \frac{1}{3}$. We have

$$\int \left(x^2 + \frac{8}{3}x - 1\right) dx = \frac{x^3}{3} + \frac{4}{3}x^2 - x + C \qquad \ldots (2)$$

Therefore, the area of ABC

$$\int_{-3}^{\frac{1}{3}} (f_1(x) - f_2(x))\, dx = \int_{-3}^{\frac{1}{3}} -\left(x^2 + \frac{8}{3}x - 1\right) dx = -\left(\frac{x^3}{3} + \frac{4}{3}x^2 - x\right)\Bigg|_{-3}^{\frac{1}{3}}$$

$$= -\left\{\left(\frac{1}{81} + \frac{4}{27} - \frac{1}{3}\right) - (-9 + 12 + 3)\right\}$$

$$= -\left(-\frac{14}{81} - 6\right) = \frac{500}{81} \qquad \ldots (3)$$

While finding the area of CDE we choose $y = x^2 + 3x - 4$ as $f_1(x)$ and $y = \left(\frac{x}{3} - 3\right)$ as $f_2(x)$. So,

$$f_1(x) - f_2(x) = \left(x^2 + 3x - 4\right) - \left(\frac{x}{3} - 3\right) = \frac{1}{3}(x+3)(3x-1) \geq 0$$

Let us check that $f_1(x) - f_2(x) \geq 0$. Since $x \geq 0$, $x + 3 \geq 3 > 0$. Also, since $x \geq \frac{1}{3}$, we have $3x - 1 \geq 0$. It follows that $f_1(x) - f_2(x) = \frac{1}{3}(x+3)(3x-1) \geq 0$ in $\left[\frac{1}{3}, 3\right]$.

We need to integrate

$$f_1(x) - f_2(x) = \frac{1}{3}(x+3)(3x-1)$$

from $x = \frac{1}{3}$ to $x = 3$. Using Eqn. (2), the area of CDE is

$$\int_{\frac{1}{3}}^{3} \left(\frac{1}{3}(x+3)(3x-1)\right) dx = \frac{x^3}{3} + \frac{4}{3}x^2 - x\Bigg|_{\frac{1}{3}}^{3} = \left\{(9 + 12 - 3) - \left(\frac{1}{81} + \frac{4}{27} - \frac{1}{3}\right)\right\}$$

$$= 18 + \frac{14}{81} = \frac{1472}{81} \qquad \ldots (4)$$

From Eqn. (3) and Eqn. (4), the required area is $\frac{1972}{81}$.

* * *

Try to solve the exercises given below to check your understanding of the examples we have discussed so far.

E1) Find the area of the region bounded by the curve $y = 9 - x^2$, the x – axis and the ordinates $x = 2$ and $x = -2$.

E2) Find the area between the curves $y = 3 - 4x - x^2$ and $y = -(x+1)$ and bounded by the lines $x = -4$ and $x = 1$.

E3) Find upper part of the area between the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and the line $y = \frac{x}{5} - 1$.

In many of the curves on the plane, there may not be any functional relationship between the x–coordinate and the y–coordinate. For example the equation of the Cardioid in cartesian form is

$$\left(x^2 + y^2 + ax\right)^2 = a^2\left(x^2 + y^2\right).$$

See Fig. 9. (Cardioid means 'Heart like shaped'.)

From what we have learnt so far, to find the area of the Cardioid, we should find functions $f_1(x)$ and $f_2(x)$ such that

1) The graph of $y = f_2(x)$ is the graph of the curve below the x– axis.
2) The graph of $y = f_1(x)$ is the graph of the curve above the x–axis.

You can see that, the line x = 0.4 intersects the curve in four points, twice above x–axis and twice below x–axis.

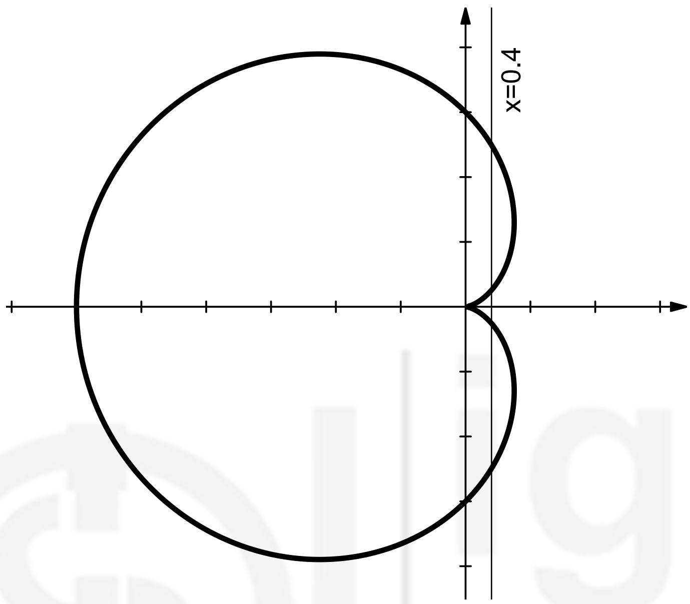


Fig. 9: Cartesian representation of the Cardioid $\left(x^2 + y^2 + ax\right)^2 = a^2\left(x^2 + y^2\right)$ when a = 3.

In this case it is not possible to find such functions $f_1(x)$ and $f_2(x)$. In such cases the cartesian representation of the curve is not very useful in finding the area. However, as we will see shortly, it is possible to find the area if we can write down the equation of the curve in polar form.

Suppose we want to find the area of a curve given by $r = f(\theta)$, between the lines $\theta = \alpha$ and $\theta = \beta$. See Fig. 10.

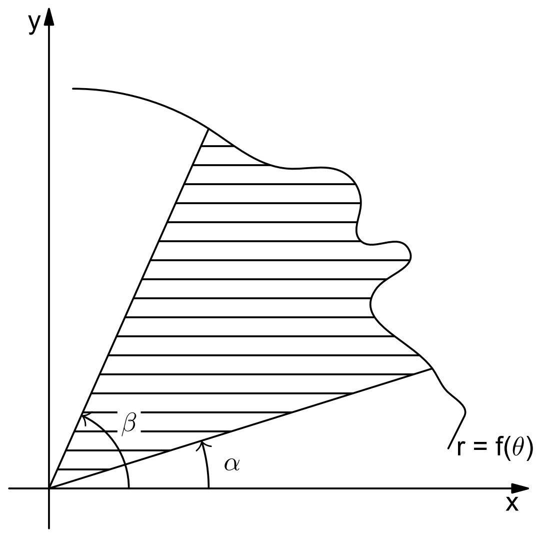


Fig. 10: Area of the curve $r = f(\theta)$ between the lines $\theta = \alpha$ and $\theta = \beta$.

As we did in the case of cartesian coordinates in Unit 17, we divide the curve into small parts. We choose

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{i-1} < \theta_i < \cdots < \theta_{n-1} < \theta_n = \beta.$$

We write $r_i = f(\theta_i)$. The lines r_i divide the area into smaller slices as you can see in Fig. 11.

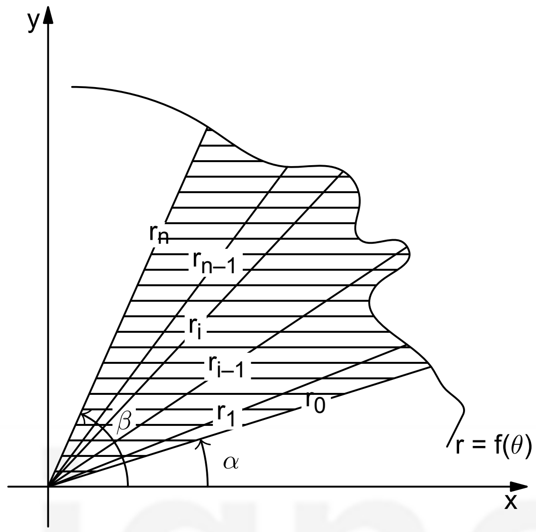


Fig. 11: Dividing the area of the curve r = f(θ) between the lines $\theta = \alpha$ and $\theta = \beta$.

Now, consider the area between lines $\theta = \theta_{i-1}$ and $\theta = \theta_i$. As we did in the case of cartesian coordinates, we can find lower and upper bounds for the slice. Let

$$m_i = \inf\{f(\theta) | \theta \in [\theta_{i-1}, \theta_i]\},\ M_i = \sup\{f(\theta) | \theta \in [\theta_{i-1}, \theta_i]\}, \Delta_i = \theta_i - \theta_{i-1}.$$

Let us denote the area between the lines r_i and r_{i-1} by A_i. Recall that the area of circular sector with radius r and sectorial angle θ radians is $\frac{r^2\theta}{2}$. Therefore, A_i is greater than the area of the sector of radius m_i and sectorial angle $\Delta_i = \theta_i - \theta_{i-1}$.

$$\text{i.e. } \frac{m_i^2 \Delta_i}{2} \le A_i \qquad \ldots(5)$$

See Fig. 12.

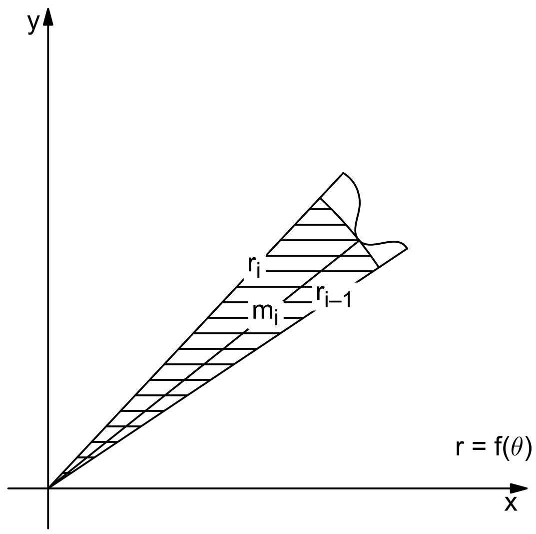


Fig. 12: Lower bound for A_i.

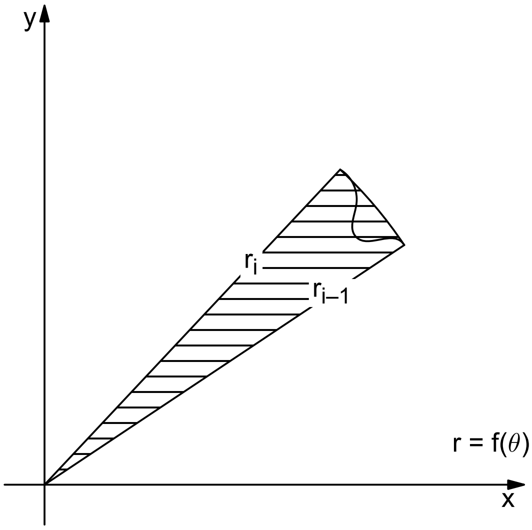


Fig. 13: Upper bound for A_i.

So, we have

$$A_i \le \frac{M_i^2 \Delta_i}{2} \qquad \ldots (6)$$

Combining Eqn. (5) and Eqn. (6) and summing up from i = 1 to n, we get

$$\sum_{i=1}^{n} \frac{m_i^2 \Delta_i}{2} \le A \le \sum_{i=1}^{n} \frac{M_i^2 \Delta_i}{2} \qquad \ldots (7)$$

since

$$A = A_1 + A_2 + \cdots + A_{i-1} + A_i + \cdots + A_{n-1} + A_n$$

is the area bounded by the lines $\theta = \alpha$ and $\theta = \beta$. Note that, the sum in the LHS of Eqn. (7) is the lower sum L(P, f) for the function $f^2(\theta)$ for the partition

$$P = \{\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{i-1} < \theta_i < \cdots < \theta_{n-1} < \theta_n = \beta\}$$

and the sum in the right is the upper sum for the function $f^2(\theta)$ for the partition P. So, we have

$$\frac{1}{2}\underline{\int_{\alpha}^{\beta}} f^2(\theta)\, d\theta \le A \le \frac{1}{2}\overline{\int_{\alpha}^{\beta}} f^2(\theta)\, d\theta$$

If $f^2(\theta)$ is integrable over $[\alpha, \beta]$, the upper and lower integrals coincide and we have

$$A = \frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta \qquad \ldots (8)$$

Let us now put to use the work we have done by calculating the area of the cardioid using its polar representation.

Example 6: Find the area enclosed by the cardioid $r = a(1 - \cos\theta)$.

Solution: We have r = 0 for $\theta = 0$ and r = 2a for $\theta = \pi$.

Since $\cos\theta = \cos(-\theta)$, the cardioid is symmetrical about the initial lines AOX. (See Fig. 14.)

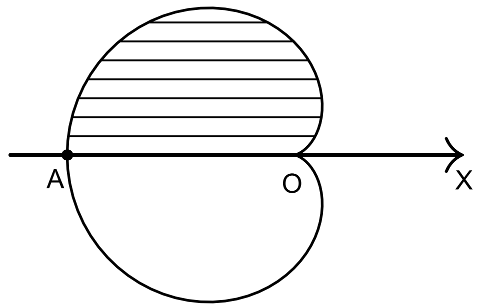


Fig. 14: Cardioid.

Hence the required area A, which is twice the area of the shaded region in Fig. 14, is given by

$$A = 2\int_0^{\pi} \frac{1}{2}r^2\, d\theta = \int_0^{\pi} a^2(1-\cos\theta)^2\, d\theta$$

Since $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$, we have

$(1-\cos\theta)^2 = \left\{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\right\}^2 = 4\sin^4\frac{\theta}{2}$. So,

$$A = 4a^2\int_0^{\pi} \sin^4\frac{\theta}{2} d\theta$$

On substituting $\phi = \frac{\theta}{2}$, we get

$$A = 8a^2\int_0^{\pi/2} \sin^4\phi\, d\phi$$

Using reduction formula in Eqn.(13), Unit 19, we have

$$A = 8a^2\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2} = \frac{3}{2}a^2\pi$$

* * *

Try to do the following exercises now to check your understanding of the example we have discussed.

E4) Find the area of a loop of the curve $r = a\sin 3\theta$, $a > 0$.

E5) Find the area enclosed by the curve $r = a\cos 2\theta$ and the radius vectors $\theta = 0$, $\theta = \pi/2$.

E6) Find the area of the region outside the circle $r = 2$ and inside the lemniscate $r^2 = 8\cos 2\theta$.

Now we shall turn our attention to closed curves whose equations are given in the parametric form.

Let the parametric equations be

$$x = \phi(t),\ \ y = \psi(t),\ \ t \in [\alpha,\ \beta],$$

where $\phi(\alpha) = \phi(\beta)$, and $\psi(\alpha) = \psi(\beta)$, represent a plane closed curve. See Fig. 15.

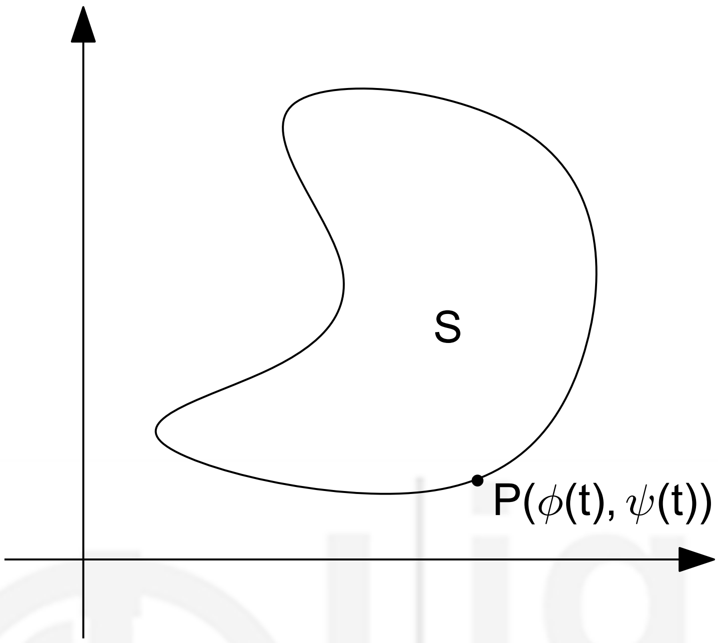


Fig. 15

If we assume that $\phi(t)$ has a continuous derivative and $\psi(t)$ is integrable, we have

$$\text{Area } S = -\int_{\alpha}^{\beta} y\frac{dx}{dt}dt \qquad \ldots(i)$$

We assume that we move in anti-clockwise direction.

If we assume that $\psi(t)$ has a continuous derivative and $\phi(t)$ is integrable, we have

$$S = \int_{\alpha}^{\beta} x\frac{dy}{dt}\,dt \qquad \ldots(ii)$$

Again, we assume that we move in anti-clockwise direction.

From (i) and (ii), we get

$$2S = \int_{\alpha}^{\beta} \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\ dt.$$

Hence, if both x(t) and y(t) have continuous derivatives, we have

$$S = \frac{1}{2}\int_{\alpha}^{\beta}(xdy - ydx) \qquad \ldots(9)$$

We will not prove any of these formulas in this course.

We can use any of the formulas (i), (ii) and Eqn. (9) above for calculating S. But in many cases you will find that formula Eqn. (9) is more convenient because of its symmetry.

Example 7: Check that the parametric form of the Astroid

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1.$$

is given by $x = a\cos^3 t,\ y = b\sin^3 t$. Use the parametric form to find its area.

Solution: By substitution, you can easily check that $x = a\cos^3 t,\ y = b\sin^3 t$' is the parametric form of the given curve. The curve lies between the lines $x = \pm a$ and $y = \pm b$ since $-1 \le \cos t \le 1$, and $-1 \le \sin t \le 1$. The curve is symmetrical about both the axes since its equation remains unchanged if we change the signs of x and y. The value t = 0 corresponds to the point (a, 0) and $t = \pi/2$ corresponds to the point (b, 0). By applying the curve tracing methods discussed in Unit 9 we can draw this curve. (See Fig. 16). The region bounded by the Astroid is shown in this figure.

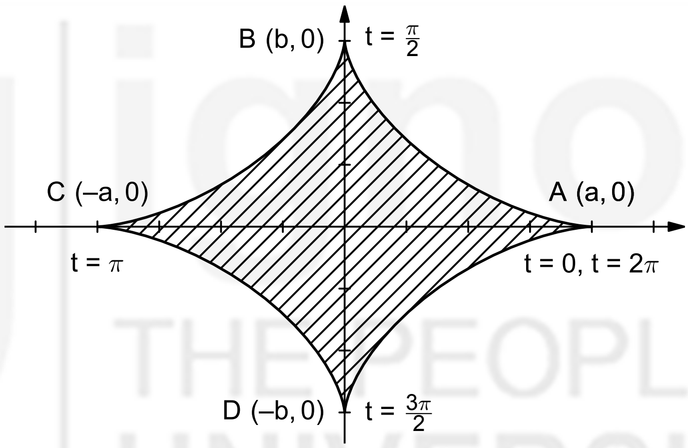


Fig. 16: Area of an Astroid with a = 4, b = 3.

The area A of the region is given by

$$A = \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} a\cos^3 t(3b\sin^2 t\cos t) - b\sin^3 t(-3a\cos^2 t\sin t)\,dt$$

$$= \frac{3ab}{2}\int_0^{2\pi}\cos^2 t\sin^2 t\left(\cos^2 t + \sin^2 t\right)\,dt = \frac{3ab}{2}\int_0^{2\pi}\cos^2 t\sin^2 t\,dt$$

$$= \frac{3ab}{8}\int_0^{2\pi}\sin^2 2t\,dt = \frac{3ab}{8}\int_0^{2\pi}\left(\frac{1-\cos 4t}{2}\right)\,dt = \frac{3ab}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8}\right)\Bigg|_0^{2\pi} = \frac{3\pi ab}{8}$$

* * *

Try these exercises to check your understanding of our discussion so far.

E7) Find the area of the curve $x = a\,(3\sin\theta - \sin^3\theta),\ y = a\cos^3\theta,\ 0 \le \theta \le 2\pi.$

E8) Find the area of the Tricuspoid given by $x = a(2\cos\theta + \cos 2\theta)$, $y = a(2\sin\theta - \sin 2\theta)$, $0 \le \theta \le 2\pi$.

E9) Find the area of one of the loops of the curve $x = a\sin 2t$, $y = a\sin t$.

We conclude this section here. In the next section, we will see how we can find the arc lengths of curves using integration.

20.3 LENGTH OF A PLANE CURVE

In this section we shall see how definite integrals can be used to find the lengths of plane curves whose equations are given in the Cartesian, polar or parametric form. A curve whose length can be found is called a **rectifiable curve** and the process of finding the length of a curve is called **rectification**. You will see here that to find the length of an arc of a curve, we shall have to integrate an expression which involves not only the given function, but also its derivative. Therefore, to ensure the existence of the integral which determines the arc length, we make an assumption that the function defining the curve is derivable, and its derivative is also continuous on the interval of integration.

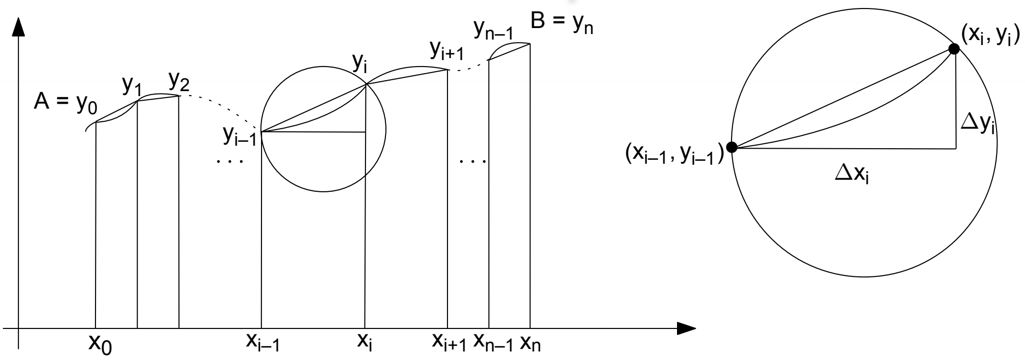
Let's first consider a curve whose equation is given in the Cartesian form.

Let $y = f(x)$ be defined on the interval $[a, b]$. We assume that f is derivable and its derivative f' is continuous. Let us consider a partition P of $[a, b]$, given by

$$P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$$

The ordinates $x = a$ and $x = b$ determine the extent of the arc AB of the curve $y = f(x)$ [Fig. 17(a)]. Let $y_i = f(x_i)$, $i = 0, 2, \ldots, n$, be the points in which the lines $x = x_i$ meet the curve.

Join the successive points $A = y_0, y_1, y_2, \ldots, B = y_n$, by straight line segments. The sum of the lengths of the line segments gives an approximation to the length of the curve.



(a) (b)

Fig. 17: Length of a curve.

Let us write ℓ_i for the length of the line segment joining the points (x_{i-1}, y_{i-1}) and (x_i, y_i). If we can find the sum ℓ_i, $1 \le i \le n$, the sum $\sum_{i=1}^{n} \ell_i$ will give us an approximation to the length of the curve.

But how do we find the length of any of these line segments? Look at the enlarged portion in Fig. 17(b). Looking at it we find that $\ell_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$,.

where $\Delta x_i = \Delta_i = x_i - x_{i-1}$, and $\Delta y_i = y_i - y_{i-1}$. In this way we can find the lengths of the chords joining (x_{i-1}, y_{i-1}) and (x_i, y_i).

Our assumptions that f is derivable on [a, b], and that f' is continuous, permit us to apply the mean value theorem. Thus, there exists a point $P_i^*(x_i^*, y_i^*)$ between the points (x_{i-1}, y_{i-1}) and (x_i, y_i) on the curve, where the tangent to the curve is parallel to the chord. That is,

$$f'(x_i^*) = \frac{\Delta y_i}{\Delta x_i}$$

or, $\Delta y_i = f'(x_i^*)\, \Delta x_i$. It follows that $\ell_i = \sqrt{1 + f\left(x_i^*\right)^2}\Delta x_i$. Writing

$$M_i = \sup\left\{\sqrt{1 + f'(x)^2} \,|x \in [x_{i-1}, x_i]\right\},\ m_i = \inf\left\{\sqrt{1 + f'(x)^2} \,|x \in [x_{i-1}, x_i]\right\}$$

we have

$$m_i \Delta x_i \le \ell_i \le M_i \Delta x_i$$

Summing up from 1 to n, it follows that

$$\sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} \ell_i \le \sum_{i=1}^{n} M_i \Delta x_i.$$

But,

$$\sum_{i=1}^{n} m_i \Delta x_i = L(P, g) \text{ and } \sum_{i=1}^{n} M_i \Delta x_i = U(P, g)$$

for the function $g(x) = \sqrt{1 + f'(x)^2}$ corresponding to the partition P. It follows that

$$\underline{\int_a^b} g(x)\, dx \le \sum_{i=1}^{n} \ell_i \le \overline{\int_a^b} g(x)\, dx$$

Since we have assumed that $f'(x)$ is continuous, it follows that g(x) is integrable and

$$\underline{\int_a^b} g(x)\, dx = \overline{\int_a^b} g(x)\, dx = \int_a^b g(x)\, dx$$

Since $\sum_{i=1}^{n} \ell_i$ is sandwiched between the upper and lower integrals and the upper and lower integrals are equal the sum $\sum_{i=1}^{n} \ell_i$ approaches a definite value as the number of subdivisions in the partition grow bigger and bigger and give the length of the curve in the limit.

Therefore

$$L_A^B = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \qquad \ldots(10)$$

Remark 1: It is sometimes convenient to express x as a single valued function of y. In this case we interchange the roles of x and y, and get the length

$$L_A^B = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy, \qquad \ldots(11)$$

where the limits of integration are with respect to y. Note that the length of an arc of a curve is invariant since it does not depend on the choice of coordinates, that is, on the frame of reference. Our assumption that f' is continuous on $[a, b]$ ensures that the integrals in Eqn. (10) and Eqn. (11) exist, and their value L_A^B is the length of the curve $y = f(x)$ between the ordinates $x = a$ and $x = b$.

The following example illustrates the use of the formulas given by Eqn. (10) and Eqn. (11).

Example 8: Find the length of the arc of the curve $y = \ln|x|$ intercepted by the ordinates $x = 1$ and $x = 2$. We have drawn the curve $y = \ln|x|$ in Fig. 18.

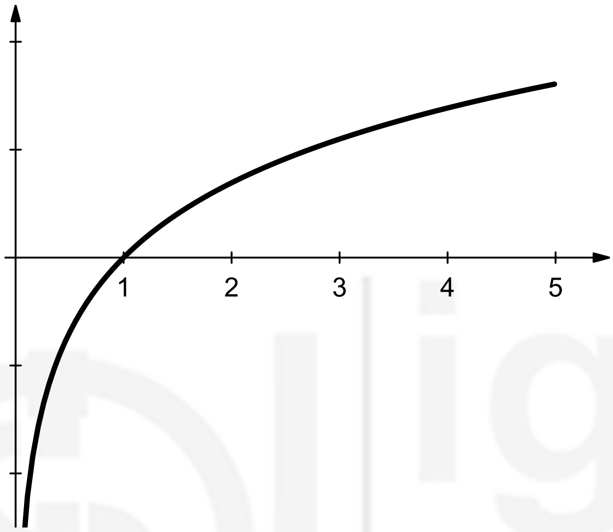


Fig. 18: Length of $y = \ln|x|$ from $x = 1$ to $x = 2$.

Using Eqn. (10), the required length L_1^2 is given by

$$L_1^2 = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{\left(1 + \frac{1}{x^2}\right)}\, dx, \text{ since } \frac{dy}{dx} = \frac{1}{x}$$

$$= \int_1^2 \frac{\sqrt{1 + x^2}}{x}\, dx$$

If we put $1 + x^2 = t^2$, we get $\frac{dx}{dt} = \frac{t}{x}$, and therefore,

$$L_1^2 = \int_{\sqrt{2}}^{\sqrt{5}} \left(1 + \frac{1}{(t^2 - 1)}\right) dt = \int_{\sqrt{2}}^{\sqrt{5}} dt + \int_{\sqrt{2}}^{\sqrt{5}} \frac{1}{t^2 - 1}\, dt = \left[t + \frac{1}{2} \ln \frac{t-1}{t+1}\right]_{\sqrt{2}}^{\sqrt{5}}$$

$$= \sqrt{5} - \sqrt{2} + \frac{1}{2} \ln \frac{\sqrt{5} - 1}{\sqrt{5} + 1} - \frac{1}{2} \ln \frac{\sqrt{2} - 1}{\sqrt{2} + 1}$$

$$= \sqrt{5} - \sqrt{2} + \ln \frac{2}{\sqrt{5} + 1} - \ln \frac{1}{\sqrt{2} + 1}$$

$$= \sqrt{5} - \sqrt{2} + \ln \frac{2\left(\sqrt{2} + 1\right)}{\sqrt{5} + 1}$$

We can also use Eqn. (11) to solve this example. For this we write the equation $y = \ln x$ as $x = e^y$. The limits $x = 1$ and $x = 2$, then correspond to the limits $y = 0$ and $y = \ln 2$, respectively. Hence, using Eqn. (11), we obtain

$$L_0^{\ln 2} = \int_0^{\ln 2} \sqrt{1 + e^{2y}}\, dy$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} \frac{u^2}{u^2 - 1} du, \text{ on putting } 1 + e^{2y} = u^2$$

$$= \int_{\sqrt{2}}^{\sqrt{5}} \left(1 + \frac{1}{u^2 - 1}\right) du, \; = \sqrt{5} - \sqrt{2} + \ln \frac{2\left(\sqrt{2} + 1\right)}{\sqrt{5} + 1}$$

as we have seen earlier. This verifies our observation in Remark 1 that both Eqn. (10) and Eqn. (11) give us the same value of arc length.

* * *

Now here are some exercises for you to solve.

E10) Find the length of the line $x = 3y$ between the points $(3, 1)$ and $(6, 2)$. Verify your answer by using the distance formula.

E11) Find the length of the curve $y = \ln \sec x$ between the points $x = 0$ and $x = \pi/3$.

E12) Find the length of the semi-cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a).

E13) Show that the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$ is $a(\ln 2 + 15/16)$.

In the next sub-section we shall consider curves whose equations are expressed in the parametric form. Here we shall derive a formula to find the length of a curve given by a pair of parametric equations.

Let $x = \phi(t),\ y = \psi(t),\ \alpha \le t \le \beta$ be the equation of a curve in parametric form. As in the previous sub-section, we assume that the functions ϕ and ψ are both derivable and have continuous derivatives ϕ' and ψ' on the interval $[\alpha, \beta]$. We have

$$\frac{dx}{dt} = \phi'(t), \text{ and } \frac{dy}{dt} = \psi'(t).$$

Hence, assuming $\phi'(t) \neq 0$ for $t \in [\alpha, \beta]$, we have

$$\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}, \text{ and } \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{\psi'(t)}{\phi'(t)}\right)^2} = \frac{\sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}}{\phi'}$$

Now, using Eqn. (11), we obtain the length

$$L = \int_{x=\phi(\alpha)}^{x=\phi(\beta)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{t=\alpha}^{t=\beta} \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\ \frac{\phi'(t)}{\phi'(t)} dt$$

Thus,

$$L = \int_{\alpha}^{\beta} \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, dt \qquad \ldots (12)$$

The following example shows that sometimes it is more convenient to express the equation of a given curve in the parametric form in order to find its length.

Example 9: Let us go back to the Astroid we discussed in Example 7. We saw that the parametric form of the curve is given by

$$x = a\cos^3 t,\ \ y = b\sin^3 t.$$

Since the curve is symmetrical about both axes, the total length of the curve is four time its length in the first quadrant.

Now,

$$\frac{dx}{dt} = -3a\cos^2 t \sin t;\ \ \frac{dy}{dt} = 3b\sin^2 t \cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t\cos^2 t(a^2\cos^2 t + b^2\sin^2 t)$$

Hence, the length of the curve is

$$L = 4\int_0^{\pi/2} 3\sin t\cos t\sqrt{a^2\cos^2 t + b^2\sin^2 t}\, dt$$

$$= 12\int_0^{\pi/2} \sin t\cos t\sqrt{a^2\cos^2 t + b^2\sin^2 t}\, dt$$

Putting $u^2 = a^2\cos t^2 t + b^2\sin^2 t$, we obtain

$$2u = (2b^2 - 2a^2)\sin t\cos t\frac{dt}{du},$$

and the limits $t = 0$, $t = \pi/2$ correspond to $u = a$, $u = b$, respectively.

Thus, we have

$$L = 12\int_a^b \frac{u^2 du}{b^2 - a^2} = \frac{12}{b^2 - a^2}\left[\frac{u^3}{3}\right]_a^b = \frac{12}{b^2 - a^2}\ \frac{b^3 - a^3}{3} = \frac{4(a^2 + b^2 + ab)}{a + b}.$$

* * *

In the next example, we will find the length of the tricuspoid using its equation in parametric form.

Example 10: Find the length of the arc of the Tricuspoid we saw in Exercise 8.

Solution: We know from Exercise 8 that

$$\frac{dx}{d\theta} = -2a(\sin\theta + \sin 2\theta) \text{ and } \frac{dy}{d\theta} = 2a(\cos\theta + \cos 2\theta).$$

$$\therefore \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 4a^2(\sin\theta + \sin 2\theta)^2 + 4a^2(\cos\theta + \cos 2\theta)^2$$

$$= 4a^2\left(\sin^2\theta + \sin^2 2\theta + 2\sin\theta\sin 2\theta\right)$$
$$+ 4a^2\left(\cos^2\theta + \cos^2 2\theta + 2\cos\theta\cos 2\theta\right)$$
$$= 4a^2\,(2 + 2\sin\theta\sin 2\theta + 2\cos\theta\cos 2\theta)$$
$$= 4a^2\left(2 + 2\left(2\sin^2\theta\cos\theta + \cos\theta(1 - 2\sin^2\theta)\right)\right)$$
$$= 8a^2(1 + \cos\theta)$$

So, the length of the arc is

$$\int_0^{2\pi}\sqrt{8a^2(1+\cos\theta)}\,d\theta = \int_0^{2\pi}\sqrt{16a^2\cos\frac{\theta}{2}}\,d\theta$$
$$= 4a\int_0^{\pi}\cos\frac{\theta}{2}d\theta - 4a\int_{\pi}^{2\pi}\cos\frac{\theta}{2}\,d\theta$$

since $\cos\frac{\theta}{2} \geq 0$ for $0 \leq \theta \leq \pi$ and $\cos\frac{\theta}{2} \leq 0$ for $\pi \leq \theta \leq 2\pi$.

$$= 4a\left(2\sin\frac{\theta}{2}\Big|_0^{\pi} + 2\sin\frac{\theta}{2}\Big|_{\pi}^{2\pi}\right) = 16a.$$

* * *

Now you can apply equation Eqn. (12) to solve the exercises that follow.

E14) Find the length of the cycloid $x = a(\theta - \sin\theta);\ y = a(1 - \cos\theta)$.

E15) Show that the length of the arc of the curve $x = e^t\sin t,\ y = e^t\cos t$ from $t = 0$ to $t = \pi/2$ is $\sqrt{2}(e^{\pi/2} - 1)$.

We shall now consider the case of a curve whose equation is given in the polar form.

Let $r = f(\theta)$ determine a curve as θ varies from $\theta = \alpha$ to $\theta = \beta$, i.e., the function f is defined in the interval $[\alpha,\ \beta]$ (See Fig. 10.) As before, we assume that the function f is derivable and its derivative f' is continuous on $[\alpha, \beta]$. This assumption ensures that the curve represented by $r = f(\theta)$ is rectifiable.

Transforming the given equation into Cartesian coordinates by taking $x = r\cos\theta,\ y = r\sin\theta$, we obtain $x = f(\theta)\cos\theta,\ y = f(\theta)\sin\theta$.

Now we proceed as in the case of parametric equations, and get

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}}{dx/d\theta}$$

Hence, changing the variable x to θ, the length of the arc of the curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$L = \int_{x=f(\alpha)\cos\alpha}^{x=f(\beta)\cos\beta}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{[f'(\theta)\cos\theta - f(\theta)\sin\theta]^2 + [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2}\, d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{[f(\theta)^2 + [f'(\theta)]^2}\; d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\; d\theta \qquad \ldots (13)$$

We shall apply this formula to find the length of the curve in the following example.

Example 11: Find the perimeter of the cardioid $r = a(1 + \cos\theta)$.

Solution: We note that the curve is symmetrical about the initial line . See Fig. 19 Therefore its perimeter is double the length of the arc of the curve lying above the x-axis.

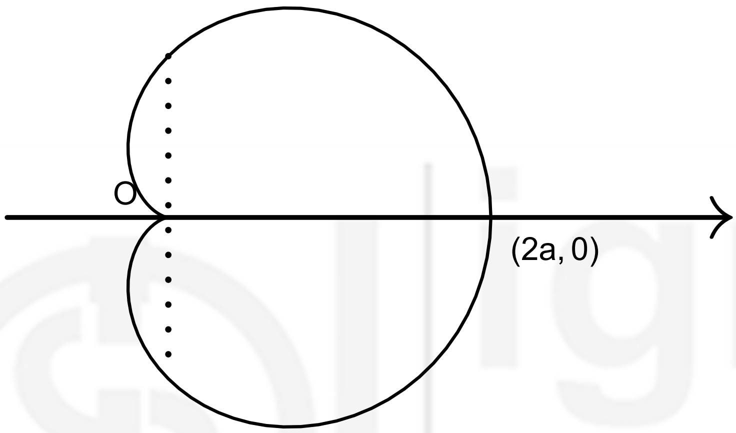


Fig. 19: Length of the arc of the Cardioid r = a(1 + cos θ).

Now, $\frac{dr}{d\theta} = -a\sin\theta$. Hence, we have

$$L = 2\int_{0}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\; d\theta = 2a\int_{0}^{\pi} \sqrt{(1 + \cos\theta)^2 + \sin^2\theta}\; d\theta$$

Since $\frac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2}$ and $\cos\frac{\theta}{2} > 0$ if $0 \leq \pi$, we have

$$= 2a\int_{0}^{\pi} \sqrt{2(1 + \cos\theta)}\, d\theta = 4a\int_{0}^{\pi} \left|\cos\frac{\theta}{2}\right|\; d\theta = 8a\sin\frac{\theta}{2}\Big|_{0}^{\pi} = 8a$$

* * *

In this section we have derived and applied the formulas for finding the length of a curve when its equation is given in either of the three forms: Cartesian, parametric or polar. Let us summarise our discussion in Table 1. Using this table you will be able to solve these exercises now.

E16) Find the length of the curve $r = a\cos^3(\theta/3)$.

E17) Find the length of the circle of radius 2 which is given by the equations $x = 2\cos t + 3,\; y = 2\sin t + 4,\; 0 \leq t \leq 2\pi$.

E18) Show that the arc of the upper half of the curve $r = a(1 - \cos\theta)$ is bisected by $\theta = 2\pi/3$.

E19) Find the length of the curve $r = a(\theta^2 - 1)$ from $\theta = -1$ to $\theta = 1$.

Table 1: Length of the arc of a curve

Equation of the Curve	Length L
$y = f(x)$	$\int_a^b \sqrt{1 + (f'(x))^2}\, dx$
$x = g(y)$	$\int_c^d \sqrt{1 + (g'(y))^2}\, dy$
$x = \phi(t),\ y = \psi(t)$	$\int_\alpha^\beta \sqrt{(\phi'(t))^2 + (\psi'(t))^2}\, dt$
$r = f(\theta)$	$\int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$

We have come to the end of this Unit. In the next section, we give a brief summary of this unit.

20.4 SUMMARY

In this Unit we have seen:

1) We say that a region between the graph of any two functions $f_1(x)$ and $f_2(x)$ is **vertically simple** if any vertical line intersects the region in either a single point or in a line segment with its lower end point on $y = f_1(x)$ and upper end point on $y = f_2(x)$.
2) The area a vertically simple region bounded by the curves $y = f_1(x)$, $y = f_2(x)$, the lines $x = a$ and $x = b$ is given by $\int_a^b (f_1(x) - f_2(x))\, dx$.
3) Given the equation of a curve in the polar form $r = f(\theta)$, the area of the region bounded by the lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta f^2(\theta)\, d\theta = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

4) If the equation of closed curve enclosing a vertically simple region is given the parametric form $x = x(t)$, $y = y(t)$, $\alpha \le t \le \beta$ and the curve is traced out in the anti-clockwise directions as t varies from α to β, the area of the enclosed region is given by the following formulas:

$$A = -\int_\alpha^\beta y\frac{dx}{dt}\, dt$$

$$A = \int_\alpha^\beta x\frac{dy}{dt}\, dt$$

$$A = \frac{1}{2}\int_\alpha^\beta \left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt}\right) dt$$

5) We also saw the formulas for the length of the arc of a curve given in different forms:

Equation of the Curve	Length L
Cartesian form y = f(x)	$\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$
cartesian form x = g(y)	$\int_c^d \sqrt{1+(g'(y))^2}\,dy$
Parametric form $x=\phi(t),\ y=\psi(t)$	$\int_\alpha^\beta \sqrt{(\phi'(t))^2+(\psi'(t))^2}\,dt$
Polar form $r=f(\theta)$	$\int_\alpha^\beta \sqrt{(f(\theta))^2+\left(f'(\theta)\right)^2}\,d\theta$

20.5 SOLUTIONS/ANSWERS

E1) Consider Fig. 20. We have to find the area of the shaded region.

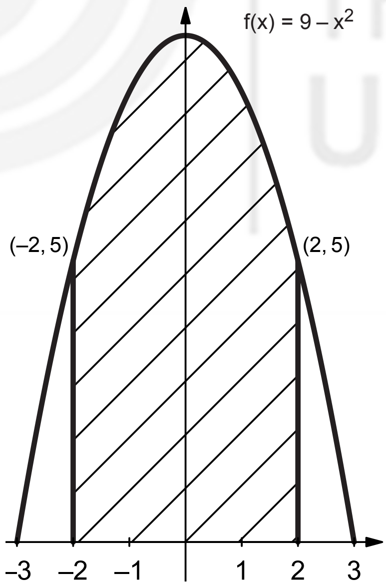


Fig. 20: Area under the curve f(x) = 9 – x².

The entire curve lies in the upper half plane. So, the area is

$$\int_{-2}^{2}\left(9-x^2\right)\,dx = 9x-\frac{x^3}{3}\Big|_{-2}^{2} = \left(18-\frac{8}{3}\right)-\left(-18+\frac{8}{3}\right) = \frac{92}{3}\text{ square units}$$

E2) See Fig. 21. The points of intersection are given by $3-4x-x^2=-(1+x)$ or $(x+4)(x-1)=0$. So, A(–4, 3) and B(1, –2) are the points of intersection of $y = 3 - 4x - x^2$ and $y = -(x + 1)$. We choose $f_1(x) = 3 - 4x - x^2$ and $f_2(x) = -(1 + x)$.

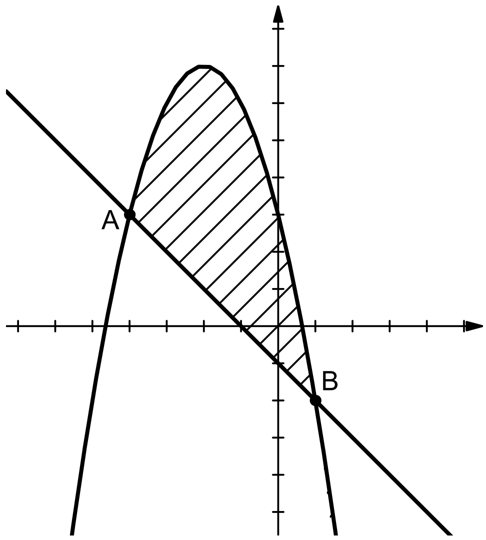


Fig. 21: Area between the curves $f_1(x) = 3 - 4x - x^2$ and $f_2(x) = -(1 + x)$.

Then,

$$f_1(x) - f_2(x) = 4 - 3x - x^2 = -(x + 4)(x - 1)$$

Let us now check $f_1(x) - f_2(x) \geq 0$. Since $x \leq 1$ in $[-4, 1]$, $x - 1 \leq 0$. Also, $x \geq -4$, so $x + 4 \geq 0$. Thus $(x - 1)(x + 4) \leq 0$ and $-(x - 1)(x + 4) \geq 0$ in $[-4, 1]$.

Therefore, the area of the region is

$$\int_{-4}^{1} \left(4 - 3x - x^2\right) dx = 4x - 3\frac{x^2}{2} - \frac{x^3}{3}\Bigg|_{-4}^{1}$$

$$= \left(4 - \frac{3}{2} - \frac{1}{3}\right) - \left(-16 - 24 - \frac{-64}{3}\right) = \frac{125}{6}.$$

E3) See Fig. 22. Here, we take $f_1(x)$ such that its graph is the arc BEC of the ellipse. We have $f_1(x) = \frac{3\sqrt{25-x^2}}{5}$. We take $f_2(x)$ to be the function

$$f_2(x) = \begin{cases} -\frac{3\sqrt{25-x^2}}{5} & \text{if } -5 \leq x \leq -4 \\ \frac{x}{5} - 1 & \text{if } -4 \leq x \leq 5 \end{cases}$$

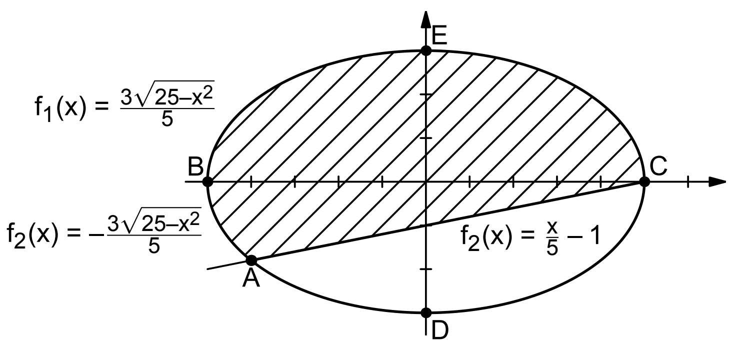


Fig. 22: Area between the curves $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and $y = \frac{x}{5} - 1$.

Note that, the graph of the portion of the arc of the ellipse from B to A, is the graph of the function $-\frac{3\sqrt{25-x^2}}{5}$. The graph of $f_2(x)$ from $x = -4$ to $x = 5$, consists of the graph of the line $y = \frac{x}{5} - 1$ from $x = -4$ to $x = 5$.

Note that, in $[-5, -4]$,

$$f_1(x) - f_2(x) = \frac{3\sqrt{25-x^2}}{5} - \left(-\frac{3\sqrt{25-x^2}}{5}\right) = \frac{6\sqrt{25-x^2}}{5} \geq 0.$$

In $[-4, 5]$,

$$f_1(x) - f_2(x) = \frac{3\sqrt{25-x^2}}{5} - \left(\frac{x}{5} - 1\right) = \frac{3\sqrt{25-x^2}}{5} + \left(1 - \frac{x}{5}\right) \geq 0.$$

The first term $\frac{3\sqrt{25-x^2}}{5} \geq 0$ since we take the positive square root of $25 - x^2$. Since $x \leq 5$ in $[-4, 5]$, $\frac{x}{5} \leq 1$ in $[-4, 5]$. So, $1 - \frac{x}{5} \geq 0$.

The required area is

$$\int_{-5}^{5} (f_1(x) - f_2(x))\, dx$$

$$= \int_{-5}^{-4} (f_1(x) - f_2(x))\, dx + \int_{-4}^{5} (f_1(x) - f_2(x))\, dx$$

$$= \int_{-5}^{-4} \left\{ \frac{3\sqrt{25-x^2}}{5} - \left(-\frac{3\sqrt{25-x^2}}{5}\right) \right\} dx$$

$$+ \int_{-4}^{5} \left\{ \frac{3\sqrt{25-x^2}}{5} - \left(\frac{x}{5} - 1\right) \right\} dx$$

$$= \frac{6}{5}\int_{-5}^{-4} \sqrt{25-x^2}\, dx + \frac{3}{5}\int_{-4}^{5} \sqrt{25-x^2}\, dx - \int_{-4}^{5} \left(\frac{x}{5} - 1\right) dx \qquad \ldots(14)$$

Recall the following formula that we derived in Unit 18, Subsection 18.3.3:

$$\int \sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C. \qquad \ldots(15)$$

Using Eqn. (15) to evaluate the first integral in Eqn. (14), we get

$$\int \sqrt{5^2 - x^2}\, dx = \frac{1}{2}x\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + C$$

where the domain of definition of $\sin^{-1} x$ is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Using the fundamental theorem of calculus, we then have,

$$\int_{-5}^{-4} \sqrt{25 - x^2}\, dx$$

$$= \frac{1}{2}x\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\Bigg|_{-5}^{-4}$$

$$= \left\{ \left(\frac{1}{2}(-4)\sqrt{25 - (-4)^2} + \frac{25}{2}\sin^{-1}\left(\frac{-4}{5}\right)\right) - \left(\frac{1}{2}(-5)\sqrt{25 - (-5)^2} + \frac{25}{2}\sin^{-1}\left(\frac{-5}{5}\right)\right) \right\}$$

$$= \left(-6 - \frac{25}{2}\alpha\right) - \left\{\left(\frac{25}{2}\right)\left(-\frac{\pi}{2}\right)\right\}$$

where $0 < \alpha < 1$ is such that $\sin\alpha = \frac{4}{5}$. Therefore,

$$\int_{-5}^{-4}\sqrt{25-x^2}\,dx = \left(\frac{25}{2}\right)\left(\frac{\pi}{2}\right) - 6 - \frac{25}{2}\alpha = \frac{25}{4}\pi - \frac{25}{2}\alpha - 6. \qquad \ldots(16)$$

Again, using Eqn. (15), we get

$$\int_{-4}^{5}\sqrt{25-x^2}\,dx$$

$$= \left\{\left(\frac{1}{2}(5)\sqrt{25-5^2} + \frac{25}{2}\sin^{-1}\left(\frac{5}{5}\right)\right) - \left(\frac{1}{2}(-4)\sqrt{25-(-4)^2} + \frac{25}{2}\sin^{-1}\left(\frac{-4}{5}\right)\right)\right\}$$

$$= \frac{25}{4}\pi + \frac{25}{2}\alpha + 6 \qquad \ldots(17)$$

Also,

$$\int_{-4}^{5}\left(\frac{x}{5} - 1\right)dx = \left.\frac{x^2}{10} - x\right|_{-4}^{5} = \left(\frac{25}{10} - 5\right) - \left(\frac{16}{10} + 4\right) = -\frac{81}{10} \qquad \ldots(18)$$

From Eqn. (16), Eqn. (17) and Eqn. (18), the required area is

$$\frac{6}{5}\left(\frac{25}{4}\pi - \frac{25}{2}\alpha - 6\right) + \frac{3}{5}\left(\frac{25}{4}\pi + \frac{25}{2}\alpha + 6\right) - \frac{81}{10} = \frac{45}{4}\pi - \frac{75}{4}\alpha - \frac{117}{10}$$

E4) See Fig. 23.

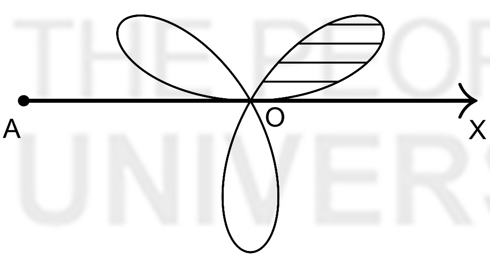


Fig. 23: Curve $r = a\sin 3\theta$.

Note that $r = 0$ for the beginning of the first loop. To find the end point of the loop, we have to find the first value of θ which is greater than 0 for which $r = a\sin 3\theta = 0$. For such a θ, $\sin 3\theta = 0$ since $a \neq 0$. The first such value of θ is $3\theta = \pi$ or $\theta = \frac{\pi}{3}$. So, the required area is

$$A = \frac{1}{2}\int_0^{\pi/3} r^2\,d\theta = \frac{a^2}{2}\int_0^{\pi/3}\sin^2 3\theta\,d\theta = \frac{a^2}{6}\int_0^{\pi}\sin^2 u\,du \text{ if } u = 3\theta.$$

$$= \frac{\pi a^2}{12}.$$

E5) See Fig. 24.

$$A = \frac{1}{2}\int_0^{\pi/2} a^2\cos^2 2\theta\,d\theta = \frac{a^2}{4}\int_0^{\pi}\cos^2\phi\,d\phi, \quad \text{on substituting } \phi = 2\theta$$

$$= \left.\left(\frac{a^2}{8}\phi + \frac{\sin 2\phi}{2}\right)\right|_0^{\pi} = \frac{\pi a^2}{8}$$

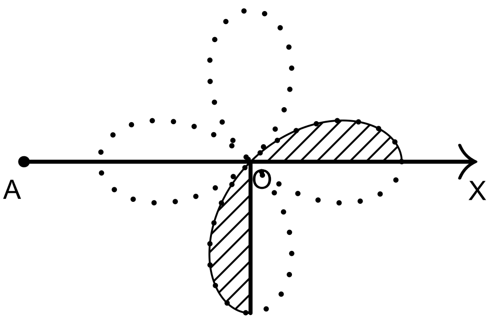


Fig. 24: Curve r = a cos 2θ.

E6) See Fig. 25.

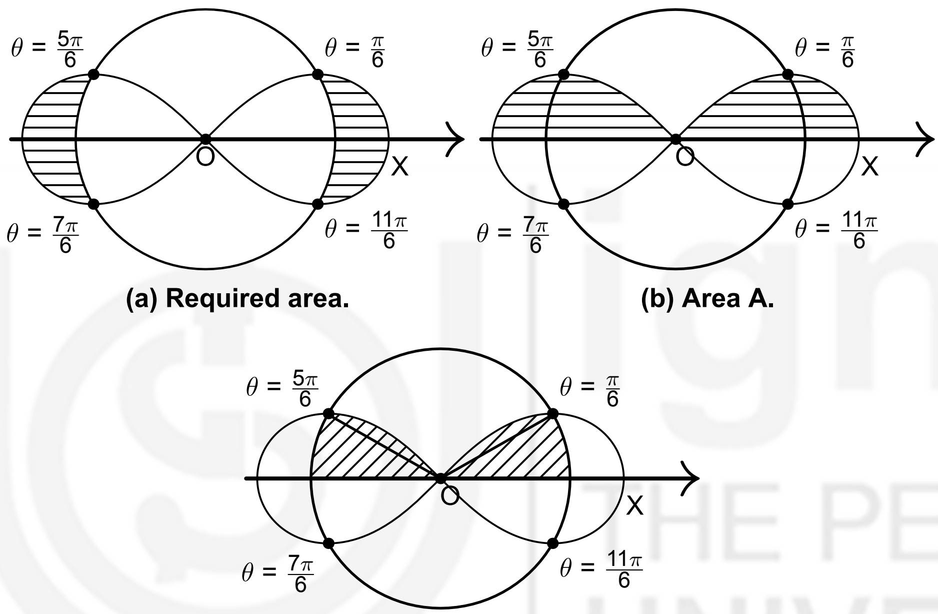


Fig. 25: Area outside r = 2 and inside $r^2 = 8\cos 2\theta$.

Points of intersection are given by $8\cos 2\theta = 4$ or $\cos 2\theta = 1/2$. The values of θ are

$$\theta = \frac{\pi}{6}, \ \frac{5\pi}{6}, \ \frac{7\pi}{6}, \ \frac{11\pi}{6}$$

We have shown the four points of intersections by four dots.

We have to find the area of the shaded region in Fig. 25a. Because of the symmetry about the initial line, the required area is 2{(A – B)} where the area A is shown in Fig. 25b and the area B is shown in Fig. 25c. The area of the region A is

$$\text{Area A} = \frac{1}{2}\int_0^{\frac{\pi}{4}} 8\cos 2\theta \, d\theta + \frac{1}{2}\int_{\frac{3\pi}{4}}^{\pi} 8\cos 2\theta \, d\theta$$

$$= 2\sin 2\theta\Big|_0^{\frac{\pi}{4}} + 2\sin 2\theta\Big|_{\frac{3\pi}{4}}^{\pi} = 2 + 2\left(0 - \sin\left(\frac{3\pi}{2}\right)\right) = 4$$

Let us now compute the area B. Note that, we have to divide each of the right and left halves into two parts. The portion in the right half is the sum

Jacob Bernoulli, a Swiss Mathemtatician and Physicist, described the Lemniscate in a paper in 1604.

Jacob Bernoulli 1655-1705

of the area of the circle r = 2 bounded by the lines $\theta = 0$ and $\theta = \frac{\pi}{6}$ and the area of the leminiscate bounded by the lines $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{4}$. So, this area is

$$\frac{1}{2}\int_0^{\frac{\pi}{6}} 4\,d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 8\cos 2\theta\,d\theta = \frac{\pi}{3} + 2\sin 2\theta\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{\pi}{3} + 2\left(1 - \frac{\sqrt{3}}{2}\right)$$

Similarly the left half of B is the sum of the area of the leminiscate bounded by the lines $\theta = \frac{3\pi}{4}$ and $\theta = \frac{5\pi}{6}$ and the area of the circle r = 2 bounded by the lines $\theta = \frac{5\pi}{6}$ and $\theta = \pi$. This area is

$$\frac{1}{2}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{6}} 8\cos 2\theta\,d\theta + \frac{1}{2}\int_{\frac{5\pi}{6}}^{\pi} 4\,d\theta = 2\sin 2\theta\Big|_{\frac{3\pi}{4}}^{\frac{5\pi}{6}} + \frac{\pi}{3} = \frac{\pi}{3} + 2\left(1 - \frac{\sqrt{3}}{2}\right)$$

So, the required area is

$$2(A - B) = 2\left\{4 - 2\left(\frac{\pi}{3} + 2 - \sqrt{3}\right)\right\} = 4\sqrt{3} - \frac{4\pi}{3}$$

E7) See Fig. 26.

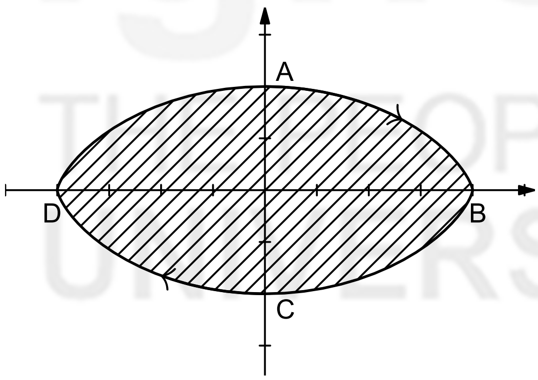


Fig. 26: Area of the curve $x = a\left(3\sin\theta - \sin^3\theta\right)$, $y = a\cos^3\theta$ when a = 2.

Note that, as θ increases from 0 to 2π the curve is traced out clockwise, starting from A. We use Eqn. (i), changing the sign from negative to positive. The required area is

$$A = \int_0^{2\pi} y\frac{dx}{d\theta}\,d\theta$$

$$\frac{dx}{d\theta} = 3a(\cos\theta - \sin^2\theta\cos\theta) = 3a\cos^3\theta$$

Using Eqn. (13) of Unit 19,

$$A = \int_0^{2\pi} y\frac{dx}{d\theta}d\theta = 3a^2\int_0^{2\pi}\cos^6\theta\,d\theta = 6a^2\cdot\frac{15}{48}2\pi = \frac{15a^2\pi}{8}.$$

E8) See Fig. 27. We have

$$\frac{dx}{d\theta} = -2a\sin\theta - 2a\sin 2\theta$$
$$\frac{dy}{d\theta} = 2a\cos\theta - 2a\cos 2\theta$$
$$x\frac{dy}{d\theta} = 4a^2\cos^2\theta - 2a^2\cos\theta\cos 2\theta - 2a^2\cos^2 2\theta$$
$$y\frac{dx}{d\theta} = -4a^2\sin^2\theta - 2a^2\sin\theta\sin 2\theta + 2a^2\sin^2 2\theta$$
$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = 4a^2(\cos^2\theta + \sin^2\theta) - 2a^2(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta)$$
$$-2a^2(\cos^2 2\theta + \sin^2 2\theta)$$
$$= 2a^2 - 2a^2\cos 3\theta$$

The Tricuspoid was first considered by Euler in connection with an optical problem.

L. Euler (1703-1783)

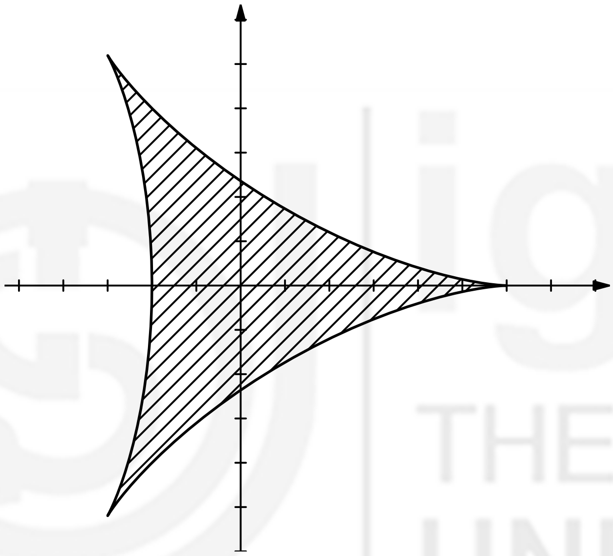


Fig. 27: Area of the Tricuspoid $x = a(2\cos\theta + \cos 2\theta)$, $y = a(2\sin\theta - \sin 2\theta)$ when $a = 2$**.**

Therefore, the required area is

$$\frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right)d\theta = \frac{1}{2}\int_0^{2\pi} 2a^2\,d\theta - \frac{1}{2}2a^2\int_0^{2\pi}\cos 3\theta\,d\theta$$
$$= 2\pi a^2 - a^2\left.\frac{\sin 3\theta}{3}\right|_0^{2\pi} = 2\pi a^2$$

E9) See Fig. 28. Note that for $t = 0$, we get the point $(0, 0)$. Putting $\sin t = 0$, $\sin 2t = 0$, we find that $t = \pi$. Also, there is no smaller value of t for which $\sin t = 0$, $\sin 2t = 0$. Note also that the y-coordinate $y = a\sin t > 0$. So, the portion of the curve where $y \geq 0$ is traced out in the anti-clockwise direction.

Using Eqn. (ii), we get

$$A = \int_0^{\pi} x\frac{dy}{dt}\,dt = a^2\int_0^{\pi}\sin 2t\cos t\,dt = 2a^2\int_0^{\pi}\sin^2\cos t\,dt$$

using $\sin 2t = 2\sin t\cos t$. On substituting $u = \cos t$

$$A = a^2 \int_{-1}^{1} u^2\, du = 2a^2 \left(\frac{u^3}{3}\right)\Bigg|_{-1}^{1} = \frac{4a^2}{3}$$

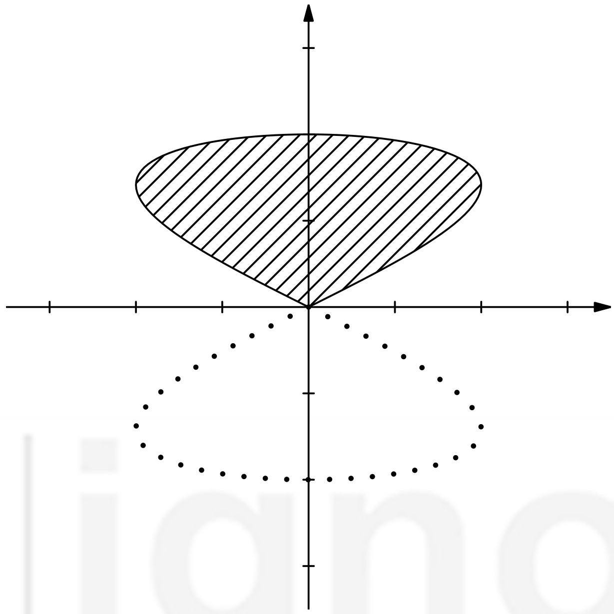


Fig. 28: Area of one loop of the curve $x = a\sin 2t$, $y = a\sin t$ when $a = 2$**.**

E10) We have

$$L = \int_{c}^{d} \sqrt{1 + (dx/dy)^2}\, dy = \int_{1}^{2} \sqrt{1 + (3)^2}\, dy = \sqrt{10} \int_{1}^{2} dy = \sqrt{10}.$$

By distance formula, $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$L = \sqrt{(3-6)^2 + (1-2)^2} = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$$

E11) We have

$$L = \int_{a}^{b} \sqrt{1 + (dy/dx)^2}\, dx \quad \left(\frac{dy}{dx} = \frac{1}{\sec x}.\sec x \tan x = \tan x\right)$$

$$= \int_{0}^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_{0}^{\pi/3} \sec x\, dx = \ln|\sec x + \tan x|\Bigg|_{0}^{\pi/3}$$

$$= \ln \left|\frac{\sec \pi/3 + \tan \pi/3}{\sec 0 + \tan 0}\right| = \ln\left(2 + \sqrt{3}\right).$$

E12) $y = \sqrt{\frac{x^3}{a}}$. $\therefore$ $\frac{dy}{dx} = (3/2)\sqrt{\frac{x}{a}}$

$$L = \int_0^a \sqrt{1+\frac{9x}{4a}}\,dx = \frac{1}{2\sqrt{a}}\int_0^a \sqrt{4a+9x}\,dx = \frac{1}{27\sqrt{a}}(4a+9x)^{3/2}\Big]_0^a$$

$$= \frac{1}{27\sqrt{a}}\left((13a)^{3/2} - (4a)^{3/2}\right) = \frac{a}{27}(13^{3/2} - 8)$$

E13) $3y = 8x$, so, $y = \frac{8x}{3}$. Substituting this in $y^2 = 4ax$ we get $\frac{64x^2}{9} = 4ax$, i.e., $64x^2 - 36ax = 0$.

$$\Rightarrow x = 0 \text{ or } x = \frac{9a}{16}$$

$$\Rightarrow y = 0 \text{ or } y = \frac{3a}{2}$$

Hence $(0, 0)$ and $\left(\frac{9a}{16}, \frac{3a}{2}\right)$ are the points of intersection. Now $4ax = y^2 \Rightarrow \frac{dx}{dy} = \frac{y}{2a}$

$$\therefore L = \int_0^{3a/2} \sqrt{1+\frac{y^2}{4a^2}}\,dy = \frac{1}{2a}\int_0^{3a/2}\sqrt{4a^2+y^2}\,dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{4a^2+y^2} + 2a^2 \ln|y + \sqrt{4a^2+y^2}|\right]_0^{3a/2}$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2\ln 2\right] = \left(\frac{15}{16} + \ln 2\right) a$$

E14) We have

$$\frac{dx}{d\theta} = a(1-\cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= a^2[1+\cos^2\theta - 2\cos\theta + \sin^2\theta] = 2a^2(1-\cos\theta) = 4a^2\sin^2(\theta/2)$$

$$\therefore L = 2a\int_0^{2\pi}\sin(\theta/2)\,d\theta = 4a\int_0^{\pi}\sin\phi\,d\phi = 8a\int_0^{\pi/2}\sin\phi\,d\phi = 8a$$

E15) We have

$$\frac{dx}{dt} = e^t(\cos t + \sin t), \quad \frac{dy}{dt} = e^t(\cos t - \sin t)$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2e^{2t}$$

$$\therefore L = \sqrt{2}\int_0^{\pi/2} e^t dt = \sqrt{2}\,e^t\Big|_0^{\pi/2} = \sqrt{2}\left(e^{\pi/2} - 1\right)$$

E16) $r = a\cos^3\frac{\theta}{3} \Rightarrow \frac{dr}{d\theta} = -a\cos^2\frac{\theta}{3}\sin\frac{\theta}{3}$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos^6\frac{\theta}{3} + a^2\cos^4\frac{\theta}{3}\sin^2\frac{\theta}{3} = a^2\cos^4\frac{\theta}{3}$$

$$\therefore L = 2a\int_0^{3\pi/2}\cos^2\frac{\theta}{3}d\theta = 6a\int_0^{\pi/2}\cos^2\phi\,d\phi = 3a\frac{\pi}{2}.$$

E17) $\frac{dx}{dt} = -2\sin t,\ \frac{dy}{dt} = 2\cos t$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2\sqrt{\sin^2 t + \cos^2 t} = 2$$

$$L = 2\int_0^{2\pi} dt = 4\pi$$

Note that $L = 2\pi r = 2\pi \times 2 = 4\pi$ since, here, $r = 2$.

E18) $r = a(1-\cos\theta),\ \frac{dr}{d\theta} = a\sin\theta$

$$\therefore \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2a\sin\frac{\theta}{2}$$

The length of the curve in the upper half is $\int_0^{\pi} 2a\sin(\theta/2)\, d\theta = 4a$.

The length from $\theta = 0$ to $\theta = 2\pi/3$

$$= \int_0^{2\pi/3} 2a\sin\frac{\theta}{2} d\theta = 2a$$

The arc of the curve in the upper half is bisected by $\theta = 2\pi/3$.

E19) $r = a(\theta^2 - 1),\ \frac{dr}{d\theta} = 2a\theta$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2[\theta^4 - 2\theta^2 + 1 + 4\theta^2]$$

$$= a^2(\theta^2+1)^2$$

$$\therefore\ L = a\int_{-1}^{1}(\theta^2+1)d\theta = a\left(\frac{\theta^3}{3}+\theta\right)\Bigg]_{-1}^{1} = a\left(\frac{1}{3}+1+\frac{1}{3}+1\right) = \frac{8a}{3}.$$

MISCELLANEOUS EXAMPLES AND EXERCISES

The examples and exercises given below cover the concepts and processes you have studied in this block. Doing them will give you a better understanding of the concepts concerned, as well as practice in solving such problems.

Example 1: Integrate the following:

i) $\frac{x^{n-1}}{x^n + a^n}$ ii) $\tan^n x \sec^2 x$ iii) $\cos x \left(\frac{1}{\sin x} + \frac{1}{\sin^2 x}\right)$

iv) $\frac{1}{x\sqrt{2x^2 - 8}}$ v) $\frac{1}{x \sec^{-1} x \sqrt{x^2 - 1}}$

Solution:

i) Subsituting $u = x^n + a^n$,we get $du = nx^{n-1}dx$or $x^{n-1}dx = \frac{du}{n}$.

$$\therefore \int \frac{x^{n-1}}{x^n + a^n}dx = \frac{1}{n}\int \frac{du}{u} = \frac{1}{n}\ln|n| + C = \frac{1}{n}\ln\left|x^n + a^n\right| + C$$

ii) Putting $u = \tan x$, $du = \sec^2 x\, dx$. The integral becomes

$$\int u^n du = \frac{u^{n+1}}{n+1} + C = \frac{\tan^{n+1} x}{n+1} + C$$

iii) Putting $u = \sin x$, we have $du = \cos x\ dx$.

$$\int \left(\frac{1}{\sin x} + \frac{1}{\sin^2 x}\right)\cos x\, dx = \int \left(\frac{1}{u} + \frac{1}{u^2}\right) du = \int \frac{du}{u} + \int \frac{du}{u^2}$$

$$= \ln|u| - \frac{1}{u} + C = \ln|\sin x| - \operatorname{cosec} x + C$$

iv) We have

$$\int \frac{dx}{x\sqrt{2x^2 - 8}} = \int \frac{dx}{x\sqrt{2(x^2 - 4)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{x\sqrt{x^2 - 2^2}}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)\right) + C = \frac{1}{2\sqrt{2}}\sec^{-1}\left(\frac{x}{2}\right) + C$$

v) Putting $u = \sec^{-1}(x)$,we have $du = \frac{1}{x\sqrt{x^2 - 1}}dx$

$$\therefore \int \frac{dx}{x \sec^{-1} x\sqrt{x^2 - 1}}dx = \int \frac{du}{u} + C = \ln\left|\sec^{-1}(x)\right| + C.$$

* * *

Example 2: Evaluate the following integrals:

i) $\int \frac{e^{\tan^{-1} x}}{1 + x^2}dx$ ii) $\int \frac{1}{\sqrt{x}}\cos\sqrt{x}\, dx$ iii) $\int_0^1 \frac{dx}{\sqrt{5 - 2x - x^2}}$

iv) $\int \frac{x^3}{1 + x^8}dx$ v) $\int \frac{dx}{2\sqrt{x}(1 + x)}$

Solution:

i) Putting $u = \tan^{-1} x$,we get $du = \frac{1}{1+x^2}dx$. Therefore

$$\int \frac{e^{\tan^{-1} x}}{1 + x^2}dx = \int e^u du = e^u + C = e^{\tan^{-1} x} + C.$$

ii) Putting $\sqrt{x} = u$, $\frac{1}{2}\frac{1}{\sqrt{x}}dx = du$. Therefore

$$\int \frac{1}{\sqrt{x}} \cos\sqrt{x}\, dx = 2\int \cos u \; du = 2\sin u + C = \sin\sqrt{x} + C$$

iii) $\displaystyle\int \frac{dx}{\sqrt{5-2x-x^2}} = \int \frac{dx}{\sqrt{6-(x+1)^2}}$. Putting $u = x+1$, $du = dx$. Therefore

$$\int \frac{dx}{\sqrt{6-(x+1)^2}} = \int \frac{du}{\sqrt{\sqrt{6}^2 - u^2}} = \sin^{-1}\left(\frac{u}{\sqrt{6}}\right) + C$$

$$= \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C$$

$$\int_0^1 \frac{dx}{\sqrt{5-2x-x^2}} = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)\Bigg|_0^1 = \sin^{-1}\left(\frac{2}{\sqrt{6}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$$

iv) Putting $x^4 = u$,we get $4x^3dx = du$. Therefore

$$\int \frac{x^3}{1+x^8}dx = \frac{1}{4}\int \frac{du}{1+u^2} = \frac{1}{4}\tan^{-1} u + c = \frac{1}{4}\tan^{-1} x^4 + C.$$

v) Putting $x = u^2$,$dx = 2u\, du = 2\sqrt{x}\, du$.

$$\therefore \int \frac{dx}{2\sqrt{x}(1+x)} = \int \frac{du}{1+u^2} = \tan^{-1} u + c = \tan^{-1}\left(\sqrt{x}\right) + C.$$

* * *

Example 3: Evaluate the following integrals:

i) $\displaystyle\int_a^{2a} \left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}}\right)^2 dx,\ a > 0$ ii) $\displaystyle\int \frac{ae^{ex} + be^{ax}}{e^{ax} + e^{bx}}dx$

iii) $\displaystyle\int \left(ax + \frac{b}{x} + c\right)^n \left(\frac{ax^2 - b}{x^2}\right) dx$ iv) $\displaystyle\int \frac{2ax^n + b}{(ax^{2n} + bx^n + c)}x^{n-1}dx$

v) $\displaystyle\int \frac{dx}{x\ln|x|}$.

Solution:

i) We have

$$\int_a^{2a} \left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}}\right)^2 dx = \int_a^{2a} \left(\frac{a}{x} + \frac{x}{a} + 2\right) dx$$

$$= a\ \ln|x| + \frac{x^2}{2a} + 2x\Bigg|_a^{2a} = \left(a\ln 2a + \frac{4a^2}{2a} + 4a\right)$$

$$-\left(a\ln a + \frac{a^2}{2a} + 2a\right)$$

$$= a\ln\frac{2a}{a} + 6a - 2a - \frac{a}{2} = a\ln 2 + \frac{7a}{2}$$

ii) Putting $u = e^{ax} + e^{bx}$, $du = ae^{ax} + be^{bx}$. Therefore, the integral becomes

$$\int \frac{du}{u} = \ln|u| + C = \ln\left|e^{ax} + e^{bx}\right| + C$$

iii) Putting $u = ax + \frac{b}{x}$, we have $\left(a - \frac{b}{x^2}\right) dx = du$ or $\left(\frac{ax^2 - b}{x^2}\right) dx = du$.
Therefore, the integral becomes

$$\int u^n du = \frac{u^{n+1}}{n+1} + C = \frac{1}{n+1}\left(ax + \frac{b}{x}\right) + C$$

iv) This is again in the form $\int \frac{f'(x)}{f(x)} dx$. The integral is $\ln\left|ax^{2n} + b^n + c\right| + C$.

v) This is in the form $\int \frac{f'(x)}{f(x)} dx$ where $f(x) = \ln|x|$. The integral is

$$\ln \ln |x| + C$$

* * *

Let us now see another method to handle the case where we have to integrate an even power of $\sin x$ or $\cos x$. Let us write $y = \cos + i \sin x$. Then, $\frac{1}{y} = \cos x - i \sin x$. Further, by De Moivre's theorem, we have

$$y^n = \cos nx + i \sin nx \text{ and } \frac{1}{y^n} = \cos nx - i \sin nx.$$

Therefore,

$$y^n + \frac{1}{y^n} = 2 \cos nx \text{ and } y^n - \frac{1}{y^n} = 2i \sin x$$

To integrate $\cos^n x$, we use the relation

$$\cos^n x = \left(\frac{1}{2}\left(y + \frac{1}{y}\right)\right)^n$$

We then use binomial theorem to expand the RHS of the above equation to get

$$\left(\frac{1}{2}\left(y + \frac{1}{y}\right)\right)^n = \frac{1}{2^n}\left(y^n + C(n,1)\, y^{n-1}\frac{1}{y} + \cdots + C(n, n-1)\, y \frac{1}{y^{n-1}} + \frac{1}{y^n}\right)$$

We can group each term of the $y^{n-2k} = y^{n-k}\frac{1}{y^k}$ with unique term of the form $y^{-(n-2k)} = y^{2k-n}$, replace $y^{2n-k} + \frac{1}{y^{2n-k}}$ by $\cos(2n-k)x$ and integrate them term by term.

To integrate $\sin^n x$, we use the relation

$$\sin^n x = \frac{1}{2i}\left(y - \frac{1}{y}\right)^n$$

As before, we use binomial theorem to expand the RHS of the above equation to get

$$\left(\frac{1}{2i}\left(y - \frac{1}{y}\right)\right)^n = \frac{1}{(2i)^n}\left(y^n - C(n,1) y^{n-1}\left(\frac{1}{y}\right)\right.$$
$$\left. + \cdots + (-1)^{n-1} C(n, n-1)\, y \left(\frac{1}{y}\right)^{n-1} + \left(\frac{1}{y}\right)^n\right)$$

We can group each term of the $\pm y^{n-2k} = \pm y^{n-k}\frac{1}{y^k}$ with unique term of the form $\pm y^{-(n-2k)} = \pm y^k \frac{1}{y^{n-k}}$. If n and k have the same parity, we get a term of the form $y^{n-2k} + \frac{1}{y^{n-2k}}$ and we replace it by $\cos(n-2k)x$. If they have different parity, we get a term of the form $\pm\left(y^{n-2k} - \frac{1}{y^{n-2k}}\right)$ and replace it by $\pm \sin(n-2k)x$.

Example 4: Evaluate $\int \sin^4 x \, dx$.

Solution: We have

$$\sin^4 x = \left(\frac{1}{2i}\left(y - \frac{1}{y}\right)\right)^4 = \frac{1}{16}\left(y - \frac{1}{y}\right)^4$$

$$= \frac{1}{16}\left(y^4 - 4y^3\frac{1}{y} + 6y^2\frac{1}{y^2} - 4y\frac{1}{y^3} + \frac{1}{y^4}\right)$$
$$= \frac{1}{16}\left\{y^4 + \frac{1}{y^4} - 4\left(y^2 + \frac{1}{y^2}\right) + 6\right\}$$
$$= \frac{1}{16}(2\cos 4x - 8\cos 2x + 6) = \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8}$$

So,

$$\int \sin^4 x\,dx = \frac{1}{8}\int \cos 4x\,dx - \frac{1}{2}\int \cos 2x\,dx + \frac{3}{8}\int dx$$
$$= \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3}{8}x + C$$

Note the difference in the answer we get by this method and the one using reduction formula. Here we get the answer in the form of multiples of sines like $\sin 2x,\ \sin 4x, \ldots$. So, depending on the form of answer we want, we can use either of the methods.

* * *

Integrals of the Type $\int e^x[f(x) + f'(x)]\,dx$

We first prove the formula $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$ and see how it can be used in integrating some functions. By the formula for integration by parts

$$\int e^x f(x)dx = \int f(x)\,e^x dx = f(x)\,e^x - \int f'(x)e^x\,dx.$$

This implies $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$

Example 5: Evaluate the following integrals:

i) $\displaystyle\int \frac{1+x}{(2+x)^2}e^x dx$ ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{1-\sin x}}{1+\cos x}e^{-x/2}dx$

Solution:

i) We have

$$\int \frac{1+x}{(2+x)^2}e^x dx = \int \frac{(2+x)-1}{(2+x)^2}e^x dx$$

Since

$$\frac{-1}{(2+x)^2} = \frac{d}{dx}\left(\frac{1}{2+x}\right)$$
$$\int \frac{1+x}{(2+x)^2}e^x\,dx = \int\left[\frac{1}{2+x} + \frac{-1}{(2+x)^2}\right]e^x\,dx = \frac{1}{2+x}e^x + C,$$

ii) Since

$$\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2 = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\cos\frac{x}{2}\sin\frac{x}{2} = 1 - \sin x,$$

We have

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{1-\sin x}}{1+\cos x}e^{-x/2}dx = \int_0^{\frac{\pi}{2}} \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{2\cos^2\frac{x}{2}}\,e^{-x/2}\,dx$$

Notice here that we have chosen $\cos\frac{x}{2} - \sin\frac{x}{2}$ as the square root $\sqrt{1-\sin x}$. This is because $\tan\frac{x}{2} \le 1$ for $0 \le x \le \frac{\pi}{2}$ and hence $\sin\frac{x}{2} \le \cos\frac{x}{2}$ in $\left[0, \frac{\pi}{2}\right]$. (Check this!) Since we always take the positive square root, we have taken $\cos\frac{x}{2} - \sin\frac{x}{2}$ as the square root of $1-\sin x$. We have

$$\int \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{2\cos^2\frac{x}{2}} e^{-x/2} dx = \frac{1}{2}\int \sec\frac{x}{2} e^{-x/2} dx - \frac{1}{2}\int \tan\frac{x}{2}\sec\frac{x}{2} e^{-x/2} dx$$

$$\int \sec\frac{x}{2} e^{-x/2} dx = \left(\sec\frac{x}{2}\right)(-2e^{-x/2}) - \int\left(\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}\right)(-2e^{-x/2})\,dx$$

$$= -2\sec\frac{x}{2}e^{-x/2} + \int \sec\frac{x}{2}\tan\frac{x}{2} e^{-x/2} dx$$

Thus,

$$\int \frac{\sqrt{1+\sin x}}{1+\cos x} e^{-x/2} = -\sec\frac{x}{2}e^{-x/2} + \frac{1}{2}\int \sec\frac{x}{2}\tan\frac{x}{2}dx - \frac{1}{2}\int \sec\frac{x}{2}\tan\frac{x}{2}e^{-x/2}dx = -\sec\frac{x}{2}e^{-x/2} + C$$

So,

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} dx = -\sec\frac{x}{2}e^{-\frac{x}{2}}\Big]_0^{\frac{\pi}{2}} = (-\sqrt{2}e^{-\frac{\pi}{4}}) - (-1) = 1 - \sqrt{2}e^{-\frac{\pi}{4}}$$

* * *

The following examples give you more practice in integration by parts.

Example 6: Evaluate the following integrals using integration by parts:

i) $\int e^{m\sin^{-1}x}dx$ ii) $\int x^2\sin^{-1}x\,dx$ iii) $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx,\ a > 0$

iv) $\int \frac{xe^x}{(x+1)^2}dx$ v) $\int e^x \sin 2x\cos x\,dx$

Solution:

i) Integrating by parts,

$$\int e^{m\sin^{-1}x}dx = \int e^{m\sin^{-1}x}(1)\,dx = x e^{m\sin^{-1}x} - \int x\frac{d(e^{m\sin^{-1}x})}{dx}dx$$

Putting $m\sin^{-1}x = u$, we have $du = \frac{m\,dx}{\sqrt{1-x^2}}$. Also, $x = \sin\left(\frac{u}{m}\right)$. Therefore

$$\int x e^{m\sin^{-1}x}\left(\frac{m}{\sqrt{1-x^2}}\right)dx = \int \sin\left(\frac{u}{m}\right)e^u du.$$

Putting $\frac{u}{m} = v$, the integral becomes $\int e^{mv}\sin v\,dv$. This is a standard from discussed in 18.3.2. You can now complete the solution.

ii) Integrating by parts,

$$\int x^2\sin^{-1}x\,dx = \frac{x^3\sin^{-1}x}{3} - \int \frac{x^3}{3}\frac{1}{\sqrt{1-x^2}}dx.$$

Consider $\int \frac{x^3}{\sqrt{1-x^2}}$ du. Putting $u = 1 - x^2$, $du = -2x\,dx$.

$$\int \frac{x^3}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}(-2x)\,dx = -\frac{1}{2}\int \frac{(1-u)}{\sqrt{u}}\,du$$
$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} + \frac{1}{2}\int \sqrt{u}\,du$$

This is a standard form. You can now complete the solution.

iii) Integrating by parts,

$$\int \sin^{-1}\left(\sqrt{\frac{x}{a+x}}\right) dx = x\sin^{-1}\left(\sqrt{\frac{x}{a+x}}\right)$$
$$-\int x\frac{1}{\sqrt{1-\frac{x}{a+x}}}\cdot\frac{1}{2}\sqrt{\frac{a+x}{x}}\left\{\frac{(a+x)-x}{(a+x)^2}\right\}dx$$

We can rewrite the integral on RHS as

$$\frac{1}{2}\int \frac{x\sqrt{a+x}}{\sqrt{a}}\sqrt{\frac{a+x}{x}}\frac{a}{(a+x)^2}dx = \frac{1}{2}\int \frac{ax(a+x)}{\sqrt{a}\sqrt{x}(a+x)^2}\,dx$$
$$= \sqrt{\frac{a}{2}}\int \frac{\sqrt{x}}{(a+x)}\,dx.$$

Substituting $u = \sqrt{x}$ we have $du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$ or $dx = 2u\,du$. $\sqrt{x}\,dx = 2u^2du$. The integral becomes

$$\frac{\sqrt{a}}{2}\int \frac{2u^2du}{(a+u^2)} = \sqrt{a}\int\left(1-\frac{a}{a+u^2}\right)du = \sqrt{a}\,u - a^{3/2}\int \frac{du}{(\sqrt{a})^2+u^2}$$

iv) Taking $\frac{1}{(1+x)^2}$ as g(x) and integrating by parts, we get

$$\int \frac{xe^x}{(x+1)^2}\,dx = xe^x\left(\frac{-1}{(x+1)}\right) - \int \frac{-1}{(x+1)}(e^x + xe^x)\,dx$$
$$= \frac{-xe^x}{x+1} + \int e^x dx = \frac{-xe^x}{x+1} + e^x + C$$

v) We have $\sin 2x\cos x = \frac{1}{2}(\sin 3x + \sin x)$. Therefore,

$$\int e^x \sin 2x\cos x\,dx = \frac{1}{2}\int e^x\sin 3x\,dx + \frac{1}{2}\int e^x \sin x\,dx$$

You can now complete the solution because both the integrals are standard types that we discussed in 18.3.2.

* * *

Example 7: Evaluate the integral $\int x\sqrt{\frac{a-x}{a+x}}$ dxusing the substitution $x = a\cos 2\theta$.

Solution: Putting $x = a\cos 2\theta$, $dx = -2a\,\sin 2\theta$. Further $\sqrt{\frac{a-x}{a+x}} = \sqrt{\frac{a(1-\cos 2\theta)}{a(1+\cos 2\theta)}}$. Using $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$1 - \cos 2\theta = 1 - \cos^2\theta + \sin^2\theta = 2\sin^2\theta.$$

Using $\cos 2\theta = 2\cos^2\theta - 1$, $1 + \cos 2\theta = 2\cos^2\theta$. Therefore,
$\sqrt{\frac{a-x}{a+x}} = \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} = \tan\theta$.

$$I = \int x\sqrt{\frac{a-x}{a+x}}\,dx = -2a^2\int \cos 2\theta\ \tan\theta\ \sin 2\theta\, d\theta$$
$$= -2a^2\int \cos 2\theta \tan\theta 2\sin\theta\cos\theta\, d\theta = -2a^2\int \cos 2\theta 2\sin^2\theta\, d\theta$$
$$= -2a^2\int \cos 2\theta(1-\cos 2\theta)\, d\theta = -2a^2\int\left(\cos 2\theta - \frac{1+\cos 4\theta}{2}\right) d\theta$$
$$= -2a^2\left(\frac{\sin 2\theta}{2} - \frac{\theta}{2} - \frac{\sin 4\theta}{8}\right) = \frac{a^2}{4}(4\theta - 4\sin 2\theta + \sin 4\theta)$$

From $x = a\cos 2\theta$,it follows that $\theta = \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$

$$\sin 2\theta = \sqrt{1-\cos^2 2\theta} = \sqrt{1-\left(\frac{x}{a}\right)^2} = \frac{\sqrt{a^2-x^2}}{a}$$

$$\sin 4\theta = 2\sin 2\theta\ \cos 2\theta = 2\frac{\sqrt{a^2-x^2}}{a}\ \left(\frac{x}{a}\right)$$

$$\therefore \int x\sqrt{\frac{a-x}{a+x}}\,dx = \frac{a^2}{4}\left[2\cos^{-1}\left(\frac{x}{a}\right) - \frac{4}{a}\sqrt{a^2-x^2} + \frac{2x}{a^2}\sqrt{a^2-x^2}\right] + C$$
$$= \frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{(x-2a)}{2}\sqrt{a^2-x^2} + C$$

* * *

Example 8: Evaluate the following integrals:

i) $\int \frac{dx}{(x-1)^2(x-2)(x-3)}$ ii) $\int \frac{dx}{(x^2+1)(x^2+4)}$ iii) $\int \frac{x\,dx}{x^4+x^2+1}$

iv) $\int \frac{(x^2+a^2)}{x^4+a^2x^2+a^4}$ v) $\int \frac{dx}{(x+1)(x^3-1)}$

Solution:

i) To split into partial fractions, we write

$$\frac{1}{(x-1)^2(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)} + \frac{D}{(x-3)}$$

Multiplying both sides by $(x-1)^2(x-2)(x-3)$,we get

$$1 = A(x-1)(x-2)(x-3) + B(x-2)(x-3) + C(x-1)^2(x-3) + D(x-1)^2(x-2)$$

Putting $x = 3$,we get $1 = D(3-1)^2(3-2) = 4D$ or $D = 1/4$.

Putting $x = 2$,we get $1 = C(2-1)^2(2-3) = -C$or $C = -1$.

Putting $x = 1$, $1 = B(1-2)(1-3) = 2B$ or $B = \frac{1}{2}$

Putting $x = 0$, we get

$$1 = A(-1)(-2)(-3) + B(-2)(-3) + C(-1)^2(-3) + D(-1)^2(-2)$$

or $1 = -6A + \frac{6}{2} + 3 - \frac{1}{2}\cdot$ or $A = \frac{9}{12} = \frac{3}{4}$.

$$\int \frac{dx}{(x-1)^2(x-2)(x-3)} = \frac{3}{4}\int\frac{dx}{x-1} + \frac{1}{2}\int\frac{dx}{(x-1)^2} - \int\frac{dx}{(x-2)} + \frac{1}{4}\int\frac{dx}{x-3}$$

ii) To split into partial fractions, we write

$$\frac{1}{(x^2+4)(x^2+1)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+1)}$$

Multiplying both sides by $(x^2+4)(x^2+1)$we get

$$1 = (Ax+B)(x^2+1) + (Cx+D)(x^2+4)$$

Putting x = i, we get

$$1 = 3(Ci + D) \qquad \ldots (19)$$

Putting x = –i, we get

$$1 = 3(-Ci + D) \qquad \ldots (20)$$

Adding Eqn. (19) and Eqn. (20), we get 2 = 6D or D = 1/3. Putting this in Eqn. (19) we get C = 0.

Putting x = 2i, we get

$$1 = -3(2Ai + B) \qquad \ldots (21)$$

Putting x = –2i, we get

$$1 = -3(-2Ai + B) \qquad \ldots (22)$$

Solving, we get B = –1/3, A = 0. Therefore,

$$\int \frac{dx}{(x^2+4)(x^2+1)} = \frac{1}{3}\left\{\int \frac{dx}{x^2+1} - \int \frac{dx}{x^2+4}\right\}$$

$$= \frac{1}{3}\left\{\tan^{-1} x - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right\} + C$$

iii) We have

$$\int \frac{xdx}{x^4+x^2+1} = \int \frac{xdx}{\left(x^2+\frac{1}{2}\right)^2 - \frac{3}{4}} = \int \frac{xdx}{\left(x^2+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2}$$

Putting $u = x^2 + \frac{1}{2}$, $du = 2x\,dx$. Therefore,

$$\int \frac{xdx}{\left(x^2+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}\int \frac{du}{u^2 - \left(\frac{\sqrt{3}}{2}\right)^2}$$

which is a standard form.

iv) We have

$$\int \frac{x^2+a^2}{x^4+a^2x^2+a^4}dx = \int \frac{x^2+a^2}{(x^2+a^2)^2 - a^2x^2}dx$$

$$= \int \frac{x^2+a^2}{(x^2+a^2+ax)\,(x^2+a^2-ax)}dx$$

$$\frac{x^2+a^2}{(x^2+a^2+ax)\,(x^2+a^2-ax)} = \frac{1}{2}\left(\frac{1}{x^2+a^2+ax} + \frac{1}{x^2+a^2-ax}\right)$$

$$\int \frac{x^2+a^2}{(x^2+a^2+ax)\,(x^2+a^2-ax)}dx = \frac{1}{2}\left(\int \frac{dx}{x^2+a^2+ax} + \int \frac{dx}{x^2+a^2-ax}\right)$$

$$\int \frac{dx}{x^2+a^2+ax} = \int \frac{dx}{(x+\frac{a}{2})^2+\left(\frac{\sqrt{3}a}{2}\right)^2}$$

Putting $u = x+\frac{a}{2}$, the integral becomes $\int \frac{du}{u^2+\left(\frac{\sqrt{3}a}{2}\right)^2}$ which is a standard form. Similarly,

$$\int \frac{dx}{x^2+a^2-ax} = \int \frac{dx}{(x-\frac{a}{2})^2+\frac{3a^2}{4}}$$

and we can reduce this to the standard form using the substitution

$$u = x-\frac{a}{2}.$$

v) We have

$$\frac{1}{(x+1)(x^3-1)} = \frac{1}{(x+1)(x-1)(x^2+x+1)}.$$

To split into partial fractions we write

$$\frac{1}{(x+1)(x-1)(x^2+x+1)} = \frac{A}{x+1}+\frac{B}{x-1}+\frac{Cx+D}{x^2+x+1}$$

Multiplying both sides by $(x+1)(x-1)(x^2+x+1)$, we get

$$1 = A(x-1)\left(x^2+x+1\right)+B(x+1)\left(x^2+x+1\right)+(Cx+D)\left(x^2-1\right)$$

Putting $x = 1$, we get $1 = 6B$ or $B = \frac{1}{6}$. Putting $x = -1$, we get $1 = -2A$ or $A = -\frac{1}{2}$. Putting $x = 0$, we get $1 = -A+B-D$ or $D = B-A-1 = \frac{1}{6}+\frac{1}{2}-1 = -\frac{1}{3}$. Putting $x = 2$gives $1 = 7A+21B+3(2C+D)$. Solving for C, we get $C = \frac{1}{3}$.

$$\begin{aligned}\int \frac{dx}{(x+1)\,(x^3-1)} &= \int\left(\frac{1}{6(x-1)}-\frac{1}{2(x+1)}+\frac{x-1}{3\,(x^2+x+1)}\right)dx\\ &= \frac{1}{6}\int\frac{dx}{x-1}-\frac{1}{2}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{x-1}{x^2+x+1}dx\\ &= \frac{1}{6}\ln|x-1|-\frac{1}{2}\ln|x+1|+\frac{1}{3}\int\frac{x-1}{x^2+x+1}dx\end{aligned}$$

We have dealt with an integral of the same type as the last integral in example 18 in Unit 18.

* * *

Example 9: Obtain the area bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$.

Solution: (See Fig. 1)The points of intersection are given by $\left(\frac{x^2}{4a}\right)^2 = 4ax$ or $x^4-64a^3x = 0$, i.e. $x\left(x^3-64a^3\right) = 0$. $\left(x^3-64a^3\right) = (x-4a)\left(x^2+4ax+16a^2\right)$.

The equation $x^2 + 4ax + 16a^2$ has complex roots since the discriminant $b^2 - 4ac = 16a^2 - 64a^2 < 0$. So, the points of intersection are given by $x = 0$ and $x = 4a$. When $x = 0$, $y = 0$. When $x = 4a$, $y = (4a)^2/4a = 4a$.

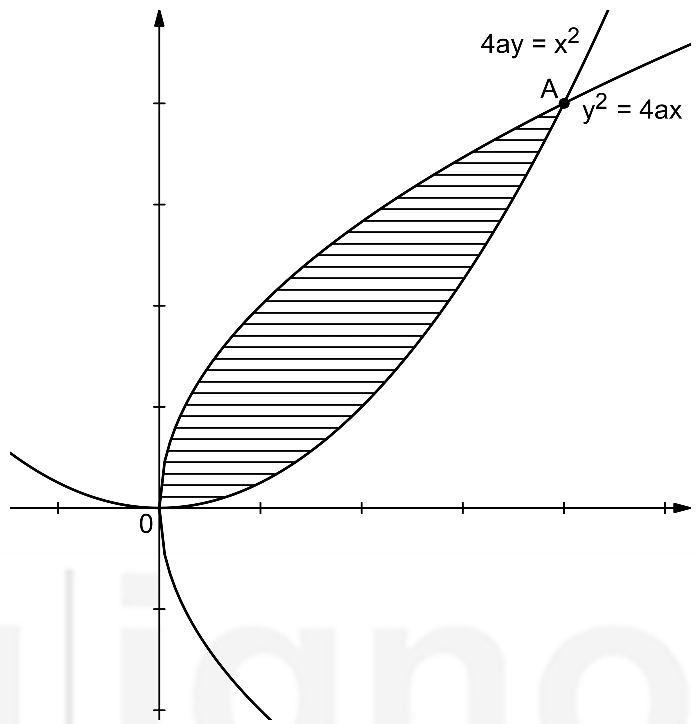


Fig. 1: Area bounded by $y^2 = 4ax$ and $x^2 = 4ay$.

So, the curves intersect at $(0, 0)$ and $(4a, 4a)$. We have $2\sqrt{ax} - \frac{x^2}{4a} = \sqrt{x}\left(2\sqrt{a} - \frac{x^{3/2}}{4a}\right)$. If $x \le 4a$, $\frac{x^{3/2}}{4a} \le \frac{(4a)^{3/2}}{4a} = \frac{8a^{3/2}}{4a} = 2\sqrt{a}$. So, if $x \le 4a$, $2\sqrt{ax} - \frac{x^2}{4a} \ge 0$. To find the area we take $f_1(x) = \sqrt{2ax}$ and $f_2(x) = \frac{x^2}{4a}$ and integrate $f_1(x) - f_2(x)$ from 0 to 4a, i.e. we evaluate the integral $\int_0^{4a}\left(\sqrt{2ax} - \frac{x^2}{4a}\right) dx$. This is an easy integral to evaluate.

* * *

Example 10: Find the area and the length of the arc of the Nephroid $x = a(3\cos t - \cos 3t)$ and $y = a(3\sin t - \sin 3t)$.

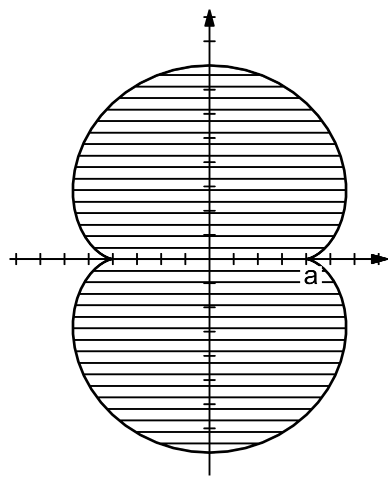


Fig. 2: Area bounded by a Nephroid.

Solution: When $t = 0$, $x = 2a$, $y = 0$. When $t = \frac{\pi}{2}$, $x = 0, y = 4a$. When $t = \pi$, $x = -2a$. When $t = \frac{3\pi}{2}$, $x = 0$, $y = -2a$. When $t = 2\pi$, $x = 2a, y = 0$. So, the Nephroid is a closed curve with $\alpha = 0$ and $\alpha = 2\pi$ as the beginning and end points. Also, the curve is traced out in the anti-clockwise directions as tvaries from 0 to 2π. The area is

$$A = \int_0^{2\pi} y\frac{dx}{dt}dt$$

$$\frac{dx}{dt} = a(-3\sin t + 3\sin 3t)$$

$$\begin{aligned}\therefore y\frac{dx}{dt} &= -a^2(3\sin t + 3\sin 3t)(-3\sin t + 3\sin 3t)\\ &= -a^2(-9\sin^2 t + 9\sin t\sin 3t + 3\sin 3t\ \sin t - 3\sin^2 3t)\\ &= a^2(9\sin^2 t - 12\sin t\sin 3t + 3\sin^2 3t)\\ &= 9a^2\frac{(1-\cos 2t)}{2} - 6a^2(\cos 2t - \cos 4t) + 3a^2\frac{(1-\cos 6t)}{2}\end{aligned}$$

$$\begin{aligned}\therefore A = \frac{9a^2}{2}\int_0^{2\pi} dt - \frac{9a^2}{2}\int_0^{2\pi}\cos 2t\,dt - 6a^2\int_0^{2\pi}\cos 2t\,dt\\ + 3a^2\int_0^{2\pi} dt - \frac{3a^2}{2}\int_0^{2\pi}\cos 6t\,dt = \frac{9a^2}{2}(2\pi) + \frac{3a^2}{2}(2\pi) = 12\pi a^2\end{aligned}$$

The length of the arc is

$$L = \int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.\ \frac{dy}{dt} = a[3\cos t - 3\cos 3t]$$

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= a^2(3\sin 3t - 3\sin t)^2 + a^2(3\cos t - 3\cos 3t)^2\\ &= a^2\Big\{9\sin^2 9t + 9\sin^2 t\\ &\quad -18\sin 3t\sin t + 9\cos^2 t + 9\cos^2 9t - 18\cos t\cos 3t\Big\}\\ &= a^2\{18 - 18(\sin 3t\sin t + \cos t\cos 3t)\} = a^2\{18 - 18\cos 2t\}\\ &= 18a^2 2\sin^2 t = 36a^2\sin^2 t\end{aligned}$$

$$\begin{aligned}L &= \int_0^{2\pi} 6a|\sin t|\,dt = 6a\int_0^{\pi}\sin t dt - 6a\int_{\pi}^{2\pi}\sin t\,dt\\ &= 6a\,(-\cos t)\Big|_0^{\pi} - 6a\,(-\cos t)\Big|_{\pi}^{2\pi} = 6a\,[1-(-1)] - 6a[-1-(1)] = 24a\end{aligned}$$

* * *

Example 11: Find the length of the arc of the Tricuspoid given by the equations $x = a(2\cos\theta + \cos 2\theta)$, $y = a(2\sin\theta - \sin 2\theta)$.

Solution: $\frac{dx}{d\theta} = -2a\sin\theta - 2a\sin 2\theta$, $\frac{dy}{d\theta} = 2a\cos\theta - 2a\cos 2\theta$

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= 4a^2\sin^2\theta + 4a^2\sin^2 2\theta + 8a^2\sin\theta\sin 2\theta + 4a^2\cos^2\theta\\ &\quad + 4a^2\cos^2 2\theta - 8a^2\cos\theta\cos 2\theta\end{aligned}$$

$$= 8a^2 + 8a(\sin\theta\sin 2\theta - \cos\theta\cos 2\theta)$$

$$= 8a^2 - 8a^2\cos 3\theta = 8a^2 . 2\sin^2\frac{3\theta}{2} = 16a^2\sin^2\frac{3\theta}{2}.$$

Therefore the arc length

$$L = 4a\int_0^{2\pi}\left|\sin\frac{3\theta}{2}\right| d\theta \ .$$

We have $\sin\frac{3\theta}{2} \geq 0$ if $0 \leq \theta \leq \frac{2\pi}{3}$, $\sin\frac{3\theta}{2} \leq 0$ if $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$, $\sin\frac{3\theta}{2} \geq 0$ if $\frac{4\pi}{3} \leq \theta \leq 2\pi$.

$$L = 4a\left\{\int_0^{2\pi/3}\sin\frac{3\theta}{2}d\theta - \int_{2\pi/3}^{4\pi/3}\sin\frac{3\theta}{2}\,d\theta + \int_{4\pi/3}^{2\pi}\sin\frac{3\theta}{2}\,d\theta\right\}$$

$$= 4a\left\{\left(\frac{-2}{3}\cos\frac{3\theta}{2}\right)\Bigg|_0^{2\pi/3} - \left(\frac{-2}{3}\cos\frac{3\theta}{2}\right)\Bigg|_{2\pi/3}^{4\pi/3} + \left(\frac{-2}{3}\cos\frac{3\theta}{2}\right)\Bigg|_{4\pi/3}^{2\pi}\right\}$$

$$= 4a\left[-\frac{2}{3}\cos\pi - \left(-\frac{2}{3}\cos 0\right)\right] - 4a\left[-\frac{2}{3}\cos 2\pi - \left(-\frac{2}{3}\cos\pi\right)\right]$$

$$+ 4a\left[-\frac{2}{3}\cos 3\pi - \left(-\frac{2}{3}\cos 2\pi\right)\right] = \frac{16a}{3}$$

* * *

Example 12: Derive the reduction formula

$$\int x^{m-1}(a+bx^n)^p\,dx = \frac{x^m(a+bx^n)^p}{m+pn} + \frac{apx}{m+pn}\int x^{m-1}(a+bx^n)^{p-1}\,dx$$

Use it to evaluate the integral $\int(c^2-x^2)^{3/2}\,dx$

Solution: Integrating by parts with x^{m-1}as g(x), we get

$$\int x^{m-1}(a+bx^n)^p\,dx$$

$$= \frac{x^m}{m}(a+bx^n)^p - \frac{1}{m}\int x^m p\,(a+bx^n)^{p-1}\,nbx^{n-1}dx$$

$$= \frac{x^m}{m}(a+bx^n)^p - \frac{px}{m}\int x^{m-1}(a+bx^n)^{p-1}\left((a+bx^n)-a\right)\,dx$$

$$= \frac{x^m}{m}(a+bx^n)^p - \frac{pn}{m}\int x^{m-1}(a+bx^n)^p\,dx + \frac{apn}{m}\int x^{m-1}(a+bx^n)^{p-1}\,dx$$

Therefore,

$$\left(1+\frac{pn}{m}\right)\int x^{m-1}(a+bx^n)^p\,dx = \frac{x^m(a+bx^n)^p}{m} + \frac{apn}{m}\int x^{m-1}(a+bx^n)^{p-1}\,dx$$

$$\int x^{m-1}(a+bx^n)^p\,dx = \frac{x^m(a+bx^n)^p}{m+pn} + \frac{apn}{m+pn}\int x^{m-1}(a+bx^n)^{p-1}\,dx$$

We apply the reduction formula with m = 1, a = c^2 , b = –1, n = 2, p = $\frac{3}{2}$.

$$\int\left(c^2-x^2\right)^{3/2}dx = \frac{x\left(c^2-x^2\right)}{4} + \frac{3c^2}{4}\int\left(c^2-x^2\right)^{1/2}dx$$

$\int\left(c^2-x^2\right)^{1/2}$ dx is a standard form we dealt with in Unit 18.

* * *

Example 13: Prove the following properties of the definite integrals:

i) $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx.$

ii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx.$ In particular if $f(2a-x) = f(x)$,

$\int_0^{2a} f(x)\,dx = 2\int_0^{2a} f(x)\,dx.$ If $f(x) = -f(2a-x)$, $\int_0^{2a} f(x)\,dx = 0$

iii) If $f(x) = f(a+x)$, $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx.$

Solution:

i) Putting $y = a - x$, $dy = -dx$. If $x = a$, $y = 0$ and if $x = 0$, $y = a$. So, the integral becomes $-\int_a^0 f(y)\,dy = \int_0^a f(y)\,dy = \int_0^a f(x)\,dx.$

ii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx.$ In the second integral, substituting $y = 2a - x$ when $x = a$, $y = a$, when $x = 2a$, $y = 0$. $dy = -dx$. So, the second integral becomes $-\int_a^0 f(2a-y)\,dy = \int_0^a f(2a-x)\,dx.$

iii) We have

$$\int_0^{na} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx + \cdots + \int_{ia}^{(i+1)a} f(x)\,dx + \cdots + \int_{(n-1)a}^{na} f(x)\,dx.$$

It is enough to show that $\int_{ia}^{(i+1)a} f(x)\,dx.$ From $f(x+a) = a$, it follows that $f(x+2a) = f((x+a)+a) = f(x+a) = f(x)$. Proceeding thus, we have $f(x+ia) = f(x)$. Substituting $x = y + ia$ in the integral $\int_{ia}^{(i+1)a} f(x)\,dx$, we have $y = 0$ when $x = ia$ and $y = a$ when $x = (i+1)a$.

Also, $dy = dx$. So, $\int_{ia}^{(i+1)a} f(x)\,dx = \int_0^a f(y+ia)\,dy = \int_0^a f(y)\,dy.$

* * *

Now, try the following exercises.

E1) Evaluate $\int \cos^6 x dx$.

E2) Evaluate the following: i) $\int \frac{(1-x)^2}{(1+x^2)^2} e^x dx$ ii) $\int \frac{x^2+2}{\sqrt{1+x^2}} e^x\,dx$

E3) Integrate the following:

i) $\int \left(ax^2 + bx + c\right)\left(\frac{a}{x^2} + \frac{b}{x} + c\right) dx$ ii) $\int \frac{ce^x}{ae^x + b} dx$

iii) $\int \frac{dx}{\left(1 + x^2\right)\left(\tan^{-1} x\right)^n}$ iv) $\int \frac{x^2 + 4}{x^2 - 4} dx$

v) $\int (\cos x - \sin x)(2 + \sin 2x) \sec^2 x \cos ec^2 x \, dx$

E4) Prove that $\int_a^b \frac{\ln x}{x} dx = \frac{1}{2} \ln(ab) \ln\left(\frac{a}{b}\right)$.

E5) Evaluate the following integrals:

i) $\int_0^1 \frac{x\,dx}{\sqrt{1 + x^2}}$

ii) $\int \frac{dx}{x\sqrt{x^{2n} - a^{2n}}}$ using the substitution $z = \left(\frac{a}{x}\right)^n$

iii) $\int \frac{(ax^2 - b)\,dx}{2e^{2x} + 2e^x + 1}$ using subsituion $e^x = \frac{z}{1-z}$

iv) $\int \frac{(ax^2 - b)}{x\sqrt{c^2x^2 - (ax^2 + b)^2}}$ using substitution $ax + \frac{b}{x} = z$.

E6) Evaluate $\int \frac{dx}{(x - 1)\left(x^2 + 1\right)\left(x^2 + 4\right)^2} dx$.

E7) Find the area between the curves $y^2 = x + 3$ and $y = \frac{x}{2}$.

E8) Find the area enclosed by the outer boundary of the Nephroid of Freeth given by the equation $r = a\left(1 + 2\sin\frac{\theta}{2}\right)$.

E9) Show that $\int_0^{\pi} \sin^n dx = 2\int_0^{\pi/2} \sin^n x dx$, $\int_0^{\pi} \cos^{2n+1} x\,dx = 0$ and

$$\int_0^{\pi} \cos^{2n} x\,dx = 2\int_0^{\pi/2} \cos^{2n} dx.$$

We have come to the end our set of miscellaneous exercises. You can refer the answers to the exercises in the next section.

SOLUTIONS/ANSWERS

E1) We have $\cos x = \frac{1}{2}\left(y + \frac{1}{y}\right)$. So, we have

$$\cos^6 x = \frac{1}{2^6}\left(y + \frac{1}{y}\right)^6 = \frac{1}{2^6}\left(y^6 + 6y^5\frac{1}{y} + 15y^4\frac{1}{y^2} + 20y^3\frac{1}{y^3} + 15y^2\frac{1}{y^4} + 6y\frac{1}{y^5} + \frac{1}{y^6}\right)$$

Grouping y^i terms with $\frac{1}{y^i}$, we get

$$\cos^6 x = \frac{1}{32}\left\{y^6 + \frac{1}{y^6} + 6\left(y^4 + \frac{1}{y^4}\right) + 15\left(y^2 + \frac{1}{y^2}\right) + 20\right\}$$
$$= \frac{1}{16}(\cos 6x + 6\cos 4x + 15\cos 2x + 10)$$

Therefore,

$$\int \cos^6 x\,dx = \frac{1}{16}\left(\frac{1}{6}\sin 6x + \frac{6}{4}\sin 4x + \frac{15}{2}\sin 2x + 10x\right) + C$$

E2) i) We have

$$\int \frac{(1-x)^2}{(1+x^2)^2}e^x\,dx = \int \frac{x^2+1-2x}{(1+x^2)^2}e^x\,dx = \int\left\{\frac{1}{(1+x^2)} + \frac{-2x}{(1+x^2)^2}\right\}dx$$

This is of the form $\int (f(x) + f'(x))\,e^x\,dx$ where $f(x) = \frac{1}{(1+x^2)}$. Hence

$$\int \frac{(1-x)^2}{(1+x^2)^2}dx = \frac{e^x}{1+x^2} + C$$

ii) We have

$$\int \frac{x^2+2}{\sqrt{1+x^2}}e^x\,dx = \int \frac{1+x^2+1}{\sqrt{1+x^2}}e^x\,dx$$
$$= \int\left\{\sqrt{1+x^2} + \frac{1}{\sqrt{1+x^2}}\right\}e^x\,dx$$

This is of the form $\int (f(x) + f'(x))\,e^x\,dx$ where $f(x) = \sqrt{1+x^2}$.

E3) i) We have

$$\int (ax^2+bx+c)\left(\frac{a}{x^2} + \frac{b}{x} + c\right)dx$$
$$= \int\left(a^2 + abx + acx^2 + \frac{ab}{x} + b^2 + bcx + \frac{ac}{x^2} + \frac{bc}{x} + c^2\right)dx$$
$$= (a^2+b^2+c^2)x + (ab+bc)\frac{x^2}{2} + ac\frac{x^3}{3} + (ab+bc)\ln|x| - \frac{ac}{x} + C.$$

ii) Putting $u = ae^x + b$, we have $du = ae^x\,dx$. Therefore,

$$\int \frac{ce^x}{ae^x+b}dx = \frac{c}{a}\int\frac{du}{u} = \frac{c}{a}\ln|u| + C = \frac{c}{a}\ln\left|ae^x+b\right| + C$$

iii) Putting $u = \tan^{-1}x$, $du = \frac{1}{1+x^2}dx$. The integral becomes

$$\int \frac{du}{u^n} = \frac{1}{(1-n)\,u^{n-1}} + C = \frac{1}{(1-n)(\tan^{-1}x)^{n-1}} + C$$

if $n > 1$. If $n = 1$, the integral is $\ln\left|\tan^{-1}x\right| + C$.

iv) We have

$$\int \frac{x^2+4}{x^2-4}dx = \int\frac{x^2-4+8}{x^2-4}dx = \int\left(1 + \frac{8}{x^2-4}\right)dx$$
$$= \int dx + \int\frac{dx}{x^2-4} = x + \frac{1}{2}\ln\left|\frac{x-2}{x+2}\right| + C$$

v) We have

$$(\cos x - \sin x)(2 + \sin 2x)(\sec^2 x \cos ec^2 x)$$
$$= 2\left(\frac{\cos x - \sin x}{\sin x \cos x}\right)\left(\frac{1 + \sin x \cos x}{\sin x \cos x}\right)$$
$$= (\operatorname{cosec} x - \sec x)(1 + \operatorname{cosec} x \sec x)$$
$$= \operatorname{cosec} x + (1 + \cot^2 x) \sec x - \sec x - (1 + \tan^2 x) \operatorname{cosec} x$$
$$= \operatorname{cosec} x + \sec x + \operatorname{cosec} x \cot x$$
$$- \sec x - \operatorname{cosec} x - \sec x \tan x$$
$$= \operatorname{cosec} x \cot x - \sec x \tan x$$

Therefore

$$\int (\cos x - \sin x)(2 + \sin 2x)(\sec^2 x \operatorname{cosec}^2 x)\, dx$$
$$= \int \operatorname{cosec} x \cot x - \int \sec x \tan x\, dx = -\operatorname{cosec} x - \sec x + C$$

E4) We have $\int_a^b \frac{\ln|x|}{x} dx = \int_{\ln|a|}^{\ln|b|} u\, du$ on substituting $u = \log x$.

$$\therefore \int_a^b \frac{\ln x}{x} dx = \left.\frac{u^2}{2}\right|_{\ln|a|}^{\ln|b|} = \frac{1}{2}\left[(\ln|b|)^2 - (\ln|a|)^2\right]$$
$$= \frac{1}{2}(\ln|b| + \ln|a|)(\ln|b| - \ln|a|) = \frac{1}{2} \ln|ab| \ln\left|\left(\frac{b}{a}\right)\right|$$

E5) i) Putting $u = 1 + x^2$, $du = 2x\,dx$. When $x = 0$, $u = 1$, when $x = 1$, $u = 2$.

$$\int_1^2 \frac{du}{2\sqrt{u}} = \frac{1}{2} \left.\frac{u^{1/2}}{1/2}\right|_1^2 = \sqrt{2} - 1$$

ii) $\int \frac{dx}{x\sqrt{x^{2n} - a^{2n}}} = \int \frac{dx}{x \, . \, x^n \sqrt{1 - \left(\frac{a}{x}\right)^{2n}}}$. Putting $z = \left(\frac{a}{x}\right)^n dz = -\frac{n\,a^n}{x^{n+1}} dx$.

Therefore

$$\int \frac{dx}{x^{n+1}\sqrt{x^{2n} - a^{2n}}} = -\frac{1}{a^{n-1}} \int \frac{dz}{\sqrt{1 - x^2}} = -\frac{1}{na^n} \sin^{-1} z$$
$$= -\frac{1}{na^n} \sin^{-1}\left(\frac{a}{b}\right)^n + C$$

iii) Putting $e^x = \frac{z}{1-z}, e^x dx = \left(\frac{(1-z)+z}{(1-z)^2}\right) dz = \left(\frac{1}{(1-z)^2}\right) dx$

$$2e^{2x} + 2e^x + 1 = e^{2x} + (e^x + 1)^2 = \left(\frac{z}{1 - z}\right)^2 + \left(\frac{1}{1 - z}\right)^2 = \frac{z^2 + 1}{(1 - z)^2}$$

$$\therefore \int \frac{e^x}{2e^{2x} + 2e^x + 1} dx = \int \frac{1/(1 - z)^2}{(z^2 + 1)/(1 - z)^2} dz = \int \frac{dz}{z^2 + 1} = \tan^{-1}(z)$$

$e^x = \frac{z}{1-z}$. $e^x - ze^x = z$. $e^x = (e^x + 1)z$ or $z = \frac{e^x}{e^x+1}$

$$\int \frac{e^x}{2e^{2x} + 2e^x + 1} dx = \tan^{-1}\left(\frac{e^x}{e^x + 1}\right) + C$$

iv) Putting $ax + \frac{b}{x} = x$, we have $\left(a - \frac{b}{x^2}\right)$ dx = dz or $\frac{(ax^2-b)}{2}dx = dz$.

$\frac{ax^2+b}{x} = z$. Rewriting the integral as $\int \frac{(ax^2-b)\,dx}{x^2\sqrt{c^2-\left(\frac{ax^2+b}{x}\right)^2}}$, the integral

becomes $\displaystyle\int \frac{dz}{\sqrt{c^2-z^2}} = \sin^{-1}\left(\frac{z}{c}\right) + C = \sin^{-1}\left(\frac{ax^2+b}{cx}\right) + C$

E6) To split into partial fractions, we write

$$\frac{1}{(x-1)\left(x^2+1\right)\left(x^2+4\right)^2} = \frac{A}{(x-1)} + \frac{Bx+C}{\left(x^2+1\right)} + \frac{Dx+E}{\left(x^2+4\right)} + \frac{Fx+G}{\left(x^2+4\right)^2}$$

Multiplying both sides by $(x-1)\left(x^2+1\right)\left(x^2+4\right)^2$, we get

$$1 = A\left(x^2+1\right)\left(x^2+4\right)^2 + (Bx+C)(x-1)\left(x^2+4\right)^2 + (Dx+E)(x-1)\left(x^2+1\right)\left(x^2+4\right) + (Fx+G)(x-1)\left(x^2+1\right)$$

Putting x = 1, 1 = 50A or $A = \frac{1}{50}$. Putting x = i, we have

$$1 = 9(Bi + C)(i-1) = -9(B+C) - 9i(B-C)$$

Comparing the real and imaginary parts both sides, we get $-9(B+C) = 1$, $-9(B-C) = 0$. Solving, we get $B = C = -\frac{1}{18}$. Putting x = 2i, we get

$$1 = (2Fi + G)(2i-1)(-3) = 3(4F+G) + 6i(F-G).$$

Comparing real and imaginary parts, we get $4F + G = \frac{1}{3}$, $F - G = 0$.

Solving, we get $F = G = \frac{1}{15}$. Comparing coefficients of x^6 both sides,

we get A + B + D = 0. Therefore, $D = -A - B = -\frac{1}{50} + \frac{1}{18} = \frac{8}{225}$.

$$\frac{1}{(x-1)\left(x^2+1\right)\left(x^2+4\right)^2} = \frac{1}{50}\frac{1}{(x-1)} - \frac{1}{18}\frac{x+1}{\left(x^2+1\right)} + \frac{8}{225}\frac{x+1}{\left(x^2+4\right)} + \frac{1}{15}\frac{x+1}{\left(x^2+4\right)^2}$$

$$\therefore \int \frac{dx}{(x-1)\left(x^2+1\right)\left(x^2+4\right)^2} = \frac{1}{50}\int \frac{dx}{(x-1)} - \frac{1}{18}\int \frac{x+1}{\left(x^2+1\right)}dx + \frac{8}{225}\int \frac{x+1}{\left(x^2+4\right)}dx + \frac{1}{15}\int \frac{x+1}{\left(x^2+4\right)^2}dx$$

All the integrals except the last one are standard ones that we have already seen. To evaluate $\displaystyle\int \frac{(x+1)}{\left(x^2+4\right)^2}dx$, we find the derivative of $x^2 + 4$ which is 2x. We write $x + 1 = \frac{1}{2}(2x) + 1$. Then,

$$\int \frac{(x+1)}{\left(x^2+4\right)^2}dx = \frac{1}{2}\int \frac{2x}{\left(x^2+4\right)^2}dx + \int \frac{dx}{\left(x^2+4\right)^2}.$$

To evaluate the first integral, we put $u = x^2 + 4$ to get $du = 2x$. The first integral becomes

$$\int \frac{du}{u^2} = u^2 + C = -\left(x^2 + 4\right) + C.$$

For the second integral, we put $x = 2\sec\theta$. Then, $dx = 2\sec\theta\tan\theta$ and $x^2 + 4 = 4\sec^2\theta + 4 = 4\tan^2\theta$.

Therefore,

$$\int \frac{dx}{\left(x^2+4\right)^2}dx = \int \frac{2\sec\theta\tan\theta}{16\tan^4\theta}d\theta = \frac{1}{8}\int \cos^4\theta\sin^{-5}\theta\, d\theta.$$

Recall the reduction formula

$$I_{m,n} = \frac{\cos^{n-1}x\sin^{m+1}x}{m+n} + \frac{n-1}{m+n}I_{m,n-2}$$

where $I_{m,n} = \int \cos^m x\sin^n x\,dx$. Applying this twice, we get

$$\begin{aligned}\int \cos^4\theta\ \sin^{-5}\theta\, d\theta &= -\cos^3\theta\sin^{-4}\theta - 2\int \cos^2\theta\sin^{-5}\theta\, d\theta\\ &= -\cos^3\theta\sin^{-4}\theta\\ &\quad -2\left(-\frac{\cos\theta\sin^{-4}\theta}{3} - \frac{1}{3}\int \sin^{-5}\theta\, d\theta\right)\end{aligned}$$

We can now use the reduction formula for $\int \operatorname{cosec}^n\theta\, d\theta$ in E9) of Unit 19 to complete the computation of the integral.

E7) The points of intersection are given by $\frac{x^2}{4} = x + 3$ or $x^2 - 4x - 12 = 0$.

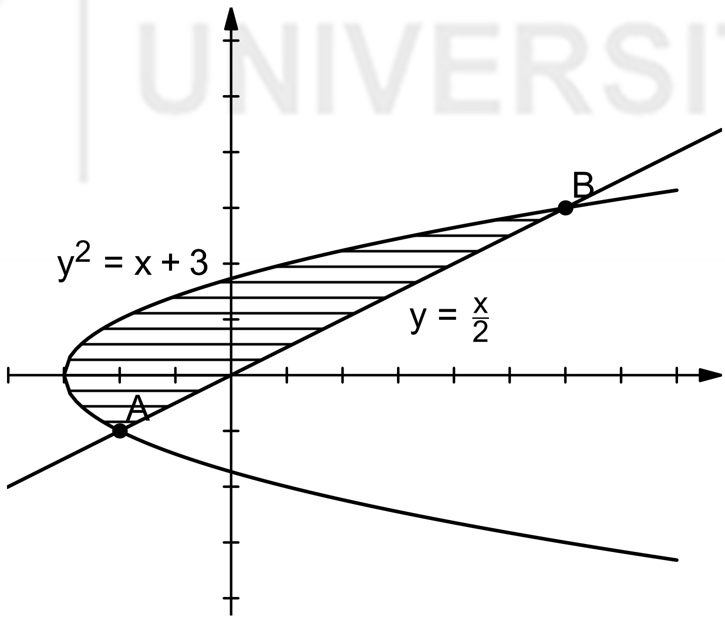


Fig. 3: Area bounded by $y^2 = x + 3$ and $y = \frac{x}{2}$.

Factorising, we get $(x+2)(x-6) = 0$. So, the points of intersection are $(-2,-1)$ and $(6,3)$. We choose $f_1(x) = \sqrt{x+3}$ and

$$f_2(x) = \begin{cases} -\sqrt{x+3} & \text{if } -3 \le x \le -2 \\ \frac{x}{2} & \text{if } -2 \le x \le 6 \end{cases}$$

and

$$f_1(x) - f_2(x) = \begin{cases} 2\sqrt{x+3} & \text{if } -3 \le x \le 2 \\ \sqrt{x+3} & \text{if } -2 \le x \le 6 \end{cases}$$

$f_1(x) - f_2(x) = 2\sqrt{x+3} \ge 0$ if $-3 \le x \le -2$. If $-2 \le x \le 6$, $f_1(x) - f_2(x) = \sqrt{x+3} - \frac{x}{2}$. We have

$$\left(x + 3 - \frac{x^2}{4}\right) = \frac{1}{4}\left(12 + 4x - x^2\right) = -\frac{1}{4}(x-6)(x+2).$$

Since $x \ge -2$, $x + 2 \ge 0$. Since $x \le 6$, $(x-6) \le 0$. Therefore, $(x-6)(x+2) \le 0$ and $-\frac{1}{4}(x-6)(x+2) \ge 0$. Therefore,

$$\left(\sqrt{x+3} + \frac{x}{2}\right)\left(\sqrt{x+3} - \frac{x}{2}\right) = -\frac{1}{4}(x-6)(x+2) \ge 0.$$

Now, $x \ge -2$, $x + 3 \ge 1$, so $\sqrt{x+3} \ge 1$, $\frac{x}{2} \ge -1$. Therefore, $\sqrt{x+3} + \frac{x}{2} \ge 0$. Since the product $\left(\sqrt{x+3} + \frac{x}{2}\right)\left(\sqrt{x+3} - \frac{x}{2}\right)$ is non-negative and one of the factors, $\left(\sqrt{x+3} + \frac{x}{2}\right)$ is non-negative, the other factor $\left(\sqrt{x+3} - \frac{x}{2}\right)$ is also non-negative, i.e. $\left(\sqrt{x+3} - \frac{x}{2}\right) \ge 0$. To find the required area, we have to evaluate the integral

$$\int_{-3}^{6} (f_1(x) - f_2(x))\, dx = 2\int_{-3}^{6} \sqrt{x+3}dx + \int_{-2}^{6} \sqrt{x+3} - \frac{1}{2}\int_{-2}^{6} x\, dx$$

and this is an easy integral to evaluate.

E8) (See Fig. 4) When $0 \le \theta \le 2\pi$, $\sin\frac{\theta}{2} \ge 0$, so, $(1 + 2\sin\theta/2) \ge 1$. So the outer boundary is traced out when $0 \le \theta \le 2\pi$. When $2\pi \le \theta \le 4\pi$, $\sin\frac{\theta}{2} \le 0$ and $(1 + \sin\theta) \le 1$. So, the two inner loops are traced out for $2\pi \le \theta \le 4\pi$.

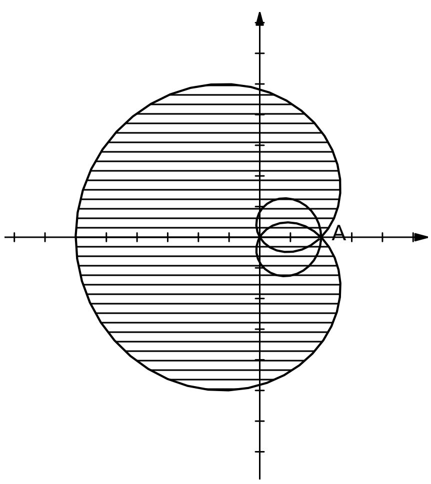


Fig. 4: Area bounded by Nephroid of Freeth

Therefore the area of the outer boundary is given by

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{a^2}{2}\int_0^{2\pi}\left(1 + 2\sin\frac{\theta}{2}\right)^2 d\theta = \frac{a^2}{2}\int_0^{2\pi}\left(1 + 4\sin^2\frac{\theta}{2} + \sin\frac{\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} d\theta + a^2\int_0^{2\pi}(1 - \cos\theta)\, d\theta + 2a^2\int_0^{2\pi}\sin\frac{\theta}{2}\, d\theta$$

E9) We have $\sin(\pi - x) = \sin x$. By the second property, we have

$$\int_0^{\pi}\sin^n x dx = \int_0^{\pi/2}\sin^n x\, dx + \int_0^{\pi/2}\sin(\pi - x) dx = 2\int_0^{\pi/2}\sin x dx$$

Similarly, since $\cos(\pi - x) = -\cos x$, so $\cos^{2n+1}(\pi - x) = -\cos x$.

$$\int_0^{\pi}\cos^{2n+1} x\, dx = \int_0^{\pi/2}\cos^{2n+1} x\, dx + \int_0^{\pi/2}\cos^{2n+1}(\pi - x)\, dx$$

$$= \int_0^{\pi/2}\cos^{2n+1} x\, dx - \int_0^{\pi/2}\cos^{2n+1} x\, dx = 0$$